第4章.

63. 设随机变量 X, Y 相互独立, 具有共同分布 $N(\mu, \sigma^2)$. 设 α, β 为两个常数.

(1) 求 $\mathrm{Cov}(\alpha X+\beta Y, \alpha X-\beta Y)$.

(2) 当 α, β 取何值时, $\alpha X+\beta Y$ 与 $\alpha X-\beta Y$ 相互独立.

$$EX = EY = \mu, \quad Var(X) = Var(Y) = \sigma^2$$

$$\mathrm{Cov}(\alpha X+\beta Y, \alpha X-\beta Y) = E[(\alpha X+\beta Y-\alpha EX-\beta EY)(\alpha X-\beta Y-\alpha EX+\beta EY)]$$

$$= E[(\alpha(X-EX)+\beta(Y-EY))(\alpha(X-EX)-\beta(Y-EY))]$$

$$= E[\alpha^2(X-EX)^2-\beta^2(Y-EY)^2]$$

$$= \alpha^2 Var(X) - \beta^2 Var(Y) = (\alpha^2-\beta^2)\sigma^2$$

(2). $\alpha X+\beta Y$ 与 $\alpha X-\beta Y$ 服从二维正态分布.

$\therefore$ 两者独立 $\Leftrightarrow$ 两者不相关 $\Leftrightarrow$ $\mathrm{Cov}(\alpha X+\beta Y, \alpha X-\beta Y)=0$

$\therefore \alpha^2-\beta^2=0$. 即 $|\alpha|=|\beta|$.

16. 设 $X_1, X_2, \ldots, X_9$ 为独立同分布的正态随机变量, 记

$$Y_1 = \frac{1}{6}(X_1+\ldots+X_6), \quad Y_2 = \frac{1}{3}(X_7+X_8+X_9), \quad S^2 = \frac{1}{2}\sum_{i=7}^{9}(X_i-Y_2)^2.$$

试求 $Z = \dfrac{\sqrt{2}(Y_1-Y_2)}{S}$ 的分布.

☆ 注意 $Var(X-Y) = Var(X)+Var(-Y)$
$= Var(X)+Var(Y)$

设 $X_i \sim N(\mu, \sigma^2)$ $(i=1,2,\ldots,9)$, 则 $EY_1=\mu$, $EY_2=\mu$, $Var Y_1 = \frac{1}{36}\sum_{i=1}^{6} Var(X_i) = \frac{1}{6}\sigma^2$

$Var Y_2 = \frac{1}{9}\sum_{i=1}^{9} Var(X_i) = \frac{1}{3}\sigma^2$

$\therefore Y_1 \sim (\mu, \frac{1}{6}\sigma^2)$, $Y_2 \sim (\mu, \frac{1}{3}\sigma^2)$

$E(X_i-Y_2)=0$, $Var(X_i-Y_2) \underset{\text{假设}i=7}{=} Var(\frac{2}{3}X_7-\frac{1}{3}X_8-\frac{1}{3}X_9) = \frac{4}{9}\cdot\sigma^2+\frac{1}{9}\sigma^2+\frac{1}{9}\sigma^2 = \frac{2}{3}\sigma^2$

$\therefore \dfrac{X_i-Y_2}{\sqrt{\frac{2}{3}}\sigma} \sim N(0,1)$ $(i=7,8,9)$

则 $\dfrac{3S^2}{\sigma^2} = \dfrac{3}{2\sigma^2}\sum_{i=7}^{9}(X_i-Y_2)^2 = \sum_{i=7}^{9}\left(\dfrac{X_i-Y_2}{\sqrt{\frac{2}{3}}\sigma}\right)^2 \sim \chi^2(3)$

$E\left(\sqrt{2}(Y_1-Y_2)\right)=0$, $Var\left(\sqrt{2}(Y_1-Y_2)\right)=2Var(Y_1)+2Var(-Y_2)=\sigma^2$

$\therefore \dfrac{\sqrt{2}(Y_1-Y_2)}{\sigma}\sim N(0,1)$

$\because$ Y_1与S^2独立,（显然） Y_2也与S^2独立（由PPT） $\therefore \dfrac{\sqrt{2}(Y_1-Y_2)}{\sigma}$ 与 $\sqrt{\dfrac{3S^2}{\sigma^2}\cdot\dfrac{1}{3}}$ 相互独立

$$\therefore Z=\frac{\dfrac{\sqrt{2}(Y_1-Y_2)}{\sigma}}{\sqrt{\dfrac{3S^2}{\sigma^2}\cdot\dfrac{1}{3}}}\sim t(3)$$

17 设 $X_1, X_2, \ldots, X_{15}$ 是独立同分布的随机变量, 服从正态分布 $N(0,2^2)$. 试求

$$Y=\frac{X_1^2+\ldots+X_{10}^2}{2(X_{11}^2+\ldots+X_{15}^2)}$$

的概率分布.

$X_i\sim N(0,2^2)$, $EX_i=0$, $VarX_i=4$

则 $\dfrac{X_i}{2}\sim N(0,1)$ $(i=1,2,\cdots,15)$

$$Y=\frac{\left(\frac{X_1^2}{4}+\cdots+\frac{X_{10}^2}{4}\right)/10}{\left(\frac{X_{11}^2}{4}+\cdots+\frac{X_{15}^2}{4}\right)/5}\overset{记为}{=}\frac{U/10}{V/5}$$

易知 $U\sim\chi^2(10)$, $V\sim\chi^2(5)$

$\therefore Y\sim F(10,5)$

20 设 $X_1,\ldots,X_n$ 为来自正态总体 $N(a,\sigma^2)$ 的一个简单随机样本, $\bar{X}$ 和 S_n^2 分别表示样本均值和样本方差, 又设 $X_{n+1}\sim N(a,\sigma^2)$ 且与 $X_1,\ldots,X_n$ 独立, 试求统计量 $\frac{X_{n+1}-\bar{X}}{S_n}\sqrt{\frac{n}{n+1}}$ 的分布.

由题可知 $\bar{X}\sim(a,\frac{\sigma^2}{n})$, $\dfrac{(n-1)S_n^2}{\sigma^2}\sim\chi^2(n-1)$

$\therefore E(X_{n+1}-\bar{X})=EX_{n+1}-E\bar{X}=a-a=0$

$Var(X_{n+1}-\bar{X})=Var(X_{n+1})+Var(-\bar{X})=\sigma^2+\frac{\sigma^2}{n}=\frac{n+1}{n}\sigma^2$

$$\therefore \frac{X_{n+1}-\bar{X}}{\sqrt{\frac{n+1}{n}}\,\sigma} \sim N(0,1)$$

$$\therefore Y = \frac{X_{n+1}-\bar{X}}{S_n}\cdot\sqrt{\frac{n}{n+1}} = \frac{\dfrac{X_{n+1}-\bar{X}}{\sqrt{\frac{n+1}{n}}\,\sigma}}{\sqrt{\dfrac{(n-1)S_n^2}{\sigma^2}\cdot\dfrac{1}{n-1}}} \sim t(n-1)$$

21. 设 $X_1,\dots,X_m$ 为来自正态总体 $N(\mu_1,\sigma^2)$ 的一个简单随机样本, $Y_1,\dots,Y_n$ 为来自正态总体 $N(\mu_2,\sigma^2)$ 的一个简单随机样本, 且 $X_1,\dots,X_n$ 和 $Y_1,\dots,Y_n$ 相互独立, $\bar{X}$ 和 $\bar{Y}$ 分别表示它们的样本均值, S_{1m}^2 和 S_{2n}^2 分别表示它们的样本方差, α 和 β是两个给定的实数, 试求

$$T=\frac{\alpha(\bar{X}-\mu_1)+\beta(\bar{Y}-\mu_2)}{\sqrt{\frac{(m-1)S_{1m}^2+(n-1)S_{2n}^2}{n+m-2}\cdot\left(\frac{\alpha^2}{m}+\frac{\beta^2}{n}\right)}}$$

的分布.

$\bar{X}\sim(\mu_1,\frac{\sigma^2}{m})$, $\bar{Y}\sim(\mu_2,\frac{\sigma^2}{n})$. $\frac{(m-1)S_{1m}^2}{\sigma^2}\sim\chi^2(m-1)$, $\frac{(n-1)S_{2n}^2}{\sigma^2}\sim\chi^2(n-1)$

$\therefore E\left[\alpha(\bar{X}-\mu_1)+\beta(\bar{Y}-\mu_2)\right]=\alpha(E\bar{X}-\mu_1)+\beta(E\bar{Y}-\mu_2)=0$.

$\therefore Var\left[\alpha(\bar{X}-\mu_1)+\beta(\bar{Y}-\mu_2)\right]=\alpha^2\,Var(\bar{X})+\beta^2\,Var(\bar{Y})=\frac{\alpha^2\sigma^2}{m^2}+\frac{\beta^2\sigma^2}{n^2}$

$\therefore$ 记 $W=\dfrac{\alpha(\bar{X}-\mu_1)+\beta(\bar{Y}-\mu_2)}{\sigma\sqrt{\frac{\alpha^2}{m^2}+\frac{\beta^2}{n^2}}}\sim N(0,1)$

又 $V=\dfrac{(m-1)S_{1m}^2}{\sigma^2}+\dfrac{(n-1)S_{2n}^2}{\sigma^2}\sim\chi^2(m+n-2)$ (卡方分布的可加性)

$$\therefore T=\frac{\alpha(\bar{X}-\mu_1)+\beta(\bar{Y}-\mu_2)}{\sigma\sqrt{\frac{\alpha^2}{m^2}+\frac{\beta^2}{n^2}}}\Bigg/\sqrt{\frac{(m-1)S_{1m}^2+(n-1)S_{2n}^2}{\sigma^2(n+m-2)}}=\frac{W}{\sqrt{\frac{V}{n+m-2}}}\sim t(n+m-2)$$

19. 设 $X_1,\dots,X_n$ 是从两点分布 $B(1,p)$ 中抽取的简单样本, $0<p<1$, 记 $\bar{X}$ 为样本均值, 求 $S_n^2=\frac{1}{n}\sum_{i=1}^{n}(X_i-\bar{X})^2$ 的期望.

$X_i\sim B(1,p)$, $EX_i=p$, $Var(X_i)=p(1-p)$ $(i=1,2,\cdots,n)$

$\bar{X}=\frac{1}{n}\sum_{i=1}^{n}X_i$, $\therefore E\bar{X}=p$, $Var(\bar{X})=\frac{1}{n^2}\sum_{i=1}^{n}Var(X_i)=\frac{1}{n}p(1-p)$

$$\therefore ES_n^2=\frac{1}{n}\sum_{i=1}^{n}E(X_i-\bar{X})^2=\frac{1}{n}\sum_{i=1}^{n}\left(EX_i^2-2EX_i\bar{X}+E\bar{X}^2\right)$$

$$=\frac{1}{n}\left(\sum_{i=1}^{n}EX_i^2-2\sum_{i=1}^{n}EX_i\bar{X}+\sum_{i=1}^{n}E\bar{X}^2\right)$$

$\because EX_i^2=Var(X_i)+(EX_i)^2=p$, $E\bar{X}^2=Var(\bar{X})+(E\bar{X})^2=\frac{p(1-p)}{n}+p^2$

$$\sum_{i=1}^{n} E X_i \bar{X} = E\left(\bar{X} \cdot \sum_{i=1}^{n} X_i\right) = n E\bar{X}^2$$

$$\therefore E S_n^2 = \frac{1}{n}\left(np - 2nE\bar{X}^2 + nE\bar{X}^2\right)$$

$$= \frac{1}{n}\left(np - \left[p(1-p) + np^2\right]\right) = p(1-p)\cdot\left(1-\frac{1}{n}\right)$$