Alternative Inverse Kinematic Solution of the UR5 Robotic Arm

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Abstract. Inverse kinematic control of industrial robotic manipulators is extensively used, for this reason it is necessary to improve current direct and inverse kinematic solutions. While some previous solutions present partial sets of equations and others have some inconsistencies, this study presents the complete derivation of an alternative set of equations for the inverse kinematic solution for Universal Robots robotic arms, particularly the UR5. The herein inverse kinematic solution is obtained by applying the conventional Denavit-Hartenberg algebraic method and is validated with the direct kinematic solution.

1 Introduction

There are different methods to study the forward or direct (DK) and inverse kinematics (IK); the DK solution allows to compute the end effector's pose depending on the position of each joint, while the IK solution computes the joint positions that take the end effector to a specified pose. A well-known technique for analysing the IK uses the Denavit-Hartenberg (DH) parameters, when using this method it is necessary to study the DK first. The DH parameters are four values that fully describe each link and how they relate to each other, there are two alternatives the classic and the modified methods [1], which are also referred in the literature as "original" and "modified" DH conventions.

The UR5 robotic arm is a 6 degree of freedom (DoF) industrial collaborative robot produced by Universal Robots whose kinematics has already been studied [2–5]; however, some works present partial solutions, which means that researchers employing these robots have to complete the analysis. Some of these solutions have been referred to in studies such as the one in [6], where the presented IK solution was adopted from [2].

This paper presents an alternative IK set of equations based on the one in [3], the classic DH parameters and the algebraic method described in [7]. This work is organized as follows: Previous solutions are presented in Sect. 2. Section 3 details the steps that were followed in order to derive the DK and IK solutions. Finally, conclusions are presented in Sect. 4.

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2 Previous Inverse Kinematic Solutions of the UR5

This section presents the IK solutions of the UR5 that were mentioned in Sect. 1. Some of the equations are written differently in the referenced works; however, the expressions are equivalent. All the solutions were obtained by using the DH parameters and homogeneous transformation matrices, and each one results in 8 different sets of angles because the equations for some joints lead to 2 possibilities that must be taken into account to compute the remaining angles.

The following notation is used: $s_i = \sin\theta_i$, $c_i = \cos\theta_i$, $s_{ij\ldots} = \sin(\theta_i + \theta_j + \ldots)$ and $c_{ij\ldots} = \cos(\theta_i + \theta_j + \ldots)$. The elements of the transformation matrix ${}^{\mathbf{0}}_{\mathbf{6}}\boldsymbol{T}$ (Sect. 3.1) are also used: p_x, p_y and p_z represent the translation; and the r_{ij} elements describe the rotation. The a_i and d_i variables are DH parameters (Fig. 1).

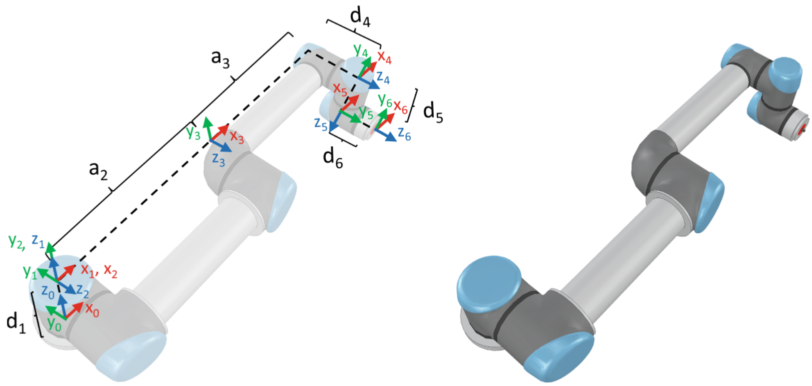

Fig. 1. Coordinate frame assignment ($\theta_i = 0$ for $i = 0, 1, 2, 3, 4, 5, 6$)

2.1 Models 1 and 2

The first two models are based on [2], which reports the complete DK solution and a partial IK solution (Eqs. 1, 2 and 3). The process to obtain the rest of the equations is only briefly explained.

$$\theta_1 = \operatorname{atan2}(p_y - d_6 r_{23}, p_x - d_6 r_{13}) + \frac{\pi}{2} \pm \arccos\left(\frac{d_4}{\sqrt{(p_y - d_6 r_{23})^2 + (p_x - d_6 r_{13})^2}}\right) \tag{1}$$

$$\theta_5 = \pm\arccos\left(\frac{p_x s_1 - p_y c_1 - d_4}{d_6}\right) \tag{2}$$

$$\theta_6 = \operatorname{atan2}\left(\frac{c_1 r_{22} - s_1 r_{12}}{s_5}, \frac{s_1 r_{11} - c_1 r_{21}}{s_5}\right) \tag{3}$$

The first model ("Model 1") completes the solution with the equations for θ_2, θ_3 and θ_4 that can be found in [4], these expressions are shown next:

$$\begin{aligned} A_3 = \quad & d_5 s_6 \left(c_1 r_{11} + s_1 r_{21}\right) + d_5 c_6 \left(c_1 r_{12} + s_1 r_{22}\right) - d_6 \left(c_1 r_{13} + s_1 r_{23}\right) \\ & + p_x c_1 + p_y s_1 \qquad (4) \\ = \quad & c_2 a_2 + c_{23} a_3 \qquad (5) \end{aligned}$$

$$\begin{aligned} B_3 &= d_5 \left(s_6 r_{31} + c_6 r_{32}\right) - d_6 r_{33} + p_z - d_1 \qquad (6) \\ &= s_2 a_2 + s_{23} a_3 \qquad (7) \end{aligned}$$

$$\theta_3 = \pm \arccos\left(\frac{A_3^2 + B_3^2 - a_2^2 - a_3^2}{2 a_2 a_3}\right) \qquad (8)$$

$$\theta_2 = \arcsin\left(\frac{-s_3 a_3}{\sqrt{A_3^2 + B_3^2}}\right) + \operatorname{atan2}\left(B_3, A_3\right) \qquad (9)$$

$$A_4 = c_1 r_{11} + s_1 r_{21} \qquad (10)$$

$$B_4 = c_1 r_{12} + s_1 r_{22} \qquad (11)$$

$$C_4 = c_1 r_{13} + s_1 r_{23} \qquad (12)$$

$$D_4 = c_6 r_{31} - s_6 r_{32} \qquad (13)$$

$$c_4 = c_{23} \left(c_5 c_6 A_4 - c_5 s_6 B_4 - s_5 C_4\right) + s_{23} \left(c_5 D_4 - s_5 r_{33}\right) \qquad (14)$$

$$s_4 = s_{23} \left(-c_5 c_6 A_4 + c_5 s_6 B_4 + s_5 C_4\right) + c_{23} \left(c_5 D_4 - s_5 r_{33}\right) \qquad (15)$$

$$\theta_4 = \operatorname{atan2}\left(s_4, c_4\right) \qquad (16)$$

A second model ("Model 2") was found in [5], only the expressions for θ_2 and θ_4 (Eqs. 18 and 21) differ from those in "Model 1", this means that Eq. 8 is used to compute θ_3.

$$A_2 = a_3 c_3 + a_2 \qquad (17)$$

$$\theta_2 = \operatorname{atan2}\left(A_2 B_3 - a_3 s_3 A_3, A_2 A_3 + a_3 s_3 B_3\right) \qquad (18)$$

$$A_4 = -s_5 \left(c_1 r_{13} + s_1 r_{23}\right) + c_5 \left[c_6 \left(c_1 r_{11} + s_1 r_{21}\right) - s_6 \left(c_1 r_{12} + s_1 r_{22}\right)\right] \qquad (19)$$

$$B_4 = c_5 \left(c_6 r_{31} - s_6 r_{32}\right) - s_5 r_{33} \qquad (20)$$

$$\theta_4 = \operatorname{atan2}\left(B_4, A_4\right) - \theta_2 - \theta_3 \qquad (21)$$

2.2 Model 3

This model was obtained from [3], said study presents analytical solutions for both, the DK and the IK; however, the expressions differ from those in Sect. 2.1. Some inconsistencies were found in this solution, these could be due to typing errors. The complete IK solution is shown next:

$$A_1 = p_x - d_6 r_{13} \tag{22}$$

$$B_1 = d_6 r_{23} - p_y \tag{23}$$

$$\theta_1 = \operatorname{atan2}(A_1, B_1) \pm \operatorname{atan2}\left(\sqrt{A_1^2 + B_1^2 - d_4^2}, d_4\right) \tag{24}$$

The first term in c_5 (Eq. 25) is different from the one in [3], this inconsistency was found while studying the solution.

$$c_5 = s_1 r_{13} - c_1 r_{23} \tag{25}$$

$$s_5 = \sqrt{(s_1 r_{11} - c_1 r_{21})^2 + (s_1 r_{12} - c_1 r_{22})^2} \tag{26}$$

$$\theta_5 = \pm \operatorname{atan2}(s_5, c_5) \tag{27}$$

$$\theta_6 = \operatorname{atan2}\left(\frac{c_1 r_{22} - s_1 r_{12}}{\operatorname{sign}(s_5)}, \frac{s_1 r_{11} - c_1 r_{21}}{\operatorname{sign}(s_5)}\right) \tag{28}$$

$$A_{234} = c_1 r_{11} + s_1 r_{21} \tag{29}$$

$$\theta_{234} = \operatorname{atan2}(c_5 c_6 r_{31} - s_6 A_{234}, c_5 c_6 A_{234} + s_6 r_{31}) \tag{30}$$

$$A_2 = 2a_2 (d_1 - p_z - d_5 c_{234} - d_6 s_5 s_{234}) \tag{31}$$

$$B_2 = 2a_2 (d_5 s_{234} - d_6 s_5 c_{234} - c_1 p_x - s_1 p_y) \tag{32}$$

$$\begin{aligned} C_2 = a_3^2 - a_2^2 - d_5^2 - (p_z - d_1)(2d_5 c_{234} + 2d_6 s_5 s_{234} + p_z - d_1) \\ + (c_1 p_x + s_1 p_y)(2d_5 s_{234} - 2d_6 s_5 c_{234} - c_1 p_x - s_1 p_y) - d_6^2 s_5^2 \end{aligned} \tag{33}$$

$$\theta_2 = \operatorname{atan2}(A_2, B_2) \pm \operatorname{atan2}\left(\sqrt{A_2^2 + B_2^2 - C_2^2}, C_2\right) \tag{34}$$

$$\theta_{34} = \theta_{234} - \theta_2 \tag{35}$$

Another inconsistency is the sign of the term $s_1 s_2 p_y$ in A_3 (Eq. 36), which is also different from the one presented in [3].

$$A_3 = \frac{-c_1 s_2 p_x - s_1 s_2 p_y + c_2 p_z - c_2 d_1 + c_{34} d_5 + s_{34} s_5 d_6}{a_3} \tag{36}$$

$$B_3 = \frac{c_1 c_2 p_x + s_1 c_2 p_y + s_2 p_z - s_2 d_1 - s_{34} d_5 + c_{34} s_5 d_6 - a_2}{a_3} \tag{37}$$

$$\theta_3 = \operatorname{atan2}(A_3, B_3) \tag{38}$$

$$\theta_4 = \theta_{34} - \theta_3 \tag{39}$$

3 Derivation of the UR5 Kinematics

In this work the robot kinematics is studied by using the classic DH parameters (Table 1). The frame assignment is the one introduced in Sect. 2 (Fig. 1) and is the same as in [2], here frame 0 represents the base and frames 1−6 the joints.

Table 1. Denavit-Hartenberg parameters

i	α_i(rad)	a_i(mm)[a]	d_i(mm)[a]	θ_i
1	$\pi/2$	0	$d_1 = 89.2$	θ_1
2	0	$a_2 = 425.0$	0	θ_2
3	0	$a_3 = 392.0$	0	θ_3
4	$\pi/2$	0	$d_4 = 109.3$	θ_4
5	$-\pi/2$	0	$d_5 = 94.75$	θ_5
6	$\sim (0)$	$\sim (0)$	$d_6 = 82.5$	θ_6

[a] The values for a_i and d_i were obtained from [8]

3.1 Direct Kinematics

The homogeneous transformation matrix (Eq. 40) represents frame i with respect to frame $i-1$, while the position and rotation of the end effector with respect to the base can be found by multiplying the 6 transformation matrices (Eq. 41). The elements of ${}^{\mathbf{0}}_{\mathbf{6}}\boldsymbol{T}$ can be found in [2] and [3].

$$
{}^{\boldsymbol{i-1}}_{\boldsymbol{i}}\boldsymbol{T} = \begin{bmatrix} \cos\theta_i & -\sin\theta_i\cos\alpha_i & \sin\theta_i\sin\alpha_i & a_i\cos\theta_i \\ \sin\theta_i & \cos\theta_i\cos\alpha_i & -\cos\theta_i\sin\alpha_i & a_i\sin\theta_i \\ 0 & \sin\alpha_i & \cos\alpha_i & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{40}
$$

$$
{}^{\mathbf{0}}_{\mathbf{6}}\boldsymbol{T} = {}^{\mathbf{0}}_{\mathbf{1}}\boldsymbol{T}\ {}^{\mathbf{1}}_{\mathbf{2}}\boldsymbol{T}\ {}^{\mathbf{2}}_{\mathbf{3}}\boldsymbol{T}\ {}^{\mathbf{3}}_{\mathbf{4}}\boldsymbol{T}\ {}^{\mathbf{4}}_{\mathbf{5}}\boldsymbol{T}\ {}^{\mathbf{5}}_{\mathbf{6}}\boldsymbol{T} = \begin{bmatrix} r_{11} & r_{12} & r_{13} & p_x \\ r_{21} & r_{22} & r_{23} & p_y \\ r_{31} & r_{32} & r_{33} & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{41}
$$

3.2 Inverse Kinematics

This section is divided in two: the derivation of the expressions for θ_1, θ_5, θ_6 and θ_{234} by using the same process as in [3]; and the derivation of alternative expressions for θ_2, θ_3 and θ_4. These derivations were done by applying the algebraic method described in [7] and the equivalency between Eqs. 43 and 44, where the following notation is used: $\alpha = c_{23}$, $\beta = s_{23}$, $\gamma = c_{234}$ and $\delta = s_{234}$.

$$
{}^{\mathbf{1}}_{\mathbf{6}}\boldsymbol{T} = {}^{\mathbf{1}}_{\mathbf{0}}\boldsymbol{T}\ {}^{\mathbf{0}}_{\mathbf{6}}\boldsymbol{T} = {}^{\mathbf{1}}_{\mathbf{2}}\boldsymbol{T}\ {}^{\mathbf{2}}_{\mathbf{3}}\boldsymbol{T}\ {}^{\mathbf{3}}_{\mathbf{4}}\boldsymbol{T}\ {}^{\mathbf{4}}_{\mathbf{5}}\boldsymbol{T}\ {}^{\mathbf{5}}_{\mathbf{6}}\boldsymbol{T} \tag{42}
$$

$$ {}^{\mathbf{1}}_{\mathbf{6}}\boldsymbol{T} = \begin{bmatrix} c_1r_{11}+s_1r_{21} & c_1r_{12}+s_1r_{22} & c_1r_{13}+s_1r_{23} & c_1p_x+s_1p_y \\ r_{31} & r_{32} & r_{33} & p_z-d_1 \\ s_1r_{11}-c_1r_{21} & s_1r_{12}-c_1r_{22} & s_1r_{13}-c_1r_{23} & s_1p_x-c_1p_y \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{43} $$

$$ = \begin{bmatrix} \gamma c_5c_6-s_6\delta & -\gamma c_5s_6-c_6\delta & -\gamma s_5 & c_2a_2+\alpha a_3+\delta d_5-\gamma s_5d_6 \\ c_5c_6\delta+\gamma s_6 & -c_5\delta s_6+\gamma c_6 & -\delta s_5 & s_2a_2+\beta a_3-\gamma d_5-\delta s_5d_6 \\ c_6s_5 & -s_5s_6 & c_5 & c_5d_6+d_4 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{44} $$

3.2.1 Expressions for $\boldsymbol{\theta_1}$, $\boldsymbol{\theta_5}$, $\boldsymbol{\theta_6}$ and $\boldsymbol{\theta_{234}}$

This section presents the steps to obtain the solutions for θ_1, θ_5, θ_6 and θ_{234}. To solve for θ_1, c_5 was obtained from ${}^{\mathbf{1}}_{\mathbf{6}}\boldsymbol{T}\,(3,3)$ (Eq. 25) and substituted into ${}^{\mathbf{1}}_{\mathbf{6}}\boldsymbol{T}\,(3,4)$ (Eq. 45), manipulating this new equality led to a system of trigonometric equations. The solution to this system is θ_1 (Eq. 24) and exists provided that $d_4^2 \leq A_1^2 + B_1^2$.

$$ s_1p_x - c_1p_y = c_5d_6 + d_4 \tag{45} $$

Solving for θ_5 (Eq. 27) was done using c_5 (Eq. 25) and s_5. Without θ_6, the latter one can be obtained by using ${}^{\mathbf{1}}_{\mathbf{6}}\boldsymbol{T}\,(3,1)$ and ${}^{\mathbf{1}}_{\mathbf{6}}\boldsymbol{T}\,(3,2)$, which are shown next:

$$ s_1r_{11} - c_1r_{21} = c_6s_5 \tag{46} $$

$$ s_1r_{12} - c_1r_{22} = -s_5s_6 \tag{47} $$

Equations 46 and 47 were also used to obtain c_6 and s_6, these were used to solve for θ_6. From the expression for θ_6 (Eq. 3), it can be seen that the solution is valid if at least one of the numerators is different from 0 and $s_5 \neq 0$. To obtain expressions for the rest of the angles, θ_{234} (Eq. 30) is used; this was computed by solving the following system of equations which is given by elements ${}^{\mathbf{1}}_{\mathbf{6}}\boldsymbol{T}\,(1,1)$ and ${}^{\mathbf{1}}_{\mathbf{6}}\boldsymbol{T}\,(2,1)$:

$$ c_1r_{11} + s_1r_{21} = c_{234}c_5c_6 - s_6s_{234} \tag{48} $$

$$ r_{31} = c_5c_6s_{234} + c_{234}s_6 \tag{49} $$

3.2.2 Expressions for $\boldsymbol{\theta_2}$, $\boldsymbol{\theta_3}$ and $\boldsymbol{\theta_4}$

This section presents the remaining expressions. ${}^{\mathbf{1}}_{\mathbf{6}}\boldsymbol{T}\,(1,4)$ and ${}^{\mathbf{1}}_{\mathbf{6}}\boldsymbol{T}\,(2,4)$ (Eqs. 50 and 51) were used to define K_C (Eq. 52) and K_S (Eq. 55). The solution for θ_3 (Eq. 58) uses c_3 and s_3, the former one was computed by adding the squares of Eqs. 53 and 56 and applying sum and difference identities, while the latter one was obtained from a Pythagorean identity. This solution is valid only if $c_3 \in [-1, 1]$.

$$ c_1p_x + s_1p_y = c_2a_2 + c_{23}a_3 + s_{234}d_5 - c_{234}s_5d_6 \tag{50} $$

$$ p_z - d_1 = s_2a_2 + s_{23}a_3 - c_{234}d_5 - s_{234}s_5d_6 \tag{51} $$

$$K_C = c_1 p_x + s_1 p_y - s_{234} d_5 + c_{234} s_5 d_6 \tag{52}$$
$$= c_2 a_2 + c_{23} a_3 \tag{53}$$
$$= c_2 \left(a_2 + c_3 a_3\right) - s_2 \left(s_3 a_3\right) \tag{54}$$

$$K_S = p_z - d_1 + c_{234} d_5 + s_{234} s_5 d_6 \tag{55}$$
$$= s_2 a_2 + s_{23} a_3 \tag{56}$$
$$= s_2 \left(a_2 + c_3 a_3\right) + c_2 \left(s_3 a_3\right) \tag{57}$$

$$\theta_3 = \pm\,\mathrm{atan2}\left(\sqrt{1 - \left(\frac{K_S^2 + K_C^2 - a_2^2 - a_3^2}{2a_2 a_3}\right)^2}, \frac{K_S^2 + K_C^2 - a_2^2 - a_3^2}{2a_2 a_3}\right) \tag{58}$$

Angle θ_2 (Eq. 59) was found by solving the system of trigonometric equations given by Eqs. 54 and 57. Finally, θ_4 is obtained as shown in Eq. 60.

$$\theta_2 = \mathrm{atan2}\left(K_S, K_C\right) - \mathrm{atan2}\left(a_3 s_3, a_3 c_3 + a_2\right) \tag{59}$$

$$\theta_4 = \theta_{234} - \theta_2 - \theta_3 \tag{60}$$

The DK solution shown in Eq. 41 was used in order to validate the herein proposed and the previous IK solutions by verifying that the computed angles led to the desired pose. The validation data is not shown due to page limitation.

4 Conclusions

A complete IK solution can help researchers and practitioners who work with the UR robotic arms due to its many practical applications such as the implementation of trajectory generators and control schemes.

In this work an alternative analytic IK solution was derived for the UR5 robotic arm by using the original DH convention. While some works present only partial solutions, this one comprises a complete set of equations and describes how it was obtained. As a part of this analytical study some inconsistencies were found and corrected in some previous solutions.

The presented solution was validated with the DK solution and can be extended to other models of the UR robots by adjusting the DH parameters; future work will be focused on comparing the four IK solutions by statistical methods and a computational complexity evaluation to find the optimal set of equations.

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