Problem 1 (35%)

A club rewards members with *prizes* for *achievements* using a model captured by the following type declarations:

```
type Prize       = string
type Achievement = int
type PrizeTable = (Prize * Achievement) list
```

The kinds of club, prizes and achievements are not important; we just need to be able to compare achievements and we assume that they are described by integers.

A *prize table* is a list of pairs: $[(p_1, a_1); (p_2, a_2); \ldots; (p_n, a_n)]$, where p_i is a prize and a_i is an achievement. For a prize table we require that achievements occur in ascending order, that is, $a_1 < a_2 < \cdots < a_n$. This requirement is called the *prize-table invariant* or just *invariant*. The following example prize table satisfies the invariant:

```
let pt = [("p1", 3); ("p2", 5);("p3", 8); ("p4", 11) ]
```

The questions 1. to 5. in this problem should be solved without using functions from the libraries `List`, `Seq`, `Set` and `Map`. That is, the requested functions should be declared using explicit recursion.

1. Declare a function `inv: PrizeTable -> bool` that can check whether the invariant holds for a given prize table.

From now on you can assume that prize-table arguments to functions satisfy the invariant; but you must ensure that prize-tables returned by functions satisfy the invariant.

2. Declare a function `prizesFor: Achievement -> PrizeTable -> Prize list` so that `prizesFor` *a pt* is the list of prizes that are associated with achievements smaller that or equal to *a* in *pt*. For example, `prizesFor 7 pt = ["p1"; "p2"]`.

3. Declare a function `increase: int -> PrizeTable -> PrizePrize`. The value of the expression `increase` *k pt* is the prize table obtained from *pt* by adding *k* to every achievement. For example, `increase 2 pt = [("p1",5);("p2",7);("p3",10);("p4",13)]`.

4. Declare a function `add: (Prize*Achievement) * PrizeTable -> PrizeTable`. The value of `add`$((p, a), pt)$ is the prize table obtained from *pt* by insertion of the pair (p, a). An exception should be raised if there is a pair in *pt*, for which the achievement is equal to *a*. For example, `[("p1",3);("p2",5);("p3",8);("p35",10);("p4",11)]` is the value of `add(("p35",10),pt)`.

5. Declare a function `merge` pt_1 pt_2 that gives the prize table consisting of all the pairs from prize table pt_1 and prize table pt_2. If a pair (p_1, a_1) in pt_1 has the same achievement as a pair (p_2, a_2) in pt_2, i.e. $a_1 = a_2$, then an exception should be raised.

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In the last question you may use the following functions from the `List` library: `filter`, `map`, `fold` and `foldBack`.

6. Give alternative declarations for the functions `prizesFor`, `increase` and `merge` from questions 2., 3. and 5. You may use the above-mentioned functions from the `List` library and other functions from this problem; but you should not use explicit recursion in the declarations.

Problem 2 (20%)

The function `choose` from the `List` library could have the following declaration:

```
let rec choose f xs =
    match xs with
    | []      -> []                                (* C1 *)
    | x::rest -> match f x with
                 | None   -> choose f rest         (* C2 *)
                 | Some y -> y::choose f rest;;    (* C3 *)
val choose : ('a -> 'b option) -> 'a list -> 'b list
```

Notice that the F# system automatically infers the type of `choose`.

1. Give an argument showing that `('a -> 'b option) -> 'a list -> 'b list` is the most general type of `choose`. That is, any other type for `choose` is an instance of `('a -> 'b option) -> 'a list -> 'b list`.

let `chEven` be declared by:

```
let chEven n = if n%2=0 then Some (string n) else None;;
```

2. Give an evaluation of the expression `choose chEven [1;2;3;4;5]`. Use the notation $e_1 \rightsquigarrow e_2$ from the textbook and include at least as many steps as there are recursive calls.

The declaration of `choose` is not tail recursive.

3. Provide a declaration of a tail-recursive variant of `choose` that is based on an accumulating parameter. Your tail-recursive declaration must be based on an explicit recursion.

4. Provide a declaration of a tail-recursive variant of `choose` that is continuation-based. Your tail-recursive declaration must be based on an explicit recursion.

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Problem 3 (15%)

Consider the following declarations:

```
type T = | One of int | Two of int * T * int * T

let rec f p t =
   match t with
   | One v when p v            -> [v]              (* C1 *)
   | Two(v1,t1,_,_) when p v1 -> v1::f p t1       (* C2 *)
   | Two(_,_,v2,t2)            -> v2::f p t2       (* C3 *)
   | _                         -> [];;             (* C4 *)
```

1. Give the type for `f` and describe what `f` computes. Your description should focus on what it computes, rather than on individual computation steps.

Notice that the declaration of `f` has a match expression with 4 clauses marked `C1` to `C4` in comments.

A *test description* for `f` consists of

- a value p_v for argument `p`,
- a value t_v for argument `t`,
- the expected value of `f` p_v t_v, and
- an enumeration of the clauses that are selected during evaluation of `f` p_v t_v. The order in which clauses are enumerated is not significant. Repeated enumeration of a clause is not necessary.

2. Give a small number (≤ 4) of test descriptions for `f`. Together they should ensure that every clause of `f` is selected during an evaluation.

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Problem 4 (30%)

A type for so-called *tries* is defined as a tree type `Trie<'a>`, where a node carries a value of type `'a`, a truth value, and an arbitrary number of child tries:

```
type Trie<'a>     = N of 'a * bool * Children<'a>
and Children<'a> = Trie<'a> list
```

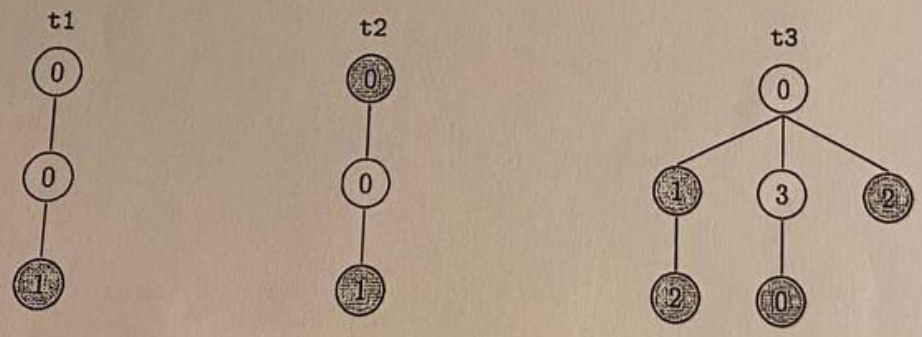
Consider the three values `t1`, `t2` and `t3` of type `Trie<int>`:

```
let t1 = N(0, false, [N(0, false, [N(1,true,[])])]);;

let t2 = N(0, true, [N(0, false, [N(1,true,[])])]);;

let ta = N(1,true,[N(2,true,[])]);;
let tb = N(3,false,[N(0,true,[])]);;
let tc = N(2,true,[]);;
let t3 = N(0,false, [ta;tb;tc]);;
```

The three values are illustrated as trees in the following figure, where each node carry an integer value, and a shaded node indicates that the truth value associated with the node is `true`. Shaded nodes are also called *accepting nodes*.



`t1` accepts [0;0;1] `t2` accepts [0] and [0;0;1] `t3` accepts [0;1], [0;1;2], [0;3;0] and [0;2]

A value in a node of a trie is called a *letter*. For example, trie `t3` contains four letters: $0, 1, 2, 3$.

A *word* is a list of letters. Furthermore, a word w is *accepted by* a trie t if there is a path from the root of t to an accepting node, so that w equals the list of letters of the nodes of the path. For example, $[0; 1; 2]$ is accepted by `t3` and the tries `t1`, `t2` and `t3` accept 1, 2 and 4 words, respectively, as shown in the figure.

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1. Declare a function that counts the number of nodes of a trie. For example, `t3` has 6 nodes.

2. Declare a function `accept` w t that can check whether word w is accepted by trie t. Give the type of `accept`.

3. Declare a function `wordsOf: Trie<'a> -> Set<'a>` that gives the set of words accepted by a trie t.

Leaves of tries have the form $\texttt{N}(v, b, [])$. Leaves where $b = \texttt{false}$ do not contribute to the words accepted by a trie and such leaves are called *useless*.

4. Declare a function that can check whether a trie contains useless leaves.

The *degree of a node* $\texttt{N}(v, b, ts)$ is the length of the list of children ts. The maximum degree of all nodes in a trie is called the *degree of a trie*.

5. Declare a function that computes the degree of a trie.