Hw4 2020/04/09

2.5(3)
$$\begin{cases} u_{tt} = a^2 u_{xx} - 2h u_t \\ 0<x<l,\ t>0,\ 0<h<\frac{\pi a}{l} \\ u(t,0)=u(t,l)=0 \\ u(0,x)=\varphi(x),\ u_t(0,x)=\psi(x) \end{cases}$$

令 $u(t,x)=T(t)X(x)$ 有

$$\frac{1}{a^2}\frac{T''}{T} + \frac{2h}{a^2}\frac{T'}{T} = \frac{X''}{X} = -\lambda$$

解固有值问题

$$\begin{cases} X''+\lambda X=0 \\ X(0)=X(l)=0 \end{cases}$$

$\lambda=k^2>0 \Rightarrow X(x)=A\cos kx + B\sin kx$

$X(0)=A=0$

$X(l)=B\sin kl=0 \Rightarrow k_n l = n\pi,\quad n=1,2,\cdots$

$k_n=\frac{n\pi}{l},\quad \lambda_n=\left(\frac{n\pi}{l}\right)^2,\quad X_n(x)=\sin\frac{n\pi x}{l}$

解关于 t 的方程 $T''(t)+2hT'(t)+a^2\lambda_n T(t)=0$

特征方程：$k^2+2hk+\left(\frac{n\pi a}{l}\right)^2=0$

$$\Rightarrow k=-h\pm i\sqrt{\left(\frac{n\pi a}{l}\right)^2-h^2},\quad \omega_n=\sqrt{\left(\frac{n\pi a}{l}\right)^2-h^2}$$

$$\Rightarrow T_n(t)=e^{-ht}\cdot(A_n\cos\omega_n t+B_n\sin\omega_n t)$$

$$\Rightarrow u(t,x)=\sum_{n=1}^{\infty}T_n(t)X_n(x)$$

$$=\sum_{n=1}^{\infty}e^{-ht}\cdot(A_n\cos\omega_n t+B_n\sin\omega_n t)\sin\frac{n\pi x}{l}$$

代入初始条件，

$$u(0,x)=\sum_{n=1}^{\infty}A_n\sin\frac{n\pi x}{l}=\varphi(x)$$

$$\Rightarrow A_n=\frac{\langle\varphi(x),\sin\frac{n\pi x}{l}\rangle}{\|\sin\frac{n\pi x}{l}\|^2}=\frac{2}{l}\int_0^l\varphi(x)\sin\frac{n\pi x}{l}dx$$

$$u_t(0,x)=\sum_{n=1}^{\infty}(\omega_n B_n-hA_n)\sin\frac{n\pi x}{l}=\psi(x)$$

$$\Rightarrow\omega_n B_n-hA_n=\frac{\langle\psi(x),\sin\frac{n\pi x}{l}\rangle}{\|\sin\frac{n\pi x}{l}\|^2}=\frac{2}{l}\int_0^l\psi(x)\sin\frac{n\pi x}{l}dx$$

$$\Rightarrow B_n = \frac{hA_n}{\omega_n} + \frac{2}{\omega_n l}\int_0^l \psi(x)\sin\frac{n\pi x}{l}\,dx$$

(4) $$\begin{cases} u_{tt} = a^2 u_{xx}, \quad 0<x<l \\ u_x(t,0)=0, \quad u_x(t,l)+hu(t,l)=0, \quad h>0 \\ u(0,x)=\varphi(x), \quad u_t(0,x)=\psi(x) \end{cases}$$

令 $u(t,x)=T(t)X(x)$，分离变量得

$$\frac{1}{a^2}\frac{T''(t)}{T(t)} = \frac{X''(x)}{X(x)} = -\lambda$$

解固有值问题

$$\begin{cases} X''+\lambda X=0 \\ X'(0)=0, \quad X'(l)+hX(l)=0 \end{cases}$$

$\lambda=k^2>0 \quad \Rightarrow \quad X(x)=A\cos kx + B\sin kx$

$X'(0)=kB=0 \quad \Rightarrow B=0$

$X'(l)+hX(l) = -kA\sin kl + hA\cos kl = 0$

$\Rightarrow \quad A\cdot(h^2+k^2)\sin(\varphi-kl)=0, \quad \varphi=\arctan\frac{h}{k}$

$\Rightarrow \quad k_n l-\arctan\frac{h}{k_n}=n\pi \quad \Rightarrow \quad \tan k_n l = \frac{h}{k_n}$

$$\Rightarrow k_n \tan k_n l = h \ , \ \lambda_n = k_n^2 \ , \ X_n(x) = \cos k_n x \ , \ n = 1, 2, \cdots$$

解关于 t 的方程 $T''(t) + a^2 \lambda_n T(t) = 0$

$$\Rightarrow T_n(t) = A_n \cos k_n a t + B_n \sin k_n a t$$

$$\Rightarrow u(t,x) = \sum_{n=1}^{\infty} T_n(t) X_n(x) = \sum_{n=1}^{\infty} (A_n \cos k_n a t + B_n \sin k_n a t) \cos k_n x$$

代入初值条件，

$$\begin{cases} u(0,x) = \sum_{n=1}^{\infty} A_n \cos k_n x = \varphi(x) \\ u_t(0,x) = \sum_{n=1}^{\infty} k_n a B_n \sin k_n x = \psi(x) \end{cases}$$

$$\|\cos k_n x\|^2 = \int_0^l \cos^2 k_n x \, dx = \int_0^l \frac{1 + \cos 2k_n x}{2} dx$$

$$= \frac{l}{2} + \frac{1}{4k_n} \sin 2k_n l = \frac{l}{2} + \frac{1}{2k_n} \sin k_n l \cos k_n l$$

$$= \frac{l}{2} + \frac{1}{2k_n} \cdot \frac{k_n h}{k_n^2 + h^2} = \frac{l}{2} + \frac{h}{2(k_n^2 + h^2)}$$

$$\Rightarrow A_n = \frac{\langle \varphi(x), \cos k_n x \rangle}{\|\cos k_n x\|^2} = \frac{1}{\|\cos k_n x\|^2} \int_0^l \varphi(x) \cos k_n x \, dx$$

$$\Rightarrow B_n = \frac{1}{k_n a} \cdot \frac{\langle \psi(x), \cos k_n x \rangle}{\| \cos k_n x \|^2} = \frac{1}{k_n a \| \cos k_n x \|^2} \int_0^L \psi(x) \cos k_n x \, dx$$

(5) $\begin{cases} \Delta_2 u = 0, & r < a \\ u_r(a, \theta) - h u(a, \theta) = f(\theta), & h > 0 \end{cases}$

圆内通解：$u(r, \theta) = A_0 + \sum_{k=1}^{\infty} r^k (C_k \cos k\theta + D_k \sin k\theta)$

代入边界条件，

$$f(\theta) = u_r(a, \theta) - h u(a, \theta)$$

$$= \sum_{k=1}^{\infty} k a^{k-1} (C_k \cos k\theta + D_k \sin k\theta) - (h A_0 + h \sum_{k=1}^{\infty} a^k (C_k \cos k\theta + D_k \sin k\theta))$$

$$= -h A_0 + \sum_{k=1}^{\infty} (k - ha) a^{k-1} (C_k \cos k\theta + D_k \sin k\theta)$$

$$\| \cos k\theta \|^2 = \int_0^{2\pi} \cos^2 k\theta \, d\theta = \int_0^{2\pi} \frac{1 + \cos 2k\theta}{2} d\theta = \pi$$

$$\| \sin k\theta \|^2 = \int_0^{2\pi} \sin^2 k\theta \, d\theta = \int_0^{2\pi} \frac{1 - \cos 2k\theta}{2} d\theta = \pi$$

$$\Rightarrow \begin{cases} C_k = \dfrac{1}{(ka^{k-1}-ha^k)\pi}\displaystyle\int_0^{2\pi} f(\theta)\cos k\theta\, d\theta \\ D_k = \dfrac{1}{(ka^{k-1}-ha^k)\pi}\displaystyle\int_0^{2\pi} f(\theta)\sin k\theta\, d\theta \end{cases}, \quad k>0$$

$$A_0 = -\frac{1}{2\pi h}\int_0^{2\pi} f(\theta)\, d\theta$$

特别地，$f(\theta) = \cos^2\theta = \frac{1}{2} + \frac{1}{2}\cos 2\theta$ 时

$$\frac{1}{2} + \frac{1}{2}\cos 2\theta = -hA_0 + \sum_{k=1}^{\infty}(k-ha)a^{k-1}(C_k\cos k\theta + D_k \sin k\theta)$$

$$\Rightarrow A_0 = -\frac{1}{2h}, \quad C_2 = \frac{1}{2(2a-ha^2)}$$，其余为0

$$u(r,\theta) = -\frac{1}{2h} + \frac{r^2\cos 2\theta}{2(2a-ha^2)}\cos 2\theta$$

(6) $\begin{cases} \Delta_2 u = 0, \ a<r<b \\ u(a,\theta)=1, \ u(b,\theta)=0 \end{cases}$

求解极坐标下的二维对称Laplace方程：

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$$

令 $u(r,\theta)=R(r)\Theta(\theta)$ 得

$$r^2\frac{R''}{R}+r\frac{R'}{R}=\frac{\Theta''}{\Theta}=-\lambda$$

解固有值问题：

$$\begin{cases}\Theta''+\lambda\Theta=0\\ \Theta(\theta)=\Theta(\theta+2\pi)\end{cases}$$

$\lambda=k^2>0 \Rightarrow \Theta=A\cos k\theta+B\sin k\theta$

$\Theta(\theta)=\Theta(\theta+2\pi) \Rightarrow k\in\mathbb{Z}$

$\Rightarrow \Theta_k(\theta)=A_k\cos k\theta+B_k\sin k\theta \quad , \quad \lambda=k^2$

$\lambda=0 \Rightarrow \Theta_0(\theta)=1$

解关于 r 的部分：$\lambda>0$ 时

$$r^2R''+rR'+\lambda R=0$$

令 $t=\ln r$，$f(t)=R(r)$，则

$$\frac{dR}{dr}=\frac{df}{dt}\cdot\frac{dt}{dr}=f'\cdot\frac{1}{r}$$

$$\frac{d^2R}{dr^2}=f''\cdot\frac{1}{r^2}-\frac{1}{r^2}f'$$

$$\Rightarrow r^2R''+rR'+\lambda R = f''+\lambda f = 0$$ 注意这里f'是$\frac{df}{dt}$

$$\Rightarrow f(t) = C_1 e^{kt} + C_2 e^{-kt}$$

$$\Rightarrow R_k(r) = C_k r^k + D_k r^{-k}$$

$\lambda = 0$时

$$r^2R''+rR' = 0 \Rightarrow (rR')' = 0$$

$$\Rightarrow rR' = D_0 \Rightarrow R_0(r) = C_0 + D_0 \ln r$$

$\Rightarrow$ 通解为

$$u(r,\theta) = C_0 + D_0 \ln r + \sum_{k=1}^{\infty} (C_k r^k + D_k r^{-k})(A_k \cos k\theta + B_k \sin k\theta)$$

代入边界条件得

$$\begin{cases} u(a,\theta) = C_0 + D_0 \ln a + \sum_{k=1}^{\infty} (C_k a^k + D_k a^{-k})(A_k \cos k\theta + B_k \sin k\theta) = 1 \\ u(a,\theta) = C_0 + D_0 \ln b + \sum_{k=1}^{\infty} (C_k b^k + D_k b^{-k})(A_k \cos k\theta + B_k \sin k\theta) = 0 \end{cases}$$

$$\Rightarrow C_k = D_k = 0, \quad k>0$$

$$\Rightarrow \begin{cases} C_0 + D_0 \ln a = 1 \\ C_0 + D_0 \ln b = 0 \end{cases} \Rightarrow \begin{cases} C_0 = \dfrac{\ln b}{\ln b - \ln a} \\ D_0 = \dfrac{-1}{\ln b - \ln a} \end{cases}$$

$$\Rightarrow u(r,\theta)=\frac{\ln b-\ln r}{\ln b-\ln a}$$

(7) $\begin{cases} \Delta_2 u=0, & r<a, \ 0<\theta<\alpha \\ u(r,0)=u(r,\alpha)=0 \\ u(a,\theta)=f(\theta) \end{cases}$

令 $u(r,\theta)=R(r)\Theta(\theta)$，得

$$r^2\frac{R''}{R}+r\frac{R'}{R}=\frac{\Theta''}{\Theta}=-\lambda$$

解固有值问题

$$\begin{cases} \Theta''+\lambda\Theta=0 \\ \Theta(0)=\Theta(\alpha)=0 \end{cases}$$

$\lambda=k^2>0 \Rightarrow \Theta=A\cos k\theta+B\sin k\theta$

$\Theta(0)=A=0$

$\Theta(\alpha)=B\sin k\alpha=0 \Rightarrow k=\frac{n\pi}{\alpha}$, $n=1,2,\cdots$

$\Rightarrow \Theta_n(\theta)=\sin\frac{n\pi\theta}{\alpha}$, $\lambda=\left(\frac{n\pi}{\alpha}\right)^2$ 解法同上题λ>0情形

关于 r 的部分解得 $R_n(r)=C_n r^n+D_n r^{-n}$

通解 $u(r,\theta)=\sum_{n=1}^{\infty}(C_n r^n+D_n r^{-n})\sin\frac{n\pi\theta}{\alpha}$

代入边界条件得，

$$\begin{cases}|u(0,\theta)|<+\infty \Rightarrow D_n=0\\ u(a,\theta)=\sum_{n=1}^{\infty}C_n a^n\sin\frac{n\pi\theta}{\alpha}=f(\theta)\end{cases}$$

$$\left\|\sin\frac{n\pi\theta}{\alpha}\right\|^2=\int_0^{\alpha}\sin^2\frac{n\pi\theta}{\alpha}\,d\theta=\int_0^{\alpha}\frac{1-\cos\frac{2n\pi\theta}{\alpha}}{2}\,d\theta=\frac{\alpha}{2}$$

$$\Rightarrow C_n=\frac{1}{a^n}\frac{\langle f(\theta),\sin\frac{n\pi\theta}{\alpha}\rangle}{\|\sin\frac{n\pi\theta}{\alpha}\|^2}=\frac{2}{\alpha}\cdot\frac{1}{a^n}\int_0^{\alpha}f(\theta)\sin\frac{n\pi\theta}{\alpha}\,d\theta$$

$$u(r,\theta)=\sum_{n=1}^{\infty}\frac{2}{\alpha}\cdot\left(\frac{r}{a}\right)^n\cdot\int_0^{\alpha}f(\xi)\sin\frac{n\pi\xi}{\alpha}\,d\xi\ \sin\frac{n\pi\theta}{\alpha}$$

2.7 (1) $$\begin{cases}u_t=a^2\Delta_3 u\\ u|_{r=R}=0,\ u(t,0)\text{有限}\\ u|_{t=0}=f(r)\end{cases}$$

球坐标系下展开：$\frac{\partial u}{\partial t}=\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial u}{\partial r})$

令 $u(t,r)=T(t)R(r)$ 得

$$\frac{T'}{T}=\frac{1}{r^2}\frac{1}{R}(r^2R')'=-\lambda$$

解固有值问题

$$\begin{cases}(r^2R')'+\lambda r^2R=0\\ |R(0)|<+\infty,\ R(R)=0\end{cases}$$

注意，$\rho(r)=r^2$

$\lambda>0$ 时，令 $R(r)=\frac{v(r)}{r}$，则

$$\frac{dR}{dr}=\frac{1}{r}\frac{dv}{dr}-\frac{1}{r^2}v$$

$$r^2\frac{dR}{dr}=r\frac{dv}{dr}-v$$

$$\frac{d}{dr}(r^2\frac{dR}{dr})=\frac{dv}{dr}+r\frac{d^2v}{dr^2}-\frac{dv}{dr}=r\frac{d^2v}{dr^2}$$

$\Rightarrow$ 原方程化为 $\frac{d^2v}{dr^2}+\lambda v=0$，$\lambda=k^2>0$

$\Rightarrow v(r)=A\cos kr+B\sin kr$

$v(0)=A=0$，$v(R)=B\sin kR=0 \Rightarrow k=\frac{n\pi}{R}$，$n=1,2,\cdots$

$$\Rightarrow v_n(r) = \sin\frac{n\pi r}{R}, \quad \lambda_n = \left(\frac{n\pi}{R}\right)^2$$

$$\Rightarrow R_n(r) = \frac{1}{r}\sin\frac{n\pi r}{R}, \quad \lambda_n = \left(\frac{n\pi}{R}\right)^2, \quad n = 1, 2, \cdots$$

解关于t的方程：$T' + \lambda T = 0$

$$\Rightarrow T_n(t) = e^{-\left(\frac{n\pi}{R}\right)^2 t}$$

$$\Rightarrow u(t,r) = \sum_{n=1}^{\infty} C_n e^{-\left(\frac{n\pi}{R}\right)^2 t} \cdot \frac{1}{r}\sin\frac{n\pi r}{R}$$

代入初始条件得，

$$u(0,r) = \sum_{n=1}^{\infty} C_n \cdot \frac{1}{r}\sin\frac{n\pi r}{R} = f(r)$$

$$\left\|\frac{1}{r}\sin\frac{n\pi r}{R}\right\|^2 = \int_0^R r^2 \cdot \left(\frac{1}{r}\sin\frac{n\pi r}{R}\right)^2 dr = \frac{R}{2}$$

$$C_n = \frac{\left\langle f(r), \frac{1}{r}\sin\frac{n\pi r}{R}\right\rangle}{\left\|\frac{1}{r}\sin\frac{n\pi r}{R}\right\|^2} = \frac{2}{R}\int_0^R \rho f(\rho)\sin\frac{n\pi\rho}{R}\,d\rho$$

$$u(t,r) = \sum_{n=1}^{\infty} \frac{2}{R}\int_0^R \rho f(\rho)\sin\frac{n\pi\rho}{R}\,d\rho\; e^{-\left(\frac{n\pi}{R}\right)^2 t} \cdot \frac{1}{r}\sin\frac{n\pi r}{R}$$

$$= \frac{2}{Rr}\sum_{n=1}^{\infty}\int_0^R \rho f(\rho)\sin\frac{n\pi\rho}{R}\,d\rho\; e^{-\left(\frac{n\pi}{R}\right)^2 t} \cdot \sin\frac{n\pi r}{R}$$

(2) $$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} = a^2 \dfrac{\partial^4 u}{\partial x^4}, \quad t>0,\ 0<x<l \\ u(0,x) = x(l-x), \quad u_t(0,x) = 0 \\ u(t,0) = u(t,l) = 0 \\ u_{xx}(t,0) = u_{xx}(t,l) = 0 \end{cases}$$

令 $u(t,x) = T(t)X(x)$ 得，

$$\frac{1}{a^2}\frac{T''}{T} = \frac{X^{(4)}}{X} = -\lambda$$

解固有值问题

$$\begin{cases} X^{(4)} + \lambda X = 0 \\ X(0) = X(l) = X''(0) = X''(l) = 0 \end{cases}$$

$$\int_0^l \lambda X^2 dx = -\int_0^l X^{(4)} X dx = -\int_0^l X\, dX^{(3)}$$

$$= -XX^{(3)}\Big|_0^l + \int_0^l X^{(3)} X' dx = \int_0^l X' dX''$$

$$= X'X''\Big|_0^l - \int_0^l (X'')^2 dx = -\int_0^l (X'')^2 dx \leqslant 0$$

$\lambda = 0$时 $X''(x) = 0 \Rightarrow X(x) = Ax + B$

由边界条件 $\Rightarrow A = B = 0 \qquad \Rightarrow \lambda < 0$

解特征方程 $\alpha^4 + \lambda = 0$

$\Rightarrow \alpha = \pm\omega, \pm i\omega, \quad \omega = |\lambda|^{\frac{1}{4}}$

$\Rightarrow X(x) = Ae^{\omega x} + Be^{-\omega x} + C\cos\omega x + D\sin\omega x$

$X''(x) = \omega^2(Ae^{\omega x} + Be^{-\omega x}) - \omega^2(C\cos\omega x + D\sin\omega x)$

$\omega^2 X + X'' = 2\omega^2(Ae^{\omega x} + Be^{-\omega x}) \xrightarrow{\text{代}\lambda B.C.} A = B = 0$

$X(0) = C = 0$

$\Rightarrow X(l) = D\sin\omega l = 0 \Rightarrow \omega = \frac{n\pi}{l}, \quad n = 1, 2, \cdots$

$X_n(x) = \sin\frac{n\pi x}{l}$ 也满足其他 B.C.

$\Rightarrow X_n(x) = \sin\frac{n\pi x}{l}, \quad \lambda_n = -\left(\frac{n\pi}{l}\right)^4$

解关于t的方程：$T'' + a^2\lambda_n T = 0$

$\Rightarrow T_n(t) = C\,\mathrm{ch}\left(\left(\frac{n\pi}{l}\right)^2 at\right) + D\,\mathrm{sh}\left(\left(\frac{n\pi}{l}\right)^2 at\right)$

$$\Rightarrow u(t,x)=\sum_{n=1}^{\infty}\left(C_n \operatorname{ch}\left(\left(\frac{n\pi}{l}\right)^2 at\right)+D_n \operatorname{sh}\left(\left(\frac{n\pi}{l}\right)^2 at\right)\right)\sin\frac{n\pi x}{l}$$

代入初始条件得

$$\begin{cases} u(0,x)=\sum\limits_{n=1}^{\infty} C_n \sin\frac{n\pi x}{l}=x(l-x) \\ u_t(0,x)=\sum\limits_{n=1}^{\infty}\left(\frac{n\pi}{l}\right)^2 a D_n \sin\frac{n\pi x}{l}=0 \quad \Rightarrow D_n=0 \end{cases}$$

$$\left\|\sin\frac{n\pi x}{l}\right\|^2=\int_0^l \sin^2\frac{n\pi x}{l}\,dx=\frac{l}{2}$$

$$\left\langle x(l-x),\sin\frac{n\pi x}{l}\right\rangle=\int_0^l x(l-x)\sin\frac{n\pi x}{l}\,dx$$

$$=l^3\cdot\left(\int_0^1 t\sin n\pi t\,dt-\int_0^1 t^2\sin n\pi t\,dt\right)$$

$$=\left(\frac{l}{n\pi}\right)^3\cdot 2\left[1-(-1)^n\right]$$ 计算过程见HW3 2.3

$$\Rightarrow C_n=\frac{\left\langle x(l-x),\sin\frac{n\pi x}{l}\right\rangle}{\left\|\sin\frac{n\pi x}{l}\right\|^2}=\frac{2}{l}\cdot\left(\frac{l}{n\pi}\right)^3\cdot 2\left[1-(-1)^n\right]$$

$$=\begin{cases}\dfrac{8l^2}{(n\pi)^3}, & n=2k+1,\ k=0,1,2,\cdots \\ 0, & n=2k\end{cases}$$

$$\Rightarrow u(t,x) = \frac{8l^2}{\pi^3}\sum_{k=0}^{\infty}\frac{1}{(2k+1)^3}\,ch\left(\left(\frac{(2k+1)\pi}{l}\right)^2 at\right)\sin\frac{(2k+1)\pi x}{l}$$

2.8 达到稳态 $\Rightarrow u = u(r,\theta)$

$$\begin{cases} \Delta_2 u = 0 \\ u(a,\theta) = T\theta(\pi-\theta) \\ u(a,0) = u(a,\pi) = 0 \end{cases}$$

令 $u(r,\theta) = R(r)\Theta(\theta)$，得

$$r^2\frac{R''}{R} + r\frac{R'}{R} = \frac{\Theta''}{\Theta} = -\lambda$$

解固有值问题

$$\begin{cases} \Theta'' + \lambda\Theta = 0 \\ \Theta(0) = \Theta(\pi) = 0 \end{cases}$$

$\lambda = k^2 > 0 \Rightarrow \Theta = A\cos k\theta + B\sin k\theta$

$\Theta(0) = A = 0$

$\Theta(\alpha) = B\sin k\pi = 0 \Rightarrow k = n \quad , n = 1, 2, \cdots$

$\Rightarrow \Theta_n(\theta) = \sin n\theta \quad , \lambda = n^2$ 解法同2.5(6) $\lambda>0$ 情形

关于 r 的部分解得 $R_n(r) = C_n r^n + D_n r^{-n}$

通解 $u(r,\theta)=\sum_{n=1}^{\infty}(C_n r^n+D_n r^{-n})\sin n\theta$

代入边界条件得，

$$\begin{cases} |u(0,\theta)|<+\infty \Rightarrow D_n=0 \\ u(a,\theta)=\sum_{n=1}^{\infty} C_n a^n \sin n\theta = T\theta(\pi-\theta) \end{cases}$$

$$\|\sin n\theta\|^2=\int_0^\pi \sin^2 n\theta\, d\theta=\int_0^\pi \frac{1-\cos 2n\theta}{2}\, d\theta=\frac{\pi}{2}$$

$$\langle T\theta(\pi-\theta),\ \sin n\theta\rangle=\int_0^\pi T\theta(\pi-\theta)\sin n\theta\, d\theta$$

$$=\pi^3 T\int_0^1 t(1-t)\sin n\pi t\, dt=\pi^3 T\cdot\frac{2}{(n\pi)^3}[1-(-1)^n]$$

$$=\begin{cases} \frac{4T}{n^3}, & n=2k+1 \\ 0, & n=2k \end{cases}, \quad k=0,1,2,\cdots$$

$$\Rightarrow C_n=\begin{cases} \frac{1}{a^n}\frac{\langle T\theta(\pi-\theta),\ \sin n\theta\rangle}{\|\sin n\theta\|^2}=\frac{1}{\pi}\cdot\frac{8T}{a^n\cdot n^3}, & n=2k+1 \\ 0, & n=2k \end{cases}$$

$$\Rightarrow u(r,\theta)=\frac{8T}{\pi}\sum_{k=1}^{\infty}\frac{1}{(2k+1)^3}\cdot\left(\frac{r}{a}\right)^{2k+1}\sin(2k+1)\theta$$

2.10 (3)
$$\begin{cases}\dfrac{\partial^2 u}{\partial x^2}-a^2\dfrac{\partial u}{\partial t}+Ae^{-2x}=0\\ u(t,0)=u(t,L)=0\\ u(0,x)=T_0\end{cases}$$

先求解对应的齐次问题

$$\begin{cases}u_t=a^2u_{xx}\\ u(t,0)=u(t,l)=0\end{cases}$$

令 $u(t,x)=T(t)X(x)$ 有

$$\frac{1}{a^2}\frac{T'}{T}=\frac{X''}{X}=-\lambda$$

解固有值问题

$$\begin{cases}X''+\lambda X=0\\ X(0)=X(l)=0\end{cases}$$

$\lambda=k^2>0 \quad\Rightarrow\quad X(x)=A\cos kx+B\sin kx$

$X(0)=A=0$

$X(l)=B\sin kl=0 \quad\Rightarrow\quad k_n l=n\pi,\quad n=1,2,\cdots$

$k_n = \frac{n\pi}{l}$, $\lambda_n = (\frac{n\pi}{l})^2$, $X_n(x) = \sin\frac{n\pi x}{l}$

则 $u(t,x) = \sum_{n=1}^{\infty} T_n(t) \sin\frac{n\pi x}{l}$

令 $f(x) = Ae^{-2x} = \sum_{n=1}^{\infty} f_n \sin\frac{n\pi x}{l}$

$T_0 = \sum_{n=1}^{\infty} t_n \sin\frac{n\pi x}{l}$

确定展开系数：

$$f_n = \frac{2}{l}\int_0^l Ae^{-2x}\sin\frac{n\pi x}{l}dx = \frac{2n\pi A}{l^2}\cdot\frac{1}{4+(\frac{n\pi}{l})^2}[1-(-1)^n e^{-2l}]$$

$$t_n = \frac{2}{l}\int_0^l T_0\sin\frac{n\pi x}{l}dx = \frac{2T_0}{n\pi}[1-(-1)^n]$$

代入原问题中得

$$\begin{cases} \sum_{n=1}^{\infty} T_n'(t)\sin\frac{n\pi x}{l} = -a^2\sum_{n=1}^{\infty} T_n(t)(\frac{n\pi}{l})^2\sin\frac{n\pi x}{l} + \sum_{n=1}^{\infty} f_n\sin\frac{n\pi x}{l} \\ \sum_{n=1}^{\infty} T_n(0)\sin\frac{n\pi x}{l} = \sum_{n=1}^{\infty} t_n\sin\frac{n\pi x}{l} \end{cases}$$

对比系数得 $\begin{cases} T_n'(t) + (\frac{n\pi a}{l})^2 T_n(t) = f_n \\ T_n(0) = t_n \end{cases}$

$$\Rightarrow \left(T_n e^{(\frac{n\pi a}{l})^2 t}\right)' = f_n e^{(\frac{n\pi a}{l})^2 t}$$

$$\Rightarrow T_n(t) = \left(\frac{L}{n\pi a}\right)^2 f_n + C_n e^{-(\frac{n\pi a}{l})^2 t}$$

又 $T_n(0) = \left(\frac{L}{n\pi a}\right)^2 f_n + C_n = t_n$

$$\Rightarrow C_n = t_n - \left(\frac{L}{n\pi a}\right)^2 f_n$$

$$\Rightarrow T_n(t) = \left[t_n - \left(\frac{L}{n\pi a}\right)^2 f_n\right] e^{-(\frac{n\pi a}{l})^2 t} + \left(\frac{L}{n\pi a}\right)^2 f_n$$

$$\Rightarrow u(t,x) = \sum_{n=1}^{\infty} T_n(t) \sin\frac{n\pi x}{L}$$

(4) $\begin{cases} u_{tt} = a^2 u_{xx} + b\,\mathrm{sh}x \\ u(t,0) = u(t,l) = 0 \\ u(0,x) = u_t(0,x) = 0 \end{cases}$

先求解对应的齐次问题

$$\begin{cases} u_{tt} = a^2 u_{xx} \\ u(t,0) = u(t,l) = 0 \end{cases}$$

令 $u(t,x) = T(t)X(x)$ 有

$$\frac{1}{a^2}\frac{T''}{T} = \frac{X''}{X} = -\lambda$$

解固有值问题

$$\begin{cases} X'' + \lambda X = 0 \\ X(0) = X(l) = 0 \end{cases}$$

$\lambda = k^2 > 0 \Rightarrow X(x) = A\cos kx + B\sin kx$

$X(0) = A = 0$

$X(l) = B\sin kl = 0 \Rightarrow k_n l = n\pi, \quad n = 1, 2, \cdots$

$k_n = \frac{n\pi}{l}, \quad \lambda_n = \left(\frac{n\pi}{l}\right)^2, \quad X_n(x) = \sin\frac{n\pi x}{l}$

则 $u(t,x) = \sum_{n=1}^{\infty} T_n(t)\sin\frac{n\pi x}{l}$

令 $f(x) = b\,\mathrm{sh}x = \sum_{n=1}^{\infty} f_n \sin\frac{n\pi x}{l}$

确定展开系数：

$$f_n = \frac{2}{l}\int_0^l b\,\mathrm{sh}x \sin\frac{n\pi x}{l}\,dx$$

计算积分 $\int_0^l \mathrm{sh}x \sin\frac{n\pi x}{l}\,dx = \int_0^l \sin\frac{n\pi x}{l}\,d\,\mathrm{ch}x$

$$= \mathrm{ch}x \sin\frac{n\pi x}{l}\Big|_0^l - \int_0^l \mathrm{ch}x \cdot \frac{n\pi}{l}\cos\frac{n\pi x}{l}\,dx$$

$$= -\frac{n\pi}{l}\int_0^l \cos\frac{n\pi x}{l}\,d\,\mathrm{sh}x$$

$$= -\frac{n\pi}{l}\cos\frac{n\pi x}{l}\,\mathrm{sh}\,x\Big|_0^l - \left(\frac{n\pi}{l}\right)^2\int_0^l \mathrm{sh}x\sin\frac{n\pi x}{l}dx$$

$$= \frac{1}{1+\left(\frac{n\pi}{l}\right)^2}\cdot\left(-\frac{n\pi}{l}\right)\cdot(-1)^n\mathrm{sh}\,l$$

$$\Rightarrow f_n = \frac{2b}{l}\cdot\frac{\left(\frac{n\pi}{l}\right)}{1+\left(\frac{n\pi}{l}\right)^2}\cdot(-1)^{n+1}\mathrm{sh}\,l$$

代入原问题中得

$$\begin{cases}\sum\limits_{n=1}^{\infty} T_n''(t)\sin\frac{n\pi x}{l} = -\left(\frac{n\pi a}{l}\right)^2\sum\limits_{n=1}^{\infty} T_n(t)\sin\frac{n\pi x}{l} + \sum\limits_{n=1}^{\infty} f_n\sin\frac{n\pi x}{l} \\ \sum\limits_{n=1}^{\infty} T_n(0) = \sum\limits_{n=1}^{\infty} T_n'(0) = 0\end{cases}$$

对比系数得

$$T_n''(t) + \left(\frac{n\pi a}{l}\right)^2 T_n(t) = f_n$$

解得 $T_n(t) = A\sin\frac{n\pi at}{l} + B\cos\frac{n\pi at}{l} + \left(\frac{l}{n\pi a}\right)^2 f_n$

$$T_n(0) = B + \left(\frac{l}{n\pi a}\right)^2 f_n \Rightarrow B = -\left(\frac{l}{n\pi a}\right)^2 f_n$$

$$T_n'(0) = \frac{n\pi a}{l}\cdot A = 0 \Rightarrow A = 0$$

$$\Rightarrow u(t,x)=\sum_{n=1}^{\infty}\left(\frac{l}{n\pi a}\right)^2 f_n\left(1-\cos\frac{n\pi at}{l}\right)\sin\frac{n\pi x}{l}$$

$$=\sum_{n=1}^{\infty}\left(\frac{l}{n\pi a}\right)^2\cdot\frac{2b}{l}\cdot\frac{\frac{n\pi}{l}}{1+\left(\frac{n\pi}{l}\right)^2}\cdot(-1)^{n+1}\operatorname{sh}l\left(1-\cos\frac{n\pi at}{l}\right)\sin\frac{n\pi x}{l}$$

$$=\frac{2bl^2\operatorname{sh}l}{\pi a^2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n(l^2+n^2\pi^2)}\left(1-\cos\frac{n\pi at}{l}\right)\sin\frac{n\pi x}{l}$$

2.12 (3)
$$\begin{cases} u_t=a^2\Delta_2 u, \quad t>0,\ 0<x<l_1,\ 0<y<l_2 \\ u|_{x=0}=u|_{x=l_1}=u|_{y=0}=u|_{y=l_2}=0 \\ u|_{t=0}=\varphi(x,y) \end{cases}$$

令 $u(t,x,y)=T(t)X(x)Y(y)$ 得

$$\frac{1}{a^2}\frac{T'}{T}-\frac{Y''}{Y}=\frac{X''}{X}=-\lambda \Rightarrow X''+\lambda X=0$$

$$\frac{1}{a^2}\frac{T'}{T}+\lambda=\frac{Y''}{Y}=-\mu \Rightarrow Y''+\mu Y=0$$

两个固有值问题

$$\begin{cases} X''+\lambda X=0 \\ X(0)=X(l_1)=0 \end{cases} \quad 和 \quad \begin{cases} Y''+\mu Y=0 \\ Y(0)=Y(l_2)=0 \end{cases}$$

解得 $X_n(x)=\sin\frac{n\pi x}{L_1}$, $\lambda_n=(\frac{n\pi}{L_1})^2$

$Y_n(y)=\sin\frac{n\pi y}{L_2}$, $\mu_n=(\frac{n\pi}{L_2})^2$

解关于 t 的方程 $T'+a^2(\lambda+\mu)T=0$

$$\Rightarrow T_{mn}(t)=e^{-a^2(\lambda_m+\mu_n)t}$$

$$\Rightarrow u(t,x,y)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}C_{mn}e^{-a^2(\lambda_m+\mu_n)t}\sin\frac{m\pi x}{L_1}\sin\frac{n\pi y}{L_2}$$

代入初值条件得

$$u(0,x,y)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}C_{mn}\sin\frac{m\pi x}{L_1}\sin\frac{n\pi y}{L_2}=\varphi(x,y)$$

$$\Rightarrow C_{mn}=\frac{4}{L_1L_2}\int_0^{L_1}\int_0^{L_2}\varphi(x,y)\sin\frac{m\pi x}{L_1}\sin\frac{n\pi y}{L_2}\,dy\,dx$$

$$\Rightarrow u(t,x,y)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}C_{mn}e^{-a^2\left((\frac{m\pi}{L_1})^2+(\frac{n\pi}{L_2})^2\right)t}\sin\frac{m\pi x}{L_1}\sin\frac{n\pi y}{L_2}$$