Tutorial of ST5215

AY2020/2021 Semester 1

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Exercise 1 Suppose that $\{f_n\}$ is a sequence of Borel functions on a measure space $(\Omega, \mathcal{F}, \nu)$ and $f_n(\omega) \to f(\omega)$ for $\omega \in \Omega$ with $\nu(\Omega) < \infty$. For any $\epsilon > 0$, define $A_k = \{x \in \Omega : |f_k(x) - f(x)| \geq \epsilon\}$, show that

$$\lim_{j \to \infty} m(\cup_{k=j}^{\infty} A_k) = 0. \tag{1}$$

Exercise 2 Prove the Egoroff's theorem:
Suppose that $\{f_n\}$ is a sequence of Borel functions on a measure space $(\Omega, \mathcal{F}, \nu)$, $f_n(\omega) \to f(\omega)$ ν-a.e., and $\nu(\Omega) < \infty$. Show that for any $\epsilon > 0$, there is a $B \in \mathcal{F}$ with $\nu(B) < \epsilon$ such that $f_n(\omega) \to f(\omega)$ uniformly on B^c, i.e., for any small $\eta > 0$, one can find an $N_\eta \in \mathbb{N}$ such that $|f_n(\omega) - f(\omega)| < \eta$, $\forall n \geq N_\eta$ and $\forall \omega \in B^c$.

Exercise 3 Prove the monotone convergence theorem:
If $0 \leq f_1 \leq \cdots$ and $\lim_n f_n = f$ a.e., then

$$\int \lim_n f_n \, \mathrm{d}\nu = \lim_n \int f_n \, \mathrm{d}\nu. \tag{2}$$