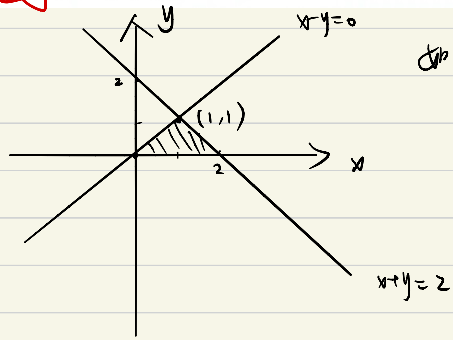
35. (2011年全国考研试题) 设 (X,Y) 在 G 上服从均匀分布，G 由 $x-y=0, x+y=2$ 与 $y=0$ 围成.

(1) 求边缘密度函数 $f_X(x)$;

(2) 求 $f_{X|Y}(x|y)$.



如图，所围区域面积 $S = 2\times\frac{1}{2}\times 1 = 1$

$$\therefore f(x,y) = \begin{cases} 1, & (x,y)\in G \\ 0, & (x,y)\notin G \end{cases}$$

(1). $$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\cdot I[G]\,dy = \begin{cases} \int_0^x 1\,dy = x, & 0<x\le 1 \\ \int_0^{2-x} 1\,dy = 2-x, & 1<x<2 \end{cases}$$

(2). $$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\, I(G)\,dx = \int_y^{2-y} 1\,dx = 2-2y, \quad 0<y<1$$

在 G 上，$f(x,y)$ 连续；在 $y\in(0,1)$，$f_Y(y)$ 连续，∴有：

$$P(X|Y=x|y) = P(X=x|Y=y) = \frac{P(X=x, Y=y)}{P(Y=y)} = \frac{f(x,y)\,dx\,dy}{f_Y(y)\,dy} = \frac{1}{2-2y}\,dx$$

$$\therefore f_{X|Y}(x|y) = \frac{1}{2-2y}, \quad 0<y<1, \quad y<x<2-y$$

36. 设随机向量 (X,Y) 服从区域 D 内的均匀分布，其中 D 是由直线 $y=x, x=0, y=1$ 所围成的区域，试求:

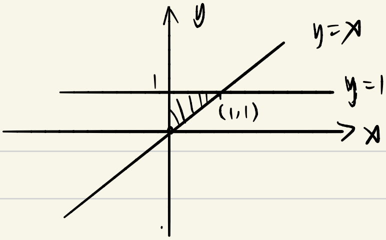
(1) (X,Y) 的联合密度函数 $f(x,y)$;

(2) (X,Y) 的边缘密度函数 $f_1(x)$ 和 $f_2(y)$;

(3) 条件密度 $f_{X|Y}(x|y)$;

(4) $\mathbb{E}(X|Y=y)$.

(1).



所围区域面积为 $S_D = 1 \times 1 \times \frac{1}{2} = \frac{1}{2}$

$$\therefore f(x,y) = \begin{cases} 2, & (x,y) \in D \\ 0, & (x,y) \notin D \end{cases}$$

(2). $f_1(x) = \int_{-\infty}^{\infty} f(x,y)\, I[D]\, dy = \int_x^1 2\, dy = 2-2x, \quad 0<x<1$

$f_2(y) = \int_{-\infty}^{\infty} f(x,y)\, I[D]\, dx = \int_0^y 2\, dx = 2y, \quad 0<y<1$

(3). $f_{X|Y}(x|y) = \dfrac{f(x,y)}{f_Y(y)} = \dfrac{2}{2y} = \dfrac{1}{y}, \quad 0<y<1,\ 0<x<y$

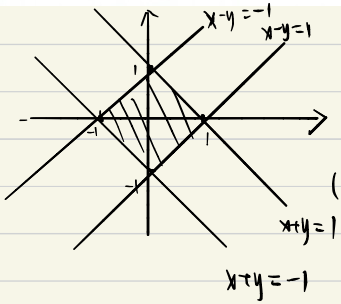
$-1 \le x+y \le 1 \quad -1 \le x-y \le 1$

37. 设随机向量 (X, Y) 服从 $\{(x, y) : |x + y| \le 1, |x - y| \le 1\}$ 内的均匀分布,

(1) 试求出 X 和 Y 的边缘分布; 求边缘密度函数.

(2) X 和 Y 是否相互独立?

(3) 求在 $X = x\ (0 < x < 1)$ 时 Y 的条件密度函数.



记 $D = \{(x,y) \mid |x+y| \le 1, |x-y| \le 1\}$, $S_D = \sqrt{2} \times \sqrt{2} = 2$

则 $f(x,y) = \begin{cases} \frac{1}{2}, & (x,y) \in D \\ 0, & (x,y) \notin D \end{cases}$ $\qquad \frac{1}{2}[(x+1)-(-1-x)]$.

(1). $f_X(x) = \int_{-\infty}^{\infty} f(x,y) \cdot I[D]\, dy = \begin{cases} \int_{-1-x}^{x+1} \frac{1}{2}\, dy = x+1, & -1 \le x \le 0 \\ \int_{x-1}^{1-x} \frac{1}{2}\, dy = 1-x, & 0 < x \le 1 \end{cases}$

$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\, I[D]\, dx = \begin{cases} \int_{-1-y}^{1+y} \frac{1}{2}\, dx = y+1, & -1 \le y \le 0 \\ \int_{y-1}^{1-y} \frac{1}{2}\, dx = 1-y, & 0 < y \le 1 \end{cases}$

$$F(x) = P(X \le x) = \begin{cases} 0, & x \le -1 \\ \int_{-1}^{x} (t+1)\, dt = \frac{1}{2}t^2 + t \Big|_{-1}^{x} = \frac{1}{2}x^2 + x + \frac{1}{2}, & -1 < x \le 0 \\ \int_{-1}^{0} (t+1)\, dt + \int_0^x (1-t)\, dt = \frac{1}{2} + (t - \frac{1}{2}t^2)\Big|_0^x = -\frac{1}{2}x^2 + x + \frac{1}{2}, & 0 < x < 1 \\ \int_{-1}^{0} (t+1)\, dt + \int_0^1 (1-t)\, dt = \frac{1}{2} + \frac{1}{2} = 1, & x \ge 1 \end{cases}$$

$$F(y)=P(Y\le y)=\begin{cases}0, & y\le -1\\ \int_{-1}^{y}(t+1)\,dt=\frac{1}{2}y^2+y+\frac{1}{2}, & -1<y\le 0\\ \int_{-1}^{0}(t+1)\,dt+\int_{0}^{y}(1-t)\,dt=-\frac{1}{2}y^2+y+\frac{1}{2}, & 0<y<1\\ \int_{-1}^{0}(t+1)\,dt+\int_{0}^{1}(1-t)\,dt=1, & y\ge 1\end{cases}$$

(2). 取定 $X=x$ $\left\{\begin{array}{l}0<x\le 1,\ \text{则}\ x-1<Y<1-x\\ -1\le x\le 0\text{时},\ \text{则}\ -1-x<Y<x+1,\end{array}\right\}\Rightarrow$ 可见 Y 的取值与 X 有关

$\therefore$ X,Y 不独立.

(或者由 $f_X(x)\cdot f_Y(y)\ne f(x,y)=\frac{1}{2}$ 说明)

$\Downarrow$ 4个象限均满足 $\ne f(x,y)$.

(3). $$f_{Y|X}(y|x)=\frac{f(x,y)}{f_X(x)}=\frac{\frac{1}{2}}{1-x}=\frac{1}{2-2x},\quad 0<x<1,\ x-1<y<1-x$$