HOMEWORK 2

1. Do the vectors $(1,2),(2,1),(2,4),(4,2)$ span ${}_{\mathbb{Q}}\mathbb{Q}^2$? What about $(1,2),(2,3),(3,6),(4,8)$?

2. Consider the vectors $(1,2,3,4),(2,3,4,5),(3,4,5,6),(4,5,6,7) \in_{\mathbb{R}} \mathbb{R}^4$.
a) Are they linearly independent?
b) Does $(-3,-2,-1,0)$ belong to the subspace of $\mathbb{R}^4$ spanned by these vectors?
c) Find the subspace of $\mathbb{R}^4$ spanned by these vectors and determine its dimension
d) Find a basis for the subspace from c).

3. Let $V = \{(a,b,c,d) \in \mathbb{Q}^4 | 2a+b=c, a+2b-3c+d=0\}$.
a) Show that $V \leq_{\mathbb{Q}} \mathbb{Q}^4$.
b) Find $\dim_{\mathbb{Q}} V$.
c) Find a basis for V.

4. Let $\mathcal{B} = ((1,1,1),(1,1,2),(1,2,3))$.
a) Is $\mathcal{B}$ a basis in ${}_{\mathbb{Q}}\mathbb{Q}^3$?
b) Is $\mathcal{B}$ a basis in ${}_{\mathbb{C}}\mathbb{C}^3$?
c) Is $\mathcal{B}$ a basis in ${}_{\mathbb{R}}\mathbb{C}^3$?

5. Let $\mathcal{B} = ((1,0,1),(0,1,1),(1,1,0))$ and $\mathcal{B}' = ((2,0,1),(1,2,0),(0,1,2))$. Show that $\mathcal{B}$ and $\mathcal{B}'$ are bases in ${}_{\mathbb{R}}\mathbb{R}^3$ and find the change of base matrices from $\mathcal{B}$ to $\mathcal{B}'$ and from $\mathcal{B}'$ to $\mathcal{B}$.

6. Which of the following functions are linear?
a) $f : \mathbb{Q} \to \mathbb{Q}$, $f(t) = t+3$
b) $g : \mathbb{R}^3 \to \mathbb{R}$, $g(a,b,c) = b$
c) $h : \mathbb{C} \to \mathbb{C}^3$, $h(u) = (u, 2u, 0)$
d) $det : \mathcal{M}_n(\mathbb{Q}) \to \mathbb{Q}$
e) $Tr : \mathcal{M}_n(\mathbb{R}) \to \mathbb{R}$
f) $d : \{f : \mathbb{R} \to \mathbb{R} | f \text{ is derivable } \} \to \{f : \mathbb{R} \to \mathbb{R}\}$, $d(f) = f'$
g) $m : \mathbb{C} \to \mathbb{R}$, $m(z) = |z|$
i) $c : \mathbb{C} \to \mathbb{C}$, $c(z) = \overline{z}$
j) $ev_3 : k[X] \to k$, $ev_3(P) = P(3)$

7. Find the kernel and the image of each linear function from exercise 6.

8. Consider the bases $\mathcal{B}_a^n = (1, X-a, (X-a)^2, \ldots, (X-a)^n)$ in ${}_k k[X]_n = \{f \in k[X] |\deg f \leq n\}$. Find the change of base matrix from $\mathcal{B}_7^3$ to $\mathcal{B}_5^3$, and, more generally, from $\mathcal{B}_\alpha^n$ to $\mathcal{B}_\beta^n$.

9. Write the change of base matrix from $\mathcal{B} = (e_3, e_4, e_5, e_6, e_7, e_1, e_2)$ to the canonical basis $(e_1, e_2, e_3, e_4, e_5, e_6, e_7)$ of ${}_{\mathbb{Q}}\mathbb{Q}^7$.

10. Write $M_{\mathcal{B}}^{\mathcal{C}}(f)$, where:
a) $f : \mathbb{R}^3 \to \mathbb{R}^2$, $f(a,b,c) = (a-b+3c, 2a+b)$, $\mathcal{B} = ((1,0,1),(1,1,0),(0,1,1))$, $\mathcal{C} = ((1,2),(3,4))$.
b) $f : k[X]_5 \to k[X]_5$, $f(P) = P'$, $\mathcal{B} = \mathcal{C} = \mathcal{B}_{-3}^5$ (see exercise 8).