Geometrie - seminar 1

I Spații vectoriale

Def. Vezi primul curs!

① $(\mathbb{R}^3/\mathbb{R}, +, \cdot)$ → Dem

$\hookrightarrow \{(x_1, x_2, x_3) / x_i \in \mathbb{R}, \forall i = \overline{1,3}\}$

$+ : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$

$(x_1, x_2, x_3) + (y_1, y_2, y_3) = (x_1 + y_1, x_2 + y_2, x_3 + y_3)$

$\cdot : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$

$\alpha \in \mathbb{R}$

$\alpha \cdot (x_1, x_2, x_3) = (\alpha x_1, \alpha x_2, \alpha x_3)$

- $(\mathbb{R}^3, +)$ - grup abelian.
  - (x, y, z)

    $(x + y) + z = x + (y + z), (\forall)\ x, y, z \in \mathbb{R}^3$

    $x = (x_1, x_2, x_3)$

    $y = (y_1, y_2, y_3)$

    $z = (z_1, z_2, z_3)$

    $(x + y) + z = (x_1 + y_1, x_2 + y_2, x_3 + y_3) + (z_1, z_2, z_3) \overset{+\text{ e as. pe } \mathbb{R}}{=}$

    $= ((x_1 + y_1) + z_1, (x_2 + y_2) + z_2, (x_3 + y_3) + z_3) =$

    $= (x_1 + (y_1 + z_1), x_2 + (y_2 + z_2), x_3 + (y_3 + z_3))$
  - comutativitate

    lejer!

- ~~$(\alpha_1+\alpha_2)x = \alpha_1 x + \alpha_2 x$~~

$(\alpha_1+\alpha_2)(x_1,x_2,x_3) = ((\alpha_1+\alpha_2)x_1, (\alpha_1+\alpha_2)x_2, (\alpha_1+\alpha_2)x_3) =$

$= (\alpha_1 x_1 + \alpha_2 x_1, \alpha_1 x_2 + \alpha_2 x_2, \alpha_1 x_3 + \alpha_2 x_3) =$

$= (\alpha_1 x_1, \alpha_1 x_2, \alpha_1 x_3) + (\alpha_2 x_1, \alpha_2 x_2, \alpha_2 x_3)$

$= \alpha_1 (x_1, x_2, x_3) + \alpha_2 (x_1, x_2, x_3)$

lejer!

Deci $(R^3/R, +, \cdot)$ sp. vect. real.

V/K

$S = \{v_1, \dots, v_n\} \subset V$

↳ s.v.l. ind.

$\forall\, \alpha_1, \dots, \alpha_n \in K$ a.î. $\alpha_1 v_1 + \dots + \alpha_n v_n = 0_V$, $\alpha_1 = \dots = \alpha_n =$

Altfel, s.v.l. dependent

(2) Stab. dacă urm. sisteme liniare sunt dependente sau independente.

R^3/R

a) $S_1 = \{v_1 = (2,-1,1), v_2 = (0,1,2), v_3 = (2,1,5)\}$

b) $S_2 = \{v_1 = (1,1,-1), v_2 = (1,-1,1), v_3 = (3,-1,1)\}$

a) Fie $\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 = 0_{R^3}$

$\alpha_1, \alpha_2, \alpha_3 \in R$. (comb. liniar. arb.)

$\alpha_1(2,-1,1) + \alpha_2(0,1,2) + \alpha_3(2,1,5) = (0,0,0)$

$\alpha_1 = 2 \quad \alpha_2 = 4, \quad \alpha_3 = -2$

Scanned by CamScanner

$$\begin{cases} 2\alpha_1 + 0 + 2\alpha_3 = 0 \\ -\alpha_1 + \alpha_2 + \alpha_3 = 0 \\ \alpha_1 + 2\alpha_2 + 5\alpha_3 = 0 \end{cases}$$

$\Downarrow$ S. liniar omogen.

$\alpha_1 = \alpha_2 = \alpha_3 = 0.$

$$A = \begin{pmatrix} 2 & 0 & 2 \\ -1 & 1 & 1 \\ 1 & 2 & 5 \end{pmatrix} \Rightarrow \det A = 10 - 4 - 2 - 4 = 0.$$

$\begin{matrix} v_1 & v_2 & v_3 \end{matrix}$

$\Downarrow$

S' are și sol. nenule.

$\Downarrow$

e sr. liniar dep.

b) Dacă $\mathbb{R}^n / \mathbb{R}$

$S = \{v_1, \dots, v_m\}$

a) S e sistem de vect. liniar ind.

$\Updownarrow$ $\qquad m \leq n$

$\operatorname{rang} \underset{(n,m)}{A} = m$

b) S e sist. — dep.

$\Updownarrow$

$\operatorname{rang} A \neq m$

$$A = \begin{pmatrix} 1 & 1 & 3 \\ 1 & -1 & -1 \\ -1 & 1 & 5 \end{pmatrix} \Rightarrow \cancel{d} \quad -5 + 1 + 3 - 3 - 5 + 1 = ?.$$

Fie $D_p = |1| = 1 \neq 0$

Bordam!

$$D_{11} = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -1 - 1 = -2 \neq 0 \Rightarrow \operatorname{Rang} A > 1$$

$$D_p = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -1-1 = -2 \neq 0$$

Bordăm!

$$D_{11} = \begin{vmatrix} 1 & 1 & 3 \\ 1 & -1 & -1 \\ -1 & 1 & 5 \end{vmatrix} = -5+1+\cancel{3}-\cancel{3}-5+1 = -8 \neq 0$$

Deci $\text{Rang } A > 2$
Dar $\text{Rang } A \leq 3$ $\Rightarrow$ $\text{Rang } A = 3$

$\Downarrow$

S e s.v. liniar. indep.

$\boxed{\text{Temă:}}$

$m \in \mathbb{R}$, $m = ?$ a.î.

$$S = \{ v_1 = (1, 1, m), v_2 = (3, 2, 1), v_3 = (1, 0, 2) \} \in \mathbb{R}^3$$

a) lin. indep.

b) lin. dep.

Sist. de generatori

$V/_K$

$S \subseteq V$

$\hookrightarrow$ sistem de gen. ai lui V

dacă $\overline{S} = V$

Adk. $(\forall)\, v \in V$, $\exists\, \alpha_1, \dots, \alpha_n \in K$

a.î.

$$v = \alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n$$

③ Stab. dacă urm. sisteme vectoriale sunt sisteme de generatori pt. spațiile vect. din care fac parte.

a) $S_1 = \{v_1 = (1,5), v_2 = (5,1)\} \subset \mathbb{R}^2$

b) $S_2 = \{v_1 = (-1,4,3), v_2 = (3,-1,1)\} \subset \mathbb{R}^3$

a) $S_1 \subset \mathbb{R}^2$

$\{v_1, v_2\}$ – s. de gen. pt. $\mathbb{R}^2$

$\Leftrightarrow (\forall) v \in \mathbb{R}^2, (\exists) \alpha, \beta \in \mathbb{R}$ a.î. $v = \alpha v_1 + \beta v_2$.

$(a,b) = \alpha(1,5) + \beta(5,1)$

$(a,b) = (\alpha + 5\beta, 5\alpha + \beta)$

$$\begin{cases} a = \alpha + 5\beta \\ b = 5\alpha + \beta \quad |\cdot(-5) \end{cases} (+)$$

$-24\alpha = a - 5b$

$$\boxed{\alpha = \frac{-a + 5b}{24}}$$

$$\boxed{\beta = \frac{5a - b}{24}}$$

S_1 e sist de gen. pt. $\mathbb{R}^2$

Pt. K^n / K

$S = \{v_1, \ldots, v_m\} \subset \mathbb{R}^n$

a) S e sist. de gen. pt. $\mathbb{R}^n$ dacă $\operatorname{rang} A = n$

$(n, m \mid (n \leq m))$

b) S nu e sist. de gen. pentru $\mathbb{R}^n/\mathbb{R}$ dacă $\text{rang } A \neq n$

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c) $A = \begin{pmatrix} -1 & 3 \\ 1 & -1 \\ 3 & 1 \end{pmatrix} \in \mathcal{M}_{(3,2)}(\mathbb{R})$

$\qquad\; v_1 \qquad v_2$

$\text{rang } A \leq 2 \neq 3 \Rightarrow S$ nu e sist. de gener. pt. $\mathbb{R}^3/\mathbb{R}$.

Baze + Coordonate

V/K sp. vect.

$B \subset V$

↳ bază pt. V/K dacă $\boxed{\begin{array}{l} 1) \text{ s.v. lin. ind.} \\ 2) \text{ s. de gen.} \end{array}}$

$\mathbb{R}^n/\mathbb{R}$

$S = \{v_1, \dots, v_m\} \subset \mathbb{R}^n$

S e bază pt. $\mathbb{R}^n/\mathbb{R} \Leftrightarrow$ 1) 2) $\Leftrightarrow$

$\Leftrightarrow \begin{cases} \text{rang } A = m \\ \text{rang } A = n \end{cases} \Leftrightarrow \text{rang } A = m = n$

$\Downarrow$

1) $A \in \mathcal{M}_n(\mathbb{R})$

2) $\det A \neq 0$

14) Stab. dacă sunt baze.

a) $S_1' = \{v_1=(1,1,1),\ v_2=(0,1,1),\ v_3=(0,0,1)\} \subset \mathbb{R}^3$

b) $S_2' = \{v_1=(5,3,1),\ v_2=(-1,2,1),\ v_3=(3,4,3)\} \subset \mathbb{R}^3$

~~a) A~~

Temă: b) $v_1=(1,2,3)$, $v_2=(2,1,-1)$ $\in \mathbb{R}^3$

det. $v_3 \in \mathbb{R}^3$ a.î. $S'=\{v_1,v_2,v_3\}$ să fie bază

$\cancel{\det}\ |A| = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix} = 1 \neq 0$

Geometrie - seminar ②

$\mathbb{R}^3/\mathbb{R}$

① $S = \{v_1 = (1,0,0), v_2 = (0,1,0), v_3 = (0,0,1)\}$

$S' = \{v_1' = (-1,2,1), v_2' = (2,0,3), v_3' = (0,-1,2)\} \in \mathbb{R}^3$ baze.

$\boxed{\text{Det } A, A'}$; $S \xrightarrow{A} S'$, $S' \xrightarrow{A'} S$.

Obs. $S = B_0 = \{e_1, e_2, e_3\} \subset \mathbb{R}^3$
↳ baza canonică

P $\mathbb{R}^n/\mathbb{R}$

$B_0 = \{e_1, \dots, e_n\}$
$B = \{f_1, \dots, f_n\} \subset \mathbb{R}^n$

$B \xrightarrow{A} B'$, $A = \begin{pmatrix} | & | & & | \\ f_1 & f_2 & & f_n \end{pmatrix}$

$B \xrightarrow{A} B' \Rightarrow B' \xrightarrow{A^{-1}} B$

În cazul nostru:

$$A = \begin{pmatrix} -1 & 2 & 0 \\ 2 & 0 & -1 \\ 1 & 3 & 2 \end{pmatrix} \qquad A' = A^{-1}.$$

② $\mathbb{R}^3$

$F = \{f_1 = (1,0,2), f_2 = (-2,-2,-2), f_3 = (0,1,2)\} \subset \mathbb{R}^3$

$G = \{g_1 = (1,1,0), g_2 = (1,0,1), g_3 = (0,1,1)\}$

a) $F, G \subset \mathbb{R}^3$ baze

b) det. coord. vect. $v = (-3,-3,0)$ în rap. cu G_F

c) $F \xrightarrow{FG} G$ și $G \xrightarrow{G_F} F$

$FG = ?$, $G_F = ?$

a) $A_F = \begin{pmatrix} 1 & -2 & 0 \\ 0 & -2 & 1 \\ 2 & -2 & 2 \end{pmatrix} \in \mathcal{M}_3(\mathbb{R}) \Rightarrow \det A_F = -2 \cdot \begin{vmatrix} 1 & +1 & 0 \\ 0 & 1 & 1 \\ 2 & 1 & 2 \end{vmatrix} =$

$= -2(2+2-1) = -6 \neq 0.$

Deci F e bază.

$A_G = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \in \mathcal{M}_3(\mathbb{R}) \Rightarrow \det . A_G = -2 \neq 0$

Deci G e bază.

b) V1 $F \subset \mathbb{R}^3$ bază $\Leftrightarrow (\forall) v \in \mathbb{R}^3, \exists \alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$, a.î.

$v = \alpha_1 f_1 + \alpha_2 f_2 + \alpha_3 f_3$

$[v]_F = (\alpha_1, \alpha_2, \alpha_3)$

$\hookrightarrow$ coord. vect. v în rap. cu baza F

$(-3,-3,0) = \alpha_1 \cdot (1,0,2) + \alpha_2 \cdot (-2,-2,-2) + \alpha_3 \cdot (0,1,2)$

$$\begin{cases} \alpha_1 - 2\alpha_2 + 0 = -3 & (1) \\ 0 - 2\alpha_2 + \alpha_3 = -3 & (2) \\ 2\alpha_1 - 2\alpha_2 + 2\alpha_3 = 0. & (3) \end{cases}$$

$(1)-(2): \quad \alpha_1 - \alpha_3 = 0$

$(1)-(3): \quad -\alpha_1 - 2\alpha_3 = -3$ (+)

$-3\alpha_3 = -3 \Rightarrow \alpha_3 = 1$

$\alpha_1 = 1$

$\alpha_2 = 2$

$\textcircled{V_2}$ $B_0 \xrightarrow{A_F} F$

$X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ $X' = \begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix}$

$X = A_F X' \mid \cdot A_F^{-1}$.

$A_F^{-1} X = X'$

$[v]_F = A_F^{-1} \cdot [v]_{B_0}$

$\begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix} = \underbrace{\begin{pmatrix} & & \end{pmatrix}^{-1}}_{A_F} \begin{pmatrix} -3 \\ -3 \\ 0 \end{pmatrix}$

Temă: în raport cu baza G.

c) $\textcircled{V_1}$ $F \xrightarrow{F_G} G$

$F_G = \begin{pmatrix} [g_1]_F & [g_2]_F & [g_3]_F \end{pmatrix}$

$G_F = \begin{pmatrix} [f_1]_G & [f_2]_G & [f_3]_G \end{pmatrix}$

$\textcircled{V_2}$ $\underbrace{B_0 \xrightarrow{A_F} F \xrightarrow{F_G} G}_{A_G}$.

$\boxed{P}$ $B \xrightarrow{A} B' \xrightarrow{A'} B'' \Rightarrow B \xrightarrow{AA'} B''$

$A_F \cdot F_G = A_G \cdot | A_F^{-1}$.

T $\boxed{F_G = A_F^{-1} \cdot A_G.}$

$\boxed{G_F = F_G^{-1} = A_G^{-1} \cdot A_F.}$

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Subspații vectoriale

Def. Fie V/K sp. vect.

$U \subseteq V \ (U \neq \emptyset)$

U ssp. vect. al lui V

dacă 1) $(\forall) v_1, v_2 \in U \Rightarrow v_1 + v_2 \in U$

2) $(\forall) v \in U, \ \alpha \in K \Big| \Rightarrow \alpha v \in U$

P $U \subseteq V$ ssp. vect. $\Leftrightarrow (\forall) v_1, v_2 \in U, \ \alpha_1, \alpha_2 \in K \Rightarrow \alpha_1 v_1 + \alpha_2 v_2 \in U$

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(1) $\mathbb{R}^3/\mathbb{R}$

$U = \{(x, y, z) \in \mathbb{R}^3 \ / \ 3x - y + z = 0\} \subset \mathbb{R}^3$

a) $U \subseteq \mathbb{R}^3$ ssp. vect.

b) $\dim U = ?$

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a) Fie $v_1, v_2 \in U$, $v_1 = (x_1, y_1, z_1)$, $3x_1 - y_1 + z_1 = 0$

$v_2 = (x_2, y_2, z_2)$, $3x_2 - y_2 + z_2 = 0$

Fie $\alpha_1, \alpha_2 \in \mathbb{R}$.

$\alpha_1 v_1 + \alpha_2 v_2$
$\parallel$
$\alpha_1 (x_1, y_1, z_1) + \alpha_2 (x_2, y_2, z_2)$
$\parallel$
$(\alpha_1 x_1, \alpha_1 y_1, \alpha_1 z_1) + (\alpha_2 x_2, \alpha_2 y_2, \alpha_2 z_2)$
$\parallel$
$(\underbrace{\alpha_1 x_1 + \alpha_2 x_2}_{x}, \underbrace{\alpha_1 y_1 + \alpha_2 y_2}_{y}, \underbrace{\alpha_1 z_1 + \alpha_2 z_2}_{z})$

$3x - y + z =$
$= 3(\alpha_1 x_1 + \alpha_2 x_2) - (\alpha_1 y_1 + \alpha_2 y_2) + (\alpha_1 z_1 + \alpha_2 z_2) =$
$= \alpha_1 (3x_1 - y_1 + z_1) + \alpha_2 (3x_2 - y_2 + z_2) = 0$

Deci $\alpha_1 v_1 + \alpha_2 v_2 \in U$
$\Downarrow$
$U \subset \mathbb{R}^3$
ssp. vectorial

b) $3x - y + z = 0 \Rightarrow y = 3x + z$

$(x, y, z) \in U$, $(x, y, z) = (x, 3x+z, z) =$
$= (x, 3x, 0) + (0, z, z) = x \cdot \underbrace{(1, 3, 0)}_{v_1} + z \cdot \underbrace{(0, 1, 1)}_{v_2} =$

$= x v_1 + z v_2 \Rightarrow S = \{v_1, v_2\} \subset U$
s. de gen.

$A = \begin{pmatrix} 1 & 0 \\ 3 & 1 \\ 0 & 1 \end{pmatrix} \in \mathcal{M}_{(3,2)}(\mathbb{R}) \Rightarrow \text{Rang } A = 2 \overset{R_1}{=}$ ... $\Downarrow R_1$
$\quad v_1 \quad v_2$

S = s. v. lin. ind.

Deci $S \subset U$
bază

Deci $\dim U = \text{card } S = 2$. / U plan vectorial (0_e) (tr. prin origine)

② $\mathbb{R}^n/\mathbb{R}$

Fie $A \in \mathcal{M}_{(m,n)}(\mathbb{R})$

$rg A = m \leq n$

$\underline{S(A) = \{x \in \mathbb{R}^n/\mathbb{R}} \mid AX = 0\} \subseteq \mathbb{R}^n$

$S(A) \subseteq \mathbb{R}^n/\mathbb{R}$ ssp. vect.

$\underline{\dim S(A)} = n - \text{rang} A = n - m$

Dem.

Fie $X, Y \in S(A) \Rightarrow AX = 0$ și $AY = 0$

$\alpha, \beta \in \mathbb{R}$

$\underline{\text{Dem. că}}$ $\alpha X + \beta Y \in S(A)$

$A(\alpha X + \beta Y) = A(\alpha X) + A(\beta Y) =$

$= \alpha AX + \beta AY = \alpha \cdot 0 + \beta \cdot 0 = 0$

Deci $\alpha X + \beta Y \in S(A)$

Deci $S(A) \subseteq \mathbb{R}^3$ ssp. vect.

$\underline{\dim S(A) = n - \text{rang} A = n - m}$

③ Precizați, conform J, toate ssp. vect. din:

a) $\mathbb{R}^2/\mathbb{R}$ b) $\mathbb{R}^3/\mathbb{R}$.

$\underline{\text{Rez.:}}$

a) $\{0_{\mathbb{R}^2}\}$, $\mathbb{R}^2 \subseteq \mathbb{R}^2$ ssp. triviale

$\{0_{\mathbb{R}^2}\}$ — 0, $\mathbb{R}^2$ — 2

$d = \{(x, y) \in \mathbb{R}^2 / ax + by = 0\} \subseteq \mathbb{R}^2$

d — dr. vectorială (O∈)

rang $(a\ b) = 1$

$\Updownarrow$ $a^2 + b^2 > 0$

b) $\{0_{\mathbb{R}^3}\}, \mathbb{R}^3$ ssp. vect. triviale.

$\downarrow$ 0 $\downarrow$ 3.

dim = 1: $d = \{(x,y,z) \in \mathbb{R}^3 / \begin{cases} a_1x + b_1y + c_1z = 0 \\ a_2x + b_2y + c_2z = 0 \end{cases}\} \subseteq \mathbb{R}^3$

$\downarrow$ dr. vect. ($\ni 0$)

$rg\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{pmatrix} = 2$

dim = 2:

$P = \{(x,y,z) \in \mathbb{R}^3 / ax + by + cz = 0\} \subseteq \mathbb{R}^3$

$\downarrow$ plan vect. ($\ni 0$)

$rg(a\ b\ c) = 1 \Leftrightarrow a^2 + b^2 + c^2 > 0$

Grassman:

Fie $V_1, V_2 \subseteq V$
2 ssp. vect.

Atunci: $\dim(V_1 + V_2) = \dim V_1 + \dim V_2 - \dim(V_1 \cap V_2)$

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① Fie $\underbrace{\{(x,y,0,0) / x,y \in \mathbb{R}\}}_{V_1} \subseteq \mathbb{R}^4$

$\underbrace{\{(0,0,z,t) / z,t \in \mathbb{R}\}}_{V_2} \subseteq \mathbb{R}^4$.

a) Dem. că $V_1, V_2 \subseteq \mathbb{R}^4$ ssp. vect.

b) $\mathbb{R}^4 = V_1 \oplus V_2$.

c) Verificați th. Grassman în acest caz.

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b) $V_1 \subseteq \mathbb{R}^4$ ssp. vect.

Fie $v, v' \in V_1$, $\alpha, \alpha' \in \mathbb{R}$ $\Rightarrow \alpha v + \alpha' v' \in V_1$.

$v=(x,y,0,0)$
$v'=(x',y',0,0)$ $\Rightarrow \alpha v+\alpha' v'=(\alpha x+\alpha' x', \alpha y+\alpha' y', 0, 0)\in\mathbb{R}^4$

Analog, $V_2 \subseteq \mathbb{R}^4$
ssp. vect.

b) $\mathbb{R}^4 = V_1 \oplus V_2$.

Avem: evident $V_1+V_2 \subseteq \mathbb{R}^4$.
Dem. că și $\mathbb{R}^4 \subseteq V_1+V_2$.

~~$(\forall)(x,y$~~

$(\forall)\ v\in\mathbb{R}^4,\ (\exists)\ v_1\in V_1,\ v_2\in V_2$ a.î. $v=v_1+v_2$.

Fie $v=(x,y,z,t)\in\mathbb{R}^4$

$= (x,y,0,0)+(0,0,z,t)$, cu $(x,y,0,0)\in V_1$ și $(0,0,z,t)\in V_2$.

$V_1\cap V_2=\{0_{\mathbb{R}^4}\}$

Fie $v\in V_1\cap V_2 \Rightarrow x,y=0$, $z,t=0$ $\Rightarrow v=0_{\mathbb{R}^4}$, unde $v=(x,y,z,t)$.

Deci $\mathbb{R}^4=V_1\oplus V_2$

Analog, $\dim V_2=2$.

c) $\underbrace{\dim(\mathbb{R}^4)}_{4=}=\dim V_1+\dim V_2-\dim(V_1\cap V_2)$

$\dim V_1=2$.
$\Uparrow$
$S\subset V_1$
$\Uparrow$
bază
$\Uparrow$

$V_1\ni(x,y,0,0)=x\underbrace{(1,0,0,0)}_{e_1}+y\underbrace{(0,1,0,0)}_{e_2}$

$=xe_1+ye_2 \Rightarrow S=\{e_1,e_2\}\subset V_1$
sist. de gen. + sist. de vect. liniar

Geometrie - seminar 3

1. Op. cu subsp. Th. dim.
2. Apl. liniare. Th. rangului
3. Endomorf. diag.

1 ① Fie $V_1 = \{A \in \mathcal{M}_n(\mathbb{R}) \mid TrA = 0\}$

$V_2 = \{A \in \mathcal{M}_n(\mathbb{R}) \mid A = \lambda I_n, \lambda \in \mathbb{R}\}$

a) $V_1, V_2 \subset \mathcal{M}_n(\mathbb{R})$ ssp. vect.

b) $V_1 \oplus V_2 = \mathcal{M}_n(\mathbb{R})$

c) Verif. th. dim (Grassman) în acest caz.

② $\boxed{\text{Temă}}$

$V_1 = \{(x, 0, 0, t) / x, t \in \mathbb{R}\}$

$V_2 = \{(0, y, z, 0) / y, z \in \mathbb{R}\}$

a) $V_1, V_2 \subseteq \mathbb{R}^4$ ssp. vect

b) $V_1 \oplus V_2 = \mathbb{R}^4$

c) Verif. th. dim. în acest caz //

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Fie $V_1 = \{(x, y, 0) / x, y \in \mathbb{R}\}$ $\in \mathbb{R}^3$

$V_2 = \{(t, 0, t) / t \in \mathbb{R}\}$

a) $V_1, V_2 \subseteq \mathbb{R}^3$ ssp. vect

b) $V_1 \cap V_2 = ?$

c) $V_1 + V_2 = ?$ //

a) ca în rețetă

b) Fie $v \in V_1 \cap V_2 \Rightarrow \underbrace{v \in V_1 \wedge v \in V_2}$

$$\Downarrow$$

$$\begin{cases} x = t \\ y = 0 \\ t = 0 \end{cases} \rightarrow x = 0$$

$V_1 \cap V_2 = \{0_{\mathbb{R}^3}\}$ $\Leftarrow$ Deci $v = (0, 0, 0) \in \mathbb{R}^3$

c) cf. Th. Grassmann $\dim(V_1 + V_2) = \dim(V_1) + \dim(V_2) - \underbrace{\dim(V_1 \cap V_2)}_{= 0}$

$(x,y,0)\in V_1 \Rightarrow (x,y,0)=xe_1+ye_2$

$\Downarrow \quad\quad B_1=\{e_1,e_2\}\subset V$ bază

$\dim V_1=2$ (plan vect.)

$(t,0,t)\in V_2 \Rightarrow (t,0,t)=t(1,0,1)=tf$, $f=(1,0,1)$

$B_2=\{f\}\subset V$ bază

$\dim V_2=1$ (dreaptă vect.)

Deci $\dim V_1+\dim V_2=3=\dim(V_1\oplus V_2)$

$V_1\oplus V_2=\mathbb{R}^3 \Leftarrow \left|\; V_1\oplus V_2\subseteq\mathbb{R}^3\right.$ ssp. vect.

$\boxed{\text{Temă}}$

Fie $V_1=\{(x,y,0)/\ x,y\in\mathbb{R}\}$

$V_2=\{(u,0,v)/\ u,v\in\mathbb{R}\}$

a) $V_1,V_2\subset\mathbb{R}^3$ / ssp. vect.

b) $V_1\cap V_2=?$

c) $V_1+V_2=?$

Def.

Fie $V, W/K$ două sp. vectoriale

O aplicatie $f: V \to W$ morfi-

s.n. aplicatie liniara daca:

(morf. de sp.)

1) $f(v_1+v_2) = f(v_1)+f(v_2), (\forall) v_1, v_2 \in V$

2) $f(\alpha v) = \alpha f(v), (\forall) v \in V$

$\alpha \in K$

$\Leftrightarrow f(\alpha_1 v_1 + \alpha_2 v_2) = \alpha_1 f(v_1) + \alpha_2 f(v_2)$

$\text{Ker} f = \{ v \in V / f(v) = 0_W \} \subseteq V$ ssp. vect.

nucleul apl. liniare f.

$\text{Im} f = \{ w \in W / (\exists) v \in V \text{ a.î. } f(v) = w \} \subseteq W$ ssp. vect.

Th. rangului

Fie $f: V \to W$ apl. lin.

Atunci $\underbrace{\dim(\text{Ker} f)}_{\text{def}(f)} + \overbrace{\dim(\text{Im} f)}^{\text{rg}(f)} = \dim V$

def(f) (defectul lui f)

1) Aratati că $f: \mathbb{R}^3 \to \mathbb{R}^3$, ~~$f((x_1, x_2, x_3)) = x_1 + 2x_2$~~

$f((x_1, x_2, x_3)) = (x_1 + 2x_2 - x_3, -2x_1 + x_2 + 2x_3, x_1 + 2x_2 + 3x_3)$

apl. liniara.

Rez.

$f(x) = AX$, $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

$$A = \begin{pmatrix} 1 & 2 & -1 \\ -2 & 1 & 2 \\ 1 & 2 & 3 \end{pmatrix}$$ m. as. lui f în raport cu b canonică.

$\begin{array}{ccc} fe_1 & fe_2 & fe_3 \end{array}$

unde $\{e_1, e_2, e_3\} \subset \mathbb{R}^3$ baza canonică

Fie ~~$A, B \in$~~ $x, y \in \mathbb{R}^3$
$\alpha, \beta \in \mathbb{R}$.

$$f(\alpha x + \beta y) = A(\alpha x + \beta y) =$$
$$= \alpha(AX) + \beta(AY) = \alpha f(x) + \beta f(y)$$
$\Downarrow$
f apl. lin.

$\boxed{\text{Temă}}$

a) $f : \mathbb{R}^3 \to \mathbb{R}^2$
$f(x_1, x_2, x_3) = (-2x_1 + x_2 + x_3), x_1 - 2x_2 - x_3)$
apl. lin.

b) rg (f).

② Fie $f: \mathbb{R}^2 \to \mathbb{R}^3$, $f(x,y) = (-y, x, 2x+y)$

a) f apl. lin.

b) $\ker f = ?$, $\operatorname{Im} f = ?$

c) f inj, surj, bij?

d) Verif. th. rangului

a) $f(X) = AX$, $X = \begin{pmatrix} x \\ y \end{pmatrix}$

$$A = \left(\underbrace{\begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}}_{fe_1} \underbrace{\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}}_{fe_2} \right) \Rightarrow \text{m. as. lui } f$$

în raport cu rep. can. pe $\mathbb{R}^2$, resp. $\mathbb{R}^3$

unde $\{e_1, e_2\} \subset \mathbb{R}^2$ baza canonică

Fie $X, Y \in \mathbb{R}^2$

$\alpha, \beta \in \mathbb{R}$

$f(\alpha X + \beta Y) = A(\alpha X + \beta Y) = A(\alpha X) + A(\beta Y) =$

$= (A\alpha)X + (A\beta)Y = (\alpha A)X + (\beta A)Y = \alpha(AX) + \beta(AY) =$

$= \alpha f(X) + \beta f(Y) \Rightarrow f$ apl. liniară

b) $\ker f = \{ \underset{(x,y)}{v} \in \mathbb{R}^2 \mid f(v) = 0_{\mathbb{R}^3} \}$

$f(x,y) = (0,0,0) \iff \begin{cases} -y = 0 \\ x = 0 \\ 2x + y = 0 \end{cases} \Rightarrow x = y = 0$ sol. unică (rang $A = 2$)

$\Downarrow$

$\ker f = \{ \underset{0_{\mathbb{R}^2}}{(0,0)} \}$

$\operatorname{Im} f = \{ \underset{(x',y',z')}{w} \in \mathbb{R}^3 \mid (\exists)\, \underset{(x,y)}{v} \in \mathbb{R}^2 \text{ a.î. } f(v) = w \}$

$f(v) = w \iff f((x,y)) = (x', y', z') \iff \begin{cases} -y = x' & (1) \\ x = y' & (2) \\ 2x + y = z' & (3) \end{cases}$

$\xrightarrow[x,y]{elimin}$ $\left| \begin{array}{l} x = x' \\ y = y' \\ \boxed{2y' - x' = z'} \end{array} \right.$

$\Updownarrow$

$-x' + 2y' - z' = 0$

$x' - 2y' + z' = 0$

$Im f = \{(x', y', z') \in \mathbb{R}^3 / x' - 2y' + z' = 0\}$

c) $Ker f = \{0_{\mathbb{R}^2}\} \Rightarrow f$ e injectiva

$Im f \subset \mathbb{R}^3$ (plan vectorial)

$\Downarrow$

f nu e surj $\Rightarrow$ f nu e bijectiva

d) Verificam ca $\underbrace{\dim Ker f}_{=0} + \dim Im f = \underbrace{\dim \mathbb{R}^2}_{=2}$

$\dim(Im f) = 2$

Deci DA ✓

$\boxed{\text{Tema}}$

Fie $A = \begin{pmatrix} 1 & 0 & 2 \\ -3 & 2 & 1 \\ 2 & 1 & -1 \end{pmatrix} \in M_3(\mathbb{R})$

a) Constr. a pl. lin. $f : \mathbb{R}^3 \to \mathbb{R}^3$ a.î. A m. a. lui f. m. rap. ...

$f(x) = AX$

b) Ker f, Im ?

c) f inj, surj, resp. bij ?

Geometrie - seminarul 4

1. Diagonalizarea endomorfismelor
2. Forme biliniare. Forme pătratice

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① **Def.** Fie $V/_K$ un spațiu vectorial ($\dim_K V = n < \infty$)

Un endomorfism $f: V \to V$ s.n. diag. dacă $\exists$ $B \subset V$ reper vectorial a.î. m. as. endm. f să aibă forma diagonală

$$D = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_1 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_r \end{pmatrix}$$

$\boxed{T}$ $f: V \to V$ endm. diag. $\Leftrightarrow$ 1. Toate rădăcinile ec. caract. sunt în K (= val. proprii)

$\hookrightarrow m_1 + \ldots + m_r = \dim_K V$

2. $\dim V_{\lambda_i} = m_i, \forall i = \overline{1,r}$

$\hookrightarrow m_g(\lambda_i) = m_a(\lambda_i), \forall i = \overline{1,r}$

Aplicații:

⊛ Fie $f: \mathbb{R}^3 \to \mathbb{R}^3$, $f(x,y,z) = (x - 2y, -2x + 2y - 2z, -2y + 3z)$

a) Arătați că f e apl. liniară

b) Scrieți m. as. lui f în raport cu reperul canonic.

c) Determinați valorile proprii și subsp. proprii coresp.

d) Stabiliți dacă f e diag.

e) Scrieți o bază în care f are forma diag. și $D = ?$

f) Verificați rezultatul obț.

g) $A^n = ?$, $n \in \mathbb{N}^*$

a) $f(x,y,z) = (x-2y, -2x+2y-2z, -2y+3z)$

$f: \mathbb{R}^3 \to \mathbb{R}^3$, $f(X) = AX$, unde $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

$$A = \begin{array}{ccc} \begin{pmatrix} 1 & -2 & 0 \\ -2 & 2 & -2 \\ 0 & -2 & 3 \end{pmatrix} \\ f(e_1) \quad f(e_2) \quad f(e_3) \end{array}$$

m. as. endm. f în rap. cu reperul canonic din $\mathbb{R}^3$ ($B_0 = \{e_1, e_2, e_3\}$)

(b)

Fie $x, y \in \mathbb{R}^3$

$\alpha, \beta \in \mathbb{R}$

$f(\alpha x + \beta y) = A(\alpha X + \beta Y) = A\alpha X + A\beta Y =$

$= \alpha AX + \beta AY = \alpha f(x) + \beta f(y)$

Deci f e apl. liniară

↳ un endomorfism al sp. vect. $\mathbb{R}^3/\mathbb{R}$.

c) Rezolv. ec. caracteristică

$\det(A - \lambda I_3) = 0$, în $\mathbb{R}$

$$\begin{vmatrix} 1-\lambda & -2 & 0 \\ -2 & 2-\lambda & -2 \\ 0 & -2 & 3-\lambda \end{vmatrix} = 0$$

$\Updownarrow$

$(1-\lambda)(2-\lambda)(3-\lambda) - 4(3-\lambda) - 4(1-\lambda) =$

$= (3-\lambda)(2 - 3\lambda + \lambda^2 - 4) - 4(1-\lambda) =$

$= (3-\lambda)(\lambda^2 - 3\lambda - 2) - 4(1-\lambda) =$

$= \cancel{3\lambda^2}\ 6\lambda^2 - \lambda^3 - 3\lambda - 10 = -\lambda^3 + 6\lambda^2 - 3\lambda - 10 = 0$

	λ^3	λ^2	λ^1	λ^0	
	-1	6	-3	-10	
-1	-1	7	-10	0	$\Rightarrow$

$(\lambda + 1)(-\lambda^2 + 7\lambda - 10) = 0$

$\Delta = 49 - 40 = 9$

$\lambda_{1,2} = \frac{-7 \pm 3}{-2}$ ↗ 5 ↘ 2

Deci $\lambda \in \{-1, 2, 5\}$ val. proprii dist.
$\Downarrow$
$m_i = 1, \forall\ i = \overline{1,3}$

Nr. ssp. vect = nr. val. distincte

Det. ssp. proprii

$V_{\lambda_1 = -1} = \{ v \in \mathbb{R}^3 / f(v) = \lambda_1 v \}$

$v = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ $\quad \underbrace{(A - \lambda_1 I_3)}_{A + I_3} v = 0_{(3,1)}$

$$\begin{vmatrix} 2 & -2 & 0 \\ -2 & 3 & -2 \\ 0 & -2 & 4 \end{vmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} 2x - 2y = 0 \\ -2x + 3y - 2z = 0 \\ 0 - 2y + 4z = 0 \end{cases}$$

$D_p = \begin{vmatrix} -2 & 3 \\ 0 & -2 \end{vmatrix} = 4 \neq 0 \Rightarrow Rg(A + \lambda_1 I_3) = 2$

x, y nec. princ.
$z = \alpha, \alpha \in \mathbb{R}$ nec. sec.

$$\begin{cases} -2x + 3y = 2\alpha \\ -2y = -4\alpha \Rightarrow y = 2\alpha = x \\ z = \alpha \in \mathbb{R} \end{cases}$$

$V_{\lambda_1} = \{ \alpha \underbrace{(2, 2, 1)}_{v_1} / \alpha \in \mathbb{R} \} = \{ \alpha v_1 / \alpha \in \mathbb{R} \}$

$(A - \lambda_2 I_3) v = 0_{(3,1)}$
$(A + 2 I_3) v = 0$

$$\begin{vmatrix} -1 & -2 & 0 \\ -2 & 0 & -2 \\ 0 & -2 & 1 \end{vmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

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$$\begin{cases} -x-2y+0=0 \Rightarrow x=-2y \\ -2x+0-2z=0 \quad 4y-2z=0 \Leftrightarrow x=-z \\ 0\ -2y+z=0 \quad -2y+z=0 \quad x=-z=-\alpha \end{cases}$$

$y=\frac{\alpha}{2}$

$2\beta(-2,1,1)$

$$V_{\lambda_2}=\{\alpha(\underbrace{-1,\tfrac{1}{2},1}_{v_2})/\alpha\in\mathbb{R}\}$$

$\cancel{A}-\lambda_3 I_3=0$

$A+5I_3=0$

$$\begin{pmatrix} -4 & -2 & 0 \\ -2 & -3 & -2 \\ 0 & -2 & -2 \end{pmatrix}\cdot\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} -4x-2y+0=0 \\ -2x-3y-2z=0 \quad \cancel{y=0} \quad \cancel{3y-4z \Rightarrow y=\tfrac{4}{3}z} \\ 0\ -2\cancel{y}-2z=0 \Rightarrow \cancel{x=z=\alpha} \end{cases}$$

(1) $2x+y=0$

(2) $2x+3y+2z=0$

(3) $y=-z \Rightarrow x=\frac{1}{2}z,\ z=\alpha$

$2\beta(1,-2,2)$

$$V_{\lambda_3}=\{\alpha(\tfrac{1}{2},-1,1)/\alpha\in\mathbb{R}\}$$

d) 1. $m_1+m_2+m_3=3=\dim_{\mathbb{R}}\mathbb{R}^3$

2. $\dim V_{\lambda_i}=m_{\lambda_i},\ \forall i=\overline{1,3}$

$\quad\quad\quad\quad\quad\quad\ \ 1$

$V_{\lambda_1}=\{\lambda v_1/\lambda\in\mathbb{R}\}$

$B_1=\{v_1\}\subset V_{\lambda_1}\Rightarrow\dim V_{\lambda_1}=1$ (bază)

$V_{\lambda_2}=\{\beta v_2/\beta\in\mathbb{R}\}$

$B_2=\{2v_2\}\subset V_{\lambda_2}\Rightarrow\dim V_{\lambda_2}=1.$

$V_{\lambda_3} = \{ \beta v_3 \mid \beta \in \mathbb{R} \} \Rightarrow \underset{\text{baza}}{B_3 = \{v_3\}} \subset V_{\lambda_3} \Rightarrow \dim V_{\lambda_3} = 1$

Deci f endomorfism diag.

e) $B = B_1 \cup B_2 \cup B_3 = \{v_1, v_2, v_3\}$

$\hookrightarrow$ baza în care se realizează f diag.

$$D = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{pmatrix}$$

$\hookrightarrow$ m. diag.

f) Verificare rezultat obținut!

$$\begin{array}{ccc} B_0 & \xrightarrow{C} & B \\ \downarrow & \text{m. de tr.} & \downarrow \\ A & & D \end{array}$$

$C = ($

$\boxed{D = C^{-1} A C}$

Temă!

e) $$C = \begin{pmatrix} 2 & -2 & 1 \\ 2 & 1 & -2 \\ 1 & 2 & 2 \end{pmatrix}$$

$\underline{\quad}$ v_1, v_2, v_3

g) $A^n = ?$, $n \in \mathbb{N}^*$

$A = C D C^{-1}$

$A^n = (C D C^{-1})(C D C^{-1}) \ldots (C D C^{-1})$, $C^{-1}C = I_3$

$A^n = C D^n C^{-1}$, unde $D^n = \begin{pmatrix} (-1)^n & 0 & 0 \\ 0 & 2^n & 0 \\ 0 & 0 & 5^n \end{pmatrix}$.

⑤ **Def:**

Fie V/K sp. vect.

O aplicație $g: V\times V \to K$ s.n. f. b. pe V dacă:

1. $g(\alpha_1 x_1 + \alpha_2 x_2, y) = \alpha_1 g(x_1, y) + \alpha_2 g(x_2, y)$, $(\forall) x_1, x_2, y, y_1, y_2 \in V$, $\alpha_1, \alpha_2, \beta_1, \beta_2 \in K$
2. $g(x, \beta_1 y_1 + \beta_2 y_2) = \beta_1 g(x, y_1) + \beta_2 g(x, y_2)$

Dacă, în plus, $g(x,y) = g(y,x)$, $(\forall) x, y \in V$

atunci g s.n. f. b. simetrică

Fie $B \subset V/K$
$B = \{e_1, \dots, e_n\}$ $\to$ reper vect.

$G = (g_{ij})_{i,j=\overline{1,n}}$, unde $g_{ij} = g(e_i, e_j)$, $(\forall) i,j = \overline{1,n}$
$\to$ m. as. f. b. s. g
$\to$ e simetrică)

$g(x,y) = {}^t x G y$, $(\forall) x, y$

Schimbare de reper

$$\begin{array}{ccc} B & \xrightarrow{C} & B' \\ \downarrow & & \downarrow \\ G & & G' \end{array}$$

$C \to$ m. de tr.

$$\boxed{G' = {}^t C \cdot G \cdot C}$$

Fie $g: V\times V \to K$ f. b. s.

$\to Q: V \to K$, unde $Q(x) = g(x,x)$,

f. pătr. pe V/K, $\forall x \in V$.

Identitatea de polarizare

$g : V \times V \to K$

$$\boxed{g(x,y) = \frac{1}{2}\{Q(x+y) - Q(x) - Q(y)\}, \quad (\forall) x, y \in V}$$

Obs. $g \longleftrightarrow Q$
f.b.s. f.p.

F. canonica: $Q(x) = \lambda_1 x_1^2 + \lambda_2 x_2^2 + \dots + \lambda_r x_r^2$
r - rangul matricei G

$B = \{e_1, \dots, e_n\}$
$x = \sum_{i=1}^{n} x_i e_i$
$\to$ reper canonic

Aducerea unei f.p. la o forma canonica

APLICAȚIE:

Fie $Q : \mathbb{R}^3 \to \mathbb{R}^3$, $Q(x) = Q((x_1, x_2, x_3)) =$
$= 2x_1^2 + 5x_2^2 + 3x_3^2 - 4x_1x_2 - 6x_2x_3$

a) Scrieți forma biliniară simetrica g asociată lui Q

b) Scrieți matricea f.b.s. ~~G și a lui Q~~ $g(Q)$ în rap. cu rep. can. din $\mathbb{R}^3$

a) $\boxed{M_1:}$ Se utilizează ident. de polarizare

Temă $\to$ $g : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$

$$g(x,y) = \frac{1}{2}\left(Q(x+y) - Q(x) - Q(y)\right), \quad (\forall) x, y \in \mathbb{R}^3$$

$Q(x_1, x_2, x_3)$, $Q(y_1, y_2, y_3)$

$\boxed{V_2}$ = DEDUBLARE

$x_1^2 \to x_1 y_1$ $\qquad$ $x_1 x_2 \to \frac{1}{2}(x_1 y_2 + x_2 y_1)$

$x_2^2 \to x_2 y_2$ $\qquad$ $x_1 x_3 \to \frac{1}{2}(x_1 y_3 + x_3 y_1)$

$x_3^2 \to x_3 y_3$ $\qquad$ $x_2 x_3 \to \frac{1}{2}(x_2 y_3 + x_3 y_2)$

$$g(x,y) = 2x_1y_1 + 5x_2y_2 + 3x_3y_3 - 2(x_1y_2 + x_2y_1) - 3(x_2y_3 + x_3y_2),$$

$$\forall x = (x_1, x_2, x_3),\ y = (y_1, y_2, y_3) \in \mathbb{R}^3$$

b)
$$G = \begin{pmatrix} 2 & -2 & 0 \\ -2 & 5 & -3 \\ 0 & 3 & 3 \end{pmatrix}$$

→ m. as. f. b. s. g în reperul canonic

$$g(x,y) = {}^T x\, G\, y = (x_1\ x_2\ x_3)\left(G \right)\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

Seminarul 5 - Geometrie

1. Aducerea unei forme pătratice la o formă canonică
2. Spații vectoriale euclidiene
   - Procedeul de ortonormalizare Gram-Schmidt

1.

Fie $g: \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$

$g(x,y) = 2x_1y_1 + x_2y_2 - 2x_1y_2 - 2x_2y_1 - 2x_2y_3 - 2x_3y_2$

$(\forall)\ x = (x_1, x_2, x_3)$
$y = (y_1, y_2, y_3) \in \mathbb{R}^3$

a) $g \in$ f.b.s.

b) Scrieți matricea lui g în raport cu B_0 din $\mathbb{R}$ $(\{e_1, e_2, e_3\})$

c) g — cu baza B,
$B = \{(1,1,1), (2,-1,2), (1,3,-3)\}$

d) să se scrie forma pătratică Q cores. lui g și să se aducă la o formă canonică utilizând toate cele 3 metode
1. metoda Gauss
2. metoda Jacobi
3. M. VP. (mat. transf. ort.)

Rez.

a) $g: \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$

1) $g(\alpha_1 x_1 + \alpha_2 x_2, y) = \alpha_1 g(x_1, y) + \alpha_2 g(x_2, y)$ $(\forall)\ x, x_1, x_2$

2) $g(x, \beta_1 y_1 + \beta_2 y_2) = \beta_1 g(x, y_1) + \beta_2 g(x, y_2)$ $y, y_1, y_2 \in \mathbb{R}^3$

3) $g(x,y) = g(y,x)$, $(\forall)\ x, y \in \mathbb{R}^3$ $\quad \alpha_{1,2}, \beta_{1,2} \in \mathbb{R}$

f.b.s. $\Rightarrow$ Fie (1) și (3) sau (2) și (3)

(1)(2) Matriceal, $g(x,y) = {}^t x\, G\, y$, $(\forall)\ x = (1\,\cancel{3})$
$\quad (-)(\ )(1)$

(3): G

$y = (1\,\cancel{3})$

b) (B) $G = \begin{pmatrix} 2 & -2 & 0 \\ -2 & 1 & -2 \\ 0 & -2 & 0 \end{pmatrix}$

$\to$ m. as. formei biliniare sim. g în raport cu B_0

G e simetrică

c) $B_0 \xrightarrow{C} B$: C – matricea de trecere de la B_0 la B.

$\downarrow \qquad \downarrow$

$G \qquad G'$

$$\boxed{G' = {}^tC \cdot G \cdot C}$$

d $C = \begin{pmatrix} 1 & 2 & 1 \\ 1 & -1 & 3 \\ 1 & 2 & -3 \end{pmatrix}$ (T)

d) $g: \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$, f. b. s.

$Q: \mathbb{R}^3 \to \mathbb{R}$

$Q(x) = g(x, x)$, $(\forall) x = (x_1, x_2, x_3) \in \mathbb{R}^3$

f. b. s. g.

$Q(x) = Q(x_1, x_2, x_3) =$

$= 2x_1^2 + x_2^2 - 2x_1x_2 - 2x_2x_1 - 2x_2x_3 - 2x_3x_2 =$

$= 2x_1^2 + x_2^2 - 4x_1x_2 - 4x_2x_3$

Deci $Q(x) = 2x_1^2 + x_2^2 - 4x_1x_2 - 4x_2x_3$

1) Gauss: Metoda construcției de pătrate

$\to$ merge mereu

$Q(x) = 2(x_1^2 - 2x_1x_2) + x_2^2 - 4x_2x_3 =$

$= 2(x_1 - x_2)^2 - 2x_2^2 + x_2^2 - 4x_2x_3 =$

$= 2(x_1 - x_2)^2 - x_2^2 - 4x_2x_3 =$

$= 2(x_1 - x_2)^2 - (x_2^2 + 4x_2x_3) =$

$= 2\underbrace{(x_1 - x_2)}_{y_1}^2 - (\underbrace{x_2 + 2x_3}_{y_2})^2 + 4\underbrace{x_3}_{y_3}^2$

$Q(x) = 2y_1^2 - y_2^2 + 4y_3^2$,

unde $x = (y_1, y_2, y_3)$ în raport cu B_2

sch. de reper $\begin{cases} y_1 = x_1 - x_2 \\ y_2 = x_2 + 2x_3 \\ y_3 = x_3 \end{cases}$

signatura: $+ - +$

$s = 2 - 1 = 1$ (nr. >0 - nr <0)

index: nr t. $<0 = 1$

2) Met. Jacobi

$Q : V \to \mathbb{R},\ Q(x) = \sum_{i,j=1}^{n} g_{ij} x_i x_j\ (\forall) x = \sum_{i=1}^{n} x_i e_i \in V$

$\Delta_1 = g_{11},\ \Delta_2 = \begin{vmatrix} g_{11} & g_{12} \\ g_{12} & g_{22} \end{vmatrix}, \dots, \Delta_n = \det G$

Dacă: $\Delta_i \neq 0,\ (\forall) i = \overline{1,n}$

Atunci $(\exists) B \subset V$ ~~a.î.~~ bază

a.î.

$Q(x) = \frac{1}{\Delta_1} y_1^2 + \frac{\Delta_1}{\Delta_2} y_2^2 + \dots + \frac{\Delta_{n-1}}{\Delta_n} y_n^2$

, unde $x = (y_1, \dots, y_n)$ în B

În cazul nostru,

$\Delta_1 = 2 \neq 0$

$\Delta_2 = \begin{vmatrix} 2 & -2 \\ -2 & 1 \end{vmatrix} = -2 \neq 0$

$\Delta_3 = \begin{vmatrix} 2 & -2 & 0 \\ -2 & 1 & -2 \\ 0 & -2 & 0 \end{vmatrix} = -8 \neq 0$

$Q(x) = \frac{1}{2} y_1^2 - 1 y_2^2 + \frac{1}{4} y_3^2$, unde $x = (y_1, y_2, y_3)$ în B

$+ - +$ $s = 1$

$i = 1$.

3) Metoda valorilor proprii (trans. ortog.)

G e m. simetrica

↳ Toate valorile proprii ale lui G sunt reale

Rezolv ec. caract. $\boxed{\det(G-\lambda I_n)=0}$ în $\mathbb{R}$

↳ obțin $\lambda_1, \dots, \lambda_r$

$$Q(x)=\lambda_1 x_1^2+\lambda_2 x_2^2+\dots+\lambda_r x_r^2,\ r=\operatorname{rg} G.$$

În cazul nostru:

Rezolv. ec. carac.:

$$\det(G-\lambda I_3)=0 \text{ în } \mathbb{R}$$

$$G=\begin{pmatrix} 2 & -2 & 0 \\ -2 & 1 & -2 \\ 0 & -2 & 0 \end{pmatrix}$$

$$\begin{vmatrix} 2-\lambda & -2 & 0 \\ -2 & 1-\lambda & -2 \\ 0 & -2 & -\lambda \end{vmatrix}=0$$

$$-\lambda(2-\lambda)(1-\lambda)+4\lambda-4(2-\lambda)=0$$

$$-\lambda^3+3\lambda^2+6\lambda-8=0$$

	λ^3	λ^2	λ	λ^0
	-1	3	6	-8
1	-1	2	8	0

R

$$(\lambda-1)(-\lambda^2+2\lambda+8)=0$$

$$-(\lambda-1)(\lambda^2-2\lambda-8)=0$$

↳ $\Delta=36$

$$\lambda_{1,2}=\frac{2\pm 6}{2}\ \begin{matrix} 4 \\ -2 \end{matrix}$$

$$Q(x)=x_1^2-2x_2^2+4x_3^2,$$

$+\ -\ +$ $\quad s=1$ $\quad x=(x_1,x_2,x_3)$ în B

$t=1$

2. **Inegalitatea C-B-S**

În $(\forall)$ sp. vect. euclidian $(E/\mathbb{R}, \langle , \rangle)$ (avem) p.s.

avem: $|\langle x,y\rangle| \leq \|x\| \|y\|, \forall x,y \in E$

" = " $\Leftrightarrow \{x,y\}$ s.v. lin. dep. ($x \| y$)

$\cos \underbrace{\sphericalangle(x,y)}_{\theta} \stackrel{def.}{=} \frac{\langle x,y\rangle}{\|x\| \|y\|}, \forall x,y \in E^*$
$\theta \in [0, \pi)$

Procedeul de orton. G-S

$\boxed{Th}$ $(\forall)$ $\{f_1, \dots, f_n\}$ c E
b. arbitrara
$\Rightarrow \exists \{e_1, \dots, e_n\}$ c E
b. orton.
$(\langle e_i, e_j\rangle = \delta_{ij}, (\forall) i,j = \overline{1,n})$

a.î. $\overline{\{f_1, \dots, f_i\}} = \overline{\{e_1, \dots, e_i\}}, (\forall) i = \overline{1,n}$

V1 cu formule de la curs

$$\begin{cases} e_1' = f_1 \\ e_i' = f_i - \sum_{j=1}^{i-1} \frac{\langle f_i, e_j'\rangle}{\|e_j'\|^2} e_j', (\forall) i = \overline{2,n} \end{cases}$$

B ortog.
$\Downarrow$
B orton. $= \{e_i = \frac{e_i'}{\|e_i'\|}\}, i = \overline{1,n}$

V2 seminar

$e_1' = f_1, e_1 = \frac{f_1}{\|f_1\|}$

$\boxed{e_i' = f_i - \sum_{j=1}^{i-1} \langle f_i, e_j\rangle e_j}, (\forall) i = \overline{2,n}$

$\boxed{e_i = \frac{e_i'}{\|e_i'\|}}$

Aplicații:

① Fie $(\mathbb{R}^3/\mathbb{R}, \langle , \rangle)$ sp. vect. euclidian p.s.c.

Să se const. o bază ortogonală, pornind de la baza
$B = \{f_1 = (1,0,2), f_2 = (2,1,1), f_3 = (0,1,1)\}$
utilizând P.o.G-S.

Aplic V1

$\|e_1'\|^2 = \langle e_1', e_1' \rangle = 5$

$e_1' = f_1 = (1,0,2)$

$e_2' = f_2 - \frac{\langle f_2, e_1' \rangle}{\|e_1'\|^2} e_1' =$

$\|e_2'\|^2 = \frac{70}{25}$

$= (2,1,1) - \frac{4}{5}(1,0,2) =$

$= \left(\frac{6}{5}, 1, -\frac{3}{5}\right) = \frac{1}{5}(6, 5, -3)$

$e_3' = f_3 - \frac{\langle f_3, e_1' \rangle}{\|e_1'\|^2} \cdot e_1' - \frac{\langle f_3, e_2' \rangle}{\|e_2'\|^2} \cdot e_2' =$

$= (0,1,1) - \frac{2}{5} \cdot (1,0,2) - \frac{\frac{2}{5}}{\frac{70}{25}} \cdot \frac{1}{5}(6,5,-3) =$

$= (0,1,1) - \frac{2}{5}(1,0,2) - \frac{1}{35}(6,5,-3) =$

$= \left(-\frac{4}{7}, \frac{6}{7}, \frac{2}{7}\right) = \frac{1}{7}(-4, 6, 2)$

$B' = \{e_1', e_2', e_3'\} \subset \mathbb{R}^3$

b. ortog. $(\langle e_i', e_j' \rangle = \delta_{ij}, (\forall) 1 \le i,j \le 3)$

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② Să se constr. o bază ortonormată pornind de la Baza

$B = \{ f_1 = (1,1,1), f_2 = (1,1,-1), f_3 = (1,-1,-1) \}$

$\boxed{\text{T: ca la ②, } B = \{ f_1 = (-1,0,2), f_2 = (1,-1,1), f_3 = (2,1,0) \}}$

V_2 $e_1' = f_1 = (1,1,1)$, $e_1 = \frac{f_1}{\|f_1\|} = \frac{1}{\sqrt{3}}(1,1,1)$

$e_2' = f_2 - \langle f_2, e_1 \rangle e_1 =$

$= (1,1,-1) - \frac{1}{\sqrt{3}} \cdot \frac{1}{\sqrt{3}} (1,1,1) =$

$\frac{1}{3}$ $\quad 1 \cdot \frac{1}{\sqrt{3}} + 1 \cdot \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{3}}$

$= \left(\frac{2}{3}, \frac{2}{3}, \frac{-4}{3} \right) = \frac{2}{3}(1,1,-2)$

$e_2 = \frac{e_2'}{\|e_2'\|} = \cancel{\frac{3}{2}} \cdot \frac{1}{\sqrt{6}} \cdot \cancel{\frac{2}{3}} (1,1,-2) = \frac{1}{\sqrt{6}}(1,1,-2)$

$e_3' = f_3 - \langle f_3, e_1 \rangle e_1 - \langle f_3, e_2 \rangle e_2 =$

$= (1,-1,-1) - \left(\frac{-1}{\sqrt{3}} \right) \cdot \frac{1}{\sqrt{3}} (1,1,1) - \frac{2}{\sqrt{6}} (1,-1,2) \cdot \frac{1}{\sqrt{6}} =$

$= (1,-1,-1) + \frac{1}{3}(1,1,1) - \frac{1}{3}(1,+1,-2) =$

$= (1,-1,0)$

$\|e_3'\| = \sqrt{2}$

$e_3 = \frac{e_3'}{\|e_3'\|} = \frac{1}{\sqrt{2}}(1,-1,0)$

B orton $= \{ e_1, e_2, e_3 \}$

3. În $E^3 = (\mathbb{R}^3/\mathbb{R}, \langle,\rangle_{p.s.c.})$

$f_1 = (3,1,2)$, $f_2 = (-2,2,-1)$

a) Calculați $\|f_1\|$, $\|f_2\|$, $(\widehat{f_1, f_2}) = ?$

b) det. un vector nenul $f_3 \in E^{3*}$ a.î. $f_3 \perp f_1, f_2$

c) pt. f_3 de la b, orton. sist. (f_1, f_2, f_3) prin G.-S.

a) $\|f_1\| = 3$

$\|f_2\| = 3$

$\cos(\underbrace{\widehat{f_1, f_2}}_{\theta}) = \dfrac{\langle f_1, f_2\rangle}{\|f_1\|\,\|f_2\|} = \dfrac{-4}{9}$

$\cos\theta = \dfrac{-4}{9} \Rightarrow \theta = \arccos\dfrac{-4}{9} = \pi - \arccos\dfrac{4}{9}$

b) $\langle f_3, f_1\rangle = 0$

$\langle f_3, f_2\rangle = 0$

$f_3 = (x, y, z)$

$$\begin{cases} 2x + \cancel{4}y + 2z = 0 \\ -2x + 2y - z = 0 \end{cases}$$

Geometrie – seminarul 5

1. Suplementul ortogonal
   Apl. ortogonale
2. Conice în planul euclidian E^2
   pe ec. generală

①

1. Def.

Fie sp. vectorial euclidian

$$(\underset{/\mathbb{R}}{E}, \underset{\text{p.s.}}{\langle , \rangle})$$

$\underset{\text{ssp. vect.}}{U \subseteq E}$

$U^\perp = \{v \in E \mid v \perp u, \forall u \in U\} \underset{\text{ssp. vect.}}{\subseteq} E$

↳ complementul ortogonal al lui U

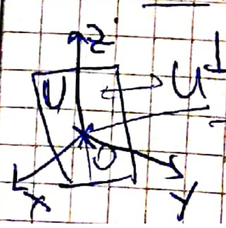
Dacă, în plus,

$U \oplus U^\perp = E$

$(U \cap U^\perp = \{0_E\})$ atunci $U^\perp$ s.n. suplementul ortogonal al lui U

$\boxed{P}$ $(\forall)\ \underset{\text{ssp. vect.}}{U \subseteq E} \Rightarrow (\exists)!\ U^\perp$ supl. său ortogonal

Ex: $E^3 = (\mathbb{R}^3/\mathbb{R}, \underset{\text{p.s.c.}}{\langle , \rangle})$ sp. vect. euclidian



$U^\perp$ – dr. perp. dusă prin origine

Def.

Fie $(E_1/_{\mathbb{R}}, \langle,\rangle_1)$
$(E_2/_{\mathbb{R}}, \langle,\rangle_2)$ → 2 sp. vect. euclidiene

O aplicație
$f: E_1 \to E_2$ s.n. apl. ortog.

dacă $\langle f(x), f(y)\rangle_2 = \langle x, y\rangle_1$, $(\forall) x, y \in E_1$

C.P.
$E_1 = E_2$ $(=E)$
Def. Un endom. $f: E \to E$ s.n. transf. ortogonală

dacă $\langle f(x), f(y)\rangle = \langle x, y\rangle$, $(\forall) x, y \in E$

$\boxed{Th}$
Un endomorfism
$f: E \to E$ este transf. ortog.
$\Updownarrow$
matricea sa asociată A, în raport cu un reper ortonormat, să fie ortogonală

Adică: ${}^tA \cdot A = I_n$ $(A^{-1} = {}^tA)$

Reper ortonormat

$(E/_{\mathbb{R}}, \langle,\rangle)$ sp. vect. euclidian
$\dim_{\mathbb{R}} E = n$

$\mathcal{R} = \{e_1, \dots, e_n\}$ s.n. reper ortonormat dacă $\langle e_i, e_j\rangle = \delta_{ij}$, $(\forall) i, j = \overline{1,n}$

Obs. Rep. canonic din $E^n = (\mathbb{R}^n/\mathbb{R}, \underset{p.s.c.}{<,>})$

este un reper ortonormat

1. Aplicații

Fie $E^3 = (\mathbb{R}^3/\mathbb{R}, \underset{p.s.c.}{<,>})$ sp. vect. eucl.

Det. supl. ortog. al următoarelor ssp. vectoriale

a) $U = \langle \overbrace{(2,1,1)}^{u_1}, \overbrace{(1,-1,0)}^{u_2} \rangle$ plan vect, dim = 2

b) $V = \langle (3,1,2) \rangle$

Rez.:

a) $\underset{\text{bază}}{\{u_1, u_2\}} \subset U$

$U^\perp = \{ v \in E^3 / v \perp u_1, v \perp u_2 \}$

↳ supl. ortog. al lui U

$\Leftrightarrow \begin{cases} \langle v, u_1 \rangle = 0 \\ \langle v, u_2 \rangle = 0 \end{cases} \Leftrightarrow \begin{cases} 2x + y + z = 0 \\ x - y = 0 \end{cases}$

$U^\perp = \{ (x,y,z) \in E^3 / \begin{cases} 2x + y + z = 0 \\ x - y = 0 \end{cases} \}$

Rezolvăm

$\begin{cases} 2x + y + z = 0 \\ x - y = 0 \end{cases}$

$rg \left(\begin{pmatrix} 2 & 1 & 1 \\ 1 & -1 & 0 \end{pmatrix} \right) = 2$

x, y nec. pr.
$z = \alpha, \alpha \in \mathbb{R}$ nec. sec.

$\begin{cases} 2x + y = -\alpha \\ x - y = 0 \end{cases}$ $(+)$

$3x = -\alpha \Rightarrow \begin{cases} x = \frac{-\alpha}{3} \\ y = -\frac{\alpha}{3} \\ \cancel{z} \in \mathbb{R} = \alpha \end{cases}$

Deci $U^{\perp} = \{(-\frac{\alpha}{3}; -\frac{\alpha}{3}; \alpha) / \alpha \in \mathbb{R}\}$

$\alpha = 3\beta$

$U^{\perp} = \{(-\beta; -\beta; 3\beta) / \beta \in \mathbb{R}\}$

$U^{\perp} = \{\beta \underline{(-1, -1, 3)} / \beta \in \mathbb{R}\}$
↳ dr. vect.

b) $U = \langle \overbrace{(3, 1, 2)}^{u} \rangle$ → dr. vect.

$\underbrace{\{u\}}_{\text{bază}} \subset U$

$\dim U = 1$ (dr. vect.)

$U^{\perp} = \{ \underset{\underset{(x,y,z)}{||}}{v} \in \mathbb{E}^3 / v \perp u \}$

$\langle v, u \rangle = 0 \Leftrightarrow 3x + y + 2z = 0$

det. + bază + pr. G-S la ea → $U^{\perp} = \{(x, y, z) \in \mathbb{E}^3 / 3x + y + 2z = 0\}$
spl. ortogonal

2. Fie $E^3 = (\mathbb{R}^3/\mathbb{R}, +, \cdot, \langle\, ,\rangle_{p.s.c.})$ sp. vect. eucl.

$B_0 = \{e_1, e_2, e_3\}$ rep. canonic din E^3

Stabiliți dacă urm. apl. liniare sunt transf. ortogonale:

$f : E^3 \to E^3$

a) $f(e_1) = e_1$, $f(e_2) = \frac{1}{2} e_2 + \frac{\sqrt{3}}{2} e_3$, $f(e_3) = -\frac{\sqrt{3}}{2} e_2 + \frac{1}{2} e_3$

b) $f(e_1) = e_1 + e_2$, $f(e_2) = e_2 + e_3$, $f(e_3) = e_3 + e_1$

c) $f(e_1) = \frac{2}{3} e_1 + \frac{2}{3} e_2 - \frac{1}{3} e_3$, $f(e_2) = \frac{2}{3} e_1 - \frac{1}{3} e_2 + \frac{2}{3} e_3$

$f(e_3) = -\frac{1}{3} e_1 + \frac{2}{3} e_2 + \frac{2}{3} e_3$

Ox = axă de rotație

Rez.

a) $A = \begin{array}{c} \\ \left(\begin{array}{ccc} 1 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ 0 & \frac{\sqrt{3}}{2} & \frac{1}{2} \end{array} \right) \\ \begin{array}{ccc} f(e_1) & f(e_2) & f(e_3) \end{array} \end{array}$ de $\frac{\pi}{3}$ → m. as. endom. f. în raport cu rep. can. (rep. ortn.)

DA!

i.e. $\langle e_i, e_j \rangle = \delta_{ij}$, $\forall i,j = \overline{1,3}$

${}^tA \cdot A = I_3$ (A m. ortog.) f transf. ortog. (izometrie)

↓

rotație în jurul axei Ox, în pl. yOz, de $\frac{\pi}{3}$ (în sens direct trigonometric)

$${}^tA = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & +\frac{\sqrt{3}}{2} \\ 0 & -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$$

$${}^tA \cdot A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I_3$$

b) $A = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$ → m. as. end. f. în rap. cu rep. can

NU! ${}^tA \cdot A \neq I_3$

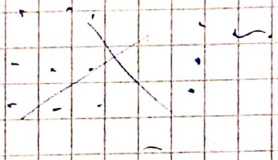
c) $A = \begin{pmatrix} \frac{2}{3} & \frac{2}{3} & -\frac{1}{3} \\ \frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \\ -\frac{1}{3} & \frac{2}{3} & \frac{2}{3} \end{pmatrix}$

$\boxed{\text{Fie } A \in O(n) = \{A \mid {}^tA \cdot A = I_n\} \quad \det A \in \{\pm 1\}}$

${}^tA \cdot A = I_3$

DA !

	Ox	Oy	Oz
Ox'	α_1	α_2	α_3
Oy'	β_1	β_2	β_3
Oz'	γ_1	γ_2	γ_3



2) <u>Def</u> $\gamma : f(x,y) = 0$, ~~unde $f(x,y) = a_{11}$~~

unde $f(x,y) = a_{11}x^2 + a_{22}y^2 + 2a_{12}xy + 2a_{13}x + 2a_{23}y + a_{33} = 0$.

$a = (a_{ij})_{i,j = \overline{1,2}}$

$A = (a_{ij})_{i,j = \overline{1,3}}$

$a_{ij} = a_{ji} \quad (\forall)\, i,j = \overline{1,3}$

${}^tb = (a_{13} \; a_{23})$

$c = (a_{33})$

Matriceal, $\gamma : f(X) = {}^tX\, a\, X + 2\, {}^tbX + c = 0$,

unde $X = \begin{pmatrix} x \\ y \end{pmatrix}$

$\delta = \det a$

$\Delta = \det A$

$\left[\begin{array}{l} \delta \neq 0 \Rightarrow \gamma \text{ are centru unic} \\ \delta = 0 \Rightarrow \underline{NU} \end{array}\right.$

$\Delta \neq 0 \to \gamma$ nedeg.
$\Delta = 0 \to \gamma$ degen.

Clasif. izom. a conicelor

I $\Delta \neq 0$ (nedeg.)
- $d > 0 \to$ ELIPSĂ (reală $\Delta < 0$ sau imag. $\Delta < 0$)
- $d = 0 \to$ PARABOLĂ
- $d < 0 \to$ HIPERBOLĂ

II $\Delta = 0$ (deg.)
- $d > 0 \to$ punct dublu
- $d = 0 \to \emptyset$ sau o per. de dr. ∥
- $d < 0 \to$ o per. de dr. conc.

Aplicații

Fie $\gamma: x^2 - 3xy + y^2 - 4x + 2y - 1 = 0$

Clasificați izometric conica dată:
a) precizați natura și genul conicei și să se reducă la forma canonică
b) să se scrie schimbarea izom. de reper
c) să se scrie ec. axelor de simetrie ale conicei date în raport cu reperul inițial

Rez.

a) $a = \begin{pmatrix} 1 & -\frac{3}{2} \\ -\frac{3}{2} & 1 \end{pmatrix} \to d = -\frac{5}{4} < 0$

$A = \begin{pmatrix} 1 & -\frac{3}{2} & -2 \\ -\frac{3}{2} & 1 & 1 \\ -2 & 1 & -1 \end{pmatrix} \to \Delta = -1 + \cancel{3} + \cancel{3} - 4 + \frac{9}{4} + 1 = \frac{9}{4} \neq 0$

$\Delta \neq 0 \to$ conică nedegenerată
$d < 0 \to$ genul = HIPERBOLA

d) Aducem conica la forma canonică!

$\delta \neq 0 \Rightarrow \gamma$ are centrul unic

$P_0(x_0, y_0)$

$$\begin{cases} \frac{\partial f}{\partial x} = 0 \\ \frac{\partial f}{\partial y} = 0 \end{cases} \Leftrightarrow \begin{cases} 2x - 3y - 4 = 0 \\ -3x + 2y + 2 = 0 \end{cases}$$

$$\Updownarrow$$

$$\begin{cases} 2x - 3y = 4 \\ -3x + 2y = -2 \end{cases}$$

$$x = -\frac{2}{5}$$

$$y = -\frac{8}{5}$$

$P_0\left(-\frac{2}{5}; -\frac{8}{5}\right)$ centrul conicei

Efectuez transl.

$$t \begin{cases} x' = x - x_0 \\ y' = y - y_0 \end{cases} \Leftrightarrow \begin{cases} x' = x + \frac{2}{5} \\ y' = y + \frac{8}{5} \end{cases} \Leftrightarrow \begin{cases} x = x' - \frac{2}{5} \\ y = y' - \frac{8}{5} \end{cases}$$

$$t(\gamma): x'^2 - 3x'y' + y'^2 + f(x_0, y_0) = 0$$

$$\frac{\Delta}{\delta} = -\frac{9}{5}$$

$$t(\gamma): x'^2 - 3x'y' + y'^2 - \frac{9}{5} = 0$$

$$P_0 \Rightarrow t(P_0) = 0.$$

Alegem două axe ortogonale ale $t(\gamma)$.

Direcțiile lor sunt date de doi vectori proprii ortogonali ai lui a.

Det. val. proprii ale lui a, rezolvând ec. caract.

$$\det(a - S I_2) = 0 \text{ în } \mathbb{R}$$

$\det(a - sI_2) = 0$

$\hookrightarrow s^2 - Is + d = 0$, unde $I = \operatorname{Tr} a$

(ec. seculară)

$$\begin{vmatrix} 1-s & -\frac{3}{2} \\ -\frac{3}{2} & 1-s \end{vmatrix} = 0 \Leftrightarrow (1-s)^2 - \left(\frac{3}{2}\right)^2 = 0$$

$$\left(-s - \frac{1}{2}\right)\left(\frac{5}{2} - s\right) = 0$$

$\hookrightarrow s_1 = \frac{5}{2}$ val. proprii

$\hookrightarrow s_2 = -\frac{1}{2}$ ale lui a

$\cancel{V_{\mathrm{I}}}$

V_{S_1}

$$\begin{pmatrix} -\frac{3}{2} & -\frac{3}{2} \\ -\frac{3}{2} & -\frac{3}{2} \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Leftrightarrow u + v = 0$$

$\hookrightarrow v = -u$

$V_{s_1} = \{u(1,-1) / u \in \mathbb{R}\}$

$\cancel{V_{S_2}}$

$$\begin{pmatrix} +\frac{3}{2} & -\frac{3}{2} \\ -\frac{3}{2} & +\frac{3}{2} \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Leftrightarrow u - v = 0$$

$\hookrightarrow u = v$

$V_{s_2} = \{u(1,1) / u \in \mathbb{R}\}$

$f_1 = \frac{u_1}{\|u_1\|} = \frac{1}{\sqrt{2}}(1,-1)$

$f_2 = \frac{u_2}{\|u_2\|} = \frac{1}{\sqrt{2}}(1,1)$

Efectuez rotația r $\begin{cases} x'' = \frac{1}{\sqrt{2}}(x' - y') \\ y'' = \frac{1}{\sqrt{2}}(x' + y') \end{cases}$

$f_1^* \to (1,0)$

$f_2 \to (0,1)$

$\begin{cases} x' = \frac{1}{\sqrt{2}}(x'' + y'') \\ y' = \frac{1}{\sqrt{2}}(-x'' + y'') \end{cases}$

$(\mathrm{rot})(\gamma)$: $s_1 x''^2 + s_2 y''^2 + \frac{\Delta}{\delta} = 0$

$\frac{5}{2}x''^2 - \frac{1}{2}y''^2 - \frac{9}{5} = 0$

$$\frac{x''^2}{\frac{18}{25}} - \frac{y''^2}{\frac{18}{5}} - 1 = 0$$

$$\frac{x''^2}{\left(\frac{3\sqrt{2}}{5}\right)^2} - \frac{y''^2}{\left(\frac{3\sqrt{2}}{\sqrt{5}}\right)^2} - 1 = 0$$

$\boxed{H}$ $\boxed{\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1}$

b) rot: $\begin{cases} x'' = \frac{1}{\sqrt{2}}\left(x - y - \frac{6}{5}\right) & (x' - y' = x - y - \frac{6}{5}) \\ y'' = \frac{1}{\sqrt{2}}\left(x + y + 2\right) & (x' + y' = x + y + 2) \end{cases}$

$\hookrightarrow$ (rototrans)

$\hookrightarrow$ sch. izom. de reper.

c) V1) R. inițial. $\rightarrow (rot)^{-1}(A_1): x - y - \frac{6}{5} = 0$

$(rot)^{-1}(A_2): x + y + 2 = 0$

R. finală A_1: $x'' = 0$

A_2: $y'' = 0$

V2) $A_2: \frac{x + \frac{2}{5}}{1} = \frac{y + \frac{8}{5}}{-1} \Leftrightarrow -x - \frac{2}{5} = y + \frac{8}{5}$ — Axa transv.

$\boxed{x + y + 2 = 0.}$ $\quad y'' = 0.$

$A_1: \frac{x + \frac{2}{5}}{1} = \frac{y + \frac{8}{5}}{1} \Leftrightarrow x + \frac{2}{5} = y + \frac{8}{5}$

Parabolă

$x - y - \frac{6}{5} = 0$

Temă: Fie $\gamma: x^2 - 2xy + y^2 - 2x + 4y + 1 = 0$

$\rightarrow$ rot $\rightarrow$ trans!

(S7) - Geometrie

1. Conice în pl. euclidian $E^2 = (R^2/R, \langle , \rangle_{p.s.c.})$
(pe cr. gen!

2. Geom. afină
Sp. afine
Subsp. afine
Op. cu subsp.

(S8) Transf. afine

1. Fie $\gamma: x^2 - 2xy + y^2 - 2x + 4y + 1 = 0$
Clasif. izom. conica γ.

<u>Rez:</u> $a = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$, $\delta = \det a = 2 \Rightarrow$ genul PARABOLĂ

$A = \begin{pmatrix} 1 & -1 & -1 \\ -1 & 1 & 2 \\ -1 & 2 & 1 \end{pmatrix}$, $\Delta = \det A = -1 \neq 0$
$\Downarrow$
γ e nedeg.

$\delta = 0 \Rightarrow \gamma$ nu are centru unic
Det. val. proprii ale lui a, rez. ec. carac.
$\det(a - \lambda I_2) = 0$, în $\mathbb{R}$

$S^2 - \underset{\substack{\parallel \\ \mathrm{Tr}\, a}}{I} S + \delta = 0$ (ec. seculară)

$S^2 - 2S = 0 \Leftrightarrow S(S-2) = 0 \quad \begin{cases} S_1 = 2 \\ S = 0 \end{cases}$ val. pr.

$V_{S_1} \begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Leftrightarrow u + v = 0$
$v = -u$

$V_{S_2} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Leftrightarrow u - v = 0$
$\parallel$
$u = v.$

$V_{S_1} = \{ u\underbrace{(+1, -1)}_{u_1} / u \in \mathbb{R} \}, \; f_1 = \frac{u_1}{\|u_1\|} = \frac{1}{\sqrt{2}}(1, -1)$

$V_{S_2} = \{ u\underbrace{(1, 1)}_{u_2} / u \in \mathbb{R} \}, \; f_2 = \frac{u_2}{\|u_2\|} = \frac{1}{\sqrt{2}}(1, 1)$

$r: \begin{cases} x' = \frac{1}{\sqrt{2}}(x - y) \\ y' = \frac{1}{\sqrt{2}}(x + y) \end{cases} \rightarrow \begin{cases} x = \frac{1}{\sqrt{2}}(x' + y') \\ y = \frac{1}{\sqrt{2}}(-x' + y') \end{cases}$

(rot) $\frac{\pi}{4}$, în sens d. trig. $R^t R = I_2 \Rightarrow R^{-1} = {}^t R$

$r(\gamma): 2x'^2 - 2 \cdot \frac{1}{\sqrt{2}}(x' + y') + 4 \cdot \frac{1}{\sqrt{2}}(-x' + y') + 1 = 0$

$2x'^2 - 3\sqrt{2}\, x' + \sqrt{2}\, y' + 1 = 0$

$$2\left(x'^2 - \frac{3}{\sqrt{2}}x'\right) + \sqrt{2}y' + 1 = 0$$

$$2\left(x' - \frac{3}{2\sqrt{2}}\right)^2 + \sqrt{2}y' + 1 - 2\cdot\frac{9}{8} = 0$$

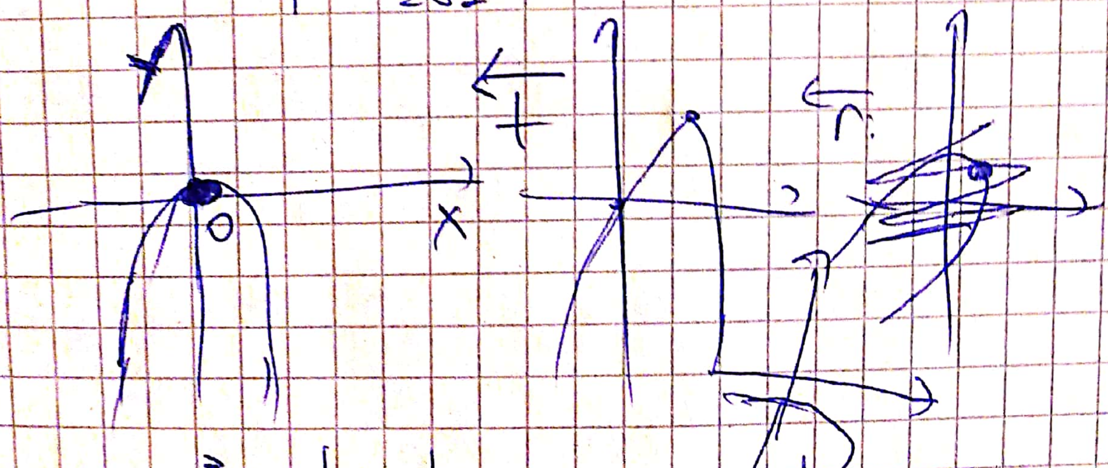
$$2\left(x' - \frac{3}{2\sqrt{2}}\right)^2 + \sqrt{2}\left(y' - \frac{5}{4\sqrt{2}}\right) = 0$$

$$t: \begin{cases} x'' = x' - \frac{3}{2\sqrt{2}} \\ y'' = y' - \frac{5}{4\sqrt{2}} \end{cases}$$

$$(t \circ r)(\gamma): 2x''^2 + \sqrt{2}y'' = 0$$

$$x''^2 = -\frac{1}{\sqrt{2}}y'' \quad (x''^2 = 2py'')$$

$$p = \frac{1}{2\sqrt{2}}$$ (PARABOLA



Scrieți schimbarea izometrică de reper care a condus la forma canonică găsită.

$$h: \begin{cases} x'' = \frac{1}{\sqrt{2}}(x - y) - \frac{3}{2\sqrt{2}} \\ y'' = \frac{1}{\sqrt{2}}(x + y) - \frac{5}{4\sqrt{2}}. \end{cases}$$

Ec. axei parabolei în $\mathcal{R}$ inițial

$\mathcal{R}_f : x'' = 0 \Leftrightarrow \frac{1}{\sqrt{2}}(x-y) - \frac{3}{2\sqrt{2}} = 0$

$\Updownarrow$

$x - y - \frac{3}{2} = 0$

$V : \begin{cases} x'' = 0 \\ y'' = 0 \end{cases} \Leftrightarrow \begin{cases} x - y - \frac{3}{2} = 0 \\ x + y - \frac{5}{4} = 0 \end{cases} \Rightarrow x = \frac{11}{8}$

$\Downarrow$

$y = -\frac{1}{8}$

2. Geometrie afină

$(\mathcal{A}, V/K, \varphi)$ - sp. afin.

$(\mathbb{R}^n, \mathbb{R}^n/\mathbb{R}, \varphi)$ - sp. afin n-dim

$\hookrightarrow$ str. can.

$\varphi : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, str. afină can.

$\varphi((x_1, \dots, x_n), (y_1, \dots, y_n))$

$\shortparallel$

$(y_1 - x_1, \dots, y_n - x_n)$

Def Fie $(\mathcal{A}, V/K, \varphi)$ sp. afin

Sn. centru de greutate (baricentru) al sistemului $\mathcal{S} = \{P_1, \dots, P_n\}$ cât cu ponderile

$\alpha_1, \dots, \alpha_n \in K$,

pct. G def. de val.: $\sum_{i=1}^{n} \alpha_i = 1$

$\overrightarrow{OG} = \alpha_1 \overrightarrow{OP_1} + \dots + \alpha_n \overrightarrow{OP_n}$, $O \in \mathcal{E}$ punct arb.

$G = \alpha_1 P_1 + \alpha_2 P_2 + \dots + \alpha_n P_n$

$\hookrightarrow$ echibaricentru: $\alpha_1 = \alpha_2 = \dots = \alpha_n = \frac{1}{n}$

Aplicație

$(\mathcal{E}, V/\mathbb{R}, \varphi)$ pl. afin geom.

$\varphi : \mathcal{E} \times \mathcal{E} \to V$

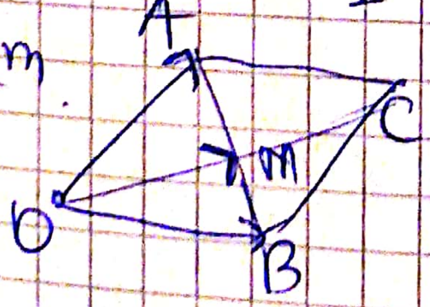
$\varphi(A, B) = \overrightarrow{AB}$, $(\forall) A, B \in \mathcal{E}$

M - mijl. seg. $[AB] \subset \mathcal{E}$

M - echib. sist. format din $\{A, B\}$

i.e. $M = \frac{1}{2} A + \frac{1}{2} B$

Dem.



$\overrightarrow{OA} + \overrightarrow{OB} = \overrightarrow{OC} = 2\overrightarrow{OM}$

$\overrightarrow{OM} = \frac{1}{2}\overrightarrow{OA} + \frac{1}{2}\overrightarrow{OB}$

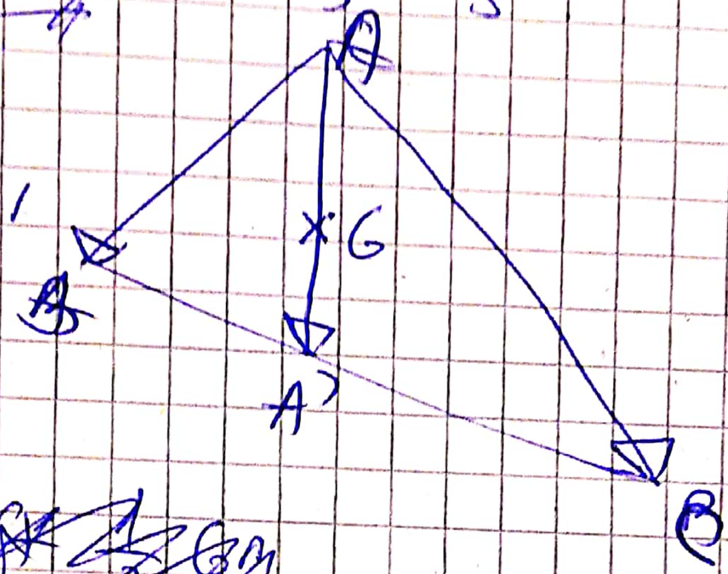
$M = \frac{1}{2} A + \frac{1}{2} B$

Apl. Δ_{ABC} CW

$G = \cap$ med. ΔABC

G este echibaricentrul sist. format din $\{A, B, C\}$

i.e. $G = \frac{1}{3}A + \frac{1}{3}B + \frac{1}{3}C$



$\overrightarrow{AG} = \frac{2}{3}\overrightarrow{AA'}$

$= \frac{2}{3} \cdot \frac{1}{2}(\overrightarrow{AB} + \overrightarrow{AC}) = \frac{1}{3}\overrightarrow{AB} + \frac{1}{3}\overrightarrow{AC}$

$= \frac{1}{3}\overrightarrow{AA} + \frac{1}{3}\overrightarrow{AB} + \frac{1}{3}\overrightarrow{AC}$

$\Downarrow$

$$\boxed{G = \frac{1}{3}A + \frac{1}{3}B + \frac{1}{3}C}$$

$\overrightarrow{MG} = \frac{1}{3}\overrightarrow{MA} + \frac{1}{3}\overrightarrow{MB} + \frac{1}{3}\overrightarrow{MC}$

Temă Teoremă

$\mathcal{S} = \{P_1, \dots, P_n\}$, $n > 1$

$(\mathcal{A}, V/K, \varphi)$, char $K \nmid n$.

G – echib. sist. $\mathcal{S}$

$1 \le m < n$ fixat.

$\mathcal{S}_m \subset \mathcal{S}$, card $\mathcal{S}_m = m$ și card $\mathcal{S}_{n-m} = \mathcal{S} \setminus \mathcal{S}_m$

G_m, ~~resp~~ G_{n-m} echib. sist. $\mathcal{S}_m$, resp $\mathcal{S}_{n-m}$,

$\to G \in [G_m G_{n-m}]$

C6.

1) $n = 3$, $m = 1$ (2)

(3 pct. necol.)

$\xrightarrow{\Leftrightarrow} \cap$ med $\triangle$.

2) $n = 4$, $m = 2$ $\to$ conc. mijl. lat. și a diag. într-un ▱

$\Leftrightarrow$

3) $n = 4$, $m = 1$ $\to$ conc. med. unui tetraedru

$$G = \frac{1}{4}A + \frac{1}{4}B + \frac{1}{4}C + \frac{1}{4}D$$

4) $n=4$, ~~$m\leq 3$~~, $m\geq 2$.
$\to \mathcal{R}$ (3-dim)

choaker.
$\cap$ bimedianelor. tetr. $\left(\frac{3}{4}, \frac{1}{4}\right)$

Aplicație

$P, Q, R \in (\mathcal{A}, V/K, \varphi)$, char $K \neq 3$

$\mathcal{R}_{af} = \{A_1, A_2, A_3\} \subset \mathcal{A}$

$P \to (p_1, p_2, p_3)$
$Q \to (q_1, q_2, q_3)$
$R \to (r_1, r_2, r_3)$

G – centrul de greut. al ΔPQR
Coord. baricentrice ale lui G.

$p_1 + p_2 + p_3 = q_1 + q_2 + q_3 = r_1 + r_2 + r_3 = 1$
(coord. bar.)

G – c. d. g.
$G = \frac{1}{3} P + \frac{1}{3} Q + \frac{1}{3} R$
$P = p_1 A_1 + p_2 A_2 + p_3 A_3$
$Q = \dots$
$R = \dots$

$\Rightarrow G = \frac{1}{3}(p_1 + q_1 + r_1) A_1$
$+ \dots$

Subspații afine

Def. Fie $(\mathcal{A}, V/K, \varphi)$ un spațiu afin

$\mathcal{A}' \subseteq \mathcal{A}$

$\hookrightarrow$ ssp. afin dacă este $\emptyset$

sau

$\exists P_0 \in \mathcal{A}'$ a.î. $V' = \{\overrightarrow{P_0P'} \mid P' \in \mathcal{A}'\}$ e ssp. vect.

Teorema dimensiunii

$\boxed{\text{Th}}$ Fie $\mathcal{A}_1, \mathcal{A}_2 \subseteq \mathcal{A}$ ssp. afine.

$$\dim(\mathcal{A}_1 \vee \mathcal{A}_2) = \begin{cases} \dim \mathcal{A}_1 + \dim \mathcal{A}_2 - \dim(\mathcal{A}_1 \cap \mathcal{A}_2), & \text{dacă } \mathcal{A}_1 \cap \mathcal{A}_2 \neq \emptyset \\ \dim \mathcal{A}_1 + \dim \mathcal{A}_2 - \dim(V_1 \cap V_2) + 1, & \text{dacă } \mathcal{A}_1 \cap \mathcal{A}_2 = \emptyset. \end{cases}$$

$(\mathbb{R}^n, \mathbb{R}^n/\mathbb{R}, \varphi)$

s. af. can.

$\mathcal{A} = \{x \in \mathbb{R}^n / AX = B\}$ (mult. sol. unui sist. lin. neomogen)

, unde $A \in \mathcal{M}_{(m,n)}(\mathbb{R})$, $m \leq n$

$rg\, A = m$

$V_p = \{x \in \mathbb{R}^n / AX = 0\}$ (mult. sol. sist. lin. omogen asociat sist. anterior)

$(m,n) \cdot (n,1)$

Atunci $\mathcal{A} \subseteq \mathbb{R}^n$ ssp. afin.

În plus, dc. $\mathcal{A} \neq \emptyset$ atunci

$\underline{\text{dir}\, \mathcal{A} = V}$

i.e. $\boxed{\dim \mathcal{A} = \dim V = n - rg\, A = n - m}$

Apl.

Fie $\mathcal{A}_1, \mathcal{A}_2 \subseteq \mathbb{R}^4$ (2 ssp. afine) $\mathbb{R}_4$ str. afină can.

$\mathcal{A}_1 = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 / 2x_1 - x_2 + x_3 - x_4 = 1\}$

$\mathcal{A}_2 = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 / \begin{cases} 2x_1 + x_2 - x_3 = 1 \\ x_2 + x_3 + x_4 = 0 \end{cases}\}$

$\mathcal{A}_1 \cap \mathcal{A}_2$, $dir(\mathcal{A}_1 \cap \mathcal{A}_2)$

$\mathcal{A}_1 \vee \mathcal{A}_2$, $dir(\mathcal{A}_1 \vee \mathcal{A}_2)$

$\vee$

$\cap$

Rez.

$\mathcal{A}_1 \cap \mathcal{A}_2 = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 /$

$$\begin{cases} 2x_1 - x_2 + x_3 - x_4 = 1 \\ 2x_1 + x_2 - x_3 = -1 \\ x_2 + x_3 + x_4 = 0 \end{cases} \}$$

$\cancel{(t+1, -t+1, -t-1, t)}$

Se elim. termenii liberi

$dir(\mathcal{A}_1 \cap \mathcal{A}_2) = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 /$

$$\begin{cases} 2x_1 - x_2 + x_3 - x_4 = 0 \\ 2x_1 + x_2 - x_3 = 0 \\ x_2 + x_3 + x_4 = 0 \end{cases} \}$$

$\dim \mathcal{A}_1 \vee \mathcal{A}_2 = \dim \mathcal{A}_1 + \dim \mathcal{A}_2 - \dim(\mathcal{A}_1 \cap \mathcal{A}_2)$

$(\mathcal{A}_1 \cap \mathcal{A}_2 \neq \emptyset)$

$\dim \mathcal{A}_1 = 4 - 1 = 3$ (hiperplan afin)

$\dim \mathcal{A}_2 = 4 - 2 = 2$ (plan afin)

$\dim \mathcal{A}_1 \cap \mathcal{A}_2 = 4 - 3 = 1$ (dr. afină)

$\dim \mathcal{A}_1 \vee \mathcal{A}_2 = 4.$

Dar $\mathcal{A}_1 \vee \mathcal{A}_2 \subset \mathbb{R}^4$ ssp. afin $\Big| \Rightarrow \mathcal{A}_1 \vee \mathcal{A}_2 = \mathbb{R}^4$

$\dim(\mathcal{A}_1 \vee \mathcal{A}_2) = \mathbb{R}^4/\mathbb{R}$

Apl.

Fie sp. afin $\mathbb{R}^3$ cu str. afină canonică

$$\mathcal{A} = \{(x, y, z) \in \mathbb{R}^3 \mid \begin{cases} 3x + 2y + 5z - 4 = 0 \\ x + y + z - 2 = 0 \end{cases}\}$$

punctul $P_0 = (0, 0, 1)$

Să se scrie $\mathcal{A}'$ ce trece prin P_0 și este $\parallel$ cu $\mathcal{A}$ și de aceeași dim. cu $\mathcal{A}$?

Rez

$\mathcal{A}' \parallel \mathcal{A} \Rightarrow V' \subseteq V$ sau $V \subseteq V'$ $\Big| \Rightarrow \dim V' = \dim V$
dar $\dim \mathcal{A}' = \dim \mathcal{A}$

$\Downarrow$

$V = V'$

$$V = \operatorname{dir} \mathcal{A} = \{(x, y, z) \in \mathbb{R}^3 \mid \begin{cases} 3x + 2y + 5z = 0 \\ x + y + z = 0 \end{cases}\}$$

$V' = \operatorname{dir}$

$$\mathcal{A}' = \{(x,y,z) \in \mathbb{R}^3 / \begin{cases} \beta x + 2y + 5z + \alpha = 0 \\ x + y + z + \beta = 0 \end{cases}, \alpha, \beta \in \mathbb{R}\}$$

$$P_0 \in \mathcal{A}' \Rightarrow \begin{cases} \alpha = -5 \\ \beta = -1 \end{cases}$$

Deci $\mathcal{A}' = \{(x,y,z) \in \mathbb{R}^3 / \begin{cases} 3x + 2y + 5z = 5 \\ x + y + z = 1 \end{cases}\}$.