Hw 5 2020/04/16

2.10 (2)
$$\begin{cases} u_t = a^2 u_{xx} \\ u(t,0) = 0, \; u_x(t,l) = -\dfrac{q}{k} \\ u(0,x) = u_0 \end{cases}$$

令 $u(t,x) = v(t,x) - \dfrac{q}{k}x$

则 $v(t,x)$ 满足

$$\begin{cases} v_t = a^2 v_{xx} \\ v(t,0) = v_x(t,l) = 0 \\ v(0,x) = u_0 + \dfrac{q}{k}x \end{cases}$$

令 $v(t,x) = T(t)X(x)$ 有

$$\frac{1}{a^2}\frac{T'}{T} = \frac{X''}{X} = -\lambda$$

解固有值问题

$$\begin{cases} X'' + \lambda X = 0 \\ X(0) = X'(l) = 0 \end{cases}$$

$$\lambda = k^2 > 0 \Rightarrow X(x) = A\cos kx + B\sin kx$$

$$X(0) = A = 0$$

$$X'(l) = kB\cos kl = 0 \Rightarrow k = \frac{2n+1}{2l}\pi \ , \ n = 0,1,2,\cdots$$

$$X_n(x) = \sin\frac{2n+1}{2l}\pi x \ , \ \lambda_n = \left(\frac{2n+1}{2l}\pi\right)^2 , \ n = 0,1,2,\cdots$$

解关于t的方程 $T' + a^2\lambda T = 0$

$$\Rightarrow T_n(t) = e^{-a^2\lambda_n t}$$

$$\Rightarrow u(t,x) = \sum_{n=0}^{\infty} C_n e^{-a^2\lambda_n t} \cdot \sin\frac{2n+1}{2l}\pi x$$

代入初始条件得

$$u(0,x) = \sum_{n=0}^{\infty} C_n \sin\frac{2n+1}{2l}\pi x = u_0 + \frac{q}{k}x$$

$$\left\|\sin\frac{2n+1}{2l}x\right\|^2 = \int_0^l \sin^2\frac{2n+1}{2l}\pi x\,dx = \int_0^l \frac{1-\cos\frac{2n+1}{l}\pi x}{2}dx = \frac{l}{2}$$

$$C_n = \frac{2}{l}\int_0^l \left(u_0 + \frac{q}{k}x\right)\sin\frac{2n+1}{2l}\pi x\,dx$$

其中 $\int_0^l \sin\frac{2n+1}{2l}\pi x\,dx = -\frac{2l}{(2n+1)\pi}\cos\frac{2n+1}{2l}\pi x\Big|_0^l = \frac{2l}{(2n+1)\pi}$

$$\int_0^l x\sin\frac{2n+1}{2l}\pi x\,dx = -\frac{2l}{(2n+1)\pi}\int_0^l x\,d\cos\frac{2n+1}{2l}\pi x$$

$$= -\frac{2l}{(2n+1)\pi}\,x\cos\frac{2n+1}{2l}\pi x\Big|_0^l + \frac{2l}{(2n+1)\pi}\int_0^l \cos\frac{2n+1}{2l}\pi x\,dx$$

$$= \frac{4l^2}{(2n+1)^2\pi^2}\sin\frac{2n+1}{2l}\pi x\Big|_0^l = \frac{4l^2}{(2n+1)^2\pi^2}(-1)^n$$

$$\Rightarrow C_n = \frac{2u_0}{l}\cdot\frac{2l}{(2n+1)\pi} + \frac{2q}{kl}\cdot\frac{4l^2}{(2n+1)^2\pi^2}(-1)^n$$

$$= \frac{4u_0}{(2n+1)\pi} + \frac{8ql}{k(2n+1)^2\pi^2}(-1)^n$$

$$\Rightarrow u(t,x) = v(t,x) - \frac{q}{k}x$$

$$= -\frac{q}{k}x + \sum_{n=0}^{\infty} C_n e^{-\left(\frac{(2n+1)\pi a}{2l}\right)^2 t}\sin\frac{2n+1}{2l}\pi x$$

$$\lim_{t\to\infty} u(t,x) = -\frac{q}{k}x$$

(5) $\begin{cases} u_{tt} = u_{xx} + g \\ u(t,0) = 0, \ u_x(t,l) = E \\ u(0,x) = Ex, \ u_t(0,x) = 0 \end{cases}$

令 $u(t,x) = v(t,x) + Ex$

则 $v(t,x)$ 满足：

$$\begin{cases} v_{tt} = v_{xx} + g \\ v(t,0) = v_x(t,l) = 0 \\ v(0,x) = v_t(0,x) = 0 \end{cases}$$

令 $v(t,x) = T(t)X(x)$ 有

$$\frac{T''}{T} = \frac{X''}{X} = -\lambda$$

解固有值问题

$$\begin{cases} X'' + \lambda X = 0 \\ X(0) = X'(l) = 0 \end{cases}$$

$\lambda = k^2 > 0 \Rightarrow X(x) = A\cos kx + B\sin kx$

$$X(0)=A=0$$

$$X'(l)=kB\cos kl=0 \Rightarrow k=\frac{2n+1}{2l}\pi \quad , n=0,1,2,\cdots$$

$$X_n(x)=\sin\frac{2n+1}{2l}\pi x \quad , \lambda_n=\left(\frac{2n+1}{2l}\pi\right)^2 , n=0,1,2,\cdots$$

则 $v(t,x)=\sum_{n=0}^{\infty} T_n(t)\sin\frac{2n+1}{2l}\pi x$

令 $g=\sum_{n=0}^{\infty} g_n \sin\frac{2n+1}{2l}\pi x$

则 $g_n=\frac{2}{l}\int_0^l g\sin\frac{2n+1}{2l}\pi x\,dx=\frac{4}{(2n+1)\pi}$

代入原问题中得

$$\begin{cases} \sum_{n=0}^{\infty} T_n''(t)\sin\frac{2n+1}{2l}\pi x+\sum_{n=0}^{\infty}\left(\frac{2n+1}{2l}\pi\right)^2 T_n(t)\sin\frac{2n+1}{2l}\pi x \\ =\sum_{n=0}^{\infty} g_n\sin\frac{2n+1}{2l}\pi x \\ \sum_{n=0}^{\infty} T_n'(t)\sin\frac{2n+1}{2l}\pi x=\sum_{n=0}^{\infty} T_n(t)\sin\frac{2n+1}{2l}\pi x=0 \end{cases}$$

对比系数得

$$\begin{cases} T_n''(t) + \left(\frac{2n+1}{2l}\pi\right)^2 T_n(t) = g_n \\ T_n'(0) = T_n(0) = 0 \end{cases}$$

解得 $T_n(t) = C_n \cos\frac{2n+1}{2l}\pi t + D_n \sin\frac{2n+1}{2l}\pi t + \frac{4l^2 g_n}{(2n+1)^2\pi^2}$

$$T_n(0) = C_n + \frac{4l^2 g_n}{(2n+1)^2\pi^2} = 0 \Rightarrow C_n = -\frac{4l^2 g_n}{(2n+1)^2\pi^2}$$

$$T_n'(0) = \frac{2n+1}{2l}\pi D_n = 0 \Rightarrow D_n = 0$$

$$\Rightarrow v(t,x) = \sum_{n=0}^{\infty} \frac{4l^2 g_n}{(2n+1)^2\pi^2}\left(1-\cos\frac{2n+1}{2l}\pi t\right)\sin\frac{2n+1}{2l}\pi x$$

$$= \frac{16gl^2}{\pi^3}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^3}\left(1-\cos\frac{2n+1}{2l}\pi t\right)\sin\frac{2n+1}{2l}\pi x$$

$$u(t,x) = v(t,x) + Ex$$

$$= Ex + \frac{16gl^2}{\pi^3}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^3}\left(1-\cos\frac{2n+1}{2l}\pi t\right)\sin\frac{2n+1}{2l}\pi x$$

2.11(2) $\begin{cases} \Delta_2 u = A, & a<r<b \\ u(a,\theta) = u_1, & \frac{\partial u(b,\theta)}{\partial n} = u_2 \end{cases}$

观察知 $u=u(r)$（边界条件不含 θ）

$$\Rightarrow \quad u''+\frac{1}{r}u'=A$$

$$\Rightarrow \quad (ru')'=Ar$$

$$\Rightarrow \quad ru'=\frac{1}{2}Ar^2+C_1$$

$$\Rightarrow \quad u'=\frac{1}{2}Ar+\frac{C_1}{r}$$

$$\Rightarrow \quad u=\frac{1}{4}Ar^2+C_1\ln r+C_2$$

代入边界条件：

$$\begin{cases} u(a)=\frac{1}{4}Aa^2+C_1\ln a+C_2=u_1 \\ u'(b)=\frac{1}{2}Ab+\frac{C_1}{b}=u_2 \end{cases}$$

$$\Rightarrow C_1=bu_2-\frac{1}{2}Ab^2,\ C_2=u_1-\frac{1}{4}Aa^2-C_1\ln a$$

$$\Rightarrow u(r)=u_1+\frac{1}{4}A(r^2-a^2)+b(u_2-\frac{1}{2}Ab)\ln\frac{r}{a}$$