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CSEN 502 Theory of Computation, Winter Term 2021
Assignment1

Exercise 1-1

Reading

Read Chapter 0 to page 20 of the text. You may skip the section on Boolean logic.

Exercise 1-2

Exercises from Textbook

Sipser (*pp* 25 - 27 International Edition): Solve exercises 0.3[1], 0.4[2] (skip e), 0.5, 0.6, and 0.7

Solution:

0.3 AxB contains all the pairs that have as their first element an element of set A and as their second element and element of set B. To compose a set containing all possible pairs, we have a possibilities for the first element of the pair and b possibilities for the second element of the pair, which makes a total of $a * b$ different pairs.

0.4 a) The infinite set of all odd natural numbers.
b) The infinite set of all even integers.
c) The set of all natural numbers divisible by two (even natural numbers).
d) The set of all natural numbers divisible by six.
e) (skip)
f) The empty set $\varnothing$

0.5 The power set P of set C has 2^c elements. If we start with an empty set C, with $P = \{\{\}\}$, then whenever C grows, the size of P gets doubled. **Example:** consider that the power set P, for a set C with n elements, has x elements. If one adds one element to C, the power set will grow to have $2x$ elements.
$C = \{a, b, c\}$
$P = \{\{\}, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$
$C_{new} = \{a, b, c, x\}$
$P_{new} = \{\{\}, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\},$
$\{x\}, \{a, x\}, \{b, x\}, \{c, x\}, \{a, b, x\}, \{a, c, x\}, \{b, c, x\}, \{a, b, c, x\}\}$
Therefore the size of P grows in power of 2 with the growth of C, since each element (subset) in P has to be joined to the new element, so the size of P gets doubled.
This, in a way looks like an induction with a basis and an induction step. However, there are other ways to explain the size of a power set. If you consider the elements of the power set as all possible combinations of C, i.e. two possibilities for each element of C, 1 if it is included and 0 if it is not included. That would make a total of 2^c possibilities.

[1]Exercise 0.4 in normal edition
[2]Exercise 0.1 in normal edition

0.6 a) 7

b) Domain is X and range is Y

c) 6

d) Domain is $X\mathrm{x}Y$ and range is Y

e) 8

0.7 a) Consider a non-empty set X consisting of two subsets X_1 and X_2, such that $X = X_1 \cup X_2$, $X_1 \not\subseteq X_2$ and $X_2 \not\subseteq X_1$. $X_1 \cap X_2$, which is also non-empty contains element a. Define a binary relation R on X (xRy) to be valid if x and y both lie in X_1 or X_2 (or both), so that if x and y lie in two different subsets xRy (or yRx) is false.
This relation is *reflexive*: every element in X lies either in X_1 or X_2 or both, therefore xRx is valid.
This relation is *symmetric*: if xRy is valid (which means that x and y lie in the same subset) then yRx is also valid.
This relation is *not transitive*: if x and y lie in two different subsets xRa is valid and aRy is also valid, but xRy is false!

b) $\leq$ (less than or equal) on $\mathcal{N}$.
This relation is *reflexive*: $n \leq n$ is valid.
This relation is *transitive*: if $n \leq s$ and $s \leq m$ then $n \leq m$ is also valid.
This relation is *not symmetric*: $2 \leq 3$ is valid but $3 \leq 2$ is false!

c) Define a binary relation R on $\mathcal{N}$ that is valid if both its arguments are even.
This relation is *symmetric*: if nRm is valid (both n and m are even natural numbers) then mRn is also valid.
This relation is *transitive*: if nRs is valid (both n and s are even) and sRm is valid as well, then nRm is necessarily valid, too.
This relation is *not reflexive*: $3R3$ is false!

Exercise 1-3

In each of the following cases, determine whether the relation ρ is reflexive, symmetric, anti-symmetric, asymmetric or transitive.

(a) $\rho \subseteq \mathbb{Z} \times \mathbb{Z}$, where $a\ \rho\ b$ if and only if there is $n \in \mathbb{Z}$ such that $a = bn$.

Solution:

The relation is reflexive and transitive.

- reflexive: for n=1 we can always have the pair $(x, x) \in \rho, x = x * 1$
- **not** symmetric: since n must be an integer so we will have for example (4,2) but we cant have (2,4)
- **not** anti-symmetric: since integer numbers contain negative integers as well so we can have pairs like (1,-1) and (-1,1) but 1≠-1.
- **not** asymmetric: since this relation is reflexive so at least we will have (x,x) pairs, also we have pairs (x,-x).
- transitive: by definition of transitivity $xRy\ and\ yRz \to xRz$, to have xRy here means that $x = y * n_1$ and to have yRz here means that $y = z * n_2$, and the question now is that can we get x in terms of z? yes $x = z * n_3 \ \equiv\ x = z * (n_1 * n_2)$. Note that they (n_1, n_2, n_3) could be different since we only care that the given pair exists in this relation.

(b) For a given universe $\mathcal{U}$ and $C \subseteq \mathcal{U}$, where $C \neq \varnothing$, define $\rho \subseteq P(\mathcal{U}) \times P(\mathcal{U})$ (ρ is a set of ordered pairs of sets over $\mathcal{U}$) such that $A\ \rho\ B$ if and only if $A \cup C = B \cup C$.

Solution:

The relation is reflexive, symmetric and transitive.
Note that: C would is fixed throughout the relation, so there exists a certain C which is not empty that would make $A \cup C = B \cup C$ for all $A\ \rho\ B$.

- reflexive: for it to be reflexive the relation should contain (A, A) for all possible sets in the $\mathcal{U}$ $A\ \rho\ B$ means that $A \cup C = B \cup C$, so $A\ \rho\ A$ means $A \cup C = A \cup C$ can we have this statement to be true for whatever value of A to belong in our relation? Yes.
- symmetric: if we have $A \cup C = B \cup C$ can we have $B \cup C = A \cup C$ for every $(A, B) \in\ \rho$? Yes.
- **not** anti-symmetric: since this relation is symmetric we know that the first part of the implication is true, that both $A\ \rho\ B$ and $B\ \rho\ A$ are true, but that does not necessarily mean that A and B are equal.
- **not** asymmetric: since this relation is symmetric.
- transitive: $A\ \rho\ B\ and\ B\ \rho\ D \to A\ \rho\ D$ since this is an equality statement, $A\ \rho\ B$ would mean that $A \cup C = B \cup C$, and $B\ \rho\ D$ would mean that $B \cup C = D \cup C$ (note that it is the same C) which could be seen as $A \cup C = B \cup C = D \cup C$ from which we get that $A \cup C = D \cup C$, which makes $A\ \rho\ D$ true.

(c) $\rho \subseteq \mathbb{Z} \times \mathbb{Z}$ where $x\ \rho\ y$ if and only if $x + y$ is odd.

Solution:

The relation is symmetric.
Note that to have $x + y$ to be odd one of them must be even and the other to be odd.

- **not** reflexive: Since we can't have $x\ \rho\ x$.
- symmetric: Since $x + y = y + x$
- **not** anti-symmetric: The relation would have pairs (x, y) and (y, x) but $x \neq y$
- **not** asymmetric: Since it is symmetric
- **not** transitive: $xRy\ and\ yRz \to xRz$, for xRy to be true the either x is even and y is odd and in this case z would be even for yRz to be true, so xRz can not be true since both of them would be even or both of them would be odd.

(d) $\rho \subseteq (\mathbb{Z} \times \mathbb{Z}) \times (\mathbb{Z} \times \mathbb{Z})$ where $(a, b)\ \rho\ (c, d)$ if and only if $a \leq c$.

Solution:

The relation is reflexive and transitive.
First note that x and y are pairs, $x = (a, b)$ and $y = (c, d)$

- reflexive: Since we can accept when $a = c$, so $x\ \rho\ x$ would be true for any $x \in \mathbb{Z}$.
- **not** symmetric: will fail for the pairs where $a < c$.
- **not** anti-symmetric: The relation would have pairs (x, y) and (y, x) but its not always the case that $x = y$, for example: $x = (1, 2)$ and $y = (1, 3)$ which will make $xRyandyRx$ true but $x \neq y$
- **not** asymmetric: will fail for the pairs where $a = c$
- transitive: $xRy\ and\ yRz \to xRz$, for xRy to be true it means that $x = (a, b)$ and $y = (c, d)$ and $a \leq c$, and having yRz means that, for the same y, $z = (e, f)$ such that $c \leq e$, Which makes $a \leq c \leq e$ therefore, $a \leq e$.

Exercise 1-4

Extra Problem

In each of the following cases, and by filling the appropriate *circles*, indicate whether the relation $\mathcal{R}$ on the *set of line segments in the plane* is reflexive (r), symmetric (s), anti-symmetric (an), asymmetric (as), or transitive (t).

a) $(a, b) \in \mathcal{R}$ if and only if a and b are not equal in length.

r ◯ s ◯ an ◯ as ◯ t ◯

Solution:

The relation is only symmetric.

- **not** reflexive: since $a \neq b$ so we can never have $(a, a) \in \mathcal{R}$ for any $a \in \mathcal{U}$.
- symmetric: for every $a\mathcal{R}b$, which means that $a \neq b$, the relation would always contain $b\mathcal{R}a$, $b \neq a$.
- **not** anti-symmetric: The relation would have pairs (a, b) and (b, a) but $a \neq b$, from the definition of the relation "a and b are not equal in length".
- **not** asymmetric: since it is symmetric.
- **not** transitive: $a\mathcal{R}b \textit{ and } b\mathcal{R}c \rightarrow a\mathcal{R}c$, for example: a could have length 1 and b could have length 2, which makes $a\mathcal{R}b$ true, and c could have length 1, which makes $b\mathcal{R}c$ true, but we can not have the pair $(a, c) \in \mathcal{R}$ because in this case both a and c have equal length.

b) $(a, b) \in \mathcal{R}$ if and only if b is longer than 10 cm.

r ◯ s ◯ an ◯ as ◯ t ◯

Solution:

The relation is only transitive

- **not** reflexive: since the only pairs of the form (a, a) that would be included in this relation are when length of a greater than 10, so this does not cover all the element in the universe.
- **not** symmetric: for every $a\mathcal{R}b$ it's not always the case that $b\mathcal{R}a$ is true, for example: when the length of a is less than or equal 10.
- **not** anti-symmetric: The relation would have pairs $(a, b) \in \mathcal{R}$ and $(b, a) \in \mathcal{R}$ when both the length of a and b are greater than 10 but it is not always the case that a and b are the same line segment.
- **not** asymmetric: will fail for the pairs when both the length of a and b are greater than 10, example when a has length equal to 12 and b has length equal to 13 which will make $a\mathcal{R}b$ true and $b\mathcal{R}a$ also true.
- transitive: $a\mathcal{R}b \textit{ and } b\mathcal{R}c \rightarrow a\mathcal{R}c$, for $a\mathcal{R}b$ to be true, it means that the length of b greater than 10, and having $b\mathcal{R}c$ means that the length of c greater than 10, therefore $(a, c) \in \mathcal{R}$ since c already satisfied the requirement to have length greater than 10 .

c) $(a, b) \in \mathcal{R}$ if and only if a and b have at least two common points.

r ◯ s ◯ an ◯ as ◯ t ◯

Solution:

The relation is reflexive and symmetric

- reflexive: $a\mathcal{R}a$ would always be true, since each line have all the points in common with iteslf so it did meet the minimum requirement, two points.
- symmetric: for every $a\mathcal{R}b$ it' always the case that $b\mathcal{R}a$ is true , so whenever a has at least two common points with b, b would have at least two common points with a, which is always true in this relation..
- **not** anti-symmetric: The relation would have pairs (a, b) and (b, a) but it is not always the case that a and b are the same line segments, two lines could have points in common but its not necessarily for them to be the same line.

- **not** asymmetric: since it is symmetric.
- **not** transitive: $a\mathcal{R}b$ *and* $b\mathcal{R}c \rightarrow a\mathcal{R}c$, for $a\mathcal{R}b$ to be true it means that a has at least two points in common with b, and having $b\mathcal{R}c$ means that b has at least two points in common with c, but that does not necessarily mean that a and c would have points in common.

Exercise 1-5

Programming

Using your favorite programming language, implement an abstract data type for sets. Your implementation should include functions/methods/clauses for checking set membership, checking subset relations among sets, and computing set intersections, unions and differences.