Solving the Fourth-Order Linear Homogeneous Differential Equation

The differential equation

$$y^{(4)} + y = 0$$

is a fourth-order linear homogeneous differential equation. To solve it, follow these steps:

Step 1: Write the Characteristic Equation

Assume a solution of the form $y = e^{rt}$. Substituting this into the differential equation gives:

$$r^4 + 1 = 0$$

Step 2: Solve the Characteristic Equation

The equation $r^4 + 1 = 0$ can be written as:

$$r^4 = -1$$

The fourth roots of -1 are:

$$r = e^{i\pi/4},\, e^{i3\pi/4},\, e^{i5\pi/4},\, e^{i7\pi/4}$$

These correspond to:

$$r = \pm\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}, \quad r = \pm\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}.$$

Step 3: Write the General Solution

The general solution to the differential equation is a linear combination of terms based on the roots of the characteristic equation. Using Euler's formula $e^{i\theta} = \cos\theta + i\sin\theta$, we express the solutions as trigonometric functions:

$$y(t) = C_1 \cos\left(\frac{\sqrt{2}}{2}t\right) + C_2 \sin\left(\frac{\sqrt{2}}{2}t\right) + C_3 \cos\left(-\frac{\sqrt{2}}{2}t\right) + C_4 \sin\left(-\frac{\sqrt{2}}{2}t\right).$$

Since $\cos(-x) = \cos(x)$ and $\sin(-x) = -\sin(x)$, this simplifies to:

$$y(t) = C_1 \cos\left(\frac{\sqrt{2}}{2}t\right) + C_2 \sin\left(\frac{\sqrt{2}}{2}t\right) + C_3 \cos\left(\frac{\sqrt{2}}{2}t\right) - C_4 \sin\left(\frac{\sqrt{2}}{2}t\right).$$

Combine terms to get:

$$y(t) = (C_1 + C_3) \cos\left(\frac{\sqrt{2}}{2}t\right) + (C_2 - C_4) \sin\left(\frac{\sqrt{2}}{2}t\right).$$

Finally, rename constants:

$$y(t) = A \cos\left(\frac{\sqrt{2}}{2}t\right) + B \sin\left(\frac{\sqrt{2}}{2}t\right).$$

General Solution

$$y(t) = A \cos\left(\frac{\sqrt{2}}{2}t\right) + B \sin\left(\frac{\sqrt{2}}{2}t\right),$$

where A and B are arbitrary constants.