Hw10 2020/05/24

5.9 (1) $\begin{cases} u_t + a u_x = f(t,x), & t>0, -\infty < x < +\infty \\ u|_{t=0} = \varphi(x) \end{cases}$

先求基本解：$\begin{cases} U_t + aU_x = 0 \\ U|_{t=0} = \delta(x) \end{cases}$

傅立叶变换：$\begin{cases} \hat{U}_t - i\lambda a \hat{U} = 0 \\ \hat{U}|_{t=0} = 1 \end{cases}$

$\Rightarrow \hat{U}(t,\lambda) = Ce^{i\lambda at}$

代入初值条件：

$\hat{U}(0,\lambda) = C = 1 \Rightarrow \hat{U}(t,\lambda) = e^{i\lambda at}$

反变换 $U(t,x) = \delta(x-at)$

$\Rightarrow u(t,x)$

$= \delta(x+at) * \varphi(x) + \int_0^t \delta(x-a(t-\tau)) * f(\tau,x)\,d\tau$

$= \varphi(x-at) + \int_0^t f(\tau, x-a(t-\tau))\,d\tau$

(2) $\begin{cases} u_{tt}+2u_t=a^2u_{xx}-2u, & t>0, -\infty<x<+\infty \\ u|_{t=0}=0, & u_t|_{t=0}=\psi(x) \end{cases}$

先求基本解：$\begin{cases} U_{tt}+2U_t=a^2U_{xx}-2U \\ U|_{t=0}=0, \quad U_t|_{t=0}=\delta(x) \end{cases}$

傅立叶变换：$\begin{cases} \hat{U}_{tt}+2\hat{U}_t=(-a^2\lambda^2-2)U \\ \hat{U}|_{t=0}=0, \quad \hat{U}_t|_{t=0}=1 \end{cases}$

特征方程 $x^2+2x+(a^2\lambda^2+2)=0$

$$\Rightarrow x_{1,2}=-1\pm i\sqrt{a^2\lambda^2+1}$$

$$\Rightarrow \hat{U}(t,\lambda)=e^{-t}\left(A\cos\sqrt{a^2\lambda^2+1}\,t+B\sin\sqrt{a^2\lambda^2+1}\,t\right)$$

代入初始条件：

$$\begin{cases} \hat{U}(0,\lambda)=A=0 \\ \hat{U}_t(0,\lambda)=B\sqrt{a^2\lambda^2+1}=1 \Rightarrow B=\dfrac{1}{\sqrt{a^2\lambda^2+1}} \end{cases}$$

$$\Rightarrow \hat{U}(t,\lambda)=\frac{1}{\sqrt{a^2\lambda^2+1}}e^{-t}\sin\sqrt{a^2\lambda^2+1}\,t$$

反变换：$U(t,x) = F^{-1}\left[e^{-t}\cdot\frac{\sin t\sqrt{a^2\lambda^2+1}}{\sqrt{a^2\lambda^2+1}}\right]$

$$= e^{-t}\cdot\frac{1}{2a}J_0\left(\frac{1}{a}\sqrt{a^2t^2-x^2}\right)H(at-|x|)$$

$$\Rightarrow u(t,x) = U(t,x) * \psi(x)$$

$$= \frac{e^{-t}}{2a}\int_{-\infty}^{+\infty} J_0\left(\frac{1}{a}\sqrt{a^2t^2-\xi^2}\right)H(at-|\xi|)\psi(x-\xi)d\xi$$

$$= \frac{e^{-t}}{2a}\int_{-at}^{at} J_0\left(\frac{1}{a}\sqrt{a^2t^2-\xi^2}\right)\psi(x-\xi)d\xi$$

5.10 $\begin{cases} u_{tt} = a^2\Delta_2 u + f(t,x,y) \\ u(0,x,y)=0,\quad u_t(0,x,y)=0 \end{cases}$

二维波动方程初值问题的基本解为：

$$U(t,x,y) = \frac{1}{2\pi a\sqrt{a^2t^2-r^2}}H(at-r),\quad r=\sqrt{x^2+y^2}$$

$$\Rightarrow u(t,x,y) = \int_0^t \frac{1}{2\pi a\sqrt{a^2(t-\tau)^2-r^2}}H(a(t-\tau)-r) * f(\tau,x,y)d\tau$$

$$= \frac{1}{2\pi a}\int_0^t d\tau \iint_{\mathbb{R}^2}\frac{1}{\sqrt{a^2(t-\tau)^2-r'^2}}H(a(t-\tau)-r')f(\tau,x-\xi,y-\eta)d\xi d\eta$$

$$= \frac{1}{2\pi a}\int_0^t d\tau \iint_D \frac{f(\tau,x-\xi,y-\eta)}{\sqrt{a^2(t-\tau)^2-r'^2}}d\xi d\eta,\quad r'=\sqrt{\xi^2+\eta^2}$$

$D: r \leq a(t-\tau)$

5.12(1) $\begin{cases} u_t = a^2 u_{xx} \\ u(0,x) = e^{-x^2} \end{cases}$

基本解：$\begin{cases} U_t = a^2 U_{xx} \\ U(0,x) = \delta(x) \end{cases}$

FT：$\begin{cases} \hat{U}_t + a^2\lambda^2 \hat{U} = 0 \\ \hat{U}(0,\lambda) = 1 \end{cases}$

$$\Rightarrow \quad \hat{U}(t,\lambda) = e^{-a^2\lambda^2 t}$$

$$\Rightarrow U(t,x) = \frac{1}{2a\sqrt{\pi t}} \exp\left(-\frac{x^2}{4a^2t}\right)$$

$$\Rightarrow u(t,x) = U(t,x) * e^{-x^2}$$

$x^2 - 2\xi x + \xi^2$

$$= \frac{1}{2a\sqrt{\pi t}} \int_{-\infty}^{+\infty} e^{-\frac{\xi^2}{4a^2t}} \cdot e^{-(x-\xi)^2} d\xi$$

$$= \frac{1}{2a\sqrt{\pi t}} \int_{-\infty}^{+\infty} e^{-\frac{4a^2t+1}{4a^2t}\left(\xi - \frac{4a^2t}{4a^2t+1}x\right)^2} d\xi \cdot e^{-x^2} \cdot e^{\frac{4a^2t}{4a^2t+1}x^2}$$

$$= \frac{1}{2a\sqrt{\pi t}} \cdot \frac{2a\sqrt{\pi t}}{\sqrt{4a^2t+1}} \cdot e^{-\frac{x^2}{4a^2t+1}}$$

$$= \frac{1}{\sqrt{4a^2t+1}} \cdot e^{-\frac{x^2}{4a^2t+1}}$$

(2) $\begin{cases} u_{tt} = a^2 \Delta_2 u \\ u(0,x,y) = x^2(x+y),\ u_t(0,x,y) = 0 \end{cases}$

设 $u(t,x,y) = x^2(x+y) + t^2 f(x,y)$

$a^2 \Delta_2 u = a^2\left[6x+2y + t^2 \Delta_2 f(x,y)\right]$

$u_{tt} = 2f(x,y)$

$\Rightarrow f(x,y) = a^2(3x+y)$ 且 $\Delta_2 f(x,y) = 0$

$\Rightarrow u(t,x,y) = x^2(x+y) + a^2t^2(3x+y)$

(3) $\begin{cases} u_{tt} = a^2 \Delta_2 u + x + y \\ u|_{t=0} = 0 \\ u_t|_{t=0} = x+y \end{cases}$

设 $u(t,x,y) = t(x+y) + t^2 f(x,y)$

$u_{tt} = 2f(x,y)$

$a^2 \Delta_2 u = a^2t^2 \Delta_2 f(x,y)$

$\Rightarrow 2f(x,y) = a^2t^2 \Delta_2 f(x,y) + x+y$

$\Rightarrow f(x,y) = \frac{1}{2}(x+y)$ 且 $\Delta_2 f(x,y) = 0$

$\Rightarrow u(t,x,y) = (x+y)(t+\frac{1}{2}t^2)$

(4) $\begin{cases} u_{tt} = a^2 \Delta_3 u + x + y + z \\ u|_{t=0} = x + y + z \\ u_t|_{t=0} = x + y + z \end{cases}$

设 $u(t,x,y,z) = (1+t+ct^2)(x+y+z)$

$u_{tt} = 2c(x+y+z)$

$a^2 \Delta_3 u = 0$

$\Rightarrow c = \frac{1}{2}$

$\Rightarrow u(t,x,y,z) = (1+t+\frac{1}{2}t^2)(x+y+z)$