概率论与数理统计B 第二次习题课

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第一次小测

有效份数: 95, 最小值:0, 最大值: 32, 平均值: 20.51, 中位数: 20, 方差: 58.27

题目1

设连续型随机变量X的概率密度函数为

$$f(x)=\begin{cases}A\cos x, & -\frac{\pi}{2}\le x<0,\\ A(1-x), & 0\le x\le 1\\ 0, & \text{其他}.\end{cases}$$

(1) 求常数A.

(2) 求X的分布函数$F(x)$.

(3) 问$P(-\frac{\pi}{4}<X<\frac{1}{2})$.

解:

(1)

因为$\int_{-\infty}^{+\infty} f(x)\,\mathrm{d}\,x=F(x)\Big|_{-\infty}^{+\infty}=1$, 所以我们有

$$\begin{aligned}1&=\int_{-\frac{\pi}{2}}^{0} A\cos x\,\mathrm{d}\,x+\int_{0}^{1} A(1-x)\,\mathrm{d}\,x\\ &=A+\frac{1}{2}A=\frac{3}{2}A.\end{aligned}$$

所以$A=\frac{2}{3}$.

(2)

因为

$$F(x)=\int_{-\infty}^{x} f(t)\,\mathrm{d}\,t$$

在本题中，我们有

$$F(x)=0,\ x<-\frac{\pi}{2}$$
$$F(x)=1,\ x\ge 1$$

注意到本题中$F(x)$是连续的，所以

$$F(x)=\begin{cases}0, & x<-\frac{\pi}{2}\\ \frac{2}{3}\sin x+\frac{2}{3}, & -\frac{\pi}{2}\le x<0,\\ -\frac{1}{3}(1-x)^2+1, & 0\le x<1\\ 1, & x\ge 1.\end{cases}$$

(3)

X是连续型随机变量，$\forall x\in\mathbb{R}, P(X=x)=0.$

$$P(-\frac{\pi}{4}<X<\frac{1}{2})=F(\frac{1}{2})-F(-\frac{\pi}{4})=\frac{11}{12}-(-\frac{\sqrt{2}}{3}+\frac{2}{3})=\frac{1}{4}+\frac{\sqrt{2}}{3}.$$

题目2

设随机变量X和Y均服从参数为λ的指数分布且相互独立，记

$$Z=\frac{X}{X+Y},\ U=\min\{X,Y\},\ V=\max\{X,Y\}-\min\{X,Y\}.$$

(1) 求Z的密度函数;

(2) 给定$U=u$时，求V的条件密度函数;

(3) 证明U和V互相独立.

(1)

已知$X\sim E(\lambda),Y\sim E(\lambda)$. 设$W=X+Y$, 我们有

$$\begin{gathered}Z=\frac{X}{W},\\ W=X+Y,\\ 0<Z<1,W>0.\end{gathered}$$

所以

$$\begin{gathered}X(Z,W)=ZW,\\ Y(Z,W)=W-X=W-ZW.\end{gathered}$$

上述变换的Jacobian行列式是

$$J(z,w)=\frac{\partial(X,Y)}{\partial(Z,W)}=\begin{vmatrix}w & z\\ -w & 1-z\end{vmatrix}=w.$$

所以

$$\begin{aligned}f_{Z,W}(z,w)&=f_{X,Y}(X(Z,W),Y(Z,W))|J(Z,W)|\\ &=\lambda\exp\{-\lambda zw\}\cdot\lambda\exp\{-\lambda(w-zw)\}\cdot w\\ &=\lambda^2\exp\{-\lambda w\}w,\ 0<z<1,w>0\end{aligned}$$

所以

$$
\begin{aligned}
f_Z(z) &= \int_0^{+\infty} f_{Z,W}(z,w)\,\mathrm{d}\,w \\
&= \int_0^{+\infty} \lambda^2 e^{-\lambda w} w\,\mathrm{d}\,w \\
&= \int_0^{+\infty} \lambda w \cdot e^{-\lambda w}\,\mathrm{d}(\lambda w) \\
&= \Gamma(2) = 1,\ 0 < z < 1.
\end{aligned}
$$

也就是

$$Z \sim U(0,1).$$

另一种解法

因为$Z = \frac{X}{X+Y}$, 我们有

$$X(X,Z) = X, \quad Y(X,Z) = \frac{X(1-Z)}{Z}$$

$$J(x,z) = \frac{\partial(x,y)}{\partial(x,z)} = \begin{vmatrix} 1 & 0 \\ \frac{1-z}{z} & \frac{-x}{z^2} \end{vmatrix} = \frac{-x}{z^2}.$$

所以

$$
\begin{aligned}
f_{X,Z}(x,z) &= f_{X,Y}(x(x,z), y(x,z))|J(x,z)| \\
&= \lambda e^{-\lambda x} \cdot \lambda e^{-\lambda(\frac{x(1-z)}{z})} \cdot \frac{x}{z^2} \\
&= \lambda^2 \frac{x}{z^2} e^{-\frac{\lambda x}{z}}.
\end{aligned}
$$

$$
\begin{aligned}
f_Z(z) &= \int_0^{+\infty} f_{X,Z}(x,z)\,\mathrm{d}\,x \\
&= \int_0^{+\infty} \lambda^2 \frac{x}{z^2} e^{-\frac{\lambda x}{z}}\,\mathrm{d}\,x \\
&= \int_0^{+\infty} \frac{\lambda x}{z} e^{-\frac{\lambda x}{z}}\,\mathrm{d}(\frac{\lambda x}{z}) \\
&= \Gamma(2) = 1,\ 0 < z < 1.
\end{aligned}
$$

(2)

$$
\begin{aligned}
P(U=u, V=v) &= P(U=X, V=Y-X)\,\mathrm{I}(X \le Y) + \\
&\quad P(U=Y, V=X-Y)\,\mathrm{I}(Y < X) \\
&= P(X=U, Y=V+U)\,\mathrm{I}(X \le Y) + \\
&\quad P(X=V+U, Y=U)\,\mathrm{I}(Y < X)
\end{aligned}
$$

所以

$$
\begin{aligned}
f_{V,U}(v,u) &= f_{X,Y}(X(U,V), Y(U,V))|J(u,v)|\,\mathrm{I}(X \le Y) \\
&\quad + f_{X,Y}(X(U,V), Y(U,V))|J(u,v)|\,\mathrm{I}(Y < X)
\end{aligned}
$$

其中

$$X(U,V)=\begin{cases}U, & X\le Y\\ V+U, & Y<X\end{cases}$$
$$Y(U,V)=\begin{cases}V+U, & X\le Y\\ U, & Y<X\end{cases}$$

注意到$|J(u,v)|=1$始终成立，又因为X,Y独立且$f_X(x)=f_Y(y)=\lambda e^{-\lambda x}$, 所以

$$\begin{aligned}f_{V,U}(v,u)&=\lambda e^{-\lambda u}\cdot\lambda e^{-\lambda(v+u)}\cdot 2\\&=2\lambda^2e^{-\lambda(v+2u)},\ v\ge 0,u>0\end{aligned}$$

所以

$$\begin{aligned}f_U(u)&=\int_0^{+\infty}f_{V,U}(v,u)\,\mathrm{d}\,v\\&=\int_0^{+\infty}2\lambda^2e^{-\lambda(v+2u)}\,\mathrm{d}\,v\\&=\left(-2\lambda e^{-\lambda(v+2u)}\right)\Big|_0^{+\infty}\\&=2\lambda e^{-2\lambda u},u>0\end{aligned}$$

$$\begin{aligned}f_{V|U}(v|u)&=\frac{f_{V,U}(v,u)}{f_U(u)}\\&=\frac{2\lambda^2e^{-\lambda(v+2u)}}{2\lambda e^{-2\lambda u}}\\&=\lambda e^{-\lambda v},\ v\ge 0.\end{aligned}$$

(3)

$$\begin{aligned}f_V(v)&=\int_0^{+\infty}f_{V,U}(v,u)\,\mathrm{d}\,u\\&=\int_0^{+\infty}2\lambda^2e^{-\lambda(v+2u)}\,\mathrm{d}\,u\\&=\left(-\lambda e^{-\lambda(v+2u)}\right)\Big|_0^{+\infty}\\&=\lambda e^{-\lambda v},\ v\ge 0.\end{aligned}$$

也就是说

$$f_V(v)=f_{V|U}(v|u).$$

所以U和V互相独立.

> *常见分布的分布律/密度函数, 期望, 方差*

> | 分布名称 | 参数 | 概率密度 | 期望 | 方差 | 特征函数 |
> | --- | --- | --- | --- | --- | --- |
> | 退化分布 | c | $\binom{c}{1}$ | c | 0 | e^{ict} |
> | 二点分布 | p
 $(0<p<1)$ | $\begin{pmatrix} 0 & 1 \\ q & p \end{pmatrix}$ | p | pq | $q+pe^{it}$ |
> | 二项分布
 $B(n,p)$ | $n \geq 1$
 $0<p<1$ | $\binom{n}{k}p^k q^{n-k}$
 $k=0,\cdots,n$ | np | npq | $(q+pe^{it})^n$ |
> | 几何分布 | p
 $(0<p<1)$ | $q^{k-1}p, k=1,2,\cdots$ | $\frac{1}{p}$ | $\frac{q}{p^2}$ | $\frac{pe^{it}}{1-qe^{it}}$ |
> | 巴斯卡分布 | r,p
 $r \in \mathbb{N}$
 $0<p<1$ | $\binom{k-1}{r-1}p^r q^{k-r}$,
 $k=r,r+1,\cdots$ | $\frac{r}{p}$ | $\frac{rq}{p^2}$ | $(\frac{pe^{it}}{1-qe^{it}})^r$ |
> | 波松分布$P(\lambda)$ | $\lambda(\lambda>0)$ | $\frac{\lambda^k}{k!}e^{-\lambda}$,
 $k=0,1,\cdots$ | λ | λ | $e^{\lambda(e^{it}-1)}$ |
> | 超几何分布 | $M,N,n \in \mathbb{N}$ | $\frac{\binom{M}{k}\binom{N-M}{n-k}}{\binom{N}{n}}$ | $\frac{nM}{N}$ | $\frac{nM}{N}\frac{(N-M)}{N}\frac{N-n}{N-1}$ | |
> | 均匀分布
 $U(a,b)$ | $a,b(a<b)$ | $\frac{1}{b-a}I_{a<x<b}$ | $\frac{a+b}{2}$ | $\frac{(b-a)^2}{12}$ | $\frac{e^{itb}-e^{ita}}{it(b-a)}$ |
> | 正态分布
 $N(a,\sigma^2)$ | a,σ^2 | $\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{(x-a)^2}{2\sigma^2}}$ | a | σ^2 | $e^{iat-\frac{1}{2}\sigma^2t^2}$ |
> | 指数分布 | $\lambda(\lambda>0)$ | $\lambda e^{-\lambda x}I_{x>0}$ | $\frac{1}{\lambda}$ | $\frac{1}{\lambda^2}$ | $(1-\frac{it}{\lambda})^{-1}$ |
> | χ^2分布 | $n(n \geq 1)$ | $\frac{1}{2^{n/2}\Gamma(n/2)}x^{n/2-1}e^{-x/2}$
 $x>0$ | n | $2n$ | $(1-2it)^{-n/2}$ |

第八周作业

(习题集第四章-28题) 设随机变量X服从区间$(-\frac{\pi}{2},\frac{\pi}{2})$上的均匀分布. 试求期望$\mathbb{E}[\sin X]$, $\mathbb{E}[\cos X]$, $\mathbb{E}[X\cos X]$.

解:

> *随机变量函数的期望 教材P109 - 110*
>
> $$E(g(X))=\sum_j g\left(a_i\right) p_i \quad\left(\text{当} \sum_i\left|g\left(a_i\right)\right| p_i<\infty \text{时}\right)$$
> $$E(g(X))=\int_{-\infty}^{\infty} g(x) f(x) \mathrm{d} x \quad\left(\text{当} \int_{-\infty}^{\infty}|g(x)| f(x) \mathrm{d} x<\infty \text{时}\right)$$

因为$X \sim U(-\frac{\pi}{2},\frac{\pi}{2})$, 所以

$$f_X(x)=\frac{1}{\pi}, x \in (-\frac{\pi}{2},\frac{\pi}{2})$$

所以

$$\mathbb{E}[\sin X]=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{\pi}\sin x \,\mathrm{d}\, x=0.$$

(奇函数$\sin x$在对称区间积分)

$$\mathbb{E}[\cos X] = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{\pi}\cos x \,\mathrm{d}\, x = \frac{2}{\pi}.$$

$$\mathbb{E}[X\cos X] = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{\pi} x\cos x \,\mathrm{d}\, x = 0.$$

($x\cos x$是奇函数)

(习题集第四章-34题) 假设有$n(n \geq 3)$个不同的盒子与m个相同的小球, 每个小球独立地以概率p_k落入第k个盒子($k = 1, 2, \cdots, n$). 分别以$X_1, X_2, \cdots, X_n$表示落入各个盒子的球数. 试求

(1) $\mathbb{E}[X_2|X_1 = k]$和$\mathrm{Var}(X_2|X_1 = k)$.

(2) $\mathbb{E}[X_1 + X_2]$和$\mathrm{Var}(X_1 + \cdots + X_k), k = 1, \cdots, n$.

解:

(1)

$$\begin{aligned} P\left(X_2 = i|X_1 = k\right) &= \frac{P\left(X_1 = k, X_2 = i\right)}{P\left(X_1 = k\right)} \\ &= \frac{\binom{m}{k}\binom{m-k}{i} p_1^k p_2^i (1 - p_1 - p_2)^{m-k-i}}{\binom{m}{k} p_1^k (1 - p_1)^{m-k}} \\ &= \frac{\binom{m-k}{i} p_2^i (1 - p_1 - p_2)^{m-k-i}}{(1 - p_1)^{m-k}} \\ &= \binom{m-k}{i}\left(\frac{p_2}{1 - p_1}\right)^i \left(1 - \frac{p_2}{1 - p_1}\right)^{m-k-i} \end{aligned}$$

也就是

$$(X_2|X_1 = k) \sim B(m - k, \frac{p_2}{1 - p_1})$$

所以

$$\begin{aligned} \mathbb{E}[X_2|X_1 = k] &= (m - k)\frac{p_2}{1 - p_1}, \\ \mathrm{Var}(X_2|X_1 = k) &= (m - k)\frac{p_2}{1 - p_1}\frac{1 - p_1 - p_2}{1 - p_1}. \end{aligned}$$

(2)

$$(X_1 + X_2) \sim B(m, p_1 + p_2)$$

$$(X_1 + \cdots + X_k) \sim B(m, p_1 + \cdots + p_k)$$

所以

$$\begin{aligned} \mathbb{E}[X_1 + X_2] &= m(p_1 + p_2), \\ \mathrm{Var}(X_1 + \cdots + X_k) &= m\sum_{i=1}^{k} p_i (1 - \sum_{i=1}^{k} p_i). \end{aligned}$$

(第三章 slides 93页 性质4) 设a, b, c是常数, $\mathbb{E}X_j = \mu_j, \mathrm{Var}(X_j) < \infty, 1 \le j \le n$.

求证当$X_1, \cdots, X_n$相互独立时,

$$\mathrm{Var}\left(\sum_{j=1}^{n} X_j\right) = \sum_{j=1}^{n} \mathrm{Var}(X_j)$$

证明:

$$\begin{aligned}
\mathrm{Var}\left(\sum_{j=1}^{n} X_j\right) &= \mathbb{E}\left[\sum_{j=1}^{n} X_j - \mathbb{E}\left[\sum_{j=1}^{n} X_j\right]\right]^2 \\
&= \mathbb{E}\left[\sum_{j=1}^{n} X_j - \sum_{j=1}^{n} \mathbb{E}X_j\right]^2 \\
&= \mathbb{E}\left[\sum_{j=1}^{n} (X_j - \mathbb{E}X_j)\right]^2 \\
&= \mathbb{E}\left[\sum_{i=1}^{n}\sum_{j=1}^{n} (X_i - \mathbb{E}X_i)(X_j - \mathbb{E}X_j)\right] \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n} \mathrm{Cov}(X_i, X_j) \\
&= \sum_{j=1}^{n} \mathrm{Var}(X_j).
\end{aligned}$$

最后一步成立因为$\forall i \ne j, X_i, X_j$独立时, $\mathrm{Cov}(X_i, X_j) = 0$.

> *一些经常使用的公式*
>
> $$\begin{gathered}
> \mathbb{E}\left[\sum_{j=1}^{n} X_j\right] = \sum_{j=1}^{n} \mathbb{E}X_j \\
> \mathbb{E}[X + c] = \mathbb{E}X + c \\
> \mathbb{E}[cX] = c\mathbb{E}X \\
> \mathbb{E}c = c \\
> \mathbb{E}\left[\prod_{j=1}^{n} X_j\right] = \prod_{j=1}^{n} \mathbb{E}X_j, \text{要求} X_1, \cdots, X_n \text{相互独立} \\
> \mathrm{Var}(X + c) = \mathrm{Var}(X) \\
> \mathrm{Var}(cX) = c^2\,\mathrm{Var}(X) \\
> \mathrm{Var}(c) = 0 \\
> \mathrm{Var}\left(\sum_{j=1}^{n} X_j\right) = \sum_{j=1}^{n} \mathrm{Var}(X_j), \text{要求} X_1, \cdots, X_n \text{相互独立} \\
> \mathrm{Cov}(X, Y) = \mathbb{E}\left[(X - \mathbb{E}X)(Y - \mathbb{E}Y)\right] = \mathbb{E}[XY] - \mathbb{E}X\mathbb{E}Y \\
> \mathrm{Var}(X) = \mathbb{E}[X - \mathbb{E}X]^2 = \mathbb{E}[X^2] - (\mathbb{E}X)^2 \\
> \mathbb{E}[X^2] = \mathrm{Var}(X\text{）} + (\mathbb{E}X)^2
> \end{gathered}$$

第九周作业

χ^2分布

定义 5.4.1. 设$X_1, X_2, \cdots, X_n\ i.i.d. \sim N(0,1)$,令$X = \sum\limits_{i=1}^{n} X_i^2$,则称$X$是自由度为$n$的$\chi^2$变量,其分布称为自由度为$n$的$\chi^2$**分布**, 记为$X \sim \chi_n^2$.

χ^2变量具有下列性质:

(1) 设随机变量$X \sim \chi_n^2$则有$E(X) = n$, $Var(X) = 2n$.

(2) 设$Z_1 \sim \chi_{n_1}^2$, $Z_2 \sim \chi_{n_2}^2$,且Z_1和Z_2独立, 则$Z_1 + Z_2 \sim \chi_{n_1+n_2}^2$.

t分布

定义 5.4.2. 设随机变量$X \sim N(0,1), Y \sim \chi_n^2$, 且$X$和$Y$独立, 则称

$$T = \frac{X}{\sqrt{Y/n}}$$

为自由为n的t变量, 其分布称为由为n的t**分布**,记为$T \sim t_n$.

t变量具有下列的性质:

(1)若随机变量$T \sim t_n$,则当$n \geq 2$时, $E(T) = 0$. 当$n \geq 3$时, $Var(T) = \frac{n}{n-2}$.

(2)当$n \to \infty$时, t变量的极限分布为$N(0,1)$.

F分布

定义 5.4.3. 设随机变量$X \sim \chi_m^2$, $Y \sim \chi_n^2$,且X和Y独立,则称

$$F = \frac{X/m}{Y/n}$$

为自由度分别是m和n的F变量, 其分布称为自由度分别是m和n的F**分布**, 记为$F \sim F_{m,n}$.

F变量具有下列的性质:

(1) 若$Z \sim F_{m,n}$,则$1/Z \sim F_{n,m}$.

(2) 若$T \sim t_n$,则$T^2 \sim F_{1,n}$

(3) $F_{m,n}(1-\alpha) = 1/F_{n,m}(\alpha)$

正态总体样本和方差的分布

定理 5.4.1. 设$X_1, X_2, \cdots, X_n\ i.i.d. \sim N(a, \sigma^2), \bar{X} = \frac{1}{n}\sum\limits_{i=1}^{n} X_i$和$S^2 = \frac{1}{n-1}\sum\limits_{i=1}^{n}(X_i - \bar{X})^2$分别为样本均值和样本方差,则有

(1) $\bar{X} \sim N(a, \frac{1}{n}\sigma^2)$;

(2) $(n-1)S^2/\sigma^2 \sim \chi_{n-1}^2$;

(3) $\bar{X}$和S^2独立.

重要的推论

> **推论 5.4.1.** 设$X_1, X_2, \cdots, X_n$相互独立相同分布$(i.i.d.) \sim N(a, \sigma^2)$, 则
> $$T = \frac{\sqrt{n}(\bar{X} - a)}{S} \sim t_{n-1}.$$
>
> **推论 5.4.2.** 设$X_1, X_2, \cdots, X_m \ i.i.d. \sim N(a_1, \sigma_1^2)$, $Y_1, Y_2, \cdots, Y_n \ i.i.d. \sim N(a_2, \sigma_2^2)$,且假定$\sigma_1^2 = \sigma_2^2 = \sigma^2$, 样本$X_1, X_2, \cdots, X_m$ 与$Y_1, Y_2, \cdots, Y_n$独立, 则
> $$T = \frac{(\bar{X} - \bar{Y}) - (a_1 - a_2)}{S_w} \cdot \sqrt{\frac{mn}{n+m}} \sim t_{n+m-2},$$
> 此处$(n+m-2)S_w^2 = (m-1)S_1^2 + (n-1)S_2^2$, 其中
> $$S_1^2 = \frac{1}{m-1}\sum_{i=1}^{m}(X_i - \bar{X})^2, \quad S_2^2 = \frac{1}{n-1}\sum_{j=1}^{n}(Y_j - \bar{Y})^2.$$

(习题集第四章-63题) 设随机变量X, Y相互独立, 具有共同分布$N(\mu, \sigma^2)$. 设α, β为两个常数.

(1) 求$\operatorname{Cov}(\alpha X + \beta Y, \alpha X - \beta Y)$.

(2) 当α, β取何值时, $\alpha X + \beta Y$ 与 $\alpha X - \beta Y$ 相互独立.

解:

(1)

$$\begin{aligned}
&\operatorname{Cov}(\alpha X + \beta Y, \alpha X - \beta Y) \\
&= \mathbb{E}\left[(\alpha X + \beta Y)(\alpha X - \beta Y)\right] - \mathbb{E}\left[\alpha X + \beta Y\right]\mathbb{E}\left[\alpha X - \beta Y\right] \\
&= \mathbb{E}\left[\alpha^2 X^2 - \beta^2 Y^2\right] - \mathbb{E}[\alpha X + \beta Y]\mathbb{E}\left[\alpha X - \beta Y\right] \\
&= \alpha^2\mathbb{E}\left[X^2\right] - \beta^2\mathbb{E}\left[Y^2\right] - (\alpha + \beta)\mu \cdot (\alpha - \beta)\mu \\
&= \left(\alpha^2 - \beta^2\right)\left(\mu^2 + \sigma^2\right) - \left(\alpha^2 - \beta^2\right)\mu^2 \\
&= \left(\alpha^2 - \beta^2\right)\sigma^2.
\end{aligned}$$

(2)

> *独立 总是可以推出不相关, 但反过来一般不成立.*
>
> *当(X, Y)是二维正态分布时, 不相关可以推出独立.*

> *关于二元正态分布的一些结论:*
>
> 1. $X \sim N(\mu, \sigma), Y = aX + b, a \neq 0$, *则* $Y \sim N(a\mu + b, a^2\sigma^2)$
> 2. $X \sim N(\mu_1, \sigma_1), Y \sim N(\mu_2, \sigma_2)$, X*与*Y*独立*, α*和*β*是不全为*0*的常数, 则* $\alpha X + \beta Y \sim N\left(\alpha\mu_1 + \beta\mu_2, \alpha^2\sigma_1^2 + \beta^2\sigma_2^2\right)$.

令$\operatorname{Cov}(\alpha X + \beta Y, \alpha X - \beta Y) = \left(\alpha^2 - \beta^2\right)\sigma^2 = 0$, 可以得到

$$\alpha = \pm\beta$$

(一般认为正态分布的方差满足$0 < \sigma^2 < \infty$)

(习题集第六章-16题) 设$X_1, X_2, \cdots, X_9$为独立同分布的正态随机变量, 记

$$Y_1=\frac{1}{6}(X_1+\ldots+X_6),\quad Y_2=\frac{1}{3}(X_7+X_8+X_9),\quad S^2=\frac{1}{2}\sum_{i=7}^{9}(X_i-Y_2)^2.$$

试求$Z=\frac{\sqrt{2}(Y_1-Y_2)}{S}$的分布.

解:

设

$$X_i\sim N(\mu,\sigma^2)$$

那么

$$Y_1\sim N(\mu,\frac{\sigma^2}{6})$$
$$Y_2\sim N(\mu,\frac{\sigma^2}{3})$$

且Y_1和Y_2独立.

所以

$$(Y_1-Y_2)\sim N(0,\frac{\sigma^2}{2})$$
$$\frac{Y_1-Y_2}{\frac{\sigma}{\sqrt{2}}}=\frac{\sqrt{2}(Y_1-Y_2)}{\sigma}\sim N(0,1)$$

又因为

$$\frac{2S^2}{\sigma^2}\sim\chi^2(2)$$

所以

$$Z=\frac{\sqrt{2}(Y_1-Y_2)}{S}=\frac{\sqrt{2}(Y_1-Y_2)}{\sigma}\Big/\sqrt{\frac{2S^2}{2\sigma^2}}\sim t(2)$$

(习题集第六章-17题) 设$X_1,X_2,\cdots,X_{15}$是独立同分布的随机变量, 服从正态分布$N(0,2^2)$. 试求

$$Y=\frac{X_1^2+\cdots+X_{10}^2}{2(X_{11}^2+\cdots+X_{15}^2)}$$

的概率分布.

解:

由题意,

$$
\begin{gathered}
\frac{X_i}{2} \sim N(0,1) \\
\frac{1}{4}(X_1^2 + \cdots + X_{10}^2) \sim \chi^2(10) \\
\frac{1}{4}(X_{11}^2 + \cdots + X_{15}^2) \sim \chi^2(5)
\end{gathered}
$$

所以

$$
Y = \frac{X_1^2 + \cdots + X_{10}^2}{2(X_{11}^2 + \cdots + X_{15}^2)} = (\frac{\frac{1}{4}(X_1^2 + \cdots + X_{10}^2)}{10}) \Big/ (\frac{\frac{1}{4}(X_{11}^2 + \cdots + X_{15}^2)}{5}) \sim F(10,5).
$$

(习题集第六章-19题) 设$X_1, \cdots, X_n$是从两点分布$B(1,p)$中抽取的简单样本, 0<p<1, 记$\bar{X}$为样本均值, 求$S_n^2 = \frac{1}{n}\sum\limits_{i=1}^{n}(X_i - \bar{X})^2$的期望.

解:

由题意,

$$
\begin{gathered}
X \sim B(1,p), \\
\mathbb{E}\bar{X} = \mathbb{E}X = p, \\
\mathrm{Var}(\bar{X}) = \frac{\sigma^2}{n} = \frac{p(1-p)}{n}.
\end{gathered}
$$

所以

$$
\begin{aligned}
\mathbb{E}[S_n^2] &= \mathbb{E}\left[\frac{1}{n}\sum_{i=1}^{n}(X_i-\bar{X})^2\right] \\
&= \frac{1}{n}\mathbb{E}\left[\sum_{i=1}^{n}(X_i-\bar{X})^2\right] \\
&= \frac{1}{n}\mathbb{E}\left[\sum_{i=1}^{n}X_i^2-2\bar{X}\sum_{i=1}^{n}X_i+\sum_{i=1}^{n}\bar{X}^2\right] \\
&= \frac{1}{n}\mathbb{E}\left[\sum_{i=1}^{n}X_i^2-2\bar{X}\cdot n\bar{X}+n\bar{X}^2\right] \\
&= \frac{1}{n}\mathbb{E}\left[\sum_{i=1}^{n}X_i^2-n\bar{X}^2\right] \\
&= \frac{1}{n}\left(\sum_{i=1}^{n}\mathbb{E}\left[X_i^2\right]-n\mathbb{E}\left[\bar{X}^2\right]\right) \\
&= \frac{1}{n}\left(\sum_{i=1}^{n}(\operatorname{Var}X_i+(\mathbb{E}X_i)^2)-n\left(\operatorname{Var}\bar{X}+(\mathbb{E}\bar{X})^2\right)\right) \\
&= \frac{1}{n}\left(\sum_{i=1}^{n}(p(1-p)+p^2)-n\left(\frac{p(1-p)}{n}+p^2\right)\right) \\
&= \frac{1}{n}\left(np-p(1-p)-np^2\right) \\
&= \frac{1}{n}\left((n-1)(p-p^2)\right) \\
&= \frac{n-1}{n}p(1-p).
\end{aligned}
$$

(习题集第六章-20题) 设$X_1,\cdots,X_n$为来自正态总体$N(a,\sigma^2)$的一个简单随机样本, $\bar{X}$和S_n^2分别表示样本均值和样本方差, 又设$X_{n+1}\sim N(a,\sigma^2)$且与$X_1,\cdots,X_n$独立, 试求统计量$\frac{X_{n+1}-\bar{X}}{S_n}\sqrt{\frac{n}{n+1}}$的分布.

解:

$$
\begin{gathered}
\bar{X}\sim N(a,\frac{\sigma^2}{n}), \\
(X_{n+1}-\bar{X})\sim N(0,\frac{n+1}{n}\sigma^2), \\
\frac{X_{n+1}-\bar{X}}{\sigma}\sqrt{\frac{n}{n+1}}\sim N(0,1), \\
\frac{(n-1)S_n^2}{\sigma^2}\sim\chi^2(n-1), \\
\frac{X_{n+1}-\bar{X}}{S_n}\sqrt{\frac{n}{n+1}} = \left(\frac{X_{n+1}-\bar{X}}{\sigma}\sqrt{\frac{n}{n+1}}\right)\Big/(\sqrt{\frac{(n-1)S_n^2}{(n-1)\sigma^2}})\sim t(n-1).
\end{gathered}
$$

(习题集第六章-21题) 设$X_1,\cdots,X_m$为来自正态总体$N(\mu_1,\sigma^2)$的一个简单随机样本, $Y_1,\cdots,Y_n$为来自正态总体$N(\mu_2,\sigma^2)$的一个简单随机样本, 且$X_1,\cdots,X_m$和$Y_1,\cdots,Y_n$相互独立, $\bar{X}$和$\bar{Y}$分别表示它们的样本均值, S_{1m}^2和S_{2m}^2分别表示它们的样本方差, α和β是给定的实数, 试求

$$T=\frac{\alpha\left(\bar{X}-\mu_1\right)+\beta\left(\bar{Y}-\mu_2\right)}{\sqrt{\frac{(m-1)S_{1m}^2+(n-1)S_{2n}^2}{n+m-2}\cdot\left(\frac{\alpha^2}{m}+\frac{\beta^2}{n}\right)}}$$

的分布.

解:

由题意,

$$\bar{X}\sim N\left(\mu_1,\frac{\sigma^2}{m}\right),\bar{Y}\sim N\left(\mu_2,\frac{\sigma^2}{n}\right),$$

所以

$$\alpha\bar{X}+\beta\bar{Y}\sim N\left(\alpha\mu_1+\beta\mu_2,\left(\frac{\alpha^2}{m}+\frac{\beta^2}{n}\right)\sigma^2\right)$$
$$U=\frac{\alpha\left(\bar{X}-\mu_1\right)+\beta\left(\bar{Y}-\mu_2\right)}{\sigma\sqrt{\frac{\alpha^2}{m}+\frac{\beta^2}{n}}}\sim N(0,1)$$

又因为

$$\frac{(m-1)S_{1m}^2}{\sigma^2}\sim\chi^2(m-1),\frac{(n-1)S_{2m}^2}{\sigma^2}\sim\chi^2(n-1)$$

所以

$$V=\frac{(m-1)S_{1m}^2+(n-1)S_{2m}^2}{\sigma^2}=\frac{(m-1)S_{1m}^2}{\sigma^2}+\frac{(n-1)S_{2m}^2}{\sigma^2}\sim\chi^2(m+n-2)$$

所以

$$T=\frac{\alpha\left(\bar{X}-\mu_1\right)+\beta\left(\bar{Y}-\mu_2\right)}{\sqrt{\frac{(m-1)S_{1m}^2+(n-1)S_{2n}^2}{n+m-2}\cdot\left(\frac{\alpha^2}{m}+\frac{\beta^2}{n}\right)}}=\frac{U}{\sqrt{\frac{V}{m+n-2}}}\sim t(m+n-2)$$