Hw 7　2020/04/30

3.22 (1) 利用递推公式

$$\int_0^1 x^m P_n(x)\,dx = \frac{m}{m+n+1}\int_0^1 x^{m-1}P_{n-1}(x)\,dx$$

$m<n$ 时：

$$\int_0^1 x^m P_n(x)\,dx = \frac{m}{m+n+1}\int_0^1 x^{m-1}P_{n-1}(x)\,dx$$

$$= \cdots = \frac{m}{m+n+1}\cdot\frac{m-1}{m+n-1}\cdots\frac{1}{n-m+3}\int_0^1 P_{n-m}(x)\,dx$$

由于 $m<n$，故 $n-m\geq 1$

$n-m$ 为偶数时（$m+n$ 与 $m-n$ 同奇偶）

$$\int_{-1}^1 x^m P_n(x)\,dx = 2\int_0^1 x^m P_n(x)\,dx = 2C\int_0^1 P_{n-m}(x)\,dx = 0$$

$n-m$ 为奇数时

$$\int_{-1}^1 x^m P_n(x)\,dx = 0$$

综上，$m<n$ 时 $\int_{-1}^1 x^m P_n(x)\,dx = 0$

$m \geq n$ 时：

$$\int_0^1 x^m P_n(x)\,dx = \frac{m}{m+n+1}\int_0^1 x^{m-1}P_{n-1}(x)\,dx$$

$$= \cdots = \frac{m}{m+n+1}\cdot\frac{m-1}{m+n-1}\cdots\frac{m-n+1}{m-n+3}\int_0^1 x^{m-n}P_0(x)\,dx$$

$$= \frac{m!}{(m-n)!}\cdot\frac{(m-n+1)!!}{(m+n+1)!!}\cdot\frac{1}{m-n+1} = \frac{m!}{(m-n)!!\,(m+n+1)!!}$$

$$\int_{-1}^1 x^m P_n(x)\,dx = \frac{m!\,[1+(-1)^{m+n}]}{(m-n)!!\,(m+n+1)!!}$$

(2) $$\int_{-1}^1 xP_m(x)P_n(x) = \frac{1}{2n+1}\int_{-1}^1 P_m(x)\left[(n+1)P_{n+1}(x)+nP_{n-1}(x)\right]dx$$

若 $m\neq n+1$ 且 $m\neq n-1$，由$\{P_n(x)\}$的正交性得

$$\int_{-1}^1 xP_m(x)P_n(x) = 0$$

若 $m=n-1$，则

$$\int_{-1}^1 xP_m(x)P_n(x) = \frac{n}{2n+1}\|P_{n-1}(x)\|^2 = \frac{n}{2n+1}\cdot\frac{2}{2n-1} = \frac{2n}{4n^2-1}$$

若 $m=n+1$，则

$$\int_{-1}^1 xP_m(x)P_n(x) = \frac{n+1}{2n+1}\|P_{n+1}(x)\|^2 = \frac{m}{2m-1}\cdot\frac{2}{2m+1} = \frac{2m}{4m^2-1}$$

3.23 Legendre多项式 $P_n(x)$ 满足

$$[(1-x^2)P_n'(x)]' + n(n+1)P_n(x) = 0$$

$$\int_{-1}^{1}(1-x^2)[P_n'(x)]^2 dx = \int_{-1}^{1}(1-x^2)P_n'(x)\,dP_n(x)$$

$$= \int_{-1}^{1}P_n(x)\,d[(1-x^2)P_n'(x)] = n(n+1)\int_{-1}^{1}P_n^2(x)\,dx = \frac{2n(n+1)}{2n+1}$$

3.24 (1) $f(x) = x^3$

$$f(x) = c_1P_1(x) + c_3P_3(x)$$

$$= c_1 x + c_3\frac{1}{2}(5x^3-3x)$$

$$\Rightarrow c_3 = \frac{2}{5}, \quad c_1 = \frac{3}{2}\cdot c_1 = \frac{3}{5}$$

$$\Rightarrow f(x) = \frac{3}{5}P_1(x) + \frac{2}{5}P_3(x)$$

(3) $f(x) = |x|$

$$f(x) = \sum_{n=0}^{\infty} C_{2n}P_{2n}(x)$$

$$\int_{-1}^{1}|x|P_{2n}(x)\,dx = 2\int_0^1 xP_{2n}(x)\,dx = \frac{2}{2n+2}\int_0^1 P_{2n-1}(x)\,dx$$

$$= \frac{1}{n+1}\cdot\frac{(-1)^{n-1}(2n-3)!!}{(2n)!!}, \quad n\geq 2$$

注：$\int_0^1 P_n(x)\,dx = \frac{1}{2n+1}\left[P_{n+1}(x) - P_{n-1}(x)\right]\Big|_0^1 \quad , \quad n \geq 2$

$$= \begin{cases} 0 \ , \quad n = 2k \\ \frac{1}{4k+3}\left[\frac{(-1)^k (2k-1)!!}{(2k)!!} - \frac{(-1)^{k+1}(2k+1)!!}{(2k+2)!!}\right] = \frac{(-1)^k (2k-1)!!}{(2k+2)!!} \ , \quad n = 2k+1 \end{cases}$$

$$C_0 = \frac{1}{2}\int_{-1}^{1} |x|\,dx = \int_0^1 x\,dx = \frac{1}{2}$$

$$C_2 = \frac{5}{2}\int_{-1}^{1} |x| P_2(x)\,dx = 5\int_0^1 x P_2(x)\,dx = \frac{5}{4}\int_0^1 P_1(x)\,dx = \frac{5}{8}$$

$$C_{2n} = \frac{4n+1}{2}\int_{-1}^{1} |x| P_{2n}(x)\,dx = \frac{(-1)^{n+1}(4n+1)(2n-3)!!}{(2n+2)!!}$$

$$\Rightarrow f(x) = \frac{1}{2} + \frac{5}{8}P_2(x) + \sum_{n=2}^{\infty} \frac{(-1)^{n+1}(4n+1)(2n-3)!!}{(2n+2)!!} P_{2n}(x)$$

$$= \frac{1}{2} + \frac{5}{8}P_2(x) + \sum_{n=2}^{\infty} \frac{(-1)^{n+1}(4n+1)(2n-2)!}{2^{2n}(n+1)!\,(n-1)!} P_{2n}(x)$$

3.25 $\begin{cases} \Delta_3 u = 0 \ , \quad r < a \\ u|_{r=a} = \cos^2\theta \end{cases}$

边界条件不含 φ，轴对称问题，通解为

$$u(r,\theta) = \sum_{n=0}^{\infty}\left[C_n r^n + D_n r^{-(n+1)}\right] P_n(\cos\theta) \ ,$$

球内问题 $D_n=0$

代入边界条件

$$u(a,\theta)=\sum_{n=0}^{\infty}C_n a^n P_n(\cos\theta)=\cos^2\theta$$

$$\Rightarrow C_2=\frac{2}{3}\cdot\frac{1}{a^2},\quad C_0=\frac{1}{3},\ \text{其余为0}$$

$$\Rightarrow u(r,\theta)=\frac{1}{3}+\frac{2}{3}\frac{r^2}{a^2}P_2(\cos\theta)$$

3.27 $\begin{cases}\Delta_3 u=0, & r>1\\ u|_{r=1}=\cos^2\theta\end{cases}$

边界条件不含 φ，轴对称问题，通解为

$$u(r,\theta)=\sum_{n=0}^{\infty}\left[C_n r^n+D_n r^{-(n+1)}\right]P_n(\cos\theta),$$

球外问题 $C_n=0,\ n\geq 1$

代入边界条件

$$u(1,\theta)=C_0+\sum_{n=0}^{\infty}D_n P_n(\cos\theta)=\cos^2\theta$$

$$\Rightarrow D_2=\frac{2}{3},\ D_0=\frac{1}{3},\ \text{其余为0}$$

$$\Rightarrow u(1,\theta)=\frac{1}{3r}+\frac{2}{3}\frac{1}{r^3}P_2(\cos\theta)$$

3.28 (1) $\begin{cases} \Delta_3 u = 0 \\ u(a,\theta) = u_0 \\ u(r,\frac{\pi}{2}) = 0, \ |u(r,0)| < +\infty \end{cases}$

令 $u(r,\theta) = R(r)\Theta(\theta)$，分离变量得固有值问题

$$\begin{cases} \frac{1}{\sin\theta}(\sin\theta\,\Theta'(\theta))' + \lambda\Theta(\theta) = 0 \\ |\Theta(0)| < +\infty, \quad \Theta(\frac{\pi}{2}) = 0 \end{cases}$$

对上述问题，对于在 $\theta = 0$ 处有界性条件，只有

$\lambda_n = n(n+1)$ 对应的 $\Theta_n(\theta) = P_n(\cos\theta)$

对于另一边界条件 $\Theta_n(\frac{\pi}{2}) = P_n(0) = 0 \Rightarrow n$ 为奇数

$\Rightarrow$ 固有值 $\lambda_n = (2n+1)(2n+2), \ n = 0, 1, 2, \cdots$

固有函数 $\Theta_n(\theta) = P_{2n+1}(\cos\theta)$

半球内通解为：

$$u(r,\theta) = \sum_{n=0}^{\infty} C_n \left(\frac{r}{a}\right)^{2n+1} P_{2n+1}(\cos\theta)$$

代入边界条件：

$$u(a,\theta) = \sum_{n=0}^{\infty} C_n P_{2n+1}(\cos\theta) = u_0$$

$$\|P_{2n+1}(\cos\theta)\|^2=\int_0^{\frac{\pi}{2}}P_{2n+1}^2(\cos\theta)\sin\theta\,d\theta$$

$$=\int_0^1 P_{2n+1}^2(x)\,dx=\frac{1}{2}\int_{-1}^1 P_{2n+1}^2(x)\,dx=\frac{1}{2}\cdot\frac{2}{4n+3}=\frac{1}{4n+3}$$

$$C_n=\frac{1}{\|P_{2n+1}(\cos\theta)\|^2}\int_0^{\frac{\pi}{2}}P_{2n+1}(\cos\theta)\,u_0\sin\theta\,d\theta$$

$$=u_0(4n+3)\int_0^1 P_{2n+1}(x)\,dx$$

$$=\begin{cases}\frac{3}{2}u_0\ , & n=0\\ \frac{(-1)^n(4n+3)(2n-1)!!}{(2n+2)!!}\cdot u_0\ , & n\geq 1\end{cases}$$

$$\Rightarrow u(r,\theta)=\frac{3r}{2a}\cos\theta+u_0\cdot\sum_{n=1}^{\infty}\frac{(-1)^n(4n+3)(2n-1)!!}{(2n+2)!!}\cdot\left(\frac{r}{a}\right)^{2n+1}\cdot P_{2n+1}(\cos\theta)$$

(2) $\begin{cases}\Delta_3 u=0\\ u(a,\theta)=u_0\\ \frac{\partial u}{\partial n}(r,\frac{\pi}{2})=0\ , \ |u(r,0)|<+\infty\end{cases}$

令 $u(r,\theta)=R(r)\Theta(\theta)$，分离变量得固有值问题

$$\begin{cases}\frac{1}{\sin\theta}(\sin\theta\,\Theta'(\theta))'+\lambda\Theta(\theta)=0\\ |\Theta(0)|<+\infty\ ,\ \Theta'(\frac{\pi}{2})=0\end{cases}$$

对上述问题，对于在 $\theta=0$ 处有界性条件，只有

$\lambda_n=n(n+1)$ 对应的 $\Theta_n(\theta)=P_n(\cos\theta)$

对于另一边界条件 $\Theta_n'(\frac{\pi}{2})=P_n'(0)=0 \Rightarrow n$ 为偶数

$\Rightarrow$ 固有值 $\lambda_n=2n(2n+1)$，$n=0,1,2,\cdots$

固有函数 $\Theta_n(\theta)=P_{2n}(\cos\theta)$

半球内通解为：

$$u(r,\theta)=\sum_{n=0}^{\infty}C_n\left(\frac{r}{a}\right)^{2n}P_{2n}(\cos\theta)$$

代入边界条件：

$$u(a,\theta)=\sum_{n=0}^{\infty}C_nP_{2n}(\cos\theta)=u_0$$

$$\|P_{2n}(\cos\theta)\|^2=\int_0^{\frac{\pi}{2}}P_{2n}^2(\cos\theta)\sin\theta\,d\theta=\int_0^1P_{2n}^2(x)\,dx$$

$$=\frac{1}{2}\int_{-1}^{1}P_{2n}^2(x)\,dx=\frac{1}{2}\cdot\frac{2}{4n+1}=\frac{1}{4n+1}$$

$$C_n=\frac{1}{\|P_{2n}(\cos\theta)\|^2}\int_0^{\frac{\pi}{2}}P_{2n}(\cos\theta)\,u_0\sin\theta\,d\theta$$

$$=u_0\cdot(4n+1)\int_0^1P_{2n}(x)\,dx=\begin{cases}u_0, & n=0\\ 0, & n\geq 1\end{cases}$$

$\Rightarrow u(r,\theta)=u_0$

3.29 $\begin{cases} \Delta_3 u=0, & \frac{R}{2}<r<R, \\ u(R,\theta)=u(\frac{R}{2},\theta)=A\sin^2\frac{\theta}{2} \\ u(r,\frac{\pi}{2})=\frac{A}{2}, & |u(r,0)|<+\infty \end{cases}$

注意到关于θ的边界条件非齐次，令$u=v+\frac{A}{2}$，则

$$\begin{cases} \Delta_3 v=0 \\ v(R,\theta)=v(\frac{R}{2},\theta)=A\sin^2\frac{\theta}{2}-\frac{A}{2}=-\frac{A}{2}\cos\theta \\ v(r,\frac{\pi}{2})=0, \quad |v(r,0)|<+\infty \end{cases}$$

令$v(r,\theta)=R(r)\Theta(\theta)$，分离变量得固有值问题

$$\begin{cases} \frac{1}{\sin\theta}(\sin\theta\,\Theta'(\theta))'+\lambda\Theta(\theta)=0 \\ |\Theta(0)|<+\infty, \quad \Theta(\frac{\pi}{2})=0 \end{cases}$$

对上述问题，对于在$\theta=0$处有界性条件，只有

$\lambda_n=n(n+1)$ 对应的$\Theta_n(\theta)=P_n(\cos\theta)$

对于另一边界条件 $\Theta_n(\frac{\pi}{2})=P_n(0)=0 \Rightarrow n$为奇数

$\Rightarrow$ 固有值 $\lambda_n=(2n+1)(2n+2)$, $n=0,1,2,\cdots$

固有函数 $\Theta_n(\theta)=P_{2n+1}(\cos\theta)$

半空心球内通解为

$$v(r,\theta)=\sum_{n=0}^{\infty}\left(C_n r^{2n+1}+D_n r^{-(2n+2)}\right)P_{2n+1}(\cos\theta)$$

代入边界条件

$$\begin{cases} v(R,\theta)=\sum_{n=0}^{\infty}\left(C_n R^{2n+1}+D_n R^{-(2n+2)}\right)P_{2n+1}(\cos\theta)=-\frac{A}{2}\cos\theta \\ v(\frac{R}{2},\theta)=\sum_{n=0}^{\infty}\left(C_n(\frac{R}{2})^{2n+1}+D_n(\frac{R}{2})^{-(2n+2)}\right)P_{2n+1}(\cos\theta)=-\frac{A}{2}\cos\theta \end{cases}$$

由于 $-\frac{A}{2}\cos\theta=-\frac{A}{2}P_1(\cos\theta)$

$\Rightarrow\quad C_n=D_n=0,\quad n\geq 1$

$$\begin{cases} C_0R+D_0R^{-2}=-\frac{A}{2} \\ C_0\frac{R}{2}+D_0(\frac{R}{2})^{-2}=-\frac{A}{2} \end{cases}\Rightarrow\begin{cases} C_0=-\frac{3A}{7R} \\ D_0=-\frac{AR^2}{14} \end{cases}$$

$$\Rightarrow v(r,\theta)=-\left(\frac{3r}{7R}+\frac{R^2}{14r^2}\right)A\cos\theta$$

$$\Rightarrow u(r,\theta)=\frac{A}{2}-\left(\frac{3r}{7R}+\frac{R^2}{14r^2}\right)A\cos\theta$$