Quaternions

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A quaternion is a four-dimensional complex number that can be used to represent the orientation of a rigid body or coordinate frame in three-dimensional space. An arbitrary orientation of frame B relative to frame A can be achieved through a rotation of angle θ around an axis ${}^A\hat{\boldsymbol{r}}$ defined in frame A. This is represented graphically in figure 1 where the mutually orthogonal unit vectors $\hat{\boldsymbol{x}}_A$, $\hat{\boldsymbol{y}}_A$ and $\hat{\boldsymbol{z}}_A$, and $\hat{\boldsymbol{x}}_B$, $\hat{\boldsymbol{y}}_B$ and $\hat{\boldsymbol{z}}_B$ define the principle axis of coordinate frames A and B respectively. The quaternion describing this orientation, ${}^A_B\hat{\boldsymbol{q}}$, is defined by equation (1) where r_x, r_y and r_z define the components of the unit vector ${}^A\hat{\boldsymbol{r}}$ in the x, y and z axes of frame A respectively. A notation system of leading super-scripts and sub-scripts adopted from Craig [1] is used to denote the relative frames of orientations and vectors. A leading sub-script denotes the frame being described and a leading super-script denotes the frame this is with reference to. For example, ${}^A_B\hat{\boldsymbol{q}}$ describes the orientation of frame B relative to frame A and ${}^A\hat{\boldsymbol{r}}$ is a vector described in frame A. Quaternion arithmetic often requires that a quaternion describing an orientation is first normalised. It is therefore conventional for all quaternions describing an orientation to be of unit length.

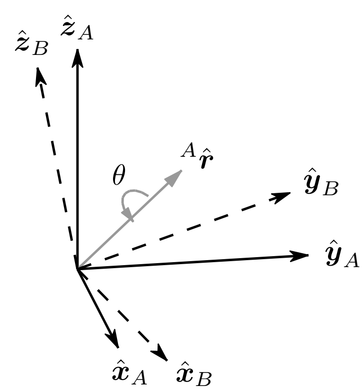

Figure 1: The orientation of frame B is achieved by a rotation, from alignment with frame A, of angle θ around the axis ${}^A\boldsymbol{r}$.

$$
{}^A_B\hat{\boldsymbol{q}} = \begin{bmatrix} q_0 & q_1 & q_2 & q_3 \end{bmatrix} = \begin{bmatrix} \cos\frac{\theta}{2} & -r_x sin\frac{\theta}{2} & -r_y sin\frac{\theta}{2} & -r_z sin\frac{\theta}{2} \end{bmatrix} \tag{1}
$$

The quaternion conjugate, denoted by *, can be used to swap the relative frames described by an orientation. For example, ${}^B_A\hat{\boldsymbol{q}}$ is the conjugate of ${}^A_B\hat{\boldsymbol{q}}$ and describes the orientation of frame A relative to frame B. The conjugate of ${}^A_B\hat{\boldsymbol{q}}$ is defined by equation (2).

$$ {}^A_B\hat{\boldsymbol{q}}^* = {}^B_A\hat{\boldsymbol{q}} = \begin{bmatrix} q_0 & -q_1 & -q_2 & -q_3 \end{bmatrix} \tag{2} $$

The quaternion product, denoted by $\otimes$, can be used to define compound orientations. For example, for two orientations described by ${}^A_B\hat{\boldsymbol{q}}$ and ${}^B_C\hat{\boldsymbol{q}}$, the compounded orientation ${}^A_C\hat{\boldsymbol{q}}$ can be defined by equation (3).

$$ {}^A_C\hat{\boldsymbol{q}} = {}^B_C\hat{\boldsymbol{q}} \otimes {}^A_B\hat{\boldsymbol{q}} \tag{3} $$

For two quaternions, $\boldsymbol{a}$ and $\boldsymbol{b}$, the quaternion product can be determined using the Hamilton rule and defined as equation (4). A quaternion product is not commutative; that is, $\boldsymbol{a} \otimes \boldsymbol{b} \neq \boldsymbol{b} \otimes \boldsymbol{a}$.

$$ \begin{aligned} \boldsymbol{a} \otimes \boldsymbol{b} &= \begin{bmatrix} a_0 & a_1 & a_2 & a_3 \end{bmatrix} \otimes \begin{bmatrix} b_0 & b_1 & b_2 & b_3 \end{bmatrix} \\ &= \begin{bmatrix} a_0b_0 - a_1b_1 - a_2b_2 - a_3b_3 \\ a_0b_1 + a_1b_0 + a_2b_3 - a_3b_2 \\ a_0b_2 - a_1b_3 + a_2b_0 + a_3b_1 \\ a_0b_3 + a_1b_2 - a_2b_1 + a_3b_0 \end{bmatrix}^T \end{aligned} \tag{4} $$

A three dimensional vector can be rotated by a quaternion using the relationship described in equation (5) [2]. ${}^A\boldsymbol{v}$ and ${}^B\boldsymbol{v}$ are the same vector described in frame A and frame B respectively where each vector contains a 0 inserted as the first element to make them 4 element row vectors.

$$ {}^B\boldsymbol{v} = {}^A_B\hat{\boldsymbol{q}} \otimes {}^A\boldsymbol{v} \otimes {}^A_B\hat{\boldsymbol{q}}^* \tag{5} $$

The orientation described by ${}^A_B\hat{\boldsymbol{q}}$ can be represented as the rotation matrix ${}^A_B\boldsymbol{R}$ defined by equation (6) [2].

$$ {}^A_B\boldsymbol{R} = \begin{bmatrix} 2q_0^2 - 1 + 2q_1^2 & 2(q_1q_2 + q_0q_3) & 2(q_1q_3 - q_0q_2) \\ 2(q_1q_2 - q_0q_3) & 2q_0^2 - 1 + 2q_2^2 & 2(q_2q_3 + q_0q_1) \\ 2(q_1q_3 + q_0q_2) & 2(q_2q_3 - q_0q_1) & 2q_0^2 - 1 + 2q_3^2 \end{bmatrix} \tag{6} $$

A quaternion may be obtain from a rotation matrix using the inverse of the relationships defined in (6); however, in some practical applications an available rotation matrix may not be orthogonal and so a more robust method is prefered. Bar-Itzhack provides a method [3] to extract the optimal, 'best fit' quaternion from an imprecise and non-orthogonal rotation matrix. The method requires the construction of the symmetric 4 by 4 matrix $\boldsymbol{K}$ (equation (7)) where r_{mn} corresponds to the element of the m^{th} row and n^{th} column of ${}^A_B\boldsymbol{R}$. The optimal quaternion, ${}^A_B\hat{\boldsymbol{q}}$, is found as the normalised Eigen vector corresponding to the maximum Eigen value of $\boldsymbol{K}$. This is defined by equation (8) where v_0 to v_3 define the elements of the normalised Eigen vector.

$$ \boldsymbol{K} = \frac{1}{3} \begin{bmatrix} r_{11} - r_{22} - r_{33} & r_{21} + r_{12} & r_{31} + r_{13} & r_{23} - r_{32} \\ r_{21} + r_{12} & r_{22} - r_{11} - r_{33} & r_{32} + r_{23} & r_{31} - r_{13} \\ r_{31} + r_{13} & r_{32} + r_{23} & r_{33} - r_{11} - r_{22} & r_{12} - r_{21} \\ r_{23} - r_{32} & r_{31} - r_{13} & r_{12} - r_{21} & r_{11} + r_{22} + r_{33} \end{bmatrix} \tag{7} $$

$$
{}^A_B\hat{\boldsymbol{q}} = \begin{bmatrix} v_3 & v_0 & v_1 & v_2 \end{bmatrix} \tag{8}
$$

The ZYX Euler angles ϕ, θ and ψ describe an orientation of frame B achieved by the sequential rotations, from alignment with frame A, of ψ around $\hat{\boldsymbol{z}}_B$, θ around $\hat{\boldsymbol{y}}_B$, and ϕ around $\hat{\boldsymbol{x}}_B$. This Euler angle representation of ${}^A_B\hat{\boldsymbol{q}}$ can be calculated [4] using equations (9) to (11).

$$
\phi = \operatorname{atan2}\left(2(q_2q_3 - q_0q_1), 2q_0^2 - 1 + 2q_3^2\right) \tag{9}
$$

$$
\theta = -\arctan\left(\frac{2(q_1q_3 + q_0q_2)}{\sqrt{1 - (2q_1q_3 + 2q_0q_2)^2}}\right) \tag{10}
$$

$$
\psi = \operatorname{atan2}\left(2(q_1q_2 - q_0q_3), 2q_0^2 - 1 + 2q_1^2\right) \tag{11}
$$

References

[1] John J. Craig. *Introduction to Robotics Mechanics and Control.* Pearson Education International, 2005.

[2] J. B. Kuipers. *Quaternions and Rotation Sequences: A Primer with Applications to Orbits, Aerospace and Virtual Reality.* Princeton University Press, 1999.

[3] Itzhack Y Bar-Itzhack. New method for extracting the quaternion from a rotation matrix. *AIAA Journal of Guidance, Control and Dynamics*, 23(6):10851087, Nov.Dec 2000. (Engineering Note).

[4] Mei Wang, Yunchun Yang, R.R. Hatch, and Yanhua Zhang. Adaptive filter for a miniature mems based attitude and heading reference system. *Position Location and Navigation Symposium, 2004. PLANS 2004*, pages 193 – 200, apr. 2004.