Solutions for exercises of Chapter 8 of "Nielesen and Chuang"

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Exercise 8.1

Under unitary evolution, a pure state transforms as

$$|\psi\rangle \to U|\psi\rangle$$

Equivalently, the density matrix $\rho = |\psi\rangle\langle\psi|$ evolves as

$$\rho \to \mathcal{E}(\rho) = U|\psi\rangle\langle\psi|U^\dagger = U\rho U^\dagger$$

and therefore ρ transforms as $\rho \to U\rho U^\dagger$.

Exercise 8.2

The state after obtaining the measurement result m is

$$|\psi_i^m\rangle = \frac{M_m|\psi_i\rangle}{\sqrt{\langle\psi_i|M_m^\dagger M_m|\psi_i\rangle}}$$

Let $\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$. Hence, for $\mathcal{E}_m(\rho) = M_m\rho M_m^\dagger$, after the measurement the final state is

$$\begin{aligned}
\rho_m &= \sum_i p(i|m)|\psi_i^m\rangle\langle\psi_i^m| = \sum_i p(i|m)\frac{M_m|\psi_i\rangle\langle\psi_i|M_m^\dagger}{\langle\psi_i|M_m^\dagger M_m|\psi_i\rangle} = \\
&= \sum_i \frac{p(m|i)p_i}{p_m}\frac{M_m|\psi_i\rangle\langle\psi_i|M_m^\dagger}{\mathrm{Tr}(M_m^\dagger M_m|\psi_i\rangle\langle\psi_i|)} = \\
&= \sum_i \frac{\mathrm{Tr}(M_m^\dagger M_m|\psi_i\rangle\langle\psi_i|)}{\mathrm{Tr}(M_m^\dagger M_m\rho)} p_i \frac{M_m|\psi_i\rangle\langle\psi_i|M_m^\dagger}{\mathrm{Tr}(M_m^\dagger M_m|\psi_i\rangle\langle\psi_i|)} = \\
&= \sum_i p_i \frac{M_m|\psi_i\rangle\langle\psi_i|M_m^\dagger}{\mathrm{Tr}(M_m^\dagger M_m\rho)} = \frac{M_m\rho M_m^\dagger}{\mathrm{Tr}(M_m^\dagger M_m\rho)} = \\
&= \frac{\mathcal{E}_m(\rho)}{\mathrm{Tr}(\mathcal{E}_m(\rho))}
\end{aligned}$$

where we used the cyclic property of the trace $\mathrm{Tr}(M_m^\dagger M_m\rho) = \mathrm{Tr}(M_m\rho M_m^\dagger)$.
For the probability of the m state, using $p(m|i) = \langle\psi_i|M_m^\dagger M_m|\psi_i\rangle = tr(M_m^\dagger M_m|\psi_i\rangle\langle\psi_i|)$, we get

$$p(m) = \sum_i p_i p(m|i) = \sum_i p_i\langle\psi_i|M_m^\dagger M_m|\psi_i\rangle = \sum_i p_i \,\mathrm{Tr}(M_m^\dagger M_m|\psi_i\rangle\langle\psi_i|) = \mathrm{Tr}(\mathcal{E}_m(\rho))$$

Exercise 8.3

Initially we have the state $\rho \otimes |0_{CD}\rangle\langle 0_{CD}|$.
Consider the action of $\mathcal{E}$, with i basis for A and j basis for D):

$$\begin{aligned}\mathcal{E}(\rho) &= tr_A(tr_D(U[\rho \otimes |0_{CD}\rangle\langle 0_{CD}|]U^\dagger)) = \\ &= \sum_i \sum_j \langle i| \langle j| U[\rho \otimes |0_{CD}\rangle\langle 0_{CD}|]U^\dagger |j\rangle |i\rangle = \\ &= \sum_i \sum_j \langle i| \langle j| U |0_{CD}\rangle \rho \langle 0_{CD}| U^\dagger |j\rangle |i\rangle = \\ &= \sum_j E_j \rho E_j^\dagger\end{aligned}$$

where $E_j = \sum_i \langle i| \langle j| U |0_{CD}\rangle$.
Also, using $\sum_i |i\rangle\langle i| = I$, we have:

$$\begin{aligned}\sum_j E_j^\dagger E_j &= \sum_i \sum_j \langle 0_{CD}| U^\dagger |j\rangle |i\rangle \langle i| \langle j| U |0_{CD}\rangle = \\ &= I \langle 0_{CD}| U^\dagger U |0_{CD}\rangle = I \langle 0_{CD}|0_{CD}\rangle = I\end{aligned}$$

Exercise 8.4

The initial state is $\rho \otimes |0\rangle\langle 0|$. When we apply the quantum operation, it becomes:

$$\mathcal{E}(\rho) = tr_{env}\left[U\left(\rho \otimes |0\rangle\langle 0|\right)U^\dagger\right]$$

where $U = P_0 \otimes \mathcal{I} + P_1 \otimes X$.
We have that $E_k = \langle k| U |0\rangle$, hence using the orthogonality of the $|0\rangle$ and $|1\rangle$ states:

$$\begin{aligned}E_k &= \langle k| U |0\rangle = \\ &= \langle k| \left(|0\rangle\langle 0| \otimes \mathcal{I} + |1\rangle\langle 1| \otimes X\right) |0\rangle = \\ &= \langle 0| |0\rangle\langle 0| \otimes \mathcal{I} |0\rangle + \langle 1| |1\rangle \otimes X |0\rangle = \\ &= \langle 0| \otimes |0\rangle + \langle 1| \otimes |1\rangle = \\ &= |0\rangle\langle 0| + |1\rangle\langle 1|\end{aligned}$$

Thus $E_0 = P_0$, $E_1 = P_1$.
Therefore

$$\mathcal{E}(\rho) = |0\rangle\langle 0| \rho |0\rangle\langle 0| + |1\rangle\langle 1| \rho |1\rangle\langle 1|$$

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