M.E. 530.646
UR5 Inverse Kinematics

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Introduction

Figure refFrames below illustrates a common assignment of Denavit-Hartenberg convention to the UR5 robot (shown with all joint angles at 0).

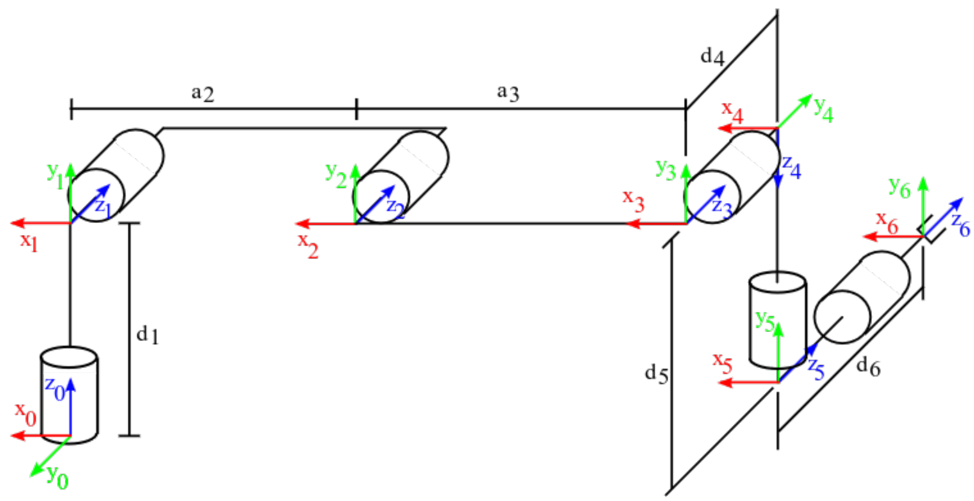


Figure 1: D-H Convention Frame Assignment

Joint	a	α	d	θ
1	0	$\pi/2$	0.089159	θ_1
2	-0.425	0	0	θ_2
3	-0.39225	0	0	θ_3
4	0	$\pi/2$	0.10915	θ_4
5	0	$-\pi/2$	0.09465	θ_5
6	0	0	0.0823	θ_6

Table 1: D-H Parameters

As with any 6-DOF robot, the homogeneous transformation from the base frame to the gripper can be defined as follows:

$$T_6^0(\theta_1, \theta_2, \theta_3, \theta_4, \theta_5, \theta_6) = T_1^0(\theta_1)\, T_2^1(\theta_2)\, T_3^2(\theta_3)\, T_4^3(\theta_4)\, T_5^4(\theta_5)\, T_6^5(\theta_6) \tag{1}$$

Also remember that a homogenous transformation T_j^i has the following form:

$$T_j^i = \begin{bmatrix} R_j^i & \vec{P_j^i} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} x_x & y_x & z_x & (P_j^i)_x \\ x_y & y_y & z_y & (P_j^i)_y \\ x_z & y_z & z_z & (P_j^i)_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{2}$$

where $\vec{P_j^i}$ is the translation from frame i to frame j and each column of R_j^i is the projection of one of the axes of frame j onto the axes of frame i (i.e. $[x_x, x_y, x_z]^T \equiv x_j^i$).

Inverse Kinematics

The first step to solving the inverse kinematics is to find θ_1. To do so, we must first find the location of the 5^{th} coordinate frame with respect to the base frame, P_5^0. As illustrated in Figure 2 below, we can do so by translating by d_6 in the negative z direction from the 6^{th} frame.

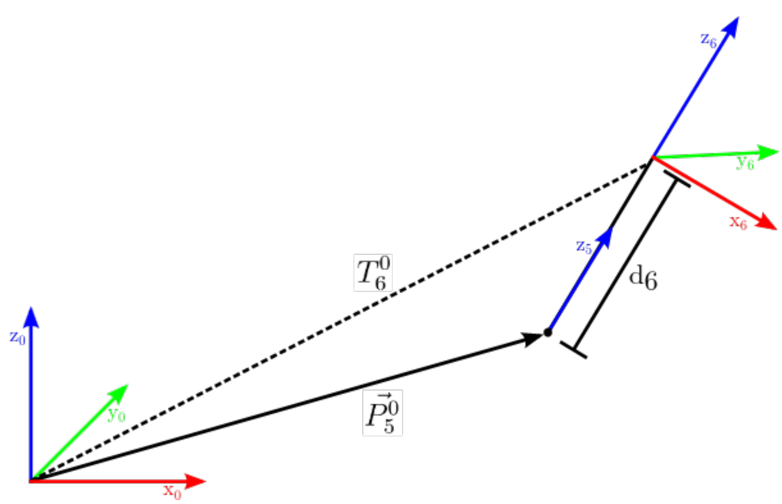


Figure 2: Finding the Origin of the 5^{th} Frame.

This is equivalent to the following operation:

$$\vec{P_5^0} = T_6^0 \begin{bmatrix} 0 \\ 0 \\ -d_6 \\ 1 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \tag{3}$$

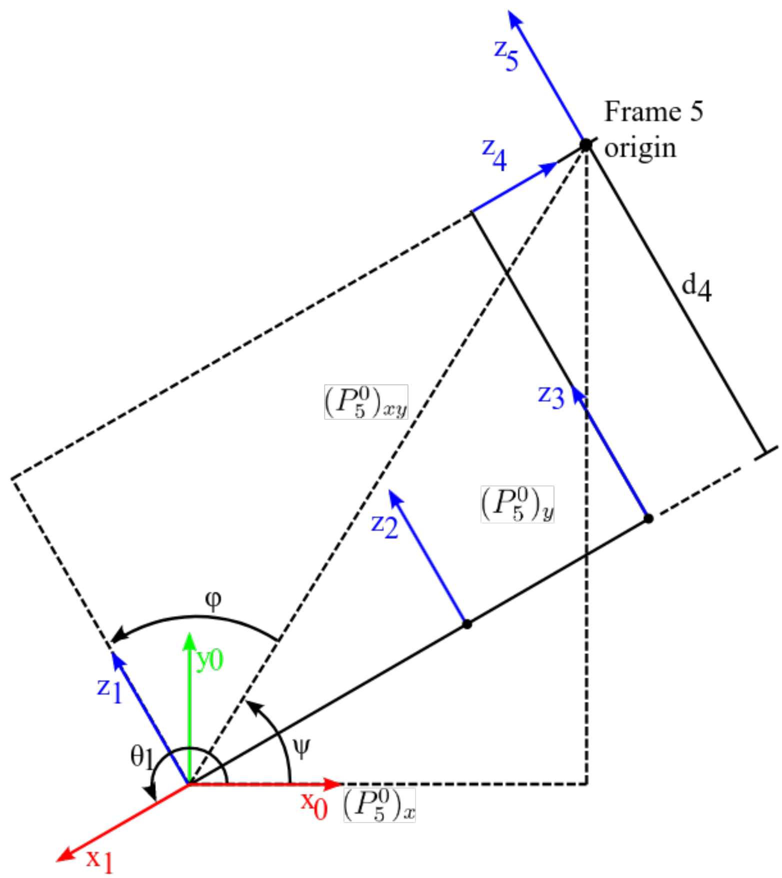


Figure 3: Finding θ_1. Note that the rotations around the axes z_1, z_2, z_3, and z_4 do *not* change the fact that the origin of Frame 5 is in the plane that is parallel to the x_1 axis, offset by d_4 as shown.

The key to this observation is that T_0^6 is *known* because it is simply the desired transformation! That is, given the desired transformation, T_0^6, we can calculate the vector from the origin of Frame 0 to the origin of Frame 5.

With the location of the 5^{th} frame, we can draw an overhead view of the robot as shown in Figure 3. From this, we can see that $\theta_1 = \psi + \phi + \frac{\pi}{2}$ where

$$\psi = \operatorname{atan2}\left((P_5^0)_y, (P_5^0)_x\right) \tag{4}$$

$$\phi = \pm \arccos\left(\frac{d_4}{(P_5^0)_{xy}}\right) = \pm \arccos\left(\frac{d_4}{\sqrt{{(P_5^0)_x}^2 + {(P_5^0)_y}^2}}\right) \tag{5}$$

The two solutions for θ_1 above correspond to the shoulder being either “left” or “right,”. Note that Equation 5 has a solution in all cases except that $d_4 > (P_5^0)_{xy}$. You can see from Figure 3 this happens when the origin of the 3^{rd} frame is close to the z axis of frame 0. This forms an unreachable cylinder in the otherwise spherical workspace of the UR5 (as shown in Figure 4 taken from the UR5 manual).

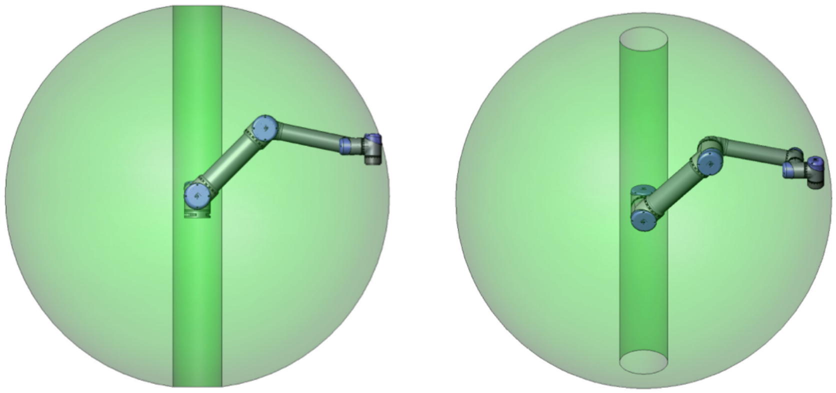


Figure 4: The Workspace of the UR5

Knowing θ_1, we can now solve for θ_5. Once again we draw an overhead view of the robot, but this time we consider location of the 6^{th} frame with respect to the 1^{st} (Figure 5.

We can see that $(P_6^1)_z = d_6 \cos(\theta_5) + d_4$, where $(P_6^1)_z = (P_6^0)_x \sin(\theta_1) - (P_6^0)_y \cos(\theta_1)$. Solving for θ_5,

$$\theta_5 = \pm \ \arccos\left(\frac{(P_6^1)_z - d_4}{d_6}\right) \tag{6}$$

Once again, there are two solutions. These solutions correspond to the wrist being "down" and "up."

Figure 6 illustrates that, ignoring translations between frames, z_1 can be represented with respect to frame 6 as a unit vector defined with spherical coordinates. We can find the x and y components of of this vector by projecting it onto the x-y plane and then onto the x or y axes.

Next we find transformation from frame 6 to frame 1,

$$T_1^6 = ((T_1^0)^{-1}\, T_6^0)^{-1} \tag{7}$$

Remembering the structure the of the first three columns of the homogenous transformation T_1^6(see Equation 2), we can form the following equalities:

$$-\sin(\theta_6)\sin(\theta_5) = z_y \tag{8}$$

$$\cos(\theta_6)\sin(\theta_5) = z_x \tag{9}$$

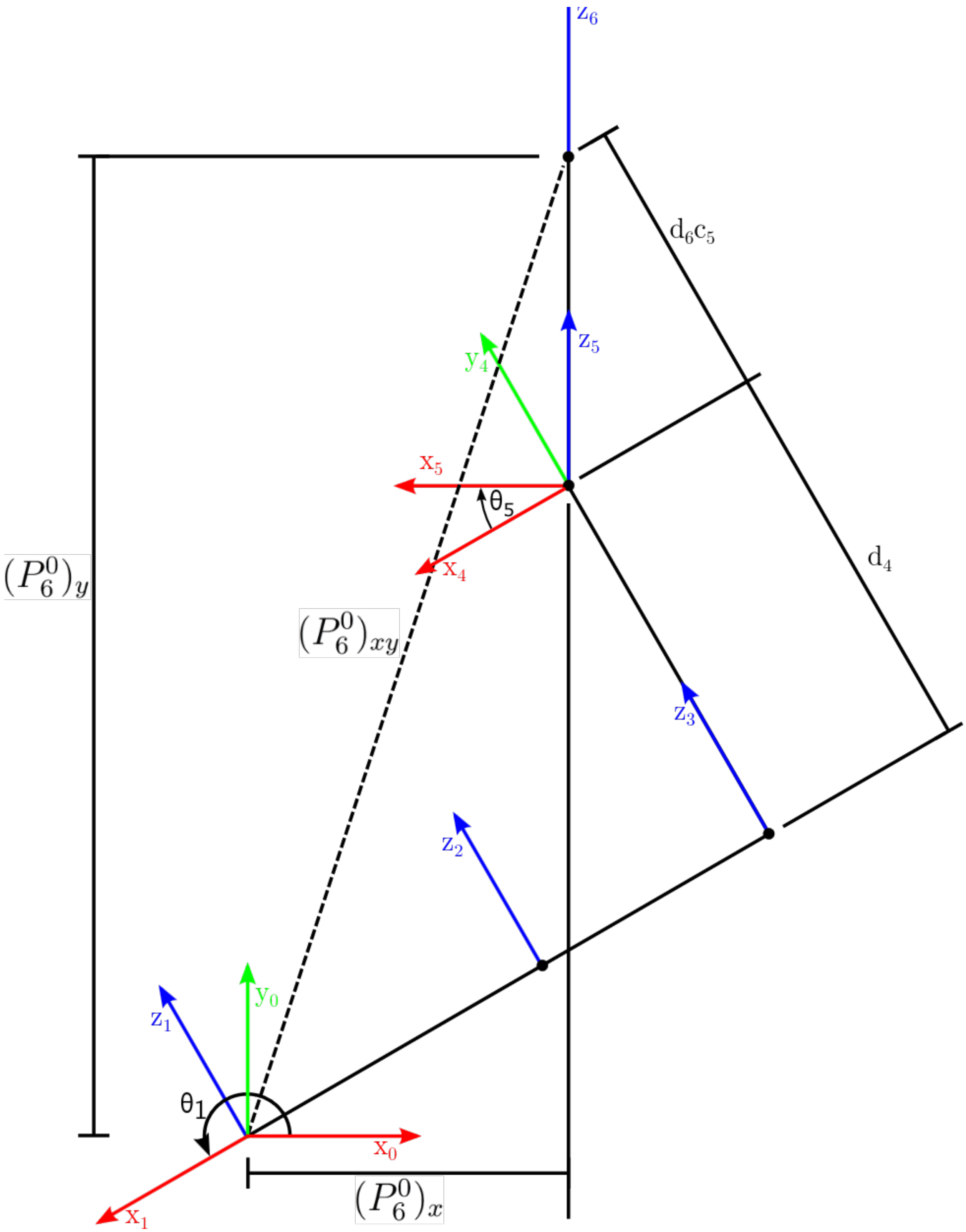


Figure 5: Finding θ_5

Solving for θ_6,

$$\theta_6 = \text{atan2}\left(\frac{-z_y}{\sin(\theta_5)}, \frac{z_x}{\sin(\theta_5)}\right) \tag{10}$$

Equation 10 shows that θ_6 is not well-defined when $\sin(\theta_5) = 0$ or when $z_x, z_y = 0$. We can see from Figure 6 that these conditions are actually the same. In this configuration joints 2, 3, 4, and 6 are parallel. As a result, there four degrees of freedom to determine the position and rotation of the end-effector in the plane, resulting in an infinite number of solutions. In this case, a desired value for q_6 can be chosen to reduce the number of degrees of freedom to three.

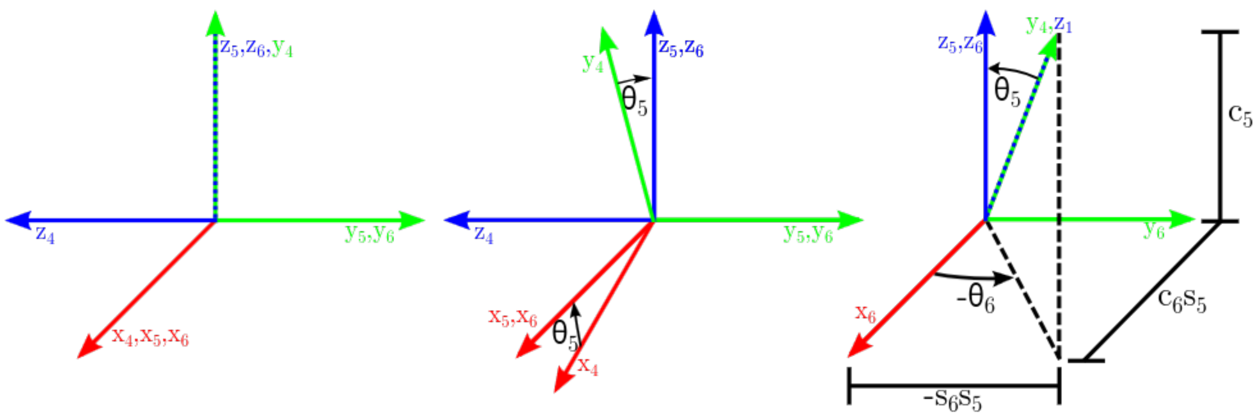


Figure 6: Finding θ_6

We can now consider the three remaining joints as forming a planar 3R manipulator. First we will find the location of frame 3 with respect to frame 1. This is done as follows:

$$T_4^1 = T_6^1 \, T_4^6 = T_6^1 \, (T_5^4 \, T_6^5)^{-1} \tag{11}$$

$$\vec{P_3^1} = T_4^1 \begin{bmatrix} 0 \\ -d_4 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \tag{12}$$

Now we can draw the plane containing frames 1-3, as shown in Figure 7. We can see that

$$\cos(\xi) = \frac{||\vec{P_3^1}||^2 - a_2^2 - a_3^2}{2a_2a_3} \tag{13}$$

with use of the law of cosines.

$$\begin{aligned}\cos(\xi) &= -\cos(\pi - \xi) \\ &= -\cos(-\theta_3) \\ &= \cos(\theta_3)\end{aligned} \tag{14}$$

Combining 13 and 14

$$\theta_3 = \pm \arccos\left(\frac{||\vec{P_3^1}||^2 - a_2^2 - a_3^2}{2a_2a_3}\right) \tag{15}$$

Equation 15 has a solution as long as the argument to arccos is $\in [-1, 1]$. We can see that large values of $||\vec{P_3^1}||$ will cause this the argument to exceed 1. The physical interpretation here is that the robot can only reach so far out in any direction (creating the spherical bound to the workspace as seen in 4).

Figure 7 also shows that

$$\theta_2 = -(\delta - \epsilon) \tag{16}$$

where $\delta = \text{atan2}((P_3^1)_y, -(P_3^1)_x)$ and ϵ can be found via law of sines:

$$\frac{\sin(\xi)}{||\vec{P_3^1}||} = \frac{\sin(\epsilon)}{a_3} \tag{17}$$

Combining 16 and 17

$$\theta_2 = -\,\text{atan2}((P_3^1)_y, -(P_3^1)_x) + \arcsin\left(\frac{a_3 \sin(\theta_3)}{||\vec{P_3^1}||}\right) \tag{18}$$

Notice that there are two solutions for θ_2 and θ_3. These solutions are known as "elbow up" and "elbow down."

The final step to solving the inverse kinematics is to solve for θ_4. First we want to find T_4^3:

$$T_4^3 = T_1^3\, T_4^1 = (T_2^1\, T_3^2)^{-1} T_4^1 \tag{19}$$

Using the first column of T_4^3,

$$\theta_4 = \text{atan2}(x_y, x_x) \tag{20}$$

Below are figures which show the eight solutions for one end-effector position/orientation.

This solution was adapted from an existing solution written by Kelsey P. Hawkins

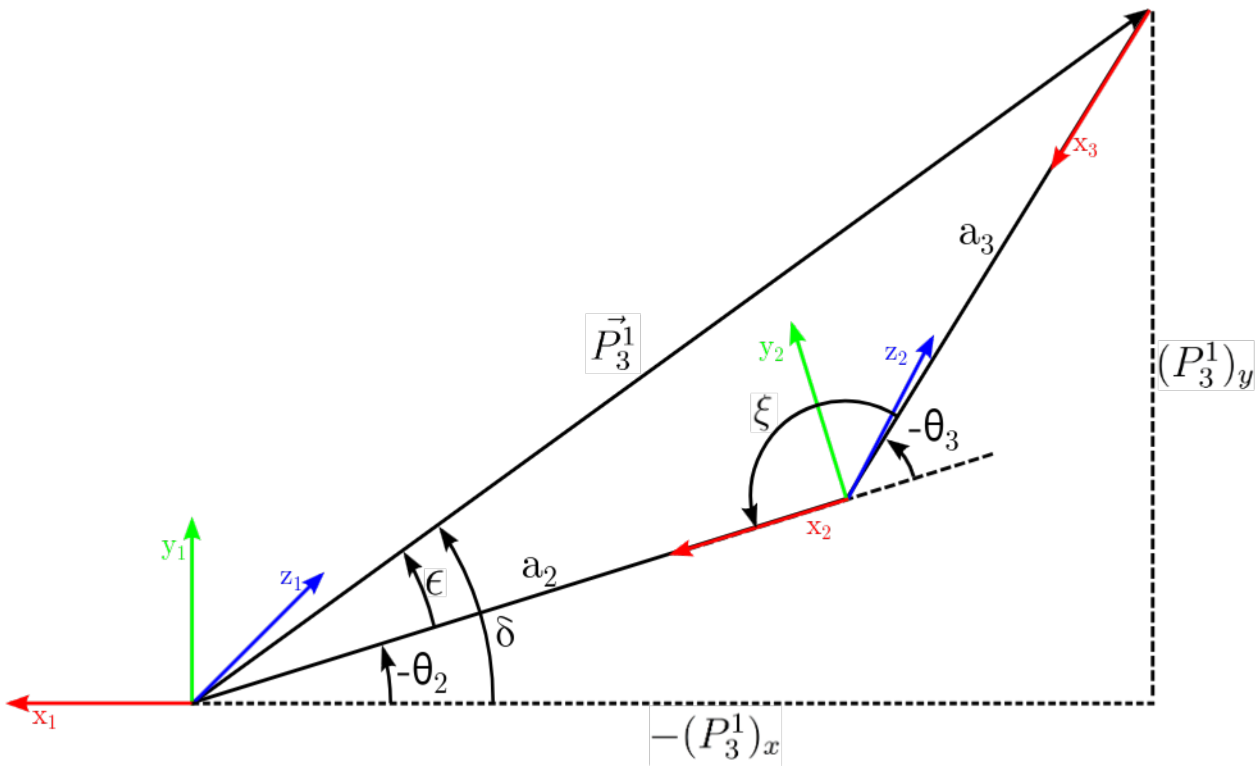
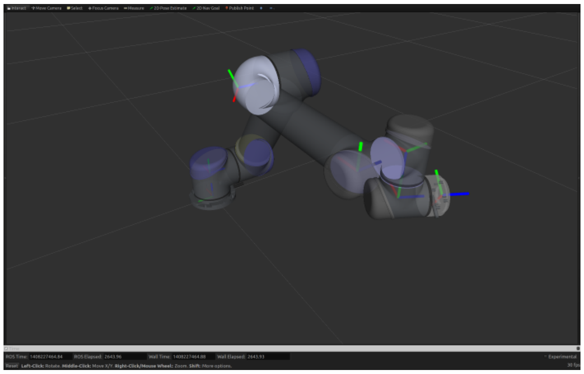
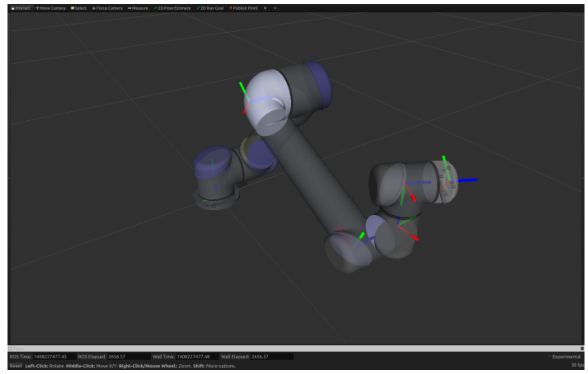


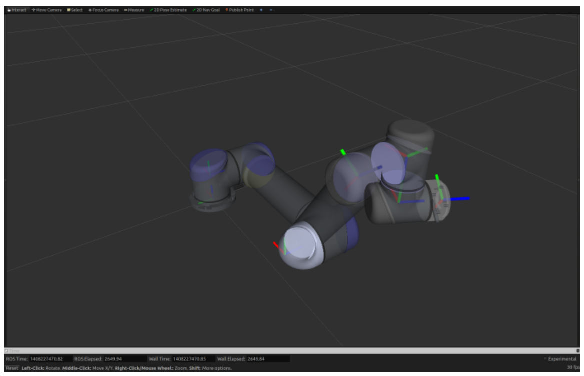
Figure 7: Finding θ_2 and θ_3



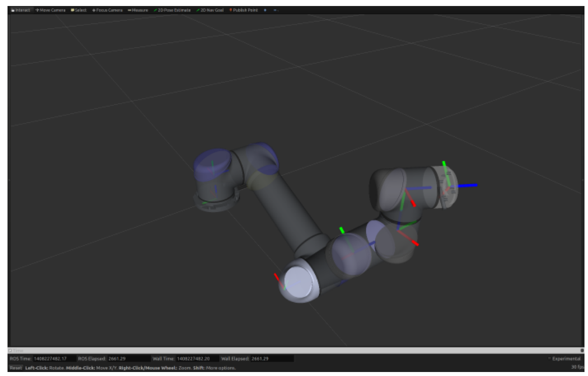
(a) Shoulder Left, Elbow Up, Wrist Down



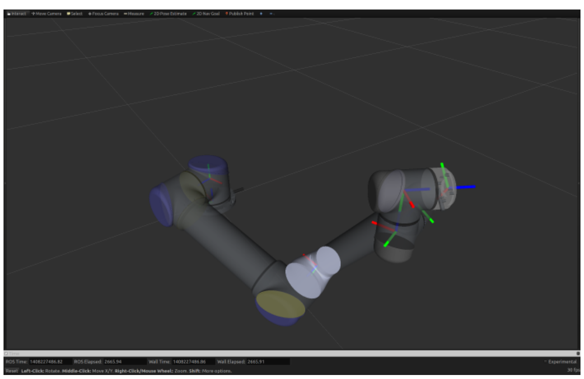
(b) Shoulder Left, Elbow Up, Wrist Up



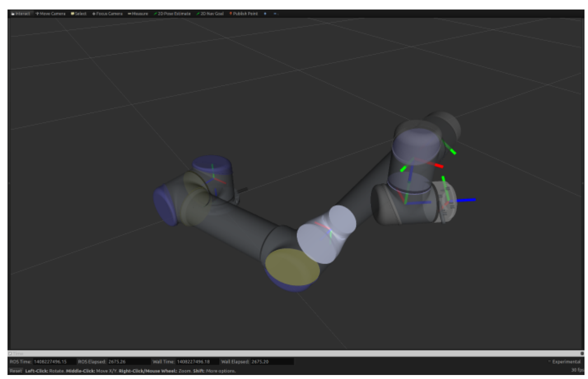
(c) Shoulder Left, Elbow Down, Wrist Down



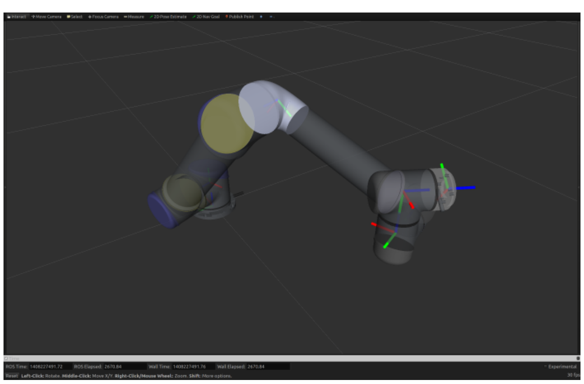
(d) Shoulder Left, Elbow Down, Wrist Up



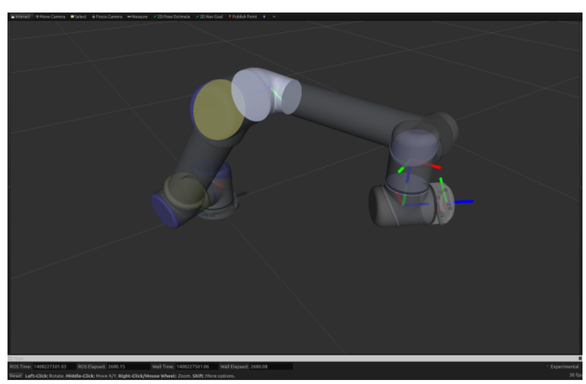
(e) Shoulder Right, Elbow Down, Wrist Up



(f) Shoulder Right, Elbow Down, Wrist Down



(g) Shoulder Right, Elbow Up, Wrist Up



(h) Shoulder Right, Elbow Up, Wrist Down