Exercise2

202005100214

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Exercise 例三. 半径为 R_0 的接地导体球置于均匀外电场 $\boldsymbol{E}_0$ 中，求电势和导体上的电荷面密度。

Solution. r 是球坐标下坐标 (r,θ,ϕ) 的分量。定义球面内势场 u_1，球面外势场 u_2。其 u_1 只在球面内有定义，u_2 只在球面外有定义，两者不能直接叠加。它们分别满足

$$\begin{cases}\nabla^2 u_1 = 0\\ u_1|_{r=R_0} = 0\\ u_1(0) < \infty\end{cases}, \quad \begin{cases}\nabla^2 u_2 = 0\\ u_2|_{r=R_0} = 0\\ \lim\limits_{r\to\infty} u_2 = u_0 - E_0 r P_1(\cos\theta)\end{cases}$$

根据方位角无关的通解公式得到 $u_1 = \sum\limits_{n=0} a_n r^n P_n(\cos\theta)$，$u_2 = u_0 + \frac{b_0}{r} - E_0 r\cos\theta + \frac{b_1}{r^2}\cos\theta + \sum\limits_{n=2}\frac{b_n}{r^{n+1}}P_n(\cos\theta)$，根据边界条件列出方程

$$\begin{aligned}&\sum_{n=0} a_n R_0^n P_n(\cos\theta) = 0\\ &u_0 + \frac{b_0}{R_0} - E_0 R_0\cos\theta + \frac{b_1}{R_0^2}\cos\theta + \sum_{n=2}\frac{b_n}{R_0^{n+1}}P_n(\cos\theta) = 0\end{aligned}$$

由于基底 $P_l(x)$ 的正交性，$\sum_{n=0} a_n R_0^n P_n(\cos\theta) = 0$ 唯一的情况是 $a_n = 0$，则 $u_1 = 0$。同时还要求函数在基底 $P_n(x)$ 上的展开系数唯一，即

$$\begin{cases}u_0 + \dfrac{b_0}{R_0} = 0\\ -E_0 R_0\cos\theta + \dfrac{b_1}{R_0^2}\cos\theta = 0\\ b_n = 0\ (n\geq 2)\end{cases}$$

所以

$$u_2 = u_0 - \frac{R_0 u_0}{r} - E_0 r\cos\theta + \frac{E_0 R_0^3}{r^2}\cos\theta$$

如果取 $u_0 = 0$，得到 $u_2 = -E_0 r\cos\theta + \frac{E_0 R_0^3}{r^2}\cos\theta$。

球坐标系下的梯度算符为 $\nabla = \frac{\partial}{\partial r}\hat{\boldsymbol{e}}_r + \frac{1}{r}\frac{\partial}{\partial\theta}\hat{\boldsymbol{e}}_\theta + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\hat{\boldsymbol{e}}_\phi$，球面上 $\boldsymbol{e}_n$ 大小与方向和 $\hat{\boldsymbol{e}}_r$ 一致，所以 $\frac{\partial u}{\partial n} = \nabla u\cdot\boldsymbol{e}_n = \frac{\partial u}{\partial r}$. 带入导体的边界条件 $\varepsilon_0\left.\frac{\partial u}{\partial n}\right|_{r=R_0} = -\sigma$，可以求得约束电荷面密度

$$\sigma = 3\varepsilon_0 E_0\cos\theta$$

□

Exercise 2.1. 一个半径为 R 的电介质球，极化强度为 $\boldsymbol{P} = K\frac{\boldsymbol{r}}{r^2}$，电容率为 ε

(1) 计算束缚电荷体密度和面密度
(2) 计算自由电荷体密度
(3) 计算球外和球内电势
(4) 求该电介质求的静电场的总能量

Solution. (1) 极化强度满足 $\nabla\cdot\boldsymbol{P}=-\rho_P$，所以

$$\rho_P=\nabla\cdot K\frac{\boldsymbol{r}}{r^2}=-K\frac{\delta^\mu{}_\mu r^2-2r^\mu r_\mu}{r^4}=-K\frac{1}{r^2}$$

球面上取一易拉罐，满足 $\boldsymbol{e}_n\cdot(\boldsymbol{P}_2-\boldsymbol{P}_1)=-\sigma_P$，但 $\boldsymbol{P}_2=0$，所以 $\sigma_p=P_1=K\dfrac{1}{R}$

(2) 考虑 $\boldsymbol{D}=\varepsilon_0+K\dfrac{\boldsymbol{r}}{r^2}$ 以及 $\boldsymbol{D}=\varepsilon\boldsymbol{E}$，得 $\boldsymbol{D}=\dfrac{\varepsilon K}{\varepsilon-\varepsilon_0}\dfrac{\boldsymbol{r}}{r^2}$，由麦克斯韦方程组 $\nabla\cdot\boldsymbol{D}=\rho$ 可得

$$\rho=\frac{\varepsilon K}{\varepsilon-\varepsilon_0}\nabla\cdot\frac{\boldsymbol{r}}{r^2}=\frac{\varepsilon K}{\varepsilon-\varepsilon_0}\frac{1}{r^2}$$

(3) 总的电荷量为

$$Q=\int_{\mathcal{V}}\rho(\boldsymbol{r})\,\mathrm{d}\tau=\frac{\varepsilon K}{\varepsilon-\varepsilon_0}\int_0^{2\pi}\int_0^{\pi}\int_0^{R}\frac{1}{r^2}r^2\sin\theta\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi=\frac{4\pi\varepsilon KR}{\varepsilon-\varepsilon_0}$$

根据高斯定理，球面外的电场满足 $E_o(r)4\pi r^2=\dfrac{4\pi\varepsilon KR}{\varepsilon_0(\varepsilon-\varepsilon_0)}$，解得 $E_o(r)=\dfrac{\varepsilon KR}{\varepsilon_0(\varepsilon-\varepsilon_0)r^2}$，从而

$$u_o(r)=-\int_\infty^r\frac{\varepsilon KR}{\varepsilon_0(\varepsilon-\varepsilon_0)t^2}\,\mathrm{d}t=\frac{\varepsilon KR}{\varepsilon_0(\varepsilon-\varepsilon_0)r}$$

上问已经求得了球内电场 $E_i(r)=\dfrac{\varepsilon K}{\varepsilon-\varepsilon_0}\dfrac{1}{r}$，因此球内电势为

$$\begin{aligned}u_i(r)&=-\int_\infty^r E\,\mathrm{d}t\\&=\int_r^R E_i(t)\,\mathrm{d}t+\int_R^\infty E_o(t)\,\mathrm{d}t\\&=\frac{K}{\varepsilon-\varepsilon_0}\left[\ln\frac{R}{r}+\frac{\varepsilon}{\varepsilon_0}\right]\end{aligned}$$

(4)

$$\begin{aligned}W&=\frac{1}{2}\int_{\mathcal{V}}\rho(r)u_i(r)\,\mathrm{d}r\\&=\frac{\varepsilon K^2}{2(\varepsilon-\varepsilon_0)^2}\left[\iiint_{\mathcal{V}}\ln\frac{R}{r}\sin\theta\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi+\iiint_{\mathcal{V}}\frac{\varepsilon}{\varepsilon_0}\sin\theta\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi\right]\\&=2\pi\varepsilon R\frac{K^2}{(\varepsilon-\varepsilon_0)^2}\left(1+\frac{\varepsilon}{\varepsilon_0}\right)\end{aligned}$$

□

Exercise 2.2. 均匀外置电场放入半径为 R_0 的导体球，用分离变量法求解下述情况
(1) 导体球接有电池，与地面保持电势差 $\varPhi_0$
(2) 导体球带有电荷 Q

Solution. (1) 球面内定义电势场 u_1，球面外定义电势场 u_2。两个标量场分别满足

$$\begin{cases}\nabla^2u_2(\boldsymbol{r})=0\\u_2|_{r=R_0}=\varPhi_0\\\lim\limits_{r\to\infty}u_2=u_0-E_0r\cos\theta\end{cases},\quad\begin{cases}\nabla^2u_1(\boldsymbol{r})=0\\u_1|_{r=R_0}=\varPhi_0\\u_1(0)<\infty\end{cases}$$

根据方位角无关的通解公式 $u_1=\sum\limits_{n=0}\left(a_nr^n+\dfrac{b_n}{r^{n+1}}\right)\mathrm{P}_n(\cos\theta)$，$u_2=\sum\limits_{n=0}\left(c_nr^n+\dfrac{d_n}{r^{n+1}}\right)\mathrm{P}_n(\cos\theta)$，列出必要条件：$a_0=u_0$，$a_1=-E_0$，$a_n=0\,(n\geq 2)$，$d_n=0$；

$$u_0+\frac{b_0}{R_0}-E_0R_0\cos\theta+\frac{b_1}{R_0^2}\cos\theta+\sum_{n=2}\frac{b_n}{R_0^{n+1}}\mathrm{P}_n(\cos\theta)=c_0+c_1R_0\cos\theta+\sum_{n=2}c_nR_0^n\mathrm{P}_n(\cos\theta)$$

$$u_0+\frac{b_0}{R_0}-E_0R_0\cos\theta+\frac{b_1}{R_0^2}\cos\theta+\sum_{n=2}\frac{b_n}{R_0^{n+1}}\mathrm{P}_n(\cos\theta)=\varPhi_0$$

解上述方程带有猜的成分，猜测 $b_n = c_n = 0\ (n \geq 2)$，如果有解，则这个解就是唯一的解。

$$u_0 + \frac{b_0}{R_0} - E_0 R_0 \cos\theta + \frac{b_1}{R_0^2}\cos\theta = \varPhi_0$$

为了使上式左边与 θ 无关，可以让 $-E_0 R_0 \cos\theta + \frac{b_1}{R_0^2}\cos\theta = 0$，得出如下方程组

$$\begin{cases} E_0 R_0 \cos\theta = \dfrac{b_1}{R_0^2}\cos\theta \\ u_0 + \dfrac{b_0}{R_0} = \varPhi_0 \\ c_0 = u_0 + \dfrac{b_0}{R_0} \\ c_1 = -E_0 R_0 \cos\theta + \dfrac{b_1}{R_0^2}\cos\theta \end{cases}$$

发现有解，所以就有

$$u_2(\boldsymbol{r}) = u_0 - E_0 r \cos\theta + \frac{R_0(\varPhi_0 - u_0)}{r} + \frac{E_0 R_0^3}{r^2}\cos\theta$$

(2) 根据高斯定理得

$$\oint_{\mathcal{S}} \boldsymbol{E}\cdot \mathrm{d}\boldsymbol{a} = \frac{Q}{\varepsilon_0}$$

球面上得单位法向量 $\boldsymbol{e}_n$ 与球坐标的基失 $\hat{\boldsymbol{e}}_r$ 是一致的，所以

$$\oint_{\mathcal{S}} \boldsymbol{E}\cdot \mathrm{d}\boldsymbol{a} = -\oint_{\mathcal{S}} \nabla u \cdot \boldsymbol{e}_n \,\mathrm{d}a = -\oint_{\mathcal{S}} \frac{\partial u}{\partial r}\,\mathrm{d}a$$

在球面上积分，得出边界条件之一 $\left.\dfrac{\partial u}{\partial r}\right|_{r=R_0} = -\dfrac{Q}{4\pi\varepsilon_0 R_0^2}$. 两个标量场分别满足

$$\begin{cases} \nabla^2 u_2(\boldsymbol{r}) = 0 \\ \left.\dfrac{\partial u}{\partial r}\right|_{r=R_0} = -\dfrac{Q}{4\pi\varepsilon_0 R_0^2} \\ \lim\limits_{r\to\infty} u_2 = u_0 - E_0 r\cos\theta \end{cases}, \qquad \begin{cases} \nabla^2 u_1(\boldsymbol{r}) = 0 \\ u_1|_{r=R_0} = u_2|_{r=R_0} = \mathrm{Const} \\ u_1(0) < \infty \end{cases}$$

同样的方法解出 $b_0 = \dfrac{Q}{4\pi\varepsilon_0}$, $b_1 = \dfrac{E_0 R_0^3}{2}$

$$u_2(\boldsymbol{r}) = u_0 - E_0 r\cos\theta + \frac{Q}{4\pi\varepsilon_0 r} + \frac{E_0 R_0^3}{2r^2}\cos\theta$$

□

Exercise 2.3. 均匀介质球的中心置一电荷 Q_f，球的电容率为 ε，球外为真空，用分离变量法球空间电势。

Solution. 定义球面内势场 u_1，球面外势场 u_2。根据电介质分界面的边界条件列出方程组

$$\begin{cases} \nabla^2 u_2 = 0 \\ \lim\limits_{r\to\infty} u_2 = 0 \end{cases}, \qquad \begin{cases} \nabla^2 u_1 = -\dfrac{Q_f \delta(r)}{\varepsilon} \\ u_2|_{R_0} = u_1|_{R_0} \\ \varepsilon_0 \left.\dfrac{\partial u_2}{\partial r}\right|_{R_0} = \varepsilon \left.\dfrac{\partial u_1}{\partial r}\right|_{R_0} \end{cases}$$

由于 $u_2(\boldsymbol{r})$ 只与长度 r 有关，其通解为 $u_2 = a + \dfrac{b}{r}$，根据条件还有 $a = 0$。可见在球面上 $u_2 = \dfrac{b}{R_0}$ 是一个常数。由此可以通过

$$\begin{cases} \nabla^2 u_1 = -\dfrac{Q_f \delta(r)}{\varepsilon} \\ u_1|_{R_0} = \mathrm{Const} \end{cases}$$

确定 u_1 的一族解 $u_1=\dfrac{Q_f}{4\pi\varepsilon r}+c$。通过 $\varepsilon_0\left.\dfrac{\partial u_2}{\partial r}\right|_{R_0}=\varepsilon\left.\dfrac{\partial u_1}{\partial r}\right|_{R_0}$ 求得 $b=\dfrac{Q_f}{4\pi\varepsilon_0}$，通过 $u_2|_{R_0}=u_1|_{R_0}$ 求出 $c=\dfrac{Q_f}{4\pi\varepsilon_0 R_0}-\dfrac{Q_f}{4\pi\varepsilon R_0}$。最终得到

$$
\begin{aligned}
u_1 &= \frac{Q_f}{4\pi\varepsilon r}+\frac{Q_f}{4\pi\varepsilon_0 R_0}-\frac{Q_f}{4\pi\varepsilon R_0} \\
u_2 &= \frac{Q_f}{4\pi\varepsilon_0 r}
\end{aligned}
$$

□

Exercise 2.6. 均匀外电场 E_0 置入一均匀带自由电荷密度 ρ_f 的绝缘介质球，电容率为 ε，求空间各点电势。

外电场不会影响自由电荷分布吗?

Solution. 假设外电场下自由电荷分布不变。设球内电势为 u_i+u_i'，u_i 是自由电荷产生的电势，u_i' 是外电场激发的束缚电荷产生的电势；球外电势为 u_e。分别满足

$$
\begin{cases}
\nabla^2 u_i = -\dfrac{\rho_f}{\varepsilon} \\
\nabla^2 u_i' = 0 \\
[u_i+u_i']_{r=0}<\infty
\end{cases},
\qquad
\begin{cases}
\nabla^2 u_e = 0 \\
\lim\limits_{r\to\infty} u_e = u_0 - E_0 r P_1(\cos\theta) \\
u_e|_{R_0} = [u_i+u_i']_{R_0} \\
\varepsilon\left.\dfrac{\partial(u_i+u_i')}{\partial r}\right|_{R_0} = \varepsilon_0\left.\dfrac{\partial u_e}{\partial r}\right|_{R_0}
\end{cases}
$$

$\nabla^2 u_i=-\dfrac{\rho_f}{\varepsilon}$ 已经包含了自身电场的极化，无需再考虑别的。而有外电场的情况下会有额外的极化，所以加上了 u_i' 这一项。

$$
\nabla^2 u_i = -\frac{\rho_f}{\varepsilon} \Rightarrow \nabla\cdot\nabla u_i = -\frac{\rho_f}{\varepsilon} \Rightarrow \oint_{\mathcal{S}} \frac{\partial u_i}{\partial r}\,\mathrm{d}a = -\int_{\mathcal{V}} \frac{\rho_f}{\varepsilon}\,\mathrm{d}\tau
$$

解得 $u_i=u_0-\dfrac{\rho_f}{6\varepsilon}r^2$。目测可知

$$
u_2 = u_0 + \frac{b_0}{r} - E_0 r P_1(\cos\theta) + \frac{b_1}{r^2}P_1(\cos\theta) + \sum_{n=2}\frac{b_n}{r^{n+1}}P_n(\cos\theta)
$$

u_i+u_i' 形如

$$
u_i+u_i' = u_0 - \frac{\rho_f}{6\varepsilon}r^2 + \sum_{n=0} c_n r^n P_n(\cos\theta)
$$

带入边界条件得

$$
\begin{aligned}
& u_0 + \frac{b_0}{R_0} - E_0 R_0 P_1(\cos\theta) + \frac{b_1}{R_0^2}P_1(\cos\theta) + \sum_{n=2}\frac{b_n}{R_0^{n+1}}P_n(\cos\theta) \\
=& u_0 - \frac{\rho_f}{6\varepsilon}R_0^2 + c_0 + c_1 R_0 P_1(\cos\theta) + \sum_{n=2} c_n R_0^n P_n(\cos\theta)
\end{aligned}
$$

$$
\begin{aligned}
& \varepsilon\left[-\frac{\rho_f R_0}{3\varepsilon} + c_1 P_1(\cos\theta) + \sum_{n=2} n c_n R_0^{n-1} P_n(\cos\theta)\right] \\
=& \varepsilon_0\left[-\frac{b_0}{R_0^2} - E_0 P_1(\cos\theta) - \frac{2b_1}{R_0^3}P_1(\cos\theta) + \sum_{n=2}\frac{-(n+1)b_n}{R_0^{n+2}}P_n(\cos\theta)\right]
\end{aligned}
$$

列出方程组

$$\begin{cases} u_0 + \dfrac{b_0}{R_0} = u_0 - \dfrac{\rho_f}{6\varepsilon}R_0^2 + c_0 \\ -E_0R_0 + \dfrac{b_1}{R_0^2} = c_1R_0 \\ -\dfrac{\rho_f R_0}{3} = -\dfrac{\varepsilon_0 b_0}{R_0^2} \\ \varepsilon c_1 = \varepsilon_0\left[-E_0 - \dfrac{2b_1}{R_0^3}\right] \\ \sum\limits_{n=2}\dfrac{b_n}{R_0^{n+1}}P_n(\cos\theta) = \sum\limits_{n=2}c_nR_0^nP_n(\cos\theta) \\ \varepsilon\sum\limits_{n=2}nc_nR_0^{n-1}P_n(\cos\theta) = \varepsilon_0\sum\limits_{n=2}\dfrac{-(n+1)b_n}{R_0^{n+2}}P_n(\cos\theta) \end{cases}$$

解得 $c_n = b_n = 0,\ (n \geq 2)$，$b_0 = \dfrac{\rho_f R_0^3}{3\varepsilon_0}$，$c_0 = \dfrac{\rho_f R_0^2}{3\varepsilon_0} + \dfrac{\rho_f R_0^2}{6\varepsilon}$，$b_1 = \dfrac{-3E_0\varepsilon_0 R_0^3}{\varepsilon + 2\varepsilon_0} + E_0R_0^3$，$c_1 = \dfrac{-3E_0\varepsilon_0}{\varepsilon + 2\varepsilon_0}$。所以

$$u_1 = u_0 - \frac{\rho_f}{6\varepsilon}r^2 + \frac{\rho_f R_0^2}{3\varepsilon_0} + \frac{\rho_f R_0^2}{6\varepsilon} + \frac{-3E_0\varepsilon_0}{\varepsilon + 2\varepsilon_0}r\cos\theta$$
$$u_2 = u_0 + \frac{\rho_f R_0^3}{3\varepsilon_0 r} - E_0r\cos\theta + \left(\frac{-3E_0\varepsilon_0R_0^3}{\varepsilon + 2\varepsilon_0} + E_0R_0^3\right)\frac{1}{r^2}\cos\theta$$

□

Exercise 2.9. 接地空心导体球内外半径分别为 R_1，R_2，在离球心为 a 处置一电荷 Q，用镜像法求电势，导体球上感应电荷有多少，分布在内表面还是外表面。

Solution. 感应电荷集中于内表面，因此在等效时也应该以内表面为准。由相似关系得 $R_1^2 = ab$，在 R_1 面上还应满足 $\dfrac{Q}{4\pi\varepsilon_0 r_0} = \dfrac{q}{4\pi\varepsilon_0 r_1}$，得 $q = \dfrac{r_1}{r_0} = \dfrac{R_1}{a}$，所以

$$u(r) = \frac{Q}{4\pi\varepsilon_0 r_0} - \frac{QR_1}{a4\pi\varepsilon_0 r_1}$$

同时用余弦定理计算 $r_0 = \sqrt{r^2 + a^2 - 2ra\cos\theta}$，$r_1 = \sqrt{r^2 + \dfrac{R_1^4}{a^2} - 2r\dfrac{R_1^2}{a}\cos\theta}$，故

$$u(r) = \frac{Q}{4\pi\varepsilon_0\sqrt{r^2 + a^2 - 2ra\cos\theta}} - \frac{QR_1}{a4\pi\varepsilon_0\sqrt{r^2 + \dfrac{R_1^4}{a^2} - 2r\dfrac{R_1^2}{a}\cos\theta}}$$

□

Exercise 2.10. 上题导体球壳不接地，而是带总电荷 Q_0，或使其有总电荷 $\varPhi_0$，求其电势。

Solution. 设球壳内电势 u_1，球壳外电势 u_2，满足

$$\begin{cases} \nabla^2 u_2 = 0 \\ \lim\limits_{r\to\infty} u_2 = 0 \\ \oint_{r=R_2}\dfrac{\partial u_2}{\partial r}\,\mathrm{d}a = -\dfrac{Q + Q_0}{\varepsilon_0} \end{cases},\quad \begin{cases} \nabla^2 u_1 = -Q\dfrac{\delta(\boldsymbol{r} - \boldsymbol{a})}{\varepsilon_0} \\ u_1|_{R_1} = u_2|_{R_2} = \mathrm{Const} \end{cases}$$

可以直接解出 $u_2 = \dfrac{Q + Q_0}{4\pi\varepsilon_0 r}$。$u_1$ 的解形如

$$u_1 = \frac{Q}{4\pi\varepsilon_0\sqrt{r^2 + a^2 - 2ra\cos\theta}} - \frac{QR_1}{a4\pi\varepsilon_0\sqrt{r^2 + \dfrac{R_1^4}{a^2} - 2r\dfrac{R_1^2}{a}\cos\theta}} + c$$

带入边界条件 $u_1|_{R_1} = \dfrac{Q+Q_0}{4\pi\varepsilon_0 R_2}$，有

$$\begin{aligned}
&\frac{Q}{4\pi\varepsilon_0\sqrt{R_1^2+a^2-2R_1 a\cos\theta}} - \frac{QR_1}{a4\pi\varepsilon_0\sqrt{R_1^2+\dfrac{R_1^4}{a^2}-2R_1\dfrac{R_1^2}{a}\cos\theta}} + c = \frac{Q+Q_0}{4\pi\varepsilon_0 R_2} \\
&\Rightarrow \frac{Q}{4\pi\varepsilon_0\sqrt{R_1^2+a^2-2R_1 a\cos\theta}} - \frac{Q}{4\pi\varepsilon_0\sqrt{R_1^2+a^2-2R_1 a\cos\theta}} + c = \frac{Q+Q_0}{4\pi\varepsilon_0 R_2} \\
&\Rightarrow c = \frac{Q+Q_0}{4\pi\varepsilon_0 R_2}
\end{aligned}$$

所以

$$u_1 = \frac{Q}{4\pi\varepsilon_0\sqrt{r^2+a^2-2ra\cos\theta}} - \frac{QR_1}{a4\pi\varepsilon_0\sqrt{r^2+\dfrac{R_1^4}{a^2}-2r\dfrac{R_1^2}{a}\cos\theta}} + \frac{Q+Q_0}{4\pi\varepsilon_0 R_2}$$

如果只给定导体的电势，也可以仿照上面的方法求出

$$\begin{aligned}
u_1 &= \frac{Q}{4\pi\varepsilon_0\sqrt{r^2+a^2-2ra\cos\theta}} - \frac{QR_1}{a4\pi\varepsilon_0\sqrt{r^2+\dfrac{R_1^4}{a^2}-2r\dfrac{R_1^2}{a}\cos\theta}} + \varPhi_0 \\
u_2 &= \frac{\varPhi_0 R_2}{r}
\end{aligned}$$

□

Exercise 2.11. 接地导体平面有一半径为 a 的半凸部，半球球心在导体平面上，点电荷 Q 位于系统的对称轴上，与平面相距为 b，求空间电势。

Solution. 为使半球面上电势为 0，可以等效为球面内有一未知电荷 q。q 在对称轴上，设其位置半径为 d，应当满足 $bd = a^2$，即 $d = \dfrac{a^2}{b}$，其与电荷 Q 的势能叠加为 0，即 $\dfrac{q}{4\pi\varepsilon_0 r_a} + \dfrac{Q}{4\pi\varepsilon_0 r_b} = 0 \Rightarrow q = -\dfrac{r_a}{r_b}Q \Rightarrow q = -\dfrac{a}{b}Q$. 此时平面的电势还不是 0，由于上述的电荷结构对球面的电势没有影响，所以考虑在平面下方对称地放置两个电荷，位置分别为 $-\dfrac{a^2}{b}$，$-b$，电荷量分别为 $\dfrac{a}{b}Q$，$-Q$。最终空间中某处的电势可以表示为

$$u(\boldsymbol{r}) = \frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{\sqrt{x^2+(y-b)^2}} - \frac{1}{\sqrt{x^2+(y+b)^2}}\right) + \frac{q}{4\pi\varepsilon_0}\left(\frac{-1}{\sqrt{x^2+(y-\frac{a^2}{b})^2}} + \frac{1}{\sqrt{x^2+(y+\frac{a^2}{b})^2}}\right)$$

□