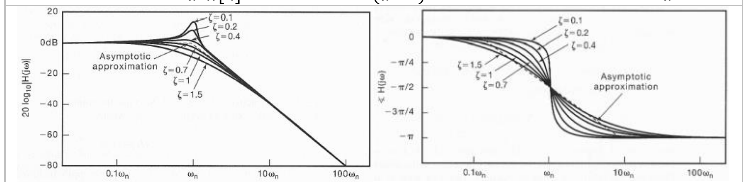
Properties of the FT (CT and DT are similar) (Ch4&5)

Property	Aperiodic signal	FT	LT	ROC (LT)	z-T	ROC (z-T)
Linearity	$ax(t)+by(t)$	$aX(j\omega)+bY(j\omega)$	$aX(s)+bY(s)$	At least $R_1\cap R_2$	$aX(z)+bY(z)$	At least $R_1\cap R_2$
Time shifting	$x(t-t_0)$ or $x[n-n_0]$	$e^{-j\omega t_0}X(j\omega)$	$e^{-st_0}X(s)$	R	$z^{-n_0}X(z)$	R reconsider origin
Freq shifting	$e^{j\omega_0 t}x(t)$ or $e^{j\omega_0 n}x[n]$	$X(j(\omega-\omega_0))$	Signal: $e^{s_0 t}x(t)$ $X(s-s_0)$	Sifted version of R $R+\Re\{s_0\}$	Signal: $e^{j\omega_0 n}x[n]$ $X(e^{-j\omega_0}z)$ rotation	R
Conjugation	$x^*(t)$ or $x^*[n]$	$X^*(-j\omega)$ or $X^*(e^{-j\omega})$	$X^*(s^*)$	R	$X^*(z^*)$	R
Time reversal	$x(-t)$ or $x[-n]$	$X(-j\omega)$ or $X(e^{-j\omega})$	$X(-s)$	Reversed R	$X(z^{-1})$	Inverted R
Time and freq scaling (CT)	$x(at)$	$\frac{1}{\lvert a\rvert}X\left(\frac{j\omega}{a}\right)$	$\frac{1}{\lvert a\rvert}X\left(\frac{s}{a}\right)$	Scaled version of R aR	-	-
Time expansion (DT)	$x_{(k)}[n]=\begin{cases}x[n/k]\\0 \text{ if irrational}\end{cases}$	$X(e^{jk\omega})$	-	-	$X(z^k)$	$R^{1/k}$
Convolution	$x(t)*y(t)$	$X(j\omega)Y(j\omega)$	$X(s)Y(s)$	At least $R_1\cap R_2$	$X(z)Y(z)$	At least $R_1\cap R_2$
Multiplication	$x(t)y(t)$	$\frac{1}{2\pi}\int_{-\infty}^{+\infty}X(j\theta)Y(j(\omega-\theta))d\theta$	-	-	-	-
Diff in time	$\frac{d}{dt}x(t)$	$j\omega X(j\omega)$	$sX(s)$	At least R	-	-
	$x[n]-x[n-1]$	$(1-e^{-j\omega})X(e^{j\omega})$	-	-	$(1-z^{-1})X(z)$	At least $R\cap\{\lvert z\rvert>0\}$
Integration (CT)	$\int_{-\infty}^{t}x(\tau)d\tau$	$\frac{1}{j\omega}X(j\omega)+\pi X(0)\delta(\omega)$	$\frac{1}{s}X(s)$	At least $R\cap\{\Re\{s\}>0\}$	-	-
Accumulation (DT)	$\sum_{k=-\infty}^{n}x[k]$	$\frac{1}{1-e^{-j\omega}}X(e^{j\omega})+$ $\pi X(e^{j0})\sum_{k=-\infty}^{+\infty}\delta(\omega-2\pi k)$	-	-	$\frac{1}{1-z^{-1}}X(z)$	At least $R\cap\{\lvert z\rvert>1\}$
Diff in freq	$tx(t)$ or $nx[n]$	$j\frac{d}{d\omega}X(j\omega)$ or $j\frac{d}{d\omega}X(e^{j\omega})$	Diff in s-dom: $-tx(t)$ $\frac{d}{ds}X(s)$	R	Diff in z-dom: $nx[n]$ $-z\frac{d}{dz}X(z)$	R

z-T scaling in the z-domian	$z_0^{\,n}x[n]$	$X(z/z_0)$	Scaled version of R	z_0R
	$a^n x[n]$	$X(a^{-1}z)$		aR



Parseval's relation for aperiodic signals (Ch4&5)

$$\int_{-\infty}^{+\infty}|x(t)|^2dt=\frac{1}{2\pi}\int_{-\infty}^{+\infty}|X(j\omega)|^2d\omega$$

$$\sum_{n=-\infty}^{+\infty}|x[n]|^2=\frac{1}{2\pi}\int_{2\pi}\left|X(e^{j\omega})\right|^2d\omega$$

Derive the DTFT for aperiodic signals (Ch5)

As $N\to\infty$, $\tilde{x}[n]=x[n]$ for any finite value of n. $\tilde{x}[n]$ is the periodic signal.
Since $\tilde{x}[n]=x[n]$ within any period $<N>$, we have

$$a_k=\frac{1}{N}\sum_{n=<N>}\tilde{x}[n]e^{-jk\omega_0 n}=\frac{1}{N}\sum_{n=-\infty}^{+\infty}x[n]e^{-jk\omega_0 n}\triangleq\frac{1}{N}X(e^{jk\omega_0})=\frac{1}{N}X(e^{j\omega})$$

Substituting this a_k to the synthesis equation yields

$$\tilde{x}[n]=\sum_{k=<N>}\frac{1}{N}X(e^{jk\omega_0})\,e^{jk\omega_0 n}=\frac{1}{2\pi}\sum_{k=<N>}X(e^{jk\omega_0})\,e^{jk\omega_0 n}\omega_0$$

Therefore, as $N\to\infty$, $\tilde{x}[n]\to x[n]$ and $\omega_0\to 0$, the above summation becomes an integral (periodic in frequency with period 2π).

$$x[n]=\frac{1}{2\pi}\int_{2\pi}X(e^{j\omega})e^{j\omega n}d\omega$$

Group delay (Ch6)

$$H(j\omega)=\frac{1+(j\omega/\omega_i)^2-2j\zeta_i(\omega/\omega_i)}{1+(j\omega/\omega_i)^2+2j\zeta_i(\omega/\omega_i)}\qquad \measuredangle H(j\omega)=-2\tan^{-1}\left[\frac{2\zeta_i(\omega/\omega_i)}{1-(\omega/\omega_i)^2}\right]$$

$|H(j\omega)|=1\to$ an all-pass system $\quad \tau(\omega)=-\frac{d}{d\omega}\{\measuredangle H(j\omega)\}\to$ group delay

First-order CT systems (Ch6)

$\tau y'(t)+y(t)=x(t)\qquad H(j\omega)=1/(j\omega\tau+1)\qquad \measuredangle H(j\omega)=-\tan^{-1}(\omega\tau)$

Second-order CT systems (Ch6)

$y''(t)+2\zeta\omega_n y'(t)+\omega_n^{\,2}y(t)=\omega_n^{\,2}x(t)$

$$H(j\omega)=\frac{\omega_n^{\,2}}{(j\omega-c_1)(j\omega-c_2)}\qquad c_i=-\zeta\omega_n\pm\omega_n\sqrt{\zeta^2-1}$$

Underdamped: $0<\zeta<1$
Critically damped: $\zeta=1$
Overdamped: $\zeta>1$
Magnitude response: when $\zeta<\sqrt{2}/2$, peak will appear.
Phase response: smoother when larger ζ

First-order DT systems (Ch6)

$y[n]-ay[n-1]=x[n], |a|<1$ When $a\to 1$, magnitude at lower frequency is larger.

$$H(e^{j\omega})=\frac{1}{1-ae^{-j\omega}}\quad |H(e^{j\omega})|=\frac{1}{\sqrt{1+a^2-2a\cos\omega}}\quad \measuredangle H(e^{j\omega})=-\tan^{-1}\left[\frac{a\sin\omega}{1-a\cos\omega}\right]$$

Second-order DT systems (Ch6)

$$y[n]-2r\cos\theta\, y[n-1]+r^2y[n-2]=x[n]\quad H(e^{j\omega})=\frac{1}{[1-(re^{j\theta})e^{-j\omega}][1-(re^{-j\theta})e^{-j\omega}]}$$

The initial-value theorem (Ch9)

If $x(t)=0$ for $t<0$ and it has contains no impulse or higher order singularities at the origin (and $x(t)$ causal),

$$x(0^+)=\lim_{s\to\infty}sX(s)$$

Proof:
by Taylor series expansion at $t=0^+$
$x(t)=0$ for $t<0\Rightarrow x(t)=x(t)u(t)$

$$e^{-at}\left(\frac{t^n}{n!}\right)u(t)\overset{\mathcal{L}}{\leftrightarrow}\frac{1}{(s+a)^{n+1}},\ \Re\{s\}>-a$$

Set $a=0$ and multiply two sides by $x^{(n)}(0^+)$

$$x^{(n)}(0^+)\left(\frac{t^n}{n!}\right)u(t)\overset{\mathcal{L}}{\leftrightarrow}\frac{x^{(n)}(0^+)}{s^{n+1}},\ \Re\{s\}>0$$

Substitute with the Taylor series, we have

$$X(s)=\sum_0^\infty\frac{x^{(n)}(0^+)}{s^{n+1}}$$

$sX(s)=x^{(0)}(0^+)+x^{(1)}(0^+)/s+\cdots$
Hence, $\lim_{s\to\infty}sX(s)=x^{(0)}(0^+)=x(0^+)$

The final-value theorem (Ch9)

If $x(t)=0$ for $t<0$ and it has a finite limit as $t\to\infty$ (and $x(t)$ causal),

$$\lim_{t\to\infty}x(t)=\lim_{s\to 0}sX(s)$$

Proof:
From $\frac{d}{dt}x(t)\overset{\mathcal{L}}{\leftrightarrow}sX(s)$ and by definition

$$sX(s)=\mathcal{L}\left\{\frac{dx(t)}{dt}\right\}=\int_{-\infty}^{\infty}x'(t)e^{-st}dt$$

$$=\int_{0-}^{\infty}x'(t)e^{-st}dt$$

$$\lim_{s\to 0}sX(s)=\lim_{s\to 0}\int_{0-}^{\infty}x'(t)e^{-st}dt$$

$$=\int_{0-}^{\infty}x'(t)dt=\lim_{t\to\infty}x(t)-x(0^-)$$

$$=\lim_{t\to\infty}x(t)-0=\lim_{t\to\infty}x(t)$$

(Causal, so $x(0^-)=0$)

Basic CTFT pairs (Ch4)

Signal	FT	FS (if periodic)	
$\sum_{k=-\infty}^{+\infty}a_ke^{jk\omega_0 t}$	$2\pi\sum_{k=-\infty}^{+\infty}a_k\delta(\omega-k\omega_0)$	a_k	
$e^{j\omega_0 t}$	$2\pi\delta(\omega-\omega_0)$	$a_1=1$	$a_k=0$ otherwise
$\cos\omega_0 t$	$\pi[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)]$	$a_1=a_{-1}=\frac{1}{2}$	
$\sin\omega_0 t$	$\frac{\pi}{j}[\delta(\omega-\omega_0)-\delta(\omega+\omega_0)]$	$a_1=-a_{-1}=\frac{1}{2j}$	
$x(t)=1$	$2\pi\delta(\omega)$	$a_0=1$	
Periodic square wave with $x(t+T)=x(t)$			
$\begin{cases}1, & \lvert t\rvert<T_1\\0, & T_1<\lvert t\rvert\le\frac{T}{2}\end{cases}$	$\sum_{k=-\infty}^{+\infty}\frac{2\sin k\omega_0T_1}{k}\delta(\omega-k\omega_0)$	$\frac{\omega_0T_1}{\pi}\operatorname{sinc}\left(\frac{k\omega_0T_1}{\pi}\right)=\frac{\sin k\omega_0T_1}{k\pi}$	
$\sum_{n=-\infty}^{+\infty}\delta(t-nT)$	$\frac{2\pi}{T}\sum_{k=-\infty}^{+\infty}\delta\left(\omega-\frac{2\pi k}{T}\right)$	$a_k=\frac{1}{T}$ for all k	

Signal	FT	Signal	FT
$x(t)=\begin{cases}1, & \lvert t\rvert<T_1\\0, & \lvert t\rvert>T_1\end{cases}$	$\frac{2\sin\omega T_1}{\omega}$	$\frac{\sin Wt}{\pi t}$	$X(j\omega)=\begin{cases}1, & \lvert\omega\rvert<W\\0, & \lvert\omega\rvert>W\end{cases}$
$\delta(t)$	1	$\delta(t-t_0)$	$e^{-j\omega t_0}$
$u(t)$	$\frac{1}{j\omega}+\pi\delta(\omega)$	$e^{-at}u(t)$ $\Re\{a\}>0$	$\frac{1}{a+j\omega}$
$te^{-at}u(t)$ $\Re\{a\}>0$	$\frac{1}{(a+j\omega)^2}$	$\frac{t^{n-1}}{(n-1)!}e^{-at}u(t)$	$\frac{1}{(a+j\omega)^n}$

Basic DTFT pairs (Ch5)

Signal	FT	FS (if periodic)	
$\sum_{k=\langle N\rangle}a_ke^{jk\left(\frac{2\pi}{N}\right)n}$	$2\pi\sum_{k=-\infty}^{+\infty}a_k\delta\left(\omega-\frac{2\pi k}{N}\right)$	a_k	$\frac{\omega_0}{2\pi}=\frac{m}{N}$ periodic; $a_k=0$ otherwise
$e^{j\omega_0 n}$	$2\pi\sum_{l=-\infty}^{+\infty}\delta(\omega-\omega_0-2\pi l)$	$a_k=1, k=m\pm sN, s\in\mathbb{Z}$	
$\cos\omega_0 n$	$\pi\sum_{l=-\infty}^{+\infty}[\delta(\omega-\omega_0-2\pi l)+\delta(\omega+\omega_0-2\pi l)]$	$a_k=\frac{1}{2}, k=\pm m\pm sN$	
$\sin\omega_0 n$	$\frac{\pi}{j}\sum_{l=-\infty}^{+\infty}[\delta(\omega-\omega_0-2\pi l)-\delta(\omega+\omega_0-2\pi l)]$	$a_k=\frac{1}{2j}, k=m\pm sN$ $a_k=-\frac{1}{2j}, k=-m\pm sN$	
$x[n]=1$	$2\pi\sum_{l=-\infty}^{+\infty}\delta(\omega-2\pi l)$	$a_k=1, k=0\pm sN$	
Periodic square wave with $x[n+N]=x[n]$		$a_k=\frac{2N_1+1}{N}, k=0\pm sN$	
$\begin{cases}1, & \lvert n\rvert<N_1\\0, & N_1<\lvert n\rvert\le\frac{N}{2}\end{cases}$	$2\pi\sum_{k=-\infty}^{+\infty}a_k\delta\left(\omega-\frac{2\pi k}{N}\right)$	$a_k=\frac{\sin[(2\pi k/N)(N_1+\frac{1}{2})]}{N\sin[2\pi k/2N]}$ otherwise	
$\sum_{k=-\infty}^{+\infty}\delta[n-kN]$	$\frac{2\pi}{N}\sum_{k=-\infty}^{+\infty}\delta\left(\omega-\frac{2\pi k}{N}\right)$	$a_k=\frac{1}{N}$ for all k	

Signal	FT	Signal	FT
$x[n]=\begin{cases}1, & \lvert n\rvert\le N_1\\0, & \lvert n\rvert>N_1\end{cases}$	$\frac{\sin[\omega(N_1+\frac{1}{2})]}{\sin(\omega/2)}$	$\frac{\sin Wn}{\pi n}$	$X(\omega)=\begin{cases}1, & 0\le\lvert\omega\rvert<W\\0, & W<\lvert\omega\rvert\le\pi\end{cases}$ Periodic with period 2π
$\delta[n]$	1	$\delta[n-n_0]$	$e^{-j\omega n_0}$
$u[n]$	$\frac{1}{1-e^{-j\omega}}+\sum_{k=-\infty}^{+\infty}\pi\delta(\omega-2\pi k)$	$a^nu[n]$ $\lvert a\rvert<1$	$\frac{1}{1-ae^{-j\omega}}$
$(n+1)a^nu[n]$ $\lvert a\rvert<1$	$\frac{1}{(1-ae^{-j\omega})^2}$	$C_n^{n+r-1}+a^nu[n]$	$\frac{1}{(1-ae^{-j\omega})^r}$

In the upper right grids in blue, $s=0,1,2\ldots$

Some LT pairs (Ch9)

Signal	LT	ROC
$\delta(t)$	1	All
$u(t)$	$\frac{1}{s}$	$\Re\{s\} > 0$
$-u(-t)$		$\Re\{s\} < 0$
$\frac{t^{n-1}}{(n-1)!}u(t)$	$\frac{1}{s^n}$	$\Re\{s\} > 0$
$-\frac{t^{n-1}}{(n-1)!}u(-t)$		$\Re\{s\} < 0$
$e^{-at}u(t)$	$\frac{1}{s+a}$	$\Re\{s\} > -a$
$-e^{-at}u(-t)$		$\Re\{s\} < -a$
$\frac{t^{n-1}}{(n-1)!}e^{-at}u(t)$	$\frac{1}{(s+a)^n}$	$\Re\{s\} > -a$
$-\frac{t^{n-1}}{(n-1)!}e^{-at}u(-t)$		$\Re\{s\} < -a$
$\delta(t-T)$	e^{-sT}	All
$[\cos\omega_0 t]u(t)$	$\frac{s}{s^2+\omega_0^2}$	$\Re\{s\} > 0$
$[\sin\omega_0 t]u(t)$	$\frac{\omega_0}{s^2+\omega_0^2}$	$\Re\{s\} > 0$
$[e^{-at}\cos\omega_0 t]u(t)$	$\frac{s+a}{(s+a)^2+\omega_0^2}$	$\Re\{s\} > -a$
$[e^{-at}\sin\omega_0 t]u(t)$	$\frac{\omega_0}{(s+a)^2+\omega_0^2}$	$\Re\{s\} > -a$
$u_n(t) = \frac{d^n\delta(t)}{dt^n}$	s^n	All
$u_{-n}(t) = u(t) * \cdots * u(t)$	$\frac{1}{s^n}$	$\Re\{s\} > 0$

Laplace transform (Ch9)

The LT for $x(t)$ is FT of $x(t)e^{-\sigma t}$

$$\mathcal{L}\{x(t)\} = X(s) = X(\sigma + j\omega) = \int_{-\infty}^{\infty} x(t)e^{-(\sigma+j\omega)t}dt$$

LT ROC properties (Ch9)

ROC	Signal
The entire s-plane	Finite length
A left-half plane	Left-sided
A right-half plane	Right-sided
A single strip	Two-sided

If LT is rational, then its ROC is bounded by poles or extends to infinity. No poles are contained in it.

Property	ROC
Causality	The right of the rightmost pole (right-sided) (rational and right-sided→causal)
Anti-causality	The left of the leftmost pole (left-sided) (rational and left-sided→anti-causal)
Stability	Include the entire $j\omega$-axis

Inverse Laplace transform (Ch9)

Direct evaluation:

$X(s) = \int_{-\infty}^{\infty} x(t)e^{-(\sigma+j\omega)t}dt$ for $s = \sigma + j\omega$ in the ROC

$$x(t)e^{-\sigma t} = \mathcal{F}^{-1}\{X(\sigma+j\omega)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\sigma+j\omega)e^{j\omega t}d\omega$$

From $s = \sigma + j\omega$, we have $ds = jd\omega$. Hence

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\sigma+j\omega)e^{j(\sigma+j\omega)t}d\omega = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} X(s)e^{st}ds$$

Partial fraction expansion:

$$X(s) = \frac{A_1}{s+a_1} + \frac{A_2}{s+a_2} + \cdots + \frac{A_m}{s+a_m}$$

$$x(t) = A_1e^{-a_1t}u(t) - A_2e^{-a_2t}u(-t) + \cdots + x_m(t)$$

if right-sided   if left-sided

Butterworth filter (Ch9)

$$|B(j\omega)| = B(j\omega)B^*(j\omega) = \frac{1}{1+(j\omega/j\omega_c)^{2N}}$$

$$B(s)B(-s) = \frac{1}{1+(s/j\omega_c)^{2N}} \Rightarrow s_p = (-1)^{\frac{1}{2N}}(j\omega_c)$$

$$\Rightarrow |s_p| = \omega_c,\ \measuredangle s_p = \frac{\pi(2k+1)}{2N} + \frac{\pi}{2},\ k \in \mathbb{Z}$$

$$\Rightarrow s_p = \omega_c e^{j\left[\frac{\pi(2k+1)}{2N}+\frac{\pi}{2}\right]}$$

Construct $B(s)$:

Poles of $B(s)B(-s)$ appear in pairs

$\Rightarrow$ choose one pole from each pair to construct $B(s)$

For the system to be stable and causal

$\Rightarrow$ all poles of $B(s)$ should be in the left half plane

Apply $B^2(s)|_{s=0} = 1$ to fix the scale factor

Unilateral Laplace transform (Ch9)

(Bilateral) LT	ULT
$X(s) \triangleq \int_{-\infty}^{\infty} x(t)e^{-st}dt$	$\mathcal{X}(s) \triangleq \int_{0^-}^{\infty} x(t)e^{-st}dt$

ULT is restricted to **causal** time functions and takes **initial conditions** into account.

ULT can be thought of as LT of $x(t)u(t)$.

Properties of ULT are similar to LT, except for

$\Rightarrow$ Diff in time: $\frac{d}{dt}x(t) \xleftrightarrow{\mathcal{UL}} s\mathcal{X}(s) - x(0^-)$

z-Transform (Ch10)

The z-T for $x[n]$ is FT of $x[n]r^{-n}$

$$\mathcal{Z}\{x[n]\} = X(z) = X(re^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n](re^{j\omega})^{-n}$$

z-T ROC properties (Ch10)

Signal	ROC
Finite duration	The entire z-plane (except possibly $z = 0$ and/or ∞)
Right-sided	$\lvert z\rvert = r_0 \in$ ROC $\Rightarrow \lvert z\rvert > r_0 \in$ ROC for all finite values of z
Left-sided	$\lvert z\rvert = r_0 \in$ ROC $\Rightarrow 0 < \lvert z\rvert < r_0 \in$ ROC for all finite values of z
Two-sided	A ring in the z-plane
$X(z)$ rational	Bounded by poles or extends to ∞

Property	ROC
Causality	The exterior of a circle (including ∞) If $H(z)$ rational $\Rightarrow$ order of numerator $\leq$ order of denumerator
Anti-causality	The interior of a circle (including 0)
Stability	Include the unit circle ($\lvert z\rvert = 1$)

Inverse z-transform (Ch10)

Direct evaluation: Using IFT to derive IZT

$X(re^{j\omega}) = \mathcal{F}\{x[n]r^{-n}\}$ for $z = re^{j\omega}$ in the ROC

$x[n]r^{-n} = \mathcal{F}^{-1}\{X(re^{j\omega})\}$

$$x[n] = r^n\mathcal{F}^{-1}\{X(re^{j\omega})\} = r^n\frac{1}{2\pi}\int_{2\pi} X(re^{j\omega})e^{j\omega n}d\omega$$

$$= \frac{1}{2\pi}\int_{2\pi} X(re^{j\omega})(re^{j\omega})^n d\omega = \frac{1}{2\pi j}\oint_{2\pi} X(z)z^{n-1}dz$$

Because $z = re^{j\omega} \Rightarrow dz = jre^{j\omega}d\omega = jzd\omega$

Partial fraction expansion:

$$X(z) = \frac{A_1}{1-a_1z^{-1}} + \frac{A_2}{1-a_2z^{-1}} + \cdots + \frac{A_m}{1-a_mz^{-1}}$$

$$x[n] = A_1a_1{}^nu[n] - A_2a_2{}^nu[-n-1] + \cdots + x_m[n]$$

if ROC outside $z = a_1$ if ROC inside $z = a_2$

Some z-T pairs (Ch10)

Signal	z-T	ROC
$\delta[n]$	1	All
$u[n]$	$\frac{1}{1-z^{-1}}$	$\lvert z\rvert > 1$
$-u[-n-1]$		$\lvert z\rvert < 1$
$\delta[n-m]$	z^{-m}	All z, except 0 if m > 0, ∞ if m < 0
$a^nu[n]$	$\frac{1}{1-az^{-1}}$	$\lvert z\rvert > \lvert a\rvert$
$-a^nu[-n-1]$		$\lvert z\rvert < \lvert a\rvert$
$na^nu[n]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert > \lvert a\rvert$
$-na^nu[-n-1]$		$\lvert z\rvert < \lvert a\rvert$
$[\cos\omega_0 n]u[n]$	$\frac{1-[\cos\omega_0]z^{-1}}{1-[2\cos\omega_0]z^{-1}+z^{-2}}$	$\lvert z\rvert > 1$
$[\sin\omega_0 n]u[n]$	$\frac{[\sin\omega_0]z^{-1}}{1-[2\cos\omega_0]z^{-1}+z^{-2}}$	$\lvert z\rvert > 1$
$[r^n\cos\omega_0 n]u[n]$	$\frac{1-[r\cos\omega_0]z^{-1}}{1-[2r\cos\omega_0]z^{-1}+r^2z^{-2}}$	$\lvert z\rvert > r$
$[r^n\sin\omega_0 n]u[n]$	$\frac{[r\sin\omega_0]z^{-1}}{1-[2r\cos\omega_0]z^{-1}+r^2z^{-2}}$	$\lvert z\rvert > r$

Unilateral z-transform (Ch10)

(Bilateral) z-T	UZT
$X(z) \triangleq \sum_{n=-\infty}^{\infty} x[n]z^{-n}$	$\mathcal{X}(z) \triangleq \sum_{n=0}^{\infty} x[n]z^{-n}$

UZT can be thought of as z-T of $x[n]u[n]$.

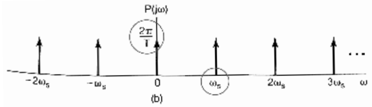
Property	Signal	UZT
Time delay	$x[n-1]$	$z^{-1}\mathcal{X}(z) + x[-1]$
Time advance	$x[n+1]$	$z\mathcal{X}(z) - zx[0]$
First difference	$x[n] - x[n-1]$	$(1-z^{-1})\mathcal{X}(z) - x[-1]$

Impulse modulation (Ch7)

$$p(t) = \sum_{n=-\infty}^{\infty}\delta(t-nT) \xleftrightarrow{\mathcal{F}} P(j\omega) = \frac{2\pi}{T}\sum_{k=-\infty}^{\infty}\delta(\omega - k\omega_s),\ \omega_s = \frac{2\pi}{T}$$

$$x_p(t) = x(t)p(t) = \sum_{n=-\infty}^{\infty} x(nT)\delta(t-nT)$$

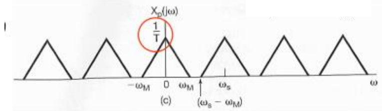
$$X_p(j\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\theta)P(j(\omega-\theta))d\theta = \frac{1}{T}\sum_{k=-\infty}^{\infty} X(j(\omega - k\omega_s))$$



The sampling theorem:

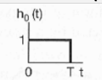
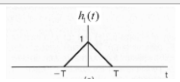
$x(t)$: band-limited with $X(j\omega) = 0$ for $|\omega| > \omega_M$, ω_M: nyquist freq.

$x(t)$ is uniquely determined by $x(nT)$ if $\omega_s > 2\omega_M$



Recovery by an ideal lowpass filter: cutoff freq ω_c (usually $\omega_c = \frac{1}{2}\omega_s$)

Reconstruction using interpolation:

Zero-order hand (ZOH)	First-order hand
	
$H_0(j\omega) = e^{-j\omega T/2}\left[\frac{\sin(\omega T/2)}{\omega/2}\right]$	$H_1(j\omega) = X^2(j\omega)$

C/D conversion (Ch7)

$$X_p(j\omega) = \sum_{n=-\infty}^{\infty} x_c(nT)e^{-j\omega nT} = X_d(e^{j\Omega}),\ \Omega = \omega T$$

$$\Rightarrow X_d(e^{j\Omega}) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c\left(\frac{j(\Omega - 2\pi k)}{T}\right)$$

DT Sampling (Ch7)

Up-sampling	Down-sampling
$x[n]_{\uparrow k} = x[n/k]$	$x[n]_{\downarrow m} = x[mn]$
Time expansion	Time contraction
$X(e^{jk\omega})$	$X_p(e^{j\omega/m})$

$$X_p(e^{j\omega}) = \frac{1}{N}\sum_{k=0}^{N-1} X(e^{j(\omega - k\omega_s)})$$

$(x[n]_{\uparrow N})_{\downarrow M} = x\left[\frac{Mn}{N}\right]$ if $n \bmod N = 0$

$(x[n]_{\downarrow M})_{\uparrow N} =$ if $Mn \bmod N = 0$

Commutative if $(M, N) = 1$

Amplitude modulation (Ch8)

Carrier	c(t) / demodulation	Y(jω)
Complex exponential	$e^{j(\omega_ct+\theta_c)}/e^{-j(\omega_ct+\theta_c)}$	$X(j(\omega-\omega_c))$ if $\theta_c = 0$
Sinusoidal	$\cos(\omega_ct+\theta_c)$ $\cos(\omega_ct+\theta_c)$&lowpass	$\frac{X(j(\omega-\omega_c)) + X(j(\omega+\omega_c))}{2}$
Pulse-train	懶得打	懶得打

Asynchronous demodulation: envelope $\approx x(t)$

if $\omega_c \gg \omega_M$ and $x(t) > 0, \forall t \Rightarrow x(t) + A \to x(t)$ make $x(t)$ positive

where $A \geq K$, K is the maximum amplitude

Modulation index: $m = K/A$

Single-sideband modulation: (SSB)

Using highpass filter or phase shifting to obtain the upper sidebands

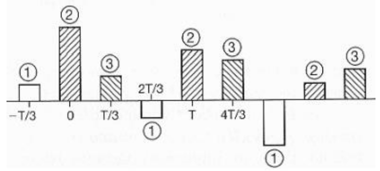
Frequency-division multiplying (FDM) (Ch8)

Different signal multiplying $\cos(\omega_it)$ with different ω_i

$$\mathrm{w}(t) \xrightarrow{\text{bandpass (demultiplex)}} y_i(t) \xrightarrow{\text{lowpass (demodulate)}} x_i(t)$$

Time-division multiplying (TDM) (Ch8)

Different signal transmitting at different time steps



←TDM-PAM Pulse-Amplitude Modulated signals

Frequency modulation (Ch8)

Sinusoidal FM: $c(t) = A\cos(\omega_ct + \theta_c) = A\cos(\theta(t))$

Phase modulation: use the modulating signal to vary the phase

$$\theta_c(t) = \theta_0 + k_px(t) \Rightarrow \frac{d\theta(t)}{dt} = \omega_c + k_p\frac{dx(t)}{dt}$$

Frequency modulation: use the modulating signal to vary $\theta'(t)$

$$\frac{d\theta(t)}{dt} = \omega_c + k_fx(t)$$

Narrowband FM: FM with $x(t) = A\cos(\omega_mt)$

$\Rightarrow \omega_i(t) = \theta'(t) = \omega_c + k_fA\cos(\omega_mt) = \omega_c + \boldsymbol{\Delta\omega}\cos(\omega_mt)$

$$\Rightarrow y(t) = \cos(\omega_ct) + k_f\int x(t)dt = \cos\left(\omega_ct + \frac{\Delta\omega}{\omega_m}\sin\omega_mt\right)$$

Modulation index: $m = \Delta\omega/\omega_m$

When $m \ll \pi/2 \Rightarrow$ narrowband FM

$y(t) \approx \cos(\omega_ct) - m\sin(\omega_mt)\sin(\omega_ct)$

Wideband FM:

$y(t) = \cos(\omega_ct)\cos(m\sin\omega_mt) - \sin(\omega_ct)\sin(m\sin\omega_mt)$