Tutorial of ST5215

AY2020/2021 Semester 1

13 Aug 2020

Exercise 1 Let $\mathcal{C}$ be a collection of subsets of Ω and $\Gamma = \{\mathcal{F} : \mathcal{F}$ is a σ-field on Ω and $\mathcal{C} \subset \mathcal{F}\}$.
Show that $\Gamma \neq \emptyset$ and $\sigma(\mathcal{C}) = \bigcap_{\mathcal{F}\in\Gamma} \mathcal{F}$.

Exercise 2 Let $(\Omega, \mathcal{F})$ be a measurable space. $f : \Omega \mapsto \mathcal{R}$ is Borel if and only if $f^{-1}(a, \infty) \in \mathcal{F}$ for all $a \in \mathcal{R}$.

Exercise 3 Show that a monotone function from $\mathcal{R}$ to $\mathcal{R}$ is Borel

Exercise 4 Let f be a Borel function on $\mathcal{R}^2$. Define a function g from $\mathcal{R}$ to $\mathcal{R}$ as $g(x) = f(x, y_0)$, where y_0 is a fixed point in $\mathcal{R}$. Show that g is Borel.

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18 Aug 2020

Exercise 1 Suppose that $\{f_n\}$ is a sequence of Borel functions on a measure space $(\Omega, \mathcal{F}, \nu)$ and $f_n(\omega) \to f(\omega)$ for $\omega \in \Omega$ with $\nu(\Omega) < \infty$. For any $\epsilon > 0$, define $A_k = \{x \in \Omega : |f_k(x) - f(x)| \geq \epsilon\}$, show that

$$\lim_{j \to \infty} m(\cup_{k=j}^{\infty} A_k) = 0. \tag{1}$$

Exercise 2 Prove the Egoroff's theorem:
Suppose that $\{f_n\}$ is a sequence of Borel functions on a measure space $(\Omega, \mathcal{F}, \nu)$, $f_n(\omega) \to f(\omega)$ ν-a.e., and $\nu(\Omega) < \infty$. Show that for any $\epsilon > 0$, there is a $B \in \mathcal{F}$ with $\nu(B) < \epsilon$ such that $f_n(\omega) \to f(\omega)$ uniformly on B^c, i.e., for any small $\eta > 0$, one can find an $N_\eta \in \mathbb{N}$ such that $|f_n(\omega) - f(\omega)| < \eta$, $\forall n \geq N_\eta$ and $\forall \omega \in B^c$.

Exercise 3 Prove the monotone convergence theorem:
If $0 \leq f_1 \leq \cdots$ and $\lim_n f_n = f$ a.e., then

$$\int \lim_n f_n \, \mathrm{d}\nu = \lim_n \int f_n \, \mathrm{d}\nu. \tag{2}$$

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20 Aug 2020

Exercise 1 (Generalization of Hölder's inequality).
For $0 < p < 1$ and $q = -p/(1-p)$

$$E|XY| \geqslant (E|X|^p)^{1/p} (E|Y|^q)^{1/q}$$

Exercise 2 (Generalization of Minkowski's inequality).
$\left(E\left(\sum_{j=1}^n |X_j|\right)^r\right)^{1/r} > \sum_{j=1}^n \left(E|X_j|^r\right)^{1/r} \quad$ for $0 < r < 1$

Exercise 3 Let Y be measurable from $(\Omega, \mathcal{F})$ to $(\Lambda, \mathcal{G})$ and Z a function from $(\Omega, \mathcal{F})$ to $\mathcal{R}^k$. If Z is Borel on $(\Omega, \sigma(Y))$, then there is a Borel function h from $(\Lambda, \mathcal{G})$ such that $Z = h \circ Y$

Exercise 4 Let ϕ_X be a ch.f. of X. Show that $|\phi_X| \leq 1$, and uniformly continuous.

Exercise 5 Find the ch.f. and m.g.f. for the Cauchy distribution (i.e., P_X has p.d.f. $f(x) = (\pi(1+x^2))^{-1}$

Exercise 6 If X_i has the Cauchy distribution $C(0,1), i = 1, \ldots, k$, then Y/k has the same distribution as X_1.

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25 Aug 2020

Exercise 1 Suppose f and g are independent and identically distributed. Show that

$$E(f \mid f+g) = (f+g)/2, \text{a.s.} \tag{1}$$

Exercise 2 Suppose $F(x)$ is a continuous CDF of P, where P is a probability measure on $(\mathcal{R}, \mathcal{B})$. Show that $\int F(x)\ \mathrm{d}P(x) = 1/2$

Exercise 3 Suppose ν is a σ-finite measure on $(\Omega, \mathcal{F})$, f is a nonnegative measurable function and $\alpha > 0$. Show that

$$\int f^\alpha\ \mathrm{d}\nu = \alpha \int_0^\infty t^{\alpha-1}\nu(f > t)\ \mathrm{d}t \tag{2}$$

Exercise 4 Suppose ν and ϕ are finite measures on $(\Omega, \mathcal{F})$. Show that there exist two measures ϕ_c and ϕ_s such that

(a) $\phi = \phi_c + \phi_s$,

(b) $\phi_c \ll \nu$, and

(c) there exists $N \in \mathcal{F}$ such that $\phi_s(N) = \nu(N^c) = 0$. (Note that in this case, we denote $\phi_s \perp \nu$ and say that ϕ_s and ν are singular with each other.)

(Hint: make use of $\frac{\mathrm{d}\phi}{\mathrm{d}(\phi+\nu)}$ by Radon-Nikodym theorem)

Remark. This result can be generalized to σ-finite measures, and is known as *Lebesgue decomposition.*

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27 Aug 2020

Exercise 1 (Stein's identity). Suppose the distribution of X has density in an exponential family whose support is $(-\infty, \infty)$. If g is any differentiable function such that $E\left|g'(X)\right| < \infty$, then

$$E\left\{\left[\frac{h'(X)}{h(X)} + \sum_{i=1}^{p} \eta_i T_i'(X)\right] g(X)\right\} = -Eg'(X),$$

where η_i's are the coordinates of $\eta(\theta)$ and T_i's are the coordinates of $T(X)$.

When $p = 1$ and $h'(X) = 0$ (e.g., the normal family with fixed σ), the identity becomes

$$E\left\{(X - \mu)g(X)\right\} = \sigma^2 Eg'(X),$$

Exercise 2 Show that if two random variables X and Y are independent, then their characteristic functions ϕ_X and ϕ_Y satisfy $\phi_X(t)\phi_Y(t) = \phi_{X+Y}(t)$ for all $t \in \mathcal{R}$.

Exercise 3 Find an example of two random variables X and Y such that X and Y are not independent but their characteristic functions ϕ_X and ϕ_Y satisfy $\phi_X(t)\phi_Y(t) = \phi_{X+Y}(t)$ for all $t \in \mathcal{R}$

Exercise 4 Let X be an integrable random variable on the probability space $(\Omega, \mathcal{F}, P)$, $\mathcal{A}$ and $\mathcal{A}_0$ be σ-fields satisfying $\mathcal{A}_0 \subset \mathcal{A} \subset \mathcal{F}$. Show that $E\left[E(X \mid \mathcal{A}) \mid \mathcal{A}_0\right] = E\left(X \mid \mathcal{A}_0\right) = E\left[E\left(X \mid \mathcal{A}_0\right) \mid \mathcal{A}\right]$ a.s.

Exercise 5 Let X be an integrable random variable on the probability space $(\Omega, \mathcal{F}, P), \mathcal{A}$ be a sub-σ-field of $\mathcal{F}$, and Y be another random variable satisfying $\sigma(Y) \subset \mathcal{A}$ and $E|XY| < \infty$. Show that

$$E(XY \mid \mathcal{A}) = YE(X \mid \mathcal{A}) \text{ a.s.}$$

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8 Sep 2020

Exercise 1 Suppose X has an exponential family distribution with density

$$p_\theta(x) = h(x)e^{\eta(\theta)T(x)-A(\theta)}$$

Derive the mean and variance formulas

$$\mathrm{E}_\theta[T(X)] = \frac{A'(\theta)}{\eta'(\theta)}, \quad \mathrm{V}_\theta[T(X)] = \frac{A''(\theta)}{[\eta'(\theta)]^2} - \frac{\eta''(\theta)A'(\theta)}{[\eta'(\theta)]^3}$$

Exercise 2 Let X and Y be two random variables such that Y has the binomial distribution $Bi(\pi, N)$ and, given $Y = y, X$ has the binomial distribution $Bi(p, y)$

(a) Suppose that $p \in (0, 1)$ and $\pi \in (0, 1)$ are unknown and N is known. Show that (X, Y) is minimal sufficient for (p, π).

(b) Suppose that π and N are known and $p \in (0, 1)$ is unknown. Show whether X is sufficient for p and whether Y is sufficient for p

Exercise 3 Let $X_1, \ldots, X_n$ be i.i.d. random variables having the Lebesgue p.d.f.

$$f_\theta(x) = \exp\left\{-\left(\frac{x-\mu}{\sigma}\right)^4 - \xi(\theta)\right\}$$

where $\theta = (\mu, \sigma) \in \Theta = \mathcal{R} \times (0, \infty)$.

Show that $\mathcal{P} = \{P_\theta : \theta \in \Theta\}$ is an exponential family, where P_θ is the joint distribution of $X_1, \ldots, X_n$ and that the statistic

$$T = \left(\sum_{i=1}^n X_i, \sum_{i=1}^n X_i^2, \sum_{i=1}^n X_i^3, \sum_{i=1}^n X_i^4\right)$$

is minimal sufficient for $\theta \in \Theta$.

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10 Sep 2020

Exercise 1 Let $X_1, \ldots, X_n$ be i.i.d. random variables having the Lebesgue p.d.f. $\theta^{-1}e^{-(x-\theta)/\theta}I_{(\theta,\infty)}(x)$, where $\theta > 0$ is an unknown parameter.
(a) Find a statistic that is minimal sufficient for θ.
(b) Show whether the minimal sufficient statistic in (a) is complete.

Exercise 2 Let $X_1, \ldots, X_n$ be i.i.d. from the $N\left(\theta, \theta^2\right)$ distribution, where $\theta > 0$ is a parameter. Find a minimal sufficient statistic for θ and show whether it is complete.

Exercise 3 Suppose that $(X_1, Y_1), \ldots, (X_n, Y_n)$ are i.i.d. random 2–vectors having the normal distribution with $EX_1 = EY_1 = 0, \operatorname{Var}(X_1) = \operatorname{Var}(Y_1) = 1$, and $\operatorname{Cov}(X_1, Y_1) = \theta \in (-1, 1)$
(a) Find a minimal sufficient statistic for θ.
(b) Show whether the minimal sufficient statistic in (a) is complete or not.
(c) Prove that $T_1 = \sum_{i=1}^n X_i^2$ and $T_2 = \sum_{i=1}^n Y_i^2$ are both ancillary but (T_1, T_2) is not ancillary.

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15 Sep 2020

Exercise 1 Let $X_1, \ldots, X_n$ be i.i.d. from a uniform distribution on $(-\theta, \theta)$, where $\theta > 0$ is an unknown parameter.
(a). Find a minimal sufficient statistic T.
(b). Define

$$V = \frac{\bar{X}}{\max_i X_i - \min_i X_i}$$

where $\bar{X}$ is the sample mean. Are T and V are independent?

Exercise 2 An object with weight θ is weighed on scales with different precision. The data $X_1, \ldots, X_n$ are independent, with $X_i \sim N\left(\theta, \sigma^2\right), i = 1, \ldots, n$ with the standard deviation σ known. Consider the absolute deviation loss $L(\theta, a) = |\theta - a|$.
(a). What is the risk of the naive estimator X_1?
(b). Use Rao-Blackwell theorem to find a better estimator.

Exercise 3 Consider an estimation problem with a parametric family $\mathcal{P} = \{P_\theta : \theta \in \Theta\}$ and the squared error loss. If $\theta_0 \in \Theta$ satisfies that $P_\theta \ll P_{\theta_0}$ for any $\theta \in \Theta$, show that the estimator $T \equiv \theta_0$ is admissible.

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17 Sep 2020

Exercise 1. Let $X_1, \ldots, X_n$ be i.i.d. random variables having the exponential distribution $E(0, \theta), \theta \in (0, \infty)$. Consider estimating θ under the squared error loss.
Calculate the risks of the sample mean $\bar{X}$ and $cX_{(1)}$, where c is a positive constant.
Is $\bar{X}$ better than $cX_{(1)}$ for some c?

Exercise 2. Consider the estimation of an unknown parameter $\theta \geq 0$ under the squared error loss. Show that if T and U are two estimators such that $T \leq U$ and $R_T(P) < R_U(P)$, then $R_{T_+}(P) < R_{U_+}(P)$, where f_+ denotes the positive part of f.

Exercise 3. Let X be a random variable having the binomial distribution $\mathrm{Bi}(p, n)$ with an unknown $p \in (0, 1)$ and a known n. Consider the problem of estimating $\theta = p^{-1}$. Show that there is no unbiased estimator of θ.