Hw 8 2020/05/09

4.1 (1) $\begin{cases} \Delta_2 u = 0, & -\infty < x < +\infty, y > 0 \\ u(x,0) = f(x) \\ \text{当 } x^2+y^2 \to \infty \text{ 时}, & u(x,y) \to 0 \end{cases}$

x 取值范围无界 $\longrightarrow$ 对 x 作 Fourier 变换

$$\begin{cases} \dfrac{d^2\hat{u}}{dy^2} - \lambda^2 \hat{u} = 0 \\ \hat{u}(\lambda, 0) = \hat{f}(\lambda) \end{cases}$$

通解 $\hat{u}(\lambda, y) = A(\lambda) e^{\lambda y} + B(\lambda) e^{-\lambda y}$

代入边界条件及有界性条件，有

$$\hat{u}(\lambda, y) = \begin{cases} \hat{f}(\lambda) e^{-\lambda y}, & \lambda \geq 0 \\ \hat{f}(\lambda) e^{\lambda y}, & \lambda < 0 \end{cases} = \hat{f}(\lambda) e^{-|\lambda| y}$$

作反变换：

$$F^{-1}[e^{-|\lambda|y}] = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-|\lambda|y} e^{i\lambda x} d\lambda = \frac{1}{\pi} \int_0^{+\infty} e^{-\lambda y} \cos\lambda x \, d\lambda$$

$$= \frac{1}{\pi} \cdot \frac{y}{x} \cdot \int_0^{+\infty} \sin\lambda x \, e^{-\lambda y} d\lambda$$

$$=\frac{1}{\pi}\cdot\frac{y}{x^2}\left(-e^{-\lambda y}\cos\lambda x\Big|_0^{+\infty}-y\int_0^{+\infty}e^{-\lambda y}\cos\lambda x\,d\lambda\right)$$

$$=\frac{1}{\pi}\cdot\frac{y}{x^2}\Big/\left(1+\frac{y^2}{x^2}\right)=\frac{1}{\pi}\cdot\frac{y}{x^2+y^2}$$

$$\Rightarrow u(x,y)=f(x)*F^{-1}[e^{-\lambda|y|}]$$

$$=f(x)*\frac{y}{\pi}\cdot\frac{1}{x^2+y^2}$$

$$=\frac{y}{\pi}\int_{-\infty}^{+\infty}\frac{f(\xi-x)\,d\xi}{\xi^2+y^2}$$

(3) $$\begin{cases} u_t=a^2u_{xx} & (0<x<+\infty,\ t>0) \\ u(t,0)=\varphi(t),\ u(0,x)=0 \\ u(t,+\infty)=u_x(t,+\infty)=0 \end{cases}$$

x取值范围在半直线上 $\longrightarrow$ 正弦变换

$$F_s\left[\frac{\partial^2u}{\partial x^2}\right]=\int_0^{+\infty}\frac{\partial^2u}{\partial x^2}\sin\lambda x\,dx=\frac{\partial u}{\partial x}\sin\lambda x\Big|_0^{+\infty}-\lambda\int_0^{+\infty}\frac{\partial u}{\partial x}\cos\lambda x\,dx$$

$$=-\lambda u\cos\lambda x\Big|_0^{+\infty}-\lambda^2\int_0^{+\infty}u\sin\lambda x\,dx$$

$$=-\lambda\varphi(t)-\lambda^2\hat{u}$$

$$\Rightarrow \begin{cases} \dfrac{d\hat{u}}{dt} + a^2\lambda^2\hat{u} = a^2\lambda\varphi(t) \\ \hat{u}(0,\lambda) = 0 \end{cases}$$

$$\Rightarrow \hat{u}(t,\lambda) = e^{-a^2\lambda^2 t}\int_0^t a^2\lambda\varphi(\tau)e^{a^2\lambda^2\tau}\,d\tau$$

作反变换：

$$u(t,x) = \frac{2}{\pi}\int_0^{+\infty} e^{-a^2\lambda^2 t}\int_0^t a^2\lambda\varphi(\tau)e^{a^2\lambda^2\tau}\,d\tau\,\sin\lambda x\,d\lambda$$

$$= \frac{2}{\pi}\int_0^t a^2\varphi(\tau)\,d\tau\int_0^{+\infty}\lambda\sin\lambda x\,e^{-a^2\lambda^2(t-\tau)}\,d\lambda$$

$$= \frac{1}{\pi}\int_0^t \frac{\varphi(\tau)}{t-\tau}\,d\tau\int_0^{+\infty}\sin\lambda x\,d\,e^{-a^2\lambda^2(t-\tau)}$$

$$= \frac{1}{\pi}\int_0^t \frac{\varphi(\tau)}{t-\tau}\,d\tau\int_0^{+\infty}x\cos\lambda x\,e^{-a^2\lambda^2(t-\tau)}\,d\lambda$$

$$= \frac{1}{\pi}\int_0^t \frac{x\varphi(\tau)}{t-\tau}\,d\tau\cdot\frac{1}{2}\int_{-\infty}^{+\infty}\cos\lambda x\,e^{-a^2\lambda^2(t-\tau)}\,d\lambda$$

$$= \frac{1}{2\pi}\int_0^t \frac{x\varphi(\tau)}{t-\tau}\,d\tau\cdot\int_{-\infty}^{+\infty}\mathrm{Re}\{e^{i\lambda x}\}\,e^{-a^2\lambda^2(t-\tau)}\,d\lambda$$

$$= \frac{1}{2\pi}\int_0^t \frac{x\varphi(\tau)}{t-\tau}\,d\tau\cdot\mathrm{Re}\left\{\int_{-\infty}^{+\infty} e^{-a^2(t-\tau)\left(\lambda-\frac{ix}{2a^2(t-\tau)}\right)^2}\cdot e^{-\frac{x^2}{4a^2(t-\tau)}}\,d\lambda\right\}$$

$$=\frac{1}{2\pi}\int_0^t \frac{x\varphi(\tau)}{t-\tau}\cdot e^{-\frac{x^2}{4a^2(t-\tau)}}\sqrt{\frac{\pi}{a^2(t-\tau)}}\,d\tau$$

$$=\frac{x}{2a\sqrt{\pi}}\int_0^t (t-\tau)^{-\frac{3}{2}}\cdot e^{-\frac{x^2}{4a^2(t-\tau)}}\cdot\varphi(\tau)\,d\tau$$

其中 $\displaystyle\int_{-\infty}^{+\infty} e^{-a^2(t-\tau)\left(\lambda-\frac{ix}{2a^2(t-\tau)}\right)^2}\cdot d\lambda$

$$=\int_{-\infty}^{+\infty} e^{-a^2(t-\tau)\lambda^2}\cdot d\lambda=\frac{1}{\sqrt{a^2(t-\tau)}}\cdot\int_{-\infty}^{+\infty} e^{-x^2}dx$$

$$\left(\int_{-\infty}^{+\infty} e^{-x^2}dx\right)^2=\int_{-\infty}^{+\infty} e^{-x^2}dx\int_{-\infty}^{+\infty} e^{-y^2}dy$$

$$=\iint_{\mathbb{R}^2} e^{-(x^2+y^2)}dxdy=\int_0^{2\pi}d\theta\int_0^{+\infty} e^{-r^2}rdr$$

$$=2\pi\int_0^{+\infty} e^{-r^2}\cdot rdr=\pi\int_0^{+\infty} e^{-r^2}dr^2=\pi e^{-r^2}\Big|_{+\infty}^{0}=\pi$$

4.2(1) $\begin{cases}\dfrac{\partial^2 u}{\partial x\partial y}=1 & (x>0,\ y>0)\\ u(0,y)=y+1,\ u(x,0)=1\end{cases}$

注：本题关于 x, y 的初值条件都给足了，因此既可以对 x 作变换，又可以对 y 作.

不使用Laplace变换：$u(x,y)=xy+f(x)+g(y)$

代入初值条件：

$$\begin{cases} u(0,y)=f(0)+g(y)=y+1 \\ u(x,0)=f(x)+g(0)=1 \end{cases}$$

$\Rightarrow f(x)=1-g(0)$

$\Rightarrow g(y)=y+1-f(0)=y+g(0)$

$\Rightarrow f(x)+g(y)=y+1$

$\Rightarrow u(x,y)=xy+y+1$

对x作Laplace变换：

$L[\frac{\partial u}{\partial x}]=p\bar{u}-u(0,y)=p\bar{u}-y-1$

$$\Rightarrow \begin{cases} \frac{\partial}{\partial y}(p\bar{u}-y-1)=\frac{1}{p} \\ \bar{u}(p,0)=\frac{1}{p} \end{cases}$$

$\Rightarrow \frac{\partial \bar{u}}{\partial y}=\frac{1}{p^2}+\frac{1}{p} \quad \Rightarrow \bar{u}(p,y)=\frac{p+1}{p^2}y+f(p)$

由 $\bar{u}(p,0)=f(p)=\frac{1}{p} \Rightarrow \bar{u}(p,y)=\frac{p+1}{p^2}y+\frac{1}{p}$

反变换得：$u(x,y)=xy+y+1$

对 y 作 Laplace 变换：

$$L[\frac{\partial u}{\partial y}]=p\bar{u}-u(x,0)=p\bar{u}-1$$

$$L[y+1]=\frac{1}{p^2}+\frac{1}{p}$$

$$\Rightarrow \begin{cases} \frac{\partial}{\partial x}(p\bar{u}-1)=\frac{1}{p} \\ \bar{u}(0,p)=\frac{1}{p}+\frac{1}{p^2} \end{cases}$$

$$\Rightarrow \frac{\partial \bar{u}}{\partial x}=\frac{1}{p^2} \quad \Rightarrow \quad \bar{u}(x,p)=\frac{x}{p^2}+f(p)$$

由 $\bar{u}(0,p)=f(p)=\frac{1}{p}+\frac{1}{p^2}$

$$\Rightarrow \bar{u}(x,p)=\frac{1}{p^2}(1+x)+\frac{1}{p}$$

反变换得 $u(x,y)=y(1+x)+1=xy+y+1$

(2) $\begin{cases} u_t = a^2 u_{xx}, & (t>0,\ 0<x<l) \\ u_x(t,0)=0,\ u(t,l)=u_0 \\ u(0,x)=u_1 \end{cases}$

对t作Laplace变换：

$$\begin{cases} p\bar{u}-u_1 = a^2\bar{u}_{xx} & (*) \\ \bar{u}_x(p,0)=0,\ \bar{u}(p,l)=\frac{u_0}{p} \end{cases}$$

(*)式等价于 $\bar{u}-\frac{u_1}{p}=\frac{a^2}{p}\left(\bar{u}-\frac{u_1}{p}\right)_{xx}$

$$\Rightarrow \bar{u}(p,x)=\frac{u_1}{p}+A\,\mathrm{sh}\frac{\sqrt{p}}{a}x+B\,\mathrm{ch}\frac{\sqrt{p}}{a}x$$

代入边界条件：

$$\begin{cases} \bar{u}_x(p,0)=A\cdot\frac{\sqrt{p}}{a}=0 \Rightarrow A=0 \\ \bar{u}(p,l)=\frac{u_1}{p}+B\,\mathrm{ch}\frac{\sqrt{p}}{a}l=\frac{u_0}{p} \Rightarrow B=\frac{u_0-u_1}{p\,\mathrm{ch}\frac{\sqrt{p}}{a}l} \end{cases}$$

$$\Rightarrow \bar{u}(p,x)=\frac{u_1}{p}+\frac{u_0-u_1}{p}\cdot\frac{\mathrm{ch}\frac{\sqrt{p}}{a}x}{\mathrm{ch}\frac{\sqrt{p}}{a}l}$$

使分母为零的p：$p=0$， $p_k=\left(\frac{(2k+1)\pi a i}{2l}\right)^2$，$k\in\mathbb{Z}$

利用留数定理计算 Laplace 反变换：

$$u(t,x) = L^{-1}[\bar{u}(p,x)]$$

$$= u_1 + \sum \mathrm{Res}\left[\frac{u_0-u_1}{p}\cdot\frac{\mathrm{ch}\frac{\sqrt{p}}{a}x}{\mathrm{ch}\frac{\sqrt{p}}{a}l}\cdot e^{pt}, p_k\right]$$

$$= u_1 + (u_0-u_1) + (u_0-u_1)\sum_k\left(\frac{p-p_k}{p}\cdot\frac{\mathrm{ch}\frac{\sqrt{p}}{a}x}{\mathrm{ch}\frac{\sqrt{p}}{a}l}\cdot e^{pt}\right)\Bigg|_{p=p_k}$$

$$= u_0 + (u_0-u_1)\sum_k -\left(\frac{2l}{(2k+1)\pi a}\right)^2\cdot\cos\frac{(2k+1)\pi x}{2l}\cdot\exp\left(-\left(\frac{(2k+1)\pi a}{2l}\right)^2 t\right)\cdot\lim_{p\to p_k}\frac{p-p_k}{\mathrm{ch}\frac{\sqrt{p}}{a}l}$$

其中 $\displaystyle \lim_{p\to p_k}\frac{p-p_k}{\mathrm{ch}\frac{\sqrt{p}}{a}l} = \lim_{p\to p_k}\frac{1}{\mathrm{sh}\frac{\sqrt{p}}{a}l\cdot\frac{l}{a}\cdot\frac{1}{2\sqrt{p}}} = \frac{2a\sqrt{p}}{l\,\mathrm{sh}\frac{\sqrt{p}}{a}l}\Bigg|_{p=p_k}$

$$= \frac{a^2(2k+1)\pi}{l^2\sin\left(\frac{\pi}{2}+k\pi\right)} = (2k+1)\pi\cdot\frac{a^2}{l^2}\cdot(-1)^k$$

$\Rightarrow u(t,x)$

$$= u_0 + \frac{4}{\pi}(u_0-u_1)\sum_{k=0}^{\infty}\frac{(-1)^{k+1}}{2k+1}\cdot\cos\frac{(2k+1)\pi x}{2l}\cdot\exp\left(-\left(\frac{(2k+1)\pi a}{2l}\right)^2 t\right)$$

(3) $\begin{cases} u_t = a^2 u_{xx} - hu \quad (x>0, t>0, h>0) \\ u(0,x) = b, \quad u(t,0) = 0 \\ \lim\limits_{x\to\infty} u_x = 0 \end{cases}$

对t作 Laplace 变换得：

$$\begin{cases} p\bar{u} - b = a^2 \bar{u}_{xx} - h\bar{u} \qquad (*) \\ \bar{u}(p,0) = 0, \quad \lim\limits_{x\to\infty} \bar{u}_x = 0 \end{cases}$$

(*)式等价于 $\bar{u} - \frac{b}{p+h} = \frac{a^2}{p+h}\left(\bar{u} - \frac{b}{p+h}\right)_{xx}$

$$\Rightarrow \bar{u}(p,x) = \frac{b}{p+h} + Ae^{\frac{\sqrt{p+h}}{a}x} + Be^{-\frac{\sqrt{p+h}}{a}x}$$

代入边界条件：

$$\begin{cases} \lim\limits_{x\to\infty} \bar{u}_x = 0 \Rightarrow A = 0 \\ \bar{u}(p,0) = \frac{b}{p+h} + B = 0 \Rightarrow B = -\frac{b}{p+h} \end{cases}$$

$$\Rightarrow \bar{u}(p,x) = \frac{b}{p+h}\left(1 - e^{-\frac{\sqrt{p+h}}{a}x}\right)$$

$$\Rightarrow u(t,x) = L^{-1}[\bar{u}(p,x)] = be^{-ht}\left[1 - \operatorname{erfc}\left(\frac{x}{2a\sqrt{t}}\right)\right]$$

补充题：$\begin{cases} \Delta_2 u = 0 \quad (x>0,\ y>0) \\ u|_{y=0} = f(x),\ u_x|_{x=0} = 0,\ u(x,y)\text{有界} \end{cases}$

添加边界条件：$\lim\limits_{x\to\infty} u(t,x) = \lim\limits_{x\to\infty} u_x(t,x) = 0$

$$F_c[\frac{\partial^2 u}{\partial x^2}] = \int_0^{+\infty} \frac{\partial^2 u}{\partial x^2} \cos\lambda x\, dx = \int_0^{+\infty} \cos\lambda x\, d(\frac{\partial u}{\partial x})$$

$$= \cos\lambda x \frac{\partial u}{\partial x}\Big|_0^{+\infty} + \int_0^{+\infty} \lambda \frac{\partial u}{\partial x} \sin\lambda x\, dx$$

$$= \int_0^{+\infty} \lambda \sin\lambda x\, du$$

$$= \lambda \sin\lambda x\, u\Big|_0^{+\infty} - \int_0^{+\infty} \lambda^2 u \cos\lambda x\, dx$$

$$= -\lambda^2 \hat{u}$$

作余弦变换得

$$\begin{cases} \dfrac{d^2\hat{u}}{dy^2} - \lambda^2 \hat{u} = 0 \\ \hat{u}(\lambda,0) = \hat{f}(\lambda) \end{cases}$$

通解 $\hat{u}(\lambda,y) = A(\lambda)e^{\lambda y} + B(\lambda)e^{-\lambda y}$

代入边界条件及有界性条件，有

$$\hat{u}(\lambda,y)=\begin{cases}\hat{f}(\lambda)e^{-\lambda y}, & \lambda\geq 0\\ \hat{f}(\lambda)e^{\lambda y}, & \lambda<0\end{cases}=\hat{f}(\lambda)e^{-|\lambda|y}$$

$$u(x,y)=\frac{2}{\pi}\int_0^{+\infty}\hat{f}(\lambda)e^{-|\lambda|y}\cos\lambda x\,d\lambda$$

$$=\frac{2}{\pi}\int_0^{+\infty}d\lambda\int_0^{+\infty}f(\xi)\cos\lambda\xi\,d\xi\;e^{-\lambda y}\cos\lambda x$$

$$=\frac{1}{\pi}\int_0^{+\infty}f(\xi)d\xi\int_0^{+\infty}[\cos(x+\xi)\lambda+\cos(x-\xi)\lambda]e^{-\lambda y}d\lambda$$

计算 $\int_0^{+\infty}\cos a\lambda\, e^{-b\lambda}d\lambda=\frac{1}{-b}\int_0^{+\infty}\cos a\lambda\, de^{-b\lambda}$

$$=\frac{1}{-b}\cos a\lambda\, e^{-b\lambda}\Big|_0^{+\infty}+\frac{a}{-b}\int_0^{+\infty}\sin a\lambda\, e^{-b\lambda}d\lambda$$

$$=\frac{1}{b}+\frac{a}{b^2}\sin a\lambda\, e^{-b\lambda}\Big|_0^{+\infty}-\frac{a^2}{b^2}\int_0^{+\infty}\cos a\lambda\, e^{-b\lambda}d\lambda$$

$$=\frac{\frac{1}{b}}{1+\frac{a^2}{b^2}}=\frac{b}{a^2+b^2}$$

$$\Rightarrow u(x,y)=\frac{1}{\pi}\int_0^{+\infty}f(\xi)\left(\frac{y}{(x-\xi)^2+y^2}+\frac{y}{(x+\xi)^2+y^2}\right)d\xi$$