Exercise1

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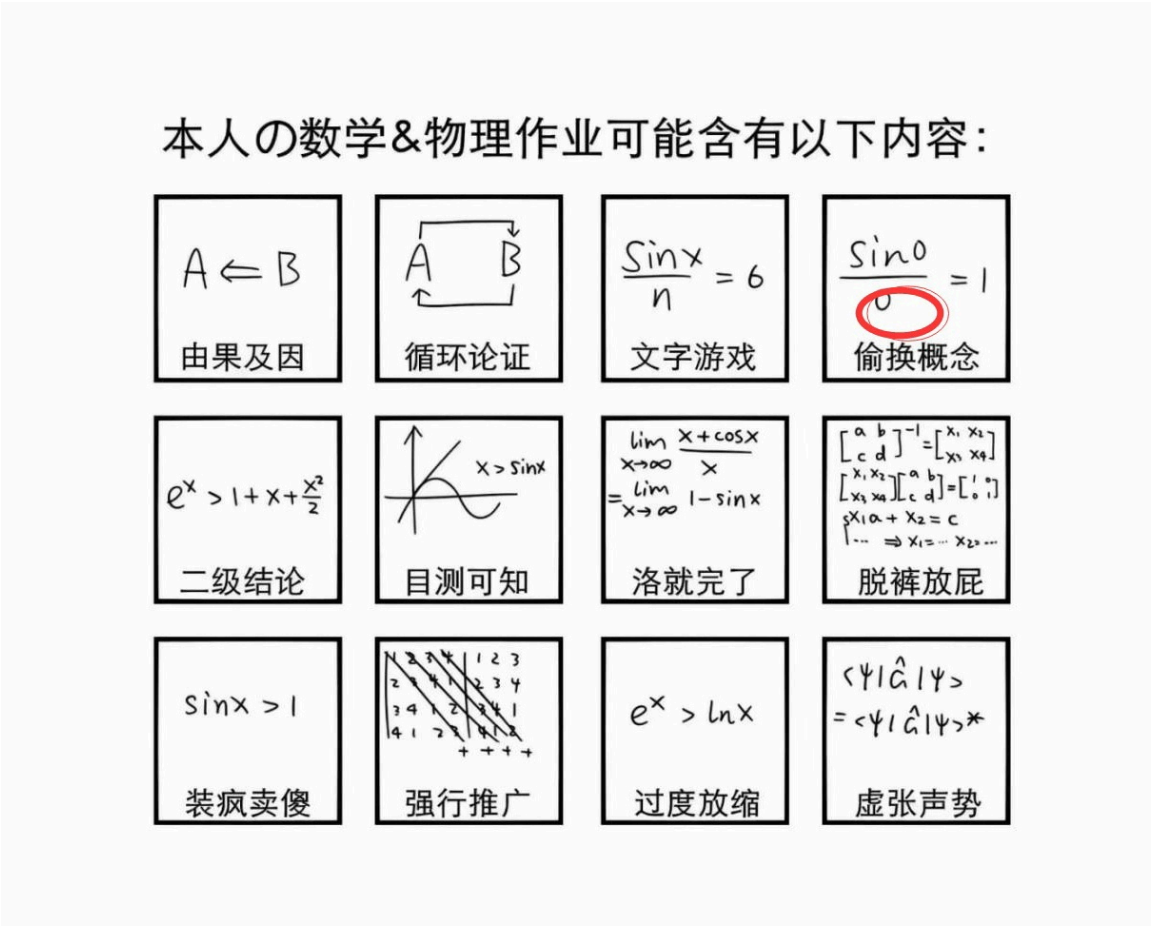

Exercise 1.2. 如果 $u:\mathbb{R}^3\to\mathbb{R}$ 是关于 (x,y,z) 的函数

$$
\begin{aligned}
\nabla f(u) &= \frac{\mathrm{d}f}{\mathrm{d}u}\nabla u \\
\nabla\cdot\boldsymbol{A}(u) &= \nabla u\cdot\frac{\mathrm{d}\boldsymbol{A}}{\mathrm{d}u} \\
\nabla\times\boldsymbol{A}(u) &= \nabla u\times\frac{\mathrm{d}\boldsymbol{A}}{\mathrm{d}u}
\end{aligned}
$$

Proof.

$$
\nabla f(u) = (\partial_\mu e^\mu)(f\circ u)(x) = \partial_\mu(f\circ u)(x)\boldsymbol{e}^\mu = \frac{\partial f}{\partial u}\frac{\partial u}{\partial x^\mu}\boldsymbol{e}^\mu = \frac{\mathrm{d}f}{\mathrm{d}u}\nabla u
$$

$$
\nabla\cdot\boldsymbol{A}(u) = \partial_\mu A^\mu = \partial_\mu(A^\mu\circ u)(x) = \frac{\partial A^\mu}{\partial u}\frac{\partial u}{\partial x^\mu} = \frac{\mathrm{d}\boldsymbol{A}}{\mathrm{d}u}\cdot\nabla u
$$

$$
\nabla\times\boldsymbol{A}(u) = \epsilon^{\mu\nu\rho}\nabla_\mu A_\nu\boldsymbol{e}_\rho = \epsilon^{\mu\nu\rho}\partial_\mu(A_\nu\circ u)(x)\boldsymbol{e}_\rho = \epsilon^{\mu\nu\rho}\frac{\partial A_\nu}{\partial u}\frac{\partial u}{\partial x^\mu}\boldsymbol{e}_\rho = \frac{\mathrm{d}\boldsymbol{A}}{\mathrm{d}u}\times\nabla u
$$

□

Exercise 1.3. 设 $\boldsymbol{r}=\boldsymbol{x}-\boldsymbol{x}'$，$r=\sqrt{\sum_i(x^i-x'^i)^2}$，定义 $\nabla=\dfrac{\partial}{\partial x^\mu}\boldsymbol{e}^\mu$，$\nabla'=\dfrac{\partial}{\partial x'^\mu}\boldsymbol{e}^\mu$，证明

$$
\begin{aligned}
\nabla r &= -\nabla' r = \frac{\boldsymbol{r}}{r} \\
\nabla\frac{1}{r} &= -\nabla'\frac{1}{r} = -\frac{\boldsymbol{r}}{r^3} \\
\nabla\times\frac{\boldsymbol{r}}{r^3} &= 0 \\
\nabla\cdot\frac{\boldsymbol{r}}{r^3} &= -\nabla'\cdot\frac{\boldsymbol{r}}{r^3} = 0
\end{aligned}
$$

同时求解 $\nabla\cdot\boldsymbol{r}$，$\nabla\times\boldsymbol{r}$，$(\boldsymbol{a}\cdot\nabla)\boldsymbol{r}$，$\nabla(\boldsymbol{a}\cdot\boldsymbol{r})$，$\nabla\cdot[E_0\sin(\boldsymbol{k}\cdot\boldsymbol{r})]$，$\nabla\times[E_0\sin(\boldsymbol{k}\cdot\boldsymbol{r})]$

Solution. 易证 $\partial_\mu\left(\dfrac{r_\nu}{r^3}\right)=\dfrac{\delta_{\mu\nu}r^2-3r_\nu r_\mu}{r^5}$，$\partial_{\mu'}\left(\dfrac{r_\nu}{r^3}\right)=-\dfrac{\delta_{\mu\nu}r^2-3r_\nu r_\mu}{r^5}$，$\partial_\mu\dfrac{1}{r}=-\dfrac{r_\mu}{r^3}$

(a)

$$
\nabla r = \partial_\mu r\boldsymbol{e}^\mu = \frac{1}{r}r_\mu\boldsymbol{e}^\mu = \frac{\boldsymbol{r}}{r}
$$

$\nabla' r$ 的结果是非常显然的。

(b)

$$
\nabla\frac{1}{r} = \partial_\mu\frac{1}{r}\boldsymbol{e}^\mu = -\frac{1}{r^3}r_\mu\boldsymbol{e}^\mu = -\frac{\boldsymbol{r}}{r^3}
$$

(c)

$$
\nabla\times\frac{\boldsymbol{r}}{r^3} = \epsilon^{\mu\nu\rho}\partial_\mu\left(\frac{\boldsymbol{r}}{r^3}\right)_\nu\boldsymbol{e}_\rho = \frac{1}{r^5}\epsilon^{\mu\nu\rho}(\delta_{\mu\nu}r^2-3r_\mu r_\nu)\boldsymbol{e}_\rho = \frac{-3}{r^5}\epsilon^{\mu\nu\rho}r_\mu r_\nu\boldsymbol{e}_\rho
$$

同时交换指标位置结果不变，即 $\epsilon^{\mu\nu\rho}r_\mu r_\nu\boldsymbol{e}_\rho=\epsilon^{\nu\mu\rho}r_\nu r_\mu\boldsymbol{e}_\rho$，但只交换 ε 的指标会产生负值，即 $\epsilon^{\nu\mu\rho}r_\mu r_\nu\boldsymbol{e}_\rho=-\epsilon^{\mu\nu\rho}r_\mu r_\nu\boldsymbol{e}_\rho$，观察到 $\epsilon^{\nu\mu\rho}r_\nu r_\mu\boldsymbol{e}_\rho=-\epsilon^{\nu\mu\rho}r_\mu r_\nu\boldsymbol{e}_\rho$，由 $\boldsymbol{r}$ 的任意性得 $\epsilon^{\mu\nu\rho}=0$.

$$
\nabla\times\frac{\boldsymbol{r}}{r^3} = 0
$$

(d)

$$
\nabla\cdot\frac{\boldsymbol{r}}{r^3} = \partial_\mu\left(\frac{\boldsymbol{r}}{r^3}\right)^\mu = \frac{1}{r^5}(\delta^\mu{}_\mu r^2-3r^\mu r_\mu) = \frac{1}{r^5}(3r^2-3r^2) = 0
$$

(f)

$$\nabla \cdot \boldsymbol{r} = 3$$

$$\nabla \times \boldsymbol{r} = \epsilon^{\mu\nu\rho}\partial_\mu r_\nu \boldsymbol{e}_\rho = \epsilon^{\mu\nu\rho}\delta_{\mu\nu}\boldsymbol{e}_\rho = 0$$

$$(\boldsymbol{a}\cdot\nabla)\boldsymbol{r} = (a^\mu\partial_\mu)\boldsymbol{r} = a^\mu \boldsymbol{e}_\mu = \boldsymbol{a}$$

$$\nabla(\boldsymbol{a}\cdot\boldsymbol{r}) = (\partial_\mu \boldsymbol{e}^\mu)(a_\nu r^\nu) = a_\nu \partial_\mu r^\nu \boldsymbol{e}^\mu = a_\nu {\delta_\mu}^\nu \boldsymbol{e}^\mu = a_\mu \boldsymbol{e}^\mu = \boldsymbol{a}$$

$$\nabla \cdot [\boldsymbol{E}_0 \sin(\boldsymbol{k}\cdot\boldsymbol{r})] = \partial_\mu\left[(E_0)_\mu \sin(k_\nu r^\nu)\right] = (E_0)_\mu \cos(k_\nu r^\nu)k_\mu = [\boldsymbol{E}_0\cos(\boldsymbol{k}\cdot\boldsymbol{r})]\cdot\boldsymbol{k}$$

$$\nabla \times [\boldsymbol{E}_0 \sin(\boldsymbol{k}\cdot\boldsymbol{r})] = \epsilon^{\mu\nu\rho}\partial_\mu[(E_0)_\nu \sin(k_a r^a)]\boldsymbol{e}_\rho = \epsilon^{\mu\nu\rho}(E_0)_\nu\cos(k_a r^a)k_\mu \boldsymbol{e}_\rho = \boldsymbol{E}_0\cos(\boldsymbol{k}\cdot\boldsymbol{r})\times\boldsymbol{k}$$

□

Exercise 1.5. 若电荷系统的偶极矩定义为

$$\boldsymbol{P}(t) = \int_{\mathcal{V}} \rho(\boldsymbol{r}', t)\boldsymbol{r}' \,\mathrm{d}\tau'$$

利用 $\nabla\cdot\boldsymbol{J} + \dfrac{\partial\rho}{\partial t} = 0$ 证明

$$\frac{\partial \boldsymbol{P}}{\partial t} = \int_{\mathcal{V}} \boldsymbol{J}(\boldsymbol{r}', t)\,\mathrm{d}\tau'$$

Proof.

$$\frac{\partial \boldsymbol{P}}{\partial t} = \int_{\mathcal{V}} \frac{\partial\rho}{\partial t}(\boldsymbol{r}', t)\boldsymbol{r}'\,\mathrm{d}\tau' = \int_{\mathcal{V}} \frac{\partial\rho}{\partial t}(\boldsymbol{r}', t)x'\,\mathrm{d}\tau'\boldsymbol{e}_x + \int_{\mathcal{V}} \frac{\partial\rho}{\partial t}(\boldsymbol{r}', t)y'\,\mathrm{d}\tau'\boldsymbol{e}_y + \int_{\mathcal{V}} \frac{\partial\rho}{\partial t}(\boldsymbol{r}', t)z'\,\mathrm{d}\tau'\boldsymbol{e}_z$$

只考察 x' 方向，有

$$\begin{aligned}\int_{\mathcal{V}} \frac{\partial\rho}{\partial t}(\boldsymbol{r}', t)x'\,\mathrm{d}\tau' &= -\int_{\mathcal{V}} \nabla\cdot\boldsymbol{J}(\boldsymbol{r}', t)x'\,\mathrm{d}\tau' \\ &= -\int_{\mathcal{V}} \nabla\cdot(x'\boldsymbol{J}(\boldsymbol{r}', t)) - \nabla x'\cdot\boldsymbol{J}(\boldsymbol{r}', t)\,\mathrm{d}\tau' \\ &= -\oint_{\mathcal{S}} (x'\boldsymbol{J}(\boldsymbol{r}', t))\cdot\mathrm{d}\boldsymbol{a} + \int_{\mathcal{V}} \nabla x'\cdot\boldsymbol{J}(\boldsymbol{r}', t)\,\mathrm{d}\tau'\end{aligned}$$

如果取 $\mathcal{S}\to\infty$，由于边界处没有电流密度，故 $\oint_{\mathcal{S}}(x'\boldsymbol{J}(\boldsymbol{r}', t))\cdot\mathrm{d}\boldsymbol{a} = 0$，另一方面 $\nabla x' = (1,0,0)^T$，所以

$$\int_{\mathcal{V}} \frac{\partial\rho}{\partial t}(\boldsymbol{r}', t)x'\,\mathrm{d}\tau' = \int_{\mathcal{V}} J^x(\boldsymbol{r}', t)\,\mathrm{d}\tau'$$

从而得出

$$\frac{\partial \boldsymbol{P}}{\partial t} = \int_{\mathcal{V}} \boldsymbol{J}(\boldsymbol{r}', t)\,\mathrm{d}\tau'$$

□

Exercise 1.6. $\boldsymbol{m}$ 是常矢量，定义矢量 $\boldsymbol{A} = \dfrac{\boldsymbol{m}\times\boldsymbol{R}}{R^3}$，标量 $\varphi = \dfrac{\boldsymbol{m}\cdot\boldsymbol{R}}{R^3}$，证明除 $R=0$ 外有

$$\nabla\times\boldsymbol{A} = -\nabla\varphi$$

Proof. $A_\nu = \epsilon_{ij\nu} m^i \dfrac{R^j}{R^3}$，$\nabla \times \boldsymbol{A} = \epsilon^{\mu\nu\rho}\partial_\mu A_\nu \boldsymbol{e}_\rho$，$\partial_i\left(\dfrac{R^\mu}{R^3}\right) = \dfrac{\delta^\mu{}_i R^2 - 3R^\mu R_i}{R^5}$

$$\begin{aligned}
\nabla \times \boldsymbol{A} &= \epsilon^{\mu\nu\rho}\partial_\mu\left(\epsilon_{ij\nu} m^i \frac{R^j}{R^3}\right)\boldsymbol{e}_\rho \\
&= \epsilon^{\mu\nu\rho}\epsilon_{ij\nu} m^i \partial_\mu\left(\frac{R^j}{R^3}\right)\boldsymbol{e}_\rho \\
&= \frac{1}{R^5}\epsilon^{\mu\nu\rho}\epsilon_{ij\nu} m^i(\delta^j{}_\mu R^2 - 3R^j R_\mu)\boldsymbol{e}_\rho \\
&= \frac{1}{R^5}(\delta^\mu{}_j\delta^\rho{}_i - \delta^\mu{}_i\delta^\rho{}_j)(\delta^j{}_\mu R^2 - 3R^j R_\mu) m^i \boldsymbol{e}_\rho \\
&= \frac{1}{R^5}\left[\delta^\mu{}_j\delta^\rho{}_i\delta^j{}_\mu R^2\boldsymbol{e}_\rho - 3\delta^\mu{}_j\delta^\rho{}_i 3R^j R_\mu m^i\boldsymbol{e}_\rho - \delta^\mu{}_i\delta^\rho{}_j\delta^j{}_\mu R^2\boldsymbol{e}_\rho + 3\delta^\mu{}_i\delta^\rho{}_j R^j R_\mu m^i \boldsymbol{e}_\rho\right] \\
&= \frac{1}{R^5}\left[-3R^2\boldsymbol{m} + 3\boldsymbol{R}(\boldsymbol{R}\cdot\boldsymbol{m})\right]
\end{aligned}$$

$$\begin{aligned}
-\nabla\varphi &= -\partial_\mu\left(m_i\frac{R^i}{R^3}\right)\boldsymbol{e}^\mu \\
&= -\frac{1}{R^5} m_i(\delta^i{}_\mu R^2 - 3R^i R_\mu)\boldsymbol{e}^\mu \\
&= -\frac{1}{R^5}\left[m_i\delta^i{}_\mu R^2\boldsymbol{e}^\mu - 3m_i R^i R_\mu \boldsymbol{e}^\mu\right] \\
&= -\frac{1}{R^5}\left[3R^2\boldsymbol{m} - 3\boldsymbol{R}(\boldsymbol{R}\cdot\boldsymbol{m})\right]
\end{aligned}$$

□

Exercise 1.7. 有一内外半径分别为 r_1，r_2 的空心介质球，介质电容率为 ε，介质内均匀带静止自由电荷密度 ρ_f，求
(1) 空间各点电场
(2) 极化电荷和极化面电荷分布

Solution. 半径内包裹的电荷量

$$Q(r) = (r^3 - r_1^3)\frac{4}{3}\pi\rho_f$$

- $r_1 < r < r_2$ 时，由 $\oint_{\mathcal{S}} \boldsymbol{D}\cdot \mathrm{d}\boldsymbol{a} = Q(r)$ 得 $D4\pi r^2 = (r^3 - r_1^3)\dfrac{4}{3}\pi\rho_f$；于是 $D = \dfrac{r^3 - r_1^3}{3r^2}\rho_f$。由 $E = \dfrac{D}{\varepsilon}$ 得 $E = \dfrac{r^3 - r_1^3}{3\varepsilon r^2}\rho_f$.
- $r > r_2$ 时，$E = \dfrac{D}{\varepsilon_0}$，故 $E = \dfrac{r_2^3 - r_1^3}{3\varepsilon_0 r_2^2}\rho_f$.
- $r < r_1$ 时，由于 $Q = 0$，故 $D = 0$，得 $E = 0$。

边界处

$$(\boldsymbol{P}_2 - \boldsymbol{P}_1)\cdot\boldsymbol{e}_n = -\sigma_P$$

成立，区别于电荷密度 ρ，电荷面密度用 σ 表示。另外 $\boldsymbol{P} = \chi_e\varepsilon_0\boldsymbol{E} \Rightarrow \boldsymbol{P} = (\varepsilon_0 - \varepsilon)\boldsymbol{E}$，$\boldsymbol{P}$ 与 $\boldsymbol{E}$ 共线时有 $P = (\varepsilon_0 - \varepsilon)E$.

- r_2 面上，$P_2 = 0$, $P_1 = (\varepsilon - \varepsilon_0)E_1 = (\varepsilon - \varepsilon_0)\dfrac{r_2^3 - r_1^3}{3\varepsilon r_2^2}\rho_f$，因此 $\sigma_P = (\varepsilon - \varepsilon_0)\dfrac{r_2^3 - r_1^3}{3\varepsilon r_2^2}\rho_f$.
- r_1 面上，$P_0 = 0$，$P_1 = (\varepsilon - \varepsilon_0)(\varepsilon - \varepsilon_0)\dfrac{r_1^3 - r_1^3}{3\varepsilon r_1^2}\rho_f = 0$，所以 $\sigma_P = 0$.

□

Exercise 1.8. 内外 i 半径分别为 r_1，r_2 的中空导体圆柱，沿着轴向有恒定均匀自由电流 $\boldsymbol{J}_f$，导体内磁导率为 μ，求磁感应强度和磁化电流。

Solution. 由 $\oint_L \boldsymbol{H}\cdot \mathrm{d}\boldsymbol{l} = I_f + \int_S \dfrac{\partial \boldsymbol{D}}{\partial t}\,\mathrm{d}\boldsymbol{a}$，以及 $\dfrac{\partial \boldsymbol{D}}{\partial t} = 0$ 得 $\oint_L \boldsymbol{H}\cdot \mathrm{d}\boldsymbol{l} = I_f$.

- $r_1 < r < r_2$ 时，$\oint_L \boldsymbol{H}\cdot \mathrm{d}\boldsymbol{l} = H2\pi r = (\pi r_2^2 - \pi r_1^2)J_f \dfrac{\pi r^2 - \pi r_1^2}{\pi r_2^2 - \pi r_1^2}$，得出 $H = \dfrac{(r^2 - r_1^2)J_f}{2r}$。目测可知 $\boldsymbol{H}$ 的方向是 $\boldsymbol{J}\times\boldsymbol{r}$ 的方向，所以上式左右分别乘 $\hat{\boldsymbol{H}}$，$\hat{\boldsymbol{J}}\times\hat{\boldsymbol{r}}$，得 $\boldsymbol{H} = \dfrac{(r^2 - r_1^2)\boldsymbol{J}_f\times\boldsymbol{r}}{2r^2}$，再用 $\boldsymbol{B} = \mu\boldsymbol{H}$ 得 $\boldsymbol{B} = \dfrac{\mu(r^2 - r_1^2)\boldsymbol{J}_f\times\boldsymbol{r}}{2r^2}$.
- $r > r_2$ 时，$H = \dfrac{(r_2^2 - r_1^2)J_f}{2r}$，即得 $\boldsymbol{H} = \dfrac{(r_2^2 - r_1^2)\boldsymbol{J}_f\times\boldsymbol{r}}{2r^2}$，$\boldsymbol{B} = \dfrac{\mu(r_2^2 - r_1^2)\boldsymbol{J}_f\times\boldsymbol{r}}{2r^2}$.
- $r < r_1$ 时，$J = 0$，故 $\boldsymbol{B} = 0$.

在边界处取一高度不太高的截面，切向方向为 $\Delta\boldsymbol{l}$，通过该截面的磁化电流为 (P.27.(5.9)).

$$I_M = (\boldsymbol{e}_n\times\Delta\boldsymbol{l})\cdot\boldsymbol{a}_M$$

为了区别电流密度与电流线密度，用 a 表示电流线密度。

$$(\boldsymbol{M}_2 - \boldsymbol{M}_1)\cdot\Delta\boldsymbol{l} = I_M = (\boldsymbol{e}_n\times\Delta\boldsymbol{l})\cdot\boldsymbol{a}_M = (\boldsymbol{a}_M\times\boldsymbol{e}_n)\cdot\Delta\boldsymbol{l}$$

$(\boldsymbol{M}_2 - \boldsymbol{M}_1) = (\boldsymbol{a}_M\times\boldsymbol{e}_n)$ 两端同时叉乘 $\boldsymbol{e}_n$，利用 123=213-312 公式以及 $\boldsymbol{e}_n$ 与 $\boldsymbol{a}_M$ 正交；得到 $\boxed{\boldsymbol{e}_n\times(\boldsymbol{M}_2 - \boldsymbol{M}_1) = \boldsymbol{a}_M}$.

- r_2 面上，$\boldsymbol{M}_2 = 0$，所以 $-\boldsymbol{e}_n\times\boldsymbol{M}_1 = \boldsymbol{a}_M$；同时 $\boldsymbol{M}_1 = \chi_M\boldsymbol{H}_1 = \left(\dfrac{\mu}{\mu_0} - 1\right)\boldsymbol{H}_1$，可见 $\boldsymbol{a}_M = \left(\dfrac{\mu}{\mu_0} - 1\right)\boldsymbol{e}_n\times\boldsymbol{H}_1$，$\boldsymbol{e}_n$ 与 $\boldsymbol{r}$ 方向一致。

$$\begin{aligned}
\left(\frac{\mu}{\mu_0} - 1\right)\boldsymbol{e}_n\times\boldsymbol{H}_1 &= \left(\frac{\mu}{\mu_0} - 1\right)\left[\boldsymbol{e}_n\times\left(\frac{1}{2}\boldsymbol{J}_f\times\boldsymbol{r} - \frac{r_1^2}{2r^2}\boldsymbol{J}_f\times\boldsymbol{r}\right)\right]_{r=r_2} \\
&= \left(\frac{\mu}{\mu_0} - 1\right)\left[\frac{r^2 - r_1^2}{2r}\right]_{r=r_2} J_f \\
&= \left(\frac{\mu}{\mu_0} - 1\right)\frac{r_2^2 - r_1^2}{2r_2}J_f
\end{aligned}$$

- r_1 面上，$\boldsymbol{M}_0 = 0$，所以 $\boldsymbol{e}_n\times\boldsymbol{M}_1 = \boldsymbol{a}_M$，借用上面的推倒得到 $\boldsymbol{a}_M = \left(\dfrac{\mu}{\mu_0} - 1\right)\dfrac{r_1^2 - r_1^2}{2r_1}J_f = 0$.
- $r_1 < r < r_2$，利用 $\dfrac{1}{\mu_0}\nabla\times\boldsymbol{B} = \boldsymbol{J}_f + \boldsymbol{J}_M + \boldsymbol{J}_P + \varepsilon_0\dfrac{\partial\boldsymbol{E}}{\partial t}$，得到 $\dfrac{1}{\mu_0}\nabla\times\boldsymbol{B} - \boldsymbol{J}_f = \boldsymbol{J}_M$，最终得 $\boldsymbol{J}_M = \left(\dfrac{\mu}{\mu_0} - 1\right)\boldsymbol{J}_f$

$$\begin{aligned}
\nabla\times\boldsymbol{B} &= \nabla\times\left[\frac{\mu(r^2 - r_1^2)\boldsymbol{J}_f\times\boldsymbol{r}}{2r^2}\right] \\
&= \frac{\mu}{2}\nabla\times(\boldsymbol{J}_f\times\boldsymbol{r}) - \frac{\mu r_1^2}{2}\nabla\times(\boldsymbol{J}\times\frac{\boldsymbol{r}}{r^2}) \\
&= \frac{\mu}{2}\left[(\boldsymbol{r}\cdot\nabla)\boldsymbol{J}_f - (\nabla\cdot\boldsymbol{J}_f)\boldsymbol{r} + (\nabla\cdot\boldsymbol{r})\boldsymbol{J}_f - (\boldsymbol{J}_f\cdot\nabla)\boldsymbol{r}\right] \\
&- \frac{\mu r_1^2}{2}\left[(\frac{\boldsymbol{r}}{r^2}\cdot\nabla)\boldsymbol{J}_f - (\nabla\cdot\boldsymbol{J}_f)\frac{\boldsymbol{r}}{r^2} + (\nabla\cdot\frac{\boldsymbol{r}}{r^2})\boldsymbol{J}_f - (\boldsymbol{J}_f\cdot\nabla)\frac{\boldsymbol{r}}{r^2}\right] \\
&= \frac{\mu}{2}\left[0 - 0 + 3\boldsymbol{J}_f - \boldsymbol{J}_f\right] - \frac{\mu r_1^2}{2}\left[0 - 0 + \frac{1}{r^2}\boldsymbol{J}_f - \frac{1}{r^2}\boldsymbol{J}_f\right] \\
&= \mu\boldsymbol{J}_f
\end{aligned}$$

□

Exercise 1.9. 证明均匀介质内部的极化电荷密度总是满足 $\rho_P = -\left(1-\dfrac{\varepsilon_0}{\varepsilon}\right)\rho_f$.

Proof. 首先从 $\nabla\cdot(\varepsilon_0\boldsymbol{E}+\boldsymbol{P})=\rho_f$ 得出 $\rho_P=\varepsilon_0\nabla\cdot\boldsymbol{E}-\rho_f$. 再者，利用 $\boldsymbol{P}=\chi_e\varepsilon_0\boldsymbol{E}=(\varepsilon-\varepsilon_0)\boldsymbol{E}$ 得到 $-\rho_P=\nabla\cdot\boldsymbol{P}=(\varepsilon-\varepsilon_0)\nabla\cdot\boldsymbol{E}$. 联立两个方程

$$\begin{cases}\rho_P=\varepsilon_0\nabla\cdot\boldsymbol{E}-\rho_f\\ \rho_P=-(\varepsilon-\varepsilon_0)\nabla\cdot\boldsymbol{E}\end{cases}$$

解得 $\rho_P=-\left(1-\dfrac{\varepsilon_0}{\varepsilon}\right)\rho_f$. □

Exercise 1.10. 证明两个闭合的恒定电流圈之间的相互作用力大小相等，方向相反。

Proof. 定义光滑环闭道路 $\Gamma_a: I_a\to\mathbb{R}^3$, $\Gamma_b: I_b\to\mathbb{R}^3$，其像点分别表示为 Γ_a, Γ_b，稳定电流分别表示为 I_a, I_b，参数分别使用 t,τ。根据 $\boldsymbol{B}(\boldsymbol{r})=\dfrac{\mu_0}{4\pi}\int_{\mathcal{V}}\dfrac{\boldsymbol{J}(\boldsymbol{r}')\times\boldsymbol{\imath}}{\imath^3}\,\mathrm{d}\tau'$，得出道路 Γ_a 在 Γ_b 上一点的磁感应强度为

$$\boldsymbol{B}_{ab}(\tau)=\frac{\mu_0 I_a}{4\pi}\oint\limits_{\Gamma_a}\frac{1}{|\Gamma_b(\tau)-\Gamma_a(t)|^3}\dot{\Gamma}_a(t)\times(\Gamma_b(\tau)-\Gamma_a(t))\,\mathrm{d}t$$

其中 $\dot{\Gamma}_a(t)=((\dot{\Gamma}_a)^1,(\dot{\Gamma}_a)^2,(\dot{\Gamma}_a)^3)(t)$，为了方便，后文用 $A(t)$, $B(\tau)$ 代替 $\Gamma_a(t)$, $\Gamma_b(\tau)$，用 $a(t)$, $b(\tau)$ 代替 $\dot{\Gamma}_a(t)$, $\dot{\Gamma}_b(\tau)$，于是

$$B_{ab}(\tau)=\frac{\mu_0 I_a}{4\pi}\oint\limits_{\Gamma_a}\frac{1}{|B-A|^3}a\times(B-A)\,\mathrm{d}t$$
$$B_{ba}(t)=\frac{\mu_0 I_b}{4\pi}\oint\limits_{\Gamma_b}\frac{1}{|A-B|^3}b\times(A-B)\,\mathrm{d}\tau$$

考察 Γ_b 的受力情况

$$F_{ab}=\oint\limits_{\Gamma_b}B_{ab}(\tau)\times(I_b\,\mathrm{d}l_b)=I_b\oint\limits_{\Gamma_b}B_{ab}(\tau)\times b(\tau)\,\mathrm{d}\tau$$

展开后得到

$$\begin{aligned}F_{ab}&=\frac{\mu_0 I_a I_b}{4\pi}\oint\limits_{\Gamma_b}\oint\limits_{\Gamma_a}\frac{1}{|B-A|^3}\epsilon^{\mu\nu\rho}\epsilon_{ij\mu}a^i(B^j-A^j)b_\nu\boldsymbol{e}_\rho\,\mathrm{d}t\,\mathrm{d}\tau\\
&=\frac{\mu_0 I_a I_b}{4\pi}\oint\limits_{\Gamma_b}\oint\limits_{\Gamma_a}\frac{1}{|B-A|^3}(\delta^\nu{}_i\delta^\rho{}_j-\delta^\nu{}_j\delta^\rho{}_i)a^i(B^j-A^j)b_\nu\boldsymbol{e}_\rho\,\mathrm{d}t\,\mathrm{d}\tau\\
&=\frac{\mu_0 I_a I_b}{4\pi}\oint\limits_{\Gamma_b}\oint\limits_{\Gamma_a}\frac{1}{|B-A|^3}(a^\nu A^\rho b_\nu\boldsymbol{e}_{rho}-a^\nu A^\rho b_\nu\boldsymbol{e}_\rho-a^\rho B^\nu b_\nu\boldsymbol{e}_\rho+a^\rho A^\nu b_{nu}\boldsymbol{e}_\rho)\,\mathrm{d}t\,\mathrm{d}\tau\\
&=\frac{\mu_0 I_a I_b}{4\pi}\left[\oint\limits_{\Gamma_b}\oint\limits_{\Gamma_a}\frac{(a\cdot b)(B-A)}{|B-A|^3}\,\mathrm{d}t\,\mathrm{d}\tau-\oint\limits_{\Gamma_a}\oint\limits_{\Gamma_b}\frac{(B-A)}{|B-A|^3}\cdot\mathrm{d}B\,\mathrm{d}A\right]\\
&=\frac{\mu_0 I_a I_b}{4\pi}\oint\limits_{\Gamma_b}\oint\limits_{\Gamma_a}\frac{(a\cdot b)(B-A)}{|B-A|^3}\,\mathrm{d}t\,\mathrm{d}\tau\end{aligned}$$

同样，考察 Γ_a 的受力情况，得到

$$F_{ba}=-\frac{\mu_0 I_a I_b}{4\pi}\oint\limits_{\Gamma_b}\oint\limits_{\Gamma_a}\frac{(a\cdot b)(B-A)}{|B-A|^3}\,\mathrm{d}t\,\mathrm{d}\tau$$

□

Exercise 1.11. 平行板电容器有两层介质，厚度分别为 l_1, l_2，电容率为 ε_1，ε_2，在两板接上电动势为 ξ 的电池
(1) 电容器两板上的自由电荷面密度 ω_f
(2) 介质分界面上的自由电荷面密度 ω_f
(3) 若介质漏电, 电导率为 σ_1，σ_2, 电流达到恒定时, 上述结果如何?

Solution.
(1) 由 $\boldsymbol{e}_n \cdot (\boldsymbol{D}_2 - \boldsymbol{D}_1) = \sigma_f$ 得, 极板 1 上满足 $D_1 - D_{10} = \omega_{f_1}$, 极板 2 上满足 $D_{20} - D_2 = \omega_{f_2}$, 介质分界面不存在自由电荷, 所以 $D_2 - D_1 = \omega_{f_3} = 0$. 同时由 $\boldsymbol{D} = \varepsilon \boldsymbol{E}$, 得下述方程

$$\begin{cases} \varepsilon_1 E_1 = \omega_{f_1} \\ \varepsilon_2 E_2 = -\omega_{f_2} \\ E_1 l_1 + E_2 l_2 = \xi \\ \varepsilon_1 E_1 - \varepsilon_2 E_2 = 0 \end{cases}$$

解得 $E_1 = \dfrac{\varepsilon_2 \xi}{\varepsilon_2 l_1 + \varepsilon_1 l_2}$, $E_2 = \dfrac{\varepsilon_1 \xi}{\varepsilon_2 l_1 + \varepsilon_1 l_2}$, $\omega_{f_1} = \dfrac{\varepsilon_1 \varepsilon_2 \xi}{\varepsilon_2 l_1 + \varepsilon_1 l_2}$, $\omega_{f_2} = -\dfrac{\varepsilon_1 \varepsilon_2 \xi}{\varepsilon_2 l_1 + \varepsilon_1 l_2}$, $\omega_{f_3} = 0$.
(2) 根据欧姆定律 $\boldsymbol{J} = \sigma \boldsymbol{E}$, 以及介质电流恒定, 得 $J = \sigma_1 E_1 = \sigma_2 E_2$, 求解下述方程组

$$\begin{cases} E_1 l_1 + E_2 l_2 = \xi \\ \omega_1 E_1 = \omega_2 E_2 \\ \varepsilon_1 E_1 = \omega_{f_1} \\ \varepsilon_2 E_2 = -\omega_{f_2} \\ \varepsilon_2 E_2 - \varepsilon_1 E_1 = \omega_{f_3} \end{cases}$$

解得 $E_1 = \dfrac{\sigma_2 \xi}{\varepsilon_2 l_1 + \varepsilon_1 l_2}$, $E_2 = \dfrac{\sigma_1 \xi}{\varepsilon_2 l_1 + \varepsilon_1 l_2}$, $\omega_{f_1} = \dfrac{\varepsilon_1 \sigma_2 \xi}{\varepsilon_2 l_1 + \varepsilon_1 l_2}$, $\omega_{f_2} = \dfrac{\varepsilon_2 \sigma_1 \xi}{\varepsilon_2 l_1 + \varepsilon_1 l_2}$.
$\omega_{f_3} = \dfrac{\varepsilon_2 \sigma_1 \xi}{\varepsilon_2 l_1 + \varepsilon_1 l_2} - \dfrac{\varepsilon_1 \sigma_2 \xi}{\varepsilon_2 l_1 + \varepsilon_1 l_2}$. □