Hw 3　　2020/04/02

2.1　由S-L定理知 $\lambda\geq 0$，$\lambda=0$ 当且仅当两端均为第Ⅱ类边界条件.

(1) $\begin{cases} y''+\lambda y=0 \\ y'(0)=0,\ y(l)=0 \end{cases}$

$\lambda=k^2>0 \Rightarrow y=A\cos kx+B\sin kx$

$y'(0)=kB=0 \Rightarrow B=0$

$y(l)=A\cos kl=0 \Rightarrow kl=\frac{\pi}{2}+n\pi,\ n=0,1,2,\cdots$

$\Rightarrow k_n=\frac{2n+1}{2l}\pi,\ \lambda_n=(\frac{2n+1}{2l}\pi)^2,\ y_n(x)=\cos\frac{2n+1}{2l}\pi x$

(2) $\begin{cases} y''+\lambda y=0 \\ y'(0)=0,\ y'(l)+hy(l)=0 \end{cases}$

$\lambda=k^2>0 \Rightarrow y=A\cos kx+B\sin kx$

$y'(0)=kB=0 \Rightarrow B=0$

$y'(l)+hy(l)=-kA\sin kl+hA\cos kl=0$

$\Rightarrow -A(k\sin kl-h\cos kl)=-A(k^2+h^2)\sin(kl-\varphi)=0$

其中 $\tan\varphi=\frac{h}{k}$，$\varphi=\arctan\frac{h}{k}$

$$\Rightarrow k_n l - \arctan\frac{h}{k_n} = n\pi,\ \lambda_n = k_n^2,\ y_n(x) = \cos k_n x$$

(3) $\begin{cases} y'' + \lambda y = 0 \\ y'(0) - k y(0) = 0,\ y'(l) + h y(l) = 0 \end{cases}$, $k, h > 0$

$$\lambda = \mu^2 > 0 \Rightarrow y = A\cos\mu x + B\sin\mu x$$

$$\begin{cases} y'(0) - k y(0) = \mu B - kA = 0 \\ y'(l) + h y(l) = -\mu A\sin\mu l + \mu B\cos\mu l \\ \qquad + hA\cos\mu l + hB\sin\mu l = 0 \end{cases}$$

$$\Rightarrow \begin{pmatrix} k & -\mu \\ h\cos\mu l - \mu\sin\mu l & h\sin\mu l + \mu\cos\mu l \end{pmatrix}\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

为使 A.B 有非零解，系数行列式为0：

$$\begin{vmatrix} k & -\mu \\ h\cos\mu l - \mu\sin\mu l & h\sin\mu l + \mu\cos\mu l \end{vmatrix}$$

$$= (kh - \mu^2)\sin\mu l + (k\mu + \mu h)\cos\mu l = 0$$

$$\Rightarrow (kh-\mu_n^2)\tan\mu_n l + \mu_n(k+h) = 0,\quad n=1,2,\cdots$$

$$\lambda_n=\mu_n^2,\quad y_n(x) = \cos\mu_n x + \frac{k}{\mu_n}\sin\mu_n x$$

2.1 (1) $\begin{cases} y''-2ay'+\lambda y=0, & 0<x<1 \\ y(0)=y(1)=0 \end{cases}$

解特征方程 $t^2-2at+\lambda=0 \Rightarrow t=a\pm\sqrt{a^2-\lambda}$

$$\Rightarrow y = Ae^{(a+\sqrt{a^2-\lambda})x} + Be^{(a-\sqrt{a^2-\lambda})x}$$

$$\begin{cases} y(0)=A+B=0 \\ y(1)=Ae^{a+\sqrt{a^2-\lambda}}+Be^{a-\sqrt{a^2-\lambda}}=0 \end{cases}$$

若 $\lambda<a^2$，则 $A=B=0$

若 $\lambda=a^2$，则 $A=-B$，$y_0(x)=0$

若 $\lambda>a^2$，则 $y=e^{ax}(A'\cos\sqrt{\lambda-a^2}x+B'\sin\sqrt{\lambda-a^2}x)$

$$y(0)=A'=0$$

$$y(1)=e^aB'\sin\sqrt{\lambda-a^2}=0 \Rightarrow \lambda-a^2=(n\pi)^2,\quad n=1,2,\cdots$$

$$\lambda_n=(n\pi)^2+a^2,\quad y_n(x)=e^{ax}\sin n\pi x$$

(2) $\begin{cases} (r^2R')' + \lambda r^2 R = 0, & 0<r<a \\ |R(0)| < +\infty, & R(a) = 0 \end{cases}$

$r^2R'' + 2rR' + \lambda r^2 R = 0$

令 $R = \frac{1}{r} y$，则

$R' = \frac{1}{r} y' - \frac{1}{r^2} y$， $R'' = \frac{1}{r} y'' - \frac{2}{r^2} y' + \frac{2}{r^3} y$

$rR'' + 2R' + \lambda r R = y'' + \lambda y = 0$

$\Rightarrow \begin{cases} y'' + \lambda y = 0 \\ y(0) = y(a) = 0 \end{cases}$

$\Rightarrow \lambda = k^2 > 0$， $y = A\cos kr + B\sin kr$

$y(0) = A = 0$

$y(a) = B\sin ka = 0 \Rightarrow k_n a = n\pi$， $n = 1, 2, \cdots$

$\Rightarrow \lambda_n = (\frac{n\pi}{a})^2$， $y_n(r) = \sin\frac{n\pi r}{a}$， $R_n(r) = \frac{1}{r}\sin\frac{n\pi r}{a}$

2.3 $\begin{cases} u_{tt} = a^2 u_{xx} \\ u(t,0) = u(t,l) = 0 \\ u(0,x) = \frac{4h}{l^2} x(l-x), \quad u_t(0,x) = 0 \end{cases}$

令 $u(t,x) = T(t)X(x)$，分离变量得

$$\frac{1}{a^2}\frac{T''(t)}{T(t)} = \frac{X''(x)}{X(x)} = -\lambda$$

解固有值问题

$$\begin{cases} X'' + \lambda X = 0 \\ X(0) = X(l) = 0 \end{cases}$$

$\lambda = k^2 > 0 \Rightarrow X(x) = A\cos kx + B\sin kx$

$X(0) = A = 0$

$X(l) = B\sin kl = 0 \Rightarrow k_n l = n\pi, \quad n = 1, 2, \cdots$

$k_n = \frac{n\pi}{l}, \quad \lambda_n = \left(\frac{n\pi}{l}\right)^2, \quad X_n(x) = \sin\frac{n\pi x}{l}$

解关于 t 的方程 $T''(t) + a^2\lambda_n T(t) = 0$

$\Rightarrow T_n(t) = A_n\cos\frac{n\pi at}{l} + B_n\sin\frac{n\pi at}{l}$

$\Rightarrow u(t,x) = \sum_{n=1}^{\infty} T_n(t)X_n(x) = \sum_{n=1}^{\infty}\left(A_n\cos\frac{n\pi at}{l} + B_n\sin\frac{n\pi at}{l}\right)\sin\frac{n\pi x}{l}$

代入初值条件，

$$
\begin{cases}
u(0,x) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{l} = \frac{4h}{l^2} x(l-x) \xlongequal{\text{记作}} f(x) \\
u_t(0,x) = \sum_{n=1}^{\infty} \frac{n\pi a}{l} B_n \sin\frac{n\pi x}{l} = 0 \qquad \Rightarrow B_n = 0
\end{cases}
$$

$$
\|X_n(x)\|^2 = \int_0^l \sin^2\frac{n\pi x}{l}\,dx = \frac{l}{n\pi}\int_0^{n\pi} \sin^2 t\,dt
$$

$$
= \frac{l}{n\pi}\int_0^{n\pi} \frac{1-\cos 2t}{2}\,dt = \frac{l}{2}
$$

$$
\langle f(x), X_n(x)\rangle = \int_0^l f(x) X_n(x)\,dx = \int_0^l \frac{4h}{l^2} x(l-x)\sin\frac{n\pi x}{l}\,dx
$$

$$
= 4hl\int_0^1 t(1-t)\sin n\pi t\,dt
$$

$$
= 4hl\int_0^1 t\sin n\pi t - t^2 \sin n\pi t\,dt
$$

其中 $\int_0^1 t\sin n\pi t\,dt = \frac{-1}{n\pi}\int_0^1 t\,d\cos n\pi t$

$$
= -\frac{1}{n\pi} t\cos n\pi t\Big|_0^1 + \frac{1}{n\pi}\int_0^1 \cos n\pi t\,dt
$$

$$
= -\frac{1}{n\pi}(-1)^n
$$

$$\int_0^1 t^2 \sin n\pi t \, dt = \frac{-1}{n\pi} \int_0^1 t^2 \, d\cos n\pi t$$

$$= -\frac{1}{n\pi} t^2 \cos n\pi t \Big|_0^1 + \frac{1}{n\pi} \int_0^1 2t \cos n\pi t \, dt$$

$$= -\frac{1}{n\pi}(-1)^n + \frac{2}{(n\pi)^2} \int_0^1 t \, d\sin n\pi t$$

$$= -\frac{1}{n\pi}(-1)^n + \frac{2}{(n\pi)^2} t \sin n\pi t \Big|_0^1 - \frac{2}{(n\pi)^2} \int_0^1 \sin n\pi t \, dt$$

$$= -\frac{1}{n\pi}(-1)^n + \frac{2}{(n\pi)^3} \cos n\pi t \Big|_0^1$$

$$= -\frac{1}{n\pi}(-1)^n + \frac{2}{(n\pi)^3}\left[(-1)^n - 1\right]$$

$$\Rightarrow A_n = \frac{\langle f(x), X_n(x) \rangle}{\| X_n(x) \|^2} = \frac{2}{l} \cdot 4hl \cdot \frac{2}{(n\pi)^3}\left[1 - (-1)^n\right]$$

$$= \frac{16h}{(n\pi)^3}\left[1 - (-1)^n\right] = \begin{cases} \frac{32h}{(n\pi)^3}, & n = 2k+1 \\ 0, & n = 2k \end{cases}$$

$$\Rightarrow u(t, x) = \frac{32h}{\pi^3} \sum_{k=0}^{+\infty} \frac{1}{(2k+1)^3} \cos \frac{(2k+1)\pi a t}{l} \sin \frac{(2k+1)\pi x}{l}$$

2.4 $\begin{cases} \Delta_2 u = 0, & r<a \\ u|_{r=a} = f \end{cases}$

圆内通解：$u(r,\theta) = A_0 + \sum_{k=1}^{\infty} r^k (C_k \cos k\theta + D_k \sin k\theta)$

(1) $f = A$

$$u(a,\theta) = A_0 + \sum_{k=1}^{\infty} a^k (C_k \cos k\theta + D_k \sin k\theta) = A$$

$$\Rightarrow C_k = D_k = 0, \quad A_0 = A, \quad u(r,\theta) = A$$

(2) $f = A\cos\theta$

$$u(a,\theta) = A_0 + \sum_{k=1}^{\infty} a^k (C_k \cos k\theta + D_k \sin k\theta) = A\cos\theta$$

$$\Rightarrow C_1 = \frac{A}{a}, \quad C_k = D_k = 0, k>1, \quad D_1 = 0, \quad A_0 = 0$$

$$u(r,\theta) = A \cdot \frac{r}{a} \cos\theta$$

(3) $f = Axy = Ar^2 \sin\theta \cos\theta = \frac{1}{2} A a^2 \sin 2\theta$

$$u(a,\theta) = A_0 + \sum_{k=1}^{\infty} a^k (C_k \cos k\theta + D_k \sin k\theta) = \frac{1}{2} A a^2 \sin 2\theta$$

$$\Rightarrow C_2 = 0, \quad D_2 = \frac{1}{2} A, \quad C_k = D_k = 0, k \neq 2, \quad A_0 = 0$$

$$u(r,\theta) = \frac{1}{2} A r^2 \sin 2\theta$$

(4) $f=\cos\theta\sin 2\theta=\frac{1}{2}(\sin 3\theta+\sin\theta)$

$$u(a,\theta)=A_0+\sum_{k=1}^{\infty}a^k(C_k\cos k\theta+D_k\sin k\theta)=\frac{1}{2}(\sin\theta+\sin 3\theta)$$

$\Rightarrow D_1=\frac{1}{2a}$，$D_3=\frac{1}{2a^3}$，其余为0

$$u(r,\theta)=\frac{1}{2}\left(\frac{r}{a}\sin\theta+\left(\frac{r}{a}\right)^3\sin 3\theta\right)$$

(5) $f=A\sin^2\theta+B\cos^2\theta=A\frac{1-\cos 2\theta}{2}+B\frac{1+\cos 2\theta}{2}$

$$=\frac{A+B}{2}+\frac{B-A}{2}\cos 2\theta$$

$$u(a,\theta)=A_0+\sum_{k=1}^{\infty}a^k(C_k\cos k\theta+D_k\sin k\theta)=\frac{A+B}{2}+\frac{B-A}{2}\cos 2\theta$$

$\Rightarrow A_0=\frac{A+B}{2}$，$C_2=\frac{B-A}{2a^2}$，其余为0

$$u(r,\theta)=\frac{A+B}{2}+\frac{B-A}{2}\frac{r^2}{a^2}\cos 2\theta$$

2.5 (1) $\begin{cases} u_{tt}=a^2u_{xx} & 0<x<l,\ t>0 \\ u(t,0)=u_x(t,l)=0 \\ u(0,x)=0,\ u_t(0,x)=x \end{cases}$

令 $u(t,x)=T(t)X(x)$，分离变量得

$$\Rightarrow \frac{1}{a^2}\frac{T''(t)}{T(t)} = \frac{X''(x)}{X(x)} = -\lambda$$

解固有值问题：

$$\begin{cases} X'' + \lambda X = 0 \\ X(0) = X'(l) = 0 \end{cases}$$

$$\lambda = k^2 > 0 \Rightarrow X = A\sin kx + B\cos kx$$

$$X(0) = B = 0$$

$$X'(l) = kA\cos kl = 0 \Rightarrow k_n l = \frac{\pi}{2} + n\pi,\ n = 0, 1, 2, \cdots$$

$$k_n = \frac{2n+1}{2l}\pi,\quad \lambda_n = \left(\frac{2n+1}{2l}\pi\right)^2,\quad X_n(x) = \sin\frac{2n+1}{2l}\pi x$$

解关于t的方程：$T''(t) + a^2\lambda_n T(t) = 0$

$$\Rightarrow T_n(t) = A_n\cos\frac{2n+1}{2l}\pi a t + B_n\sin\frac{2n+1}{2l}\pi a t$$

$$\Rightarrow u(t,x) = \sum_{n=0}^{\infty} T_n(t) X_n(x)$$

$$= \sum_{n=0}^{\infty}\left(A_n\cos\frac{2n+1}{2l}\pi a t + B_n\sin\frac{2n+1}{2l}\pi a t\right)\sin\frac{2n+1}{2l}\pi x$$

代入初始条件，

$$u(0,x)=\sum_{n=0}^{\infty}A_n\sin\frac{2n+1}{2l}\pi x=0 \quad\Rightarrow\quad A_n=0$$

$$u_t(0,x)=\sum_{n=0}^{\infty}\frac{2n+1}{2l}\pi a B_n\sin\frac{2n+1}{2l}\pi x=x$$

$$\left\|\sin\frac{2n+1}{2l}\pi x\right\|^2=\int_0^l\sin^2\frac{2n+1}{2l}\pi x\,dx$$

$$=\int_0^l\frac{1-\cos\frac{2n+1}{l}\pi x}{2}dx=\frac{l}{2}-\frac{l}{2(2n+1)\pi}\sin\frac{2n+1}{l}\pi x\Big|_0^l=\frac{l}{2}$$

$$\left\langle x,\sin\frac{2n+1}{2l}\pi x\right\rangle=\int_0^l x\sin\frac{2n+1}{2l}\pi x\,dx$$

$$=\frac{-2l}{(2n+1)\pi}\int_0^l x\,d\cos\frac{2n+1}{2l}\pi x$$

$$=\frac{-2l}{(2n+1)\pi}x\cos\frac{2n+1}{2l}\pi x\Big|_0^l+\frac{2l}{(2n+1)\pi}\int_0^l\cos\frac{2n+1}{2l}\pi x\,dx$$

$$=\frac{4l^2}{(2n+1)^2\pi^2}\sin\frac{2n+1}{2l}\pi x\Big|_0^l=\frac{4l^2}{(2n+1)^2\pi^2}\cdot(-1)^n$$

$$B_n = \frac{2l}{(2n+1)\pi a} \cdot \frac{\langle x, \sin\frac{2n+1}{2l}\pi x \rangle}{\| \sin\frac{2n+1}{2l}\pi x\|^2}$$

$$= \frac{2l}{(2n+1)\pi a} \cdot \frac{2}{l} \cdot \frac{4l^2}{(2n+1)^2\pi^2} \cdot (-1)^n$$

$$= \frac{16l^2}{(2n+1)^3\pi^3 a} \cdot (-1)^n$$

$$\Rightarrow u(t,x) = \frac{16l^2}{\pi^3 a} \cdot \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^3} \sin\frac{2n+1}{2l}\pi a t \sin\frac{2n+1}{2l}\pi x$$

(2) $$\begin{cases} u_t = a^2 u_{xx}, \quad 0<x<l, \ t>0 \\ u(t,0) = u(t,l) = 0 \\ u(0,x) = x(l-x) \end{cases}$$

令 $u(t,x) = T(t)X(x)$，分离变量得

$$\Rightarrow \frac{1}{a^2}\frac{T'(t)}{T(t)} = \frac{X''(x)}{X(x)} = -\lambda$$

解固有值问题：

$$\begin{cases} X'' + \lambda X = 0 \\ X(0) = X(l) = 0 \end{cases}$$

$$\lambda = k^2 > 0 \Rightarrow X = A\sin kx + B\cos kx$$

$$X(0) = B = 0$$

$$X(l) = A\sin kl = 0 \Rightarrow k_n l = n\pi,\ n = 1, 2, \cdots$$

$$k_n = \frac{n\pi}{l},\quad \lambda_n = \left(\frac{n\pi}{l}\right)^2,\quad X_n(x) = \sin\frac{n\pi x}{l}$$

解关于t的方程：$T'(t) + a^2\lambda_n T(t) = 0$

$$\Rightarrow T_n(t) = e^{-a^2\lambda_n t}$$

$$\Rightarrow u(t,x) = \sum_{n=1}^{\infty} T_n(t) X_n(x)$$

$$= \sum_{n=1}^{\infty} C_n e^{-\left(\frac{n\pi a}{l}\right)^2 t} \sin\frac{n\pi x}{l}$$

代入初始条件，

计算、化简过程同2.3

$$u(0,x) = \sum_{n=1}^{\infty} C_n \sin\frac{n\pi x}{l} = x(l-x)$$

$$\Rightarrow C_n = \frac{\langle x(l-x), \sin\frac{n\pi x}{l}\rangle}{\|\sin\frac{n\pi x}{l}\|^2} = \frac{2}{l}\cdot\frac{2l^3}{(n\pi)^3}\cdot\left[1-(-1)^n\right]$$

$$u(t,x) = \frac{8l^2}{\pi^3}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^3} e^{-\left(\frac{2n+1}{l}\pi a\right)^2 t} \sin\frac{2n+1}{l}\pi x$$