1. 矢量分析

1.1. 叉积公式

Proposition 1.1.1.

$$(u \times v) \cdot w = (w \times u) \cdot v$$

Proof. $(u \times v) \cdot w = \sum_{ijk} u^i v^j w^k \epsilon_{ijk}$，如果 i, j, k 是偶排列，$\epsilon_{ijk} = J$，如果是奇排列 $\epsilon_{ijk} = -J$。比如

$$(v \times u) \cdot w = \sum_{ijk} v^i u^j w^k \epsilon_{ijk} = \sum_{ijk} v^j u^i w^k \epsilon_{jik} = -\sum_{ijk} v^j u^i w^k \epsilon_{ijk}$$

□

Proposition 1.1.2.

$$u \times (v \times w) = v(u \cdot w) - w(u \cdot v)$$

Proof.

$$\begin{aligned}
(u \times (v \times w))_k &= (u \times (v \times w)) \cdot e_k \\
&= u^i (v \times w)^j (e_i \times e_j) \cdot e_k \\
&= u^i (v \times w)^j \epsilon_{ijk} \\
&= u^i v_p w_q \epsilon^{pqj} \epsilon_{ijk} \\
&= u^i v_p w_q (\delta^p_k \delta^q_i - \delta^p_i \delta^q_k) \\
&= u^i w_i v_k - u^i v_i w_k
\end{aligned}$$

其中用到了一个常用结论

$$\epsilon^{ijk}\epsilon_{pqr} = \begin{vmatrix} \delta^i_p & \delta^i_q & \delta^i_r \\ \delta^j_p & \delta^j_q & \delta^j_r \\ \delta^k_p & \delta^k_q & \delta^k_r \end{vmatrix}$$

□

1.2. 梯度相关

Define 导数算符.

$$\begin{gathered}
\nabla \cdot \boldsymbol{u} = \partial_\mu v^\mu \\
\nabla \times \boldsymbol{u} = \epsilon^{\mu\nu\rho} \partial_\mu u_\nu \boldsymbol{e}_\rho \\
(\nabla \times \boldsymbol{u})^k = \boldsymbol{e}^k (\epsilon^{\mu\nu\rho} \partial_\mu u_\nu \boldsymbol{e}_\rho) = \epsilon^{\mu\nu k} \partial_\mu u_\nu
\end{gathered}$$

Proposition 1.2.1.

$$\nabla \cdot (\boldsymbol{u} \times \boldsymbol{v}) = (\nabla \times \boldsymbol{u}) \cdot \boldsymbol{v} - (\nabla \times \boldsymbol{v}) \cdot \boldsymbol{u}$$

Proof.

$$\begin{aligned}
\nabla \cdot (\boldsymbol{u} \times \boldsymbol{v}) &= \partial_i (\boldsymbol{u} \times \boldsymbol{v})^i \\
&= \partial_i (\epsilon^{\mu\nu i} u_\mu v_\nu) \\
&= \epsilon^{\mu\nu i} (\partial_i u_\mu) v_\nu + \epsilon^{\mu\nu i} u_\mu (\partial_i v_\nu) \\
&= \epsilon^{i\mu\nu} (\partial_i u_\mu) v_\nu - \epsilon^{i\nu\mu} (\partial_i v_\nu) u_\mu \\
&= (\nabla \times \boldsymbol{u}) \cdot \boldsymbol{v} - (\nabla \times \boldsymbol{v}) \cdot \boldsymbol{u}
\end{aligned}$$

□

Proposition 1.2.2.

$$\nabla \times (\boldsymbol{u} \times \boldsymbol{v}) = (\boldsymbol{v} \cdot \nabla)\boldsymbol{u} - (\nabla \cdot \boldsymbol{u})\boldsymbol{v} + (\nabla \cdot \boldsymbol{v})\boldsymbol{u} - (\boldsymbol{u} \cdot \nabla)\boldsymbol{v}$$

Proof.

$$
\begin{aligned}
\nabla \times (\boldsymbol{u} \times \boldsymbol{v}) &= \epsilon^{ijk} \partial_i (\boldsymbol{u} \times \boldsymbol{v})_j \boldsymbol{e}_k \\
&= \epsilon^{ijk} \epsilon_{\mu\nu j} \partial_i (u^\nu v^\nu) \boldsymbol{e}_k \\
&= (\delta^i{}_\nu \delta^k{}_\mu - \delta^i{}_\mu \delta^k{}_\nu)(\partial_i u^\mu) v^\nu \boldsymbol{e}_k + (\delta^i{}_\nu \delta^k{}_\mu - \delta^i{}_\mu \delta^k{}_\nu)(\partial_i v^\nu) u^\mu \boldsymbol{e}_k \\
&= (\partial_\nu u^\mu) v^\nu \boldsymbol{e}_\mu - (\partial_\mu u^\mu) v^\nu \boldsymbol{e}_\nu + (\partial_\nu v^\nu) u^\mu \boldsymbol{e}_\mu - (\partial_\mu v^\nu) u^\mu \boldsymbol{e}_\nu \\
&= v^\nu \partial_\nu u^\mu \boldsymbol{e}_\mu - (\partial_\mu u^\mu) v^\nu \boldsymbol{e}_\nu + (\partial_\nu v^\nu) u^\mu \boldsymbol{e}_\mu - u^\mu \partial_\mu v^\nu \boldsymbol{e}_\nu \\
&= (\boldsymbol{v} \cdot \nabla)\boldsymbol{u} - (\nabla \cdot \boldsymbol{u})\boldsymbol{v} + (\nabla \cdot \boldsymbol{v})\boldsymbol{u} - (\boldsymbol{u} \cdot \nabla)\boldsymbol{v}
\end{aligned}
$$

□

Define 散度. 对于一个 vector function $v : \mathbb{R}^3 \to \mathbb{R}^3$.

$$\text{Divergence} := \nabla \cdot v = \partial_x v^x + \partial_y v^y + \partial_z v^z$$

Define 旋度. 同样对于一个 vector function v.

$$\text{Crul} := \nabla \times v = \begin{vmatrix} \boldsymbol{e}_x & \boldsymbol{e}_y & \boldsymbol{e}_z \\ \partial_x & \partial_y & \partial_z \\ v_x & v_y & v_z \end{vmatrix}$$

Proposition 1.2.3. 梯度的旋度永远是零.

$$\nabla \times (\nabla T) = 0$$

Proof. 用行列式展开

$$\nabla \times (\nabla T) = (\partial_{yz} - \partial_{zy}) T \boldsymbol{e}_x - (\partial_{xz} - \partial_{zx}) T \boldsymbol{e}_y + (\partial_{xy} - \partial_{yx}) T \boldsymbol{e}_z = 0$$

□

Proposition 1.2.4. 旋度的散度永远是零.

$$\nabla \cdot (\nabla \times v) = 0$$

Proof.

$$\nabla \cdot (\nabla \times v) = \partial_x (\partial_y v^z - \partial_z v^y) - \partial_y (\partial_x v^z - \partial_z v^x) + \partial_z (\partial_x v^y - \partial_y v^x) = 0$$

□

1.3. 积分

Define 线积分. 类似于变力沿曲线做功.

$$\int_a^b \boldsymbol{v} \cdot \mathrm{d}\boldsymbol{l}$$

其中 $\boldsymbol{v} = v^x \boldsymbol{e}_x + v^y \boldsymbol{e}_y + v^z \boldsymbol{e}_z$, $\mathrm{d}\boldsymbol{l} = \mathrm{d}x \boldsymbol{e}_x + \mathrm{d}y \boldsymbol{e}_y + \mathrm{d}z \boldsymbol{e}_z$.

Define 通量. 需要注意 $\mathrm{d}\boldsymbol{a}$ 的方向是它的法线方向.

$$\int_{\mathcal{S}} \boldsymbol{v} \cdot \mathrm{d}\boldsymbol{a}$$

Define 体积分. T 是一个标量场，在 Cartesian coordinates 下 $\mathrm{d}\tau = \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$.

$$\int_{\mathcal{V}} t \,\mathrm{d}\tau$$

Proposition 1.3.1. 梯度的积分:

$$\int_a^b \nabla T \cdot \mathrm{d}\boldsymbol{l} = T(b) - T(a)$$

具体原因目前不清楚，可以类比牛顿莱布尼兹公式:

$$\int_a^b \frac{\partial f}{\partial x} \,\mathrm{d}x = f(b) - f(a)$$

Theorem 高斯定理. 散度定理，格林公式，随便怎么叫。可以直观理解为体积散度的累加等于表面的通量。

$$\int_{\mathcal{V}} \nabla \cdot \boldsymbol{v} \, \mathrm{d}\tau = \oint_{\mathcal{S}} \boldsymbol{v} \cdot \mathrm{d}\boldsymbol{a}$$

当体积足够小时，通量密度可定义为散度.

$$\operatorname{div} \boldsymbol{v} = \lim_{\Delta V \to 0} \frac{\oint_{\mathcal{S}} \boldsymbol{v} \cdot \mathrm{d}\boldsymbol{a}}{\Delta V}$$

实际上通过高等数学的近似计算可以得到

$$\lim_{\Delta V \to 0} \frac{1}{\Delta V} \left[\int_{\mathcal{S}_1} \boldsymbol{v} \boldsymbol{e}_x \, \mathrm{d}y \, \mathrm{d}z + \int_{\mathcal{S}_2} \boldsymbol{v}(-\boldsymbol{e}_x) \, \mathrm{d}y \, \mathrm{d}z \right] = \frac{\partial \boldsymbol{v}^x}{\partial x}$$

Theorem 斯托克斯定理. 旋度定理。可以直观理解为面里旋度的累加等于边界的环量.

$$\int_{\mathcal{S}} \nabla \times \boldsymbol{v} \, \mathrm{d}\boldsymbol{a} = \oint_{\mathcal{P}} \boldsymbol{v} \cdot \mathrm{d}\boldsymbol{l}$$

面积足够小时，环量密度可定义为旋度.

$$\operatorname{curl} \boldsymbol{v} = \lim_{\Delta S \to 0} \frac{\oint_{\mathcal{P}} \boldsymbol{v} \cdot \mathrm{d}\boldsymbol{l}}{\Delta S}$$

可证明

$$\operatorname{curl} \boldsymbol{v} = \left(\frac{\partial \boldsymbol{v}^y}{\partial z} - \frac{\partial \boldsymbol{v}^z}{\partial y} \right) \boldsymbol{e}_x + \left(\frac{\partial \boldsymbol{v}^z}{\partial x} - \frac{\partial \boldsymbol{v}^x}{\partial z} \right) \boldsymbol{e}_y + \left(\frac{\partial \boldsymbol{v}^y}{\partial x} - \frac{\partial \boldsymbol{v}^y}{\partial x} \right) \boldsymbol{e}_z$$

1.4. 狄拉克函数

有一个非常特殊的向量函数:

$$\boldsymbol{v} = \frac{1}{|r|^3} \boldsymbol{r}$$

其除了该点处以外散度是 0，通量是 4π

$$\nabla \cdot \boldsymbol{v} = 0$$

$$\oint \boldsymbol{v} \cdot \mathrm{d}\boldsymbol{a} = \int \frac{1}{R^3} \boldsymbol{r} R \boldsymbol{r} \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\phi = \int_0^{\pi} \sin\theta \, \mathrm{d}\theta \int_0^{2\pi} \mathrm{d}\phi = 4\pi$$

可以看到高斯定理好像失效了，其实并没有，因为如果说

$$\int \nabla \cdot \boldsymbol{v} \, \mathrm{d}\tau = 0$$

其实是忽略了 $r = 0$ 那一点的情况，此处的散度并没有定义，但普遍认为上述积分的结果正是由高斯定理推导得出的 4π。

Define Delta 函数.

$$\int_{-\infty}^{+\infty} \delta(x) \, \mathrm{d}x = 1$$

Proposition 1.4.1.

$$\nabla \cdot \left(\frac{\boldsymbol{r}}{|r|^3} \right) = 4\pi \delta^3(x)$$