1. 证明二维随机游动的常返性.

设 $2n$ 步转移中有 k 步向前, k 步向后, $n-k$ 步向左, $n-k$ 步向右

$$\therefore P_{00}^{(2n)}=\sum_{k=0}^{n}\frac{(2n)!}{k!\,k!\,(n-k)!\,(n-k)!}\left(\frac{1}{4}\right)^{2n}$$

$$=\left(\frac{1}{4}\right)^{2n}\sum_{k=0}^{n}\frac{(2n)!}{(n!)^2}\left(\frac{n!}{k!\,(n-k)!}\right)^2$$

$$=\left(\frac{1}{4}\right)^{2n}\sum_{k=0}^{n}\binom{2n}{n}\binom{n}{k}^2=\left(\frac{1}{4}\right)^{2n}\binom{2n}{n}\sum_{k=0}^{n}\binom{n}{k}\binom{n}{n-k}$$

$$=\left(\frac{1}{4}\right)^{2n}\binom{2n}{n}^2=\left(\frac{1}{4}\right)^{2n}\frac{[(2n)!]^2}{(n!)^4}$$

由Stirling公式知 $n!\sim n^{n+\frac{1}{2}}e^{-n}\sqrt{2\pi}$

$$\text{上式}\sim\left(\frac{1}{4}\right)^{2n}\frac{\left[(2n)^{2n+\frac{1}{2}}e^{-2n}\sqrt{2\pi}\right]^2}{\left(n^{n+\frac{1}{2}}e^{-n}\sqrt{2\pi}\right)^4}$$

$$=\left(\frac{1}{4}\right)^{2n}\frac{2^{4n+1}}{n\cdot 2\pi}=\frac{1}{n\pi}$$

$\because \sum_{n=1}^{+\infty}\frac{1}{n\pi}$ 发散 $\therefore \sum_{n=1}^{+\infty}P_{00}^{(2n)}$ 发散

$\therefore$ 二维对称随机游动是常返的.

17. 试计算转移概率矩阵的极限分布

$$P=\begin{pmatrix}\frac{1}{2} & \frac{1}{2} & 0\\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3}\\ \frac{1}{6} & \frac{1}{2} & \frac{1}{3}\end{pmatrix}$$

∴其Markov链不可约、所有状态都是遍历的 ∴极限存在.

$\lim\limits_{n\to\infty} P_{ij}^{(n)} = \pi_j \quad \sum_j \pi_j = 1 \quad \sum_i \pi_i P_{ij} = \pi_j$

$\therefore \pi_0 = \pi_0 P_{00} + \pi_1 P_{10} + \pi_2 P_{20} = \frac{1}{2}\pi_0 + \frac{1}{3}\pi_1 + \frac{1}{6}\pi_2$

$\pi_1 = \frac{1}{2}\pi_0 + \frac{1}{3}\pi_1 + \frac{1}{2}\pi_2$

$\pi_2 = \frac{1}{3}\pi_1 + \frac{1}{3}\pi_2$

$\pi_0 + \pi_1 + \pi_2 = 1$

∴解得 $\pi_0 = \frac{5}{14} \quad \pi_1 = \frac{6}{14} \quad \pi_2 = \frac{3}{14}$

∴极限分布

$$\lim_{n\to\infty} P^n = \begin{pmatrix} \frac{5}{14} & \frac{6}{14} & \frac{3}{14} \\ \frac{5}{14} & \frac{6}{14} & \frac{3}{14} \\ \frac{5}{14} & \frac{6}{14} & \frac{3}{14} \end{pmatrix}$$

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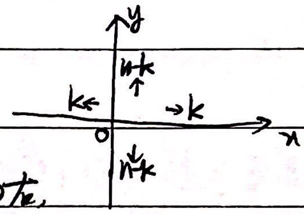
习题3

13. 证明：



对二维情况的对称随机游动，显然有 $P_{00}^{(2n+1)}=0\ (n\in N)$，考虑 $P_{00}^{(2n)}$，设有k步向右，k步向左，n-k步向上，n-k步向下.

则有 $P_{00}^{(2n)} = \sum_{k=0}^{n}\binom{2n}{2k}\binom{2k}{k}\binom{2n-2k}{n-k}\left(\frac{1}{4}\right)^{2n}$

$$=\left(\frac{1}{4}\right)^{2n}\sum_{k=0}^{n}\frac{(2n)!}{(2k)!(2n-2k)!}\frac{(2k)!(2n-2k)!}{k!k!(n-k)!(n-k)!}$$

$$=\left(\frac{1}{4}\right)^{2n}\frac{(2n)!}{n!n!}\sum_{k=0}^{n}\frac{n!}{k!(n-k)!}\frac{n!}{k!(n-k)!}$$

$$=\left(\frac{1}{4}\right)^{2n}\binom{2n}{n}\sum_{k=0}^{n}\binom{n}{k}\binom{n}{n-k}$$

$(1+x)^{2n}=(1+x)^n(1+x)^n$ x^n系数相等

$$=\left(\frac{1}{4}\right)^{2n}\binom{2n}{n}\cdot\binom{2n}{n}$$

Stirling公式 $\approx\left(\frac{1}{4}\right)^{2n}\left(\frac{2^{2n}}{\sqrt{\pi n}}\right)^2$

$\sum_{n=1}^{\infty}p_{00}^{(n)}$ $=\frac{1}{\pi n}$.

从而得 $\sum_{n=1}^{\infty}P_{00}^{(2n)}\sim\sum_{n=1}^{\infty}\frac{1}{\pi n}=\infty$、进而二维对称随机游动是(零)常返的(由不可约性)

三维情况：$P_{00}^{(2n+1)}=0$，$P_{00}^{(2n)}=\sum_{\substack{n_1,n_2,n_3\geq 0\\ n_1+n_2+n_3=n}}\frac{(2n)!}{(n_1!\,n_2!\,n_3!)^2}\left(\frac{1}{6}\right)^{2n}$

$$=\left(\frac{1}{6}\right)^{2n}\binom{2n}{n}\sum_{\substack{n_1,n_2,n_3\geq 0\\ n_1+n_2+n_3=n}}\left(\frac{n!}{n_1!\,n_2!\,n_3!}\right)^2$$

(参考《d(d≥3)维对称随机游动非常返的证明》)

$$\leq\left(\frac{1}{6}\right)^{2n}\binom{2n}{n}\max_{n_1+n_2+n_3=n}\left\{\frac{n!}{n_1!\,n_2!\,n_3!}\right\}\left(\sum_{\substack{n_1,n_2,n_3\geq 0\\ n_1+n_2+n_3=n}}\frac{n!}{n_1!\,n_2!\,n_3!}\right)$$

$$=\left(\frac{1}{6}\right)^{2n}\binom{2n}{n}\cdot 3^n\max_{n_1+n_2+n_3=n}\left\{\frac{n!}{n_1!\,n_2!\,n_3!}\right\}$$

$$\sim\left(\frac{1}{6}\right)^{2n}\frac{2^{2n}}{\sqrt{\pi n}}\cdot 3^n\cdot\frac{\left(\frac{n}{e}\right)^n\sqrt{2\pi n}}{\left(\left(\frac{n}{3e}\right)^{\frac{n}{3}}\sqrt{2\pi\frac{n}{3}}\right)^3}=\frac{3\sqrt{3}}{2}(\pi n)^{-\frac{3}{2}}.$$

从而由 $\sum_{n=1}^{\infty} p_{00}^{(2n)} = \sum_{n=1}^{\infty} p_{00}^{(2n)} \sim \sum_{n=1}^{\infty} \frac{3\sqrt{3}}{2\pi\sqrt{\pi}} \cdot \frac{1}{n^{\frac{3}{2}}} \sim \frac{3\sqrt{3}}{2\pi\sqrt{\pi}} \int_{0}^{+\infty} x^{-\frac{3}{2}} dx = \left(\frac{3}{\pi}\right)^{\frac{3}{2}} < \infty$ 知

三维随机游动是瞬过的。 Q.E.D.

17. 解：

令 $|\lambda I - P| = 0$ 解得 $\lambda_1 = 1, \lambda_2 = \frac{1}{3}, \lambda_3 = -\frac{1}{6}$，对应的特征向量为

$$\vec{x}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \vec{x}_2 = \begin{bmatrix} 3 \\ -1 \\ -3 \end{bmatrix}, \vec{x}_3 = \begin{bmatrix} 3 \\ -4 \\ 3 \end{bmatrix} \Rightarrow P = \begin{bmatrix} 1 & 3 & 3 \\ 1 & -1 & -4 \\ 1 & -3 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & -\frac{1}{6} \end{bmatrix} \begin{bmatrix} 1 & 3 & 3 \\ 1 & -1 & -4 \\ 1 & -3 & 3 \end{bmatrix}^{-1}$$

$$\Rightarrow P^n = \begin{bmatrix} 1 & 3 & 3 \\ 1 & -1 & -4 \\ 1 & -3 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{3^n} & 0 \\ 0 & 0 & (-\frac{1}{6})^n \end{bmatrix} \begin{bmatrix} 1 & 3 & 3 \\ 1 & -1 & -4 \\ 1 & -3 & 3 \end{bmatrix}^{-1}$$

$$\lim_{n\to\infty} P^n = \begin{bmatrix} 1 & 3 & 3 \\ 1 & -1 & -4 \\ 1 & -3 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 3 & 3 \\ 1 & -1 & -4 \\ 1 & -3 & 3 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{5}{14} & \frac{3}{7} & \frac{3}{14} \\ \frac{5}{14} & \frac{3}{7} & \frac{3}{14} \\ \frac{5}{14} & \frac{3}{7} & \frac{3}{14} \end{bmatrix}$$

即极限分布为 $\pi = \left[\frac{5}{14}\ \frac{3}{7}\ \frac{3}{14}\right]$.

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