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Numerical Integration

Course : Scientific Computing
Year : 2023
By : Adhe Lingga Dewi

Outlines

- Newton-Core Formulas
- Romberg Integration
- Gaussian Integration

The Antiderivative of a Function

Definition

Function F is called an **antiderivative** of f on an interval I if $F'(x) = f(x)$ for all x in I.

Theorem

If F is an antiderivative of f on an interval I, then the most general antiderivative of f on I is

$$F(x) + C$$

where C is an arbitrary constant.

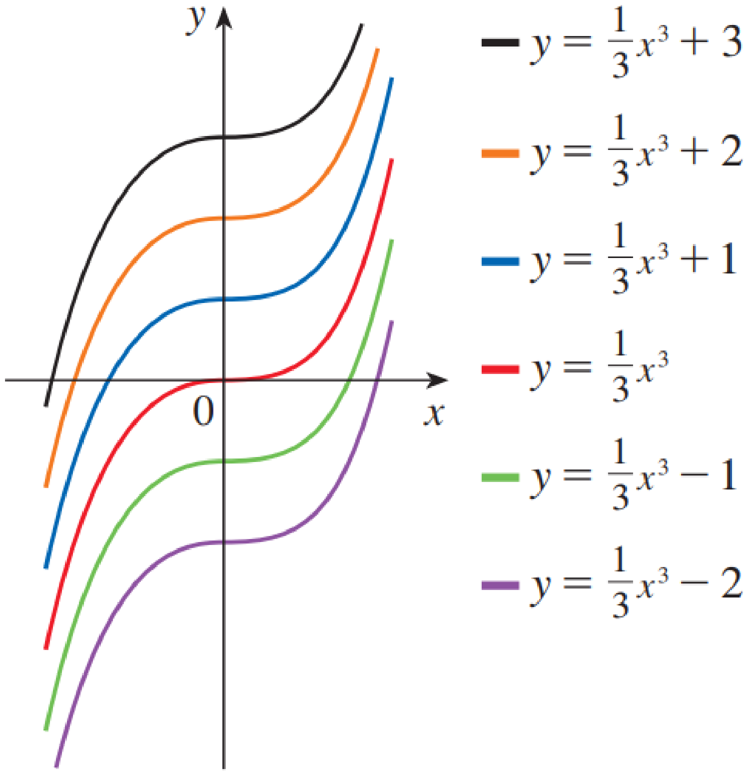


FIGURE 1
Members of the family of antiderivatives of $f(x) = x^2$

The Antiderivative of a Function

$$f(x) = x^3 \quad \rightarrow \quad f'(x) = \frac{df}{dx} = 3x^2$$

Derivative

$$f'(x) = f(x)\ ?$$

$$f(x) = \int f'(x)\,dx$$

Integral
Antiderivative

Definite

$$\int_a^b g(x)\,dx$$

Intrepetation

Indefinite

$$\int f(x)\,dx$$

formula

The Antiderivative of a Function

Table of Antidifferentiation Formulas

Function	Particular antiderivative	Function	Particular antiderivative
$cf(x)$	$cF(x)$	$\sin x$	$-\cos x$
$f(x) + g(x)$	$F(x) + G(x)$	$\sec^2 x$	$\tan x$
$x^n \quad (n \neq -1)$	$\dfrac{x^{n+1}}{n+1}$	$\sec x \tan x$	$\sec x$
$\dfrac{1}{x}$	$\ln \lvert x \rvert$	$\dfrac{1}{\sqrt{1-x^2}}$	$\sin^{-1} x$
e^x	e^x	$\dfrac{1}{1+x^2}$	$\tan^{-1} x$
b^x	$\dfrac{b^x}{\ln b}$	$\cosh x$	$\sinh x$
$\cos x$	$\sin x$	$\sinh x$	$\cosh x$

Integral Function

- $\int ax^n \, dx = \frac{a}{n+1} \; x^{n+1} + C$
- $\int e^{ax} \, dx = \frac{1}{a} e^{ax} + C$
- $\int \sin(ax + b) dx = -\frac{1}{a} \cos(ax + b) + C$
- $\int \cos(ax + b) dx = \frac{1}{a} \sin(ax + b) + C$
- $\int \frac{1}{x} dx = x \mid \ln x \mid + C$
- $\int \ln|x| \, dx = x \ln|x| - x + C$

Example 1

Find all functions t such that

$$\boldsymbol{g'(x) = 4\sin x + \frac{2x^5 - \sqrt{x}}{x}}$$

Solution:

1. Rewrite the given function as follows:

$$g'(x) = 4\sin x + \frac{2x^5}{x} - \frac{\sqrt{x}}{x} = 4\sin x + 2x^4 - x^{-\frac{1}{2}}$$

2. Using the formulas obtain

$$g(x) = 4(-\cos x) + \frac{2x^5}{4} + C$$

$$g(x) = -4\cos x + \frac{2}{5}x^5 - 2\sqrt{x} + C$$

Example 2

Find f if $f''(x) = 12x^2 + 6x - 4$, $f(0) = 4$, and $f(1) = 1$

Solution:

1. The general antiderivative of $f''(x)$ is:

$$f'(x) = 12\frac{x^3}{3} + 6\frac{x^2}{2} - 4x + C = 4x^3 + 3x^2 - 4x + C$$

2. Using the antidifferentiation rules once more,obtain

$$f(x) = 4\frac{x^4}{4} + 3\frac{x^3}{3} - 4\frac{x^2}{2} + Cx + D = x^4 + x^3 - 2x^2 + Cx + D$$

3. To determine *C* and *D,* use the given conditions that f (0) = 4 and f (1) = 1, so that C = -3 and D = 4. Therefore, the required function is $\boldsymbol{f(x) = x^4 + x^3 - 2x^2 - 3x + 4}$

Indefinite Integral

$$g(x) = 3x^2 \quad \rightarrow \int 3x^2\,dx = x^3$$

Derivative of a constant function is 0

$$g(x) = 3x^2 + 0 \quad \rightarrow \quad \int 3x^2\,dx = x^3 + c, c \in \mathbb{R}$$

Why $+\,\mathrm{C}$

$$= x^3 + 1000\,dx$$
$$= x^3 + 0{,}001\,dx$$

Exercise

Find f if:

1. $f''(x) = x^2 - 4$
2. $f''(x) = 4x^3 + 24x - 1$
3. $f''(x) = 6x - x^4 + 3x^5$
4. $f'''(x) = 2x + 3x^2$

The Area Problem

Definite Integral

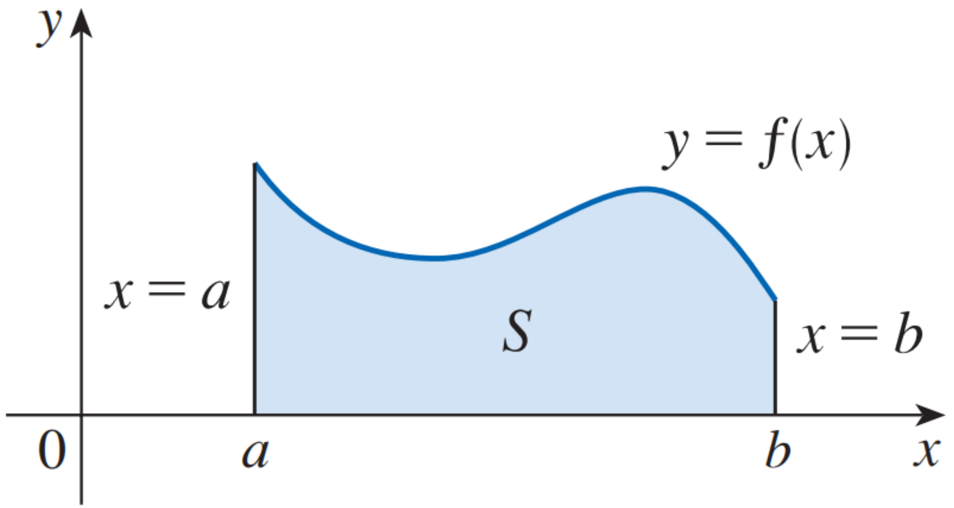
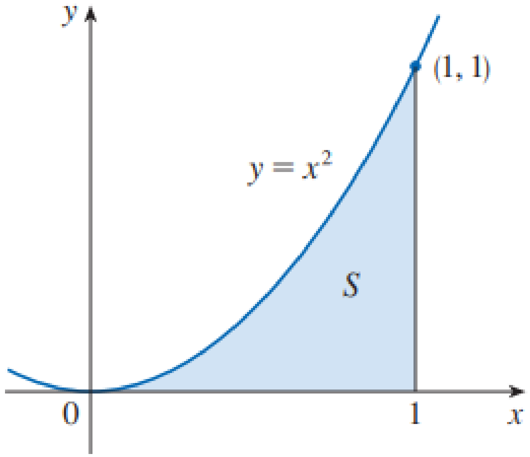


FIGURE 1
$S = \{(x, y) \mid a \leqslant x \leqslant b, 0 \leqslant y \leqslant f(x)\}$

Find the area of the region *S* that lies under the curve *y* = *f*(*x*) from *a* to *b*. This means that *S*, illustrated in Figure, is bounded by the graph of a continuous function *f* [where $f(x) \geq 0$], the vertical lines *x* = *a* and *x* = *b*, and the *x*-axis.

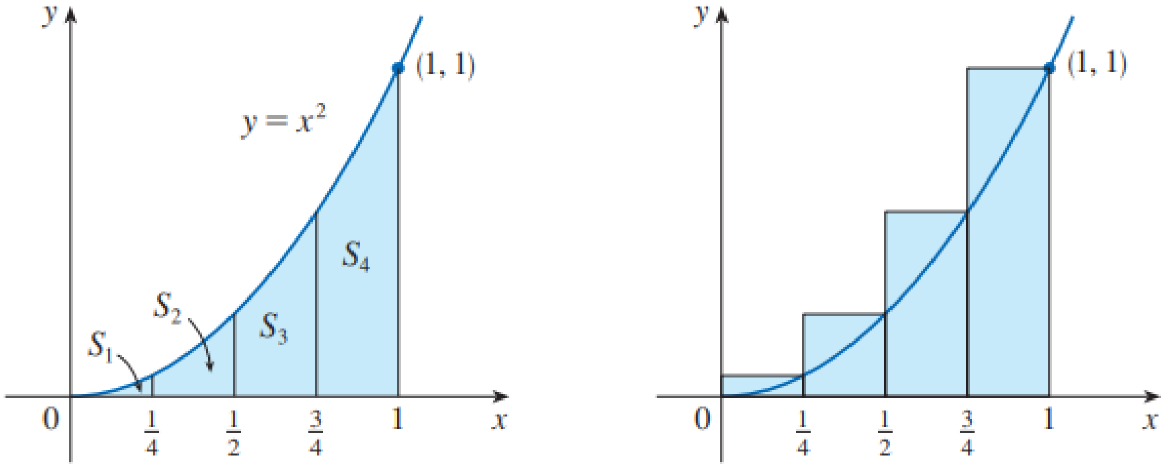
It isn't so easy, however, to find the area of a region with curved sides. We all have an intuitive idea of what the area of a region is. But part of the area problem is to make this intuitive idea precise by giving an exact definition of area.

Introduction to Area (1/2)



Use rectangles to estimate the area under the parabola $y = x^2$ from 0 to 1.

Suppose we divide *S* into four strips *S*1, *S*2, *S*3, and *S*4 as in Figure.



(a) (b)

Let R_4 be the sum of the areas of these approximating rectangles, we get

$$R_4 = \frac{1}{4} \cdot \left(\frac{1}{4}\right)^2 + \frac{1}{4} \cdot \left(\frac{1}{2}\right)^2 + \frac{1}{4} \cdot \left(\frac{3}{4}\right)^2 + \frac{1}{4} \cdot 1^2$$

$$R_4 = \frac{14}{32} = 0.46875$$

Introduction to Area (1/2)

$$A = \lim_{n \to \infty} R_n = \lim_{n \to \infty} L_n = \frac{1}{3}$$

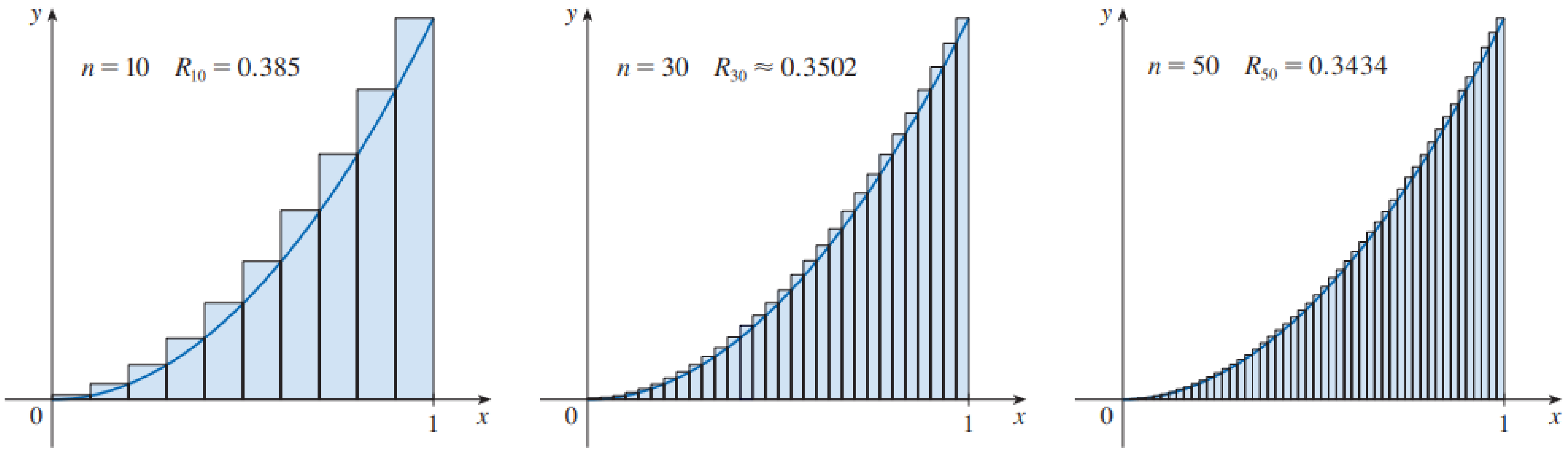


FIGURE 8 Right endpoints produce upper estimates because $f(x) = x^2$ is increasing.

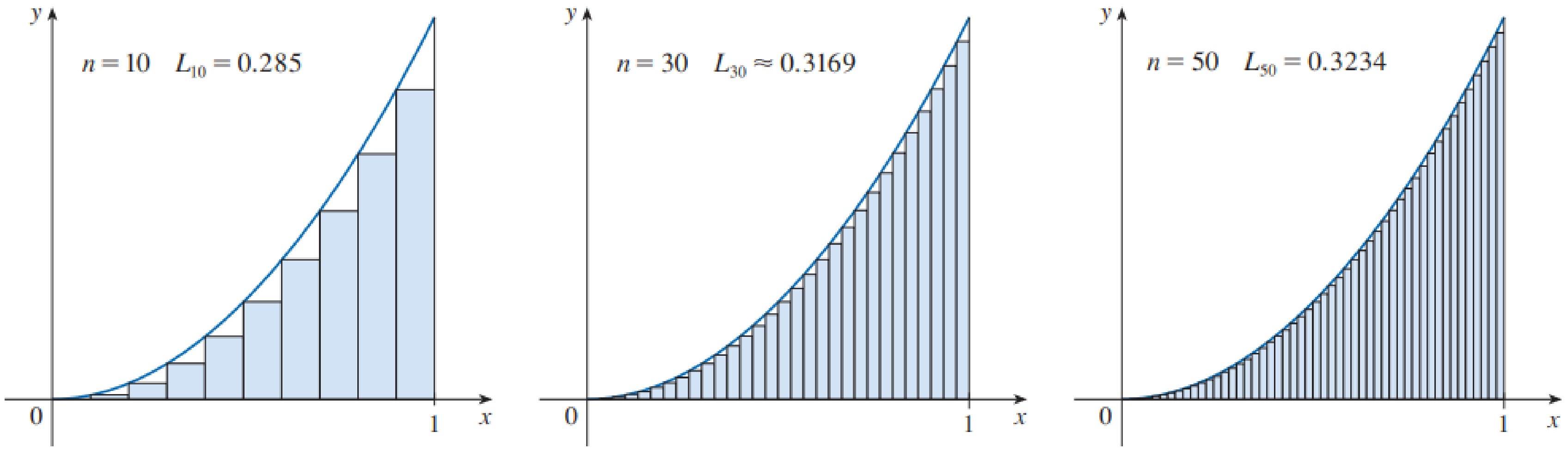
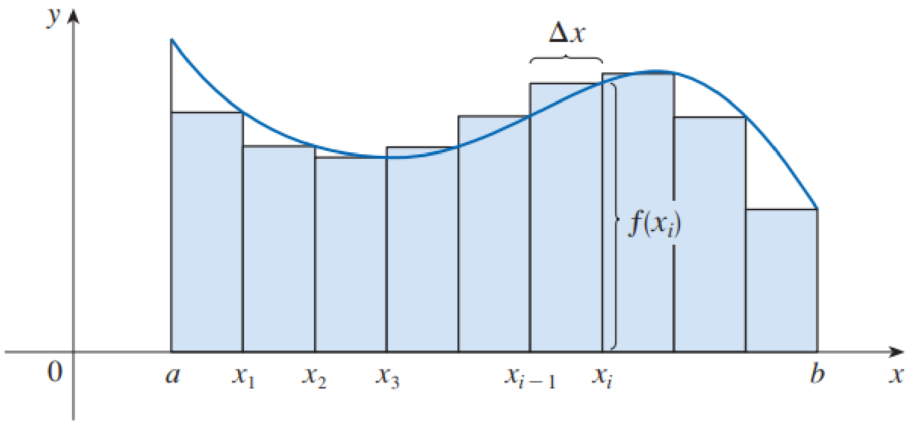
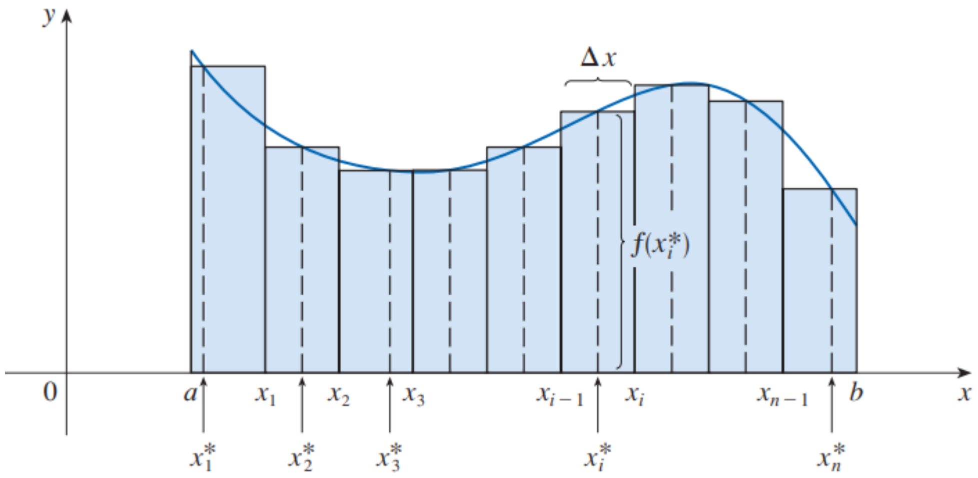


FIGURE 9 Left endpoints produce lower estimates because $f(x) = x^2$ is increasing.

The Area





Definition

The **area** A of the region S that lies under the graph of the continuous function f is the limit of the sum of the areas of approximating rectangles:

$$A = \lim_{n\to\infty} R_n = \lim_{n\to\infty}[f(x_1)\Delta x + f(x_2)\Delta x + \cdots . + f(x_n)\Delta x]$$

Properties of the Integral

Assume that *f* and g are continuous functions.

Properties of the Integral

1. $\int_a^b c\,dx = c(b-a)$, where c is any constant

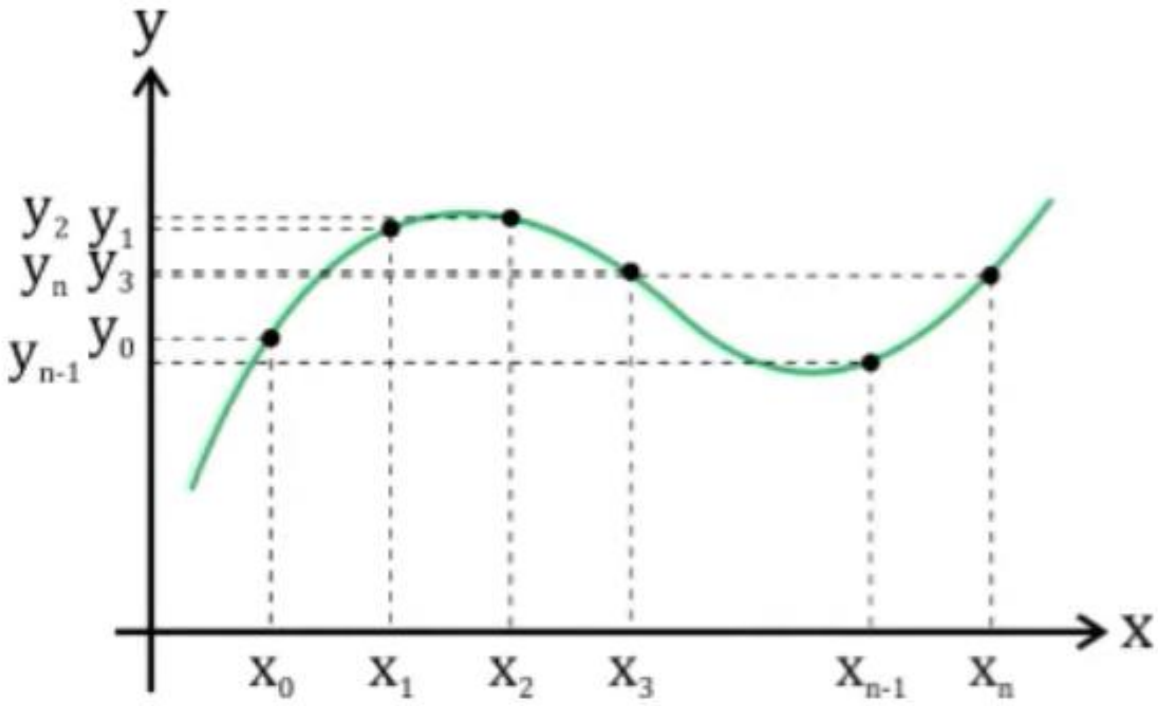
2. $\int_a^b [f(x)+g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$

3. $\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$, where c is any constant

4. $\int_a^b [f(x)-g(x)]\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$

Numerical Integration

Numerical integration is a method used to obtain approximations of integrations that cannot be solved analytically



- Integration process → perform integration on small parts.
- Numerical integration → the integration process is faster and closer to the exact answer

Numerical Integration

- Newton-Core Formulas
- Romberg Integration
- Gaussian Integration

Newton-Core Formulas →

- Reimann Integration
- Trapezoid Rule
- Simpson 1/3
- Simpson 3/8

Gaussian Integration →

Gauss Legendre Integration

Riemann Integration

Integrable

Let f be a function defined on the interval [a,b].

If $\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$

exists, we say *f* that is **integrable** on [*a*, *b*].

Definite Integral

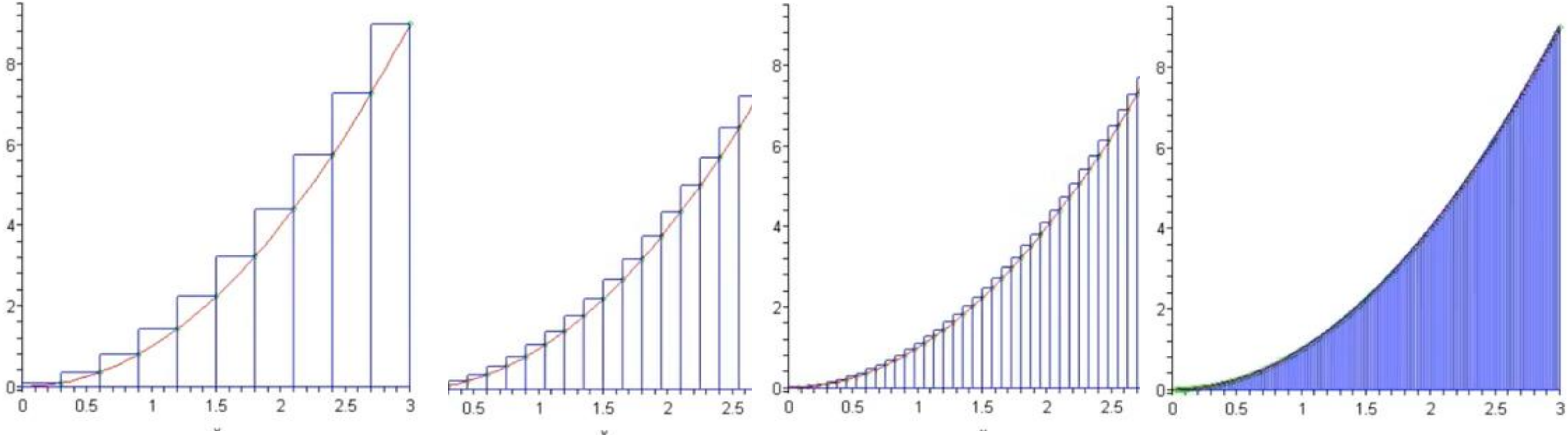
Moreover, $\int_a^b f(x)dx$ called the **definite integral** (or Riemann integral) of *f* from *a* to *b*, is then given by

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

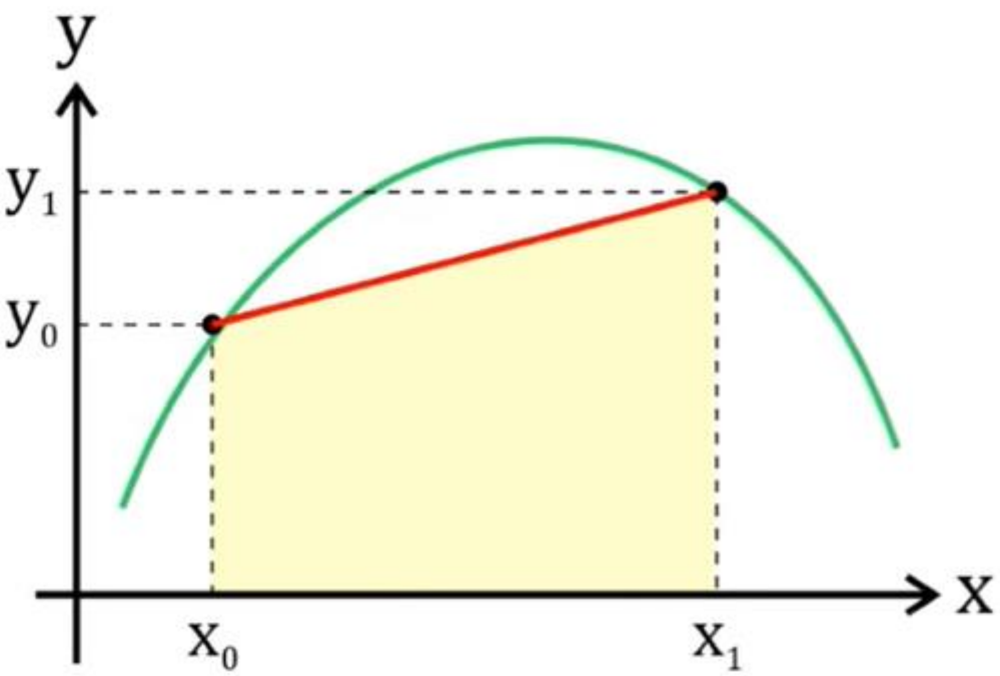
Reimann Integration

Integral basis

$$\int_{a}^{b} f(x)\,dx \approx \sum_{i=0}^{n} c_i f(x_i) = c_0 f(x_0) + c_1 f(x_1) + c_3 f(x_3) + \cdots . + c_n f(x_n)$$



Trapezoida Rule’s

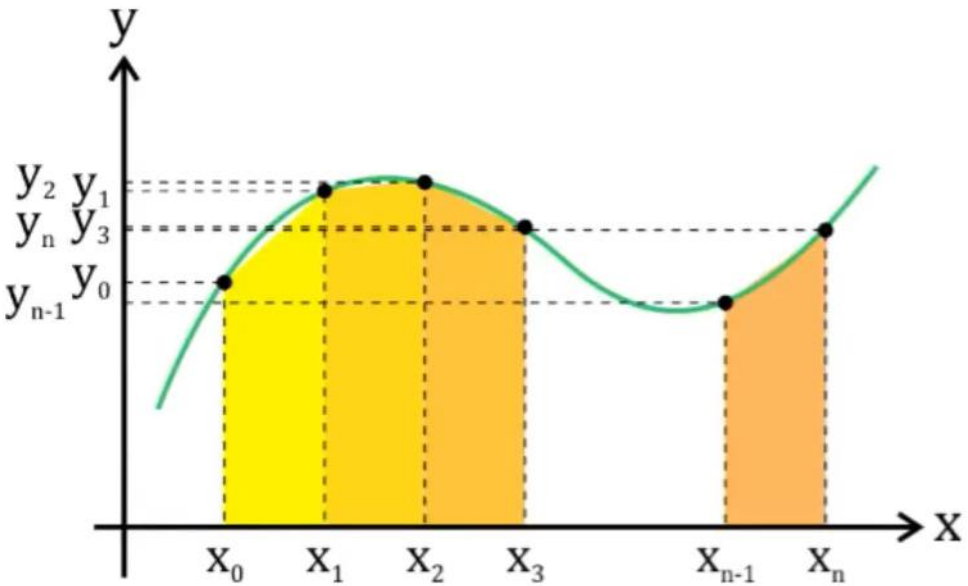


$$\int_a^b f(x)dx \approx \sum_{i=0}^{1} c_i f(x_i)$$

$$= c_0 f(x_0) + c_1 f(x_1)$$

$$= \frac{x_1 - x_0}{2}[f(x_0) + f(x_1)]$$

Trapezoida Rule’s



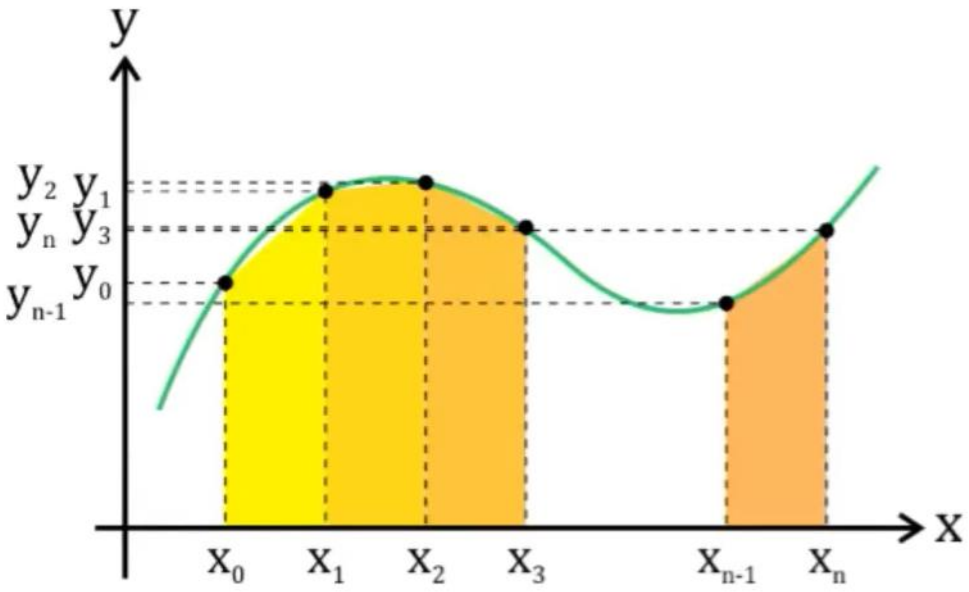
$$\int_a^b f(x)dx \approx \int_{x_0}^{x_1} f(x)dx + \int_{x_1}^{x_2} f(x)dx + \ldots + \int_{x_{n-1}}^{x_n} f(x)dx$$

$$= \frac{\Delta x}{2}[f(x_0) + f(x_1)] + \frac{\Delta x}{2}[f(x_1) + f(x_2)] + \cdots + \frac{\Delta x}{2}[f(x_{n-1}) + f(x_n)]$$

$$= \frac{\Delta x}{2}[f(x_0) + f(x_1) + f(x_1) + f(x_2) + \cdots + f(x_{n-1}) + f(x_n)]$$

$$= \frac{\Delta x}{2}[f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)]$$

Trapezoida Rule’s



$$\int_a^b f(x)dx \approx \frac{\Delta x}{2}[f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)]$$

Because, $L_i = \frac{\Delta x}{2}[f(x_0) + f(x_1)] \quad \rightarrow \Delta x = \frac{\text{b}-\text{a}}{n}$

Then,

$$L = \frac{\Delta x}{2}\left[f(x_0) + 2\sum_{i=1}^{n=1} f(x_i) + f(x_n)\right]$$

Example 1

Calculate $\int_0^1 3x^2\,dx$ using the trapezoidal method with an interval $h = \Delta \boldsymbol{x} = 0.1$

x	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
$f(x)$	0	0.03	0.12	0.27	0.48	0.75	1.08	1.47	1.92	2.43	3

Solution :

$$\boldsymbol{L = \frac{\Delta x}{2}\left[f(x_0) + 2\sum_{i=1}^{n=1} f(x_i) + f(x_n)\right]}$$

$$\boldsymbol{L = \frac{0.1}{2}[0 + 2(0.03 + 0.12 + 0.27 + \ \dots + 2.43) + 3]}$$

$$\boldsymbol{L = \frac{0.1}{2}(20,1)}$$

$$\boldsymbol{L = \frac{2.01}{2} = 1.005}$$

Example 2

Approximate $\int_0^1 \sqrt{1+x^3}\,dx$ using the trapezoidal method with 5 strips

Solution :

In this question, n = 5, a = 0, and b=1, so $\Delta \boldsymbol{x} = \frac{b-a}{n} = \frac{1}{5} = 0{,}2$

Using the formula above we get :

x	0	0.2	0.4	0.6	0.8	1
$f(x)$	1	1.00399	1.03150	1.100272	1.22963	1.41421

$$\int_0^1 \sqrt{1+x^3}\,dx \approx \frac{\Delta x}{2}\left[f(x_0) + 2\sum_{i=1}^{n=1} f(x_i) + f(x_n)\right]$$

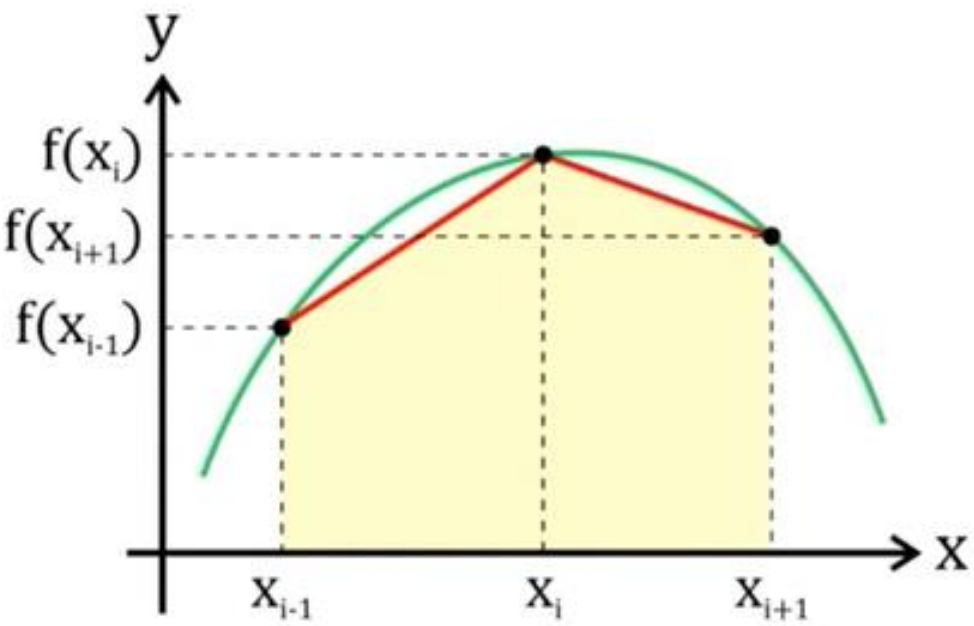
$$= \frac{0.2}{2}[f(0) + 2[f(0.2) + f(0.4) + f(0.6) + f(0.8)] + f(1)]$$

$$= 0.1\,[1 + 2(1.00399 + 1.03150 + 1.100272 + 1.22963) + 1.41421]$$

$$= 1.11499$$

Simpson 1/3 Rule’s

- The approach used is a parabolic function
- This method is an extension of the trapezoidal integration method → using two heavy → weighted trapezoids at the midpoint



- Trapezoida :

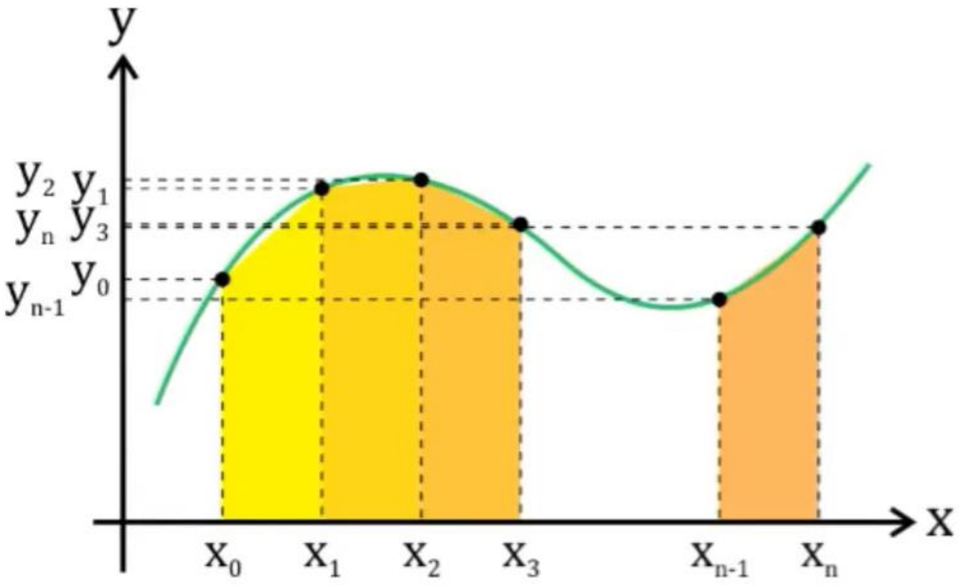
$$L = \frac{\Delta x}{2}[f(x_{i-1}) + f(x_i)] + \frac{\Delta x}{2}[f(x_i) + f(x_{i+1})]$$

- Simpson 1/8, midpoint weighted 2 :

$$\boldsymbol{L = \frac{\Delta x}{3}\left[f(x_{i-1}) + 2f(x_i)] + \frac{\Delta x}{3}[2f(x_i) + f(x_{i+1})\right]}$$

$$\boldsymbol{L = \frac{\Delta x}{3}[f(x_{i-1}) + 4f(x_i) + f(x_{i+1})]}$$

Simpson 1/3 Rule's



$$\int_a^b f(x)dx \approx \int_{x_0}^{x_2} f(x)dx + \int_{x_2}^{x_4} f(x)dx + \ldots + \int_{x_{n-2}}^{x_n} f(x)dx$$

$$= \frac{\Delta x}{3}[f(x_0) + 2f(x_1)] + \frac{\Delta x}{3}[2f(x_1) + f(x_2)] + \frac{\Delta x}{3}[f(x_2) + 2f(x_3)] + \frac{\Delta x}{3}[2f(x_3) + f(x_4)] + \cdots$$

$$+ \frac{\Delta x}{3}[f(x_{n-2}) + 2f(x_{n-1})] + \frac{\Delta x}{3}[2f(x_{n-1}) + f(x_n)]$$

$$= \frac{\Delta x}{3}[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + \cdots + 4f(x_{n-2}) + 2f(x_{n-1}) + f(x_n)]$$

$$\boldsymbol{L} = \frac{\Delta x}{3}\left[f(x_0) + 4\sum_{i=ganjil}^{n=1} f(x_i) + 2\sum_{i=genap}^{n=2} f(x_i) + f(x_n)\right]$$

Example

Calculate $\int_0^1 3x^2\, dx$ using the simpson 1/3 method with an interval $h = \Delta \boldsymbol{x} = 0.1$

x	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
$f(x)$	0	0.03	0.12	0.27	0.48	0.75	1.08	1.47	1.92	2.43	3

Solution :

$$\boldsymbol{L} = \frac{\Delta x}{3}\left[f(x_0) + 4 \sum_{i=ganjil}^{n=1} f(x_i) + 2 \sum_{i=genap}^{n=2} f(x_i) + f(x_n) \right]$$

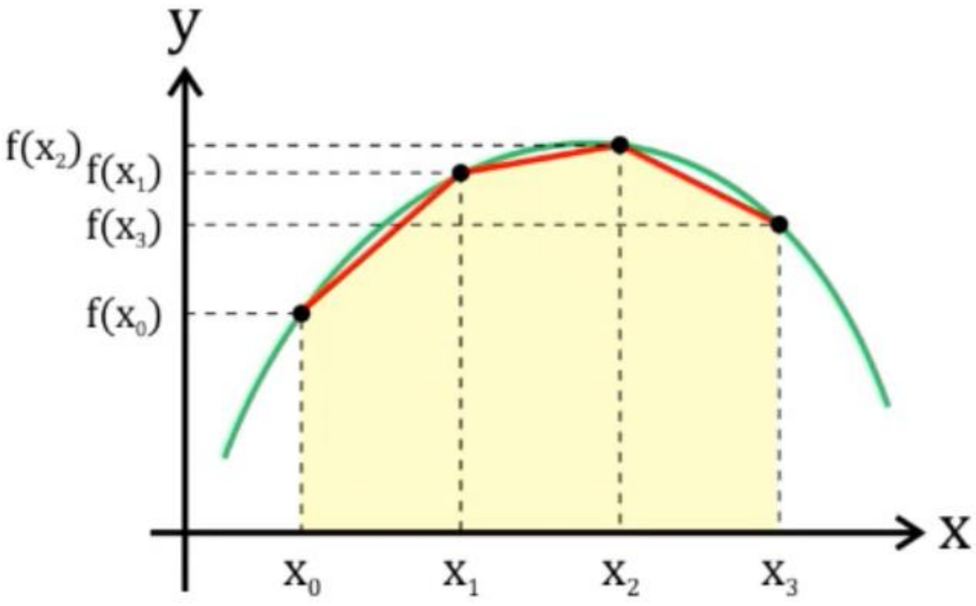
$$L = \frac{0.1}{3}[0 + 4(0.03 + 0.27 + 0.75 + 1.47 + 2.43) + 2(0.12 + 0.48 + 1.08 + 1.92) + 3]$$

$$L = \frac{0.1}{3}[0 + 4(4.95) + 2(3.6) + 3]$$

$$L = 1$$

Simpson 3/8 Rule’s

- The approach used is a cubic function
- This method is an extension of the trapezoidal integration method → weighted trapezoids at the midpoint



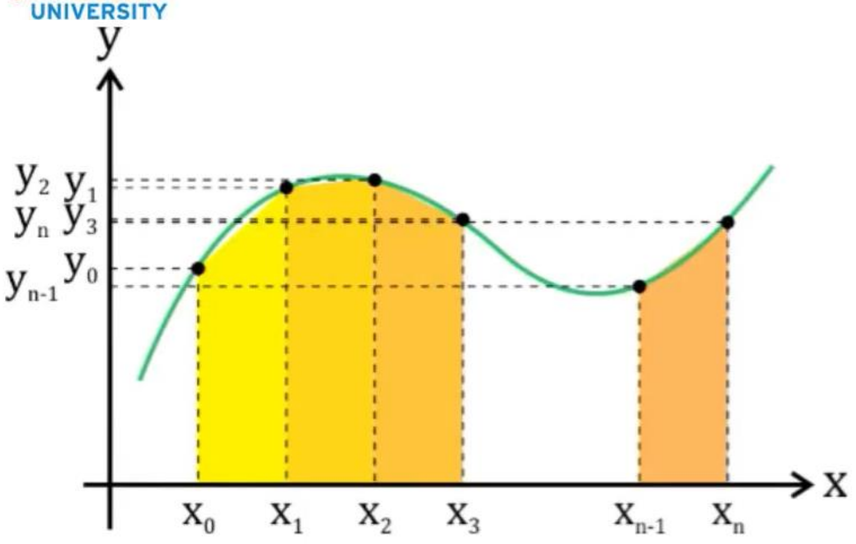
- Trapezoida :

$$L = \frac{\Delta x}{2}[f(x_0) + f(x_1)] + \frac{\Delta x}{2}[f(x_1) + f(x_2)] + \frac{\Delta x}{2}[f(x_2) + f(x_3)]$$

- Simpson 3/8, midpoint weighted 3 :

$$\boldsymbol{L = \frac{3}{8}\Delta x\,[f(x_0 + 3f(x_1) + 3f(x_2) + f(x_3)\,]}$$

Simpson 3/8 Rule’s



$$\boxed{\int_a^b f(x)dx \approx \int_{x_0}^{x_3} f(x)dx + \int_{x_3}^{x_6} f(x)dx + \ldots + \int_{x_{n-3}}^{x_n} f(x)dx}$$

$$= \frac{3}{8}\Delta x[f(x_0) + 3f(x_1) + 3f(x_2) + f(x_3)] + \frac{3}{8}\Delta x[f(x_3) + 3f(x_4) + 3f(x_5) + f(x_6)] + \cdots$$

$$+ \frac{3}{8}\Delta x[f(x_{n-3}) + 3f(x_{n-2}) + 3f(x_{n-1}) + f(x_n)]$$

$$= \frac{3}{8}\Delta x[f(x_0) + 3f(x_1) + 3f(x_2) + 2f(x_3) + 3f(x_4) + 3f(x_5) + 3f(x_6) + \cdots + 2f(x_{n-3}) + 3f(x_{n-2})$$

$$+ 3f(x_{n-1}) + f(x_n)]$$

$$\boldsymbol{L} = \frac{3}{8}\Delta x\left[f(x_0) + 3\sum_{\substack{i=1 \\ 1=3,6,9}}^{n=1} f(x_i) + 2\sum_{i=3,6,9}^{n=3} f(x_i) + f(x_n)\right]$$

Example

Calculate $\int_0^3 3x^2\,dx$ using the simpson 3/8 method with an interval $h = \Delta \boldsymbol{x} = 0.2$

x	0	0.2	0.4	0.6	0.8	1	1.2	1.4	1.6	1.8	2	2.2	2.4	2.6	2.8	3
$f(x)$	0	0.12	0.48	1.08	1.92	3	4.32	5.88	7.68	9.72	12	14.52	17.28	20.28	23.52	27

Solution :

$$\boldsymbol{L} = \frac{3}{8}\Delta x\left[f(x_0) + 3\sum_{\substack{i=1 \\ 1=3,6,9}}^{n=1} f(x_i) + 2\sum_{i=3,6,9}^{n=3} f(x_i) + f(x_n)\right]$$

$$L = \frac{3}{8}(0.2)[0 + 3(0.12 + 0.48 + 1.92 + \cdots + 23.52) + 2(1.08 + 4.32 + 9.72 + 17.28) + 27$$

$$L = \frac{3}{8}(0.2)[0 + 3(89.4) + 2(32.4) + 27$$

$$L = 27$$

Romberg Integration

The composite Trapezoidal Rule

$$\int_a^b f(x)dx \approx \frac{\Delta x}{2}\left[f(x_0) + 2\sum_{i=1}^{n=1} f(x_i) + f(x_n)\right] \rightarrow \Delta x = h = \frac{b-a}{n}$$

Romberg integration combines the trapezoidal rule with Richardson extrapolation.
Let us first introduce the notation $R_{i,1} = I_i$

where, as before, I_i represents the approximate value of computed by the recursive trapezoidal rule using 2^{i-1} panels.

Romberg integration starts with the computation of $R_{1,1} = I_1$ (one panel) and $R_{2,1} = I_2$ (two panels) from the trapezoidal rule.

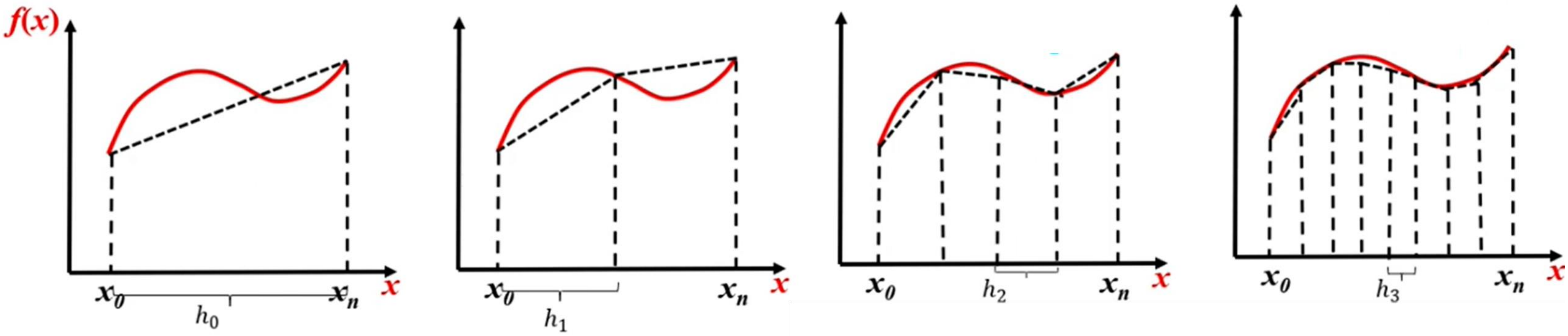
Romberg Integration

$$\int_a^b f(x)dx \approx \frac{\Delta x}{2}\left[f(x_0) + 2\sum_{i=1}^{n=1} f(x_i) + f(x_n)\right] \rightarrow \Delta x = \mathrm{h} = \frac{\mathrm{b-a}}{n}$$

$$R_{1,1} = \frac{h}{2}\left[f(a) + f(b)\right]$$

$$R_{2,1} = \frac{h}{4}\left[f(a) + 2f\left(a + \frac{h}{2}\right) + f(b)\right]$$

$$R_{3,1} = \frac{h}{8}\left[f(a) + f\left(a + \frac{h}{4}\right) + \cdots + f(b)\right]$$



Romberg Integration

Romberg integration starts with the computation of $R_{1,1} = I_1$ (one panel) and $R_{2,1} = I_2$ (two panels) from the trapezoidal rule.

The array has now expanded to

$$\begin{array}{cccc} \multicolumn{4}{c}{\begin{bmatrix} \boxed{\begin{matrix} R_{1,1} \\ R_{2,1} \\ R_{3,1} \\ R_{4,1} \end{matrix}} & \boxed{\begin{matrix} \\ R_{2,2} \\ R_{3,2} \\ R_{4,2} \end{matrix}} & \boxed{\begin{matrix} \\ \\ R_{3,3} \\ R_{4,3} \end{matrix}} & \boxed{\begin{matrix} \\ \\ \\ R_{4,4} \end{matrix}} \end{bmatrix}} \\ O(h^2) & O(h^4) & O(h^6) & O(h^8) \end{array}$$

$O(h^2)$	$O(h^4)$	$O(h^6)$	$O(h^8)$
$R_{1,1}$			
$R_{2,1}$	$R_{2,2}$		
$R_{3,1}$	$R_{3,2}$	$R_{3,3}$	
$R_{4,1}$	$R_{4,2}$	$R_{4,3}$	$R_{4,4}$

$$R_{k,2} = R_{k,1} + \frac{R_{k,1} - R_{k-1,1}}{2^2 - 1}$$

$$R_{k,3} = R_{k,2} + \frac{R_{k,2} - R_{k-1,2}}{2^4 - 1}$$

$$R_{k,4} = R_{k,3} + \frac{R_{k,3} - R_{k-1,3}}{2^6 - 1}$$

$O(h^i)$ = error

Example

Calculate $\int_0^1 \frac{1}{1+x}\,dx$ using the Romberg method with $\mathrm{n} = 8$

n	0	1	2	3	4	5	6	7	8
x	0	0.125	0.25	0.375	0.5	0.625	0.75	0.875	1
$f(x)$	1	0.89	0.8	0.727	0.667	0.615	0.5174	0.533	0.5

Solution :

$O(h^2)$	$O(h^4)$	$O(h^6)$	$O(h^8)$
$R_{1,1}$			
$R_{2,1}$	$R_{2,2}$		
$R_{3,1}$	$R_{3,2}$	$R_{3,3}$	
$R_{4,1}$	$R_{4,2}$	$R_{4,3}$	$R_{4,4}$

$$R_{1,1} = \frac{h_0}{2}\left[f(a) + f(b)\right]$$

$$R_{1,1} = \frac{1}{2}\left[f(0) + f(1)\right]$$

$$R_{1,1} = \frac{1}{2}(1 + 0.5) = 0.75$$

$$R_{2,1} = \frac{h_1}{2}\left[f(a) + 2f\left(a + \frac{h}{2}\right) + f(b)\right]$$

$$R_{2,1} = \frac{0.5}{2}\left[f(0) + 2f(0.5) + f(1)\right]$$

$$R_{2,1} = \frac{0.5}{2}\left[1 + 2(0.667) + 0.5\right] = 0.70833$$

$$R_{3,1} = \frac{h_2}{2}\left[f(0) + 2[f(0.25) + f(0.5) + f(0.75)] + f(1)\right]$$

$$R_{3,1} = 0.69702$$

*Find $R_{4,1}$ with n = 8

Example

Solution :

$O(h^2)$	$O(h^4)$	$O(h^6)$	$O(h^8)$
0.75			
0.70833	$R_{2,2}$		
0.6972	$R_{3,2}$	$R_{3,3}$	
0.69412	$R_{4,2}$	$R_{4,3}$	$R_{4,4}$

$$R_{k,2} = R_{k,1} + \frac{R_{k,1} - R_{k-1,1}}{2^2 - 1}$$
$$R_{k,3} = R_{k,2} + \frac{R_{k,2} - R_{k-1,2}}{2^4 - 1}$$
$$R_{k,4} = R_{k,3} + \frac{R_{k,3} - R_{k-1,3}}{2^6 - 1}$$

$$R_{2,2} = R_{2,1} + \frac{R_{2,1} - R_{1,1}}{2^2 - 1} = 0.69445$$

$$R_{3,2} = R_{3,1} + \frac{R_{3,1} - R_{2,1}}{2^2 - 1} = 0.69325 \qquad R_{3,3} = R_{3,2} + \frac{R_{3,2} - R_{2,2}}{2^4 - 1} = 0.69317$$

$$R_{4,2} = R_{4,1} + \frac{R_{4,1} - R_{3,1}}{2^2 - 1} = 0.69315 \qquad R_{4,3} = R_{4,2} + \frac{R_{4,2} - R_{3,2}}{2^4 - 1} = 0.69314$$

$$\boxed{R_{4,4} = R_{4,3} + \frac{R_{4,3} - R_{3,3}}{2^6 - 1} = 0.69314}$$

$O(h^2)$	$O(h^4)$	$O(h^6)$	$O(h^8)$	$O(h^{10})$	$O(h^{12})$	$O(h^{14})$
A_0						
A_1	B_1					
A_2	B_2	C_2				
A_3	B_3	C_3	D_3			
A_4	B_4	C_4	D_4	E_4		
A_5	B_5	C_5	D_5	E_5	F_5	
A_6	B_6	C_6	D_6	E_6	F_6	G_6

Best integration value

Gauss Integration

Gaussian formulas are also good at estimating integrals of the form :

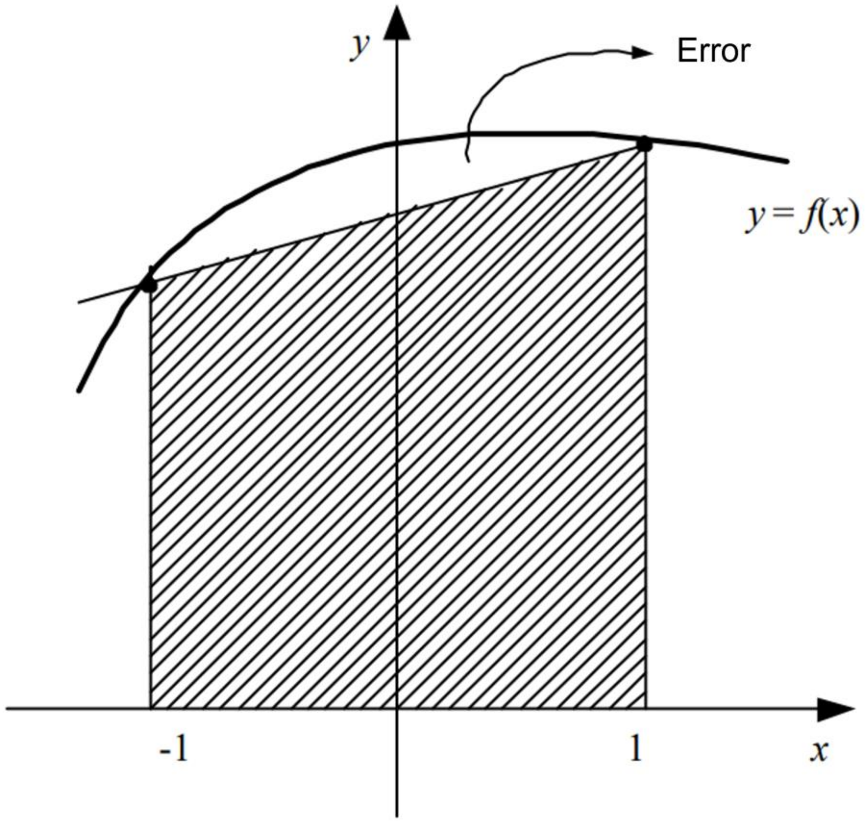
$$\int_a^b W(x)f(x)dx = \sum_{i=1}^{n} w_i f(x_i)$$

$W(x) = weight\ function,$
$w_i = weight,$
$x_i = nodes$

The Other Gaussian Integration :	
1. Gauss–Chebyshev Quadrature	$\int_{-1}^{1}(1-x^2)^{-\frac{1}{2}}f(x)dx \approx \frac{\pi}{n+1}\sum_{i=o}^{n} f(x_i)$
2. Gauss–Laguerre Quadrature	$\int_{0}^{\infty} e^{-x}f(x)dx \approx \sum_{i=o}^{n} A_i f(x_i)$
3. Gauss–Hermite Quadrature	$\int_{-\infty}^{\infty} e^{-x^2}f(x)dx \approx \sum_{i=o}^{n} A_i f(x_i)$
4. Gauss Quadrature with Logarithmic Singularity	$\int_{0}^{1} f(x)\ln(x)\,dx \approx -\sum_{i=o}^{n} A_i f(x_i)$
5. Gauss – Legendre	$\int_{-1}^{1} f(x)dx$

Gauss – Legendre Integration

Find $\int_0^1 f(x)dx$ with Trapezoide Method.

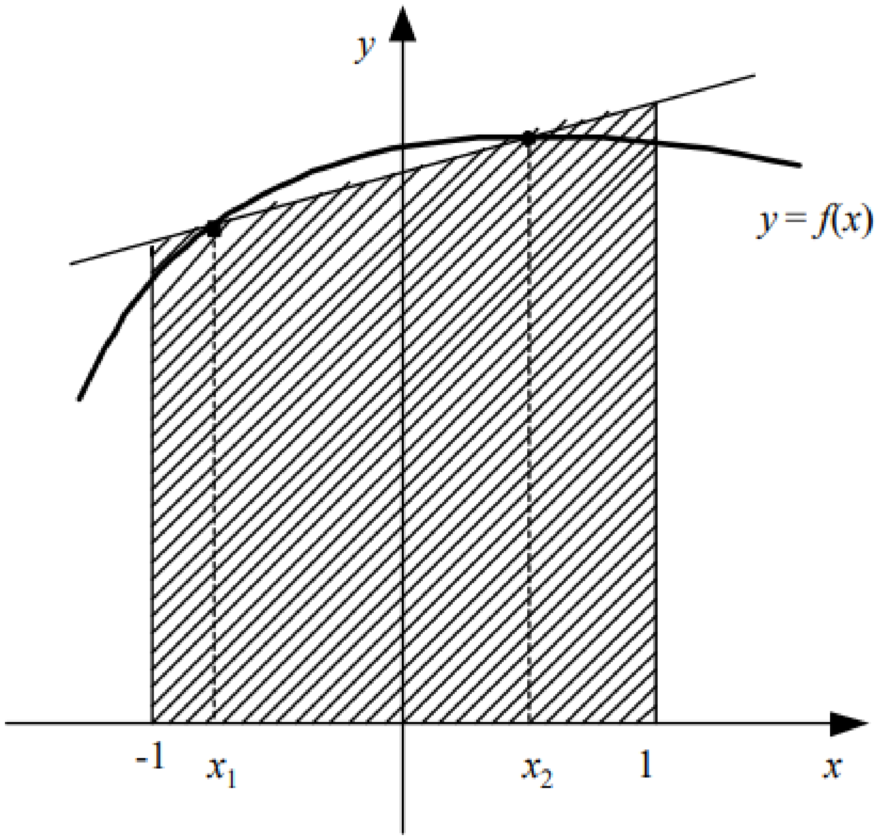


$$\int_{-1}^{1} f(x)dx \approx \boldsymbol{\frac{h}{2}[f(x_0) + f(x_1)] = \frac{h}{2}[f(1) - f(-1)]} \rightarrow \boldsymbol{h = \frac{b-a}{n}}$$

$$\int_{-1}^{1} f(x)dx \approx \boldsymbol{[f(1) + f(-1)]} \qquad \boldsymbol{\ldots(1)}$$

$$I \approx c_1 f(a) + c_2(b) \rightarrow a = -1, b = 1, c_1 = c_2 = \frac{h}{2} \qquad \ldots(2)$$

Gauss – Legendre Integration

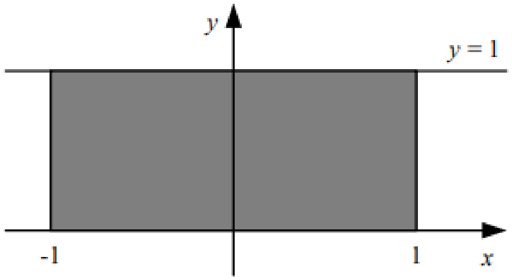


To give an idea of Gaussian quadrature, consider Figure. A straight line is drawn connecting any two points on the curve $y = f(x)$. The points are arranged so that the straight line balances the positive and negative errors. The area calculated now is the area under the straight line, expressed as:

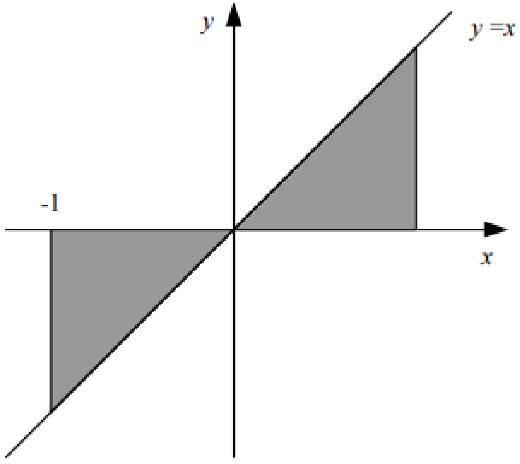
$$\boldsymbol{I = \int_{-1}^{1} f(x)dx \approx c_1 f(x_1) + c_2 f(x_2)}$$

$\rightarrow with\ c_1, c_2, x_1, and\ x_2\ are\ any\ values$

Gauss – Legendre Integration



(a) $\int_{-1}^{1} dx$



(b) $\int_{-1}^{1} x dx$

$$I = \int_{-1}^{1} f(x)dx \approx c_1 f(x_1) + c_2 f(x_2)$$

Since there are four unknowns, we must have four simultaneous equations containing x_1, x_2, c_1, and c_2.

For $f(x) = 1,\ f(x) = x, f(x) = x^2$, and $f(x) = x^3$

$$f(x) = 1 \rightarrow c_1 + c_2 = \int_{-1}^{1} 1\, dx = x \Big|_{-1}^{1} = 1 - (-1) = 2$$

$$f(x) = x \rightarrow c_1 x_1 + c_2 x_2 = \int_{-1}^{1} x\, dx = \frac{1}{2} x^2 \Big|_{-1}^{1} = 0$$

$$f(x) = x^2 \rightarrow c_1 x_1^2 + c_2 x_2^2 = \int_{-1}^{1} x^2\, dx = \frac{1}{3} x^3 \Big|_{-1}^{1} = \frac{2}{3}$$

$$f(x) = x^3 \rightarrow c_1 x_1^3 + c_2 x_2^3 = \int_{-1}^{1} x^3\, dx = \frac{1}{4} x^4 \Big|_{-1}^{1} = 0$$

Gauss – Legendre Integration

Now, we have four simultaneous equations

$$\boldsymbol{c_1 + c_2 = 2}$$
$$\boldsymbol{c_1 x_1 + c_2 x_2 = 0}$$
$$\boldsymbol{c_1 x_1^2 + c_2 x_2^2 = \frac{2}{3}}$$
$$\boldsymbol{c_1 x_1^3 + c_2 x_2^3 = 0}$$

So that,

$$c_1 = c_2 = 1, x_1 = \frac{1}{\sqrt{3}}, \text{and } \; x_2 = -\frac{1}{\sqrt{3}}$$

$$\boldsymbol{I = \int_{-1}^{1} f(x)dx \approx f\left(\frac{1}{\sqrt{3}}\right) + f\left(-\frac{1}{\sqrt{3}}\right)}$$

The above equation is called the 2-point Gauss-Legendre rule. With this rule, calculating the integral $f(x)$ in the interval [-1, 1] simply evaluates the value of the function f at $x = \frac{1}{\sqrt{3}}$ and at $x = -\frac{1}{\sqrt{3}}$

Transformasi $\int_a^b f(x)dx$ to $\int_{-1}^1 f(t)dt$

We must do the transformation:

1. Interval [a, b] becomes interval [-1, 1]
2. The x variable becomes the t variable
3. Differential dx to dt

The intervals [a, b] and [-1, 1] are depicted by the following line diagram:

From the two lines charts we make a comparison:

$$\frac{x-a}{b-a} = \frac{t-(-1)}{1-(-1)} \rightarrow \frac{x-a}{b-a} = \frac{t+1}{2}$$

$$\rightarrow 2x - 2a = (t+1)(b-a)$$

$$\rightarrow x = \frac{bt - at + b - a + 2b}{2}$$

$$\rightarrow \ x = \frac{a + b + bt - at}{2}$$

$$\rightarrow x = \frac{(a+b)+(b-a)t}{2} \quad \rightarrow dx = \frac{b-a}{2}dt$$

Transformasi $\int_a^b f(x)dx$ to $\int_{-1}^1 f(t)dt$

$$\rightarrow x = \frac{(a+b)+(b-a)t}{2}$$

$$\rightarrow dx = \frac{b-a}{2}dt$$

Transformation $\int_a^b f(x)dx$ to $\int_{-1}^1 f(t)dt$ is performed by substituting x and dx into $\int_a^b f(x)dx$

$$\int_a^b f(x)dx = \int_{-1}^{1} f\left(\frac{(a+b)+(b-a)t}{2}\right)\frac{b-a}{2}dt = \frac{b-a}{2}dt \int_{-1}^{1} f\left(\frac{(a+b)+(b-a)t}{2}\right)dt$$

Example

Find $\int_1^2 (x^2+1)\,dx$ with Gauss – Legendre two point.

Solution :

#step 1 (find a and b)

$a = 1$

$b = 2$

#step 2 (find x and dx)

$$\rightarrow x = \frac{(a+b)+(b-a)t}{2}$$

$$x = \frac{(1+2)+(2-1)t}{2} = 1.5 + 0.5t$$

$$\rightarrow dx = \frac{b-a}{2}dt = \frac{2-1}{2}dt$$

$$dx = 0.5\ d$$

Example

#step 3 (transformation)

Transformation $\int_1^2 f(x)dx$ to $\int_{-1}^1 f(t)dt$

$$\int_1^2 (x^2+1)\,dx = \int_{-1}^1 [(1.5+0.5t)^2+1]\,0.5dt = 0.5\int_{-1}^1 (1.5+0.5t)^2+1)\,dt$$

$f(x)$= $(1.5+0.5t)^2+1)$

$$f\left(\frac{1}{\sqrt{3}}\right) = \left(1.5+0.5\left(\frac{1}{\sqrt{3}}\right))^2+1\right) = 4.199$$

$$f\left(-\frac{1}{\sqrt{3}}\right) = \left(1.5+0.5\left(-\frac{1}{\sqrt{3}}\right))^2+1\right) = 2.467$$

$$\int_1^2 (x^2+1)\,dx = 0.5\int_{-1}^1 (1.5+0.5t)^2+1)\,dt = 0.5\left(f\left(\frac{1}{\sqrt{3}}\right)+f\left(-\frac{1}{\sqrt{3}}\right)\right) \approx \mathbf{3.333}$$

Metode Gauss-Legendre n-titik

$$\int_{-1}^{1} f(x)dt \approx c_1\, f(x_1) + c_2\, f(x_2) + \ldots + c_n f(x_n)$$

n	Faktor bobot	Argumen fungsi	Galat pemotongan
2	$c_1 = 1.000000000$ $c_2 = 1.000000000$	$x_1 = -0.577350269$ $x_2 = 0.577350269$	$\approx f^{(4)}(c)$
3	$c_1 = 0.555555556$ $c_2 = 0.888888889$ $c_3 = 0.555555556$	$x_1 = -0.774596669$ $x_2 = 0$ $x_1 = 0.774596669$	$\approx f^{(6)}(c)$
4	$c_1 = 0.347854845$ $c_2 = 0.652145155$ $c_3 = 0.652145155$ $c_3 = 0.347854845$	$x_1 = -0.861136312$ $x_2 = -0.339981044$ $x_3 = 0.339981044$ $x_4 = 0.861136312$	$\approx f^{(8)}(c)$
5	$c_1 = 0.236926885$ $c_2 = 0.478628670$ $c_3 = 0.568888889$ $c_4 = 0.478628670$ $c_5 = 0.236926885$	$x_1 = -0.906179846$ $x_2 = -0.538469310$ $x_3 = 0$ $x_4 = 0.538469310$ $x_5 = 0.906179846$	$\approx f^{(10)}(c)$
6	$c_1 = 0.171324492$ $c_2 = 0.360761573$ $c_3 = 0.467913935$ $c_4 = 0.467913935$ $c_5 = 0.360761573$ $c_6 = 0.171324492$	$x_1 = -0.932469514$ $x_2 = -0.661209386$ $x_3 = -0.238619186$ $x_4 = 0.238619186$ $x_5 = 0.661209386$ $x_6 = 0.932469514$	$\approx f^{(12)}(c)$

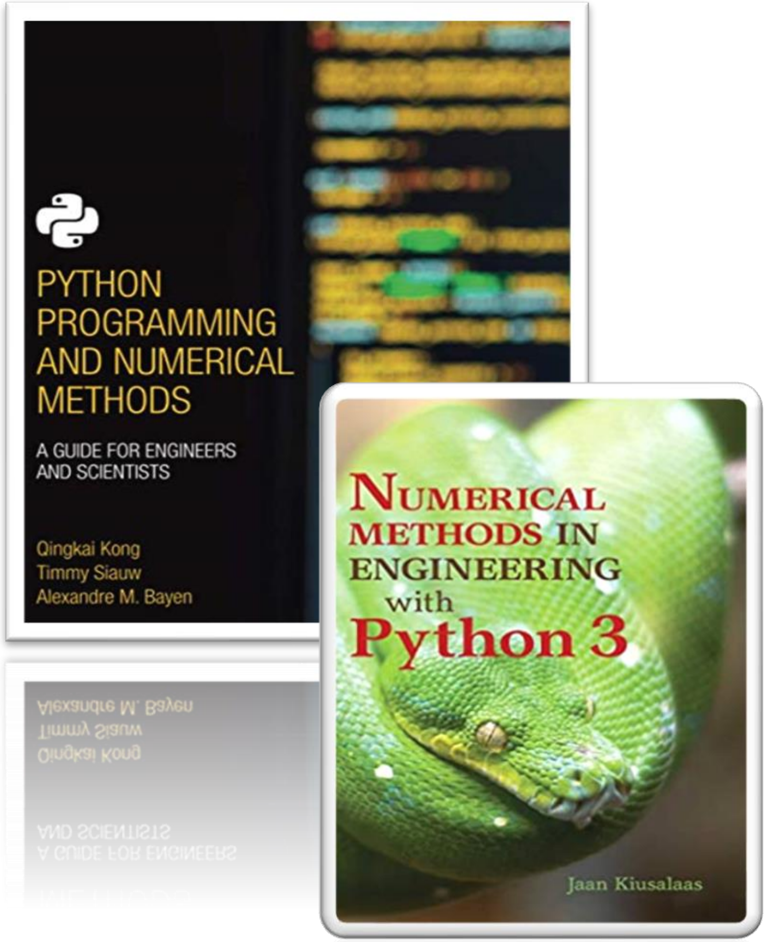
Exercise

1. Approximate $\int_0^{10} x^2\, dx$ using the trapezoidal method with 5 strips

2. Approximate $\int_0^{20} x^3\, dx$ using the simpson 1/3 method with n=4

3. Approximate $\int_0^{1} \frac{1}{1+x^2}\, dx$ using the simpson 3/8 method with n=6

Acknowledgement



These slides have been adapted from:

Kong, Q., Siauw, T., & Bayen, A. M. (2021). Python Programming and Numerical Methods: A Guide for Engineers and Scientists. Academic Press. ISBN: 978-0-12-819549-9

Kiusalaas, J. (2013). Numerical Methods in Engineering with Python 3. United Kingdom: Cambridge University Press. ISBN:9781107033856

additional materials

Chapra, S.C (2015). Numerical Methods for Engineers. 6st Edition. McGraw-Hill Companies, Inc . New York. ISBN. 978-981-4670-87