Tutorial of ST5215

AY2020/2021 Semester 1

10 Nov 2020

Exercise 1. Let $(X_1, \ldots, X_n)$ be a random sample from the exponential distribution on (a, ∞) with scale parameter θ, where $a \in \mathcal{R}$ and $\theta > 0$ are unknown. Show that $T = (X_{(1)}, \sum_{i=1} X_i - nX_{(1)})$ is a complete statistic.
Hint: Use the Rényi representation (see the reading material "Representation_BlitzsteinMorris.pdf" on LumiNUS).

Exercise 2. Consider a linear model in matrix form $X_{n\times 1} = Z_{n\times p}\boldsymbol{\beta}_{p\times 1} + \epsilon_{n\times 1}$ with $p \leq n$ and with the assumption that $\epsilon \sim N(\mathbf{0}_n, \sigma^2 \boldsymbol{I}_n)$. Show that if each coordinate of $\boldsymbol{\beta}$ is estimable, then the rank of Z is p.

Exercise 3. (James-Stein estimator) Suppose X is a p-random vector from $N(\theta, \boldsymbol{I}_p)$ with an unknown $\theta \in \mathcal{R}^p$. Consider the squared loss function for estimating θ:

$$L(\theta, a) = \|a - \theta\|^2 = \sum_{i=1}^{p} (a_i - \theta_i)^2 ,$$

where a_i and θ_i are the ith coordinates of the estimator and the estimand. Show that for any $p \geq 3$, the risk of the following estimator

$$\hat{\theta} = \left(1 - \frac{(p-2)}{\|X\|^2}\right) X$$

is strictly smaller than X.

Can you extend this result to the case where $X \sim N(\theta, D)$ with some known $p \times p$ positive definite matrix D?

Hint: Use the Stein's identity (Ex 1 in Tutorial 6).