The second Vassiliev measure

1 The second Vassiliev invariant of a knot

The second Vassiliev invariant of an oriented knot diagram, K_ξ is defined as:

$$v_2(K_\xi) = \frac{1}{2} \sum_{i<j<k<l \in I} \epsilon_{ik}\epsilon_{jl} \tag{1}$$

where I is the set of pairs of crossings in the diagram which are "alternating". In practice:

To calculate the second Vassiliev invariant of a knot, we:

- Choose a starting point on the knot (we will consider the first coordinate)
- Project the knot on a plane to obtain an oriented knot diagram
- start traveling along the knot diagram with its orientation, starting with the projection of the starting point on the diagram
- when you pass through a crossing, label the point you are on "1" and the other point on the knot "3"
- continue travelling through the knot and if you pass through a crossing before you go through point "3", then label the point you are on "2" and the other point on the crossing "4"
- this pair of crossings is a candidate for contributing to the sum of v_2
- if "1" is the over (resp. under) arc at the crossing between "1" and "3" and "2" is the under (resp. over) arc at the crossing between "2" and "4", then the pair qualifies
- if the pair qualifies, multiply their signs of crossings and add the result to v_2
- continue looking for such pairs along the knot, until you reach the endpoint.

v_2 is a topological invariant of knots.

2 The second Vassiliev measure of an open curve in 3-space

For an open curve in 3-space, l, v_2 is defined as

$$v_2(l) = \frac{1}{4\pi} \int_{\vec{\xi} \in S^2} v_2(l_{\vec{\xi}}) dS \tag{2}$$

where $l_{\vec{\xi}}$ denotes the projection of l on the plane with normal vector $\vec{\xi}$.

To calculate v_2 in practice, we approximate it by:

- Generate a random vector $\vec{\xi}$
- project l to the plane with normal vector $\vec{\xi}$, $l_{\vec{\xi}}$
- calculate $v_2(l_{\vec{\xi}})$
- do this for 500 random vectors and take the average

v_2 is a continuous function of the chain coordinates (not a topological invariant). As the endpoints of the chain tend to coincide, it tends to the Vassiliev invariant of the resulting knot.

3 Calculation of v_2

Use the code in VasDefs.py to calculate v_2 for the trefoil knot with coordinates
$((1,0,0),(4,0,0),(1,6,2),(0,2,-5),(5,2,5),(4,6,-2))$.

To check the code: try to plot the trefoil (at least on paper) to determine what crossings will appear in at least some projections and verify the calculation.

For the knot, the result should not depend on the projection direction.

Try opening the knot by a small amount that does not undo the knotting. What is v_2 then?