Hw 6 2020/04/23

3.1 柱坐标系下，Laplace 方程写为

$$\Delta_3 u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

令 $u(r,\theta,z) = R(r)\Theta(\theta)Z(z)$ 有

$$\frac{R''(r)}{R(r)} + \frac{1}{r}\frac{R'(r)}{R(r)} + \frac{1}{r^2}\frac{\Theta''(\theta)}{\Theta(\theta)} + \frac{Z''(z)}{Z(z)} = 0$$

分离变量：

$$Z''(z) - \lambda Z(z) = 0$$

$$\frac{R''(r)}{R(r)} + \frac{1}{r}\frac{R'(r)}{R(r)} + \frac{1}{r^2}\frac{\Theta''(\theta)}{\Theta(\theta)} = -\frac{Z''(z)}{Z(z)} = -\lambda$$

$$\Rightarrow \frac{R''(r)}{R(r)} + \frac{1}{r}\frac{R'(r)}{R(r)} + \frac{1}{r^2}\frac{\Theta''(\theta)}{\Theta(\theta)} = -\lambda \qquad \Theta''(\theta) + m\Theta(\theta) = 0$$

$$\Rightarrow r^2\frac{R''(r)}{R(r)} + r\frac{R'(r)}{R(r)} + \lambda r^2 = -\frac{\Theta''(\theta)}{\Theta(\theta)} = m$$

$$r^2 R''(r) + rR'(r) + (\lambda r^2 - m)R(r) = 0$$

$$\Rightarrow \begin{cases} Z''(z) - \lambda Z(z) = 0 \\ \Theta''(\theta) + m\Theta(\theta) = 0 \\ r^2 R''(r) + rR'(r) + (\lambda r^2 - m)R(r) = 0 \end{cases}$$

3.2(2) $\frac{d}{dx}[xJ_1(ax)] = J_1(ax) + axJ_1'(ax) = axJ_0(ax)$

3.6(1) 利用 $J_0' = -J_1$ 以及

$$xJ_\nu' - \nu J_\nu = -xJ_{\nu+1} \quad (\nu=1)$$

$$\Rightarrow xJ_1' - J_1 = -xJ_2$$

$$\Rightarrow J_2(x) = \frac{1}{x}J_1(x) - J_1'(x)$$

$$= J_0''(x) - \frac{1}{x}J_0'(x)$$

3.9 $\int J_3(x)\,dx = \int \frac{4}{x}J_2(x) - J_1(x)\,dx$

$$= \int -4\,d\left(\frac{1}{x}J_1(x)\right) + J_0(x)$$

$$= J_0(x) - \frac{4}{x}J_1(x) + C$$

3.12

$$f(x) = \sum_{n=1}^{\infty} C_n J_0(\omega_n x) = \begin{cases} 1, & 0<x<1 \\ \frac{1}{2}, & x=1 \\ 0, & 1<x<2 \end{cases}$$

令 $\omega_n' = 2\omega_n$，则 $J_0(\omega_n') = 0$

即 ω_n' 为 $J_0(x) = 0$ 的正实根，$J_0(\omega_n x) = J_0(\omega_n' \cdot \frac{x}{2})$

令

$$f_0(x) = \sum_{n=1}^{\infty} C_n' J_0(\omega_n' x) = \begin{cases} 1, & 0 < x < \frac{1}{2} \\ \frac{1}{2}, & x = \frac{1}{2} \\ 0, & \frac{1}{2} < x < 1 \end{cases}$$

模平方 $N_{0ln}^2 = \frac{1}{2} J_1^2(\omega_n')$

$$\int_0^1 x f_0(x) J_0(\omega_n' x)\, dx = \int_0^{\frac{1}{2}} x J_0(\omega_n' x)\, dx$$

$$= \frac{1}{\omega_n'^2} \int_0^{\frac{\omega_n'}{2}} x J_0(x)\, dx = \frac{1}{\omega_n'^2} \cdot x J_1(x) \Big|_0^{\frac{\omega_n'}{2}}$$

$$= \frac{1}{\omega_n'^2} \cdot \frac{\omega_n'}{2} \cdot J_1\left(\frac{\omega_n'}{2}\right) = \frac{1}{2\omega_n'} \cdot J_1\left(\frac{\omega_n'}{2}\right)$$

$$C_n' = \frac{1}{N_{0ln}^2} \int_0^1 x f_0(x) J_0(\omega_n' x)\, dx$$

$$= \frac{2}{J_1(\omega_n')^2} \cdot \frac{1}{2\omega_n'} \cdot J_1\left(\frac{\omega_n'}{2}\right) = \frac{J_1\left(\frac{\omega_n'}{2}\right)}{\omega_n' J_1^2(\omega_n')}$$

$$\Rightarrow f_0(x) = \sum_{n=1}^{\infty} C_n J_0(\omega_n' x) = \sum_{n=1}^{\infty} \frac{J_1(\frac{\omega_n'}{2})}{\omega_n' J_1^2(\omega_n')} J_0(\omega_n' x)$$

$$\Rightarrow f(x) = f_0(\frac{x}{2}) = \sum_{n=1}^{\infty} \frac{J_1(\frac{\omega_n'}{2})}{\omega_n' J_1^2(\omega_n')} J_0(\omega_n' \frac{x}{2})$$

$$= \sum_{n=1}^{\infty} \frac{J_1(\omega_n)}{2\omega_n J_1^2(2\omega_n)} J_0(\omega_n x)$$

3.13 $f(x) = \sum_{n=1}^{\infty} C_n J_1(\omega_n x) = x \quad , \quad 0<x<1$

模平方 $N_{1n}^2 = \frac{1}{2} \cdot J_2^2(\omega_n)$

$$\int_0^1 x f(x) J_1(\omega_n x) = \int_0^1 x^2 J_1(\omega_n x) dx$$

$$= \frac{1}{\omega_n^3} \int_0^{\omega_n} x^2 J_1(x) dx = \frac{1}{\omega_n^3} x^2 J_2(x) \Big|_0^{\omega_n}$$

$$= \frac{1}{\omega_n} J_2(\omega_n)$$

$$C_n = \frac{1}{N_{1n}^2} \cdot \int_0^1 x f(x) J_1(\omega_n x) dx = \frac{2}{J_2^2(\omega_n)} \cdot \frac{1}{\omega_n} J_2(\omega_n) = \frac{2}{\omega_n J_2(\omega_n)}$$

$$\Rightarrow f(x) = \sum_{n=1}^{\infty} \frac{2}{\omega_n J_2(\omega_n)} J_1(\omega_n x)$$

3.16 $$\begin{cases} \dfrac{\partial u}{\partial t} = a^2 \Delta_3 u, \quad t>0,\ 0<r<R \\ u(t,R) = u_0 \\ u(0,r) = 0 \end{cases}$$

边界条件齐次化：令 $v = u - u_0$，则

$$\begin{cases} \dfrac{\partial v}{\partial t} = a^2 \Delta_3 v = a^2 \cdot \dfrac{1}{r} \dfrac{\partial}{\partial r}\left(\dfrac{\partial v}{\partial r}\right), \quad t>0, 0<r<R \\ v(t,R) = 0 \\ v(0,r) = -u_0 \end{cases}$$

令 $v(t,r) = T(t)R(r)$，则

$$\frac{1}{a^2} \cdot \frac{T'(t)}{T(t)} = \frac{(rR'(r))'}{rR(r)} = -\lambda$$

固有值问题

$$\begin{cases} (rR'(r))' + \lambda r R(r) = 0 \\ |R(0)| < +\infty, \quad R(R) = 0 \end{cases}$$

$\Rightarrow$ 固有函数 $R_n(r)=J_0(\omega_{1n}r)$，固有值 $\lambda_n=\omega_{1n}^2$

ω_{1n} 为 $J_0(\omega R)=0$ 的第n个正根

解关于t的方程 $T'(t)+a^2\lambda T(t)=0$

$$\Rightarrow\quad T_n(t)=e^{-a^2\lambda_n t}$$

$$\Rightarrow\quad v(t,r)=\sum_{n=1}^{\infty}T_n(t)R_n(r)=\sum_{n=1}^{\infty}C_n e^{-a^2\lambda_n t}J_0(\omega_{1n}r)$$

代入初始条件：

$$v(0,r)=\sum_{n=1}^{\infty}C_nJ_0(\omega_{1n}r)=-u_0$$

$$N_{01n}^2=\frac{R^2}{2}J_1^2(\omega_{1n}R)$$

$$C_n=\frac{1}{N_{01n}^2}\int_0^R -u_0 r J_0(\omega_{1n}r)dr=\frac{-2u_0}{R^2J_1^2(\omega_{1n}R)}\int_0^R rJ_0(\omega_{1n}r)dr$$

$$=\frac{-2u_0}{\omega_{1n}^2R^2J_1^2(\omega_{1n}R)}\int_0^{\omega_{1n}R}xJ_0(x)\,dx$$

$$=\frac{-2u_0}{\omega_{1n}^2R^2J_1^2(\omega_{1n}R)}\cdot\omega_{1n}RJ_1(\omega_{1n}R)=\frac{-2u_0}{\omega_{1n}RJ_1(\omega_{1n}R)}$$

$$\Rightarrow u(t,r) = v(t,r) + u_0$$

$$= u_0 - \frac{2u_0}{R}\sum_{n=1}^{\infty}\frac{1}{\omega_{1n}J_1(\omega_{1n}R)} e^{-a^2\omega_{1n}^2 t}\cdot J_0(\omega_{1n}r)$$

3.18 (1) $\begin{cases} u_{rr} + \frac{1}{r}u_r + u_{zz} = 0, & 0<r<a,\ 0<z<l \\ u(a,z) = 0 \\ u(r,0) = 0,\ u(r,l) = T_0 \end{cases}$

齐次边界条件，直接分离变量. 令 $u(r,z) = R(r)Z(z)$

$$\Rightarrow \frac{R''(r)}{R(r)} + \frac{1}{r}\frac{R'(r)}{R(r)} = -\frac{Z''(z)}{Z(z)} = -\lambda$$

解固有值问题

$$\begin{cases} rR'' + R' + \lambda rR = (rR')' + \lambda rR = 0 \\ |R(0)| < +\infty,\ R(a) = 0 \end{cases}$$

I类边界条件 $\Rightarrow$ 固有值 $\lambda_n = \omega_{1n}^2$,

固有函数 $R_n(r) = J_0(\omega_{1n}r)$,

ω_{1n} 为 $J_0(\omega a) = 0$ 第 n 个正根

解关于z的方程 $Z''(z)-\lambda Z(z)=0$

$$\Rightarrow Z_n(z)=A_n \operatorname{ch}\omega_{1n}z+B_n\operatorname{sh}\omega_{1n}z$$

$$\Rightarrow u(r,z)=\sum_{n=1}^{\infty} Z_n(z)R_n(r)$$

$$=\sum_{n=1}^{\infty}(A_n\operatorname{ch}\omega_{1n}z+B_n\operatorname{sh}\omega_{1n}z)J_0(\omega_{1n}r)$$

代入边界条件，

$$\begin{cases} u(r,0)=\sum_{n=1}^{\infty}A_nJ_0(\omega_{1n}r)=0 \Rightarrow A_n=0 \\ u(r,l)=\sum_{n=1}^{\infty}B_n\operatorname{sh}\omega_{1n}l\, J_0(\omega_{1n}r)=T_0 \end{cases}$$

$$N_{01n}^2=\frac{a^2}{2}J_1^2(\omega_{1n}a)$$

$$\int_0^a rT_0J_0(\omega_{1n}r)\,dr=\frac{T_0}{\omega_{1n}^2}\cdot\omega_{1n}a\cdot J_1(\omega_{1n}a)=\frac{T_0a}{\omega_{1n}}\cdot J_1(\omega_{1n}a)$$

$$B_n=\frac{1}{\operatorname{sh}\omega_{1n}l}\cdot\frac{1}{N_{01n}^2}\int_0^a rT_0J_0(\omega_{1n}r)\,dr$$

$$=\frac{1}{\operatorname{sh}\omega_{1n}l}\cdot\frac{2}{J_1^2(\omega_{1n}a)\cdot a^2}\cdot\frac{T_0a}{\omega_{1n}}J_1(\omega_{1n}a)$$

$$=\frac{2T_0}{\omega_{1n}\operatorname{sh}\omega_{1n}l\;J_1(\omega_{1n}a)\cdot a}$$

$$\Rightarrow u(r,z)=\frac{2T_0}{a}\sum_{n=1}^{\infty}\frac{1}{\omega_{1n}J_1(\omega_{1n}a)}\cdot\frac{\operatorname{sh}\omega_{1n}z}{\operatorname{sh}\omega_{1n}l}\cdot J_0(\omega_{1n}r)$$

(2) $$\begin{cases} u_{tt}+2hu_t=a^2\left(u_{rr}+\frac{1}{r}u_r\right),\ h\ll 1,\ t>0,\ 0<r<l \\ |u(t,0)|<+\infty,\quad u_r(t,l)=0 \\ u(0,r)=\varphi(r),\quad u_t(0,r)=0 \end{cases}$$

令 $u(t,r)=T(t)R(r)$，分离变量得

$$\frac{1}{a^2}\left(\frac{T''(t)}{T(t)}+2h\frac{T'(t)}{T(t)}\right)=\frac{R''(r)}{R(r)}+\frac{1}{r}\frac{R'(r)}{R(r)}=-\lambda$$

解固有值问题

$$\begin{cases} rR''+R'+\lambda rR=(rR')'+\lambda rR=0 \\ |R(0)|<+\infty,\quad R'(l)=0 \end{cases}$$

II类边界条件 $\Rightarrow$ 固有值 $\lambda_n=\omega_{2n}^2$，$\lambda_0=0$

固有函数 $R_n(r)=J_0(\omega_{2n}r)$，$R_0(r)=1$

ω_{2n} 为 $J_0'(\omega l)=0$ 第 n 个正根

解关于t的方程 $T''(t)+2hT'(t)+a^2\lambda T(t)=0$

特征方程：$x^2+2hx+a^2\lambda=0 \quad (\lambda>0)$

$$\Rightarrow x_n=-h\pm i\sqrt{a^2\lambda_n-h^2}=-h\pm i\mu_n$$

$$\Rightarrow T_n(t)=e^{-ht}(A_n\cos\mu_n t+B_n\sin\mu_n t)$$

$\lambda=0$时：$T''(t)+2hT'(t)=0$

$$T'(t)+2hT(t)=C_1$$

$$(Te^{2ht})'=C_1e^{2ht}$$

$$\Rightarrow T_0(t)=A_0e^{-2ht}+B_0$$

$$\Rightarrow u(t,r)=\sum_{n=0}^{\infty}T_n(t)R_n(r)$$

$$=(A_0e^{-2ht}+B_0)+\sum_{n=1}^{\infty}e^{-ht}(A_n\cos\mu_n t+B_n\sin\mu_n t)J_0(\omega_{2n}r)$$

代入初始条件，

$$\begin{cases} u(0,r)=A_0+B_0+\sum_{n=1}^{\infty}A_nJ_0(\omega_{2n}r)=\varphi(r) \\ u_t(0,r)=-2hA_0+\sum_{n=1}^{\infty}(-hA_n+\mu_nB_n)J_0(\omega_{2n}r)=0 \end{cases}$$

$$N_{02n}^2 = \frac{l^2}{2} J_0^2(\omega_{2n} l) \quad , \quad N_{020}^2 = \int_0^l r\,dr = \frac{l^2}{2}$$

$$A_n = \frac{1}{N_{02n}^2} \int_0^l r\varphi(r) J_0(\omega_{2n} r)\,dr \quad , \quad n \geq 1$$

$$B_n = \frac{hA_n}{\mu_n} \quad , \quad A_0 = 0 \; , \quad B_0 = \frac{2}{l^2} \int_0^l r\varphi(r)\,dr$$

3.19
$$\begin{cases} u_{rr} + \frac{1}{r} u_r + u_{zz} = 0 \\ |u(0,z)| < +\infty \; , \quad u_r(R,z) + k\,u(R,z) = 0 \\ u(r,0) = 0 , \quad u(r,h) = f(r) \end{cases}$$

令 $u(r,z) = R(r) Z(z)$，有

$$\frac{R''(r)}{R(r)} + \frac{1}{r} \frac{R'(r)}{R(r)} = - \frac{Z''(z)}{Z(z)} = -\lambda$$

解固有值问题

$$\begin{cases} rR'' + R' + \lambda rR = (rR')' + \lambda rR = 0 \\ |R(0)| < +\infty \; , \quad R'(R) + kR(R) = 0 \end{cases}$$

Ⅲ类边界条件 $\Rightarrow$ 固有值 $\lambda_n = \omega_{3n}^2$，

固有函数 $R_n(r) = J_0(\omega_{3n} r)$，

ω_{3n} 为 $kJ_0(\omega R)+\omega J_0'(\omega R)=0$ 第 n 个正根

解关于 z 的方程 $Z''-\lambda Z=0$

$$\Rightarrow \quad Z_n(z)=A_n \operatorname{ch}\omega_{3n}z+B_n\operatorname{sh}\omega_{3n}z$$

$$\Rightarrow u(r,z)=\sum_{n=1}^{\infty}Z_n(z)R_n(r)$$

$$=\sum_{n=1}^{\infty}(A_n\operatorname{ch}\omega_{3n}z+B_n\operatorname{sh}\omega_{3n}z)J_0(\omega_{3n}r)$$

代入初始条件：

$$\begin{cases} u(r,0)=\sum_{n=1}^{\infty}A_nJ_0(\omega_{3n}r)=0 \quad \Rightarrow A_n=0 \\ u(r,h)=\sum_{n=1}^{\infty}B_n\operatorname{sh}\omega_{3n}h\,J_0(\omega_{3n}r)=f(r) \end{cases}$$

$$N_{03n}^2=\frac{R^2}{2}\left(1+\frac{k^2}{\omega_{3n}^2}\right)J_0^2(\omega_{3n}R)$$

$$B_n=\frac{1}{\operatorname{sh}\omega_{3n}h}\cdot\frac{1}{N_{03n}^2}\int_0^R rf(r)J_0(\omega_{3n}r)\,dr$$

$$\Rightarrow u(r,z)=\frac{2}{R^2}\sum_{n=1}^{\infty}\frac{\int_0^R rf(r)J_0(\omega_{3n}r)\,dr}{\left(1+\frac{k^2}{\omega_{3n}^2}\right)J_0^2(\omega_{3n}R)}\cdot\frac{\operatorname{sh}\omega_{3n}z}{\operatorname{sh}\omega_{3n}h}\cdot J_0(\omega_{3n}r)$$

3.20 Legendre 多项式

$$P_n(x)=\sum_{k=0}^{[\frac{n}{2}]}\frac{(-1)^k(2n-2k)!}{2^n\cdot k!(n-k)!(n-2k)!}x^{n-2k},\quad n\geq 0$$

$n=0$时，$P_0(x)=1 \Rightarrow P_0(0)=1,\ P_0'(0)=0$

$n=2m,\ m\geq 1$时，

$$P_n(x)=\sum_{k=0}^{m}\frac{(-1)^k(4m-2k)!}{2^{2m}\cdot k!(2m-k)!(2m-2k)!}x^{2m-2k}$$

$$P_n'(x)=\sum_{k=0}^{m}\frac{(-1)^k(4m-2k)!(2m-2k)}{2^{2m}\cdot k!(2m-k)!(2m-2k)!}x^{2m-2k-1}$$

$$P_n(0)=\frac{(-1)^m\cdot(2m)!}{2^{2m}\cdot(m!)^2}=\frac{(-1)^m\cdot(2m-1)!!}{(2m)!!}$$

$$P_n'(0)=0$$

$n=2m+1,\ m\geq 0$时，

注意 $P_n(x)$ 表达式中 x^{n-2k} 指数全为奇数

$\Rightarrow P_n(0)=0$

由递推关系，$P_n'(0)=-(n+1)P_{n+1}(0)=\dfrac{(-1)^m\cdot(2m+1)!!}{(2m)!!}$