Hw 9 2020/05/21

5.3 (1) $\begin{cases} u_t = a^2 u_{xx} \quad (0<x<l,\ t>0) \\ u(t,0) = u(t,l) = 0 \\ u(0,x) = \delta(x-\xi),\quad 0<\xi<l \end{cases}$

令 $u(t,x) = T(t)X(x)$，有

$$\frac{1}{a^2}\frac{T'(t)}{T(t)} = \frac{X''(x)}{X(x)} = -\lambda$$

解固有值问题

$$\begin{cases} X'' + \lambda X = 0 \\ X(0) = X(l) = 0 \end{cases}$$

$\lambda = k^2 > 0 \Rightarrow X(x) = A\cos kx + B\sin kx$

$X(0) = A = 0$

$X(l) = B\sin kl = 0 \Rightarrow k_n l = n\pi,\quad n = 1, 2, \cdots$

$k_n = \frac{n\pi}{l}$，$\lambda_n = (\frac{n\pi}{l})^2$，$X_n(x) = \sin\frac{n\pi x}{l}$

解关于t的方程 $T'(t) + \lambda a^2 T(t) = 0$

$\Rightarrow T_n(t) = e^{-a^2\lambda_n t}$

$$\Rightarrow u(t,x)=\sum_{n=1}^{\infty} C_n e^{-a^2\lambda_n t} \sin\frac{n\pi x}{l}$$

代入初始条件：

$$u(0,x)=\sum_{n=1}^{\infty} C_n \sin\frac{n\pi x}{l}=\delta(x-\xi)$$

$$\Rightarrow C_n=\frac{1}{\|\sin\frac{n\pi x}{l}\|^2}\int_0^l \sin\frac{n\pi x}{l}\,\delta(x-\xi)\,dx=\frac{2}{l}\sin\frac{n\pi\xi}{l}$$

$$\Rightarrow u(t,x)=\sum_{n=1}^{\infty}\frac{2}{l}\sin\frac{n\pi\xi}{l}\sin\frac{n\pi x}{l}\,e^{-(\frac{n\pi a}{l})^2 t}$$

(2) $\begin{cases} u_{tt}=a^2u_{xx} & (0<x<l,\ t>0) \\ u_x(t,0)=u_x(t,l)=0 \\ u(0,x)=0,\ u_t(0,x)=\delta(x-\xi),\ 0<\xi<l \end{cases}$

令 $u(t,x)=T(t)X(x)$，有

$$\frac{1}{a^2}\frac{T''(t)}{T(t)}=\frac{X''(x)}{X(x)}=-\lambda$$

解固有值问题

$$\begin{cases} X''+\lambda X=0 \\ X'(0)=X'(l)=0 \end{cases}$$

II类边界条件： $\lambda_0=0$ ， $X_0(x)=1$

$$\lambda_n = k^2 \Rightarrow X_n(x) = A\sin kx + B\cos kx$$

$$X'(0) = kA = 0 \Rightarrow A = 0$$

$$X'(l) = -kB\sin kl = 0 \Rightarrow k_n = \frac{n\pi}{l}$$

$$\Rightarrow \lambda_n = \left(\frac{n\pi}{l}\right)^2, \quad X_n(x) = \cos\frac{n\pi x}{l}, \quad n = 1, 2, 3, \dots$$

解关于t的方程 $T''(t) + a^2\lambda T(t) = 0$

$$\Rightarrow \begin{cases} T_n(t) = A_n\cos\frac{n\pi a t}{l} + B_n\sin\frac{n\pi a t}{l} \\ T_0(t) = A_0 + B_0 t \end{cases}$$

$$\Rightarrow u(t,x) = A_0 + B_0 t + \sum_{n=1}^{\infty}\left(A_n\cos\frac{n\pi a t}{l} + B_n\sin\frac{n\pi a t}{l}\right)\cos\frac{n\pi x}{l}$$

代入初始条件：

$$\begin{cases} u(0,x) = A_0 + \sum_{n=1}^{\infty} A_n\cos\frac{n\pi x}{l} = 0 \Rightarrow A_n = 0, \quad n = 0, 1, 2, \cdots \\ u_t(0,x) = B_0 + \sum_{n=1}^{\infty}\frac{n\pi a}{l} B_n\cos\frac{n\pi x}{l} = \delta(x-3) \end{cases}$$

$$\Rightarrow \quad B_0 = \frac{1}{\|X_0(x)\|^2}\int_0^l \delta(x-3)\,dx = \frac{1}{l}$$

$$B_n = \frac{1}{\|\cos\frac{n\pi x}{l}\|^2}\int_0^l \cos\frac{n\pi x}{l}\,\delta(x-3)\,dx \cdot \frac{l}{n\pi a}$$

$$= \frac{2}{l}\cdot\cos\frac{n\pi 3}{l}\cdot\frac{l}{n\pi a} = \frac{2}{n\pi a}\cos\frac{n\pi 3}{l}$$

$$\Rightarrow u(t,x)=\frac{t}{l}+\sum_{n=1}^{\infty}\frac{2}{n\pi a}\cos\frac{n\pi\xi}{l}\cos\frac{n\pi x}{l}\sin\frac{n\pi a t}{l}$$

5.4 (1) 求解 $u_{xx}+\beta^2 u_{yy}=\delta(x,y)$

作代换 $s=x$，$t=\frac{y}{\beta}$，则　注：$\Delta_{st}=\frac{\partial^2}{\partial s^2}+\frac{\partial^2}{\partial t^2}$

$$u_{xx}+\beta^2 u_{yy}=u_{ss}+u_{tt}=\Delta_{st}u=\delta(s,\beta t)=\frac{1}{\beta}\delta(s,t)$$

由二维 Laplace 方程的基本解：

$$u=\frac{1}{\beta}\cdot\frac{1}{2\pi}\ln\sqrt{s^2+t^2}$$

$$=\frac{1}{\beta}\cdot\frac{1}{4\pi}\ln\left(x^2+\left(\frac{y}{\beta}\right)^2\right)$$

(2) 求解 $\Delta_2\Delta_2 u=0$

由二维 Laplace 方程的基本解：

$$\Delta_2 u=\frac{1}{2\pi}\ln r+C$$

$$\Rightarrow \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)=\frac{1}{2\pi}\ln r+C$$

$$\int r\ln r\,dr=\int\frac{1}{2}\ln r\,dr^2=\frac{1}{2}r^2\ln r-\frac{1}{2}\int r\,dr=\frac{1}{4}(2r^2\ln r-r^2)+C$$

$$\Rightarrow r\frac{\partial u}{\partial r}=\frac{1}{8\pi}\cdot r^2(2\ln r+C)$$ 在计算过程中，我们通过调整任意常数 C，使之只剩下在 $\Delta_2\Delta_2$ 下有奇性的部分。

$$\Rightarrow u = \frac{1}{8\pi} r^2 \ln r$$

5.5 利用 Fourier Transform 求解 $\Delta_3 u + k^2 u = \delta(x,y,z)$

$$\Rightarrow -(\lambda^2+\mu^2+\nu^2)\hat{u} + k^2\hat{u} = 1$$

$$\Rightarrow \hat{u} = \frac{1}{k^2-(\lambda^2+\mu^2+\nu^2)} = \frac{1}{k^2-\rho^2}$$

作逆变换：

$$u = F^{-1}[\hat{u}] = \frac{1}{(2\pi)^3}\iiint_{\mathbb{R}^3} \frac{1}{k^2-\rho^2} e^{i(\lambda x+\mu y+\nu z)} d\lambda d\mu d\nu$$

$$= \frac{1}{(2\pi)^3}\int_0^{2\pi} d\varphi \int_0^{\pi} \sin\theta\, d\theta \int_0^{\infty} \rho^2 \cdot \frac{e^{i\vec{\rho}\cdot\vec{r}}}{k^2-\rho^2} d\rho$$

$$= \frac{1}{(2\pi)^3}\cdot 2\pi \cdot \int_0^{\infty} \frac{\rho^2}{k^2-\rho^2} d\rho \int_0^{\pi} e^{i\rho r\cos\theta} \sin\theta\, d\theta$$

$$= \frac{1}{(2\pi)^3}\cdot 2\pi \cdot \int_0^{\infty} \frac{\rho^2}{k^2-\rho^2} d\rho \int_0^{\pi} e^{i\rho r\cos\theta} \cdot d(i\rho r\cos\theta) \cdot \frac{i}{\rho r}$$

$$= \frac{1}{(2\pi)^3}\cdot 2\pi \cdot \int_0^{\infty} \frac{\rho^2}{k^2-\rho^2} d\rho\, (e^{-i\rho r} - e^{i\rho r}) \cdot \frac{i}{\rho r}$$

$$= \frac{1}{4\pi^2 r}\int_0^{\infty} \frac{\rho}{k^2-\rho^2}(-2i\sin\rho r)\cdot i\, d\rho$$

$$= \frac{1}{2\pi^2 r}\int_0^{\infty} \frac{\rho\sin\rho r}{k^2-\rho^2} d\rho$$

$$= \frac{1}{2\pi^2 r}\int_0^{\infty} \frac{x\sin x}{(kr)^2-x^2} dx = \frac{1}{4\pi r^2}\int_{-\infty}^{+\infty} \frac{x\sin x}{(kr)^2-x^2} dx$$

$$=\frac{1}{4\pi^2 r}\mathrm{Im}\left\{\int_{-\infty}^{\infty}\frac{xe^{ix}}{(kr)^2-x^2}dx\right\}$$

$$=\frac{1}{4\pi^2 r}\mathrm{Im}\left\{\pi i\left(\mathrm{Res}\left[\frac{xe^{ix}}{(kr)^2-x^2},kr\right]+\mathrm{Res}\left[\frac{xe^{ix}}{(kr)^2-x^2},-kr\right]\right)\right\}$$

$$=\frac{1}{4\pi^2 r}\mathrm{Im}\left\{\pi i\left(-\left.\frac{xe^{ix}}{kr+x}\right|_{x=kr}+\left.\frac{xe^{ix}}{kr-x}\right|_{x=-kr}\right)\right\}$$

$$=\frac{1}{4\pi^2 r}\mathrm{Im}\left\{\pi i\left(-\frac{1}{2}e^{ikr}-\frac{1}{2}e^{-ikr}\right)\right\}$$

$$=\frac{1}{4\pi^2 r}\mathrm{Im}\left\{-\pi i\cos kr\right\}=-\frac{\cos kr}{4\pi r}$$

5.6 (1) 四分之一空间

$$\begin{cases}\Delta_3 G=-\delta(M-M_0), & x>0,y>0\\ G|_{\partial v}=0 & (\text{即 } G|_{x=0}=G|_{y=0}=0)\end{cases}$$

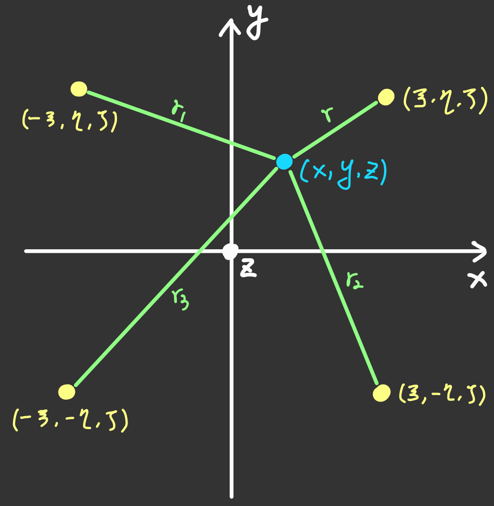
$$G(x,y,z;\xi,\eta,\zeta)=\frac{1}{4\pi}\left(\frac{1}{r}-\frac{1}{r_1}-\frac{1}{r_2}+\frac{1}{r_3}\right)$$

其中：$r=[(x-\xi)^2+(y-\eta)^2+(z-\zeta)^2]^{\frac{1}{2}}$

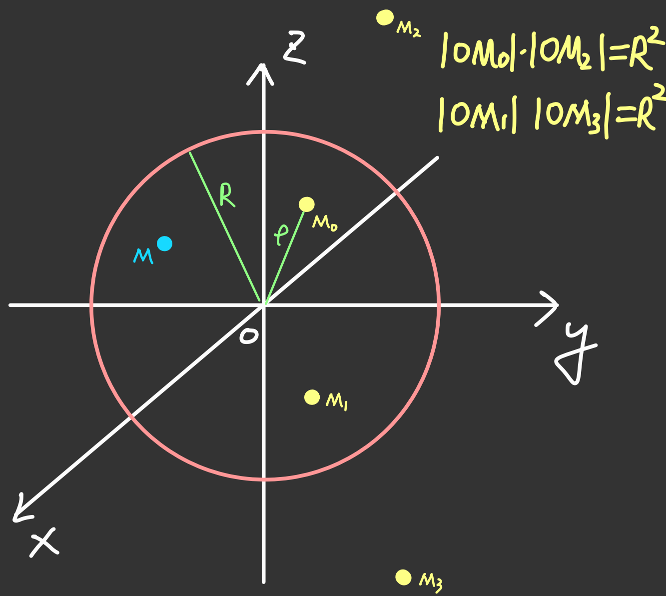
$r_1=[(x+\xi)^2+(y-\eta)^2+(z-\zeta)^2]^{\frac{1}{2}}$

$r_2=[(x-\xi)^2+(y+\eta)^2+(z-\zeta)^2]^{\frac{1}{2}}$

$r_3=[(x+\xi)^2+(y+\eta)^2+(z-\zeta)^2]^{\frac{1}{2}}$



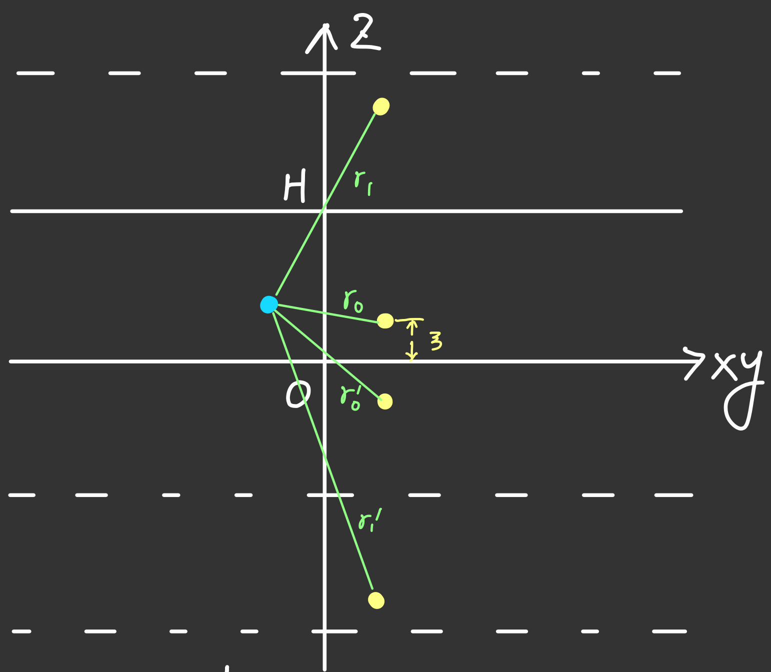
(2) 上半球



$$\begin{cases}\Delta_3 G = -\delta(M-M_0)\\ G|_{x^2+y^2+z^2=R^2}=0,\quad G|_{z=0}=0\end{cases}$$

$$G(M;M_0)=\frac{1}{4\pi}\left(\frac{1}{r(M,M_0)}-\frac{1}{r(M,M_1)}-\frac{R}{\rho}\frac{1}{r(M,M_2)}+\frac{R}{\rho}\frac{1}{r(M,M_3)}\right)$$

(3) 层状空间



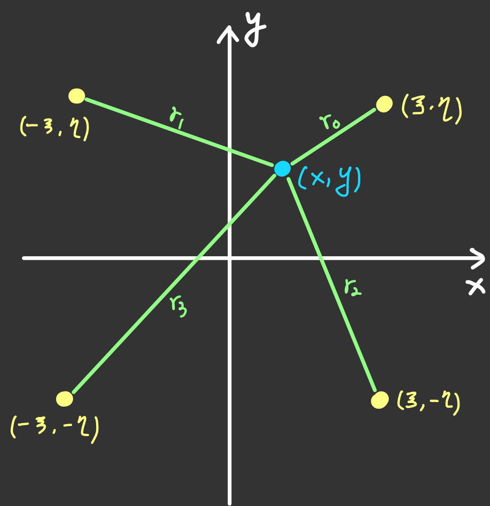
$$\begin{cases}\Delta_3 G = -\delta(M-M_0)\\ G|_{z=0}=G|_{z=H}=0\end{cases}$$

$$G(M;M_0)=\frac{1}{4\pi}\sum_{n=0}^{+\infty}\left(\frac{1}{r_n}-\frac{1}{r_n'}\right)$$

其中 $r_n=\left[(x-\xi)^2+(y-\eta)^2+(z-2nH-\zeta)^2\right]^{\frac{1}{2}}$

$r_n'=\left[(x-\xi)^2+(y-\eta)^2+(z-2nH+\zeta)^2\right]^{\frac{1}{2}}$

5.7 (1) 四分之一平面 $x>0,\ y>0$

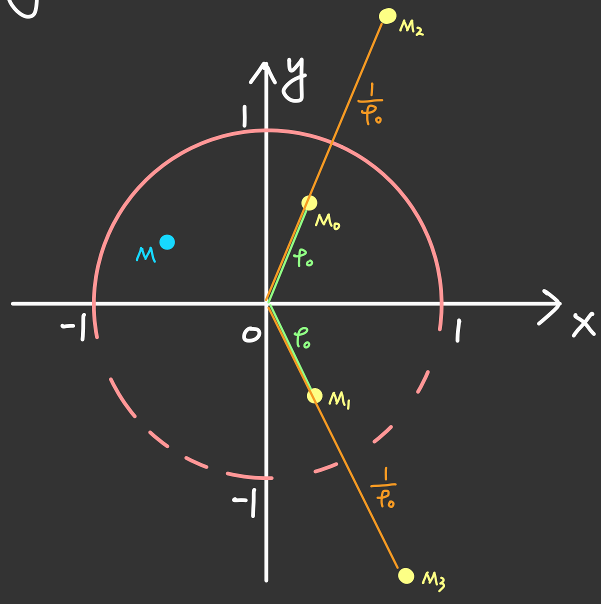


$$\begin{cases}\Delta_2 G = -\delta(M-M_0)\\ G|_{\partial v}=0\end{cases}$$

$$G(x,y;\xi,\eta)=\frac{1}{2\pi}\ln\frac{r_0 r_3}{r_1 r_2}$$

(2) 二分之一单位圆 $x^2+y^2<1,\ y>0$

$$\begin{cases} \Delta_2 G = -\delta(M-M_0) \\ G|_{\partial v} = 0 \end{cases}$$



$$G(x,y;\xi,\eta)$$

$$= \frac{1}{2\pi}\left(\ln\frac{1}{r_0} - \ln\frac{1}{\rho_0 r_2}\right) - \frac{1}{2\pi}\left(\ln\frac{1}{r_1} - \ln\frac{1}{\rho_0 r_3}\right)$$

$$= \frac{1}{2\pi}\ln\frac{r_1 r_2}{r_0 r_3}$$ ，其中 $r_i = |MM_i|$，$i=0,1,2,3$