Tutorial of ST5215

AY2020/2021 Semester 1

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Exercise 1 (Stein's identity). Suppose the distribution of X has density in an exponential family whose support is $(-\infty, \infty)$. If g is any differentiable function such that $E\left|g'(X)\right| < \infty$, then

$$E\left\{\left[\frac{h'(X)}{h(X)} + \sum_{i=1}^{p} \eta_i T_i'(X)\right] g(X)\right\} = -Eg'(X),$$

where η_i's are the coordinates of $\eta(\theta)$ and T_i's are the coordinates of $T(X)$.

When $p = 1$ and $h'(X) = 0$ (e.g., the normal family with fixed σ), the identity becomes

$$E\left\{(X - \mu)g(X)\right\} = \sigma^2 Eg'(X),$$

Exercise 2 Show that if two random variables X and Y are independent, then their characteristic functions ϕ_X and ϕ_Y satisfy $\phi_X(t)\phi_Y(t) = \phi_{X+Y}(t)$ for all $t \in \mathcal{R}$.

Exercise 3 Find an example of two random variables X and Y such that X and Y are not independent but their characteristic functions ϕ_X and ϕ_Y satisfy $\phi_X(t)\phi_Y(t) = \phi_{X+Y}(t)$ for all $t \in \mathcal{R}$

Exercise 4 Let X be an integrable random variable on the probability space $(\Omega, \mathcal{F}, P)$, $\mathcal{A}$ and $\mathcal{A}_0$ be σ-fields satisfying $\mathcal{A}_0 \subset \mathcal{A} \subset \mathcal{F}$. Show that $E\left[E(X \mid \mathcal{A}) \mid \mathcal{A}_0\right] = E\left(X \mid \mathcal{A}_0\right) = E\left[E\left(X \mid \mathcal{A}_0\right) \mid \mathcal{A}\right]$ a.s.

Exercise 5 Let X be an integrable random variable on the probability space $(\Omega, \mathcal{F}, P), \mathcal{A}$ be a sub-σ-field of $\mathcal{F}$, and Y be another random variable satisfying $\sigma(Y) \subset \mathcal{A}$ and $E|XY| < \infty$. Show that

$$E(XY \mid \mathcal{A}) = YE(X \mid \mathcal{A}) \text{ a.s.}$$