Project wiskundige ingenieurstechniek

$$\begin{cases} \frac{\partial u(x,y,z)}{\partial t} = D_u \cdot \nabla^2 u(x,y,z) - R_u(u,v) \\ \frac{\partial v}{\partial t}(x,y,z) = D_v \cdot \nabla^2 v(x,y,z) + R_v(u,v) \end{cases}$$

$\downarrow$

stationair

$$\begin{cases} 0 = D_u \nabla^2 u(x,y,z) - R_u(u,v) \\ 0 = D_v \nabla^2 v(x,y,z) + R_v(u,v) \end{cases}$$

randvoorwaarden:

$$\begin{cases} -\vec{n} \cdot (D_u \nabla u(x,y,z) = h_u (u - C_{u,amb}) \\ -\vec{n} \cdot (D_v \nabla u(x,y,z) = h_v (v - C_{v,amb}) \end{cases}$$

$\downarrow$ transformatie naar cilinder coordinaten:

$$\begin{cases} 0 = D_u \cdot \left(\frac{1}{r} \frac{\partial u}{\partial r}(r,z) + \frac{\partial^2 u}{\partial r^2}(r,z) + \frac{\partial^2 u}{\partial z^2} \right) - R_u(u,v) \\ 0 = D_v \cdot \left(\frac{1}{r} \frac{\partial v}{\partial r}(r,z) + \frac{\partial^2 v}{\partial r^2}(r,z) + \frac{\partial^2 v}{\partial z^2} \right) + R_v(u,v) \end{cases}$$

maal r $\downarrow$

$$\begin{cases} 0 = D_u \left(\frac{\partial}{\partial r} \left(r \cdot \frac{\partial u}{\partial r}(r,z) \right) + r \frac{\partial^2 u(r,z)}{\partial z^2} \right) - r R_u(u,v) \\ 0 = D_v \left(\frac{\partial}{\partial r} \left(r \frac{\partial v}{\partial r}(r,z) \right) + r \frac{\partial^2 v(r,z)}{\partial z^2} \right) + r R_v(u,v) \end{cases}$$

randvoorwaarden

$$\begin{cases} -\vec{n} \cdot (D_u \cdot \nabla u(r,z)) = h_u (u(r,z) - C_{u,amb}) \\ -\vec{n} \cdot (D_v \cdot \nabla v(r,z)) = h_v (v(r,z) - C_{v,amb}) \end{cases}$$

Galerkin residu stelling   basisfunties $\underset{\text{weging}}{N_j}(r,z)$ $j=1\ldots M$

$$\begin{cases} 0 = \displaystyle\int_\Omega \left[\left(\frac{\partial}{\partial r}\left(r\,\frac{\partial u}{\partial r}\right) + r\,\frac{\partial^2 u(r,z)}{\partial z^2}\right)D_u - r\cdot R_u(u,v)\right] N_j(r,z)\, d\Omega \\ 0 = \displaystyle\int_\Omega \left[\left(\frac{\partial}{\partial r}\left(r\,\frac{\partial v(r,z)}{\partial r}\right) + r\,\frac{\partial^2 v(r,z)}{\partial z^2}\right)D_v + r\,R_v(u,v)\right] N_j(r,z)\, dz \end{cases}$$

$$+D_u\int_\Gamma r\, N_j \Big[\underbrace{\vec{n}\cdot\nabla u}_{=\,-\frac{h_u}{D_u}(u(r,z)-C_{u,amb})}\Big]\, d\Gamma$$

$$0 = -D_u\left[\int_\Omega \left(r\cdot\frac{\partial u(r,z)}{\partial r}\frac{\partial N_j}{\partial r} + r\,\frac{\partial u(r,z)}{\partial z}\frac{\partial N_j}{\partial z}\right)d\Omega\right] - \int_\Omega r\,R_u(u,v)\,N_j(r,z)\,d\Omega$$

$$0 = -D_v\left[\int_\Omega \left(r\cdot\frac{\partial v(r,z)}{\partial r}\frac{\partial N_j}{\partial r} + r\,\frac{\partial v(r,z)}{\partial z}\frac{\partial N_j}{\partial z}\right)d\Omega\right] +$$

$$\int_\Omega r\,R_v(u,v)\,N_j(r,z)\,d\Omega + D_v\int_\Gamma r\,N_j(r,z)\Big[\underbrace{\vec{n}\cdot\nabla v}_{=\,-\frac{h_v}{\cancel{D_v}}(v(r,z)-C_{v,amb})}\Big]\,d\Gamma$$

$$0 = +D_u\int_\Omega r\left[\frac{\partial u(r,z)}{\partial r}\cdot\frac{\partial N_j(r,z)}{\partial r} + \frac{\partial u(r,z)}{\partial z}\cdot\frac{\partial N_j(r,z)}{\partial z}\right]d\Omega + \int_\Omega r\cdot N_j(r,z)\,R_u(u,v)\,d\Omega$$

$$+h_u\int_\Gamma r\,N_j(r,z)\,(u(r,z)-C_{u,amb})\,d\Gamma$$

$\forall j=1\ldots M$

$$0 = -D_v\int_\Omega r\left[\frac{\partial v(r,z)}{\partial r}\cdot\frac{\partial N_j(r,z)}{\partial r} + \frac{\partial v(r,z)}{\partial z}\frac{\partial N_j(r,z)}{\partial z}\right]d\Omega +$$

$$\int_\Omega r\cdot R_v(u,v)\,N_j(r,z)\,d\Omega - h_v\int_\Gamma r\,(v(r,z)-C_{v,amb})\,N_j(r,z)\,d\Gamma$$

$$u(r,z) \approx u_M(r,z) = \sum_{L=1}^{M} \hat{u}_L \cdot N_L(r,z)$$

$$v(r,z) \approx v_M(r,z) = \sum_{L=1}^{M} \hat{v}_L \cdot N_L(r,z)$$

$$\begin{cases} 0 = D_u \sum_{L=1}^{M} \hat{u}_L \int_\Omega r \cdot \left(\nabla N_j(r,z) \cdot \nabla N_L(r,z)\right) d\Omega + \int_\Omega r \cdot N_j(r,z) \cdot R_u(u,v) d\Omega \\ \quad + h_u \int_\Gamma r \cdot N_j(r,z) \cdot \left(\sum_{L=1}^{M} \hat{u}_L \cdot N_L(r,z) - C_{u,amb}\right) d\Gamma \\ 0 = -D_v \sum_{L=1}^{M} \hat{v}_L \int_\Omega r \left(\nabla N_j(r,z) \cdot \nabla N_L(r,z)\right) d\Omega + \int_\Omega r \cdot N_j(r,z) \cdot R_v(u,v) d\Omega \\ \quad - h_v \int_\Gamma r \cdot N_j(r,z) \cdot \left(\sum_{L=1}^{M} \hat{v}_L N_L(r,z) - C_{v,amb}\right) d\Gamma \end{cases}$$

$$\begin{cases} 0 = D_u \, A \, \hat{u} + f_1(u,v) + h_u \int_\Gamma r \cdot N \left(\left(N^T \cdot \hat{v}\right) - C_{u,amb}\right) d\Gamma \\ 0 = -D_v \, A \, \hat{v} + f_2(u,v) - h_v \int_\Gamma r \cdot N \left(\left(N^T \cdot \hat{u}\right) - C_{v,amb}\right) d\Gamma \end{cases}$$

$$A = \int_\Omega r \quad \nabla N(r,z) \cdot \nabla N(r,z)^T d\Omega$$

↓ per triangel $T \rightarrow$ hoekpunten i, j, k

$$A_{local} = \int_T r \begin{bmatrix} \nabla N_i \\ \nabla N_j \\ \nabla N_k \end{bmatrix} \begin{bmatrix} \nabla N_i & \nabla N_j & \nabla N_k \end{bmatrix} d\Omega$$

→ locale coordinaten

$$\xi = \frac{1}{2|T|} \left((z_3 - z_1)(r - r_1) - (r_3 - r_1)(z - z_1) \right)$$

$$\eta = \frac{1}{2|T|} \left(-(z_2 - z_1)(r - r_1) + (r_2 - r_1)(z - z_1) \right)$$

$$N_i = 1 - \xi - \eta \qquad r = r_1 + (r_2 - r_1)\,\xi + (r_3 - r_1)\,\eta$$

$$N_j = \xi$$

$$N_k = \eta \qquad A_{local} = \frac{2|T|}{(2|T|)^2} \int_0^1 \int_0^{1-\xi} r(\xi,\eta) \begin{bmatrix} \cdots & \nabla N_i \\ \cdots & \nabla N_j \\ \cdots & \nabla N_k \end{bmatrix} \cdot \begin{bmatrix} \cdots \end{bmatrix} d\Omega$$

$$\frac{\partial N_i}{\partial r} = \frac{1}{2|T|} \left[(\cancel{z_1} - z_3) + (z_2 - \cancel{z_1}) \right]$$

$$\frac{\partial N_j}{\partial r} = \frac{1}{2|T|} (z_3 - z_1)$$

$$\frac{\partial N_k}{\partial r} = \frac{1}{2|T|} (z_1 - z_2)$$

$$\frac{\partial N_i}{\partial z} = \frac{1}{2|T|} \left[(r_3 - \cancel{r_1}) + (\cancel{r_1} - r_2) \right]$$

$$\frac{\partial N_j}{\partial z} = \frac{1}{2|T|} (r_1 - r_3)$$

$$\frac{\partial N_k}{\partial z} = \frac{1}{2|T|} (r_2 - r_1)$$

$$= \frac{1}{2|T|} \frac{(r_1 + r_2 + r_3)}{6} \begin{bmatrix} (z_2 - z_3) & (r_3 - r_2) \\ (z_3 - z_1) & (r_1 - r_3) \\ (z_1 - z_2) & (r_2 - r_1) \end{bmatrix} \begin{bmatrix} \cdots \end{bmatrix}$$

OK gecheckt in boek

$$\begin{bmatrix} z_2 - z_3 & r_3 - r_2 \\ z_3 - z_1 & r_1 - r_3 \\ z_1 - z_2 & r_2 - r_1 \end{bmatrix}$$

$$f_1(u,v) = \int_\Omega r\, N(r,z)\, R(u,v)\, d\Omega$$

$$\simeq \underbrace{\int_\Omega r \cdot N(r,z) \cdot N(r,z)^T \, d\Omega}_{B} \begin{bmatrix} R_u(\hat{u}_1, \hat{v}_1) \\ \vdots \\ R_v(\hat{u}_M, \hat{v}_M) \end{bmatrix}$$

$$B_{local} = 2|T| \int_0^1 \int_0^{1-\xi} r(\xi,\eta) \begin{bmatrix} 1-\xi-\eta \\ \xi \\ \eta \end{bmatrix} \begin{bmatrix} 1-\xi-\eta & \xi & \eta \end{bmatrix} d\eta\, d\xi$$

$$= \frac{|T|}{60} \begin{bmatrix} 6r_1+2r_2+2r_3 & 2r_1+2r_2+r_3 & 2r_1+r_2+2r_3 \\ & 2r_1+6r_2+2r_3 & r_1+2r_2+2r_3 \\ \text{Symmetric} & & 2r_1+2r_2+6r_3 \end{bmatrix}$$

OK in book

$$f_2(u,v) \approx \underbrace{\int_\Omega r\, N(r,z)\, N(r,z)^T \, d\Omega}_{= B} \begin{bmatrix} R_u(u_1, v_1) \\ \vdots \\ R_v(u_n, v_n) \end{bmatrix}$$

rand $=$ $\int_\Gamma r \cdot N(N^T \hat{u} - C_{u,amb}) \, d\Gamma$

$\downarrow$ op 1 triangle rand

$$= |\Gamma| \left[\int_0^1 [r_1 + \gamma(r_2 - r_1)] \begin{bmatrix} 1-\gamma \\ \gamma \end{bmatrix} [\cancel{1}-\gamma \quad \cancel{\gamma}] \, d\gamma \, \hat{u} \right.$$

$\hookrightarrow r = r_1 + \gamma(r_2 - r_1)$
$z = z_1 + \gamma(z_2 - z_1)$

$$\left. + \int_0^1 [r_1 + \gamma(r_2 - r_1)] \begin{bmatrix} 1-\gamma \\ \gamma \end{bmatrix} d\gamma \, C_{u,amb} \right]$$

$$= |\Gamma| \left(\begin{bmatrix} & \end{bmatrix} \begin{bmatrix} \hat{u}_1 \\ \hat{u}_2 \end{bmatrix} + \begin{pmatrix} \frac{5r_1}{6} + \frac{r_2}{6} \\ \frac{r_2}{3} - \frac{r_1}{3} \end{pmatrix} C_{u,amb} \right)$$

$$0 = D_u A \hat{u} + B\, R_u(\hat{u}, \hat{v}) + h_u \left(C \hat{u} + D\, c_{u,amb} \right)$$

$$0 = -D_v A \hat{v} + B\, R_v(\hat{u}, \hat{v}) \mp h_v \left(C \hat{v} + D\, c_{v,amb} \right)$$

Jacobian

$$J = \begin{bmatrix} D_u A^T + B \dfrac{\partial R_u}{\partial u} + h_u C^T & B \dfrac{\partial R_u}{\partial v} \\ B \dfrac{\partial R_v}{\partial u} & -D_v A^T + B \dfrac{\partial R_v}{\partial v} \mp h_v C^T \end{bmatrix}$$

Newton-Raphson

$$x_{k+1} = x_k - J^{-1}_{(x_k)} f(x_k)$$