

Solutions for exercises of Chapter 2 of 'Nielesen and Chuang'

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Exercise 2.1

$$\begin{bmatrix} 1 \\ -1 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Exercise 2.2

$$\begin{aligned} A|0\rangle &= A_{11}|0\rangle + A_{21}|1\rangle = |1\rangle \Rightarrow A_{11} = 0, A_{21} = 1 \\ A|1\rangle &= A_{12}|0\rangle + A_{22}|1\rangle = |0\rangle \Rightarrow A_{12} = 1, A_{22} = 0 \\ \Rightarrow A &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \end{aligned}$$

now we choose the input base $\{|0\rangle, |1\rangle\}$ and the output base $\{|1\rangle, |0\rangle\}$:

$$\begin{aligned} A|0\rangle &= A_{11}|1\rangle + A_{21}|0\rangle = |1\rangle \Rightarrow A_{11} = 1, A_{21} = 0 \\ A|1\rangle &= A_{12}|1\rangle + A_{22}|0\rangle = |0\rangle \Rightarrow A_{12} = 0, A_{22} = 1 \\ A &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{aligned}$$

Exercise 2.3

From eq (2.12):

$$\begin{aligned} A|v_i\rangle &= \sum_j A_{ji}|w_j\rangle \\ B|w_j\rangle &= \sum_k B_{kj}|x_k\rangle \end{aligned}$$

Thus

$$\begin{aligned}
 BA|v_i\rangle &= B\left(\sum_j A_{ji}|w_j\rangle\right) \\
 &= \sum_j A_{ji}B|w_j\rangle \\
 &= \sum_{j,k} A_{ji}B_{kj}|x_k\rangle \\
 &= \sum_k \left(\sum_j B_{kj}A_{ji}\right)|x_k\rangle \\
 &= \sum_k (BA)_{ki}|x_k\rangle \\
 \therefore (BA)_{ki} &= \sum_j B_{kj}A_{ji}
 \end{aligned}$$

so the matrix representation for the linear transformation BA is the matrix product of the matrix representation for B and A.

Exercise 2.4

$$\begin{aligned}
 I|v_j\rangle &= \sum_i I_{ij}|v_i\rangle = |v_j\rangle, \quad \forall j. \\
 &\Rightarrow I_{ij} = \delta_{ij}
 \end{aligned}$$

Exercise 2.5

Defined inner product on \mathcal{C}^n is

$$((y_1, \dots, y_n), (z_1, \dots, z_n)) = \sum_i y_i^* z_i.$$

Verify (1) of eq (2.13).

$$\begin{aligned}
 \left((y_1, \dots, y_n), \sum_i \lambda_i (z_{i1}, \dots, z_{in})\right) &= \sum_i y_i^* \left(\sum_j \lambda_j z_{ji}\right) \\
 &= \sum_{i,j} y_i^* \lambda_j z_{ji} \\
 &= \sum_j \lambda_j \left(\sum_i y_i^* z_{ji}\right) \\
 &= \sum_j \lambda_j ((y_1, \dots, y_n), (z_{j1}, \dots, z_{jn})) \\
 &= \sum_i \lambda_i ((y_1, \dots, y_n), (z_{i1}, \dots, z_{in})).
 \end{aligned}$$

Verify (2) of eq (2.13),

$$((y_1, \dots, y_n), (z_1, \dots, z_n))^* = \left(\sum_i y_i^* z_i \right)^* \quad (1)$$

$$= \left(\sum_i y_i z_i^* \right) \quad (2)$$

$$= \left(\sum_i z_i^* y_i \right) \quad (3)$$

$$= ((z_1, \dots, z_n), (y_1, \dots, y_n)) \quad (4)$$

Verify (3) of eq (2.13),

$$\begin{aligned} ((y_1, \dots, y_n), (y_1, \dots, y_n)) &= \sum_i y_i^* y_i \\ &= \sum_i |y_i|^2 \end{aligned}$$

Since $|y_i|^2 \geq 0$ for all i . Thus $\sum_i |y_i|^2 = ((y_1, \dots, y_n), (y_1, \dots, y_n)) \geq 0$.

Now I will show the following statement,

$$((y_1, \dots, y_n), (y_1, \dots, y_n)) = 0 \text{ iff } (y_1, \dots, y_n) = 0.$$

(\Leftarrow) This is obvious.

(\Rightarrow) Suppose $((y_1, \dots, y_n), (y_1, \dots, y_n)) = 0$. Then $\sum_i |y_i|^2 = 0$.

Since $|y_i|^2 \geq 0$ for all i , if $\sum_i |y_i|^2 = 0$, then $|y_i|^2 = 0$ for all i . Therefore $|y_i|^2 = 0 \Leftrightarrow y_i = 0$ for all i . Thus,

$$(y_1, \dots, y_n) = 0.$$

Exercise 2.6

$$\begin{aligned} \left(\sum_i \lambda_i |w_i\rangle, |v\rangle \right) &= \left(|v\rangle, \sum_i \lambda_i |w_i\rangle \right)^* \\ &= \left[\sum_i \lambda_i (|v\rangle, |w_i\rangle) \right]^* \quad (\because \text{linearity in the 2nd arg.}) \\ &= \sum_i \lambda_i^* (|v\rangle, |w_i\rangle)^* \\ &= \sum_i \lambda_i^* (|w_i\rangle, |v\rangle) \end{aligned}$$

Exercise 2.7

$$\begin{aligned} \langle w|v \rangle &= \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = 1 - 1 = 0 \\ \frac{|w\rangle}{\| |w\rangle \|} &= \frac{|w\rangle}{\sqrt{\langle w|w \rangle}} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ \frac{|v\rangle}{\| |v\rangle \|} &= \frac{|v\rangle}{\sqrt{\langle v|v \rangle}} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \end{aligned}$$

Exercise 2.8

If $k = 1$,

$$\begin{aligned} |v_2\rangle &= \frac{|w_2\rangle - \langle v_1|w_2\rangle |v_1\rangle}{\| |w_2\rangle - \langle v_1|w_2\rangle |v_1\rangle \|} \\ \langle v_1|v_2\rangle &= \langle v_1| \left(\frac{|w_2\rangle - \langle v_1|w_2\rangle |v_1\rangle}{\| |w_2\rangle - \langle v_1|w_2\rangle |v_1\rangle \|} \right) \\ &= \frac{\langle v_1|w_2\rangle - \langle v_1|w_2\rangle \langle v_1|v_1\rangle}{\| |w_2\rangle - \langle v_1|w_2\rangle |v_1\rangle \|} \\ &= 0. \end{aligned}$$

Suppose $\{v_1, \dots, v_n\}$ ($n \leq d-1$) is a orthonormal basis. Then

$$\begin{aligned} \langle v_j|v_{n+1}\rangle &= \langle v_j| \left(\frac{|w_{n+1}\rangle - \sum_{i=1}^n \langle v_i|w_{n+1}\rangle |v_i\rangle}{\| |w_{n+1}\rangle - \sum_{i=1}^n \langle v_i|w_{n+1}\rangle |v_i\rangle \|} \right) \quad (j \leq n) \\ &= \frac{\langle v_j|w_{n+1}\rangle - \sum_{i=1}^n \langle v_i|w_{n+1}\rangle \langle v_j|v_i\rangle}{\| |w_{n+1}\rangle - \sum_{i=1}^n \langle v_i|w_{n+1}\rangle |v_i\rangle \|} \\ &= \frac{\langle v_j|w_{n+1}\rangle - \sum_{i=1}^n \langle v_i|w_{n+1}\rangle \delta_{ij}}{\| |w_{n+1}\rangle - \sum_{i=1}^n \langle v_i|w_{n+1}\rangle |v_i\rangle \|} \\ &= \frac{\langle v_j|w_{n+1}\rangle - \langle v_j|w_{n+1}\rangle}{\| |w_{n+1}\rangle - \sum_{i=1}^n \langle v_i|w_{n+1}\rangle |v_i\rangle \|} \\ &= 0. \end{aligned}$$

Thus Gram-Schmidt procedure produces an orthonormal basis.

Exercise 2.9

Outer product representation of Pauli operators:

$$\begin{aligned} \sigma_0 &= I = |0\rangle \langle 0| + |1\rangle \langle 1| \\ \sigma_1 &= X = |0\rangle \langle 1| + |1\rangle \langle 0| \\ \sigma_2 &= Y = -i |0\rangle \langle 1| + i |1\rangle \langle 0| \\ \sigma_3 &= Z = |0\rangle \langle 0| - |1\rangle \langle 1| \end{aligned}$$

Exercise 2.10

$$\begin{aligned} |v_j\rangle \langle v_k| &= I_V |v_j\rangle \langle v_k| I_V \\ &= \left(\sum_p |v_p\rangle \langle v_p| \right) |v_j\rangle \langle v_k| \left(\sum_q |v_q\rangle \langle v_q| \right) \\ &= \sum_{p,q} |v_p\rangle \langle v_p|v_j\rangle \langle v_k|v_q\rangle \langle v_q| \\ &= \sum_{p,q} \delta_{pj} \delta_{kq} |v_p\rangle \langle v_q| \end{aligned}$$

Thus

$$(|v_j\rangle \langle v_k|)_{pq} = \delta_{pj} \delta_{kq}$$

Exercise 2.11

$$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \det(X - \lambda I) = \det \left(\begin{bmatrix} -\lambda & 1 \\ 1 & -\lambda \end{bmatrix} \right) = 0 \Rightarrow \lambda = \pm 1$$

If $\lambda = -1$,

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} -c_1 \\ -c_2 \end{bmatrix} \Rightarrow \\ |\lambda = -1\rangle = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

If $\lambda = 1$

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \Rightarrow \\ |\lambda = 1\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

The diagonal representation of X is:

$$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \sum_i \lambda_i |\lambda_i\rangle \langle \lambda_i| = |\lambda = 1\rangle \langle \lambda = 1| - |\lambda = -1\rangle \langle \lambda = -1|$$

$$Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \det(Y - \lambda I) = \det \left(\begin{bmatrix} -\lambda & -i \\ i & -\lambda \end{bmatrix} \right) = 0 \Rightarrow \lambda = \pm 1$$

If $\lambda = -1$,

$$\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} -c_1 \\ -c_2 \end{bmatrix} \Rightarrow \\ |\lambda = -1\rangle = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -i \end{bmatrix}$$

If $\lambda = 1$

$$\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \Rightarrow \\ |\lambda = 1\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ i \end{bmatrix}$$

The diagonal representation of Y is:

$$Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = \sum_i \lambda_i |\lambda_i\rangle \langle \lambda_i| = |\lambda = 1\rangle \langle \lambda = 1| - |\lambda = -1\rangle \langle \lambda = -1|$$

$$Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \det(Z - \lambda I) = \det \left(\begin{bmatrix} 1-\lambda & 0 \\ 0 & -1-\lambda \end{bmatrix} \right) = 0 \Rightarrow \lambda = \pm 1$$

If $\lambda = -1$,

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} -c_1 \\ -c_2 \end{bmatrix} \Rightarrow \\ |\lambda = -1\rangle = |0\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

If $\lambda = 1$

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \Rightarrow \\ |\lambda = 1\rangle = |1\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

The diagonal representation of Z is:

$$Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \sum_i \lambda_i |\lambda_i\rangle \langle \lambda_i| = |\lambda = 1\rangle \langle \lambda = 1| - |\lambda = -1\rangle \langle \lambda = -1| = |0\rangle \langle 0| - |1\rangle \langle 1|$$

Exercise 2.12

$$\det \left(\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} - \lambda I \right) = (1 - \lambda)^2 = 0 \Rightarrow \lambda = 1$$

Therefore the eigenvector associated with eigenvalue $\lambda = 1$ is

$$|\lambda = 1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\text{Because } |\lambda = 1\rangle \langle \lambda = 1| = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix},$$

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \neq c |\lambda = 1\rangle \langle \lambda = 1| = \begin{bmatrix} 0 & 0 \\ 0 & c \end{bmatrix}$$

I may assume -but this has to be proven- that this matrix is non-diagonalizable because the eigenspace is degenerate (λ has multiplicity 2).

Exercise 2.13

Suppose $|\psi\rangle, |\phi\rangle$ are arbitrary vectors in V .

$$\begin{aligned} (|\psi\rangle, (|w\rangle \langle v|) |\phi\rangle)^* &= ((|w\rangle \langle v|)^\dagger |\psi\rangle, |\phi\rangle)^* \\ &= (|\phi\rangle, (|w\rangle \langle v|)^\dagger |\psi\rangle) \\ &= \langle \phi | (|w\rangle \langle v|)^\dagger | \psi \rangle. \end{aligned}$$

On the other hand,

$$\begin{aligned} (|\psi\rangle, (|w\rangle \langle v|) |\phi\rangle)^* &= (\langle \psi | w \rangle \langle v | \phi \rangle)^* \\ &= \langle \phi | v \rangle \langle w | \psi \rangle. \end{aligned}$$

Thus

$$\begin{aligned} \langle \phi | (|w\rangle \langle v|)^\dagger | \psi \rangle &= \langle \phi | v \rangle \langle w | \psi \rangle \text{ for arbitrary vectors } |\psi\rangle, |\phi\rangle \\ \therefore (|w\rangle \langle v|)^\dagger &= |v\rangle \langle w| \end{aligned}$$

Exercise 2.14

$$\begin{aligned}
 ((\sum_i a_i A_i)^\dagger |\phi\rangle, |\psi\rangle) &= (|\phi\rangle, \sum_i a_i A_i |\psi\rangle) \\
 &= \sum_i a_i (|\phi\rangle, A_i |\psi\rangle) \\
 &= \sum_i a_i (A_i^\dagger |\phi\rangle, |\psi\rangle) \\
 &= (\sum_i a_i^* A_i^\dagger |\phi\rangle, |\psi\rangle) \\
 \therefore (\sum_i a_i A_i)^\dagger &= \sum_i a_i^* A_i^\dagger
 \end{aligned}$$

where in the last passage we have used the result of Exercise 2.6 (the inner product is conjugate-linear in the first argument).

Exercise 2.15

$$\begin{aligned}
 ((A^\dagger)^\dagger |\psi\rangle, |\phi\rangle) &= (|\psi\rangle, A^\dagger |\phi\rangle) \\
 &= (A^\dagger |\phi\rangle, |\psi\rangle)^* \\
 &= (|\phi\rangle, A |\psi\rangle)^* \\
 &= (A |\psi\rangle, |\phi\rangle) \\
 \therefore (A^\dagger)^\dagger &= A
 \end{aligned}$$

Exercise 2.16

$$\begin{aligned}
 P &= \sum_i |i\rangle \langle i|. \\
 P^2 &= \left(\sum_i |i\rangle \langle i| \right) \left(\sum_j |j\rangle \langle j| \right) \\
 &= \sum_{i,j} |i\rangle \langle i|j\rangle \langle j| \\
 &= \sum_{i,j} |i\rangle \langle j| \delta_{ij} \\
 &= \sum_i |i\rangle \langle i| \\
 &= P
 \end{aligned}$$

Exercise 2.17

Proof. (\Rightarrow) Suppose A is Hermitian. Then $A = A^\dagger$. Let $|\lambda\rangle$ be eigenvectors of A with eigenvalues λ , that is,

$$A|\lambda\rangle = \lambda|\lambda\rangle.$$

Therefore

$$\langle \lambda | A | \lambda \rangle = \lambda \langle \lambda | \lambda \rangle = \lambda.$$

On the other hand,

$$\lambda^* = \langle \lambda | A | \lambda \rangle^* = \langle \lambda | A^\dagger | \lambda \rangle = \langle \lambda | A | \lambda \rangle = \lambda \langle \lambda | \lambda \rangle = \lambda.$$

Hence eigenvalues of Hermitian matrix are real.

(\Leftarrow) Suppose eigenvalues of A are real. From spectral theorem, normal matrix A can be written by

$$A = \sum_i \lambda_i |\lambda_i\rangle \langle \lambda_i|$$

where λ_i are real eigenvalues with eigenvectors $|\lambda_i\rangle$. By taking adjoint, we get

$$\begin{aligned} A^\dagger &= \sum_i \lambda_i^* |\lambda_i\rangle \langle \lambda_i| \\ &= \sum_i \lambda_i |\lambda_i\rangle \langle \lambda_i| \quad (\because \lambda_i \text{ are real}) \\ &= A \end{aligned}$$

Thus A is Hermitian. □

Exercise 2.18

Suppose $|v\rangle$ is a eigenvector with corresponding eigenvalue λ .

$$\begin{aligned} U |v\rangle &= \lambda |v\rangle. \\ 1 &= \langle v | v \rangle \\ &= \langle v | I | v \rangle \\ &= \langle v | U^\dagger U | v \rangle \\ &= \lambda \lambda^* \langle v | v \rangle \\ &= |\lambda|^2 \\ \therefore \lambda &= e^{i\theta} \end{aligned}$$

Exercise 2.19

$$X^2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Exercise 2.20

$$\begin{aligned}
 U &\equiv \sum_i |w_i\rangle \langle v_i| \\
 A'_{ij} &= \langle v_i | A | v_j \rangle \\
 &= \langle v_i | U U^\dagger A U U^\dagger | v_j \rangle \\
 &= \sum_{p,q,r,s} \langle v_i | w_p \rangle \langle v_p | v_q \rangle \langle w_q | A | w_r \rangle \langle v_r | v_s \rangle \langle w_s | v_j \rangle \\
 &= \sum_{p,q,r,s} \langle v_i | w_p \rangle \delta_{pq} A''_{qr} \delta_{rs} \langle w_s | v_j \rangle \\
 &= \sum_{p,r} \langle v_i | w_p \rangle \langle w_r | v_j \rangle A''_{pr}
 \end{aligned}$$

Exercise 2.21

Suppose M is Hermitian. Then $M = M^\dagger$.

$$\begin{aligned}
 M &= I M I \\
 &= (P + Q) M (P + Q) \\
 &= P M P + Q M P + P M Q + Q M Q
 \end{aligned}$$

Now $P M P = \lambda P$, $Q M P = 0$, $P M Q = P M^\dagger Q = (Q M P)^* = 0$. Thus $M = P M P + Q M Q$. Next prove $Q M Q$ is normal.

$$\begin{aligned}
 Q M Q (Q M Q)^\dagger &= Q M Q Q M^\dagger Q \\
 &= Q M^\dagger Q Q M Q \quad (M = M^\dagger) \\
 &= (Q M^\dagger Q) Q M Q
 \end{aligned}$$

Therefore $Q M Q$ is normal. By induction, $Q M Q$ is diagonal ... (following is same as Box 2.2)

Exercise 2.22

Suppose A is a Hermitian operator and $|v_i\rangle$ are eigenvectors of A with eigenvalues λ_i . Then

$$\langle v_i | A | v_j \rangle = \lambda_j \langle v_i | v_j \rangle.$$

On the other hand,

$$\langle v_i | A | v_j \rangle = \langle v_i | A^\dagger | v_j \rangle = \lambda_i^* \langle v_i | v_j \rangle$$

Thus

$$(\lambda_i^* - \lambda_j) \langle v_i | v_j \rangle = 0.$$

If $\lambda_i^* \neq \lambda_j$, then $\langle v_i | v_j \rangle = 0$.

Exercise 2.23

Suppose P is projector and $|\lambda\rangle$ are eigenvectors of P with eigenvalues λ . Then $P^2 = P$.

$$P|\lambda\rangle = \lambda|\lambda\rangle \text{ and } P|\lambda\rangle = P^2|\lambda\rangle = \lambda P|\lambda\rangle = \lambda^2|\lambda\rangle.$$

Therefore

$$\begin{aligned}\lambda &= \lambda^2 \\ \lambda(\lambda - 1) &= 0 \\ \lambda &= 0 \text{ or } 1.\end{aligned}$$

Exercise 2.24

Def of positive operator: $\langle v|A|v\rangle$ is real, non-negative number and $\langle v|A|v\rangle > 0$ for all $|v\rangle \neq 0$.

Suppose A is a positive operator. A can be decomposed as follows.

$$\begin{aligned}A &= \frac{A + A^\dagger}{2} + i \frac{A - A^\dagger}{2i} \\ &= B + iC \quad \text{where } B = \frac{A + A^\dagger}{2}, \quad C = \frac{A - A^\dagger}{2i}.\end{aligned}$$

Now operators B and C are Hermitian.

$$\begin{aligned}\langle v|A|v\rangle &= \langle v|B + iC|v\rangle \\ &= \langle v|B|v\rangle + i \langle v|C|v\rangle \\ &= \alpha + i\beta \quad \text{where } \alpha = \langle v|B|v\rangle, \quad \beta = \langle v|C|v\rangle.\end{aligned}$$

Since B and C are Hermitian, $\alpha, \beta \in \mathcal{R}$. From def of positive operator, β should vanish because $\langle v|A|v\rangle$ is real. Hence $\beta = \langle v|C|v\rangle = 0$ for all $|v\rangle$, i.e. $C = 0$.

Therefore $A = A^\dagger$.

Reference: MIT 8.05 Lecture note by Prof. Barton Zwiebach.

https://ocw.mit.edu/courses/physics/8-05-quantum-physics-ii-fall-2013/lecture-notes/MIT8_05F13_Chap_03.pdf

Proposition 1. Let T be a linear operator in a complex vector space V .

If $(u, Tv) = 0$ for all $u, v \in V$, then $T = 0$.

Proof. Suppose $u = Tv$. Then $(Tv, Tv) = 0$ for all v implies that $Tv = 0$ for all v . Therefore $T = 0$. \square

Theorem 1. If $(v, Av) = 0$ for all $v \in V$, then $A = 0$.

Proof. First, we show that $(u, Tv) = 0$ if $(v, Av) = 0$. Then apply proposition 1. Suppose $u, v \in V$. Then (u, Tv) is decomposed as

$$\begin{aligned}(u, Tv) &= \frac{1}{4} \left[(u + v, T(u + v)) - (u - v, T(u - v)) + \frac{1}{i}(u + iv, T(u + iv)) \right. \\ &\quad \left. - \frac{1}{i}(u - iv, T(u - iv)) \right].\end{aligned}$$

If $(v, Tv) = 0$ for all $v \in V$, the right hand side of above eqn vanishes. Thus $(u, Tv) = 0$ for all $u, v \in V$. Then $T = 0$. \square

Exercise 2.25

$$\langle \psi | A^\dagger A | \psi \rangle = \|A | \psi \rangle\|^2 \geq 0 \text{ for all } | \psi \rangle.$$

Thus $A^\dagger A$ is positive.

Exercise 2.26

$$\begin{aligned} |\psi\rangle^{\otimes 2} &= \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \\ &= \frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle) \\ &= \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} |\psi\rangle^{\otimes 3} &= \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \\ &= \frac{1}{2\sqrt{2}}(|000\rangle + |001\rangle + |010\rangle + |011\rangle + |100\rangle + |101\rangle + |110\rangle + |111\rangle) \\ &= \frac{1}{2\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \end{aligned}$$

Exercise 2.27

$$\begin{aligned} X \otimes Z &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} I \otimes X &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \end{aligned}$$

$$\begin{aligned}
 X \otimes I &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\
 &= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}
 \end{aligned}$$

In general, the tensor product is not commutative.

Exercise 2.28

$$\begin{aligned}
 (A \otimes B)^* &= \begin{bmatrix} A_{11}B & \cdots & A_{1n}B \\ \vdots & \ddots & \vdots \\ A_{m1}B & \cdots & A_{mn}B \end{bmatrix}^* \\
 &= \begin{bmatrix} A_{11}^*B^* & \cdots & A_{1n}^*B^* \\ \vdots & \ddots & \vdots \\ A_{m1}^*B^* & \cdots & A_{mn}^*B^* \end{bmatrix} \\
 &= A^* \otimes B^*.
 \end{aligned}$$

$$\begin{aligned}
 (A \otimes B)^T &= \begin{bmatrix} A_{11}B & \cdots & A_{1n}B \\ \vdots & \ddots & \vdots \\ A_{m1}B & \cdots & A_{mn}B \end{bmatrix}^T \\
 &= \begin{bmatrix} A_{11}B^T & \cdots & A_{m1}B^T \\ \vdots & \ddots & \vdots \\ A_{1n}B^T & \cdots & A_{mn}B^T \end{bmatrix} \\
 &= A^T \otimes B^T.
 \end{aligned}$$

$$\begin{aligned}
 (A \otimes B)^\dagger &= ((A \otimes B)^*)^T \\
 &= (A^* \otimes B^*)^T \\
 &= (A^*)^T \otimes (B^*)^T \\
 &= A^\dagger \otimes B^\dagger.
 \end{aligned}$$

Exercise 2.29

Suppose U_1 and U_2 are unitary operators. Then

$$\begin{aligned}
 (U_1 \otimes U_2)(U_1 \otimes U_2)^\dagger &= U_1 U_1^\dagger \otimes U_2 U_2^\dagger \\
 &= I \otimes I.
 \end{aligned}$$

Similarly,

$$(U_1 \otimes U_2)^\dagger (U_1 \otimes U_2) = I \otimes I.$$

Exercise 2.30

Suppose A and B are Hermitian operators. Then

$$(A \otimes B)^\dagger = A^\dagger \otimes B^\dagger = A \otimes B. \quad (5)$$

Thus $A \otimes B$ is Hermitian.

Exercise 2.31

Suppose A and B are positive operators. Then

$$\langle \psi | \otimes \langle \phi | (A \otimes B) | \psi \rangle \otimes | \phi \rangle = \langle \psi | A | \psi \rangle \langle \phi | B | \phi \rangle.$$

Since A and B are positive operators, $\langle \psi | A | \psi \rangle \geq 0$ and $\langle \phi | B | \phi \rangle \geq 0$ for all $|\psi\rangle, |\phi\rangle$. Then $\langle \psi | A | \psi \rangle \langle \phi | B | \phi \rangle \geq 0$. Thus $A \otimes B$ is positive if A and B are positive.

Exercise 2.32

Suppose P_1 and P_2 are projectors. Then

$$\begin{aligned} (P_1 \otimes P_2)^2 &= P_1^2 \otimes P_2^2 \\ &= P_1 \otimes P_2. \end{aligned}$$

Thus $P_1 \otimes P_2$ is also a projector.

Exercise 2.33

$$H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \quad (6)$$

$$H^{\otimes 2} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \otimes \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

Exercise 2.34

Suppose $A = \begin{bmatrix} 4 & 3 \\ 3 & 4 \end{bmatrix}$.

$$\begin{aligned} \det(A - \lambda I) &= (4 - \lambda)^2 - 3^2 \\ &= \lambda^2 - 8\lambda + 7 \\ &= (\lambda - 1)(\lambda - 7) \end{aligned}$$

Eigenvalues of A are $\lambda = 1, 7$. Corresponding eigenvectors are $|\lambda = 1\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$,
 $|\lambda = 7\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Thus

$$A = \sum_i \lambda_i |\lambda_i\rangle \langle \lambda_i| = \frac{7}{2} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$\begin{aligned} \sqrt{A} &= \sum_i \sqrt{\lambda_i} |\lambda_i\rangle \langle \lambda_i| = \\ &= \frac{1}{2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + \frac{\sqrt{7}}{2} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \\ &= \frac{1}{2} \begin{bmatrix} 1 + \sqrt{7} & -1 + \sqrt{7} \\ -1 + \sqrt{7} & 1 + \sqrt{7} \end{bmatrix} \end{aligned}$$

$$\begin{aligned} \log(A) &= \sum_i \log \lambda_i |\lambda_i\rangle \langle \lambda_i| = \\ &= \frac{\log(7)}{2} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \end{aligned}$$

Exercise 2.35

$$\begin{aligned} \vec{v} \cdot \vec{\sigma} &= \sum_{i=1}^3 v_i \sigma_i \\ &= v_1 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + v_2 \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} + v_3 \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \\ &= \begin{bmatrix} v_3 & v_1 - iv_2 \\ v_1 + iv_2 & -v_3 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} \det(\vec{v} \cdot \vec{\sigma} - \lambda I) &= (v_3 - \lambda)(-v_3 - \lambda) - (v_1 - iv_2)(v_1 + iv_2) \\ &= \lambda^2 - (v_1^2 + v_2^2 + v_3^2) \\ &= \lambda^2 - 1 \quad (\because |\vec{v}| = 1) \end{aligned}$$

Eigenvalues are $\lambda = \pm 1$. Let $|\lambda_{\pm 1}\rangle$ be eigenvectors with eigenvalues ± 1 .

Since $\vec{v} \cdot \vec{\sigma}$ is Hermitian, $\vec{v} \cdot \vec{\sigma}$ is diagonalizable. Then

$$\vec{v} \cdot \vec{\sigma} = |\lambda_1\rangle \langle \lambda_1| - |\lambda_{-1}\rangle \langle \lambda_{-1}|$$

Thus

$$\begin{aligned} \exp(i\theta \vec{v} \cdot \vec{\sigma}) &= e^{i\theta} |\lambda_1\rangle \langle \lambda_1| + e^{-i\theta} |\lambda_{-1}\rangle \langle \lambda_{-1}| \\ &= (\cos \theta + i \sin \theta) |\lambda_1\rangle \langle \lambda_1| + (\cos \theta - i \sin \theta) |\lambda_{-1}\rangle \langle \lambda_{-1}| \\ &= \cos \theta (|\lambda_1\rangle \langle \lambda_1| + |\lambda_{-1}\rangle \langle \lambda_{-1}|) + i \sin \theta (|\lambda_1\rangle \langle \lambda_1| - |\lambda_{-1}\rangle \langle \lambda_{-1}|) \\ &= \cos(\theta) I + i \sin(\theta) \vec{v} \cdot \vec{\sigma}. \end{aligned}$$

\therefore Since $\vec{v} \cdot \vec{\sigma}$ is Hermitian, $|\lambda_1\rangle$ and $|\lambda_{-1}\rangle$ are orthogonal. Thus

$$|\lambda_1\rangle \langle \lambda_1| + |\lambda_{-1}\rangle \langle \lambda_{-1}| = I.$$

Exercise 2.36

$$\begin{aligned}\mathrm{Tr}(\sigma_1) &= \mathrm{Tr} \left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right) = 0 \\ \mathrm{Tr}(\sigma_2) &= \mathrm{Tr} \left(\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \right) = 0 \\ \mathrm{Tr}(\sigma_3) &= \mathrm{Tr} \left(\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \right) = 1 - 1 = 0\end{aligned}$$

Exercise 2.37

$$\begin{aligned}\mathrm{Tr}(AB) &= \sum_i \langle i|AB|i\rangle \\ &= \sum_i \langle i|AIB|i\rangle \\ &= \sum_{i,j} \langle i|A|j\rangle \langle j|B|i\rangle \\ &= \sum_{i,j} \langle j|B|i\rangle \langle i|A|j\rangle \\ &= \sum_j \langle j|BA|j\rangle \\ &= \mathrm{Tr}(BA)\end{aligned}$$

Exercise 2.38

$$\begin{aligned}\mathrm{Tr}(A+B) &= \sum_i \langle i|A+B|i\rangle \\ &= \sum_i (\langle i|A|i\rangle + \langle i|B|i\rangle) \\ &= \sum_i \langle i|A|i\rangle + \sum_i \langle i|B|i\rangle \\ &= \mathrm{Tr}(A) + \mathrm{Tr}(B).\end{aligned}$$

$$\begin{aligned}\mathrm{Tr}(zA) &= \sum_i \langle i|zA|i\rangle \\ &= \sum_i z \langle i|A|i\rangle \\ &= z \sum_i \langle i|A|i\rangle \\ &= z \mathrm{Tr}(A).\end{aligned}$$

Exercise 2.39

$$(1) (A, B) \equiv \text{Tr}(A^\dagger B).$$

(i)

$$\begin{aligned} \left(A, \sum_i \lambda_i B_i \right) &= \text{Tr} \left[A^\dagger \left(\sum_i \lambda_i B_i \right) \right] \\ &= \text{Tr}(A^\dagger \lambda_1 B_1) + \cdots + \text{Tr}(A^\dagger \lambda_n B_n) \quad (\because \text{Exercise 2.38}) \\ &= \lambda_1 \text{Tr}(A^\dagger B_1) + \cdots + \lambda_n \text{Tr}(A^\dagger B_n) \\ &= \sum_i \lambda_i \text{Tr}(A^\dagger B_i) \end{aligned}$$

(ii)

$$\begin{aligned} (A, B)^* &= (\text{Tr}(A^\dagger B))^* \\ &= \left(\sum_{i,j} \langle i|A^\dagger|j\rangle \langle j|B|i\rangle \right)^* \\ &= \sum_{i,j} \langle i|A^\dagger|j\rangle^* \langle j|B|i\rangle^* \\ &= \sum_{i,j} \langle j|B|i\rangle^* \langle i|A^\dagger|j\rangle^* \\ &= \sum_{i,j} \langle i|B^\dagger|j\rangle \langle j|A|i\rangle \\ &= \sum_i \langle i|B^\dagger A|i\rangle \\ &= \text{Tr}(B^\dagger A) \\ &= (B, A). \end{aligned}$$

(iii)

$$\begin{aligned} (A, A) &= \text{Tr}(A^\dagger A) \\ &= \sum_i \langle i|A^\dagger A|i\rangle \end{aligned}$$

Since $A^\dagger A$ is positive, $\langle i|A^\dagger A|i\rangle \geq 0$ for all $|i\rangle$.

Let a_i be i -th column of A . If $\langle i|A^\dagger A|i\rangle = 0$, then

$$\langle i|A^\dagger A|i\rangle = a_i^\dagger a_i = |a_i|^2 = 0 \text{ iff } a_i = \mathbf{0}.$$

Therefore $(A, A) = 0$ iff $A = \mathbf{0}$.

(2)

(3)

Exercise 2.40

$$\begin{aligned}[X, Y] &= XY - YX \\&= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} - \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \\&= \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} - \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix} \\&= \begin{bmatrix} 2i & 0 \\ 0 & -2i \end{bmatrix} \\&= 2iZ\end{aligned}$$

$$\begin{aligned}[Y, Z] &= \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \\&= \begin{bmatrix} 0 & 2i \\ 2i & 0 \end{bmatrix} \\&= 2iX\end{aligned}$$

$$\begin{aligned}[Z, X] &= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \\&= 2i \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \\&= 2iY\end{aligned}$$

Exercise 2.41

$$\begin{aligned}\{\sigma_1, \sigma_2\} &= \sigma_1\sigma_2 + \sigma_2\sigma_1 \\&= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} + \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \\&= \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} + \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix} \\&= 0\end{aligned}$$

$$\begin{aligned}\{\sigma_2, \sigma_3\} &= \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \\&= 0\end{aligned}$$

$$\begin{aligned}\{\sigma_3, \sigma_1\} &= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \\&= 0\end{aligned}$$

$$\begin{aligned}\sigma_0^2 &= I^2 = I \\ \sigma_1^2 &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^2 = I \\ \sigma_2^2 &= \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}^2 = I \\ \sigma_3^2 &= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}^2 = I\end{aligned}$$

Exercise 2.42

$$\frac{[A, B] + \{A, B\}}{2} = \frac{AB - BA + AB + BA}{2} = AB$$

Exercise 2.43

From eq (2.75) and eq (2.76), $\{\sigma_j, \sigma_k\} = 2\delta_{jk}I$. From eq (2.77),

$$\begin{aligned}\sigma_j \sigma_k &= \frac{[\sigma_j, \sigma_k] + \{\sigma_j, \sigma_k\}}{2} \\ &= \frac{2i \sum_{l=1}^3 \epsilon_{jkl} \sigma_l + 2\delta_{jk}I}{2} \\ &= \delta_{jk}I + i \sum_{l=1}^3 \epsilon_{jkl} \sigma_l\end{aligned}$$

Exercise 2.44

By assumption, $[A, B] = 0$ and $\{A, B\} = 0$, then $AB = 0$. Since A is invertible, multiply by A^{-1} from left, then

$$\begin{aligned}A^{-1}AB &= 0 \\ IB &= 0 \\ B &= 0.\end{aligned}$$

Exercise 2.45

$$\begin{aligned}[A, B]^\dagger &= (AB - BA)^\dagger \\ &= B^\dagger A^\dagger - A^\dagger B^\dagger \\ &= [B^\dagger, A^\dagger]\end{aligned}$$

Exercise 2.46

$$\begin{aligned}[A, B] &= AB - BA \\ &= -(BA - AB) \\ &= -[B, A]\end{aligned}$$

Exercise 2.47

$$\begin{aligned}
 (i[A, B])^\dagger &= -i[A, B]^\dagger \\
 &= -i[B^\dagger, A^\dagger] \\
 &= -i[B, A] \\
 &= i[A, B]
 \end{aligned}$$

Exercise 2.48

- (Positive)

Since P is positive, it is diagonalizable. Then $P = \sum_i \lambda_i |\lambda_i\rangle \langle \lambda_i|$, ($\lambda_i \geq 0$).

$$J = \sqrt{P^\dagger P} = \sqrt{PP} = \sqrt{P^2} = \sum_i \sqrt{\lambda_i^2} |\lambda_i\rangle \langle \lambda_i| = \sum_i \lambda_i |\lambda_i\rangle \langle \lambda_i| = P.$$

Therefore polar decomposition of P is $P = UP$ for all P . Thus $U = I$, then $P = P$.

- (Unitary)

Suppose unitary U is decomposed by $U = WJ$ where W is unitary and J is positive, $J = \sqrt{U^\dagger U}$.

$$J = \sqrt{U^\dagger U} = \sqrt{I} = I$$

Since unitary operators are invertible, $W = UJ^{-1} = UI^{-1} = UI = U$. Thus polar decomposition of U is $U = U$.

- (Hermitian)

Suppose $H = UJ$.

$$J = \sqrt{H^\dagger H} = \sqrt{HH} = \sqrt{H^2}.$$

Thus $H = U\sqrt{H^2}$. In general, $H \neq \sqrt{H^2}$.

From spectral decomposition, $H = \sum_i \lambda_i |\lambda_i\rangle \langle \lambda_i|$, $\lambda_i \in \mathcal{R}$.

$$\sqrt{H^2} = \sqrt{\sum_i \lambda_i^2 |\lambda_i\rangle \langle \lambda_i|} = \sum_i \sqrt{\lambda_i^2} |\lambda_i\rangle \langle \lambda_i| = \sum_i |\lambda_i| |\lambda_i\rangle \langle \lambda_i| \neq H$$

Exercise 2.49

Normal matrix is diagonalizable, $A = \sum_i \lambda_i |\lambda_i\rangle \langle \lambda_i|$.

$$J = \sqrt{A^\dagger A} = \sum_i |\lambda_i| |\lambda_i\rangle \langle \lambda_i|$$

$$U = \sum_i |e_i\rangle \langle \lambda_i|$$

$$A = UJ = \sum_i |\lambda_i| |e_i\rangle \langle \lambda_i|$$

Exercise 2.50

Define $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$. $A^\dagger A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$.

Characteristic equation of $A^\dagger A$ is $\det(A^\dagger A - \lambda I) = \lambda^2 - 3\lambda + 1 = 0$. Eigenvalues of $A^\dagger A$ are $\lambda_{\pm} = \frac{3 \pm \sqrt{5}}{2}$ and associated eigenvectors are $|\lambda_{\pm}\rangle = \frac{1}{\sqrt{10 \mp 2\sqrt{5}}} \begin{bmatrix} 2 \\ -1 \pm \sqrt{5} \end{bmatrix}$.

$$A^\dagger A = \lambda_+ |\lambda_+\rangle \langle \lambda_+| + \lambda_- |\lambda_-\rangle \langle \lambda_-|.$$

$$\begin{aligned} J &= \sqrt{A^\dagger A} = \sqrt{\lambda_+} |\lambda_+\rangle \langle \lambda_+| + \sqrt{\lambda_-} |\lambda_-\rangle \langle \lambda_-| = \\ &= \sqrt{\frac{3+\sqrt{5}}{2}} \cdot \frac{5-\sqrt{5}}{40} \begin{bmatrix} 4 & 2\sqrt{5}-2 \\ 2\sqrt{5}-2 & 6-2\sqrt{5} \end{bmatrix} + \sqrt{\frac{3-\sqrt{5}}{2}} \cdot \frac{5+\sqrt{5}}{40} \begin{bmatrix} 4 & -2\sqrt{5}-2 \\ -2\sqrt{5}-2 & 6+2\sqrt{5} \end{bmatrix} \end{aligned}$$

$$J^{-1} = \frac{1}{\sqrt{\lambda_+}} |\lambda_+\rangle \langle \lambda_+| + \frac{1}{\sqrt{\lambda_-}} |\lambda_-\rangle \langle \lambda_-|.$$

$$U = AJ^{-1}$$

...

Exercise 2.51

$$H^\dagger H = \left(\frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \right)^\dagger \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = I.$$

Exercise 2.52

$$H^\dagger = \left(\frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \right)^\dagger = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = H.$$

Thus

$$H^2 = I.$$

Exercise 2.53

$$\begin{aligned} \det(H - \lambda I) &= \left(\frac{1}{\sqrt{2}} - \lambda \right) \left(-\frac{1}{\sqrt{2}} - \lambda \right) - \frac{1}{2} \\ &= \lambda^2 - \frac{1}{2} - \frac{1}{2} \\ &= \lambda^2 - 1 \end{aligned}$$

Eigenvalues are $\lambda_{\pm} = \pm 1$ and associated eigenvectors are $|\lambda_{\pm}\rangle = \frac{1}{\sqrt{4 \mp 2\sqrt{2}}} \begin{bmatrix} 1 \\ -1 \pm \sqrt{2} \end{bmatrix}$.

Exercise 2.54

Since $[A, B] = 0$, A and B are simultaneously diagonalize, $A = \sum_i a_i |i\rangle \langle i|$, $B = \sum_i b_i |i\rangle \langle i|$.

$$\begin{aligned}
 \exp(A) \exp(B) &= \left(\sum_i \exp(a_i) |i\rangle \langle i| \right) \left(\sum_i \exp(b_i) |i\rangle \langle i| \right) \\
 &= \sum_{i,j} \exp(a_i + b_j) |i\rangle \langle i| j\rangle \langle j| \\
 &= \sum_{i,j} \exp(a_i + b_j) |i\rangle \langle j| \delta_{i,j} \\
 &= \sum_i \exp(a_i + b_i) |i\rangle \langle i| \\
 &= \exp(A + B)
 \end{aligned}$$

Exercise 2.55

$$H = \sum_E E |E\rangle \langle E|$$

$$\begin{aligned}
 U(t_2 - t_1) U^\dagger(t_2 - t_1) &= \exp\left(-\frac{iH(t_2 - t_1)}{\hbar}\right) \exp\left(\frac{iH(t_2 - t_1)}{\hbar}\right) \\
 &= \sum_{E,E'} \left(\exp\left(-\frac{iE(t_2 - t_1)}{\hbar}\right) |E\rangle \langle E| \right) \left(\exp\left(\frac{iE'(t_2 - t_1)}{\hbar}\right) |E'\rangle \langle E'| \right) \\
 &= \sum_{E,E'} \left(\exp\left(-\frac{i(E - E')(t_2 - t_1)}{\hbar}\right) |E\rangle \langle E'| \delta_{E,E'} \right) \\
 &= \sum_E \exp(0) |E\rangle \langle E| \\
 &= \sum_E |E\rangle \langle E| \\
 &= I
 \end{aligned}$$

Similarly, $U^\dagger(t_2 - t_1) U(t_2 - t_1) = I$.

Exercise 2.56

$U = \sum_i \lambda_i |\lambda_i\rangle \langle \lambda_i|$ ($|\lambda_i| = 1 \rightarrow \lambda = e^{i\theta_i}$).

$$\begin{aligned}
 \log(U) &= \sum_j \log(\lambda_j) |\lambda_j\rangle \langle \lambda_j| = \sum_j i\theta_j |\lambda_j\rangle \langle \lambda_j| \text{ where } \theta_j = \arg(\lambda_j) \\
 K &= -i \log(U) = \sum_j \theta_j |\lambda_j\rangle \langle \lambda_j|.
 \end{aligned}$$

$$K^\dagger = (-i \log U)^\dagger = \left(\sum_j \theta_j |\lambda_j\rangle \langle \lambda_j| \right)^\dagger = \sum_j \theta_j^* |\lambda_j\rangle \langle \lambda_j| = \sum_j \theta_j |\lambda_j\rangle \langle \lambda_j| = K$$

Exercise 2.57

After the first measurement, we have the following state:

$$|\phi\rangle \equiv \frac{L_l |\psi\rangle}{\sqrt{\langle\psi|L_l^\dagger L_l|\psi\rangle}} \quad \text{then the probability that result } m \text{ occurs is}$$

$$\langle\phi|M_m^\dagger M_m|\phi\rangle = \frac{\langle\psi|L_l^\dagger M_m^\dagger M_m L_l|\psi\rangle}{\langle\psi|L_l^\dagger L_l|\psi\rangle} \quad \text{so the final state is}$$

$$\frac{M_m |\phi\rangle}{\sqrt{\langle\phi|M_m^\dagger M_m|\phi\rangle}} = \frac{M_m L_l |\psi\rangle}{\sqrt{\langle\psi|L_l^\dagger L_l|\psi\rangle}} \cdot \frac{\sqrt{\langle\psi|L_l^\dagger L_l|\psi\rangle}}{\sqrt{\langle\psi|L_l^\dagger M_m^\dagger M_m L_l|\psi\rangle}} = \frac{M_m L_l |\psi\rangle}{\sqrt{\langle\psi|L_l^\dagger M_m^\dagger M_m L_l|\psi\rangle}} = \frac{N_{lm} |\psi\rangle}{\sqrt{\langle\psi|N_{lm}^\dagger N_{lm}|\psi\rangle}}$$

Exercise 2.58

$$\begin{aligned} \langle M \rangle &= \langle\psi|M|\psi\rangle = \langle\psi|m|\psi\rangle = m \langle\psi|\psi\rangle = m \\ \langle M^2 \rangle &= \langle\psi|M^2|\psi\rangle = \langle\psi|m^2|\psi\rangle = m^2 \langle\psi|\psi\rangle = m^2 \\ \Delta(M) &= \langle M^2 \rangle - \langle M \rangle^2 = m^2 - m^2 = 0. \end{aligned}$$

Exercise 2.59

$$\begin{aligned} \langle X \rangle &= \langle 0|X|0\rangle = \langle 0|1\rangle = 0 \\ \langle X^2 \rangle &= \langle 0|X^2|0\rangle = \langle 0|X|1\rangle = \langle 0|0\rangle = 1 \\ \sigma &= \sqrt{\langle X^2 \rangle - \langle X \rangle^2} = 1 \end{aligned}$$

Exercise 2.60

$$\begin{aligned} \vec{v} \cdot \vec{\sigma} &= \sum_{i=1}^3 v_i \sigma_i \\ &= v_1 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + v_2 \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} + v_3 \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \\ &= \begin{bmatrix} v_3 & v_1 - iv_2 \\ v_1 + iv_2 & -v_3 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} \det(\vec{v} \cdot \vec{\sigma} - \lambda I) &= (v_3 - \lambda)(-v_3 - \lambda) - (v_1 - iv_2)(v_1 + iv_2) \\ &= \lambda^2 - (v_1^2 + v_2^2 + v_3^2) \\ &= \lambda^2 - 1 \quad (\because |\vec{v}| = 1) \end{aligned}$$

Eigenvalues are $\lambda = \pm 1$.

(i) if $\lambda = 1$

$$\begin{aligned} \vec{v} \cdot \vec{\sigma} - \lambda I &= \vec{v} \cdot \vec{\sigma} - I \\ &= \begin{bmatrix} v_3 - 1 & v_1 - iv_2 \\ v_1 + iv_2 & -v_3 - 1 \end{bmatrix} \end{aligned}$$

Normalized eigenvector is $|\lambda_1\rangle = \sqrt{\frac{1+v_3}{2}} \begin{bmatrix} 1 \\ \frac{1-v_3}{v_1-iv_2} \end{bmatrix}$.

$$\begin{aligned} |\lambda_1\rangle \langle \lambda_1| &= \frac{1+v_3}{2} \begin{bmatrix} 1 \\ \frac{1-v_3}{v_1-iv_2} \end{bmatrix} \begin{bmatrix} 1 & \frac{1-v_3}{v_1+iv_2} \end{bmatrix} \\ &= \frac{1+v_3}{2} \begin{bmatrix} 1 & \frac{v_1-iv_2}{1+v_3} \\ \frac{v_1+iv_2}{1+v_3} & \frac{1-v_3}{1+v_3} \end{bmatrix} \\ &= \frac{1}{2} \begin{bmatrix} 1+v_3 & v_1-iv_2 \\ v_1+iv_2 & 1-v_3 \end{bmatrix} \\ &= \frac{1}{2} \left(I + \begin{bmatrix} v_3 & v_1-iv_2 \\ v_1+iv_2 & -v_3 \end{bmatrix} \right) \\ &= \frac{1}{2} (I + \vec{v} \cdot \vec{\sigma}) \end{aligned}$$

(ii) If $\lambda = -1$.

$$\begin{aligned} \vec{v} \cdot \vec{\sigma} - \lambda I &= \vec{v} \cdot \vec{\sigma} + I \\ &= \begin{bmatrix} v_3 + 1 & v_1 - iv_2 \\ v_1 + iv_2 & -v_3 + 1 \end{bmatrix} \end{aligned}$$

Normalized eigenvector is $|\lambda_{-1}\rangle = \sqrt{\frac{1-v_3}{2}} \begin{bmatrix} 1 \\ -\frac{1+v_3}{v_1-iv_2} \end{bmatrix}$.

$$\begin{aligned} |\lambda_{-1}\rangle \langle \lambda_{-1}| &= \frac{1-v_3}{2} \begin{bmatrix} 1 \\ -\frac{1+v_3}{v_1-iv_2} \end{bmatrix} \begin{bmatrix} 1 & -\frac{1+v_3}{v_1+iv_2} \end{bmatrix} \\ &= \frac{1-v_3}{2} \begin{bmatrix} 1 & -\frac{v_1-iv_2}{1-v_3} \\ -\frac{v_1+iv_2}{1-v_3} & \frac{1+v_3}{1-v_3} \end{bmatrix} \\ &= \frac{1}{2} \begin{bmatrix} 1-v_3 & -(v_1-iv_2) \\ -(v_1+iv_2) & 1+v_3 \end{bmatrix} \\ &= \frac{1}{2} \left(I - \begin{bmatrix} v_3 & v_1-iv_2 \\ v_1+iv_2 & -v_3 \end{bmatrix} \right) \\ &= \frac{1}{2} (I - \vec{v} \cdot \vec{\sigma}). \end{aligned}$$

While I review my proof, I notice that my proof has a defect. The case $(v_1, v_2, v_3) = (0, 0, 1)$, second component of eigenstate, $\frac{1-v_3}{v_1-iv_2}$, diverges. So I implicitly assume $v_1-iv_2 \neq 0$. Hence my proof is incomplete.

Since the exercise doesn't require an explicit form of projector, we should prove the problem more abstractly. In order to prove, we use the following properties of $\vec{v} \cdot \vec{\sigma}$

- $\vec{v} \cdot \vec{\sigma}$ is Hermitian
- $(\vec{v} \cdot \vec{\sigma})^2 = I$ where \vec{v} is a real unit vector.

We can easily check the above conditions.

$$\begin{aligned} (\vec{v} \cdot \vec{\sigma})^\dagger &= (v_1\sigma_1 + v_2\sigma_2 + v_3\sigma_3)^\dagger \\ &= v_1\sigma_1^\dagger + v_2\sigma_2^\dagger + v_3\sigma_3^\dagger \\ &= v_1\sigma_1 + v_2\sigma_2 + v_3\sigma_3 \quad (\because \text{Pauli matrices are Hermitian.}) \\ &= \vec{v} \cdot \vec{\sigma} \end{aligned}$$

$$\begin{aligned}
 (\vec{v} \cdot \vec{\sigma})^2 &= \sum_{j,k=1}^3 (v_j \sigma_j)(v_k \sigma_k) \\
 &= \sum_{j,k=1}^3 v_j v_k \sigma_j \sigma_k \\
 &= \sum_{j,k=1}^3 v_j v_k \left(\delta_{jk} I + i \sum_{l=1}^3 \epsilon_{jkl} \sigma_l \right) \quad (\because \text{eqn(2.78) page78}) \\
 &= \sum_{j,k=1}^3 v_j v_k \delta_{jk} I + i \sum_{j,k,l=1}^3 \epsilon_{jkl} v_j v_k \sigma_l \\
 &= \sum_{j=1}^3 v_j^2 I \\
 &= I \quad \left(\because \sum_j v_j^2 = 1 \right)
 \end{aligned}$$

Proof. Suppose $|\lambda\rangle$ is an eigenstate of $\vec{v} \cdot \vec{\sigma}$ with eigenvalue λ . Then

$$\begin{aligned}
 \vec{v} \cdot \vec{\sigma} |\lambda\rangle &= \lambda |\lambda\rangle \\
 (\vec{v} \cdot \vec{\sigma})^2 |\lambda\rangle &= \lambda^2 |\lambda\rangle
 \end{aligned}$$

On the other hand $(\vec{v} \cdot \vec{\sigma})^2 = I$,

$$\begin{aligned}
 (\vec{v} \cdot \vec{\sigma})^2 |\lambda\rangle &= I |\lambda\rangle = |\lambda\rangle \\
 \therefore \lambda^2 |\lambda\rangle &= |\lambda\rangle.
 \end{aligned}$$

Thus $\lambda^2 = 1 \Rightarrow \lambda = \pm 1$. Therefore $\vec{v} \cdot \vec{\sigma}$ has eigenvalues ± 1 .

Let $|\lambda_1\rangle$ and $|\lambda_{-1}\rangle$ are eigenvectors with eigenvalues 1 and -1 , respectively. I will prove that $P_{\pm} = |\lambda_{\pm 1}\rangle \langle \lambda_{\pm 1}|$.

In order to prove the above equation, all we have to do is prove the following condition: (see Theorem 1)

$$\langle \psi | (P_{\pm} - |\lambda_{\pm 1}\rangle \langle \lambda_{\pm 1}|) | \psi \rangle = 0 \text{ for all } |\psi\rangle \in \mathcal{C}^2. \quad (7)$$

Since $\vec{v} \cdot \vec{\sigma}$ is Hermitian, $|\lambda_1\rangle$ and $|\lambda_{-1}\rangle$ are orthonormal vector (\because Exercise 2.22). Let $|\psi\rangle \in \mathcal{C}^2$ be an arbitrary state. $|\psi\rangle$ can be written as

$$|\psi\rangle = \alpha |\lambda_1\rangle + \beta |\lambda_{\pm 1}\rangle \quad (|\alpha|^2 + |\beta|^2 = 1, \alpha, \beta \in \mathcal{C}).$$

$$\begin{aligned}
 \langle \psi | (P_{\pm} - |\lambda_{\pm 1}\rangle \langle \lambda_{\pm 1}|) | \psi \rangle &= \langle \psi | P_{\pm} | \psi \rangle - \langle \psi | \lambda_{\pm} \rangle \langle \lambda_{\pm} | \psi \rangle . \\
 \langle \psi | P_{\pm} | \psi \rangle &= \langle \psi | \frac{1}{2} (I \pm \vec{v} \cdot \vec{\sigma}) | \psi \rangle \\
 &= \frac{1}{2} \pm \frac{1}{2} \langle \psi | \vec{v} \cdot \vec{\sigma} | \psi \rangle \\
 &= \frac{1}{2} \pm \frac{1}{2} (|\alpha|^2 - |\beta|^2) \\
 &= \frac{1}{2} \pm \frac{1}{2} (2|\alpha|^2 - 1) \quad (\because |\alpha|^2 + |\beta|^2 = 1) \\
 \langle \psi | \lambda_1 \rangle \langle \lambda_1 | \psi \rangle &= |\alpha|^2 \\
 \langle \psi | \lambda_{-1} \rangle \langle \lambda_{-1} | \psi \rangle &= |\beta|^2 = 1 - |\alpha|^2
 \end{aligned}$$

Therefore $\langle \psi | (P_{\pm} - |\lambda_{\pm 1}\rangle \langle \lambda_{\pm 1}|) | \psi \rangle = 0$ for all $|\psi\rangle \in \mathcal{C}^2$. Thus $P_{\pm} = |\lambda_{\pm 1}\rangle \langle \lambda_{\pm 1}|$. \square

Exercise 2.61

$$\begin{aligned}
 p(+1) &= \langle 0 | \lambda_1 \rangle \langle \lambda_1 | 0 \rangle = \langle 0 | \frac{1}{2} (I + \vec{v} \cdot \vec{\sigma}) | 0 \rangle \\
 &= \frac{1}{2} (1 + v_3)
 \end{aligned}$$

Post-measurement state is

$$\begin{aligned}
 \frac{|\lambda_1\rangle \langle \lambda_1 | 0 \rangle}{\sqrt{\langle 0 | \lambda_1 \rangle \langle \lambda_1 | 0 \rangle}} &= \frac{1}{\sqrt{\frac{1}{2}(1 + v_3)}} \cdot \frac{1}{2} \begin{bmatrix} 1 + v_3 \\ v_1 + iv_2 \end{bmatrix} \\
 &= \sqrt{\frac{1}{2}(1 + v_3)} \begin{bmatrix} 1 \\ \frac{v_1 + iv_2}{1 + v_3} \end{bmatrix} \\
 &= \sqrt{\frac{1 + v_3}{2}} \begin{bmatrix} 1 \\ \frac{1 - v_3}{v_1 - iv_2} \end{bmatrix} \\
 &= |\lambda_1\rangle .
 \end{aligned}$$

Exercise 2.62

Suppose M_m is a measurement operator. From the assumption, $E_m = M_m^\dagger M_m = M_m$.

Then

$$\langle \psi | E_m | \psi \rangle = \langle \psi | M_m | \psi \rangle \geq 0.$$

for all $|\psi\rangle$.

Since M_m is positive operator, M_m is Hermitian. Therefore,

$$E_m = M_m^\dagger M_m = M_m M_m = M_m^2 = M_m.$$

Thus the measurement is a projective measurement.

Exercise 2.63

$$\begin{aligned} M_m^\dagger M_m &= \sqrt{E_m} U_m^\dagger U_m \sqrt{E_m} \\ &= \sqrt{E_m} I \sqrt{E_m} \\ &= E_m. \end{aligned}$$

Since E_m is POVM, for arbitrary unitary U , $M_m = U_m \sqrt{E_m}$ is POVM.

Exercise 2.64

Read the following paper:

- Lu-Ming Duan, Guang-Can Guo. Probabilistic cloning and identification of linearly independent quantum states. Phys. Rev. Lett., 80:4999-5002, 1998. arXiv:quant-ph/9804064
<https://journals.aps.org/prl/abstract/10.1103/PhysRevLett.80.4999>
<https://arxiv.org/abs/quant-ph/9804064>
- Stephen M. Barnett, Sarah Croke, Quantum state discrimination, arXiv:0810.1970 [quant-ph]
<https://arxiv.org/abs/0810.1970>
<https://www.osapublishing.org/DirectPDFAccess/67EF4200-CBD2-8E68-1979E37886263936.176580/aop-1-2-238.pdf>

Exercise 2.65

$$|+\rangle \equiv \frac{|0\rangle + |1\rangle}{\sqrt{2}}, \quad |-\rangle \equiv \frac{|0\rangle - |1\rangle}{\sqrt{2}}$$

Exercise 2.66

$$X_1 Z_2 \left(\frac{|00\rangle + |11\rangle}{\sqrt{2}} \right) = \frac{|10\rangle - |01\rangle}{\sqrt{2}}$$

$$\langle X_1 Z_2 \rangle = \left(\frac{\langle 00| + \langle 11|}{\sqrt{2}} \right) X_1 Z_2 \left(\frac{|00\rangle + |11\rangle}{\sqrt{2}} \right) = \frac{\langle 00| + \langle 11|}{\sqrt{2}} \cdot \frac{|10\rangle - |01\rangle}{\sqrt{2}} = 0$$

Exercise 2.67

Suppose W^\perp is the orthogonal complement of W . Then $V = W \oplus W^\perp$. Let $|w_i\rangle, |w'_j\rangle, |u'_j\rangle$ be orthonormal bases for $W, W^\perp, (\text{image}(U))^\perp$, respectively.

Define $U' : V \rightarrow V$ as $U' = \sum_i |u_i\rangle \langle w_i| + \sum_j |u'_j\rangle \langle w'_j|$, where $|u_i\rangle = U |w_i\rangle$.

Now

$$\begin{aligned} (U')^\dagger U' &= \left(\sum_{i=1}^{\dim W} |w_i\rangle \langle u_i| + \sum_{j=1}^{\dim W^\perp} |w'_j\rangle \langle u'_j| \right) \left(\sum_i |u_i\rangle \langle w_i| + \sum_j |u'_j\rangle \langle w'_j| \right) \\ &= \sum_i |w_i\rangle \langle w_i| + \sum_j |w'_j\rangle \langle w'_j| = I \end{aligned}$$

and

$$\begin{aligned} U'(U')^\dagger &= \left(\sum_i |u_i\rangle \langle w_i| + \sum_j |u'_j\rangle \langle w'_j| \right) \left(\sum_i |w_i\rangle \langle u_i| + \sum_j |w'_j\rangle \langle u'_j| \right) \\ &= \sum_i |u_i\rangle \langle u_i| + \sum_j |u'_j\rangle \langle u'_j| = I. \end{aligned}$$

Thus U' is a unitary operator. Moreover, for all $|w\rangle \in W$,

$$\begin{aligned} U' |w\rangle &= \left(\sum_i |u_i\rangle \langle w_i| + \sum_j |u'_j\rangle \langle w'_j| \right) |w\rangle \\ &= \sum_i |u_i\rangle \langle w_i|w\rangle + \sum_j |u'_j\rangle \langle w'_j|w\rangle \\ &= \sum_i |u_i\rangle \langle w_i|w\rangle \quad (\because |w'_j\rangle \perp |w\rangle) \\ &= \sum_i U |w_i\rangle \langle w_i|w\rangle \\ &= U |w\rangle. \end{aligned}$$

Therefore U' is an extension of U .

Exercise 2.68

$$|\psi\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}.$$

Suppose $|a\rangle = a_0 |0\rangle + a_1 |1\rangle$ and $|b\rangle = b_0 |0\rangle + b_1 |1\rangle$.

$$|a\rangle |b\rangle = a_0 b_0 |00\rangle + a_0 b_1 |01\rangle + a_1 b_0 |10\rangle + a_1 b_1 |11\rangle.$$

If $|\psi\rangle = |a\rangle |b\rangle$, then $a_0 b_0 = 1$, $a_0 b_1 = 0$, $a_1 b_0 = 0$, $a_1 b_1 = 1$ since $\{|ij\rangle\}$ is an orthonormal basis.

If $a_0 b_1 = 0$, then $a_0 = 0$ or $b_1 = 0$.

When $a_0 = 0$, this is contradiction to $a_0 b_0 = 1$. When $b_1 = 0$, this is contradiction to $a_1 b_1 = 1$.

Thus $|\psi\rangle \neq |a\rangle |b\rangle$.

Exercise 2.69

Define Bell states as follows.

$$\begin{aligned} |\psi_1\rangle &\equiv \frac{|00\rangle + |11\rangle}{\sqrt{2}} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \\ |\psi_2\rangle &\equiv \frac{|00\rangle - |11\rangle}{\sqrt{2}} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} \\ |\psi_3\rangle &\equiv \frac{|01\rangle + |10\rangle}{\sqrt{2}} = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \\ |\psi_4\rangle &\equiv \frac{|01\rangle - |10\rangle}{\sqrt{2}} = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix} \end{aligned}$$

First, we prove $\{|\psi_i\rangle\}$ is a linearly independent basis.

$$\begin{aligned} a_1 |\psi_1\rangle + a_2 |\psi_2\rangle + a_3 |\psi_3\rangle + a_4 |\psi_4\rangle &= 0 \\ \therefore \frac{1}{\sqrt{2}} \begin{bmatrix} a_1 + a_2 \\ a_3 + a_4 \\ a_3 - a_4 \\ a_1 - a_2 \end{bmatrix} &= 0 \end{aligned}$$

$$a_1 + a_2 = 0$$

$$a_3 + a_4 = 0$$

$$a_3 - a_4 = 0$$

$$a_1 - a_2 = 0$$

$$\therefore a_1 = a_2 = a_3 = a_4 = 0$$

Thus $\{|\psi_i\rangle\}$ is a linearly independent basis.

Moreover $\| |\psi_i\rangle \| = 1$ and $\langle \psi_i | \psi_j \rangle = \delta_{ij}$ for $i, j = 1, 2, 3, 4$. Therefore $\{|\psi_i\rangle\}$ forms an orthonormal basis.

Exercise 2.70

For any Bell states we get $\langle \psi_i | E \otimes I | \psi_i \rangle = \frac{1}{2}(\langle 0 | E | 0 \rangle + \langle 1 | E | 1 \rangle)$.

Suppose Eve measures the qubit Alice sent by measurement operators M_m . The probability that Eve gets result m is $p_i(m) = \langle \psi_i | M_m^\dagger M_m \otimes I | \psi_i \rangle$. Since $M_m^\dagger M_m$ is positive, $p_i(m)$ are same values for all $|\psi_i\rangle$. Thus Eve can't distinguish Bell states.

Exercise 2.71

From spectral decomposition,

$$\begin{aligned}\rho &= \sum_i p_i |\psi_i\rangle \langle \psi_i|, \quad p_i \geq 0, \quad \sum_i p_i = 1. \\ \rho^2 &= \sum_{i,j} p_i p_j |i\rangle \langle i| j\rangle \langle j| \\ &= \sum_{i,j} p_i p_j |i\rangle \langle j| \delta_{ij} \\ &= \sum_i p_i^2 |i\rangle \langle i|\end{aligned}$$

$$\text{Tr}(\rho^2) = \text{Tr} \left(\sum_i p_i^2 |i\rangle \langle i| \right) = \sum_i p_i^2 \text{Tr}(|i\rangle \langle i|) = \sum_i p_i^2 \langle i|i\rangle = \sum_i p_i^2 \leq \sum_i p_i = 1 \quad (\because p_i^2 \leq p_i)$$

Suppose $\text{Tr}(\rho^2) = 1$. Then $\sum_i p_i^2 = 1$. Since $p_i^2 < p_i$ for $0 < p_i < 1$, only a single p_i should be 1 and otherwise have to vanish. Therefore $\rho = |\psi_i\rangle \langle \psi_i|$. It is a pure state.

Conversely if ρ is pure, then $\rho = |\psi\rangle \langle \psi|$.

$$\text{Tr}(\rho^2) = \text{Tr}(|\psi\rangle \langle \psi| \psi\rangle \langle \psi|) = \text{Tr}(|\psi\rangle \langle \psi|) = \langle \psi|\psi\rangle = 1.$$

Exercise 2.72

(1) Since the density matrix is Hermitian, the matrix representation is $\rho = \begin{bmatrix} a & b \\ b^* & d \end{bmatrix}$, $a, d \in \mathcal{R}$ and $b \in \mathcal{C}$ w.r.t. standard basis. Because ρ is density matrix, $\text{Tr}(\rho) = a + d = 1$.

Define $a = (1 + r_3)/2$, $d = (1 - r_3)/2$ and $b = (r_1 - ir_2)/2$, ($r_i \in \mathcal{R}$).

In this case,

$$\rho = \begin{bmatrix} a & b \\ b^* & d \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 + r_3 & r_1 - ir_2 \\ r_1 + ir_2 & 1 - r_3 \end{bmatrix} = \frac{1}{2}(I + \vec{r} \cdot \vec{\sigma}).$$

Thus for arbitrary density matrix ρ can be written as $\rho = \frac{1}{2}(I + \vec{r} \cdot \vec{\sigma})$.

Next, we derive the condition that ρ is positive.

If ρ is positive, all eigenvalues of ρ should be non-negative.

$$\begin{aligned}\det(\rho - \lambda I) &= (a - \lambda)(b - \lambda) - |b|^2 = \lambda^2 - (a + d)\lambda + ad - |b|^2 = 0 \\ \lambda &= \frac{(a + d) \pm \sqrt{(a + d)^2 - 4(ad - |b|^2)}}{2} \\ &= \frac{1 \pm \sqrt{1 - 4\left(\frac{1-r_3^2}{4} - \frac{r_1^2+r_2^2}{4}\right)}}{2} \\ &= \frac{1 \pm \sqrt{1 - (1 - r_1^2 - r_2^2 - r_3^2)}}{2} \\ &= \frac{1 \pm \sqrt{|\vec{r}|^2}}{2} \\ &= \frac{1 \pm |\vec{r}|}{2}\end{aligned}$$

Since ρ is positive, $\frac{1-|\vec{r}|}{2} \geq 0 \rightarrow |\vec{r}| \leq 1$.

Therefore an arbitrary density matrix for a mixed state qubit is written as $\rho = \frac{1}{2}(I + \vec{r} \cdot \vec{\sigma})$.

(2)

$\rho = I/2 \rightarrow \vec{r} = 0$. Thus $\rho = I/2$ corresponds to the origin of Bloch sphere.

(3)

$$\begin{aligned} \rho^2 &= \frac{1}{2}(I + \vec{r} \cdot \vec{\sigma}) \frac{1}{2}(I + \vec{r} \cdot \vec{\sigma}) \\ &= \frac{1}{4} \left[I + 2\vec{r} \cdot \vec{\sigma} + \sum_{j,k} r_j r_k \left(\delta_{jk} I + i \sum_{l=1}^3 \epsilon_{jkl} \sigma_l \right) \right] \\ &= \frac{1}{4} (I + 2\vec{r} \cdot \vec{\sigma} + |\vec{r}|^2 I) \\ \text{Tr}(\rho^2) &= \frac{1}{4} (2 + 2|\vec{r}|^2) \end{aligned}$$

If ρ is pure, then $\text{Tr}(\rho^2) = 1$.

$$\begin{aligned} 1 = \text{Tr}(\rho^2) &= \frac{1}{4} (2 + 2|\vec{r}|^2) \\ \therefore |\vec{r}| &= 1. \end{aligned}$$

Conversely, if $|\vec{r}| = 1$, then $\text{Tr}(\rho^2) = \frac{1}{4} (2 + 2|\vec{r}|^2) = 1$. Therefore ρ is pure.

Exercise 2.73

Theorem 2.6

$$\rho = \sum_i p_i |\psi_i\rangle \langle \psi_i| = \sum_j q_j |\varphi_j\rangle \langle \varphi_j| \Leftrightarrow |\psi_i\rangle = \sum_j u_{ij} |\varphi_j\rangle$$

where u is a unitary matrix.

The-transformation in theorem 2.6, $|\tilde{\psi}_i\rangle = \sum_j u_{ij} |\tilde{\varphi}_j\rangle$, corresponds to

$$\left[|\tilde{\psi}_1\rangle \cdots |\tilde{\psi}_k\rangle \right] = \left[|\tilde{\varphi}_1\rangle \cdots |\tilde{\varphi}_k\rangle \right] U^T$$

where $k = \text{rank}(\rho)$.

$$\sum_i |\tilde{\psi}_i\rangle \langle \tilde{\psi}_i| = \left[|\tilde{\psi}_1\rangle \cdots |\tilde{\psi}_k\rangle \right] \begin{bmatrix} \langle \tilde{\psi}_1| \\ \vdots \\ \langle \tilde{\psi}_k| \end{bmatrix} \quad (8)$$

$$= \left[|\tilde{\varphi}_1\rangle \cdots |\tilde{\varphi}_k\rangle \right] U^T U^* \begin{bmatrix} \langle \tilde{\varphi}_1| \\ \vdots \\ \langle \tilde{\varphi}_k| \end{bmatrix} \quad (9)$$

$$= \left[|\tilde{\varphi}_1\rangle \cdots |\tilde{\varphi}_k\rangle \right] \begin{bmatrix} \langle \tilde{\varphi}_1| \\ \vdots \\ \langle \tilde{\varphi}_k| \end{bmatrix} \quad (10)$$

$$= \sum_j |\tilde{\varphi}_j\rangle \langle \tilde{\varphi}_j|. \quad (11)$$

From spectral theorem, density matrix ρ is decomposed as $\rho = \sum_{k=1}^d \lambda_k |k\rangle \langle k|$ where $d = \dim \mathcal{H}$. Without loss of generality, we can assume $p_k > 0$ for $k = 1 \dots l$ where $l = \text{rank}(\rho)$ and $p_k = 0$ for $k = l+1, \dots, d$. Thus $\rho = \sum_{k=1}^l p_k |k\rangle \langle k| = \sum_{k=1}^l |\tilde{k}\rangle \langle \tilde{k}|$, where $|\tilde{k}\rangle = \sqrt{\lambda_k} |k\rangle$.

Suppose $|\psi_i\rangle$ is a state in support ρ . Then

$$|\psi_i\rangle = \sum_{k=1}^l c_{ik} |k\rangle, \quad \sum_k |c_{ik}|^2 = 1.$$

Define $p_i = \frac{1}{\sum_k \frac{|c_{ik}|^2}{\lambda_k}}$ and $u_{ik} = \frac{\sqrt{p_i} c_{ik}}{\sqrt{\lambda_k}}$.

Now

$$\sum_k |u_{ik}|^2 = \sum_k \frac{p_i |c_{ik}|^2}{\lambda_k} = p_i \sum_k \frac{|c_{ik}|^2}{\lambda_k} = 1.$$

Next, prepare a unitary operator¹ such that i -th row of U is $[u_{i1} \dots u_{ik} \dots u_{il}]$. Then we can define another ensemble such that

$$\left[|\tilde{\psi}_1\rangle \dots |\tilde{\psi}_i\rangle \dots |\tilde{\psi}_l\rangle \right] = \left[|\tilde{k}_1\rangle \dots |\tilde{k}_l\rangle \right] U^T$$

where $|\tilde{\psi}_i\rangle = \sqrt{p_i} |\psi_i\rangle$. From theorem 2.6,

$$\rho = \sum_k |\tilde{k}\rangle \langle \tilde{k}| = \sum_k |\tilde{\psi}_k\rangle \langle \tilde{\psi}_k|.$$

Therefore we can obtain a minimal ensemble for ρ that contains $|\psi_i\rangle$.

Moreover since $\rho^{-1} = \sum_k \frac{1}{\lambda_k} |k\rangle \langle k|$,

$$\langle \psi_i | \rho^{-1} | \psi_i \rangle = \sum_k \frac{1}{\lambda_k} \langle \psi_i | k \rangle \langle k | \psi_i \rangle = \sum_k \frac{|c_{ik}|^2}{\lambda_k} = \frac{1}{p_i}.$$

Hence, $\frac{1}{\langle \psi_i | \rho^{-1} | \psi_i \rangle} = p_i$.

Exercise 2.74

$$\begin{aligned} \rho_{AB} &= |a\rangle_A \langle a|_A \otimes |b\rangle_B \langle b|_B \\ \rho_A &= \text{Tr}_B \rho_{AB} = |a\rangle \langle a| \text{Tr}(|b\rangle \langle b|) = |a\rangle \langle a| \\ \text{Tr}(\rho_A^2) &= 1 \end{aligned}$$

Thus ρ_A is pure.

¹By Gram-Schmidt procedure construct an orthonormal basis $\{\mathbf{u}_j\}$ (row vector) with $\mathbf{u}_i =$

$$[u_{i1} \dots u_{ik} \dots u_{il}]. \text{ Then define unitary } U = \begin{bmatrix} \mathbf{u}_1 \\ \vdots \\ \mathbf{u}_i \\ \vdots \\ \mathbf{u}_l \end{bmatrix}.$$

Exercise 2.75

Define $|\Phi_{\pm}\rangle = \frac{1}{\sqrt{2}}(|00\rangle \pm |11\rangle)$ and $|\Psi_{\pm}\rangle = \frac{1}{\sqrt{2}}(|01\rangle \pm |10\rangle)$.

$$\begin{aligned} |\Phi_{\pm}\rangle_{AB} \langle\Phi_{\pm}|_{AB} &= \frac{1}{2}(|00\rangle \langle 00| \pm |00\rangle \langle 11| \pm |11\rangle \langle 00| + |11\rangle \langle 11|) \\ \text{Tr}_B(|\Phi_{\pm}\rangle_{AB} \langle\Phi_{\pm}|_{AB}) &= \frac{1}{2}(|0\rangle \langle 0| + |1\rangle \langle 1|) = \frac{I}{2} \\ |\Psi_{\pm}\rangle_{AB} \langle\Psi_{\pm}|_{AB} &= \frac{1}{2}(|01\rangle \langle 01| \pm |01\rangle \langle 10| \pm |10\rangle \langle 01| + |10\rangle \langle 10|) \\ \text{Tr}_B(|\Psi_{\pm}\rangle_{AB} \langle\Psi_{\pm}|_{AB}) &= \frac{1}{2}(|0\rangle \langle 0| + |1\rangle \langle 1|) = \frac{I}{2} \end{aligned}$$

Exercise 2.76

Exercise 2.77

$$\begin{aligned} |\psi\rangle &= |0\rangle |\Phi_{+}\rangle \\ &= |0\rangle \left[\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \right] \\ &= (\alpha |\phi_0\rangle + \beta |\phi_1\rangle) \left[\frac{1}{\sqrt{2}}(|\phi_0\phi_0\rangle + |\phi_1\phi_1\rangle) \right] \end{aligned}$$

where $|\phi_i\rangle$ are arbitrary orthonormal states and $\alpha, \beta \in \mathcal{C}$. We cannot vanish cross-term. Therefore $|\psi\rangle$ cannot be written as $|\psi\rangle = \sum_i \lambda_i |i\rangle_A |i\rangle_B |i\rangle_C$.

Exercise 2.78

Proof. Former part.

If $|\psi\rangle$ is product, then there exist a state $|\phi_A\rangle$ for system A , and a state $|\phi_B\rangle$ for system B such that $|\psi\rangle = |\phi_A\rangle |\phi_B\rangle$.

Obviously, this Schmidt number is 1.

Conversely, if the Schmidt number is 1, the state is written as $|\psi\rangle = |\phi_A\rangle |\phi_B\rangle$. Hence this is a product state. \square

Proof. Later part.

(\Rightarrow) Proved by exercise 2.74.

(\Leftarrow) Let a pure state be $|\psi\rangle = \sum_i \lambda_i |i_A\rangle |i_B\rangle$. Then $\rho_A = \text{Tr}_B(|\psi\rangle \langle\psi|) = \sum_i \lambda_i^2 |i\rangle \langle i|$. If ρ_A is a pure state, then $\lambda_j = 1$ and otherwise 0 for some j . It follows that $|\psi_j\rangle = |j_A\rangle |j_B\rangle$. Thus $|\psi\rangle$ is a product state. \square

Exercise 2.79

The procedure of Schmidt decomposition \Rightarrow Goal: $|\psi\rangle = \sum_i \sqrt{\lambda_i} |i_A\rangle |i_B\rangle$

- Diagonalize reduced density matrix $\rho_A = \sum_i \lambda_i |i_A\rangle \langle i_A|$.

- Derive $|i_B\rangle \rightarrow |i_B\rangle = \frac{(I \otimes \langle i_A|) |\psi\rangle}{\sqrt{\lambda_i}}$

- Construct $|\psi\rangle$.

(i)

$$\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \Rightarrow \text{This is already decomposed.}$$

(ii)

$$\frac{|00\rangle + |01\rangle + |10\rangle + |11\rangle}{2} = \left(\frac{|0\rangle + |1\rangle}{\sqrt{2}} \right) \otimes \left(\frac{|0\rangle + |1\rangle}{\sqrt{2}} \right) = |\psi\rangle |\psi\rangle \text{ where } |\psi\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}$$

(iii)

$$|\psi\rangle_{AB} = \frac{1}{\sqrt{3}}(|00\rangle + |01\rangle + |10\rangle)$$

$$\rho_{AB} = |\psi\rangle_{AB} \langle\psi|_{AB}$$

$$\begin{aligned} \rho_A &= \text{Tr}_B(\rho_{AB}) = \frac{1}{3} (2|0\rangle\langle 0| + |0\rangle\langle 1| + |1\rangle\langle 0| + |1\rangle\langle 1|) \\ \det(\rho_A - \lambda I) &= \left(\frac{2}{3} - \lambda \right) \left(\frac{1}{3} - \lambda \right) - \frac{1}{9} = 0 \\ \lambda^2 - \lambda + \frac{1}{9} &= 0 \\ \lambda &= \frac{1 \pm \sqrt{5}/3}{2} = \frac{3 \pm \sqrt{5}}{6} \end{aligned}$$

$$\text{Eigenvector with eigenvalue } \lambda_0 \equiv \frac{3 + \sqrt{5}}{6} \text{ is } |\lambda_0\rangle \equiv \frac{1}{\sqrt{\frac{5+\sqrt{5}}{2}}} \begin{bmatrix} \frac{1+\sqrt{5}}{2} \\ 1 \end{bmatrix}.$$

$$\text{Eigenvector with eigenvalue } \lambda_1 \equiv \frac{3 - \sqrt{5}}{6} \text{ is } |\lambda_1\rangle \equiv \frac{1}{\sqrt{\frac{5-\sqrt{5}}{2}}} \begin{bmatrix} \frac{1-\sqrt{5}}{2} \\ 1 \end{bmatrix}.$$

$$\rho_A = \lambda_0 |\lambda_0\rangle \langle\lambda_0| + \lambda_1 |\lambda_1\rangle \langle\lambda_1|.$$

$$\begin{aligned} |a_0\rangle &\equiv \frac{(I \otimes \langle\lambda_0|) |\psi\rangle}{\sqrt{\lambda_0}} \\ |a_1\rangle &\equiv \frac{(I \otimes \langle\lambda_1|) |\psi\rangle}{\sqrt{\lambda_1}} \end{aligned}$$

Then

$$|\psi\rangle = \sum_{i=0}^1 \sqrt{\lambda_i} |a_i\rangle |\lambda_i\rangle.$$

(It's too tiresome to calculate $|a_i\rangle$)

Exercise 2.80

Let $|\psi\rangle = \sum_i \lambda_i |\psi_i\rangle_A |\psi_i\rangle_B$ and $|\varphi\rangle = \sum_i \lambda_i |\varphi_i\rangle_A |\varphi_i\rangle_B$.
 Define $U = \sum_j |\psi_j\rangle \langle \varphi_j|_A$ and $V = \sum_j |\psi_j\rangle \langle \varphi_j|_B$.
 Then

$$\begin{aligned} (U \otimes V) |\varphi\rangle &= \sum_i \lambda_i U |\varphi_i\rangle_A V |\varphi_i\rangle_B \\ &= \sum_i \lambda_i |\psi_i\rangle_A |\psi_i\rangle_B \\ &= |\psi\rangle. \end{aligned}$$

Exercise 2.81

Let the Schmidt decomposition of $|AR_1\rangle$ be $|AR_1\rangle = \sum_i \sqrt{p_i} |\psi_i^A\rangle |\psi_i^R\rangle$ and let $|AR_2\rangle = \sum_i \sqrt{q_i} |\phi_i^A\rangle |\phi_i^R\rangle$.

Suppose ρ^A has orthonormal decomposition $\rho^A = \sum_i p_i |i\rangle \langle i|$.

Since $|AR_1\rangle$ and $|AR_2\rangle$ are purifications of the ρ^A , we have

$$\begin{aligned} \text{Tr}_{R_1}(|AR_1\rangle \langle AR_1|) &= \text{Tr}_{R_2}(|AR_2\rangle \langle AR_2|) = \rho^A \\ \therefore \sum_i p_i |\psi_i^A\rangle \langle \psi_i^A| &= \sum_i q_i |\phi_i^A\rangle \langle \phi_i^A| = \sum_i \lambda_i |i\rangle \langle i|. \end{aligned}$$

The $|i\rangle$, $|\psi_i^A\rangle$, and $|\psi_i^R\rangle$ are orthonormal bases and they are eigenvectors of ρ^A . Hence without loss of generality, we can consider

$$\lambda_i = p_i = q_i \text{ and } |i\rangle = |\psi_i^A\rangle = |\phi_i^A\rangle.$$

Then

$$\begin{aligned} |AR_1\rangle &= \sum_i \lambda_i |i\rangle |\psi_i^R\rangle \\ |AR_2\rangle &= \sum_i \lambda_i |i\rangle |\phi_i^R\rangle \end{aligned}$$

Since $|AR_1\rangle$ and $|AR_2\rangle$ have same Schmidt numbers, there are two unitary operators U and V such that $|AR_1\rangle = (U \otimes V) |AR_2\rangle$ from exercise 2.80.

Suppose $U = I$ and $V = \sum_i |\psi_i^R\rangle \langle \phi_i^R|$. Then

$$\begin{aligned} \left(I \otimes \sum_j |\psi_j^R\rangle \langle \phi_j^R| \right) |AR_2\rangle &= \sum_i \lambda_i |i\rangle \left(\sum_j |\psi_j^R\rangle \langle \phi_j^R| \phi_i^R \right) \\ &= \sum_i \lambda_i |i\rangle |\psi_i^R\rangle \\ &= |AR_1\rangle. \end{aligned}$$

Therefore there exists a unitary transformation U_R acting on system R such that $|AR_1\rangle = (I \otimes U_R) |AR_2\rangle$.

Exercise 2.82

(1)

Let $|\psi\rangle = \sum_i \sqrt{p_i} |\psi_i\rangle |i\rangle$.

$$\begin{aligned} \text{Tr}_R(|\psi\rangle \langle\psi|) &= \sum_{i,j} \sqrt{p_i} \sqrt{p_j} |\psi_i\rangle \langle\psi_j| \text{Tr}_R(|i\rangle \langle j|) \\ &= \sum_{i,j} \sqrt{p_i} \sqrt{p_j} |\psi_i\rangle \langle\psi_j| \delta_{ij} \\ &= \sum_i p_i |\psi_i\rangle \langle\psi_i| = \rho. \end{aligned}$$

Thus $|\psi\rangle$ is a purification of ρ .

(2)

Define the projector P by $P = I \otimes |i\rangle \langle i|$. The probability we get the result i is

$$\text{Tr}[P|\psi\rangle \langle\psi|] = \langle\psi|P|\psi\rangle = \langle\psi|(I \otimes |i\rangle \langle i|)|\psi\rangle = p_i \langle\psi_i|\psi_i\rangle = p_i.$$

The post-measurement state is

$$\frac{P|\psi\rangle}{\sqrt{p_i}} = \frac{(I \otimes |i\rangle \langle i|)|\psi\rangle}{\sqrt{p_i}} = \frac{\sqrt{p_i} |\psi_i\rangle |i\rangle}{\sqrt{p_i}} = |\psi_i\rangle |i\rangle.$$

If we only focus on the state of the system A ,

$$\text{Tr}_R(|\psi_i\rangle |i\rangle) = |\psi_i\rangle.$$

(3)

($\{|\psi_i\rangle\}$ is not necessary an orthonormal basis.)

Suppose $|AR\rangle$ is a purification of ρ and its Schmidt decomposition is $|AR\rangle = \sum_i \sqrt{\lambda_i} |\phi_i^A\rangle |\phi_i^R\rangle$.

From assumption

$$\text{Tr}_R(|AR\rangle \langle AR|) = \sum_i \lambda_i |\phi_i^A\rangle \langle\phi_i^A| = \sum_i p_i |\psi_i\rangle \langle\psi_i|.$$

By theorem 2.6, there exists a unitary matrix u_{ij} such that $\sqrt{\lambda_i} |\phi_i^A\rangle = \sum_j u_{ij} \sqrt{p_j} |\psi_j\rangle$.
Then

$$\begin{aligned} |AR\rangle &= \sum_i \left(\sum_j u_{ij} \sqrt{p_j} |\psi_j\rangle \right) |\phi_i^R\rangle \\ &= \sum_j \sqrt{p_j} |\psi_j\rangle \otimes \left(\sum_i u_{ij} |\phi_i^R\rangle \right) \\ &= \sum_j \sqrt{p_j} |\psi_j\rangle |j\rangle \\ &= \sum_i \sqrt{p_i} |\psi_i\rangle |i\rangle \end{aligned}$$

where $|i\rangle = \sum_k u_{ki} |\phi_k^R\rangle$.

About $|i\rangle$,

$$\begin{aligned}
 \langle k|l\rangle &= \sum_{m,n} u_{mk}^* u_{nl} \langle \phi_m^R | \phi_n^R \rangle \\
 &= \sum_{m,n} u_{mk}^* u_{nl} \delta_{mn} \\
 &= \sum_m u_{mk}^* u_{ml} \\
 &= \delta_{kl}, \quad (\because u_{ij} \text{ is unitary.})
 \end{aligned}$$

which implies $|i\rangle$ is an orthonormal basis for system R .

Therefore if we measure system R w.r.t $|i\rangle$, we obtain i with probability p_i and post-measurement state for A is $|\psi_i\rangle$ from (2). Thus for any purification $|AR\rangle$, there exists an orthonormal basis $|i\rangle$ which satisfies the assertion.

Problem 2.1

From Exercise 2.35, $\vec{n} \cdot \vec{\sigma}$ is decomposed as

$$\vec{n} \cdot \vec{\sigma} = |\lambda_1\rangle \langle \lambda_1| - |\lambda_{-1}\rangle \langle \lambda_{-1}|$$

where $|\lambda_{\pm 1}\rangle$ are eigenvector of $\vec{n} \cdot \vec{\sigma}$ with eigenvalues ± 1 .

Thus

$$\begin{aligned}
 f(\theta \vec{n} \cdot \vec{\sigma}) &= f(\theta) |\lambda_1\rangle \langle \lambda_1| + f(-\theta) |\lambda_{-1}\rangle \langle \lambda_{-1}| \\
 &= \left(\frac{f(\theta) + f(-\theta)}{2} + \frac{f(\theta) - f(-\theta)}{2} \right) |\lambda_1\rangle \langle \lambda_1| + \left(\frac{f(\theta) + f(-\theta)}{2} - \frac{f(\theta) - f(-\theta)}{2} \right) |\lambda_{-1}\rangle \langle \lambda_{-1}| \\
 &= \frac{f(\theta) + f(-\theta)}{2} (|\lambda_1\rangle \langle \lambda_1| + |\lambda_{-1}\rangle \langle \lambda_{-1}|) + \frac{f(\theta) - f(-\theta)}{2} (|\lambda_1\rangle \langle \lambda_1| - |\lambda_{-1}\rangle \langle \lambda_{-1}|) \\
 &= \frac{f(\theta) + f(-\theta)}{2} I + \frac{f(\theta) - f(-\theta)}{2} \vec{n} \cdot \vec{\sigma}
 \end{aligned}$$

Problem 2.2

Unsolved

Problem 2.3

Unsolved