Solutions for exercises of Chapter 11 of “Nielesen and Chuang”

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Exercise 11.1

The entropy associated to a fair coin:

$$H(X) = -2 \times \frac{1}{2} \log \frac{1}{2} = \log 2 = 1$$

The entropy associated to a fair die:

$$H(X) = -6 \times \frac{1}{6} \log \frac{1}{6} = \log 6 = 1 + \log 3$$

For an unfair coin we can write:

$$H(X) = -p \log p - (1-p) \log (1-p)$$

and for the unfair die:

$$\begin{aligned} H(X) = &- p_1 \log p_1 - p_2 \log p_2 - p_3 \log p_3 - p_4 \log p_4 - p_5 \log p_5 + \\ &- (1 - p_1 - p_2 - p_3 - p_4 - p_5) \log (1 - p_1 - p_2 - p_3 - p_4 - p_5) \end{aligned}$$

Differentiating both of these we see that for both the global maxima is when all the probabilities are equal, therefore for the unfair coin and die the entropy will decrease, which means we would have less uncertainty about X before we know its value.

Exercise 11.2

- $I(p) = k \log p$ is a function of probability alone.
- $\log p$ is smooth for $0 < p \leq 1$
- $I(pq) = k \log (pq) = k(\log p + \log q) = I(p) + I(q)$

Exercise 11.3

$$H_{bin}(p) = -p \log p - (1-p) \log (1-p)$$

We search for the maximum by imposing the derivative equal to zero:

$$\frac{dH_{bin}}{dp} = -\frac{1}{\ln 2} - \log p + \frac{1}{\ln 2} + \log (1-p) = 0$$

$$\frac{1-p}{p} = 1 \;\Rightarrow\; p = \frac{1}{2}$$

Exercise 11.4

For a function $f(x)$ to be concave we require $f''(x) < 0$.

$$\frac{d^2 H_{bin}}{dp^2} = \frac{d}{dp}(\log(1-p) - \log p) = \frac{1}{\ln 2(1-p)p} < 0$$

Hence, H_{bin} is concave.

Exercise 11.5

$$\begin{aligned}
H(p(x,y)||p(x)p(y)) &= \sum_{xy} p(x,y) \log \frac{p(x,y)}{p(x)p(y)} = \\
&= \sum_{xy} p(x,y) \log p(x,y) - \sum_{xy} p(x,y) \log p(x) - \sum_{xy} p(x,y) \log p(y) = \\
&= \sum_{xy} p(x,y) \log p(x,y) - \sum_{y} p(y) \sum_{x} p(x) \log p(x) - \sum_{x} p(x) \sum_{y} p(y) \log p(y) = \\
&= \sum_{xy} p(x,y) \log p(x,y) - \sum_{x} p(x) \log p(x) - \sum_{y} p(y) \log p(y) = \\
&= H(p(x)) + H(p(y)) - H(p(x,y)) = H(p(x,y)||p(x)p(y)) \geq 0
\end{aligned}$$

Therefore,

$$H(p(x)) + H(p(y)) - H(p(x,y)) = H(X) + H(Y) - H(X,Y) \geq 0$$

$$\longrightarrow H(X,Y) \leq H(X) + H(Y)$$

If X and Y are independent then $p(x,y) = p(x)p(y)$. Therefore,

$$\begin{aligned}
H(X,Y) &= -\sum_{xy} p(x,y) \log p(x,y) = -\sum_{xy} p(x)p(y) \log p(x)p(y) = \\
&= -\sum_{y} p(y) \sum_{x} p(x) \log p(x) - \sum_{x} p(x) \sum_{y} p(y) \log p(y) = \\
&= -\sum_{x} p(x) \log p(x) - \sum_{y} p(y) \log p(y) = \\
&= H(X) + H(Y)
\end{aligned}$$

Therefore, equality holds if and only if X and Y are independent.

Exercise 11.10

If $X \to Y \to Z$ is a Markov chain. then

$$p(Z|Y,X) = p(Z|Y)$$

Using $p(X|Y) = \frac{p(X,Y)}{p(Y)}$ on both sides,

$$\frac{p(Z,Y,X)}{p(Y,X)} = \frac{p(Z,Y)}{p(Y)} \to \frac{p(Z,Y,X)}{p(Z,Y)} = \frac{p(Y,X)}{p(Y)}$$

$$\Rightarrow p(X|Y,Z) = p(X|Y)$$

Therefore, $Z \to Y \to X$ is also a Markov chain.

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Exercise 11.11

$\rho = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \longrightarrow S(\rho) = -1\log 1 = 0$

$\rho = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, $\lambda = 1$ or $0 \longrightarrow S(\rho) = -1\log 1 = 0$

$\rho = \frac{1}{3}\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$, $\lambda = \frac{1}{2} \pm \frac{\sqrt{5}}{6} \longrightarrow S(\rho) = -(\frac{1}{2}+\frac{\sqrt{5}}{6})\log(\frac{1}{2}+\frac{\sqrt{5}}{6}) - (\frac{1}{2}-\frac{\sqrt{5}}{6})\log(\frac{1}{2}-\frac{\sqrt{5}}{6}) \approx 0.55$

Exercise 11.12

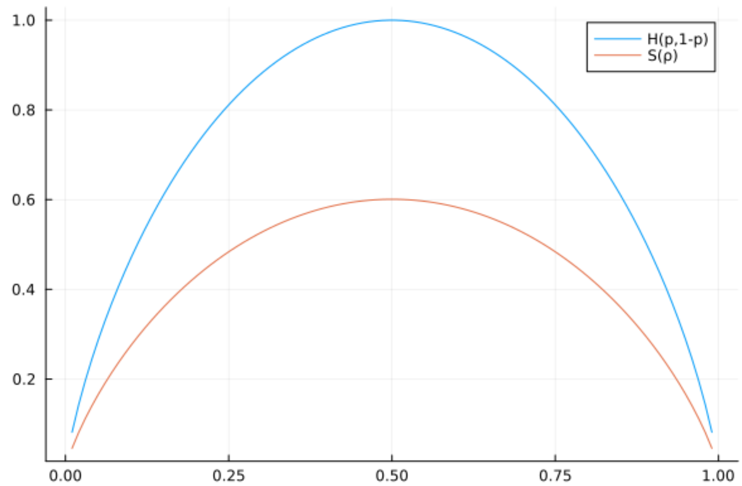
$\rho = p\,|0\rangle\langle 0| + (1-p)\frac{(|0\rangle+|1\rangle)(\langle 0|+\langle 1|)}{2} \longrightarrow \rho = \frac{1}{2}\begin{bmatrix} 1+p & 1-p \\ 1-p & 1-p \end{bmatrix} \longrightarrow \lambda = \frac{1}{2} \pm \frac{1}{2}\sqrt{1-2p(1-p)}$

$$\begin{aligned} S(\rho) = -\frac{1}{2}&((1+\sqrt{1-2p(1-p)})\log\frac{1}{2}(1+\sqrt{1-2p(1-p)}) \\ &+(1-\sqrt{1-2p(1-p)})\log\frac{1}{2}(1-\sqrt{1-2p(1-p)})) \end{aligned}$$

$$H(p,1-p) = -p\log p - (1-p)\log(1-p)$$

Therefore, as we can see from Fig. 1, we have $S(\rho) \leq H(p,1-p)$.

Figure 1: comparison of classical and quantum entropies



Exercise 11.13

First note that for $\rho = \sum_i p_i\,|i\rangle\langle i|$,

$$H(p_i) = -\sum_i p_i \log p_i = S(\rho)$$

Then using the joint entropy theorem (equation 11.58) for $\rho_i = \sigma\ \forall i$, we have

$$S(\rho\otimes\sigma) = S(\sum_i p_i\,|i\rangle\langle i|\otimes\sigma) = H(p_i) + \sum_i p_i S(\sigma) = S(\rho) + S(\sigma)$$

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Otherwise from the definition of entropy for $\rho = \sum_i p_i |i\rangle \langle i|$ and $\sigma = \sum_j q_i |j\rangle \langle j|$, we have

$$S(\rho \otimes \sigma) = S\left(\sum_{ij} p_i q_j |i\rangle |j\rangle \langle i| \langle j|\right) = -\sum_{ij} p_i q_j \log p_i q_j =$$
$$= -\sum_i p_i \log p_i - \sum_j q_j \log q_j = S(\rho) + S(\sigma)$$

Exercise 11.14

If $|AB\rangle$ is a pure state of the composite system then $|A\rangle$ is a pure state if and only if there's no entanglement. Hence, $S(A) \neq 0$ if and only if $|AB\rangle$ is entangled.
As $|AB\rangle$ is a pure state $S(A,B) = 0$, we have $S(B|A) = S(A,B) - S(A)$ and therefore $S(B|A) = -S(A)$.
Thus we obtain that $S(B|A) < 0$ if and only if $|AB\rangle$ is entangled, that is $S(A) \geq 0$.

Exercise 11.15

Let the state of the qubit be $\rho = \frac{I + \vec{r}\cdot\vec{\sigma}}{2}$. Then, being $M_1 = |0\rangle \langle 0|$ and $M_2 = |1\rangle \langle 0|$, we have

$$\rho' = M_1 \rho M_1^\dagger + M_2 \rho M_2^\dagger = \frac{1 + r_z}{2} |0\rangle \langle 0| + \frac{1 - r_z}{2} |0\rangle \langle 0| = |0\rangle \langle 0|$$

Hence,

$$S(\rho') = -\log 1 = 0$$

Therefore, $S(\rho) \geq S(\rho')$.

Exercise 11.21

Consider density matrices $\rho = \sum_i p_i |i\rangle \langle i|$ and $\sigma = \sum_i q_i |i\rangle \langle i|$.
As $|i\rangle \langle i|$ are pure states, we have

$$H(\lambda p_i + (1-\lambda) q_i) = S(\lambda\rho + (1-\lambda)\sigma) \geq \lambda S(\rho) + (1-\lambda) S(\sigma) = \lambda H(p_i) + (1-\lambda) H(q_i)$$

hence from the concavity of the von Neumann entropy, we have deduced the concavity of the Shannon entropy in probability distributions.

Problem 11.1

Problem 11.2

Problem 11.3

Problem 11.4

Problem 11.5

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