1 Implementation of Ridge Regression

Ridge Regression is a linear model for regression that takes into account a regularization term in the objective function. Starting from the linear regression objective function, We include the L2-regularization term[1] and we obtain an objective function of the form: $argmin_w \Lambda(w) + \lambda R(w) = argmin_w (y - Xw)^T (y - Xw) + \lambda w^T w$.
From this we can then obtain the closed form solution for w: (**Ridge regressor estimate**) $\hat{w} = (X^T X + \lambda I)^{-1} X^T y$.
In this way, we are able to keep the values of w low, since very large values of w could make our model very sensitive, thus leading to poor generalization. Furthermore, constraints added via regularization may help us in solving ill-posed problems, when $X^T X$ is not possible to invert (when $D >> N$). Therefore, λ becomes an hyper-parameter of our model controlling the strength of the regularization. After obtaining the vector of weights $\hat{w}$ we can perform regression in the same way as a non regularized linear regression in the form: $y = \hat{w}^T x$

1.1 The train function

In the training function, based on a boolean value (`normalize=False`) that can be set to True when our model is initialized, we normalize our training set using z-score normalization. Z-score normalization allows us to avoid that features with larger scale dominate the others, and to lower the strength of outliers. In the Olympics 100m dataset, since we have only one feature, we wouldn't need to worry about features dominating the others, but in case we pass different datasets, then the normalization may become very useful.

For the training phase of our model we then simply need to compute the closed form solution to obtain our weights. We only need to pay attention in transforming our data in order to absorb the bias term and then set $I(0,0) = 0$ before computing the solution, so that we do not regularize the bias term w_0

1.2 The predict function

In the predict function we just need to perform z-score normalization using the values that we saved as class attributes during training if `normalize=True`, then transform the data in order to take into consideration the bias term, and then compute[2] $Y = Xw$

2 Validation of the model

Since we are working on a dataset (Olympics 100m) with very few samples (29), after splitting it in training set (80%) and test set(20%), we performed **cross validation** with a **coarse-to-fine** approach to choose the best value of λ using the training set. However, since we are in the case of time series, cross validation and train/test splitting are not trivial, we cannot choose random samples and assign them to either the test set or the train because it makes no sense to use values from the future to forecast values in the past, we need to preserve the temporal dependency between observations. Thus we performed the first split in train/test taking into account this issue (by not shuffling). Then, to perform validation, for each value of λ, we start with a small subset of data for training (in our case we chose half of our training set), predict the temporally next sample, and compute MSE on this prediction[3]. Then we include the same predicted sample as part of the next training dataset and subsequent point is forecasted, and so on (see figure 1a for a general idea). Then we average the MSEs obtained for these to obtain a an average value of MSE that we use to evaluate the goodness of the given λ. Doing so, we manage to do a cross validation consistent with time series.
We performed this same cross-validation procedure using both `normalize=True` and `normalize=False` and we got the best results with `normalize=True` and $\lambda = 5.193$ obtaining an $MSE = 0.0208$.

3 Test of the model

We also tested our best configuration on a never-seen test set (20% of our dataset) after training on our full training set, and we obtained an $MSE = 0.0112$. We can say that our model generalized well.

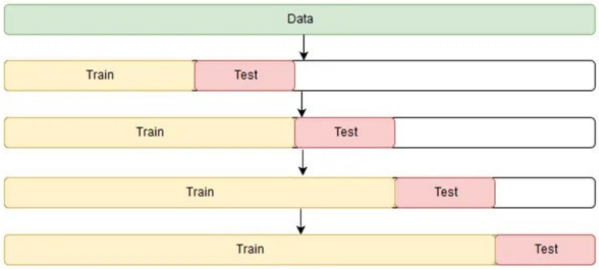
4 Comparison with scikit-learn

We then also compared our implementation of the Ridge Regression with the one of SciKit, as we can see in the plots the two implementations are equal, the only main difference is that SciKit does not perform normalization inside the class, but we had to manually normalize the dataset in experiments before passing it to the Ridge model.

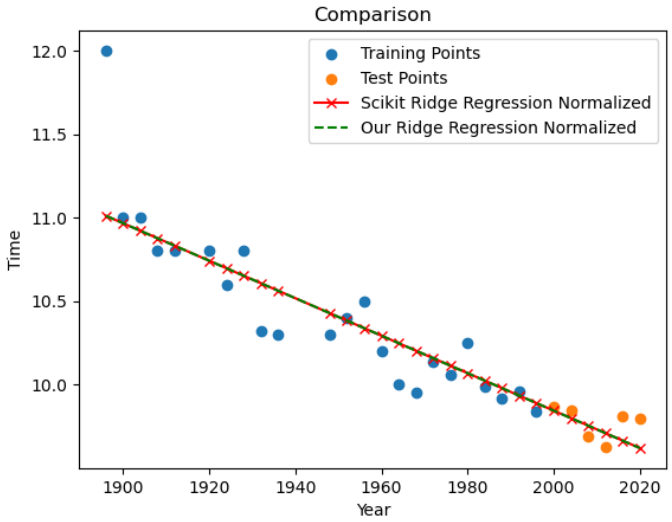
- Our implementation: $w_0 = 10.432, w_1 = -0.350, mse_{test} = 0.0112$
- Scikit: $w_0 = 10.432, w_1 = -0.350, mse_{test} = 0.0112$

The results are different only starting from the 10th decimal.

[1] Ridge Regression uses L2-regularization term
[2] We are doing matrix multiplications, thus we obtain an array of predicted values
[3] source for this method: https://medium.com/@soumyachess1496/cross-validation-in-time-series-566ae4981ce4



(a) Cross validation for time series



(b) Comparison with SciKit

Figure 1

5 ChatGPT Policy

We used ChatGPT to generate a first draft of our code documentation for the Ridge class.