Tutorial of ST5215

AY2020/2021 Semester 1

22 Oct 2020

Exercise 1. Let $X_1, X_2, \ldots$ be random variables. Show that $\{|X_n|\}$ is uniformly integrable if one of the following condition holds:

(i) $\sup_n E\,|X_n|^{1+\delta} < \infty$ for a $\delta > 0$

(ii) $P\,(|X_n| \geq c) \leq P(|X| \geq c)$ for all n and $c > 0$, where X is an integrable random variable.

Exercise 2. Let $X_1, \ldots, X_n, \ldots$ be i.i.d. observations. Suppose that $T_n = T(X_{1:n})$ is an unbiased estimator of ϑ based on $X_1, \ldots, X_n$ such that for any n,

(a) $\mathrm{Var}\,(T_n) < \infty$,

(b) $\mathrm{Var}\,(T_n) \leq \mathrm{Var}\,(U_n)$ for any other unbiased estimator U_n of ϑ based on $X_1, \ldots, X_n$.

Then T_n is consistent in mse.

Hint. Here $X_{1:n}$ is defined as $(X_1, \ldots, X_n)$. Consider $T(X_{1:n})$ and $T(X_{(n+1):(2n)})$.

Exercise 3. Let $(X_1, \ldots, X_n)$ be a random sample of random variables from a population P with $EX_1^2 < \infty$ and $\bar{X}$ be the sample mean. Consider the estimation of $\mu = EX_1$.

(i) Let $T_n = \bar{X} + \xi_n$, where ξ_n is a random variable satisfying $\xi_n = 0$ with probability $1 - n^{-1}$ and $\xi_n = n$ with probability n^{-1}. Show that the bias of T_n is not the same as the asymptotic bias of T_n for any P.

(ii) Let $T_n = \bar{X} + \eta_n$, where η_n is a random variable that is independent of $X_1, \ldots, X_n$ and equals 0 with probability $1 - 2n^{-1}$ and ± 1 with probability n^{-1}. Show that the asymptotic mean squared error of T_n, the asymptotic mean squared error of $\bar{X}$, and the mean squared error of $\bar{X}$ are the same, but the mean squared error of T_n is larger than the mean squared error of $\bar{X}$ for any P.