Hw 1 2020/03/19

1.1 (1) 极坐标下 $\Delta_2 u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}$

对形如 $u = u(r)$ 的解 $\Delta_2 u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} = 0$

$$\Rightarrow \quad r\frac{\partial^2 u}{\partial r^2} + \frac{\partial u}{\partial r} = \frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) = 0$$

$$\Rightarrow \quad r\frac{\partial u}{\partial r} = C_1$$

$$\Rightarrow \quad u = C_1 \ln r + C_2$$

(2) 球坐标下 $\Delta_3 u = \frac{1}{r^2}\left[\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 u}{\partial\varphi^2}\right]$

对形如 $u = u(r)$ 的解

$$\Delta_3 u + k^2 u = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + k^2 u = 0$$

$$\Rightarrow \quad \frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} + k^2 u = 0$$

令 $u(r) = \frac{1}{r}v(r)$，则

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial r} - \frac{1}{r^2}v \quad , \quad \frac{\partial^2 u}{\partial r^2} = \frac{1}{r}\frac{\partial^2 v}{\partial r^2} - \frac{2}{r^2}\frac{\partial v}{\partial r} + \frac{2}{r^3}v$$

$$\Rightarrow \frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} + k^2 u = \frac{1}{r}\frac{\partial^2 v}{\partial r^2} + \frac{k^2}{r}v = 0$$

$$\Rightarrow \frac{\partial^2 v}{\partial r^2} + k^2 v = 0 \quad \Rightarrow \quad v = C_1 \sin kr + C_2 \cos kr$$

$$\Rightarrow u(r) = \frac{1}{r} v(r) = \frac{1}{r}(C_1 \sin kr + C_2 \cos kr)$$

1.2 $$u = F(x+\lambda_1 y) + G(x+\lambda_2 y)$$

$$u_{xx} = F''(x+\lambda_1 y) + G''(x+\lambda_2 y)$$

$$u_{xy} = \lambda_1 F''(x+\lambda_1 y) + \lambda_2 G''(x+\lambda_2 y)$$

$$u_{yy} = \lambda_1^2 F''(x+\lambda_1 y) + \lambda_2 G''(x+\lambda_2 y)$$

$$\Rightarrow u_{yy} - (\lambda_1+\lambda_2) u_{xy} + \lambda_1 \lambda_2 u_{xx}$$

$$= \lambda_1^2 F'' + \lambda_2^2 G'' - (\lambda_1+\lambda_2)(\lambda_1 F'' + \lambda_2 G'') + \lambda_1 \lambda_2 (F'' + G'')$$

$$= 0$$

1.3 $$u = \frac{1}{\sqrt{t}} \exp\left\{-\frac{(x-\xi)^2}{4a^2 t}\right\} \quad t>0$$

$$u_t = \left(-\frac{1}{2} t^{-\frac{3}{2}} + \frac{1}{\sqrt{t}} \cdot \frac{(x-\xi)^2}{4a^2 t^2}\right) \exp\left(-\frac{(x-\xi)^2}{4a^2 t}\right)$$

$$u_x = \frac{1}{\sqrt{t}} \cdot \left(-\frac{(x-\xi)}{2a^2 t}\right) \cdot \exp\left(-\frac{(x-\xi)^2}{4a^2 t}\right)$$

$$u_{xx} = \frac{1}{\sqrt{t}} \cdot \left(-\frac{1}{2a^2 t} + \left(\frac{x-\xi}{2a^2 t}\right)^2\right) \exp\left(-\frac{(x-\xi)^2}{4a^2 t}\right)$$

$$= \frac{1}{a^2}\left(-\frac{1}{2t\sqrt{t}} + \frac{1}{\sqrt{t}} \cdot \frac{(x-\xi)^2}{4a^2 t^2}\right) \exp\left(-\frac{(x-\xi)^2}{4a^2 t}\right)$$

$\Rightarrow \quad u_t = a^2 u_{xx}$

$$\lim_{t\to 0} u(t,x) = \lim_{t\to 0} \frac{1}{\sqrt{t}} \exp\left(-\frac{(x-\xi)^2}{4a^2 t}\right) \overset{s=\frac{1}{\sqrt{t}}}{=} \lim_{s\to\infty} s\cdot \exp\left(-\frac{(x-\xi)^2 s^2}{4a^2}\right) = 0$$

1.4 把 $u = axe^{2x+y}$ 代入 $u_{xx} - 4u_{yy} = e^{2x+y}$ 得

$$u_x = a(1+2x)e^{2x+y}$$

$$u_{xx} = a(2+2(1+2x))e^{2x+y} = 4a(1+x)e^{2x+y}$$

$$u_{yy} = axe^{2x+y}$$

$$\Rightarrow (4a(1+x) - 4ax)e^{2x+y} = e^{2x+y}$$

$$\Rightarrow 4a = 1 \Rightarrow a = \frac{1}{4} \quad \Rightarrow u = \frac{1}{4}xe^{2x+y}$$

1.5 $u = f(xy)$

$$u_x = f'(xy)\cdot y, \quad u_y = f'(xy)\cdot x$$

$$\Rightarrow xu_x - yu_y = 0$$

1.6 (1) $\dfrac{\partial u}{\partial y} + a(x,y)u = 0$

$$\frac{\partial}{\partial y}\left(\exp\left(\int a(x,y)dy\right)\cdot u\right) = 0$$

$$\Rightarrow u\cdot \exp\left(\int a(x,y)dy\right) = C(x,z)$$

$$\Rightarrow u = C(x,z)\cdot \exp\left(-\int a(x,y)\,dy\right), \quad C(x,z)\in C^1$$

(2) $u_{xy} + u_y = 0$

$$\Rightarrow u_x + u = c_1(x,z)$$

$$\Rightarrow \frac{\partial}{\partial x}(ue^x) = c_1(x,z)\cdot e^x$$

$$\Rightarrow ue^x = c_2(x,z) + c_3(y,z)$$

$$\Rightarrow u(x,y,z) = f(x,z) + e^{-x}g(y,z), \quad f,g\in C^1$$

(3) $u_{tt} = a^2 u_{xx} + 3x^2$

设 $u(t,x) = v(t,x) + w(x)$，且满足

$$\begin{cases} v_{tt} = a^2 v_{xx} & ① \\ 0 = a^2 w_{xx} + 3x^2 & ② \end{cases}$$

由①，$v(t,x) = f(x+at) + g(x-at)$

由②，$w(x) = -\frac{1}{4a^2}x^4 + c_1 x + c_2$

$$\Rightarrow u(t,x) = f(x+at) + g(x-at) - \frac{1}{4a^2}x^4 + c_1 x + c_2$$

1.7 Fourier 热传导定律：$\vec{q} \propto \nabla T$

(1) $$\begin{cases} \dfrac{\partial u}{\partial t} = a^2 \dfrac{\partial^2 u}{\partial x^2} \\ \text{绝热 } q = 0 \Rightarrow \dfrac{\partial u}{\partial x}\Big|_{x=0} = 0, \text{ 恒温 } u\Big|_{x=l} = u_0 \\ u(0,x) = \varphi(x) \end{cases}$$

(2) $$\begin{cases} \dfrac{\partial u}{\partial t} = a^2 \dfrac{\partial^2 u}{\partial x^2} \\ \dfrac{\partial u}{\partial n}\Big|_{x=0} = -\dfrac{\partial u}{\partial x}\Big|_{x=0} = -\dfrac{q_1}{k} \Rightarrow \dfrac{\partial u}{\partial x}\Big|_{x=0} = \dfrac{q_1}{k} \\ \dfrac{\partial u}{\partial n}\Big|_{x=l} = \dfrac{\partial u}{\partial x}\Big|_{x=l} = -\dfrac{q_2}{k}, \quad u(0,x) = \varphi(x) \end{cases}$$

(3) $$\begin{cases} \dfrac{\partial u}{\partial t} = a^2 \dfrac{\partial^2 u}{\partial x^2} \\ u\Big|_{x=0} = \mu(t), \quad u(0,x) = \varphi(x) \\ h(u-\theta) = -k\dfrac{\partial u}{\partial n}\Big|_{x=l} \Rightarrow \left(k\dfrac{\partial u}{\partial x} + hu\right)\Big|_{x=l} = h\,\theta(t) \end{cases}$$

1.8 弦自由振动 $u = u(t,x)$

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} = a^2 \dfrac{\partial^2 u}{\partial x^2} \\ u(t,0) = u(t,l) = 0 \\ u(0,x) = \begin{cases} \dfrac{2h}{l}x, & x \in [0, \frac{l}{2}] \\ \dfrac{2h}{l}(l-x), & x \in (\frac{l}{2}, l] \end{cases}, \quad u_t(0,x) = 0 \end{cases}$$