Appendix: Mathematical proof for the independence of path α and β

Consider a simple mediation model where the relation between independent variable *X* and dependent variable *Y* is mediated by *M*. This relation is presented by the following set of linear regression equations:

$$Y_i = b_{0(1)} + tX_i + bM_i + e_{(1),}$$

$$M_i = b_{0(2)} + aX_i + e_{(2),}$$

where subscript *i* identifies the participant, *t* represents the relation between X and Y after adjusting for the effects of the mediator M, *a* represents the relation between X and M, and *b* represents the relation between M and Y. Furthermore, $b_{0(1)}$ and $b_{0(2)}$ are the intercepts, and $e_{(1)}$, $e_{(2)}$, and $e_{(3)}$ are assumed to be conditionally normally distributed, independent, homoscedastic residuals. See also Figure 1 for a graphical representation.

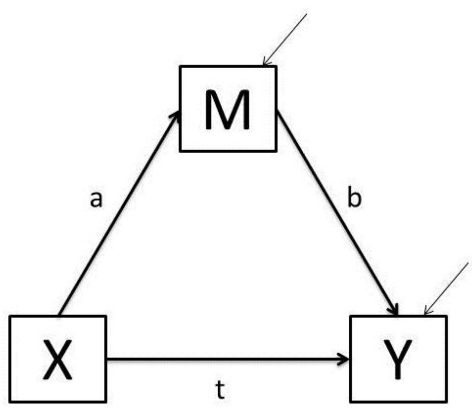

Figure 1. Diagram of the standard mediation model. The figure shows partial mediation. Diagonal arrows indicate that the graphical node is perturbed by an error term.

This Appendix provides the mathematical proof that path *a* and *b* in this simple mediation model are independent. This can be proven by taking the second derivative of the loglikelihood function with respect to *a*, *b*, and *t*, which renders the information matrix, or the Hessian. The inverse of the Hessian can be considered the covariance matrix of to *a*, *b*, and *t*.

The loglikelihood ratio function is given by

$$\mathrm{f(q)} = \mathrm{N} - 1\left(\log\left(\det(\mathrm{S})\right) + \mathrm{trace}(\mathrm{S}^{-1}\,\mathrm{So}) - \log(\mathrm{So}) - \mathrm{p}\right),$$

where *N* is the sample size, *S* the expected covariance matrix, So the observed covariance matrix, and *p* is the number of variables. Since *N* and *p* are constant, and *So* is observed, the relevant part is of the loglikelihood function is:

$$\log(\det(\mathrm{S})) + \ \mathrm{trace}(\mathrm{S}^{-1}\ \mathrm{So}).$$

We first calculate the expected covariance matrix of X, M, Y: matrix *S*.

Matrix P contains the paths *a*, *b*, and *t*, and is given by:

$$P = \begin{bmatrix} 0 & 0 & 0 \\ a & 0 & 0 \\ t & b & 0 \end{bmatrix}$$

Next, consider that $\mathrm{B} \ = \ (\mathrm{I} - \mathrm{P})^{-1}$, where *I* is a 3x3 identity matrix, then matrix B is given by:

$$B := \begin{bmatrix} 1 & 0 & 0 \\ -a & 1 & 0 \\ -t & -b & 1 \end{bmatrix}$$

Matrix Y contains the variances of X, M, and Y, and is given by:

$$Y := \begin{bmatrix} vx & 0 & 0 \\ 0 & vm & 0 \\ 0 & 0 & vy \end{bmatrix}$$

Take the inverse of B, matrix *iB*:

$$iB := \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ a\,b + t & b & 1 \end{bmatrix}$$

The expected covariance matrix *S* can be obtained by multiplying iB* Y* t(iB):

$$S := \begin{bmatrix} vx & a\,vx & (a\,b+t)\,vx \\ a\,vx & a^2\,vx + vm & a\,vx\,(a\,b+t) + b\,vm \\ (a\,b+t)\,vx & a\,vx\,(a\,b+t) + b\,vm & (a\,b+t)^2\,vx + b^2\,vm + vy \end{bmatrix}$$

Now consider the first relevant part of the loglikelihood function, $\log(\det(\mathrm{S}))$. Note that

$$\begin{aligned} &\det(\mathrm{S}) \\ &= \det(iB * Y * t(iB)) \\ &= \det(iB) * \det(Y) * \det(t(iB)). \end{aligned}$$

Furthermore,

$$\begin{aligned} &\det(iB) = 1, \\ &\det(Y) = 1, \\ &\det(t(iB)) = 1, \end{aligned}$$

so $\log(\det(S)) = 0$, which leaves us with the second relevant part of the loglikelihood function: $\mathrm{trace}(S^{-1}\,So)$.

Take the inverse of S, matrix *Sinv*:

$$Sinv := \begin{bmatrix} \frac{a^2\,vx\,vy + vm\,vx\,t^2 + vm\,vy}{vx\,vm\,vy} & -\frac{a\,vy - t\,b\,vm}{vm\,vy} & -\frac{t}{vy} \\ -\frac{a\,vy - t\,b\,vm}{vm\,vy} & \frac{b^2\,vm + vy}{vm\,vy} & -\frac{b}{vy} \\ -\frac{t}{vy} & -\frac{b}{vy} & \frac{1}{vy} \end{bmatrix}$$

Consider the following observed covariance matrix *So*:

$$So := \begin{bmatrix} 1 & .4 & .08 \\ .4 & 1 & .2 \\ .08 & .2 & 1 \end{bmatrix}$$

Next we multiply *Sinv* by *So*, which gives us matrix p1:

$$p1 :=$$

$$\left[\frac{a^2\,vx\,vy + vm\,vx\,t^2 + vm\,vy}{vx\,vm\,vy} - \frac{.4\,(a\,vy - t\,b\,vm)}{vm\,vy} - \frac{.08\,t}{vy},\right.$$
$$.4\,\frac{a^2\,vx\,vy + vm\,vx\,t^2 + vm\,vy}{vx\,vm\,vy} - \frac{a\,vy - t\,b\,vm}{vm\,vy} - \frac{.2\,t}{vy},$$
$$\left..08\,\frac{a^2\,vx\,vy + vm\,vx\,t^2 + vm\,vy}{vx\,vm\,vy} - \frac{.2\,(a\,vy - t\,b\,vm)}{vm\,vy} - \frac{t}{vy}\right]$$
$$\left[-\frac{a\,vy - t\,b\,vm}{vm\,vy} + \frac{.4\,(b^2\,vm + vy)}{vm\,vy} - \frac{.08\,b}{vy}, -.4\,\frac{a\,vy - t\,b\,vm}{vm\,vy} + \frac{b^2\,vm + vy}{vm\,vy} - \frac{.2\,b}{vy},\right.$$
$$\left.-.08\,\frac{a\,vy - t\,b\,vm}{vm\,vy} + \frac{.2\,(b^2\,vm + vy)}{vm\,vy} - \frac{b}{vy}\right]$$
$$\left[-\frac{t}{vy} - \frac{.4\,b}{vy} + \frac{.08}{vy}, -.4\,\frac{t}{vy} - \frac{b}{vy} + \frac{.2}{vy}, -.08\,\frac{t}{vy} - \frac{.2\,b}{vy} + \frac{1}{vy}\right]$$

We calculate $f1 = \mathrm{trace}(S^{-1}\,So)$:

$$f1 := \frac{a^2\,vx\,vy + vm\,vx\,t^2 + vm\,vy}{vx\,vm\,vy} - \frac{.8\,(a\,vy - t\,b\,vm)}{vm\,vy} - \frac{.16\,t}{vy} + \frac{b^2\,vm + vy}{vm\,vy} - \frac{.4\,b}{vy} + \frac{1}{vy}$$

Take the second order derivatives.

$$daa := 2 \frac{1}{vm}$$

$$dbb := 2 \frac{1}{vy}$$

$$dtt := 2 \frac{1}{vy}$$

$$dab := 0$$

$$dat := 0$$

$$dbt := .8 \frac{1}{vy}$$

The Hessian is then given by:

$$H := \begin{bmatrix} 2 \frac{1}{vm} & 0 & 0 \\ 0 & 2 \frac{1}{vy} & .8 \frac{1}{vy} \\ 0 & .8 \frac{1}{vy} & 2 \frac{1}{vy} \end{bmatrix}$$

The inverse of the Hessian can be seen as a covariance matrix of the parameters.

$$\begin{bmatrix} .5000000000\ vm & -0. & 0. \\ -0. & .5952380952\ vy & -.2380952381\ vy \\ 0. & -.2380952381\ vy & .5952380952\ vy \end{bmatrix}$$

From these covariances we can conclude that parameter *b* and *t* are correlated, and the parameters *a* and *t* and *a* and *b* are uncorrelated.