Mobius Multiplicative Functions

The following functions are all multiplicative functions, where p is a prime number and k is a positive integer:

- The constant function: $I(p^k) = 1$.
- The identity function: $\mathrm{Id}(p^k) = p^k$.
- The power function: $\mathrm{Id}_a(p^k) = p^{ak}$, where a is a constant.
- The unit function: $\chi(p^k) = [p^k = 1]$.
- The divisor function: $\sigma_a(p^k) = \sum_{i=0}^{k} p^{ai}$, denoting the sum of the a-th powers of all the positive divisors of the number.
- The Möbius function: $\mu(p^k) = [k = 0] - [k = 1]$.
- Euler's totient function: $\varphi(p^k) = p^k - p^{k-1}$.

Note: $[P]$ refers to the boolean expression, i.e., $[P] = 1$ when P is true, and 0 otherwise.

Mobius Inclusion Exclusion Example

How many numbers are there less than or equal to n that are free of squares?. Contrainsts: $1 <= n <= 10^{12}$

Change the statement to count the reverse and then sustract: How many numbers are ... that can be divided by a square of a prime. So the answer will be $n-$ Summatory with Inclusion-Exclusion of $f(prime)$

$$f(prime) = floor(\frac{n}{p * p})$$

So in those cases of summatory with primes, you can use Mobius to adding or substracting. Final answer is

$$n - \Sigma_{i=1}^{\sqrt{n}} \mu(i) \left\lfloor \frac{n}{i^2} \right\rfloor$$

Or in programming terms:

```
long long ans = n;
for (int i =1;i<=sqrt(n);i++) {
    ans -= mo[i] * n/(i*i);
}
```

Some Geometry Formulas

Volume of Glass with Water (Volumen del Vaso)

Given:

- p is the height of the water
- r_1 is the big radius at the water's surface
- r_2 is the small radius of the base

The volume of the glass with water is given by:

$$\text{Volume} = p \cdot \pi \cdot \frac{(r_1^2 + r_2^2 + r_1 \cdot r_2)}{3}$$

Heron's Formula

Finding the area of a triangle by the length of its sides, also applicable for points using Euclidean distance. You can use the following code to get the area; if the square root is negative, then the triangle is not valid.

```
long double triangle_area(long double a, long double b, long double c) {
    long double s = (a + b + c) / 2;
    return sqrtl(s * (s - a) * (s - b) * (s - c));
}
```

Sine and Cosine Laws

Let a, b, and c be the sides of the triangle, and A, B, and C the angles opposite to these sides, respectively.
The Sine Law:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

The Cosine Law:

$$c^2 = a^2 + b^2 - 2ab \cdot \cos C$$

Some Theorems

Erdős–Szekeres Theorem

This theorem is related to increasing and decreasing sequences.
Suppose $a, b \in \mathbb{N}$, $n = ab + 1$, and $x_1, x_2, \ldots, x_n$ is a sequence of n real numbers. Then this sequence contains a monotonic increasing (decreasing) subsequence of $a + 1$ terms or a monotonic decreasing (increasing) subsequence of $b+1$ terms. Dilworth's lemma is a generalization of this theorem.

Grundy Numbers in Game Theory

Grundy numbers are used in game theory to analyze games that can be represented as directed state graphs. In these graphs, if a player loses in a state, its Grundy number is zero; otherwise, it is a positive number. The Grundy number for each vertex is defined as:

$$\text{Grundy(losing state with no moves)} = 0$$

$$\text{Grundy(vertex)} = \text{MEX(adjacent\_nodes[vertex])}$$

where MEX stands for the "minimum excludant," which is the smallest non-negative integer not present in the set of Grundy numbers of adjacent nodes.
If you have multiple independent games, the final Grundy number is calculated as:

$$\text{Grundy}(\text{game}_1) \oplus \text{Grundy}(\text{game}_2) \oplus \text{Grundy}(\text{game}_3) \oplus \text{Grundy}(\text{game}_4) \oplus \cdots \oplus \text{Grundy}(\text{game}_n)$$

where $\oplus$ denotes the bitwise XOR operation.