3.1 Derivatives for Polynomials and Exponential Functions

Short cut rules for derivatives.

- Derivative of constant functions

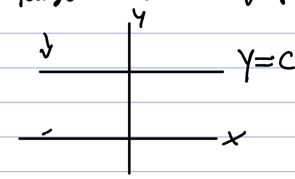
Let $f(x)=c$ Find $f'(x)$

$f'(x)=\frac{d}{dx}(c)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{c-c}{h}$

$=\lim_{h\to 0}\frac{0}{h}=\lim_{h\to 0}0=0$

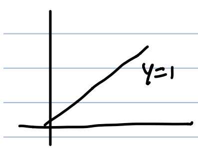
$\frac{d}{dx}[c]=0$

Tangent line will be graph itself



Power functions:

$f(x)=x^n$



$f'=1$

1. $\frac{d}{dx}(x)=1$

2. $\frac{d}{dx}(x^2)=2x$ $\quad \frac{d}{dx}(x^3)=3x^2$

3. $\frac{d}{dx}(x^4)=4x^3$

$\frac{d}{dx}(x^n)=nx^{n-1}$

Power rule: $\frac{d}{dx}[x^n]=nx^{n-1}$

Evaluate

a

ⓐ $\frac{d}{dx}(x^6)=6x^5$

ⓑ $\frac{d}{dx}(\sqrt{x})=\frac{d}{dx}(x^{\frac{1}{2}})=\frac{1}{2}x^{\frac{1}{2}-1}$

$=\frac{1}{2}x^{-\frac{1}{2}}=\frac{1}{2\sqrt{x}}$

ⓒ $\frac{d}{dx}\left(\frac{1}{x^3}\right)=\frac{d}{dx}(x^{-3})=-3x^{-4}$

$=\frac{-3}{x^4}$

The constant multiple Rule: we can differentiate x & multiply it by c

Ex: $\frac{d}{dx}[5x^2]=5(2)x=10x$

Tricker question evaluate $\frac{d}{dx}[e^3]=C$

Constant to a power $=0$ $\quad 3e^2=X$

Test → chapter 1 & 2 → 2.8

3.1

$\frac{d}{dx}[f(x)\pm g(x)]=f'(x)\pm g'(x)$

Ex: differentiate

ⓐ $F(x)=3x^6-x^4$

$F'(x)=\frac{d}{dx}(3x^6)-\frac{d}{dx}(x^4)$

→ take constant & put it in front of differentiation operator

$=3\frac{d}{dx}(x^6)-\frac{d}{dx}(x^4)$

$=3(6x^5)-4x^3=18x^5-4x^3$

ⓑ $h(x)=9x^4-3x^2$

$h'(x)=36x^3-6x$

Derivative of e^x <- Can't use power rule on e^x

Find $\frac{d}{dx}(e^x) = \lim_{h\to0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to0} \frac{e^{x+h}-e^x}{h} = \lim_{h\to0} \frac{e^x e^h - e^x}{h} = \lim_{h\to0} \frac{e^x(e^h-1)}{h}$

$= e^x \lim_{h\to0} \frac{e^h-1}{h}$ $\left[\frac{d}{dx} e^x (1) = e^x\right]$

e is the number such that $\lim_{h\to0} \frac{e^h-1}{h} = 1$

Ex: Differentiate $f(x) = 3e^x + \frac{4}{\sqrt{x}}$

$= 3e^x + 4x^{-\frac{1}{2}}$ $f'(x) = 3e^x - 4\left(\frac{1}{2}\right)x^{-\frac{3}{2}}$

Section 3.2 Product Rule

Let $f(x) = x$ & $g(x) = x^2$ power rule -> $f'(x) = 1$ & $g'(x) = 2x$

The Product Rule - If f and g are both differentiable then,

$(fg)' = fg' + f'g$

Ex: find f' and f''

$f(x) = x^3 e^x$

$f'(x) = x^3(e^x)' + (x^3)' e^x$
$= x^3 e^x + 3x^2 e^x$

$f''(x) = (x^3 e^x)' + (3x^2 e^x)'$ apply product rule to each term
$= [x^3(e^x)' + e^x(x^3)']$
$+ 3[x^2(e^x)' + e^x(x^2)']$ $= x^3 e^x + 3x^2 e^x + 3(x^2 e^x + 2xe^x)$
$= x^3 e^x + 6x^2 e^x + 6xe^x$

Quotient Rule - denominator times the derivative

$\left[\frac{f(x)}{g(x)}\right]' = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$ low is $g(x)$, high is $f(x)$] $\frac{\text{low D high} - \text{high D low}}{\text{low low}}$

Ex: given $h(x) = \frac{x^3}{2+5x}$, find $h'(x)$ & simplify

$h'(x) = \frac{(2+5x)(x^3)' - x^3(2+5x)'}{(2+5x)^2}$ -> $\frac{3x^2(2+5x) - 5x^3}{(2+5x)^2} = \frac{x^2(3(2+5x) - 5x)}{(2+5x)^2}$

$= \frac{x^2[6+15x-5x]}{(2+5x)^2} = \frac{x^2(6+10x)}{(2+5x)^2} = \boxed{\frac{2x^2(3+5x)}{(2+5x)^2}}$

Table of Differentiation formulas ↓

3.3

Two limits we will need:

• $\lim_{\theta\to0} \frac{\sin\theta}{\theta} = 1$ & $\lim_{\theta\to0} \frac{\cos\theta - 1}{\theta} = 0$

Clicker = given $f(1) = 1$ & $f'(1) = 5$ find $\frac{d}{dx}\left[\frac{f(x)}{x^3}\right]$

$\frac{(1^3)(5)' - (1)(x^3)'}{x^3} = \frac{x^3 f'(x) - f(x)(3x^2)}{(x^3)^2}$

$\boxed{=2}$

Additional formula for sin
$\sin(x+h) = \sin x \cos h + \cos x \sin h$

1. $\frac{d}{dx}[\sin x] = \lim_{h\to0} \frac{\sin(x+h) - \sin x}{h}$

2. $= \lim_{h\to0} \frac{[\sin x \cos h + \cos x \sin h] - \sin x}{h}$

3. $= \lim_{h\to0} \left[\frac{\sin x \cos h - \sin x}{h} + \frac{\cos x \sin h}{h}\right]$

4. $= \lim_{h\to0} \frac{\sin x(\cos h - 1)}{h} + \lim_{h\to0} \frac{\cos x \sin h}{h}$

5. $= \sin x \underbrace{\lim_{h\to0} \frac{\cos h - 1}{h}}_{0} + \cos x \underbrace{\lim_{h\to0} \frac{\sin h}{h}}_{0}$

6. $= (\sin x) + (\cos x)(1) = \cos x$

$\boxed{\frac{d}{dx}(\sin x) = \cos x}$

$\frac{d}{dx}[\cos x] = -\sin x$

Find $\frac{d}{dx}[\tan x]$

use quotient rule

$\frac{d}{dx}[\tan x] = \frac{d}{dx}\left[\frac{\sin x}{\cos x}\right] = \frac{(\cos x)(\sin x)' - (\sin x)(\cos x)'}{\cos^2 x} = \frac{\cos^2 x - (\sin x)(-\sin x)}{\cos^2 x}$

$= \frac{\cos^2 x + \sin x^2 x}{\cos^2 x}$ $\frac{1}{\cos^2 x} = \sec^2 x$ $\frac{d}{dx}$

$\lim_{x \to \infty} \frac{9x-3}{4x+1}$

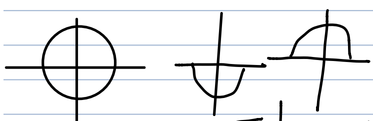
Clicker find derivative for $f(t)=\sin(e^{2t})+e^{\sin t} = 2e^{u}\cos t$] 2/13

3.5 Implicit Differentiation

$y=\sqrt{x^3+1}$ or $y=x\sin x$ in general $y=f(x)$

Ex: $x^2+y^2=25$ or $x^3+y^3=6xy$

$y=\pm\sqrt{25-x^2}$



$x^2+y^2=25$ $-\sqrt{25-x^2}$ | $\sqrt{25-x^2}$

$\frac{d}{dx}$ operator $\frac{dy}{dx}$ | y'

Two methods for finding y' for implicitly defined functions

1) Ex: Find $y' \frac{dy}{dx}$ $xy+1=x$

- Solve for y: $xy+1=x$
- $xy=x-1$
- differentiate directly: $y=\frac{x-1}{x}$

$y'=\frac{x(1)-(x-1)(1)}{x^2}=\frac{1}{x^2}$

note: this method is not always possible to use

Ex: $y=\cos(xy)$

2) Implicit Differentiation

$xy+1=x$

I) differentiate both sides of equations WRT x.

$\frac{d}{dx}(xy+1)=\frac{d}{dx}(x)$] * y represents a function of x *

II) use the product rule to differentiate xy

$\left[x\left(\frac{d}{dx}y\right)+y\left(\frac{d}{dx}x\right)\right]+0=1$

$x\frac{dy}{dx}+y(1)=1$ III Solve the equation $\frac{dy}{dx}$

$x\frac{dy}{dx}=1-y=$ $\boxed{\frac{dy}{dx}=\frac{1-y}{x}}$ — leave answer in terms of x and y

note: our answers above are equivalent $y=\frac{x-1}{x}$ | $\frac{dy}{dx}=\frac{1-y}{x}$

$\frac{dy}{dx}=\frac{1-y}{x}=\frac{1-\frac{(x-1)}{x}}{x}=\left(\frac{1-\left(\frac{x-1}{x}\right)}{x}\right)\left(\frac{x}{x}\right)$

$=\frac{x-(x-1)}{x^2}=\frac{1}{x^2}$

Another View:

let $\square$ hold a function of x

- find general formula $\frac{d}{dx}(\square^3)$
  - use the chain rule $\frac{d}{dx}(\square^3)=3\square^2\cdot\square'$ derivative
    - Replace $\square$ with y

so,

$\frac{d}{dx}(y^3)=3y^2\cdot y'$ $\frac{d}{dx}y=\frac{dy}{dx}=y'$ ← function

Ex: use implicit diff to find y' $y^3+\cos y=3x^2$

$\frac{d}{dx}(y^3+\cos y)=\frac{d}{dx}(3x^2)$

$3y^2\frac{dy}{dx}+(-\sin y)\left[\frac{d}{dx}(y)\right]=6x$

$3y^2\frac{dy}{dx}-\frac{dy}{dx}\sin y=6x$

$\frac{dy}{dx}=\frac{6x}{3y^2-\sin y}$

Ex: Find y'' by implicit diff. $x^4+y^4=16$

First, find y'

$\frac{d}{dx}(x^4+y^4)=\frac{d}{dx}(16)$

$4x^3+4y^3\frac{dy}{dx}=0$

$\frac{dy}{dx}=\frac{-4x^3}{4y^3}=-\frac{x^3}{y^3}=y'/\frac{dy}{dx}$

Differentiate y' to find y''

$y''=\frac{d}{dx}(y')=\frac{d}{dx}\left(\frac{-x^3}{y^3}\right)=-\left(\frac{y^3(3x^2)-x^3(3y^2)y'}{(y^3)^2}\right)$

$=-\left(\frac{3x^2y^3-3x^3y^2y'}{y^6}\right)$ $y'=\frac{-x^3}{y^3}$ $=-\left(\frac{3x^2y^3-3x^3y^2\left(\frac{-x^3}{y^3}\right)}{y^6}\right)$

$=-\left(\frac{3x^2y^3+\frac{3x^6}{y}}{y^6}\right)\left(\frac{y}{y}\right)=-\left(\frac{3x^2y^4+3x^6}{y^7}\right)=-\left(\frac{3x^2(y^4+x^4)}{y^7}\right)$

$=-\left(\frac{3x^2\overbrace{(y^4+x^4)}^{16}}{y^7}\right)=\boxed{\frac{-48x^2}{y^7}}$

Clicker question: The slope of the tangent line to the graph of $x = \sin y$ at the point $(0, \pi)$

Slope of tangent line comes from the derivative

$y = x - \sin$

$\pi = -\sin$

$\frac{d}{dx}[x] = \frac{d}{dx}[\sin y]$

$1 = y' \cos y \qquad y' = \frac{1}{\cos y}$

$y'\Big|_{(0,\pi)} = \frac{1}{\cos \pi} = \frac{1}{-1} = -1$

Clicker question: $y = 2015 \sec \theta \tan \theta$

$y' = 2015(\sec \theta (\sec^2 \theta + \tan^2 \theta)$

Derivatives of Inverse Trigonometric Functions:

Find derivative WRT x $\quad \frac{d}{dx}[\arcsin x]$

$y = \arcsin x$ means $x = \sin y$

outside func

inside func

$\frac{d}{dx}(x) = \frac{d}{dx}(\sin y) = \quad 1 = (\cos y)\frac{dy}{dx}$

$\frac{dy}{dx} = \frac{1}{\cos y}$ since $y = \arcsin x$, $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$ (range of $y = \arcsin x$)

For this int $\cos y \geq 0$

$\cos^2 y + \sin^2 y = 1 \qquad \cos^2 y = 1 - \sin^2 y$

$\cos y = \sqrt{1 - \sin^2 y}$ ignore your negative root b/c $\cos y \geq 0$

$\cos y = \sqrt{1 - x^2}$

This means $\frac{d}{dx}[\arcsin x] = \frac{1}{\cos y} = \boxed{\frac{1}{\sqrt{1-x}}}$ know these formulas

$\frac{d}{dx}[\arctan x] = \frac{1}{1+x^2}$

$\frac{d}{dx}[\arccos x] = \frac{-1}{\sqrt{1-x^2}}$

$\frac{d}{dx}[\operatorname{arcsec} x] = \frac{1}{x\sqrt{x^2-1}}$

$\frac{d}{dx}[\operatorname{arccsc} x] = \frac{-1}{x\sqrt{x^2-1}}$

$\frac{d}{dx}[\operatorname{arccot} x] = \frac{-1}{1+x^2}$

Ex: diff $\quad y = \arctan x^2$

chain rule $\quad y' = \left[\frac{1}{1+(x^2)^2}\right] 2x = \frac{2x}{1+x^4}$

$y = \frac{2}{\ln x} = -\frac{2}{x(\ln x)^2}$

$(\ln x)^{-1} - y' = -2(\ln x)^{-2} \cdot (\ln x)' = -\frac{2}{x(\ln x)^2}$

If $e^{\frac{1}{2}}$ is approximated by using the tangent line to the graph of $f(x) = e^x$
at $(0, 1)$ we know that $f'(0) = 1$ approximate value of $e^{\frac{1}{2}}$ is