financiële wiskunde wpo 2

Take $X \sim \mathcal{N}(0,\, \sigma^2) \Longrightarrow X = \sigma\, Z$ with $Z \sim \mathcal{N}(0, 1)$.

$$\begin{aligned}
\mathbb{E}\left[X^4\right] &= \mathbb{E}\left[(\sigma\, Z)^4\right] \\
&= \sigma^4\, \mathbb{E}\left[Z^4\right] \\
&= \sigma^4 \int_{-\infty}^{+\infty} z^4\, \frac{1}{\sqrt{2\pi}}\, e^{\frac{-z^2}{2}}\, dz \\
&= \sigma^4\, \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} z^3\, e^{\frac{-z^2}{2}}\, z\, dz
\end{aligned}$$

Integration by parts:

$$\begin{aligned}
f &= z^3, \\
df &= 3\, z^2\, dz \\
dg &= z\, e^{\frac{-z^2}{2}}\, dz \\
g &= -e^{\frac{-z^2}{2}}
\end{aligned}$$

$$\begin{aligned}
\mathbb{E}\left[X^4\right] &= \sigma^4\, \frac{1}{\sqrt{2\pi}}\, \left[-z^3\, e^{\frac{-z^2}{2}}\right]_{-\infty}^{+\infty} - \sigma^4\, \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} -e^{\frac{-z^2}{2}}\, 3\, z^2\, dz \\
&= 0 + 3\, \sigma^4 \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{\frac{-z^2}{2}}\, z^2\, dz
\end{aligned}$$

Integration by parts:

$$\begin{aligned}
f &= z \\
df &= dz \\
dg &= z\, e^{\frac{-z^2}{2}}\, dz \\
g &= -e^{\frac{-z^2}{2}}
\end{aligned}$$

$$\begin{aligned}
\mathbb{E}\left[X^4\right] &= 3\, \sigma^4 \frac{1}{\sqrt{2\pi}}\, \left[-z\, e^{\frac{-z^2}{2}}\right]_{-\infty}^{+\infty} - 3\, \sigma^4 \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} -e^{\frac{-z^2}{2}}\, dz \\
&= 0 + 3\, \sigma^4 \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{\frac{-z^2}{2}}\, dz
\end{aligned}$$

This integral is the Gauss-integral:

$$\int_{-\infty}^{+\infty} e^{\frac{-z^2}{2}}\, dz = \sqrt{2\pi}$$

And thus:

$$\mathbb{E}\left[X^4\right] = 3\, \sigma^4\, \frac{1}{\sqrt{2\pi}}\, \sqrt{2\pi} = 3\, \sigma^4$$