Hw 2 2020/03/26

1.6 (3) $u_{tt} = a^2 u_{xx} + 3x^2$

设 $u(t,x) = v(t,x) + w(x)$，且满足

$$\begin{cases} v_{tt} = a^2 v_{xx} & ① \\ 0 = a^2 w_{xx} + 3x^2 & ② \end{cases}$$

由①， $v(t,x) = f(x+at) + g(x-at)$

由②， $w(x) = -\frac{1}{4a^2}x^4 + C_1 x + C_2$

$\Rightarrow u(t,x) = f(x+at) + g(x-at) - \frac{1}{4a^2}x^4 + C_1 x + C_2$

1.9 (1) $\begin{cases} u_t = x^2 \\ u(0,x) = x^2 \end{cases}$

$u_t = x^2 \Rightarrow u(t,x) = tx^2 + f(x)$

$u(0,x) = f(x) = x^2 \Rightarrow u(t,x) = (t+1)x^2$

(2) $\begin{cases} u_{tt} = a^2 \Delta_3 u \quad 球对称\ u = u(t,r) \\ u|_{t=0} = \varphi(r) \\ u_t|_{t=0} = \psi(r) \end{cases}$

$$\Rightarrow u_{tt} = a^2 \Delta_3 u = a^2 \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial u}{\partial r}\right)$$

$$= a^2\left(\frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r}\right)$$

令 $u = \frac{v}{r}$，则

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial r} - \frac{1}{r^2} v$$

$$\frac{\partial^2 u}{\partial r^2} = \frac{1}{r}\frac{\partial^2 v}{\partial r^2} - \frac{2}{r^2}\frac{\partial v}{\partial r} + \frac{2}{r^3} v$$

$$\frac{\partial^2 u}{\partial t^2} = \frac{1}{r}\frac{\partial^2 v}{\partial t^2}$$

$\Rightarrow$ $u_{tt} = a^2\left(\frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r}\right)$ 化为 $v_{tt} = a^2 v_{rr}$

为使 $u|_{r=0}$ 有限，应有 $v(t,0)=0$

问题化为
$$\begin{cases} v_{tt} = a^2 v_{rr}, & t>0,\ r>0 \\ v(t,0) = 0 \\ v(0,r) = r\varphi(r), & v_t(0,r) = r\psi(r) \end{cases}$$

作奇延拓，令

$$\phi(r) = \begin{cases} r\varphi(r), & r\geq 0 \\ r\varphi(-r), & r<0 \end{cases},\quad \Psi(r) = \begin{cases} r\psi(r), & r\geq 0 \\ r\psi(-r), & r<0 \end{cases}$$

由 d'Alembert 公式：

$$v(t,r)=\frac{1}{2}\left[\phi(r+at)+\phi(r-at)\right]+\frac{1}{2a}\int_{r-at}^{r+at}\Psi(\xi)d\xi$$

$$=\begin{cases}\frac{1}{2}\left[(r+at)\varphi(r+at)+(r-at)\varphi(r-at)\right] \\ \quad +\frac{1}{2a}\int_{r-at}^{r+at}\xi\psi(\xi)d\xi, & t\leqslant\frac{r}{a} \\ \frac{1}{2}\left[(r+at)\varphi(-r-at)+(r-at)\varphi(at-r)\right] \\ \quad +\frac{1}{2a}\int_{r-at}^{r+at}\xi\psi(-\xi)d\xi, & t>\frac{r}{a}\end{cases}$$

取 $r\geqslant 0$ 的部分有

$$u(t,r)=\frac{v}{r}$$

$$=\begin{cases}\frac{1}{r}\left\{\frac{1}{2}\left[(r+at)\varphi(r+at)+(r-at)\varphi(r-at)\right]\right. \\ \quad \left.+\frac{1}{2a}\int_{r-at}^{r+at}\xi\psi(\xi)d\xi\right\}, & r\geqslant at \\ \frac{1}{r}\left\{\frac{1}{2}\left[(r+at)\varphi(-r-at)+(r-at)\varphi(at-r)\right]\right. \\ \quad \left.+\frac{1}{2a}\int_{r-at}^{r+at}\xi\psi(-\xi)d\xi\right\}, & 0\leqslant r<at\end{cases}$$

(3) $\begin{cases} \Delta_3 u = 0, & x^2+y^2+z^2<1 \\ u\big|_{x^2+y^2+z^2=1} = (5+4y)^{-\frac{1}{2}} \end{cases}$

$$u = \left[(x-x_0)^2+(y-y_0)^2+(z-z_0)^2\right]^{-\frac{1}{2}}$$

代入B.C.得

$$1-2(x_0x+y_0y+z_0z)+(x_0^2+y_0^2+z_0^2) = 5+4y$$

$$\Rightarrow \begin{cases} x_0^2+y_0^2+z_0^2=4 \\ x_0=z_0=0, \quad y_0=-2 \end{cases}$$

由解的唯一性，得

$$u = \left[x^2+(y+2)^2+z^2\right]^{-\frac{1}{2}}$$

(4) $\begin{cases} u_{tt} = u_{xx} \\ u\big|_{t+x=0} = \varphi(x), \quad \varphi(0)=\psi(0) \\ u\big|_{t-x=0} = \psi(x) \end{cases}$

令 $\xi = t+x$，$\eta = t-x$，得

$$u_t = u_\xi + u_\eta$$

$$u_{tt} = u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}$$

$$u_x = u_\xi - u_\eta$$

$$u_{xx} = u_{\xi\xi} - 2u_{\xi\eta} + u_{\eta\eta}$$

$$\varphi(0) = \psi(0)$$

$$\Longrightarrow \begin{cases} u_{\xi\eta} = 0 \\ u|_{\xi=0} = \varphi\left(\frac{1}{2}(\xi-\eta)\right) = \varphi\left(-\frac{1}{2}\eta\right) \\ u|_{\eta=0} = \psi\left(\frac{1}{2}(\xi-\eta)\right) = \psi\left(\frac{1}{2}\xi\right) \end{cases}$$

$$\Longrightarrow \quad u = f(\xi) + g(\eta)$$

代入B.C.得

$$\begin{cases} u|_{\xi=0} = f(0) + g(\eta) = \varphi\left(-\frac{1}{2}\eta\right) \\ u|_{\eta=0} = g(0) + f(\xi) = \psi\left(\frac{1}{2}\xi\right) \end{cases}$$

$$\begin{cases} g(\eta) = \varphi\left(-\frac{1}{2}\eta\right) - f(0) \\ f(\xi) = \psi\left(\frac{1}{2}\xi\right) - g(0) \end{cases} \Rightarrow f(0) + g(0) = \varphi(0)$$

$$\Rightarrow u(\xi,\eta) = \varphi(-\tfrac{1}{2}\eta) + \psi(\tfrac{1}{2}\xi) - (f(0)+g(0))$$

$$\Rightarrow u(t,x) = \varphi(\tfrac{x-t}{2}) + \psi(\tfrac{x+t}{2}) - \varphi(0)$$

1.10 $\begin{cases} \dfrac{\partial u}{\partial t} + a\dfrac{\partial u}{\partial x} = f(t,x) \quad (t>0,\ -\infty<x<+\infty) \\ u(0,x) = \varphi(x),\quad a\neq 0 \end{cases}$

令 $u = u_1 + u_2$，u_1，u_2 分别满足

$$\begin{cases} \dfrac{\partial u_1}{\partial t} + a\dfrac{\partial u_1}{\partial x} = 0 \\ u_1(0,x) = \varphi(x) \end{cases} \qquad \begin{cases} \dfrac{\partial u_2}{\partial t} + a\dfrac{\partial u_2}{\partial x} = f(t,x) \\ u_2(0,x) = 0 \end{cases}$$

先解 u_1：

令 $\xi = x - at$，$\eta = t$ 得

$$\frac{\partial u_1}{\partial t} = -a\frac{\partial u_1}{\partial \xi} + \frac{\partial u_1}{\partial \eta},\quad \frac{\partial u_1}{\partial x} = \frac{\partial u_1}{\partial \xi}$$

$$\Rightarrow \begin{cases} \dfrac{\partial u_1}{\partial \eta} = 0 \\ u_1\big|_{\eta=0} = \varphi(\xi + a\eta) = \varphi(\xi) \end{cases}$$

$$\Rightarrow u_1(\xi,\eta) = C_1(\xi)$$

又 $u_1\big|_{\eta=0}=\varphi(\xi) \Rightarrow C_1(\xi)=\varphi(\xi)$

$\Rightarrow u_1(t,x)=\varphi(x-at)$

再解 u_2：

先解问题 $\begin{cases}\dfrac{\partial w}{\partial t}+a\dfrac{\partial w}{\partial x}=0\\ w\big|_{t=\tau}=f(\tau,x)\end{cases}$

令 $\xi=x-at$，$\eta=t$，类似地有

$w(\xi,\eta)=C_2(\xi)$

由 B.C. $w\big|_{\eta=\tau}=C_2(\xi)=f(\tau,\xi+a\eta)=f(\tau,\xi+a\tau)$

$\Rightarrow w(\xi,\eta)=f(\tau,\xi+a\tau)$

$w(t,x)=f(\tau,x-a(t-\tau))$

由齐次化原理

$$u_2(t,x)=\int_0^t w(t,x;\tau)\,d\tau=\int_0^t f(\tau,x-a(t-\tau))\,d\tau$$

$$\Rightarrow u(t,x)=u_1(t,x)+u_2(t,x)$$
$$=\varphi(x-at)+\int_0^t f(\tau,x-a(t-\tau))\,d\tau$$

思考题 $\begin{cases} u_{tt}=a^2u_{xx} \\ u_x(t,0)=0 \\ u(0,x)=\varphi(x),\ u_t(0,x)=\psi(x) \end{cases}$

注意 $u_x(t,0)=0$，波形在 $x=0$ 处始终保持水平，
不妨对初始条件作偶延拓：

$$\phi(x)=\begin{cases} \varphi(x), & x\geq 0 \\ \varphi(-x), & x<0 \end{cases} \qquad \Psi(x)=\begin{cases} \psi(x), & x\geq 0 \\ \psi(-x), & x<0 \end{cases}$$

由 d'Alembert 公式得

$$u(t,x)=\frac{1}{2}[\phi(x+at)+\phi(x-at)]+\frac{1}{2a}\int_{x-at}^{x+at}\Psi(\xi)d\xi$$

$$=\begin{cases} \frac{1}{2}[\varphi(x+at)+\varphi(x-at)]+\frac{1}{2a}\int_{x-at}^{x+at}\psi(\xi)d\xi, & x\geq at \\ \frac{1}{2}[\varphi(x+at)+\varphi(at-x)]+\frac{1}{2a}\int_{x-at}^{0}\psi(-\xi)d\xi + \frac{1}{2a}\int_{0}^{x+at}\psi(\xi)d\xi, & 0\leq x<at \end{cases}$$