Solutions for exercises of Chapter 9 of 'Nielesen and Chuang'

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Exercise 9.1

$D\left((1,0),(\frac{1}{2},\frac{1}{2})\right) = \frac{1}{2} * 2 * \frac{1}{2} = \frac{1}{2}$

$D\left((\frac{1}{2},\frac{1}{3},\frac{1}{6}),(\frac{3}{4},\frac{1}{8},\frac{1}{8})\right) = \frac{1}{2}(\frac{1}{4} + \frac{5}{24} + \frac{1}{24}) = \frac{1}{4}$

Exercise 9.2

$D\left((p,1-p),(q,1-q)\right) = \frac{1}{2}(|p-q| + |1-p-1+q|) = \frac{1}{2}(|p-q| + |p-q|) = |p-q|$

Exercise 9.3

$F\left((1,0),(\frac{1}{2},\frac{1}{2})\right) = \frac{1}{\sqrt{2}}$

$F\left((\frac{1}{2},\frac{1}{3},\frac{1}{6}),(\frac{3}{4},\frac{1}{8},\frac{1}{8})\right) = \sqrt{\frac{3}{8}} + \sqrt{\frac{1}{24}} + \sqrt{\frac{1}{48}} = 0.96$

Exercise 9.4

$D(p_x,q_x) = \frac{1}{2}\sum_x |p_x - q_x| = \dfrac{1}{2}\left(\sum_{p_x>q_x}(p_x-q_x) - \sum_{p_x<q_x}(p_x-q_x)\right)$

$\sum_{p_x<q_x}(p_x-q_x) = \sum_{p_x<q_x} p_x - \sum_{p_x<q_x} q_x = 1 - \sum_{p_x>q_x} p_x - 1 + \sum_{p_x>q_x} q_x = -\sum_{p_x>q_x}(p_x-q_x)$

Therefore,

$$D(p_x,q_x) = \sum_{p_x>q_x}(p_x-q_x)$$

Looking at the last term, if we add an other $(p_{x'},q_{x'})$ pair to the sum, the overall sum will decrease as $(p_{x'}-q_{x'})$ is negative. Hence,

$$D(p_x,q_x) = \sum_{p_x>q_x}(p_x-q_x) = \max_S \left|\sum_{x\in S}(p_x-q_x)\right|$$

Exercise 9.5

Exploiting the proof of the previous exercise, we have that

$$D(p_x, q_x) = \max_S \left| \sum_{x\in S} (p_x - q_x) \right| = \sum_{p_x > q_x} (p_x - q_x)$$

Thus, because the sum runs over the $p_x > q_x$, we can remove the absolute value signs and write

$$D(p_x, q_x) = \max_S \left(\sum_{x\in S} p_x - \sum_{x\in S} q_x \right)$$

Exercise 9.6

$D\left(\frac{3}{4}|0\rangle\langle 0| + \frac{1}{4}|1\rangle\langle 1|, \frac{2}{3}|0\rangle\langle 0| + \frac{1}{3}|1\rangle\langle 1|\right) = \frac{1}{2}tr\left|\frac{1}{12}|0\rangle\langle 0| - \frac{1}{12}|1\rangle\langle 1|\right| = \frac{1}{12}$

$D\left(\frac{3}{4}|0\rangle\langle 0| + \frac{1}{4}|1\rangle\langle 1|, \frac{2}{3}|+\rangle\langle +| + \frac{1}{3}|-\rangle\langle -|\right) =$

$= D\left(\frac{3}{4}|0\rangle\langle 0| + \frac{1}{4}|1\rangle\langle 1|, \frac{1}{2}(|0\rangle\langle 0| + |1\rangle\langle 1|) + \frac{1}{6}(|0\rangle\langle 1| + |1\rangle\langle 0|)\right) =$

$= \frac{1}{2}tr\left|\frac{1}{4}|0\rangle\langle 0| - \frac{1}{4}|1\rangle\langle 1| + \frac{1}{6}(|0\rangle\langle 1| + |1\rangle\langle 0|)\right| = \frac{\sqrt{13}}{12}$ where we have used $|A| = \sqrt{A^\dagger A}$

Exercise 9.7

Thanks to Theorem 2.4 (singular value decomposition), let $\rho - \sigma = UDU^\dagger = U(\Lambda_+ + \Lambda_-)U^\dagger$, where Λ_+ and Λ_- are the diagonal matrices of the positive and negative eigenvalues of $\rho - \sigma$. Hence, we can write

$$\rho - \sigma = U\Lambda_+U^\dagger + U\Lambda_-U^\dagger = Q - S$$

where $Q = U\Lambda_+U^\dagger$ and $S = -U\Lambda_-U^\dagger$ are positive operators, with their support being the partial eigenbasis of $\rho - \sigma$, which is orthogonal.

Exercise 9.8

Using $\sum_i p_i = 1$ we have,

$$D\left(\sum_i p_i\rho_i, \sigma\right) = D\left(\sum_i p_i\rho_i, \sum_i p_i\sigma\right)$$

From the joint convexity of the trace distance in its inputs established by eq. 9.50: $D\left(\sum_i p_i\rho_i, \sum_i p_i\sigma\right) \leq \sum_i p_i D(\rho_i, \sigma_i)$, it follows that,

$$D\left(\sum_i p_i\rho_i, \sigma\right) = D\left(\sum_i p_i\rho_i, \sum_i p_i\sigma\right) \leq \sum_i p_i D(\rho_i, \sigma)$$

Exercise 9.9

The set of the density matrices (positive, trace one, Hermitian) is convex and compact. Hence, as the CPTP maps are continuous, they have a fixed point.

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Exercise 9.10

Let ρ and σ, $\rho \neq \sigma$, both be fixed points of $\mathcal{E}$. Therefore, $D(\mathcal{E}(\rho), \mathcal{E}(\sigma)) = D(\rho, \sigma)$ from the definition of a fixed point. However, $D(\mathcal{E}(\rho), \mathcal{E}(\sigma)) < D(\rho, \sigma)$, hence we have a contradiction, therefore, $\rho = \sigma$, i.e there's a unique fixed point.

Exercise 9.11

$$\begin{aligned} D(\mathcal{E}(\rho), \mathcal{E}(\sigma)) &= D(p\rho_0 + (1-p)\mathcal{E}'(\rho), p\rho_0 + (1-p)\mathcal{E}'(\sigma)) \\ &\leq pD(\rho_0, \rho_0) + (1-p)D(\mathcal{E}'(\rho), \mathcal{E}'(\sigma)) \\ &\leq (1-p)D(\rho, \sigma) \end{aligned}$$

Therefore, as $0 \leq (1-p) < 1$, we have $D(\mathcal{E}(\rho), \mathcal{E}(\sigma)) < D(\rho, \sigma)$, i.e. $\mathcal{E}$ is strictly contractive.

Exercise 9.12

$$\begin{aligned} D(\mathcal{E}(\rho), \mathcal{E}(\sigma)) &= \frac{1}{2} tr \left| \frac{pI}{2} - (1-p)\rho - \frac{pI}{2} + (1-p)\sigma \right| \\ &= \frac{1}{2}(1-p) tr|\rho - \sigma| \\ &= (1-p)D(\rho, \sigma) \end{aligned}$$

Therefore, as $0 \leq (1-p) < 1$, we have $D(\mathcal{E}(\rho), \mathcal{E}(\sigma)) < D(\rho, \sigma)$.

Exercise 9.13

The bit flip channel is $\mathcal{E}(\rho) = p\rho + (1-p)X\rho X$
Using that $D(X\rho X, X\sigma X) = D(\rho, \sigma)$ (because X unitary and the trace distance is preserved under unitary transformations, as shown in eq. 9.21) and Theorem 9.3 with eq. 9.50, i.e.

$$D\left(\sum_i p_i \rho_i, \sum_i p_i \sigma_i\right) \leq \sum_i p_i D(\rho_i, \sigma_i)$$

we have,

$$\begin{aligned} D(\mathcal{E}(\rho), \mathcal{E}(\sigma)) &= D(p\rho + (1-p)X\rho X, p\sigma + (1-p)X\sigma X) \\ &\leq pD(\rho, \sigma) + (1-p)D(X\rho X, X\sigma X) \\ &= pD(\rho, \sigma) + (1-p)D(\rho, \sigma) = D(\rho, \sigma) \end{aligned}$$

Hence, $\mathcal{E}$ is contractive but not strictly contractive.

Exercise 9.14

Using the fact that density matrices are positive operators and the given identity, so exploiting the property $\sqrt{UAU^\dagger} = U\sqrt{A}U^\dagger$, we have,

$$\begin{aligned} F(U\rho U^\dagger, U\sigma U^\dagger) &= tr\sqrt{(U\rho U^\dagger)^{1/2} U\sigma U^\dagger (U\rho U^\dagger)^{1/2}} \\ &= tr\sqrt{U\rho^{1/2}U^\dagger U\sigma U^\dagger U\rho^{1/2}U^\dagger} \\ &= tr\sqrt{U\rho^{1/2}\sigma\rho^{1/2}U^\dagger} \\ &= tr(U\sqrt{\rho^{1/2}\sigma\rho^{1/2}}U^\dagger) = tr\sqrt{\rho^{1/2}\sigma\rho^{1/2}} = F(\rho, \sigma) \end{aligned}$$

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Exercise 9.16

$\langle m| (A \otimes B) |m\rangle = \sum_{i,j} = \langle i_R| A |j_R\rangle \langle i_Q| B |j_Q\rangle = \sum_{i,j} A^{\dagger}_{ji} B_{ij} = tr(A^{\dagger}B)$

Exercise 9.17

$0 \leq F(\rho, \sigma) \leq 1$, hence for $A(\rho, \sigma) = \arccos F(\rho, \sigma)$ we have $0 \leq A(\rho, \sigma) \leq \frac{\pi}{2}$
$A(\rho, \sigma) = 0$ if and only if $F(\rho, \sigma) = 1$, which is only true if and only if $\rho = \sigma$.

Exercise 9.18

In the range $0 \leq x \leq 1$, $\arccos x$ is a decreasing function, hence from $F(\mathcal{E}(\rho), \mathcal{E}(\sigma)) \geq F(\rho, \sigma)$ we have

$$\arccos F(\mathcal{E}(\rho), \mathcal{E}(\sigma)) \leq \arccos F(\rho, \sigma)$$

Therefore,

$$A(\mathcal{E}(\rho), \mathcal{E}(\sigma)) = \arccos F(\mathcal{E}(\rho), \mathcal{E}(\sigma)) \leq \arccos F(\rho, \sigma) = A(\rho, \sigma)$$

Exercise 9.19

Using Theorem 9.7 and letting $q_i = p_i$ we have

$$F\left(\sum_i p_i \rho_i, \sum_i p_i \sigma_i\right) \geq \sum_i \sqrt{p_i p_i} F(\rho_i, \sigma_i) = \sum_i p_i F(\rho_i, \sigma_i)$$

Exercise 9.20

Using Theorem 9.7 and letting $\sigma_i = \sigma$ we have,

$$F\left(\sum_i p_i \rho_i, \sigma\right) = F\left(\sum_i p_i \rho_i, \sum_i p_i \sigma\right) \geq \sum_i p_i F(\rho_i, \sigma)$$

Exercise 9.21

Exercise 9.22

Exercise 9.23

The ensemble average fidelity $\bar{F}$ is given by

$$\bar{F} = \sum_j p_j F(\rho_j, \mathcal{E}(\rho_j))^2$$

For $\mathcal{E}(\rho_j) = \rho_j$ we have that $F(\rho_j, \rho_j) = 1$, thus the ensemble average fidelity becomes

$$\bar{F} = \sum_j p_j F(\rho_j, \rho_j)^2 = \sum_j p_j 1^2 = 1$$

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Problem 1

Problem 2

Problem 3

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