Midterm Exam, ST5215

AY2020/2021 Semester 1

1 Oct 2020 From 14:00 to 16:00

Instruction

- Write down your name and student ID clearly on the header of Page 1 of your solution papers.
- For each solution, write down its problem number. If a solution spread over multiple pages, also indicates the order of the pages.
- If you need to communicate with the supervisor, raise a hand first.
- At the end of the exam, stop writing. Take photos of your exam papers by a scanner App; make sure the photos are clear enough. Export to a PDF file with name [Your name displayed on Zoom]+[Your ID].pdf, using + for space
  For example: Zhang+San+A10101010.pdf
  Upload to LumiNUS at File/Submissions/Midterm, and confirm your submission via private Zoom chat with the supervisor.
- There are 5 problems in this exam. Spend your time wisely.

Problems

Problem 1. (20 pts) Let $(\Omega, \mathcal{F}, P)$ be a probability space. Suppose $A_n \in \mathcal{F}$ $(n = 1, 2, \ldots)$ are disjoint events such that $\Omega = \cup_n A_n$. Let $\mathcal{G} = \sigma(\{A_n : n = 1, 2, \ldots\})$. Suppose $B \in \mathcal{F}$ and $X = I_B$.

(a) (10 pts) Explain the definition of $E(X \mid \mathcal{G})$ and why it exists.

(b) (10 pts) Compute $E(X \mid \mathcal{G})$.

Problem 2. (15 pts) Let $(\Omega, \mathcal{F}, P)$ be a probability space, $\mathcal{G}$ be a sub-σ-field of $\mathcal{F}$. Suppose X is a r.v. such that $E(X^2) < \infty$. Suppose g is a $\mathcal{G}$-measurable function with finite second moment.
Show that

$$E[(X - g)^2] \geq E[(X - E[X \mid \mathcal{G}])^2].$$

Problem 3. (25 pts) Suppose $X = (X_1, X_2, \ldots, X_n)$ where X_i's are i.i.d. observations from some unknown population P_θ in the family $\mathcal{P}$ with Lebesgue density functions

$$f_\theta(x) = \exp\left[-(x-\theta)^4\right]/c_\theta, \quad x \in \mathcal{R}, \tag{1}$$

where θ lies in $\Theta = \mathcal{R}$ and $c_\theta = \int_{\mathcal{R}} \exp\left(-(x-\theta)^4\right)\, \mathrm{d}x$.

(a) (10 pts) Show that $T(X) = (\sum_i X_i^3, \sum_i X_i^2, \sum_i X_i)$ is sufficient for θ.

(b) (10 pts) Show that $T(X)$ is minimal sufficient.

(c) (5 pts) Suppose $n > 10$. Without knowing the true value of θ, propose a way to generate some fake data $\tilde{X} = (\tilde{X}_1, \ldots, \tilde{X}_n)$, such that $\tilde{X} \neq X$ almost surely and the distribution of $\tilde{X}$ is the same as the distribution of X. Briefly explain the steps. You can ignore the details of sampling from any fixed distribution.

Problem 4. (25 pts) Suppose $\{X_i\}_{i=1}^n$ are i.i.d. observations from Unif(a, b) where $a < b$ are unknown real numbers.

(a) (5 pts) Show that $T(X) = (X_{(1)}, X_{(n)})$ is sufficient for $\theta = (a, b)$.

(b) (10 pts) Show that $T(X)$ is complete for θ by using the following results (statements):
(R1). Let f and F be the p.d.f. and c.d.f. of X_1. The joint p.d.f. of $(X_{(1)}, X_{(n)})$ is

$$f_{X_{(1)},X_{(n)}}(x, y) = n(n-1)\left[F(y) - F(x)\right]^{n-2} f(x)f(y), \text{ where } x \leq y.$$

(R2). If $f(x, y)$ is a Borel function on $\mathcal{R}^2$ and is integrable on $[c, d]^2$ then

$$F(x, y) = \int_x^y \int_x^t f(s, t)\, \mathrm{d}s\, \mathrm{d}t,\ \forall x < y,\ x, y \in (c, d)$$

is almost everywhere differentiable and $\frac{\partial^2}{\partial x \partial y} F(x, y) = -f(x, y)$, a.e. $x < y,\ x, y \in (c, d)$

(c) (5 pts) Define $S(X) = (\sum_{i=1}^n X_i - nX_{(1)})/(X_2 - X_1)$. Prove that $X_{(1)} + X_{(n)}$ and $S(X)$ are independent.

(d) (5 pts) Find a UMVUE of $g(\theta) := a + b$.

Problem 5. (15 pts) Suppose $\{X_i\}_{i=1}^n$ are i.i.d. observations from P_θ, which is an unknown population in $\mathcal{P}_\Theta$ with Lebesgue density

$$f_\theta(x) = \theta^{-1} \exp\left(-x/\theta\right), \quad x > 0, \tag{2}$$

where $\theta \in \Theta = (0, \infty)$. Note that for f_θ, the mean is θ and the variance is θ^2.

(a) (10 pts) Find the maximum likelihood estimator $\hat{\theta}_1$. Is it the same as the method of moment estimator?

(b) (5 pts) Consider the squared error loss. Let $\mathfrak{J}$ be the class of estimators $\hat{\theta}_c := c \cdot \hat{\theta}_1$ for all $c \in (0, 1]$. Prove or disprove: $\hat{\theta}_1$ is $\mathfrak{J}$-admissible.