Tutorial of ST5215

AY2020/2021 Semester 1

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Exercise 1 Suppose f and g are independent and identically distributed. Show that

$$E(f \mid f+g) = (f+g)/2, \text{a.s.} \tag{1}$$

Exercise 2 Suppose $F(x)$ is a continuous CDF of P, where P is a probability measure on $(\mathcal{R}, \mathcal{B})$. Show that $\int F(x)\ \mathrm{d}P(x) = 1/2$

Exercise 3 Suppose ν is a σ-finite measure on $(\Omega, \mathcal{F})$, f is a nonnegative measurable function and $\alpha > 0$. Show that

$$\int f^\alpha\ \mathrm{d}\nu = \alpha \int_0^\infty t^{\alpha-1}\nu(f > t)\ \mathrm{d}t \tag{2}$$

Exercise 4 Suppose ν and ϕ are finite measures on $(\Omega, \mathcal{F})$. Show that there exist two measures ϕ_c and ϕ_s such that

(a) $\phi = \phi_c + \phi_s$,

(b) $\phi_c \ll \nu$, and

(c) there exists $N \in \mathcal{F}$ such that $\phi_s(N) = \nu(N^c) = 0$. (Note that in this case, we denote $\phi_s \perp \nu$ and say that ϕ_s and ν are singular with each other.)

(Hint: make use of $\frac{\mathrm{d}\phi}{\mathrm{d}(\phi+\nu)}$ by Radon-Nikodym theorem)

Remark. This result can be generalized to σ-finite measures, and is known as *Lebesgue decomposition.*