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Rotation Vector in Attitude Estimation

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An alternative derivation of the spacecraft attitude determination filter is developed to avoid questions of quaternion normalization or attitude matrix orthogonality constraints, quaternion covariance, and subterfuges used to circumvent these problems. This derivation is based on the Bortz equation for the rotation vector. Because the rotation vector is an unconstrained representation of attitude, the aforementioned questions do not arise. Singularities in the state dynamics equation are avoided by maintaining the predicted body attitude as the inertial reference for the filter. A simple discrete solution to the Bortz equation provides accurate attitude propagation for highly maneuverable spacecraft and also in the presence of jitter.

Introduction

SEQUENTIAL attitude estimators are most often based on the extended Kalman filter (EKF) where the attitude model is based on the quaternion kinematics equation operating in a model replacement mode,[1] where the gyro measurements replace the Euler equation that would otherwise model the dynamics of the angular rate. The first comprehensive exposition of the quaternion-based attitude determination filter is the renowned Lefferts–Markley–Shuster paper[1] in which both the full-quaternion and the reduced-order body-referenced filters are derived. Other approaches to attitude parameterization have been considered by various authors, for example, Euler angle parameterization,[2,3] Rodrigues parameters[4] (also called the Gibbs vector), and modified Rodrigues parameters.[5] Because these state vectors are allowed to represent any attitude, they can approach a singularity at certain attitudes. The covariance grows large near the singularity even though the attitude estimation error (in terms of body-axis rotation error) might be small, and so numerical difficulties can be expected. State estimate propagation also requires the solution of a nonlinear differential equation, which is also problematic near a singularity. Integrated-rate parameters were used in Refs. 6 and 7 for propagation and update of a direction cosine matrix (DCM), but the algorithm requires more computation in addition to a costly DCM orthogonalization step.

The various kinematic representations of attitude necessitate implementation in an EKF. As a result, the full- and reduced-order quaternion estimators perform an additive quaternion update[1] (which can be shown to be equivalent to a nonnormalized multiplicative update). Quaternion normalization is lost in the update and must be restored in a normalization step.[8,9] One difficulty with the derivation in Ref. 1 is that the quaternion covariance was assumed a priori to be singular, which might not be correct and makes the derivation less rigorous.

The objective of this paper is to present an alternative derivation of the spacecraft attitude determination filter that is not subject to conceptual difficulties of quaternion estimation and questions of quaternion normalization or attitude matrix orthogonality constraints and the various subterfuges that attempt to circumvent these problems. The derivation in this paper is based on the Bortz equation for the rotation vector.[10–12] The idea in this approach is to maintain the attitude information in nonsingular storage (either a quaternion or a direction cosine matrix) and the attitude error in the three-dimensional rotation vector in the filter. The expected value of the a priori estimate of the rotation vector is maintained at zero by moving the attitude information from the rotation vector to the nonsingular storage immediately after each update. The estimation error is the difference between the true rotation vector and the estimate, the estimate being the zero rotation vector. The covariance matrix represents the uncertainty in this estimate. Because the estimated rotation vector is zero before a measurement update and is a small incremental attitude estimate after a measurement update (under normal operating conditions), it is always far away from a kinematic singularity, a little less than 2π radians away. Thus, both the state and the covariance are well behaved. The development in this paper can be considered a special case of the development in Ref. 13, where the Gibbs vector is favored as the three-dimensional attitude representation. Because all small angle attitude error representations are equivalent to first order, with the exception of the Euler angles,[13] the rotation vector filter might be familiar to the reader. Fundamental differences lie only in its derivation, namely, issues of covariance constraints and quaternion normalization or attitude matrix orthogonality are avoided entirely.

A generally minor concern is the effect of coning error, which occurs when the angular rate vector is not aligned with the rotation vector that describes the change in attitude over the gyro sample interval and when this effect is not taken into account in propagation of the attitude estimate. Coning is generated by arbitrary spacecraft maneuvers and spacecraft jitter. Although coning is negligible for most spacecraft, it is a nonnegligible effect for fast, highly maneuverable precision-pointing spacecraft. This can be of importance also when performing alignment calibration[14,15] for maneuvering spacecraft because state propagation errors can bias the calibration estimates. The advantage to using the rotation vector as the attitude parameterization is that accurate attitude propagation algorithms have been developed and proven in aircraft navigation systems where coning error is a major concern.[11] Note that some gyros such as the Honeywell YG9666C compensate internally for coning and output a rotation vector rather than an integrated rate vector. Coning compensation can be adapted to other attitude determination filters such as those in Ref. 1.

A brief overview of some notation and definitions is given in the next section. We then introduce the rotation vector, whose kinematics are governed by the Bortz equation.[10–12,16] An attitude determination filter based on the Bortz equation is then derived, and we present discrete solutions to the Bortz equation for accurate attitude propagation.

Notation and Definitions

Let $\boldsymbol{p}$, $\boldsymbol{q}$, and $\boldsymbol{r}$ be unit quaternions, and let $\mathbf{T}_p$, $\mathbf{T}_q$, and $\mathbf{T}_r$ be corresponding transformation matrices (also called DCMs). The vector part of $\boldsymbol{q} = (q_x, q_y, q_z, q_s)^T$ is $\boldsymbol{q}_v = (q_x, q_y, q_z)^T$, and the scalar part

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is q_s. The operator $\otimes$ and operator matrix $[\boldsymbol{p}\otimes]$ are defined such that

$$\boldsymbol{r} = \boldsymbol{p} \otimes \boldsymbol{q} = [\boldsymbol{p}\otimes]\boldsymbol{q} = \begin{bmatrix} p_s & p_z & -p_y & p_x \\ -p_z & p_s & p_x & p_y \\ p_y & -p_x & p_s & p_z \\ -p_x & -p_y & -p_z & p_s \end{bmatrix} \begin{bmatrix} q_x \\ q_y \\ q_z \\ q_s \end{bmatrix} \tag{1}$$

and $\boldsymbol{T}_r = \boldsymbol{T}_p \boldsymbol{T}_q$. An $\circledast$ operator and operator matrix $[\boldsymbol{q}\circledast]$ are also defined such that $\boldsymbol{r} = \boldsymbol{q} \circledast \boldsymbol{p} = [\boldsymbol{q}\circledast]\boldsymbol{p}$ gives the same result, only the order of the quaternions is reversed. Likewise, the vector cross-product operator matrix $[\boldsymbol{u}\times]$ is defined such that $[\boldsymbol{u}\times]\boldsymbol{v} = \boldsymbol{u} \times \boldsymbol{v}$. A unit quaternion is one with unit norm, that is, $\boldsymbol{q}$ is a unit quaternion if $|\boldsymbol{q}| = 1$. The conjugate $\boldsymbol{p}^*$ of a quaternion $\boldsymbol{p} = (p_x, p_y, p_z, p_s)^T$ is $\boldsymbol{p}^* = (-p_x, -p_y, -p_z, p_s)^T$. For a unit quaternion $\boldsymbol{p}^* = \boldsymbol{p}^{-1}$ so that $\boldsymbol{p}^* \otimes \boldsymbol{p} = \boldsymbol{p} \otimes \boldsymbol{p}^* = (0, 0, 0, 1)^T$ and similarly for the $\circledast$ operator. All quaternions herein are unit quaternions. Note that $[\boldsymbol{q}\otimes]$ is an orthogonal matrix because $[\boldsymbol{q}\otimes][\boldsymbol{q}^*\otimes] = [\boldsymbol{q}\otimes][\boldsymbol{q}\otimes]^T = \boldsymbol{I}$ and $[\boldsymbol{q}\otimes]^{-1} = [\boldsymbol{q}\otimes]^T$. The $[\boldsymbol{q}\circledast]$ operator matrix is also an orthogonal matrix.

As discussed in many books and papers on kinematics, for example, Refs. 17 and 18, a transformation from one coordinate frame to another can be represented as a rotation about a single axis. Let $\boldsymbol{\phi}$ be a rotation vector, $\varphi = |\boldsymbol{\phi}|$, and define $\breve{\boldsymbol{\phi}} = [\begin{smallmatrix}\boldsymbol{\phi}\\0\end{smallmatrix}]$. For any quaternion $\boldsymbol{q}$ there exists a rotation vector $\boldsymbol{\phi}$ such that

$$\boldsymbol{q}(\boldsymbol{\phi}) = \begin{bmatrix} \dfrac{1}{2}\boldsymbol{\phi}\dfrac{\sin(\varphi/2)}{\varphi/2} \\ \cos(\varphi/2) \end{bmatrix} \tag{2}$$

$$[\boldsymbol{q}(\boldsymbol{\phi})\otimes] = \cos(\varphi/2)\boldsymbol{I} + \frac{\sin(\varphi/2)}{\varphi}[\breve{\boldsymbol{\phi}}\otimes] \tag{3}$$

Also for any attitude matrix A there exists a rotation vector $\boldsymbol{\phi}$ such that

$$\mathbf{A}(\boldsymbol{\phi}) = (\cos\varphi)\mathbf{I} - \frac{\sin\varphi}{\varphi}[\boldsymbol{\phi}\times] + \frac{1-\cos\varphi}{\varphi^2}\boldsymbol{\phi}\boldsymbol{\phi}^T \tag{4}$$

We will denote a 3×3 identity matrix by $\mathbf{I}$ and all other identity matrices by $\boldsymbol{I}$.

A vector $\boldsymbol{v}$ is represented in a frame a by the notation $\boldsymbol{v}^a$. A reference frame transformation from a frame a to a frame b is written $\mathbf{T}_a^b$. The same notation is used for other types of attitude representations, for example the quaternion $\boldsymbol{q}_a^b$ and the rotation vector $\boldsymbol{\phi}_a^b$.

Let $\boldsymbol{\phi}_a^b$ represent the rotation vector describing the attitude of a frame b with respect to a frame a, and let $\boldsymbol{\phi}_b^c$ represent the rotation vector describing the attitude of a frame c with respect to a frame b. The attitude matrix representing the attitude of frame c with respect to frame a is given by

$$\mathbf{A}(\boldsymbol{\phi}_a^c) = \mathbf{A}(\boldsymbol{\phi}_b^c)\mathbf{A}(\boldsymbol{\phi}_a^b) \tag{5}$$

Similarly, the attitude quaternion representing the attitude of frame c with respect to frame a is given by

$$\boldsymbol{q}(\boldsymbol{\phi}_a^c) = \boldsymbol{q}(\boldsymbol{\phi}_b^c) \otimes \boldsymbol{q}(\boldsymbol{\phi}_a^b) \tag{6}$$

When reference is made to attitude, we generally mean the orientation of the body frame with respect to an inertial reference frame (e.g., J2000), and the frame notation is omitted. A particular attitude quaternion or attitude matrix will be tagged with a subscript, for example, $\boldsymbol{q}_k$ or $\mathbf{A}_k$, or will have a tagged argument, for example, $\boldsymbol{q}(\boldsymbol{\phi}_k)$ or $\mathbf{A}(\boldsymbol{\phi}_k)$.

The notation $\mathcal{E}\{\cdot\}$ means to take the expected value of the argument. The underlying density function is implied by the context in which the expectation operator is used. An estimate of the state $\boldsymbol{x}_k$ at time t_k conditioned on measurements $\{\boldsymbol{y}_j, \boldsymbol{y}_{j-1}, \ldots, \boldsymbol{y}_1\}$ will be written $\hat{\boldsymbol{x}}_{k|j}$, whereas an estimate with no specific time or conditioning will be written with only a caret (e.g., $\hat{\boldsymbol{x}}$). The notation $\hat{\boldsymbol{x}}_{k|j}$ is redundant because the subscript indicates an estimate, but the caret is retained for clarity to aid the reader. We realize that there is some ambiguity in the notation for the estimate of attitude, for example, $\widehat{\boldsymbol{q}_i^b}$ vs $\hat{\boldsymbol{q}}_i^b$ vs $\boldsymbol{q}_{\hat{i}}^b$ vs $\hat{\boldsymbol{q}}_i^{\hat{b}}$. Because it is the orientation of an attitude frame that is of interest, perhaps $\boldsymbol{q}_i^{\hat{b}}$ (the superscript is $\hat{b}$) is the most precise notation. However, we will use the more common notation $\hat{\boldsymbol{q}}_i^b$ with the understanding that it is the attitude frame b that is estimated. Similarly, for vectors we will write $\hat{\boldsymbol{v}}^s$ with the understanding that the frame s is estimated.

Rotation Vector in Attitude Estimation

Gyro Measurement Model

For simplicity, we will assume that we have three single-axis rate-integrating gyros whose sensitive axes are aligned with the orthogonal axes of the spacecraft body reference frame, and we will assume that the scale factor error is zero. We will refer to this set of gyros as simply "the gyro." The true angular rate vector is denoted $\boldsymbol{\omega}$, and the measurement is denoted $\boldsymbol{\omega}_g$. The rate measurement is related to the true angular rate by

$$\boldsymbol{\omega}_g = \boldsymbol{\omega} - \boldsymbol{b} - \boldsymbol{\eta}_a \tag{7}$$

so that

$$\boldsymbol{\omega} = \boldsymbol{\omega}_g + \boldsymbol{b} + \boldsymbol{\eta}_a \tag{8}$$

where $\boldsymbol{b}$ is the gyro bias vector and $\boldsymbol{\eta}_a$ is a white-noise process with autocovariance $\mathcal{E}\{\boldsymbol{\eta}_a(t)\boldsymbol{\eta}_a^T(\tau)\} = \Sigma_a\delta(t-\tau) = \sigma_a^2\mathbf{I}\delta(t-\tau)$, where σ_a^2 is the "angle random walk variance" and $\delta(t)$ is the Dirac delta function. [The bias is defined in Eq. (7) with the opposite of the usual sign convention to eliminate negative signs in later derivations.] The gyro bias can be modeled by the rate random walk model

$$\frac{\mathrm{d}\boldsymbol{b}}{\mathrm{d}t} = \boldsymbol{\eta}_r \tag{9a}$$

or by the first-order Gauss–Markov model

$$\frac{\mathrm{d}\boldsymbol{b}}{\mathrm{d}t} = -\frac{1}{\tau}\boldsymbol{b} + \boldsymbol{\eta}_r \tag{9b}$$

where $\boldsymbol{\eta}_r$ is a white-noise process with autocovariance $\varepsilon\{\boldsymbol{\eta}_r(t)\boldsymbol{\eta}_r^T(\tau)\} = \Sigma_r\delta(t-\tau) = \sigma_r^2\mathbf{I}\delta(t-\tau)$ and where σ_r^2 is the "rate random walk variance." The rate random walk model (9a) is chosen for simplicity.

Kinematics of the Rotation Vector

Rate-integrating gyros produce an angular increment vector $\boldsymbol{\theta} = \int_{t_{k-1}}^{t_k} \boldsymbol{\omega}\,\mathrm{d}t$ over a short interval of time $[t_{k-1}, t_k]$, whereas the attitude frame rotation is $\boldsymbol{\phi} = \int_{t_{k-1}}^{t_k} \dot{\boldsymbol{\phi}}\,\mathrm{d}t$. But $\boldsymbol{\phi} \neq \boldsymbol{\theta}$ unless $\boldsymbol{\omega}$ is constant in direction. The time derivative of a rotation vector $\boldsymbol{\phi}$ as a function of $\boldsymbol{\omega}$ is given by the Bortz equation[10–12]:

$$\dot{\boldsymbol{\phi}} = \boldsymbol{\omega} + \frac{1}{2}\boldsymbol{\phi} \times \boldsymbol{\omega} + \frac{1}{\varphi^2}\left[1 - \frac{\varphi\sin\varphi}{2(1-\cos\varphi)}\right]\boldsymbol{\phi} \times (\boldsymbol{\phi} \times \boldsymbol{\omega}) \tag{10}$$

It is evident that the Bortz equation exhibits a pole at $\varphi = 2n\pi$, $n = \pm 1, \pm 2, \ldots$. This is simply a result of the fact[16] that there is no three-dimensional parameterization of attitude that is both global and nonsingular. This presents no difficulty for us because we will design an attitude determination filter based on small rotations (so that the rotation vector is a locally nonsingular three-dimensional parameterization) and we will accumulate the incremental attitude in a quaternion so that φ never gets large. (A DCM could also be used for storage, but the quaternion is easier to use.) For small φ it can be shown that the Bortz equation is well approximated by[11]

$$\dot{\boldsymbol{\phi}} = \boldsymbol{\omega} + \tfrac{1}{2}\boldsymbol{\phi} \times \boldsymbol{\omega} + \tfrac{1}{12}\boldsymbol{\phi} \times (\boldsymbol{\phi} \times \boldsymbol{\omega}) \tag{11}$$

which we have defined as an equality for practical use. This equation with the last term omitted is known as the coning equation[11]

$$\dot{\boldsymbol{\phi}} = \boldsymbol{\omega} + \tfrac{1}{2}\boldsymbol{\phi} \times \boldsymbol{\omega} \tag{12}$$

Observe also that although a composition rule $\phi = \phi_1 \circ \phi_2$ for rotation vectors can be written (see Appendix), it is not globally nonsingular. However, $\boldsymbol{q}(\phi) = \boldsymbol{q}(\phi_1 \circ \phi_2) = \boldsymbol{q}(\phi_1) \otimes \boldsymbol{q}(\phi_2)$ and $\mathbf{A}(\phi) = \mathbf{A}(\phi_1 \circ \phi_2) = \mathbf{A}(\phi_1)\mathbf{A}(\phi_2)$ can be regarded as nonsingular equivalents. We will make use of this fact in choosing the linearization point for the Bortz equation and also in formulating the attitude propagation and update equations.

Linearized State Equation

The Bortz equation is augmented with the bias state to form the state equation

$$\begin{bmatrix} \dot{\phi} \\ \dot{\boldsymbol{b}} \end{bmatrix} = \begin{bmatrix} \omega + \frac{1}{2}\phi \times \omega + \frac{1}{12}\phi \times (\phi \times \omega) \\ \mathbf{0}_{3 \times 1} \end{bmatrix} \tag{13}$$

where ω is given by Eq. (8). The augmented state equation can be written $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x})$.

The approximated Bortz equation (11) must be linearized about some state trajectory $\bar{\boldsymbol{x}}$ in order to build an EKF. Thus we seek the linearized state equation $\delta\dot{\bar{\boldsymbol{x}}} = F(\bar{\boldsymbol{x}})\delta\bar{\boldsymbol{x}}$, where

$$F(\bar{\boldsymbol{x}}) = \left.\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}^T}\right|_{x=\bar{x}} = \left.\begin{bmatrix} \dfrac{\partial \dot{\phi}}{\partial \phi^T} & \dfrac{\partial \dot{\phi}}{\partial \boldsymbol{b}^T} \\ \dfrac{\partial \dot{\boldsymbol{b}}}{\partial \phi^T} & \dfrac{\partial \dot{\boldsymbol{b}}}{\partial \boldsymbol{b}^T} \end{bmatrix}\right|_{x=\bar{x}} \tag{14}$$

We can consider the estimated body frame to be an instantaneously fixed inertial reference at the time of each measurement update (think of taking a snapshot of a moving frame). The Bortz equation then describes the evolution of the attitude relative to this inertial reference starting at $\phi(t_{k-1}) = \mathbf{0}$, where t_{k-1} is the time of the measurement update.

We seek a state transition matrix for the purpose of covariance propagation. The simplest linearization is about $\bar{\phi} = \mathbf{0}$, but with some effort we can define a better linearization than that. Consider the reference trajectory $\bar{\phi}(t)$ given by the solution to the Bortz equation for small rotations

$$\frac{\mathrm{d}}{\mathrm{d}t}\bar{\phi} = \hat{\omega} + \frac{1}{2}\bar{\phi} \times \hat{\omega} + \frac{1}{12}\bar{\phi} \times (\bar{\phi} \times \hat{\omega}) \tag{15}$$

in the time interval $t \in [t_{k-1}, t_k]$, where $\bar{\phi}(t_{k-1}) = \mathbf{0}$, $\omega = \omega_g + \hat{\boldsymbol{b}}$, is the angular rate reference, and $\hat{\boldsymbol{b}}$ is the bias reference. The bias reference trajectory $\hat{\boldsymbol{b}}$ will be the a priori estimated bias, as is customary in an EKF.

For the purpose of linearization, we can ignore the second-order term $\bar{\phi} \times (\bar{\phi} \times \omega)$ in Eq. (11) and $\bar{\phi} \times (\bar{\phi} \times \hat{\omega})$ in Eq. (15), and so the reference trajectory is approximately the solution to the coning equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\bar{\phi} = \hat{\omega} + \frac{1}{2}\bar{\phi} \times \hat{\omega} \tag{16}$$

The rotation error between the true rotation vector ϕ from Eq. (11) and the reference $\bar{\phi}$ from Eq. (15) is obtained from the composition rule for small rotations [Eq. (A4)]:

$$\begin{aligned} \delta\phi &= \phi \circ (-\bar{\phi}) \\ &\simeq \phi - \bar{\phi} + \tfrac{1}{2}\phi \times \bar{\phi} \end{aligned} \tag{17}$$

with $\delta\phi(t_{k-1}) = \mathbf{0}$. Differentiating this and simplifying (and omitting second-order terms) yields, with $\delta\boldsymbol{b} = \boldsymbol{b} - \hat{\boldsymbol{b}}$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\delta\phi = -\hat{\omega} \times \delta\phi + \delta\boldsymbol{b} + \boldsymbol{\eta}_a \tag{18}$$

This copacetic result is equivalent to Eq. (135) of Ref. 1, Sec. XI, where $\delta\phi = 2\delta\boldsymbol{q}$ for the $\delta\boldsymbol{q}$ defined therein. In fact, this result can be derived by using quaternions parameterized by the small-angle rotation vectors ϕ and $\bar{\phi}$; the procedure is the same as in Ref. 1, Sec. XI. Note that in both approaches the attitude perturbation is defined by a composition, $\delta\phi = \phi \circ (-\bar{\phi})$ and $\delta\boldsymbol{q} = \boldsymbol{q} \otimes \hat{\boldsymbol{q}}^*$. Therefore, the attitude update should be defined by a composition $\widehat{\delta\phi} \circ \hat{\phi}$ or $\widehat{\delta\boldsymbol{q}} \otimes \hat{\boldsymbol{q}}$ rather than an additive update. The estimation error for the rotation vector is, however, defined by $\delta\tilde{\phi} = \delta\phi - \widehat{\delta\phi}$. The a priori estimated $\widehat{\delta\phi}$ is zero and is updated additively by the EKF.

Equation (18) and the bias equation together in matrix form are

$$\begin{bmatrix} \delta\dot{\phi} \\ \delta\dot{\boldsymbol{b}} \end{bmatrix} = \begin{bmatrix} -[\hat{\omega}\times] & \mathbf{I} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \delta\phi \\ \delta\boldsymbol{b} \end{bmatrix} + \begin{bmatrix} \boldsymbol{\eta}_a \\ \boldsymbol{\eta}_r \end{bmatrix} \tag{19}$$

where, to first order, the coefficient matrix is $F(\bar{\boldsymbol{x}})$ in Eq. (14). The discrete-time solution to this equation is

$$\delta\boldsymbol{x}_k = \Phi_{k-1}\delta\boldsymbol{x}_{k-1} + \Gamma_{k-1}\boldsymbol{w}_{k-1} \tag{20}$$

where the state transition matrix $\Phi_{k-1} = \Phi_{k-1}(T_k) = \Phi(t_k, t_{k-1})$ is given by

$$\Phi_{k-1}(\tau) = \Phi(t_{k-1} + \tau, t_{k-1}) = \begin{bmatrix} R(\phi_{k-1}(\tau)) & S(\phi_{k-1}(\tau)) \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \tag{21}$$

evaluated at $\tau = T_k$, where $T_k = t_k - t_{k-1}$. The rotation vector $\phi_{k-1}(\tau)$ is, to first order, the solution to Eq. (16). This approximation is removed by letting $\phi_{k-1}(\tau)$ be the solution $\bar{\phi}(t_{k-1} + \tau)$ to Eq. (15) with initial condition $\bar{\phi}(t_{k-1}) = \mathbf{0}$. Define $\phi = \phi_{k-1}(\tau)$ and $\varphi = |\phi|$. The matrices R and S are given by

$$R(\phi) = (\cos\varphi)\mathbf{I} - \left(\frac{\sin\varphi}{\varphi}\right)[\phi\times] + \left(\frac{1-\cos\varphi}{\varphi^2}\right)\phi\phi^T \tag{22a}$$

$$S(\phi) = \tau\left[\left(\frac{\sin\varphi}{\varphi}\right)\mathbf{I} - \left(\frac{1-\cos\varphi}{\varphi^2}\right)[\phi\times] + \left(\frac{\varphi - \sin\varphi}{\varphi^3}\right)\phi\phi^T\right] \tag{22b}$$

The state transition matrix (21) is used only for propagating the EKF covariance according to

$$P_{k|k-1} = \Phi_{k-1}P_{k-1|k-1}\Phi_{k-1}^T + Q_{I_{k-1}} \tag{23}$$

where $Q_{I_{k-1}}$ is the discrete-time process noise covariance (of $\Gamma_{k-1}\boldsymbol{w}_{k-1}$). This is related to the continuous-time process noise by

$$\begin{aligned} Q_{I_{k-1}} = {} & \int_0^{T_{k-1}} \Phi_{k-1}(t)G_{k-1}(t)Q_{k-1}(t)G_{k-1}^T(t)\Phi_{k-1}^T(t)\,\mathrm{d}t \\ & + Q_{\mathrm{AWN}} \end{aligned} \tag{24}$$

where $Q_{k-1}(t) = \mathrm{diag}(\Sigma_a, \Sigma_r)$, $G_{k-1} = \boldsymbol{I}_6$, and

$$Q_{\mathrm{AWN}} = \begin{bmatrix} \sigma_w^2\mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \tag{25}$$

where σ_w^2 the variance of the angle white noise at the output of the gyro. The discrete-time process noise covariance matrix is, to first order,

$$Q_{I_{k-1}} = \begin{bmatrix} \left(\sigma_w^2 + T\sigma_a^2 + \dfrac{T^3}{3}\sigma_r^2\right)\mathbf{I} & \dfrac{T^2}{2}\sigma_r^2\mathbf{I} \\ \dfrac{T^2}{2}\sigma_r^2\mathbf{I} & T\sigma_r^2\mathbf{I} \end{bmatrix} \tag{26}$$

where $T = T_k$.

Filter State Propagation

In general, $\boldsymbol{\omega}$ is not colinear with $\boldsymbol{\phi}$ because its direction can vary with time. This is especially true of highly maneuverable spacecraft. Coning motion will reduce the accuracy of the attitude estimate unless the attitude reference is propagated at a very high rate. This is usually accomplished by propagating the attitude quaternion with each gyro sample, which is inefficient if it is performed at a high rate. Savage[11] gives general algorithms for numerically integrating the Bortz equation for attitude propagation using the gyro measurements. We present one of the simplest of such algorithms to propagate the reference state equation (15).

In this algorithm the gyro sampling and the attitude propagation are performed at a fast rate, and the attitude measurement and filter update are performed at a slow rate. Let T_f be the fast cycle interval, T_s the slow cycle interval, and m the number of fast cycles per slow cycle such that $T_f = T_s/m$. Let $\boldsymbol{\theta}_{k,\ell}$ be the gyro measurement at time $t_k + (\ell - 1)T_f$ and $\hat{\boldsymbol{\theta}}_{k,0} = \hat{\boldsymbol{\theta}}_{k-1,m}$. The following propagation algorithm from Ref. 11, Eqs. (46) and (47), is one of the simplest and most efficient that can account for coning. For $1 \le \ell \le m$,

$$\hat{\boldsymbol{\theta}}_{k,\ell} = \boldsymbol{\theta}_{k,\ell} + \hat{\boldsymbol{b}}_{k|k-1} T_f \tag{27a}$$

$$\boldsymbol{\alpha}_\ell = \boldsymbol{\alpha}_{\ell-1} + \hat{\boldsymbol{\theta}}_{k,\ell}, \qquad \boldsymbol{\alpha}_0 = 0 \tag{27b}$$

$$\boldsymbol{\beta}_\ell = \boldsymbol{\beta}_{\ell-1} + \tfrac{1}{2}\left(\boldsymbol{\alpha}_{\ell-1} + \tfrac{1}{6}\hat{\boldsymbol{\theta}}_{k,\ell-1}\right) \times \hat{\boldsymbol{\theta}}_{k,\ell}, \qquad \boldsymbol{\beta}_0 = 0 \tag{27c}$$

$$\hat{\boldsymbol{\phi}}^k_{k-1} = \boldsymbol{\alpha}_m + \boldsymbol{\beta}_m \tag{27d}$$

The predicted bias estimate is given by $\hat{\boldsymbol{b}}_{k|k-1} = \hat{\boldsymbol{b}}_{k-1|k-1}$. The term $\hat{\boldsymbol{b}}_{k|k-1} T_f$ compensates for bias in the gyro measurements. The already updated attitude reference is predicted according to either

$$\hat{\boldsymbol{q}}_{k|k-1} = \boldsymbol{q}\left(\hat{\boldsymbol{\phi}}^k_{k-1}\right) \otimes \hat{\boldsymbol{q}}_{k-1|k-1} \tag{28a}$$

or

$$\hat{\mathbf{A}}_{k|k-1} = \mathbf{A}\left(\hat{\boldsymbol{\phi}}^k_{k-1}\right)\hat{\mathbf{A}}_{k-1|k-1} \tag{28b}$$

The attitude prediction $\hat{\boldsymbol{\phi}}^k_{k-1}$ is thereby immediately put into nonsingular storage to avoid the potential nonsingularity of the rotation vector. Therefore the reference trajectory $\boldsymbol{\phi}_k$ is zero immediately after storage.

Remark: Although the subscripts $k|k-1$ and $k-1|k-1$ normally indicate a conditional expectation, it is emphasized here that $\hat{\boldsymbol{q}}_{k|k-1}$, $\hat{\mathbf{A}}_{k|k-1}$, etc., are not expectations of $\boldsymbol{q}(\boldsymbol{\phi})$ or $\mathbf{A}(\boldsymbol{\phi})$ because for any nonlinear function $f(x)$, $\hat{f}(x) \ne f(\hat{x})$.

A special case of the propagation algorithm suitable for most highly maneuverable spacecraft is for $m = 1$, where the fast cycle rate equals the slow cycle rate. In this case the gyro data can be simply summed to a slower rate, which is the same as integrating the angular rate over a longer interval. The incremental attitude is then

$$\hat{\boldsymbol{\phi}}^k_{k-1} = \hat{\boldsymbol{\theta}}_k + \tfrac{1}{12}\hat{\boldsymbol{\theta}}_{k-1} \times \hat{\boldsymbol{\theta}}_k \tag{29}$$

where $\hat{\boldsymbol{\theta}}_k = \boldsymbol{\theta}_k + \hat{\boldsymbol{b}}_{k|k-1} T_s$. Integration algorithms of higher order than the preceding are found in Refs. 10, 11, and 19 of Ref. 11.

Remark: The attitude determination filters developed in Refs. 6 and 7 utilize an attitude state transition matrix[19] that is closely related to Eq. (29) but which is nonorthogonal and requires a costly orthogonalization step, which is prohibitive at typical gyro sample rates. A quaternion version of the algorithm in Refs. 6 and 7 can be derived from Eq. (17-23) on p. 565 of Ref. 17 by approximating the derivative of the angular rate by a first-order difference of angular rates. The propagated attitude quaternion then requires normalization.

Filter State Update

The rotation vector estimate and bias perturbation estimate are updated according to

$$\begin{bmatrix} \widehat{\delta\boldsymbol{\phi}}_{k|k} \\ \widehat{\delta\boldsymbol{b}}_{k|k} \end{bmatrix} = \begin{bmatrix} \widehat{\delta\boldsymbol{\phi}}_{k|k-1} \\ \widehat{\delta\boldsymbol{b}}_{k|k-1} \end{bmatrix} + K_k \boldsymbol{\nu}_k \tag{30}$$

where

$$\begin{bmatrix} \widehat{\delta\boldsymbol{\phi}}_{k|k-1} \\ \widehat{\delta\boldsymbol{b}}_{k|k-1} \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \end{bmatrix} \tag{31}$$

is the a priori estimate of the rotation vector relative to the attitude frame represented by $\boldsymbol{q}_{k|k-1}$. We also have the Kalman gain matrix $K_k = P_{k|k-1} H_k^T (H_k P_{k|k-1} H_k^T + R_k)^{-1}$, where R_k is the measurement covariance. The residual $\boldsymbol{\nu}_k$ and the measurement sensitivity matrix H_k correspond to one of the measurement types defined in a later section.

The attitude reference is updated according to either

$$\hat{\boldsymbol{q}}_{k|k} = \boldsymbol{q}(\widehat{\delta\boldsymbol{\phi}}_{k|k}) \otimes \hat{\boldsymbol{q}}_{k|k-1} \tag{32a}$$

or

$$\hat{\mathbf{A}}_{k|k} = \mathbf{A}(\widehat{\delta\boldsymbol{\phi}}_{k|k})\hat{\mathbf{A}}_{k|k-1} \tag{32b}$$

As with the attitude prediction, the attitude update $\widehat{\delta\boldsymbol{\phi}}_{k|k}$ is immediately put into nonsingular storage to avoid the potential nonsingularity of the rotation vector, and $\widehat{\delta\boldsymbol{\phi}}_{k|k}$ is then reset to zero. For efficiency reasons multiple updates for the same measurement time may be accumulated in $\widehat{\delta\boldsymbol{\phi}}_{k|k}$ before the attitude information is moved into the quaternion for storage. The bias is updated according to $\hat{\boldsymbol{b}}_{k|k} = \widehat{\delta\boldsymbol{b}}_{k|k} + \hat{\boldsymbol{b}}_{k|k-1}$. There is no concern about quaternion approximation or normalization in the prediction and update except for machine roundoff.

Covariance Update

The covariance update is given by the usual Kalman filter equation

$$P_{k|k} = (I - K_k H_k) P_{k|k-1} \tag{33}$$

(See Ref. 20 for details of the UD factorized calculation of this update equation.) Note that the frame of reference for $P_{k|k-1}$ is the predicted attitude given by the solution (28b) [or (28a)] to Eq. (27d). The updated covariance $P_{k|k}$ is also referenced to the predicted attitude frame. The reference frame for $P_{k|k}$ should be updated from $\mathbf{A}_{k|k-1}$ to $\mathbf{A}_{k|k}$ by applying the rotation correction $\widehat{\delta\boldsymbol{\phi}}_{k|k}$. The frame correction is

$$P_{k|k} := \Psi_k P_{k|k} \Psi_k^T \tag{34}$$

where Ψ_k is given by

$$\Psi_k = \begin{bmatrix} R(\widehat{\delta\boldsymbol{\phi}}_{k|k}) & S(\widehat{\delta\boldsymbol{\phi}}_{k|k}) \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \tag{35}$$

For efficiency, this frame update can be combined with the attitude prediction [Eqs. (21) and (23)] on the next filter cycle by computing R and S from the combined rotation $\hat{\boldsymbol{\phi}}^{k+1}_k \circ \widehat{\delta\boldsymbol{\phi}}_{k|k}$.

Measurement Sensitivity Matrix

In this section we derive the measurement sensitivity matrix for quaternion measurements and for vector measurements. Because the measurements are independent of the gyro bias, the sensitivity matrices will have the form

$$H = [H^{(i)} \quad \mathbf{0}_{n\times 3}] \tag{36}$$

where $i = 1, 2, 3, \ldots$ is a tag for each type of sensor, as defined in what follows, and n is the number of scalar components in the measurement. For quaternion measurements we will have $n = 3$.

Sensitivity Matrix: Quaternion Measurements

The error in the quaternion "measurement" from a star tracker can be modeled as a rotational error $\boldsymbol{\epsilon}$ in the tracker frame. A small body attitude perturbation $\delta\boldsymbol{\phi}$ in the body frame is transformed to the tracker frame and added to the measurement error $\boldsymbol{\epsilon}$ to form the derived measurement

$$\boldsymbol{\phi}_{\text{meas}} = \mathbf{T}_b^s \delta\boldsymbol{\phi} + \boldsymbol{\epsilon} \tag{37}$$

where $\mathbf{T}_b^s$ is the body-to-sensor transformation matrix. A rigorous but rather lengthy derivation can be made by using quaternion algebra.[14,15] Equation (37) is accurate to first order, which is sufficient to compute the measurement sensitivity matrix

$$H^{(1)} = \mathbf{T}_b^s \tag{38}$$

The derived measurement $\boldsymbol{\phi}_{\text{meas}}$ is easily and accurately computed from the quaternion quotient

$$\boldsymbol{q}(\boldsymbol{\phi}_{\text{meas}}) = \boldsymbol{q}_i^m \otimes \left(\boldsymbol{q}_b^s \otimes \hat{\boldsymbol{q}}_i^b\right)^* \tag{39}$$

where $\boldsymbol{q}_i^m$ is the quaternion measurement at time k (which represents the attitude of the measured sensor frame m), $\boldsymbol{q}_b^s$ is the body-to-sensor transformation quaternion, and $\hat{\boldsymbol{q}}_i^b = \hat{\boldsymbol{q}}_{k|k-1}$ is the a priori estimated attitude quaternion given by Eq. (28a). Note that the derived measurement $\boldsymbol{\phi}_{\text{meas}}$ is also the residual $\boldsymbol{\nu}_k$ (also called the prediction error or innovations) because the a priori estimate of $\delta\boldsymbol{\phi}$ is zero. (The a priori estimate is zero unless multiple updates are performed without moving the estimate to nonsingular storage for efficiency reasons, in which case the residual is simply $\boldsymbol{\nu}_k = \boldsymbol{\phi}_{\text{meas}} - \widehat{\delta\boldsymbol{\phi}}$.)

If the filter is initialized such that $H_1 P_{1|0} H_1^T \gg R_1$, then $\widehat{\delta\boldsymbol{\phi}} \simeq (\mathbf{T}_b^s)^T \boldsymbol{\phi}_{\text{meas}}$ so that after moving the attitude information into the quaternion we get

$$\begin{aligned}
\hat{\boldsymbol{q}}_{1|1} &= \boldsymbol{q}(\widehat{\delta\boldsymbol{\phi}}) \otimes \hat{\boldsymbol{q}}_{1|0} \\
&= \boldsymbol{q}\left(\left(\boldsymbol{T}_b^s\right)^T \boldsymbol{\phi}_{\text{meas}}\right) \otimes \hat{\boldsymbol{q}}_{1|0} \\
&= \left(\boldsymbol{q}_b^s\right)^* \otimes \boldsymbol{q}_i^m
\end{aligned} \tag{40}$$

where the difference between the m frame and the s frame is the measurement noise. This filter is well behaved because the estimated quaternion equals the measured quaternion converted to the body frame when the initial reference attitude is far from the first measured attitude and the initial covariance is very large (ideally infinite). In general, nonlinear filters should be initialized with an estimate near the true state to avoid possible divergence problems. For vector sensors, which do not observe attitude about their line of sight, it is a matter of good general practice to initialize the attitude determination filter as described in the Initialization section.

Sensitivity Matrix: Focal Plane Measurements

Consider a vector $\boldsymbol{v}^s = [v_x^s,\ v_y^s,\ v_z^s]^T$ in the sensor reference frame s that is measured by some generally nonlinear function with additive noise $\boldsymbol{\epsilon}$:

$$\boldsymbol{y} = \boldsymbol{h}(\boldsymbol{v}^s) + \boldsymbol{\epsilon} \tag{41}$$

For a focal plane measurement

$$\boldsymbol{y} = \begin{bmatrix} v_x^s / v_z^s \\ v_y^s / v_z^s \end{bmatrix} + \boldsymbol{\epsilon} \tag{42}$$

We need to compute the measurement sensitivity matrix

$$H^{(2)} = \frac{\partial \boldsymbol{h}(\boldsymbol{v}^s)}{\partial \boldsymbol{\phi}^T} = \frac{\partial \boldsymbol{h}}{\partial (\boldsymbol{v}^s)^T} \frac{\partial \boldsymbol{v}^s}{\partial \boldsymbol{\phi}^T} \tag{43}$$

The vector $\boldsymbol{v}^s$ in the sensor reference frame is related to a vector $\boldsymbol{v}^i$ in the inertial reference frame i by

$$\boldsymbol{v}^s = \mathbf{T}_b^s \mathbf{A}\left(\boldsymbol{\phi}_i^b\right) \boldsymbol{v}^i \tag{44}$$

Taking the partial derivative with respect to $\boldsymbol{\phi}_i^b$ by direct partial differentiation is a rather laborious process, and so we will take a different tact. We choose a reference attitude $\mathbf{A}(\boldsymbol{\phi}_i^r)$ arbitrarily close to the body attitude $\mathbf{A}(\boldsymbol{\phi}_i^b)$ such that $\mathbf{A}(\boldsymbol{\phi}_i^b) = \mathbf{A}(\boldsymbol{\phi})\mathbf{A}(\boldsymbol{\phi}_i^r)$, where $\boldsymbol{\phi}$ is a small rotation vector from the reference attitude to the body attitude. Knowing that second-order terms will vanish in the limit as $\boldsymbol{\phi}_i^r \to \boldsymbol{\phi}_i^b$ (frame $r \to$ frame b), we can use the first-order approximation $\mathbf{A}(\boldsymbol{\phi}) \simeq \mathbf{I} - [\boldsymbol{\phi}\times]$, and so we have

$$\begin{aligned}
\boldsymbol{v}^s &= \mathbf{T}_b^s \mathbf{A}\left(\boldsymbol{\phi}_i^b\right) \boldsymbol{v}^i \\
&= \mathbf{T}_b^s \mathbf{A}(\boldsymbol{\phi}) \mathbf{A}\left(\boldsymbol{\phi}_i^r\right) \boldsymbol{v}^i \\
&= \mathbf{T}_b^s \mathbf{A}(\boldsymbol{\phi}) \boldsymbol{v}^r \\
&\simeq \mathbf{T}_b^s (\mathbf{I} - [\boldsymbol{\phi}\times]) \boldsymbol{v}^r \\
&= \mathbf{T}_b^s \boldsymbol{v}^r + \mathbf{T}_b^s [\boldsymbol{v}^r \times] \boldsymbol{\phi}
\end{aligned} \tag{45}$$

Noting that the r frame coincides with the b frame when $\boldsymbol{\phi} = 0$, we obtain

$$\left.\frac{\partial \boldsymbol{v}^s}{\partial \boldsymbol{\phi}^T}\right|_{\boldsymbol{\phi}=0} = \mathbf{T}_b^s [\boldsymbol{v}^b \times] \tag{46}$$

The measurement sensitivity matrix for the vector sensor is then

$$H^{(2)} = \begin{bmatrix} 1/v_z^s & 0 & -v_x^s/\left(v_z^s\right)^2 \\ 0 & 1/v_z^s & -v_x^s/\left(v_z^s\right)^2 \end{bmatrix} \mathbf{T}_b^s [\boldsymbol{v}^b \times] \tag{47}$$

where the vectors $\boldsymbol{v}^s = (v_x^s, v_y^s, v_z^s)^T$ and $\boldsymbol{v}^b$ are computed using the a priori estimate of the attitude quaternion. The predicted measurement is given by

$$\hat{\boldsymbol{y}}_k = \boldsymbol{h}\left(\hat{\boldsymbol{v}}_b^s\right) \tag{48}$$

where

$$\hat{\boldsymbol{v}}_b^s = \mathbf{T}_b^s \mathbf{A}\left(\hat{\boldsymbol{\phi}}_i^b\right) \boldsymbol{v}^i \tag{49}$$

and $\mathbf{A}(\hat{\boldsymbol{\phi}}_i^b) = \hat{\mathbf{A}}_{k|k-1}$ is the a priori estimated attitude from Eq. (28b). (Note that $\hat{\boldsymbol{v}}_b^s$ can be computed directly from $\hat{\boldsymbol{q}}_{k|k-1}$ so that $\hat{\mathbf{A}}_{k|k-1}$ does not also have to be computed.) The residual is simply $\boldsymbol{\nu}_k = \boldsymbol{y}_k - \hat{\boldsymbol{y}}_k$.

Sensitivity Matrix: Vector Measurements

The measurement function for a three-axis magnetometer is

$$\boldsymbol{y} = \boldsymbol{v}^s + \boldsymbol{\epsilon} \tag{50}$$

Because the three-axis magnetometer is a true vector sensor, all three axes are measured, and so $\partial\boldsymbol{h}/\partial(\boldsymbol{v}^s)^T = \mathbf{I}$. The measurement sensitivity matrix is

$$H^{(3)} = \mathbf{T}_b^s [\boldsymbol{v}^b \times] \tag{51}$$

The predicted measurement is given by

$$\hat{\boldsymbol{y}}_k = \hat{\boldsymbol{v}}_b^s = \mathbf{A}\left(\hat{\boldsymbol{\phi}}_i^b\right) \boldsymbol{v}^i \tag{52}$$

and the residual is simply $\boldsymbol{\nu}_k = \boldsymbol{y}_k - \hat{\boldsymbol{y}}_k$.

Initialization

A correct implementation of any nonlinear filter, including an attitude determination filter—correct in the sense of good design practice and numerically robust—would initialize the reference attitude with a quaternion measurement or a quaternion derived from vector measurements so that the initial attitude error is no greater than the measurement error. This avoids linearization problems and concomitant convergence problems. The attitude covariance is initialized to the measurement error covariance. The measurement update is then nowhere near a singularity, and subsequent prediction errors and state updates are small. A large initial bias error is of no consequence as long as the bias covariance is initialized accordingly.

Conclusions

The differential equation for the evolution of the rotation vector, known as the Bortz equation, was introduced as a kinematic model for attitude determination. Although the Bortz equation exhibits a singularity, as do all three-dimensional attitude parameterizations, singularity is avoided by storing the attitude information in a quaternion or direction cosine matrix (DCM) and by maintaining the rotation vector near zero. The quaternion or DCM itself is not an intrinsic part of the design of the filter—the attitude information could just as easily have been stored in any nonsingular attitude representation—and so technical issues associated with other approaches do not arise in the derivation in this paper.

In effect the filter does not know anything about the attitude quaternion or DCM. It knows only that before each update it starts at zero rotation with an a priori covariance of attitude error. Conceptually, this means that the attitude quaternion or DCM carries no error and the covariance matrix represents the error in the filter's estimate of attitude, where the filter's estimate always has zero mean but represents attitude relative to the quaternion or DCM. Said another way, once it is computed, the attitude quaternion or DCM is simply a perfectly known and convenient inertial reference frame for the filter.

Various numerical integration algorithms for the Bortz equation are available that account for coning, which occurs when the angular rate vector is not parallel to the rotation axis. Coning can be caused by certain spacecraft maneuvers or by vibration (jitter). Efficient low-order algorithms that effectively account for coning were introduced.

Appendix: Composition Rule for Rotation Vectors

A composition rule $\boldsymbol{\phi} = \boldsymbol{\phi}_1 \circ \boldsymbol{\phi}_2$ for two rotation vectors $\boldsymbol{\phi}_1$ and $\boldsymbol{\phi}_2$ is easily derived from the quaternion product $\boldsymbol{q}\boldsymbol{\phi} = \boldsymbol{q}(\boldsymbol{\phi}_1) \otimes \boldsymbol{q}(\boldsymbol{\phi}_2)$, where

$$\boldsymbol{q}(\boldsymbol{\phi}) = \begin{bmatrix} \boldsymbol{r} \\ s \end{bmatrix} = \begin{bmatrix} \dfrac{1}{2}\boldsymbol{\phi}\dfrac{\sin(\varphi/2)}{\varphi/2} \\ \cos(\varphi/2) \end{bmatrix} \tag{A1}$$

where $\varphi = |\boldsymbol{\phi}|$ and similarly for $\boldsymbol{q}(\boldsymbol{\phi}_1)$ and $\boldsymbol{q}(\boldsymbol{\phi}_2)$. The quaternion product is defined by

$$\boldsymbol{q}(\boldsymbol{\phi}_1) \otimes \boldsymbol{q}(\boldsymbol{\phi}_2) = \begin{bmatrix} s_1\boldsymbol{I} - [\boldsymbol{r}_1\times] & \boldsymbol{r}_1 \\ -\boldsymbol{r}_1^T & s_1 \end{bmatrix} \begin{bmatrix} \boldsymbol{r}_2 \\ s_2 \end{bmatrix} \tag{A2}$$

From these definitions it is easily shown that

$$\boldsymbol{\phi}\frac{\sin(\varphi/2)}{\varphi/2} = \boldsymbol{\phi}_1\frac{\sin(\varphi_1/2)}{\varphi_1/2}\cos(\varphi_2/2) + \boldsymbol{\phi}_2\frac{\sin(\varphi_2/2)}{\varphi_2/2}\cos(\varphi_1/2) - \frac{1}{2}\boldsymbol{\phi}_1 \times \boldsymbol{\phi}_2\frac{\sin(\varphi_1/2)}{\varphi_1/2}\frac{\sin(\varphi_2/2)}{\varphi_2/2} \tag{A3}$$

from which the composition rule can be written. The coefficient of $\boldsymbol{\phi}$ on the left renders $\boldsymbol{\phi}$ undefined when the angle of rotation φ reaches a nonzero multiple of 2π. This is a consequence of the fact that there is no three-parameter representation of attitude that is globally nonsingular.

For small φ_1 and small φ_2, we can make the approximation

$$\boldsymbol{\phi} = \boldsymbol{\phi}_1 + \boldsymbol{\phi}_2 - \tfrac{1}{2}\boldsymbol{\phi}_1 \times \boldsymbol{\phi}_2 \tag{A4}$$

Based on this result, a first-order attitude propagation (integration) algorithm can be written for rate-integrating gyros. It is only first order because the rotation vector is not equal to the integral of the angular rate in the presence of coning motion. For small angular increments this first-order algorithm is equivalent to accumulating the angular increments with the computationally more demanding quaternion product $\boldsymbol{q}(\boldsymbol{\phi}) = \boldsymbol{q}(\boldsymbol{\phi}_1) \otimes \boldsymbol{q}(\boldsymbol{\phi}_2)$.

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References

[1]Lefferts, E. J., Markley, F. L., and Shuster, M. D., "Kalman Filtering for Spacecraft Attitude Estimation," *Journal of Guidance, Control, and Dynamics*, Vol. 5, No. 5, 1982, pp. 417–429.

[2]Kau, S., Kumar, K., and Greenly, G., "Attitude Determination via Nonlinear Filtering," *IEEE Transactions on Aerospace and Electronic Systems*, Vol. AES-5, Nov. 1969, pp. 906–911.

[3]Bar-Itzhack, I. Y., and Idan, M., "Recursive Attitude Determination from Vector Observations: Euler Angle Estimation," *Journal of Guidance, Control, and Dynamics*, Vol. 10, No. 2, 1987, pp. 152–157.

[4]Idan, M., "Estimation of Rodrigues Parameters from Vector Observations," *IEEE Transactions on Aerospace and Electronic Systems*, Vol. AES-32, No. 2, 1996, pp. 578–586.

[5]Crassidis, J. L., and Markley, F. L., "Attitude Estimation Using Modified Rodrigues Parameters," *Flight Mechanics and Estimation Theory Symposium*, NASA CP-3333, 1996, pp. 71–83.

[6]Oshman, Y., and Markley, F. L., "Minimal-Parameter Attitude Matrix Estimation from Vector Observations," *Journal of Guidance, Control, and Dynamics*, Vol. 21, No. 2, 1998, pp. 595–602.

[7]Oshman, Y., and Markley, F. L., "Sequential Attitude and Attitude-Rate Estimation Using Integrated-Rate Parameters," *Journal of Guidance, Control, and Dynamics*, Vol. 22, No. 3, 1999, pp. 385–394.

[8]Deutschmann, J., Markley, F. L., and Bar-Itzhack, I., "Quaternion Normalization in Spacecraft Attitude Determination," *Proceedings of the AIAA Guidance, Navigation, and Control Conference*, AIAA, Washington, DC, 1991, pp. 908–916.

[9]Bar-Itzhack, I., Deutschmann, J., Markley, F. L., and "Quaternion Normalization in Spacecraft Attitude Determination," *Flight Mechanics and Estimation Theory Symposium*, 1992, pp. 441–454.

[10]Bortz, J. E., "A New Mathematical Formulation for Strapdown Inertial Navigation," *IEEE Transactions on Aerospace and Electronic Systems*, Vol. AES-7, No. 1, 1971, pp. 61–66.

[11]Savage, P. G., "Strapdown Inertial Navigation Integration Algorithm Design Part 1: Attitude Algorithms," *Journal of Guidance, Control, and Dynamics*, Vol. 21, No. 1, 1998.

[12]Shuster, M. D., "The Kinematic Equation of the Rotation Vector," *IEEE Transactions on Aerospace and Electronic Systems*, Vol. AES-29, No. 1, 1993, pp. 263–267.

[13]Markley, F. L., "Attitude Error Representations for Kalman Filtering," *Journal of Guidance, Control, and Dynamics*, Vol. 26, No. 2, 2003, pp. 311–317.

[14]Pittelkau, M. E., "Kalman Filtering for Spacecraft System Alignment Calibration," *Journal of Guidance, Control, and Dynamics*, Vol. 24, No. 6, 2001, pp. 1187–1195.

[15]Pittelkau, M. E., "Everything is Relative in System Alignment Calibration," *Journal of Spacecraft and Rockets*, Vol. 39, No. 3, 2002, pp. 460–466.

[16]Stuelpnagel, J., "On the Paramerization of the Three-Dimensional Rotational Group," *SIAM Review*, Vol. 6, No. 4, 1964, pp. 422–430.

[17]Wertz, J. R. (ed.), *Spacecraft Attitude Determination and Control*, Kluwer Academic, Boston, 1978, p. 565.

[18]Shuster, M. D., "A Survey of Attitude Representations," *Journal of the Astronautical Sciences*, Vol. 41, No. 4, 1993, pp. 439–517.

[19]Ronen, M., and Oshman, Y., "A Third-Order, Minimal-Parameter Solution of the Orthogonal Matrix Differential Equation," *Journal of Guidance, Control, and Dynamics*, Vol. 20, No. 3, 1997, pp. 516–521.

[20]Bierman, G. J., *Factorization Methods for Discrete Sequential Estimation*, Vol. 128, Mathematics in Science and Engineering, Academic Press, New York, 1981.