定理3.5.2 设a, b, c 是常数，$\mathrm{E}\,X = \mu$, $\mathrm{Var}\,(X) < \infty$, $\mu_j = \mathrm{E}\,X_j$, $\mathrm{Var}\,(X_j) < \infty (1 \leqslant j \leqslant n)$, 则

(1) $\mathrm{Var}\,(a + bX) = b^2 \mathrm{Var}\,(X)$; 方差的线性性质

(2) $\mathrm{Var}\,(X) = \mathrm{E}\,(X - \mu)^2 < \mathrm{E}\,(X - c)^2$, 只要常数 $c \neq \mu$;

(3) $\mathrm{Var}\,(X) = 0$的充分必要条件是 $P(X = \mu) = 1$; 退化分布.

(4) 当 $X_1, X_2, \cdots, X_n$ 相互独立时，

$$\mathrm{Var}\,\Big(\sum_{j=1}^{n} X_j\Big) = \sum_{j=1}^{n} \mathrm{Var}\,(X_j).$$

(4)

$$\mathrm{Var}\Big(\sum_{j=1}^n X_j\Big) = E\Big(\sum_{j=1}^n X_j - E\big(\sum_{j=1}^n X_j\big)\Big)^2 = E\Big(\sum_{j=1}^n X_j - \sum_{j=1}^n EX_j\Big)^2 = E\Big[(X_1+X_2+\cdots+X_n)^2 + (EX_1+\cdots+EX_n)^2 - 2(X_1+X_2+\cdots+X_n)(EX_1+\cdots+EX_n)\Big]$$

$$= \sum_{j=1}^n EX_j^2 + \sum_{j=1}^n (EX_j)^2 + 2\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\big(E(X_iX_j) + EX_i\cdot EX_j\big) - 2\sum_{i=1}^n\sum_{j=1}^n EX_j\cdot EX_i$$

$$= \sum_{j=1}^n EX_j^2 + \sum_{j=1}^n (EX_j)^2 + 4\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} EX_i\cdot EX_j - 4\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} EX_i\cdot EX_j - 2\sum_{i=1}^n (EX_i)^2$$

$$= \sum_{j=1}^n \big[EX_j^2 - (EX_j)^2\big] = \sum_{j=1}^n \mathrm{Var}(X_j)$$

28. 设随机变量 X 服从区间 $(-\frac{\pi}{2}, \frac{\pi}{2})$ 上的均匀分布. 试求期望 $\mathbb{E}[\sin X], \mathbb{E}[\cos X]$ 及 $\mathbb{E}[X\cos X]$.

$$f(x) = \begin{cases} \dfrac{1}{\pi}, & -\dfrac{\pi}{2} < x < \dfrac{\pi}{2} \\ 0, & x \leq -\dfrac{\pi}{2} \text{ 或 } x \geq \dfrac{\pi}{2} \end{cases}$$

$$E(\sin x) = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{\pi}\sin x\,dx = -\frac{1}{\pi}\cos x\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = 0$$

$$E(\cos x) = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{\pi}\cos x\,dx = \frac{1}{\pi}\sin x\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \frac{2}{\pi}$$

$$E(X\cos X) = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x\cos x\,dx = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x\,d\sin x = x\sin x\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} - \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin x\,dx$$

$$= \frac{\pi}{2} - \Big(\frac{\pi}{2}\Big) - \quad -\cos x\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

$$= 0$$

34. 假设有 $n\ (n \geq 3)$ 个不同的盒子与 m 个相同的小球, 每个小球独立地以概率 p_k 落入第 k 个盒子 $(k=1,2,\ldots,n)$. 分别以 $X_1, X_2, \ldots, X_n$ 表示落入各个盒子的球数. 试求

(1) $\mathbb{E}[X_2|X_1=k]$ 和 $\mathrm{Var}(X_2|X_1=k)$.

(2) $\mathbb{E}[X_1+X_2]$ 和 $\mathrm{Var}(X_1+\cdots+X_k)$, $k=1,\ldots,n$.

(1). $P(X_2=i, X_1=k) = C_m^k \cdot p_1^k \cdot C_{m-k}^i p_2^i \cdot (1-p_1-p_2)^{m-k-i}$

$P(X_1=k) = C_m^k p_1^k \cdot (1-p_1)^{m-k}$

$\therefore P(X_2=i | X_1=k) = \dfrac{P(X_2=i, X_1=k)}{P(X_1=k)} = C_{m-k}^i \left(\dfrac{p_2}{1-p_1}\right)^i \left(1-\dfrac{p_2}{1-p_1}\right)^{m-k-i} \quad (i=0,1,2,\cdots,m-k)$

$\therefore X_2 | X_1=k \sim B\left(m-k, \dfrac{p_2}{1-p_1}\right)$

$\therefore E[X_2|X_1=k] = \dfrac{(m-k)p_2}{1-p_1} \qquad Var[X_2|X_1=k] = (m-k)\cdot\dfrac{p_2}{1-p_1}\cdot\left(1-\dfrac{p_2}{1-p_1}\right)$

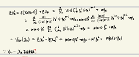
(2) $\because X_1 \sim (m, p_1), X_2 \sim (m, p_2), \cdots, X_k \sim (m, p_k) \quad (k=1,2,\cdots,n)$

$\therefore EX_1 = mp_1, \quad EX_2 = mp_2 \quad \cdots \quad EX_k = mp_k \quad (k=1,2,\cdots,n)$

$\therefore E(X_1+X_2) = EX_1 + EX_2 = m(p_1+p_2)$

令 $Y = X_1 + \cdots + X_k$

则 $Y \sim B(m, p_1+p_2+\cdots+p_k) \quad (k=1,2,\cdots,n)$



$\therefore Var(X_1+\cdots+X_k) = Var(Y) = m(p_1+p_2+\cdots+p_k)\cdot(p_{k+1}+\cdots+p_n)$

$(k=1,2,\cdots,n)$