Signals & Systems

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Ch. 10 The Z Transform

- Section 10.1 The z-Transform
- Section 10.2 The Region of Convergence for z-Transforms
- Section 10.3 The Inverse z-Transform
- Section 10.4 Geometric Evaluation of the Fourier Transform from Pole-Zero Plot
- Section 10.5 Properties of the z-Transform
- Section 10.6 Some z-Transform Pairs
- Section 10.7 Analysis & Characterization of LTI Systems Using z-Transforms
- Section 10.8 DT All-Pass, Minimum Phase System, and Spectral Factorization
- Section 10.9 System Function Algebra and Block Diagram Representations
- Section 10.10 The Unilateral z-Transform
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10.1 The z-Transform

- DTFT vs. z-Transform

DT Fourier Transform	z-Transform
$z = e^{j\omega}$	$z = re^{j\omega}$
$X(e^{j\omega}) \triangleq \sum_{n=-\infty}^{+\infty} x[n]e^{-j\omega n}$	$X(z) \triangleq \sum_{n=-\infty}^{+\infty} x[n]z^{-n}$
$x[n] \overset{\mathcal{F}}{\longleftrightarrow} X(e^{j\omega})$	$x[n] \overset{\mathcal{Z}}{\longleftrightarrow} X(z)$
$X(e^{j\omega}) = \mathcal{F}\{x[n]\}$	$X(z) = \mathcal{Z}\{x[n]\}$
$x[n] = \mathcal{F}^{-1}\{X(e^{j\omega})\}$	$x[n] = \mathcal{Z}^{-1}\{X(z)\}$

Evaluating the z-transform on the unit circle $z = e^{j\omega}$ yields the Fourier transform:

$$X(z)\Big|_{z=e^{j\omega}} = \mathcal{Z}\{x[n]\}\Big|_{z=e^{j\omega}} = \mathcal{F}\{x[n]\} = X(e^{j\omega})$$

10.7 Analysis & Characterization of LTI Systems Using z-Transforms

- Many properties of a system are tied to characteristics of the poles, zeroes, and ROC of the system.

$$H(z) = \mathcal{Z}\{h[n]\}$$

$$X(z) \longrightarrow \boxed{\text{LTI System}} \longrightarrow Y(z) = X(z)H(z)$$

$H(z)$: system function

or transfer function

10.7 Analysis & Characterization of LTI Systems Using z-Transforms

- Causality

For a causal LTI system, h[n] = 0 for n < 0. Thus h[n] is right-sided.

Since $H(z) = \sum_{n=0}^{\infty} h[n]z^{-n}$ does not include any positive power of z, the ROC of the system must include infinity.

- A DT LTI system is causal if and only if the ROC of the system function $H(z)$ is the exterior of a circle in the z-plane, including infinity
- A DT LTI system with a rational $H(z)$ is causal if and only if
  (a) ROC is exterior of a circle outside the outermost pole;
  and infinity must be in the ROC
  (b) Order of numerator ≤ order of denominator

$$H(z) = \frac{\sum a_m z^m}{\sum b_k z^k}$$

$$m \le k$$

Because H(z) must be finite as z approaches infinity.

10.7 Analysis & Characterization of LTI Systems Using z-Transforms

- Example 10.21 Causality Analysis

$$H(z) = \frac{1}{1-\frac{1}{2}z^{-1}} + \frac{1}{1-2z^{-1}}, \qquad |z| > 2$$

Since the ROC is the exterior of a circle outside the outermost pole, the impulse response is right-sided.

$$H(z) = \frac{2-\frac{5}{2}z^{-1}}{\left(1-\frac{1}{2}z^{-1}\right)\left(1-2z^{-1}\right)} = \frac{2z^2-\frac{5}{2}z}{z^2-\frac{5}{2}z+1}$$

Numerator degree = Denominator degree

$\Rightarrow$ The system is causal, but not stable

Check: $h[n] = \left[\left(\frac{1}{2}\right)^n + 2^n\right]u[n] \Rightarrow h[n] = 0$ for $n < 0$

10.7 Analysis & Characterization of LTI Systems Using z-Transforms

- Stability
  - A DT LTI system is stable if and only if the ROC of H(z) includes the unit circle of $|z| = 1$.
  - A causal LTI system with rational H(z) is stable if and only if all poles of H(z) lie inside the unit circle, i.e., all of the poles have magnitudes < 1.

10.7 Analysis & Characterization of LTI Systems Using z-Transforms

- Example 10.22

$$H(z) = \frac{1}{1-\frac{1}{2}z^{-1}} + \frac{1}{1-2z^{-1}}, \quad |z| > 2$$

Since the ROC does not include the unit circle $\Rightarrow$ unstable

We can check this result by noting that

$$h[n] = \left[\left(\frac{1}{2}\right)^n + 2^n\right]u[n] \to \infty, \quad \text{as } n \to \infty$$

If ROC is the region $1/2 < |z| < 2 \;\Rightarrow\; h[n] = \left(\frac{1}{2}\right)^n u[n] - 2^n u[-n-1]$

$\Rightarrow$ The system is NOT causal, but stable

If $ROC = |z| < \frac{1}{2} \;\Rightarrow\; h[n] = -\left[\left(\frac{1}{2}\right)^n + 2^n\right]u[-n-1]$

$\Rightarrow$ The system is neither causal nor stable

10.7 Analysis & Characterization of LTI Systems Using z-Transforms

- Example 10.24

$$H(z) = \frac{1}{1-(2r\cos\theta)z^{-1}+r^2z^{-2}}$$

$$\Rightarrow z_1 = re^{j\theta}, \quad z_2 = re^{-j\theta}$$

To be causal $\Rightarrow$ $|z|>|r|$.

To be stable $\Rightarrow$ $r<1$.

If $r>1$, the poles are outside the unit circle. In this case, since the ROC does not include the unit circle, the system is unstable.

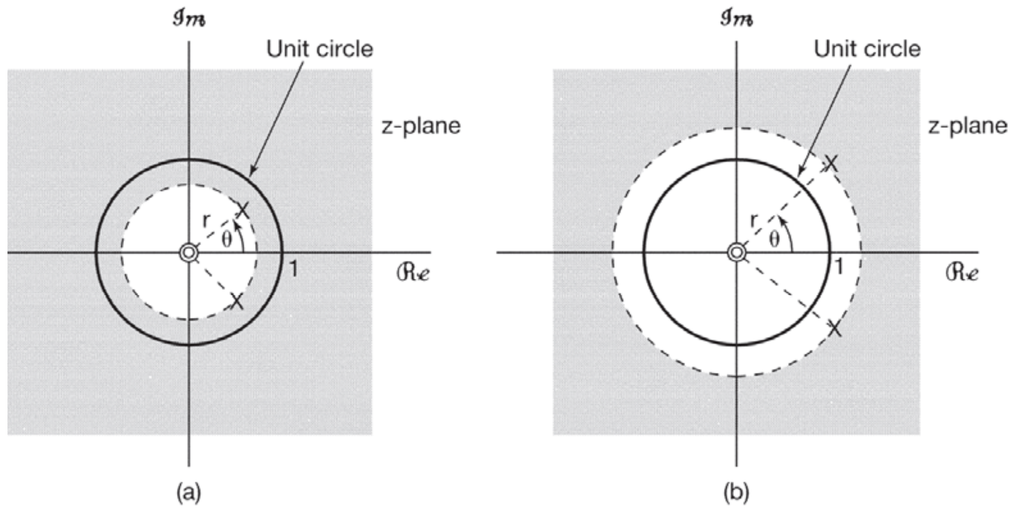


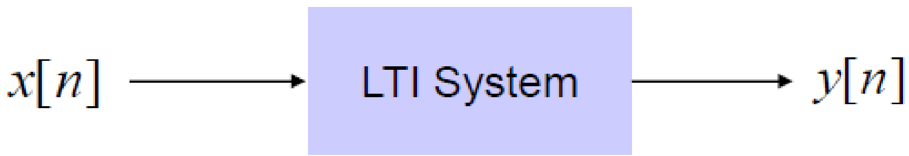
Figure 10.16 Pole-zero plot for a second-order system with complex poles: (a) $r<1$; (b) $r>1$.

10.7 Analysis & Characterization of LTI Systems Using z-Transforms

- LTI Systems by Linear Constant-Coeff. Difference Equations

$$a_0 y[n] + a_1 y[n-1] + \cdots + a_{N-1} y[n-N+1] + a_N y[n-N]$$
$$= b_0 x[n] + b_1 x[n-1] + \cdots + b_{M-1} x[n-M+1] + b_M x[n-M]$$

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$



$$Y(z) = X(z)H(z) \qquad\qquad H(z) = \frac{Y(z)}{X(z)}$$

10.7 Analysis & Characterization of LTI Systems Using z-Transforms

- LTI Systems by Linear Constant-Coeff. Difference Equations (cont'd)

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k] \qquad x[n-n_0] \xleftrightarrow{\ z\ } z^{-n_0}\boxed{X(z)}$$

$$z^{-n_0}\left(\sum_{n=-\infty}^{\infty} x[n-n_0]\, z^{-(n-n_0)}\right)$$

$$\mathcal{Z}\left\{\sum_{k=0}^{N} a_k y[n-k]\right\} = \mathcal{Z}\left\{\sum_{k=0}^{M} b_k x[n-k]\right\}$$

$$\sum_{k=0}^{N} a_k \mathcal{Z}\{y[n-k]\} = \sum_{k=0}^{M} b_k \mathcal{Z}\{x[n-k]\}$$

$$\sum_{k=0}^{N} a_k z^{-k} Y(z) = \sum_{k=0}^{M} b_k z^{-k} X(z)$$

$$\Rightarrow H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}} \quad \begin{matrix}\text{zeros} \\ \text{poles}\end{matrix}$$

10.7 Analysis & Characterization of LTI Systems Using z-Transforms

- Example 10.25

$$y[n]-\frac{1}{2}y[n-1]=x[n]+\frac{1}{3}x[n-1]$$

$$Y(z)-\frac{1}{2}z^{-1}Y(z)=X(z)+\frac{1}{3}z^{-1}X(z)\Rightarrow H(z)=(1+\tfrac{1}{3}z^{-1})\frac{1}{1-\frac{1}{2}z^{-1}} = \frac{1}{1-\frac{1}{2}z^{-1}}+\frac{1}{3}\frac{z^{-1}}{1-\frac{1}{2}z^{-1}}$$

Two choices of ROC:

- If $|z|>1/2 \;\Rightarrow\; h[n]$ is right-sided
- If $|z|<1/2 \;\Rightarrow\; h[n]$ is left-sided

If ROC is the region $|z|>1/2$, $\quad h[n]=\left(\frac{1}{2}\right)^{n}u[n]+\frac{1}{3}\left(\frac{1}{2}\right)^{n-1}u[n-1]$

If ROC is the region $|z|<1/2$, $\quad h[n]=-\left(\frac{1}{2}\right)^{n}u[-n-1]-\frac{1}{3}\left(\frac{1}{2}\right)^{n-1}u[-n]$,

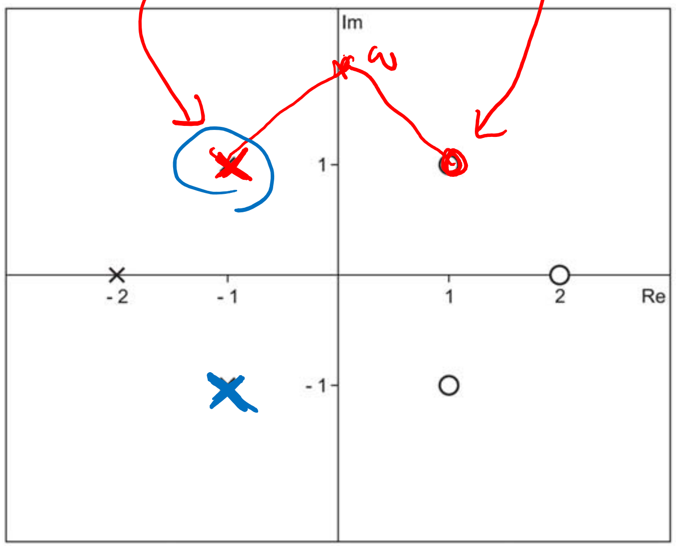
which is anticausal and unstable.

Revisit of Sect. 9.8 All-Pass System, Min. Phase System, and Spectral Factorization*

- Recall that for a CT All-Pass System
  - Assuming that the transfer function H(s) is rational in s, it will have the following form:

$$H_{ap}(s) = A \prod_{k=1}^{M} \frac{s + a_k^*}{s - a_k}. \tag{9.155}$$

  - For an all-pass system, it there is a pole at $s = \sigma_k + j\omega_k$, then there should be a zero at $s = -\sigma_k + j\omega_k$.
  - For a real all-pass system, if there is a pole at $s = \sigma_k + j\omega_k$, then there should also be a pole at $s = \sigma_k - j\omega_k$ and a zero at $s = -\sigma_k + j\omega_k$ and $s = -\sigma_k - j\omega_k$.



10.8 DT All-Pass, Min.-Phase Systems, & Spectral Factorization

- All-Pass System
  - For a DT all-pass system, if its associated transfer function $H(z)$ is rational in z, it will have the form:

$$H_{ap}(z) = A \prod_{k=1}^{M} \frac{z^{-1} - b_k^*}{1 - b_k z^{-1}} = A \prod_{k=1}^{M} \boxed{(-b_k^*)} \frac{z - 1/b_k^*}{z - b_k}.$$

  - The poles and zeros in this case occur at conjugate reciprocal locations: for each pole at $z = b_k$, there is a zero at $z = \boxed{1/b^*_k}$.

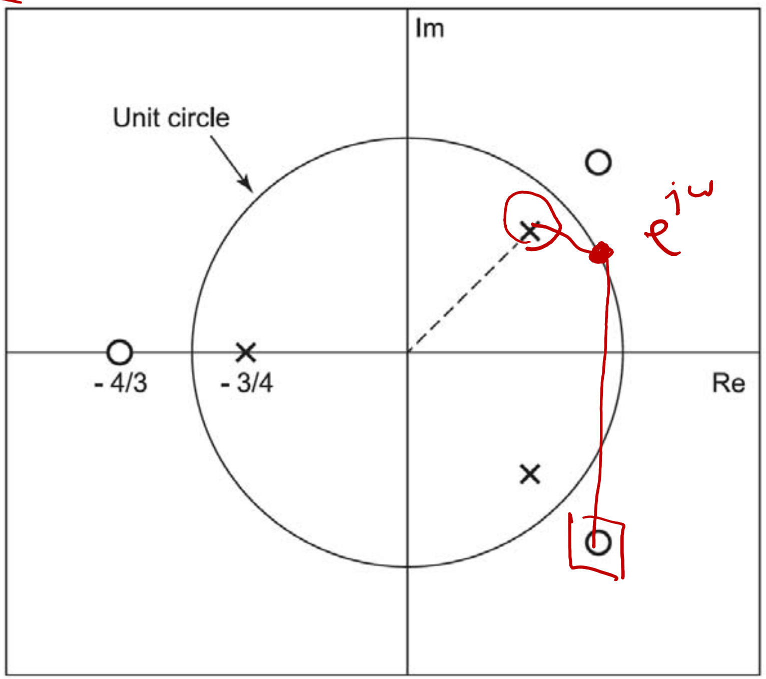


Figure 10.17 Typical pole-zero plot for a DT all-pass system.

10.8 DT All-Pass, Min.-Phase Systems, & Spectral Factorization

- Min.-Phase System
  - For a DT LTI system with a rational transfer function, we call the system a *minimum phase system* if it is stable, causal (i.e., if p_k is a pole, then $| p_k | < 1$), and all these zeros are strictly inside the unit circle (i.e., if z_k is a zero, then $| z_k | < 1$).

- Spectral Factorization
  - If $|H(e^{j\omega})|^2$ is known but $H(e^{j\omega})$ and h[n] are unknown, we can use the following way to retrieve $H(e^{j\omega})$ and h[n] with the assumption that h[n] is (i) causal, (ii) stable, (iii) minimum phase, and (iv) real.

    Since

$$|H(e^{j\omega})|^2 = H(e^{j\omega})H^*(e^{j\omega}),$$

    if h[n] is real (i.e., $H^*(e^{j\omega}) = H(e^{-j\omega})$)

$$|H(e^{j\omega})|^2 = H(e^{j\omega})H(e^{-j\omega}),$$

$$|H(z)|^2 = H(z)\boxed{H(1/z)}\big|_{z=e^{j\omega}}.$$

10.8 DT All-Pass, Min.-Phase Systems, & Spectral Factorization

For example, if we have known that

$$|H(e^{j\omega})|^2 = \frac{1}{2\cos(2\omega) + 17/4},$$

$$|H(e^{j\omega})|^2 = \frac{1}{e^{j2\omega} + 17/4 + e^{-j2\omega}}, \qquad (10.118)$$

from the fact that the discrete-time Fourier transform is equivalent to the *z*-transform where $z = e^{j\omega}$, we have

$$|H(z)|^2 = \frac{1}{z^2 + 17/4 + z^{-2}} = \frac{z^2}{z^4 + 17z^2/4 + 1} \qquad (10.119)$$

$$= \frac{z^2}{(z - j/2)(z + j/2)(z - j2)(z + j2)}.$$

poles: $\pm j/2$, $\pm 2j$

10.8 DT All-Pass, Min.-Phase Systems, & Spectral Factorization

If $h[n]$ is stable and causal, $H(z)$ should has poles at $j/2$ and $-j/2$, i.e.,

$$H(z) = \frac{Cz^k}{(z-j/2)(z+j/2)} \tag{10.120}$$

After some calculation, we find that $C = \pm 1/2$ and k can be any integer. Therefore, we obtain

$$H(z) = \frac{\pm z^k}{2(z-j/2)(z+j/2)} = \frac{\pm z^{k-2}}{2(1-(j/2)z^{-1})(1+(j/2)z^{-1})} \tag{10.121}$$

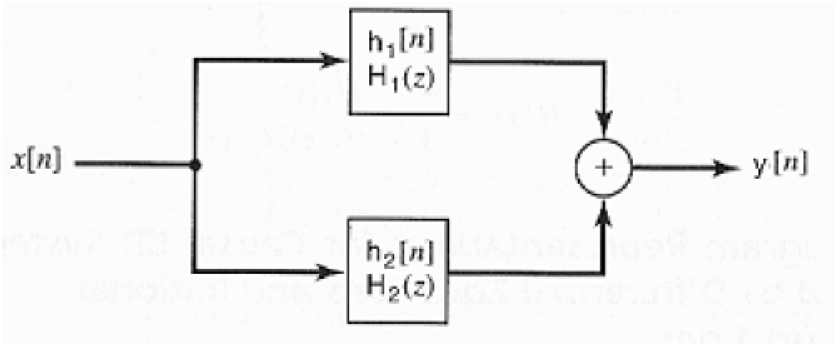
$$= \pm\frac{1}{4}\left[\frac{z^{k-2}}{1-(j/2)z^{-1}} + \frac{z^{k-2}}{1+(j/2)z^{-1}}\right].$$

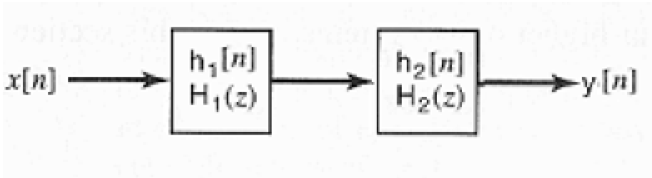
Then, from Table 10.2 together with some calculation, its inverse z transform is

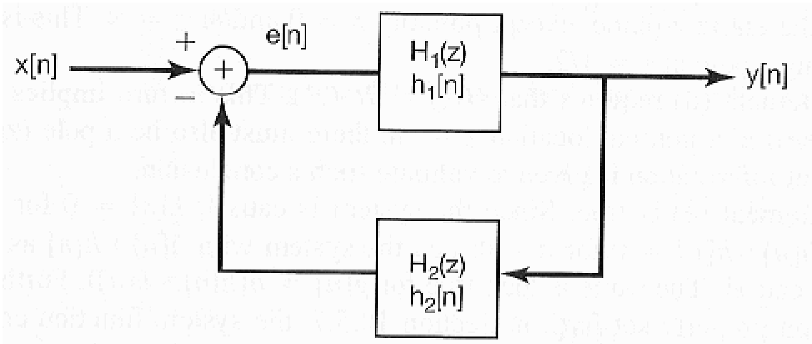
$$h[n] = \pm\left\{\left(\frac{j}{2}\right)^{n+k} u[n+k-2] + \left(-\frac{j}{2}\right)^{n+k} u[n+k-2]\right\}. \tag{10.122}$$

10.9 System Function Algebra and Block Diagram Representations

- Block diagram of interconnected systems







Parallel interconnection

$$h[n] = h_1[n] + h_2[n]$$

$$H(z) = H_1(z) + H_2(z)$$

Series interconnection

$$h[n] = h_1[n] * h_2[n]$$

$$H(z) = H_1(z)H_2(z)$$

Feedback interconnection

$$Y(z) = H_1(z)E(z)$$

$$Z(z) = H_2(z)Y(z)$$

$$E(z) = X(z) - Z(z)$$

$$H(z) = \frac{H_1(z)}{1 + H_1(z)H_2(z)}$$

10.9 System Function Algebra and Block Diagram Representations

- Example 10.28

$$H(z) = \frac{1}{1 - \frac{1}{4}z^{-1}} := \frac{Y(z)}{X(z)}$$

$$y[n] - \frac{1}{4}y[n-1] = x[n],$$

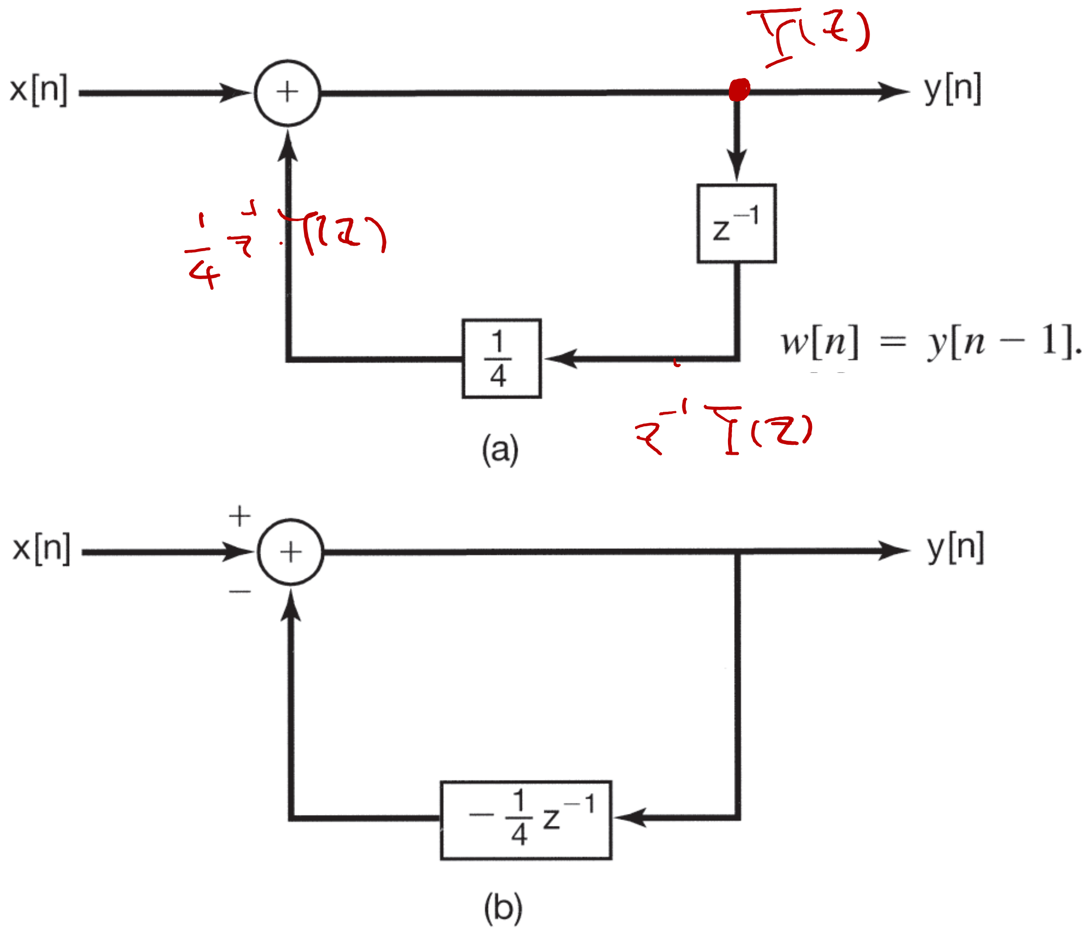


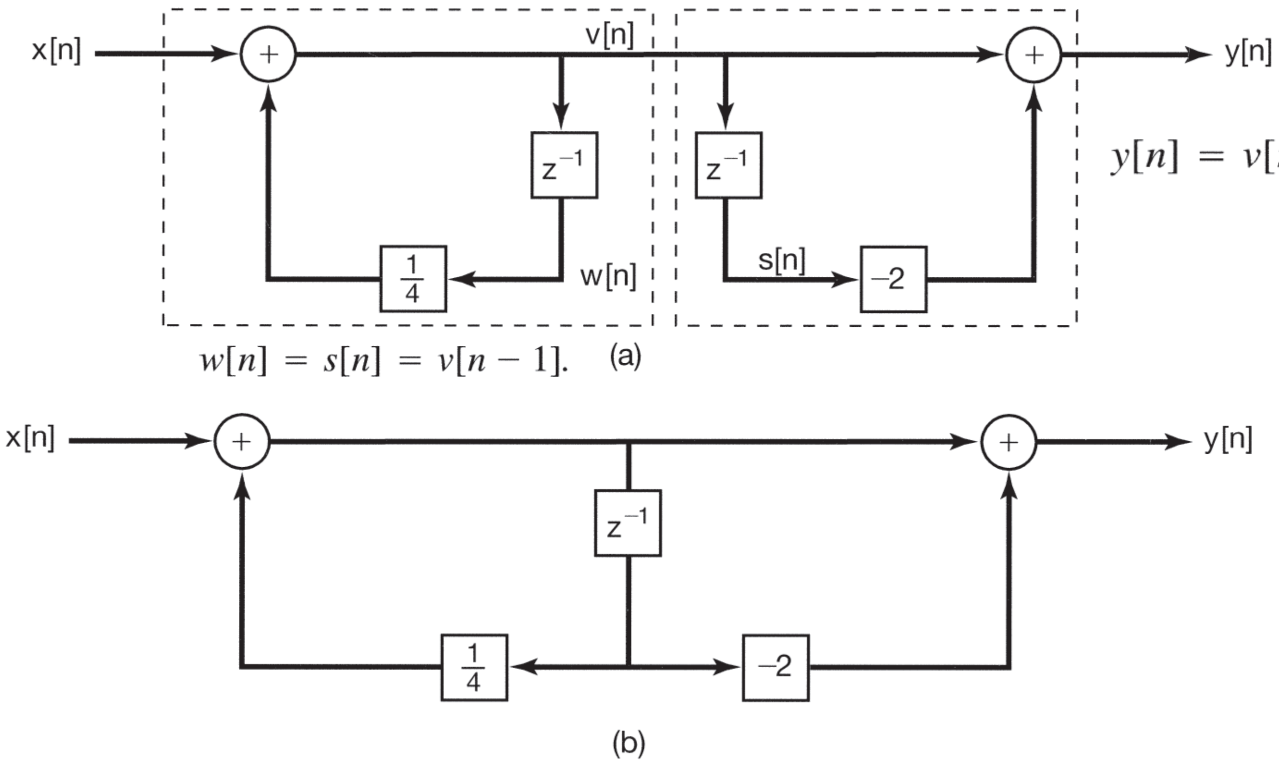
Figure 10.19 (a) Block diagram representations of the causal LTI system in Example 10.28; (b) equivalent block diagram representation.

- Example 10.29

$$\frac{Y(z)}{X(z)} = H(z) = \frac{1-2z^{-1}}{1-\frac{1}{4}z^{-1}} = \left(\frac{1}{1-\frac{1}{4}z^{-1}}\right)\left(1-2z^{-1}\right)$$

$$Y(z) = (1-2z^{-1})V(z)$$

$$V(z) = \left(\frac{1}{1-\frac{1}{4}z^{-1}}\right)X(z)$$



$$y[n] = v[n] - 2v[n-1].$$

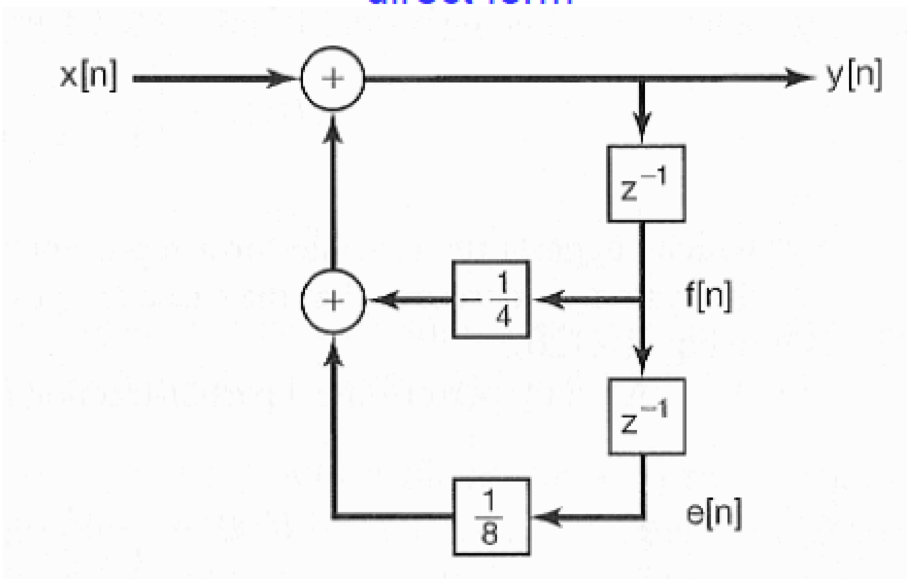
$$w[n] = s[n] = v[n-1].$$

Figure 10.20 (a) Block-diagram representations for the system in Example 10.29; (b) equivalent block-diagram representation using only one unit delay element.

- Example 10.30

$$H(z)=\frac{1}{\left(1+\frac{1}{2}z^{-1}\right)\left(1-\frac{1}{4}z^{-1}\right)}=\frac{1}{1+\frac{1}{4}z^{-1}-\frac{1}{8}z^{-2}} \leftrightarrow \frac{Y(z)}{X(z)}$$

direct form



(a)

$$\Rightarrow y[n]+\frac{1}{4}y[n-1]-\frac{1}{8}y[n-2]=x[n]$$

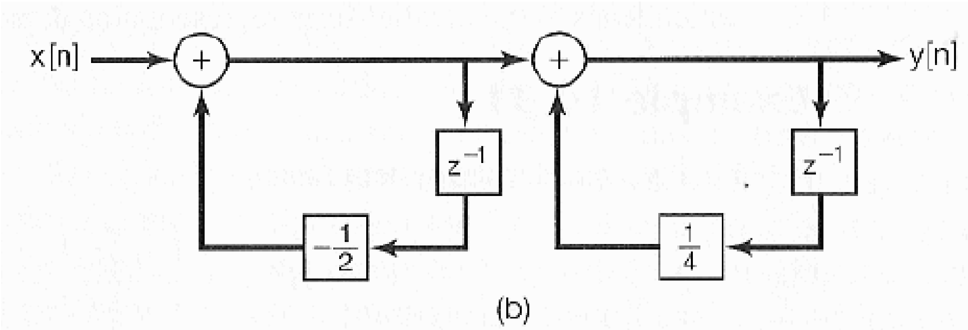
$$\Rightarrow y[n]=-\frac{1}{4}y[n-1]+\frac{1}{8}y[n-2]+x[n]$$

$$\Rightarrow \begin{cases} f[n]=y[n-1] \\ e[n]=f[n-1]=y[n-2] \end{cases}$$

- Example 10.30 (cont'd)

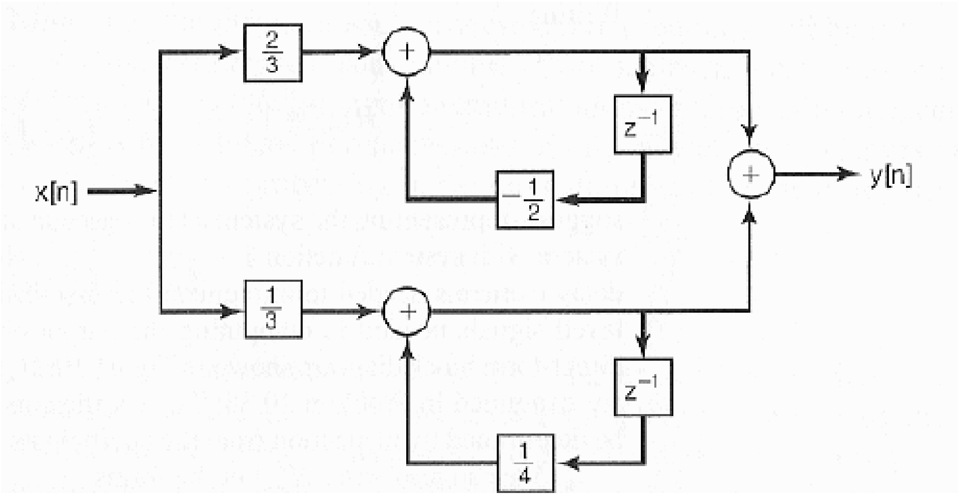
$$H(z)=\left(\frac{1}{1+\frac{1}{2}z^{-1}}\right)\left(\frac{1}{1-\frac{1}{4}z^{-1}}\right)$$

cascade form



$$H(z)=\frac{2/3}{\left(1+\frac{1}{2}z^{-1}\right)}+\frac{1/3}{\left(1-\frac{1}{4}z^{-1}\right)}$$

parallel form



10.10 Unilateral z-Transform

Bilateral z-transform	Unilateral z-transform
$X(z) \triangleq \sum_{n=-\infty}^{+\infty} x[n]z^{-n}$	$\mathcal{X}(z) \triangleq \sum_{n=0}^{+\infty} x[n]z^{-n}$
$x[n] \overset{z}{\longleftrightarrow} X(z)$	$x[n] \overset{uz}{\longleftrightarrow} \mathcal{X}(z)$

- The unilateral z-transform differs from the bilateral z-transform in that the summation is carried out over nonnegative values of n.
- The unilateral z-transform is useful in analyzing causal systems specified by linear constant-coefficient difference equations with nonzero initial conditions.
- If x[n]=0 for n<0, X(z)= $\mathcal{X}$(z).
- The unilateral z-transform of x[n] can be thought of as the bilateral z-transform of x[n]u[n], which is a right-sided signal. Therefore, the ROC of $\mathcal{X}$(z) is always the exterior of a circle.

10.10 Unilateral z-Transform

- Example 10.3

$$x[n] = a^{n+1}u[n+1]$$

$$\text{since } x[-1] = 1 \neq 0, \quad X(z) \neq \mathcal{X}(z)$$

Bilateral trasnform:

$$X(z) = \frac{z}{1 - az^{-1}}, \qquad |z| > |a|$$

Unilateral trasnform:

$$\mathcal{X}(z) = \sum_{n=0}^{+\infty} x[n]z^{-n} = \sum_{n=0}^{+\infty} a^{n+1}z^{-n}$$

$$= \frac{a}{1 - az^{-1}}, \qquad |z| > |a|$$

10.10 Unilateral z-Transform

- Time Delay of Unilateral z-Transform

$$y[n] = x[n-1]$$

$$\mathcal{Y}(z) = \sum_{n=0}^{\infty} x[n-1] z^{-n}$$

$$= x[-1] + \sum_{n=1}^{\infty} x[n-1] z^{-n} \qquad n' = n-1$$

$$= x[-1] + \sum_{n'=0}^{\infty} x[n'] z^{-(n'+1)}$$

$$= x[-1] + z^{-1} \sum_{n'=0}^{\infty} x[n'] z^{-n'}$$

$$= x[-1] + z^{-1} \mathcal{X}(z)$$

10.10 Unilateral z-Transform

- Time Advance of Unilateral z-Transform

$$y[n] = x[n+1]$$

$$\begin{aligned}
\mathcal{Y}(z) &= \sum_{n=0}^{\infty} x[n+1] z^{-n} \\
&= \sum_{n=0}^{\infty} x[n] z^{-(n-1)} - x[0]z \\
&= z\sum_{n=0}^{\infty} x[n] z^{-n} - zx[0] \\
&= z\mathcal{X}(z) - zx[0]
\end{aligned}$$

Chapter 8 Communication Systems

Sect. 8.1 Complex Exponential and Sinusoidal Amplitude Modulation
Sect. 8.2 Demodulation for Sinusoidal AM
Sect. 8.3 Frequency Division Multiplexing
Sect. 8.4 Single-Sideband Sinusoidal Amplitude Modulation
Sect. 8.5 Amplitude Modulation with a Pulse-Train Carrier
Sect. 8.6 Pulse Amplitude Modulation
Sect. 8.7 Sinusoidal Frequency Modulation
Sect. 8.8 Discrete-Time Modulation
Sect. 8.9 Frequency-Shift Keying, Phase-Shift Keying,
and Quadrature Amplitude Modulation

Communication:
Transmission / receiving a signal

Modulation:
The general process of embedding an information-bearing signal into a second signal.

Amplitude modulation (AM):
Info of the input signal is embedded in the amplitude of the modulation output

Frequency modulation (FM):
Info of the input signal is embedded in the frequency of the modulation output.

Demodulation:
Extracting the information-bearing signal from the modulation output.

Multiplexing:
Making possible the simultaneous transmission of more than one signal.

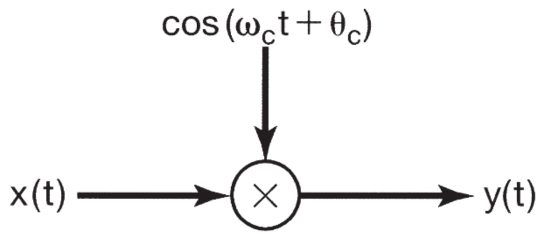
Modulation Methods

- Continuous-Time Modulation
  - Amplitude Modulation
    - with a Complex Exponential Carrier (Sec. 8-1, demodulation: Sec. 8-2)
    - with a Sinusoid Carrier (Sec. 8-1, demodulation: Sec. 8-2)
      - Double Sided Band (Sec. 8-1, Sec. 8-2)
      - Single Sided Band (Sec. 8-4)
    - with a Pulse-Train Carrier (Sec. 8-5)
    - Pulse-Amplitude Modulation (PAM) (Sec. 8-6)
    - Quadrature Amplitude Modulation(QAM) (Sec. 8-9)
  - Angle Modulation
    - Phase Modulation (Sec. 8-7)
      Phase-Shift Keying (PSK) (Sec. 8-9)
    - Frequency Modulation (Sec. 8-7)
      Frequency-Shift Keying (FSK) (Sec. 8-9)
- Discrete-Time Modulation (Sec. 8-8, Sec. 8-11)

Multiplexing Methods

- Frequency-Division Multiplexing
  - Continuous-Time, AM with a Sinusoid Carrier (Sec. 8-3)
  - Discrete-Time (Sec. 8-8)
    - Orthogonal Frequency-Division Multiplexing (OFDM) (Sec. 8-11)
- Time-Division Multiplexing
  - Continuous-Time (Sec. 8-5, Sec. 8.6 (for PAM))
  - Discrete-Time (Sec. 8-8)
- Code Division Multiple Access (CDMA) (Sec. 8.11)

Sect. 8.1 Complex Exponential & Sinusoidal Amplitude Modulation & Demodulation



- Amplitude Modulation

  $y(t) = x(t)c(t)$

  $x(t)$: modulating signal (the input of modulation)

  $c(t)$: carrier signal with carrier frequency ω_c

  $y(t)$: modulated signal (the output of modulation)

  $c(t) = e^{j(\omega_c t + \theta_c)}$ or $c(t) = \cos(\omega_c t + \theta_c).$

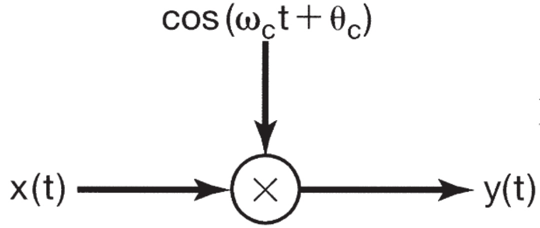
  choose $\theta_c = 0$, so that the modulated signal is $y(t) = x(t)e^{j\omega_c t}.$

  denoting the Fourier transforms of *x*(*t*), *y*(*t*), and *c*(*t*), respectively,

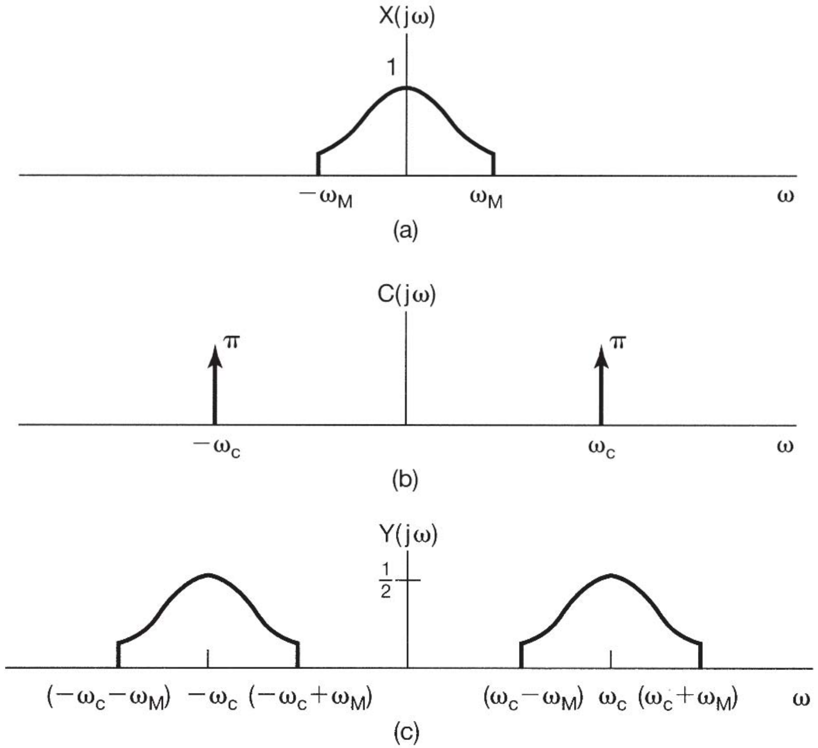
$$Y(j\omega) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\theta)C(j(\omega - \theta))d\theta.$$

- Amplitude Modulation (cont'd)

$$C(j\omega) = \pi[\delta(\omega - \omega_c) + \delta(\omega + \omega_c)], \quad \theta_c = 0$$



$$Y(j\omega) = \frac{1}{2}[X(j\omega - j\omega_c) + X(j\omega + j\omega_c)].$$



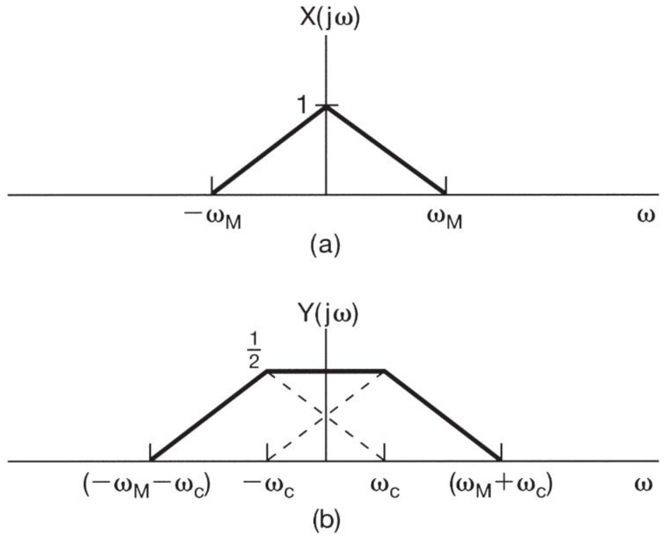


Figure 8.5 Sinusoidal amplitude modulation with carrier cos $\omega_c t$ for which $\omega_c = \omega_M/2$: (a) spectrum of modulating signal; (b) spectrum of modulated signal.

Sect. 8.2 Demodulation for Sinusoidal AM

cos(θ1 ± θ2)

= cos θ1 cos θ2 ∓ sin θ1 sin θ2

- Synchronous Demodulation

Assuming that $\omega_c > \omega_M$, demodulation of a signal that was modulated with a sinusoidal carrier is relatively straightforward, consider the signal

$$y(t) = x(t) \cos \omega_c t.$$

the original signal can be recovered by modulating $y(t)$ with the same sinusoidal carrier and applying a lowpass filter to the result. To see this, consider

$$w(t) = y(t) \cos \omega_c t.$$

$$w(t) = x(t) \cos^2 \omega_c t, \qquad \cos^2 \omega_c t = \frac{1}{2} + \frac{1}{2} \cos 2\omega_c t,$$

cos(wct − wct)

cos(wct + wct)

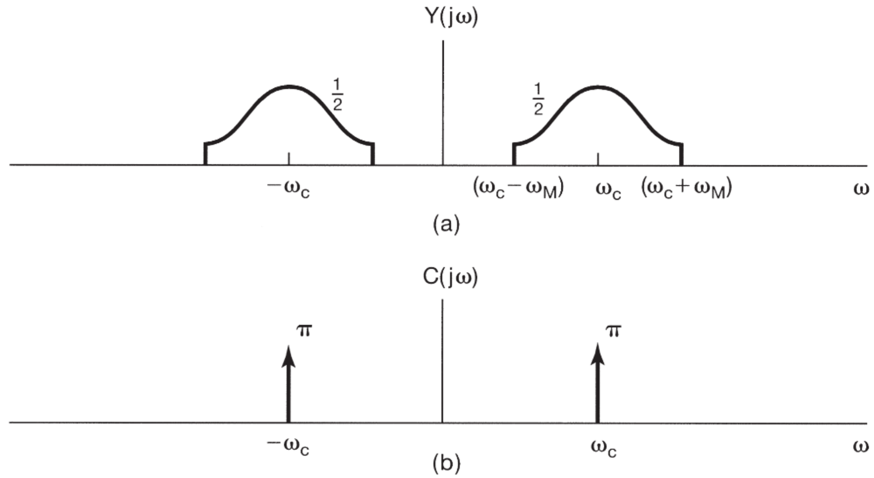
$$w(t) = \frac{1}{2} x(t) + \frac{1}{2} x(t) \cos 2\omega_c t.$$

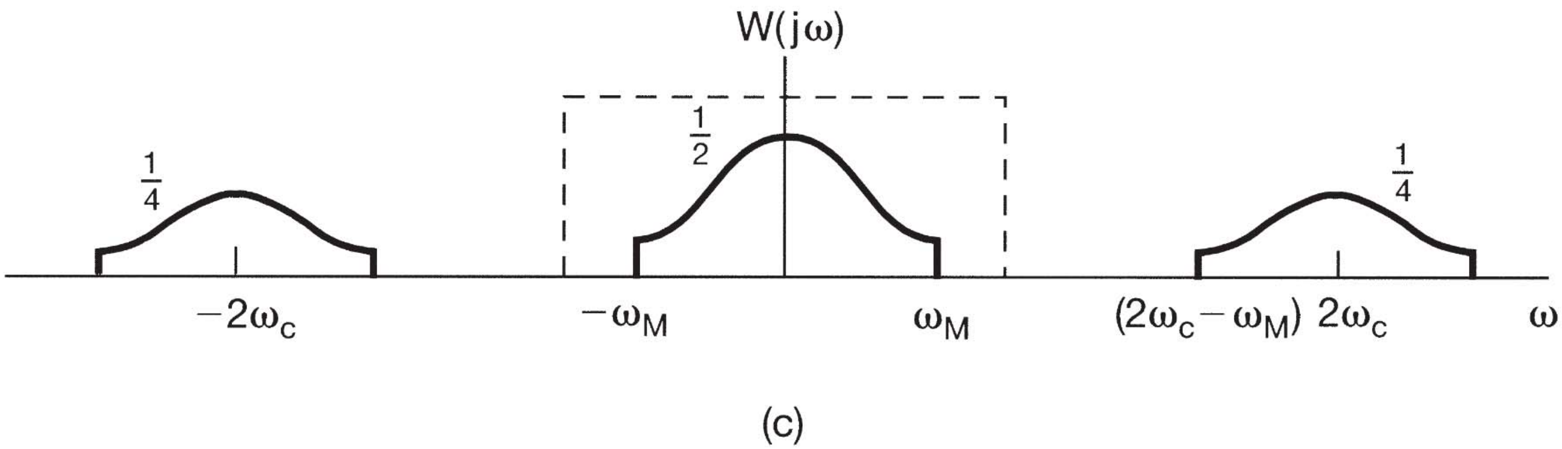
Sect. 8.2 Demodulation for Sinusoidal AM

- Synchronous Demodulation of $y(t) = x(t)\cos\omega_c t.$

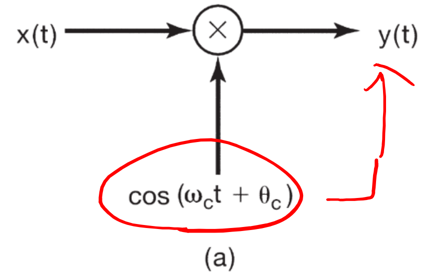
$$w(t) = y(t)\cos\omega_c t.$$

$$w(t) = \frac{1}{2}x(t) + \frac{1}{2}x(t)\cos 2\omega_c t.$$



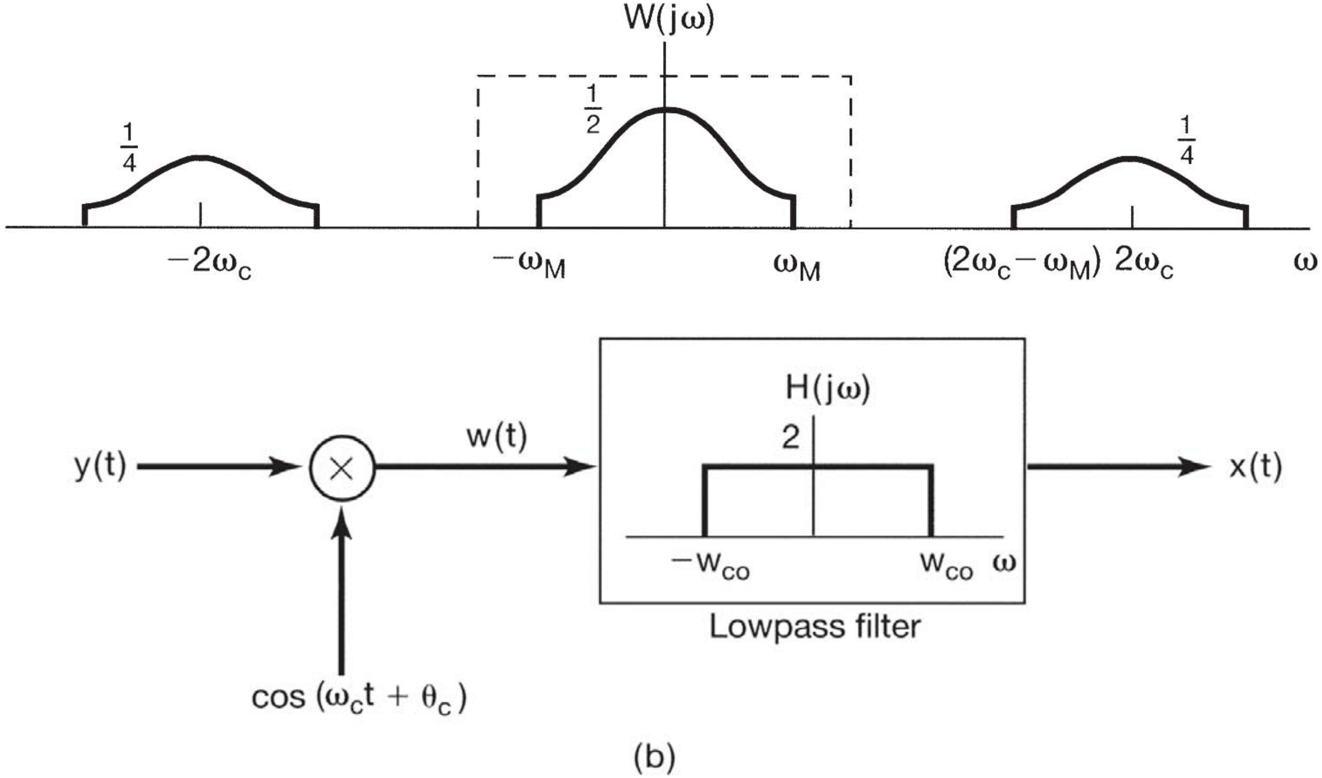


Sect. 8.2 Demodulation for Sinusoidal AM



- Synchronous Demodulation of $y(t) = x(t)\cos\omega_c t.$

$$w(t) = \frac{1}{2}x(t) + \frac{1}{2}x(t)\cos 2\omega_c t.$$

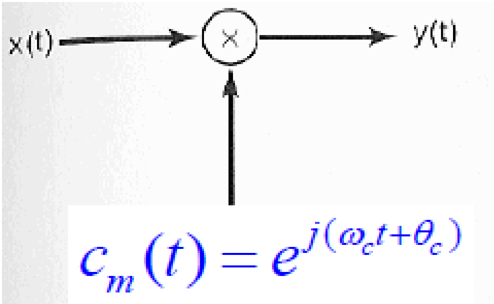


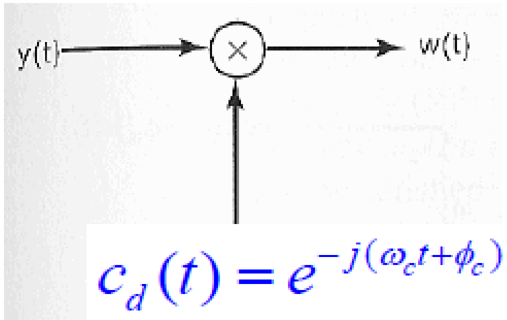
Sect. 8.2 Demodulation for Sinusoidal AM

- Synchronous or Asynchronous Demodulation

$$y(t) = e^{j(\omega_c t+\theta_c)}x(t),$$

$$w(t) = e^{-j(\omega_c t+\phi_c)}y(t), \qquad w(t) = e^{j(\theta_c-\phi_c)}x(t).$$

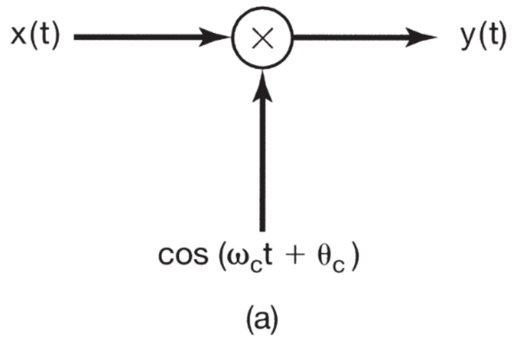




$\Rightarrow$ Only ensure $|x(t)| = |\omega(t)|$

$\Rightarrow$ If $x(t) > 0$, we get $x(t) = |\omega(t)|$

Sect. 8.2 Demodulation for Sinusoidal AM



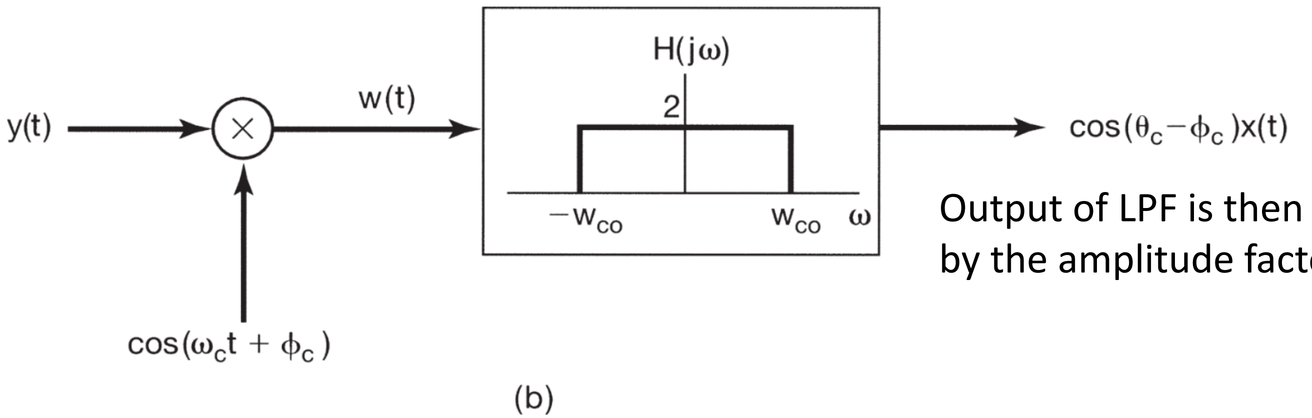
- Synchronous or Asynchronous Demodulation

$$y(t) = x(t)\cos(\omega_c t + \theta_c)$$

$$w(t) = x(t)\cos(\omega_c t + \theta_c)\cos(\omega_c t + \phi_c),$$

$$\cos(\omega_c t + \theta_c)\cos(\omega_c t + \phi_c) = \frac{1}{2}\cos(\theta_c - \phi_c) + \frac{1}{2}\cos(2\omega_c t + \theta_c + \phi_c),$$

$$w(t) = \frac{1}{2}\cos(\theta_c - \phi_c)x(t) + \frac{1}{2}x(t)\cos(2\omega_c t + \theta_c + \phi_c),$$



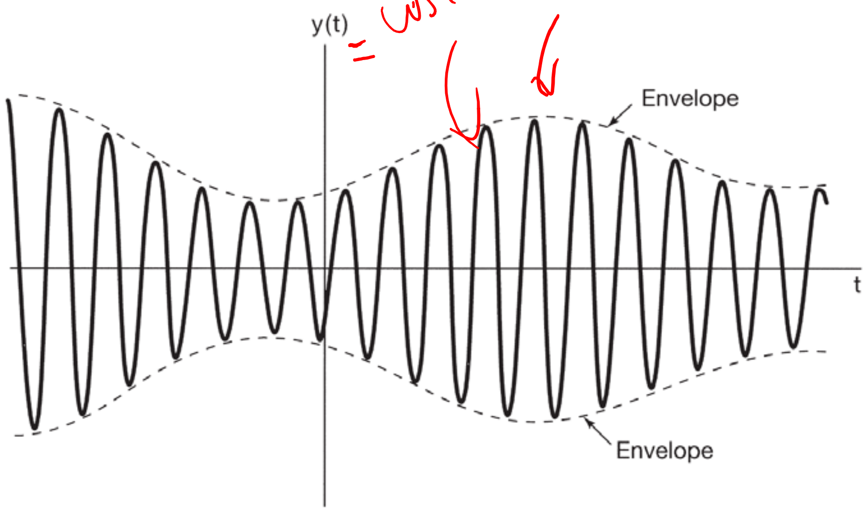
Output of LPF is then $x(t)$ multiplied by the amplitude factor $\cos(\vartheta_c - \phi_c)$.

Sect. 8.2 Demodulation for Sinusoidal AM

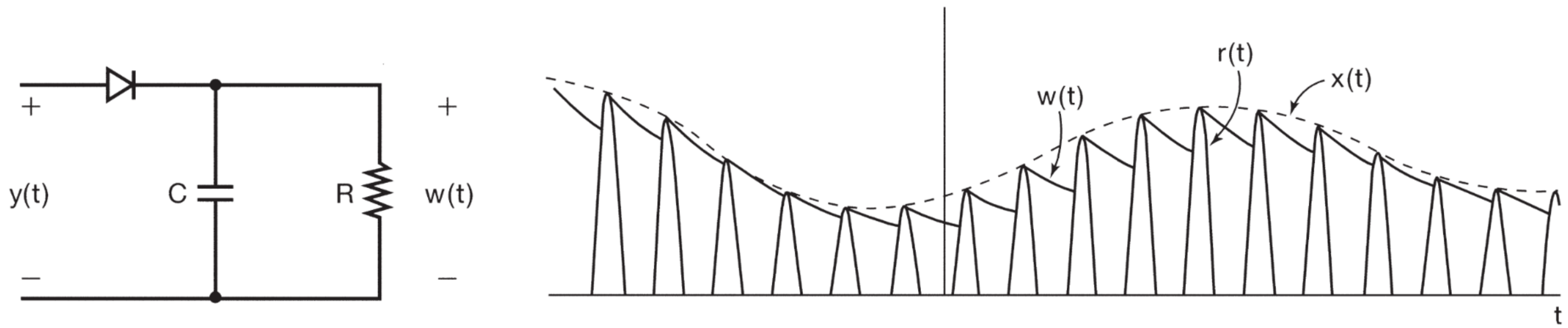
- Asynchronous Demodulation
  - Avoids the sync need between mod/demodulator
  - If the following 2 conditions:
    - ✓ $\omega_c >> \omega_M$
    - ✓ $x(t) > 0$ for all t

    are satisfied, x(t) can be approx. recovered from y(t) by an envelope detector.

$= \cos(\omega_c t) \cdot x(t)$



- Demodulation by Envelope Detector



(a) (b)

Sect. 8.2 Demodulation for Sinusoidal AM

- Asynchronous Demodulation (cont'd)

We can make x(t) positive by adding a constant to it,

$$\boxed{x(t) + A} \rightarrow x(t).$$

For $x(t) + A$ to be positive,

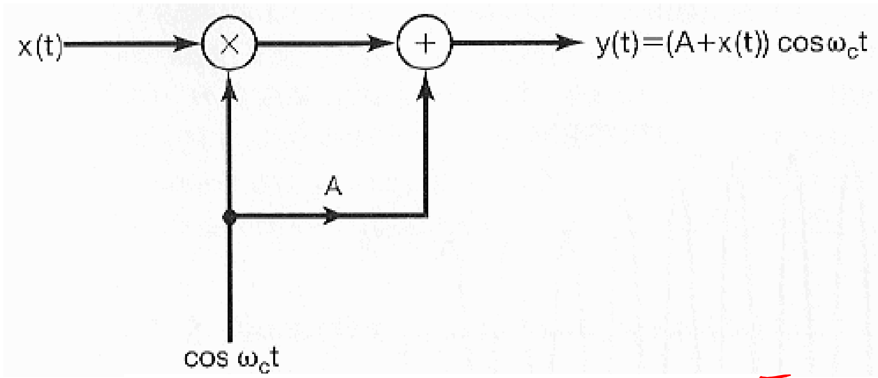
$$A \geq K$$

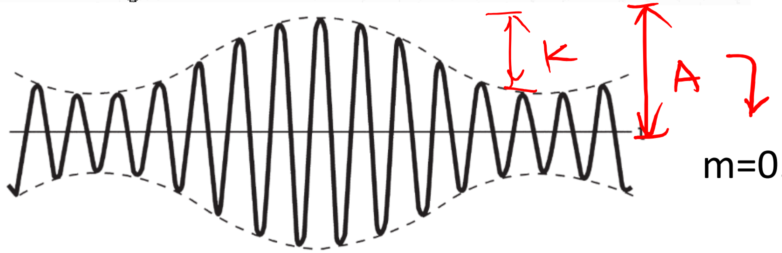
where K is the maximum amplitude of $x(t)$. The ratio $m = \dfrac{K}{A}$ is called the modulation index.

Tradeoff:

A $\downarrow$, m $\uparrow$ $\Rightarrow$ power requirement $\downarrow$

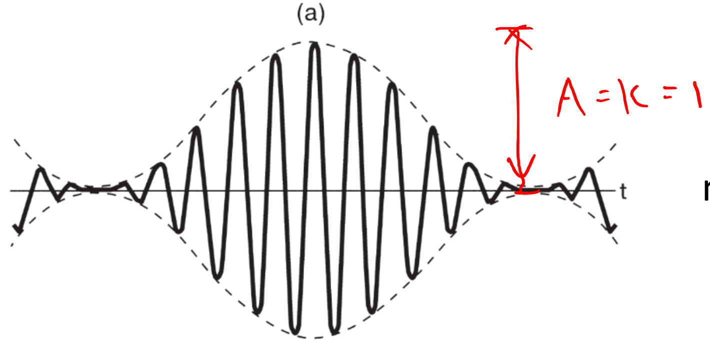
A $\uparrow$, m $\downarrow$ $\Rightarrow$ perf. of env. detector $\uparrow$





m=0.5

(a)



m = 1

(b)

Sect. 8.2 Demodulation for Sinusoidal AM

- Comparisons of Sync & Async Demodulation

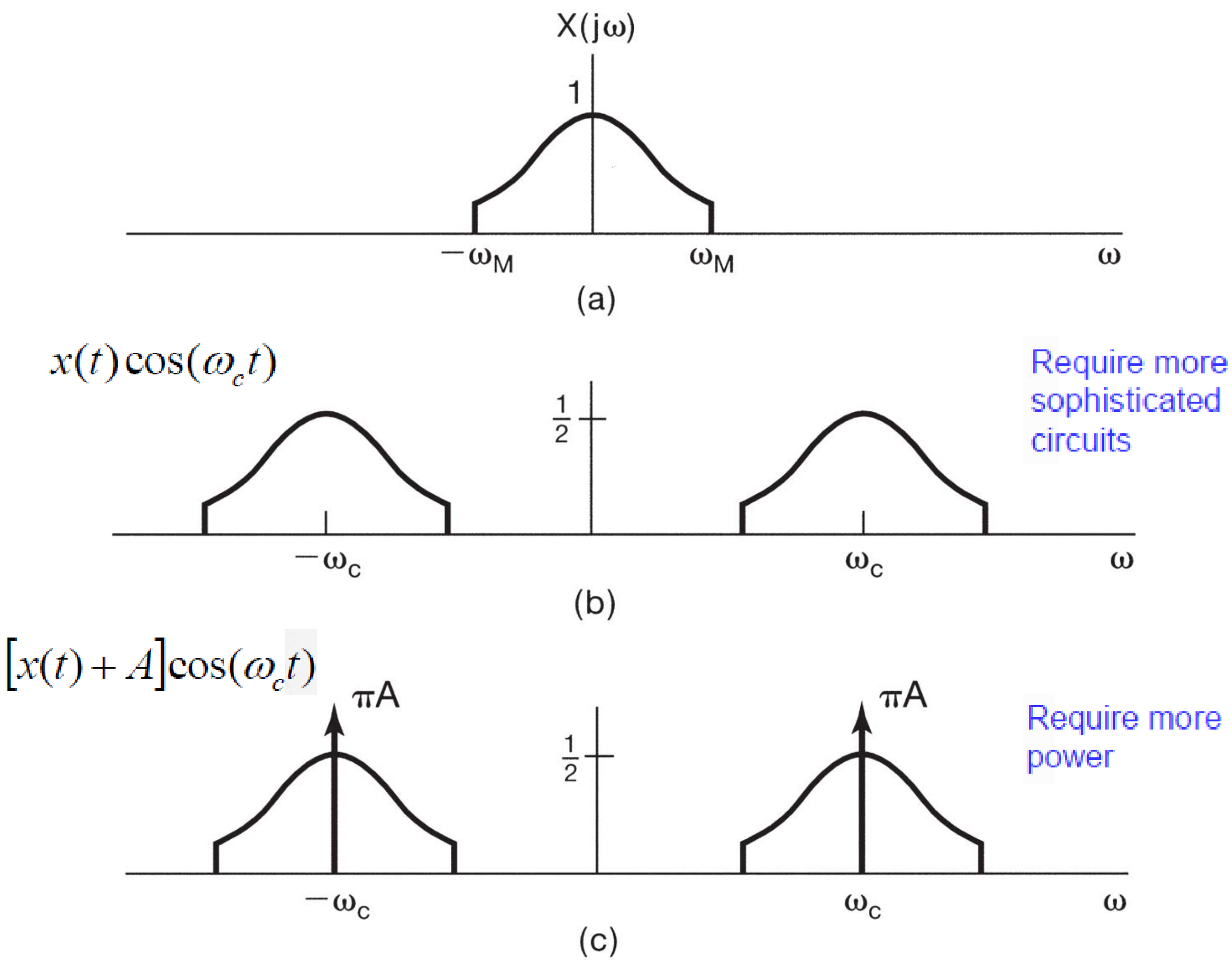
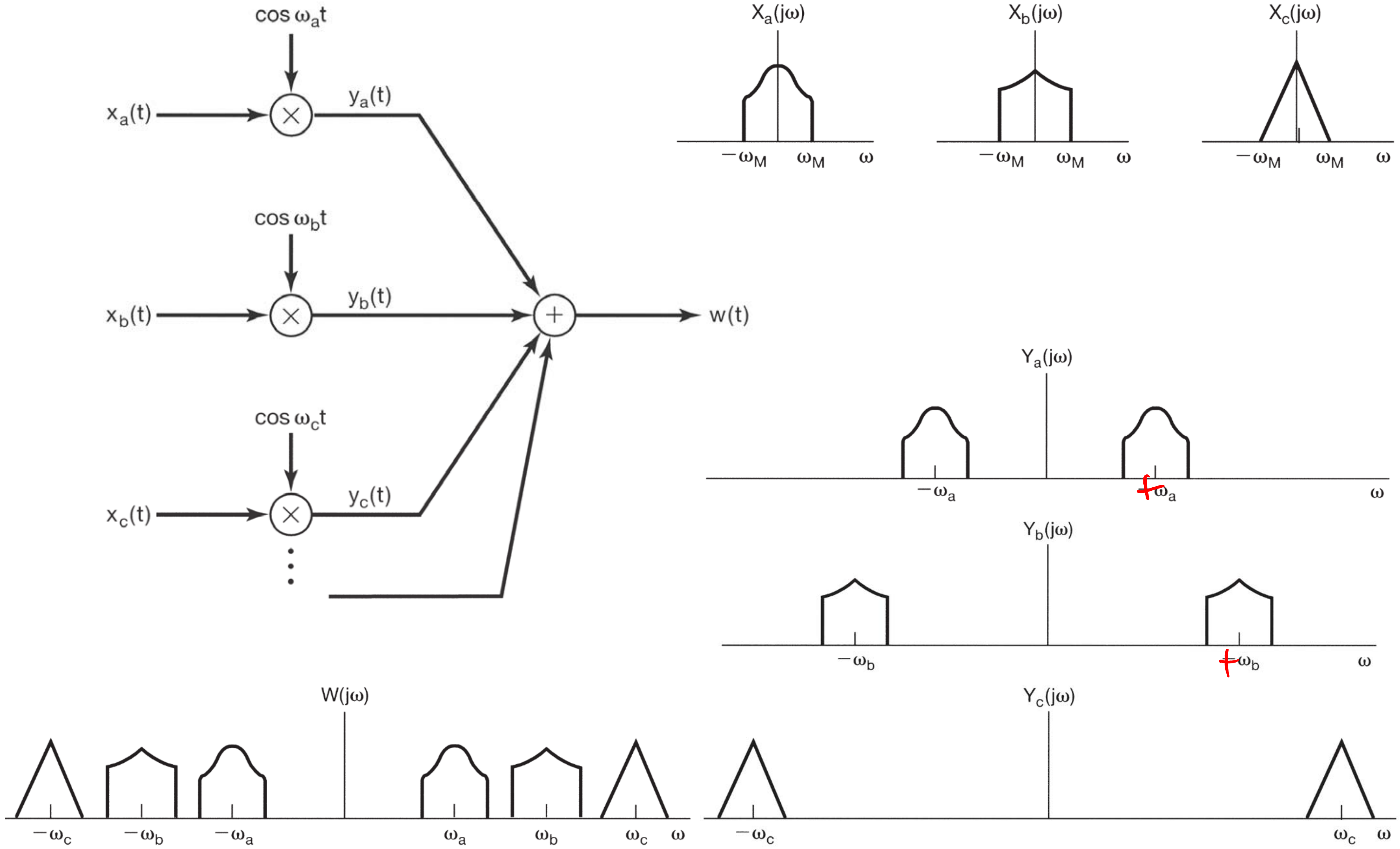


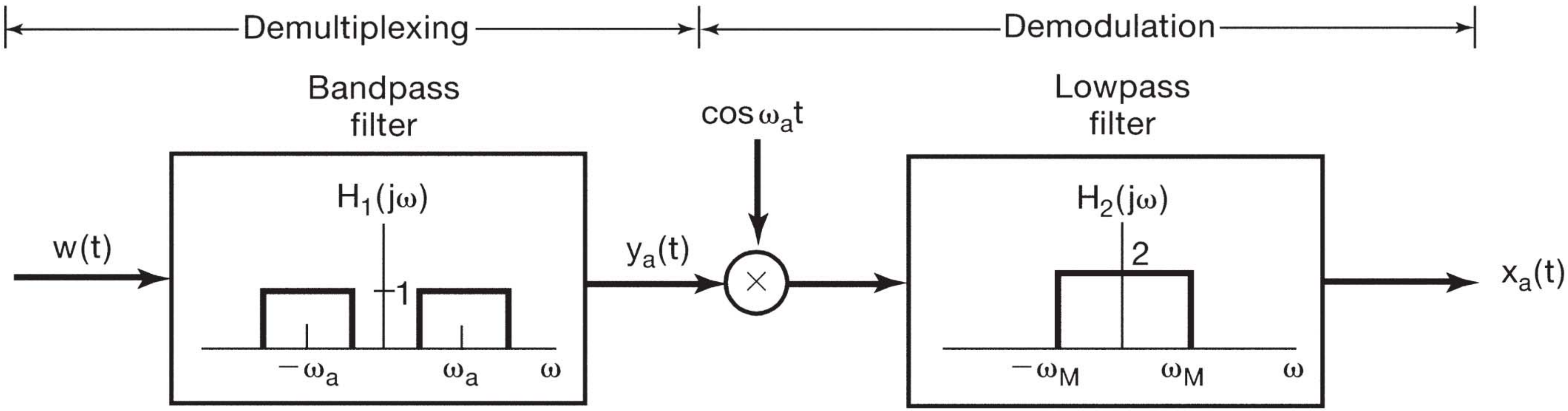
Figure 8.14 Comparison of spectra for synchronous and asynchronous sinusoidal amplitude modulation systems: (a) spectrum of modulating signal; (b) spectrum of $x(t)\cos\omega_c t$ representing modulated signal in a synchronous system; (c) spectrum of $[x(t) + A]\cos\omega_c t$ representing modulated signal in an asynchronous system.

Sect. 8.3 Frequency Division Multiplexing



Sect. 8.3 Frequency Division Multiplexing

- Telephone comm. is one important application of frequency-division multiplexing.
- Another is the transmission of signals through the atmosphere in the RF band.
- In principle, at the receiver, an individual radio station can be selected by demultiplexing and demodulating, as illustrated below.



Sect. 8.4 Single-Sideband Sinusoidal AM

Reduce the redundancy in the modulated signal

- Single Sideband (SSB) vs. Double-Sideband (DSB) Modulation
- Upper Sideband vs. Lower Sideband

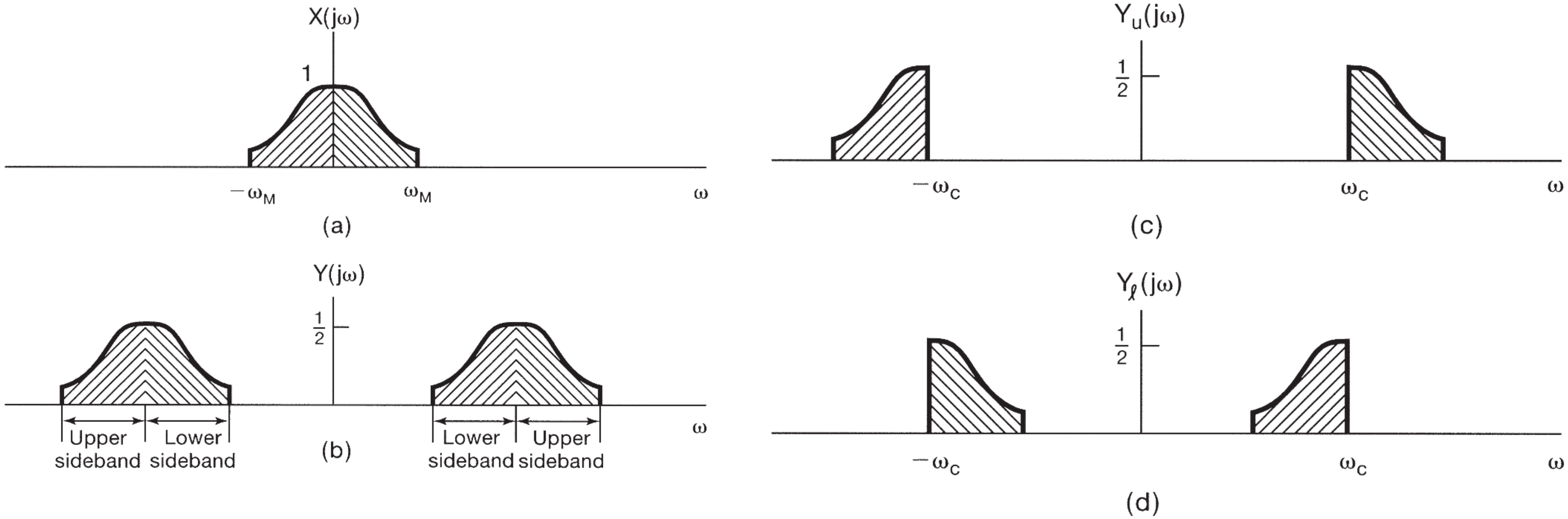
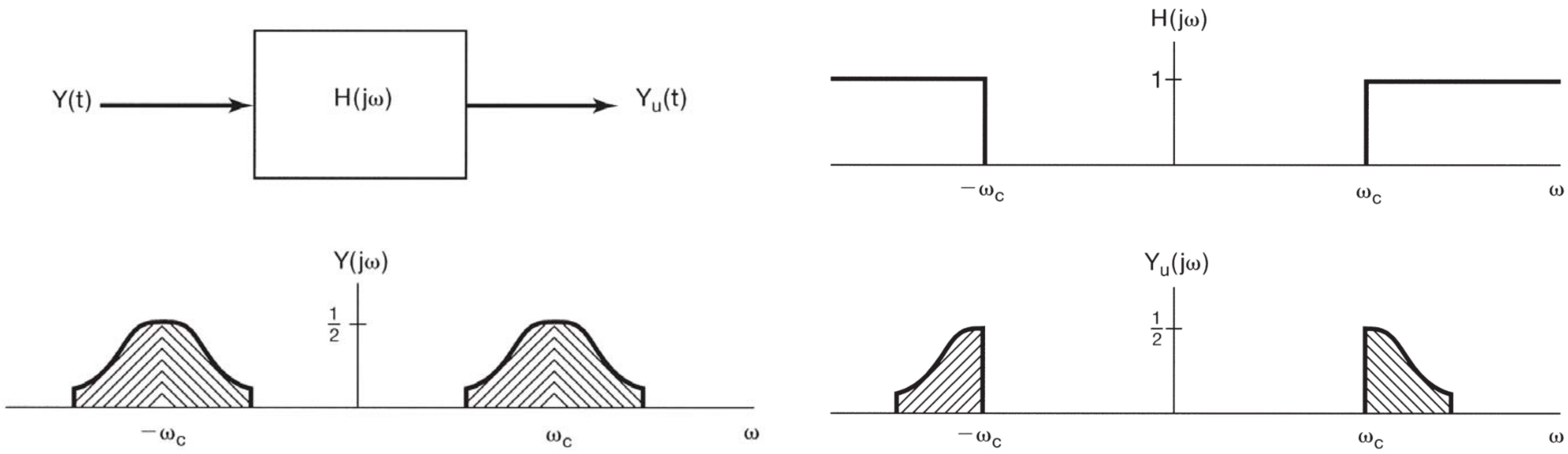


Figure 8.19 Double- and single-sideband modulation: (a) spectrum of modulating signal; (b) spectrum after modulation with a sinusoidal carrier; (c) spectrum with only the upper sidebands; (d) spectrum with only the lower sidebands.

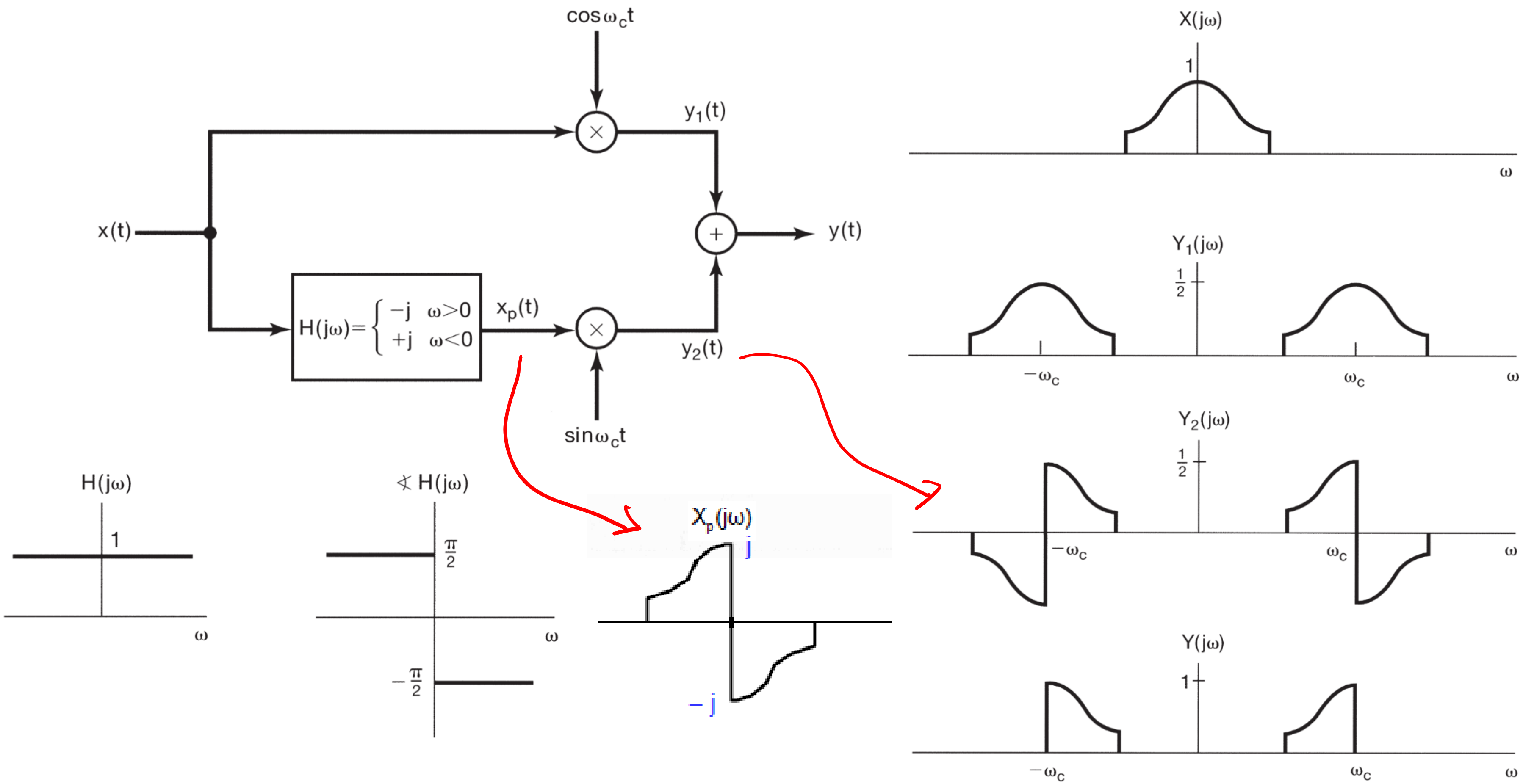
Sect. 8.4 Single-Sideband Sinusoidal AM

- Generating Sidebands Using **Ideal HPF**

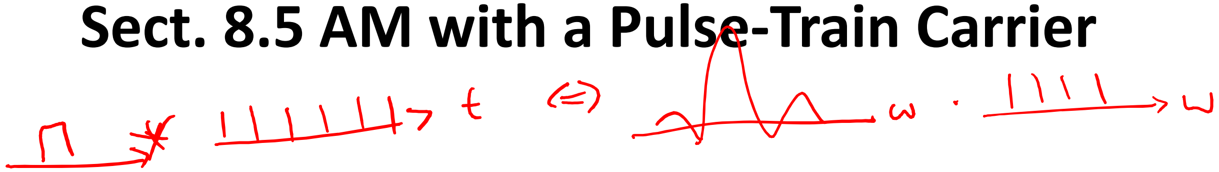


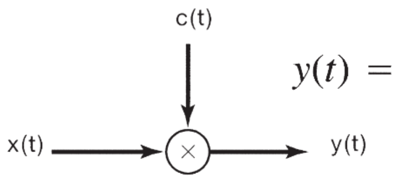
Sect. 8.4 Single-Sideband Sinusoidal AM

- Generating Sidebands Using **Phase Shifting**

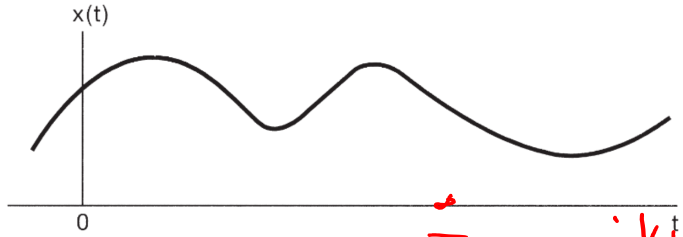


Sect. 8.5 AM with a Pulse-Train Carrier

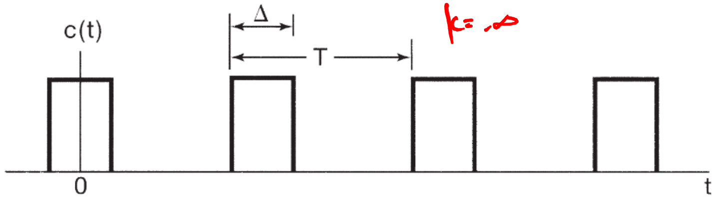


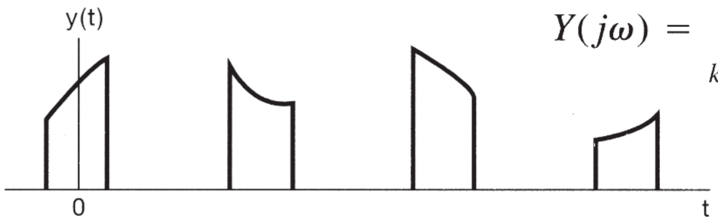


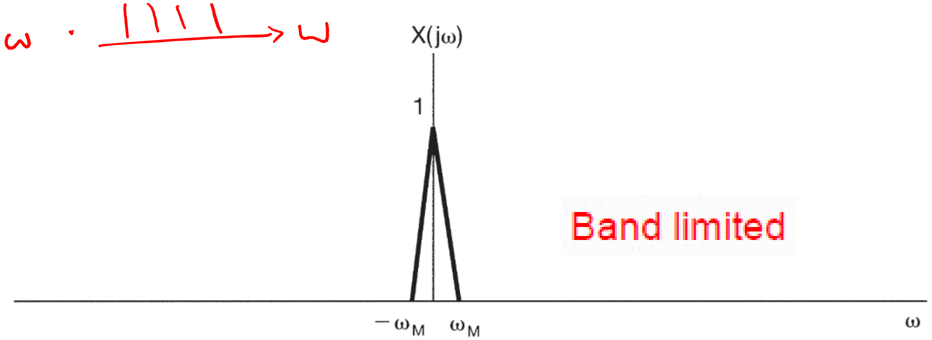
$$y(t) = x(t)c(t);$$



$$X(t) = \sum_{k=-\infty} a_k e^{jk\omega_c t}$$





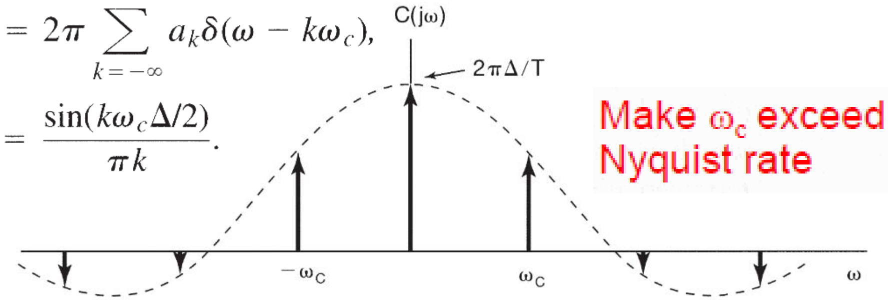


(a)

Band limited

$$C(j\omega) = 2\pi \sum_{k=-\infty}^{\infty} a_k \delta(\omega - k\omega_c),$$

$$a_k = \frac{\sin(k\omega_c\Delta/2)}{\pi k}.$$

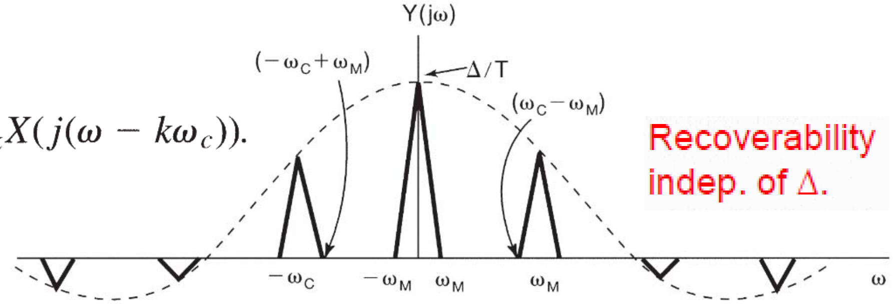


(b)

(b)

Make ω_c exceed Nyquist rate

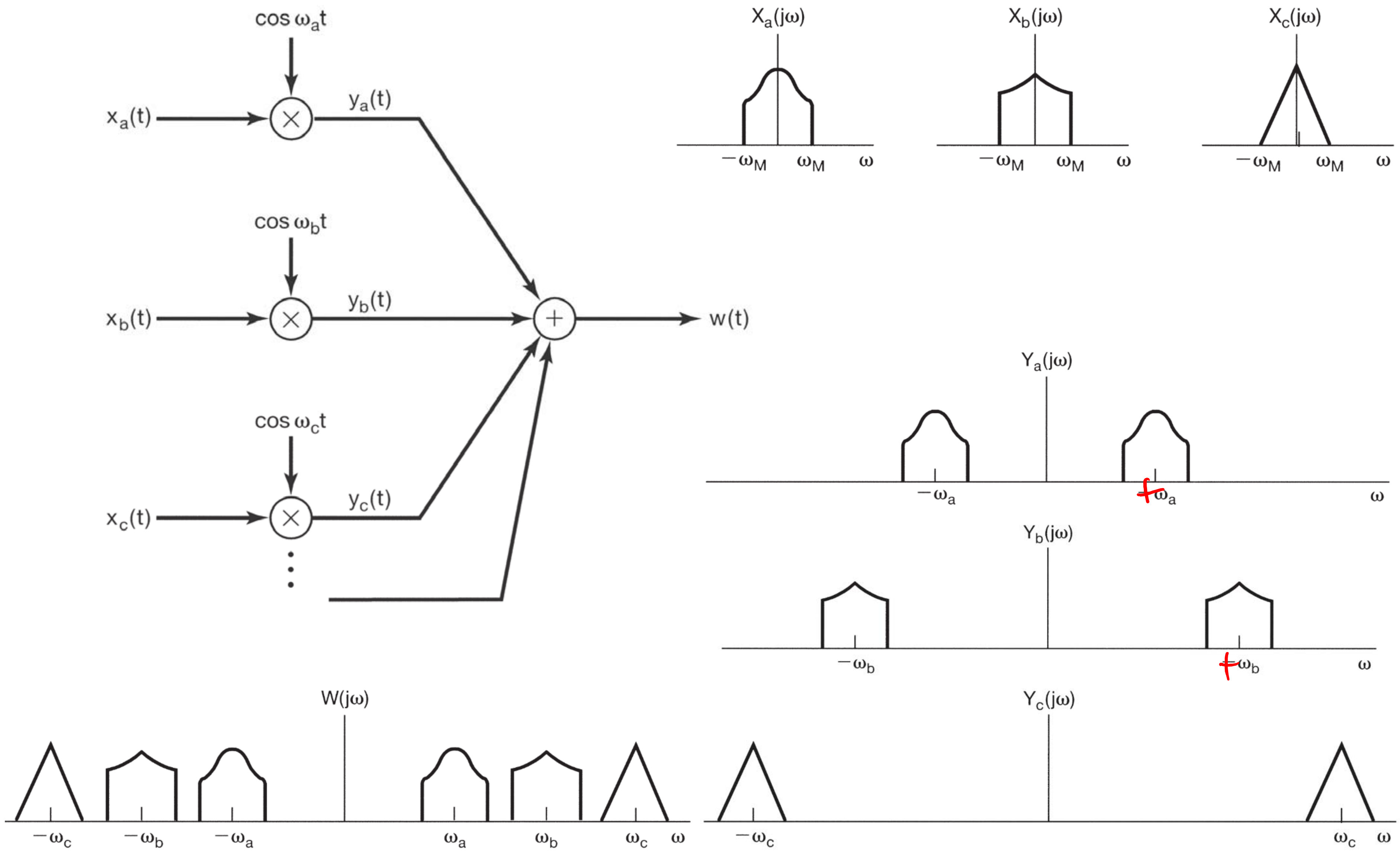
$$Y(j\omega) = \sum_{k=-\infty}^{+\infty} a_k X(j(\omega - k\omega_c)).$$



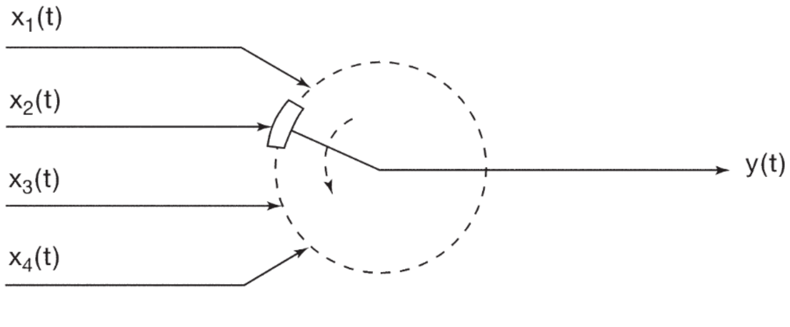
(c)

Recoverability indep. of Δ.

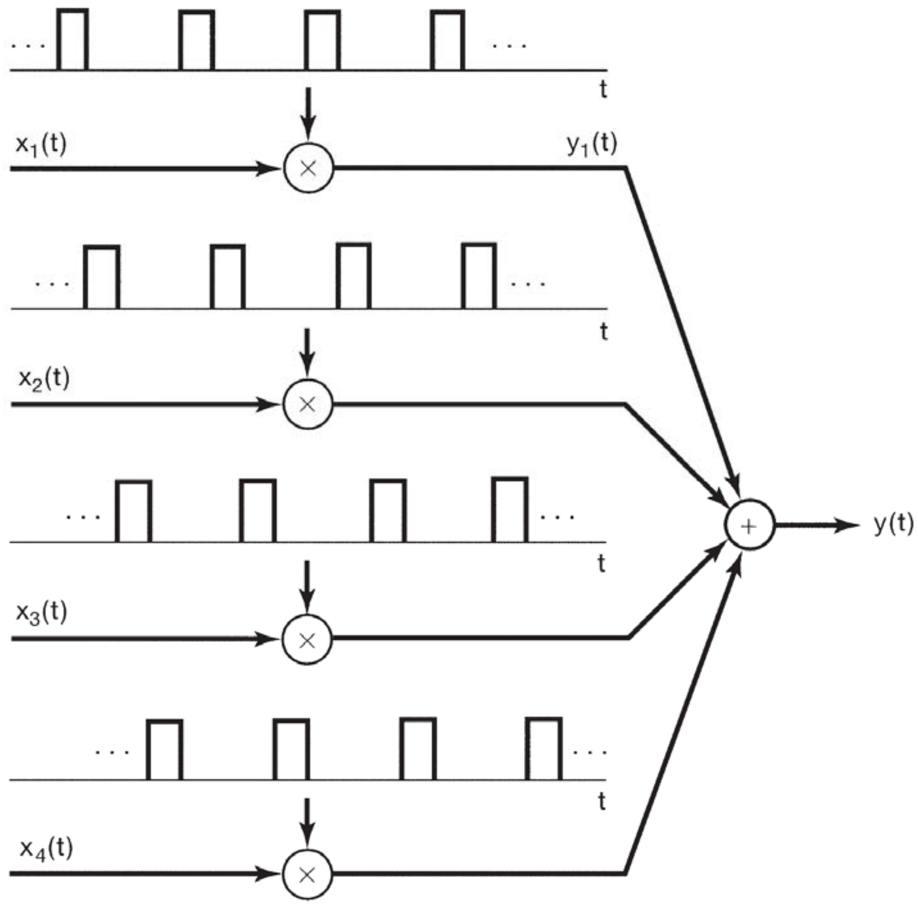
Revisit of Frequency Division Multiplexing (FDM)



8.5.2 Time-Division Multiplexing (TDM)



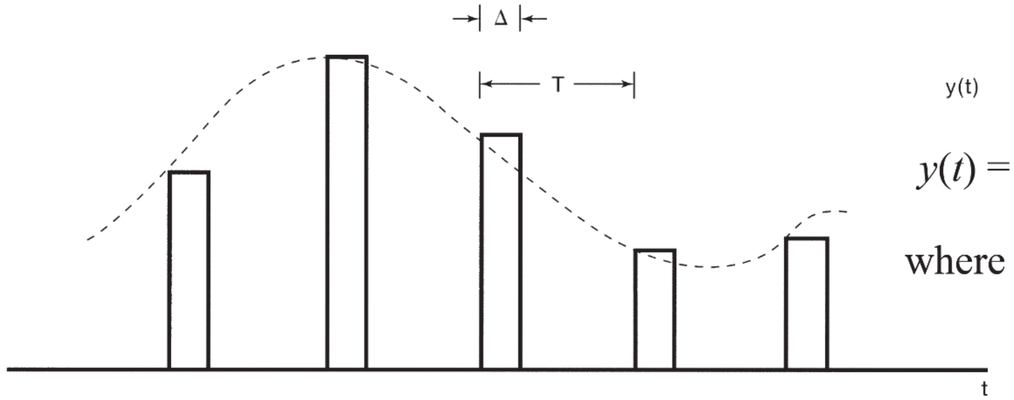
(a)



(b)

Sect. 8.6 Pulse Amplitude Modulation

- Pulse-Amplitude Modulated Signals



$$y(t) = \sum_n a[n]p(t - nT) \quad \text{where} \quad a[n] = x(nT).$$

where $p(t) = 1$ for $|t| < \Delta/2$, $p(t) = 0$ otherwise.

- TDM-PAM

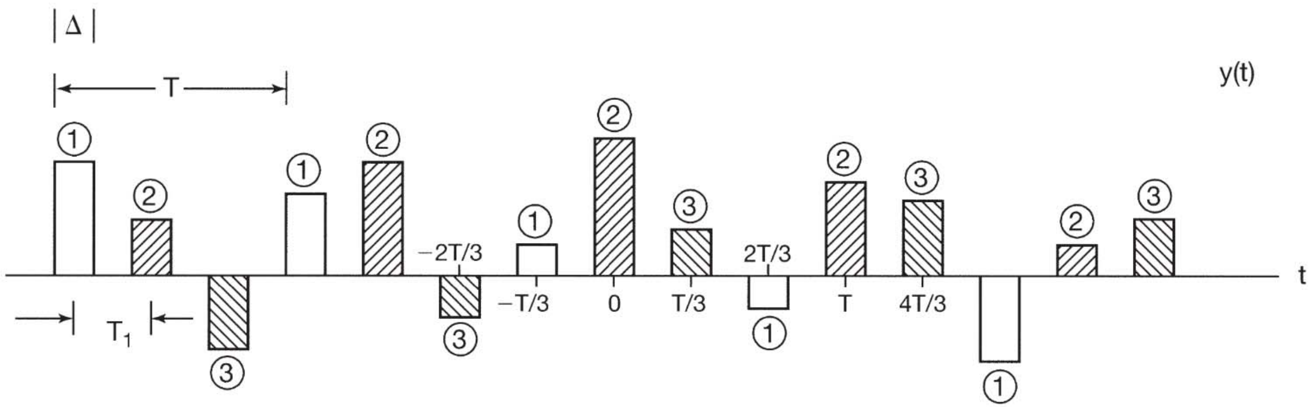
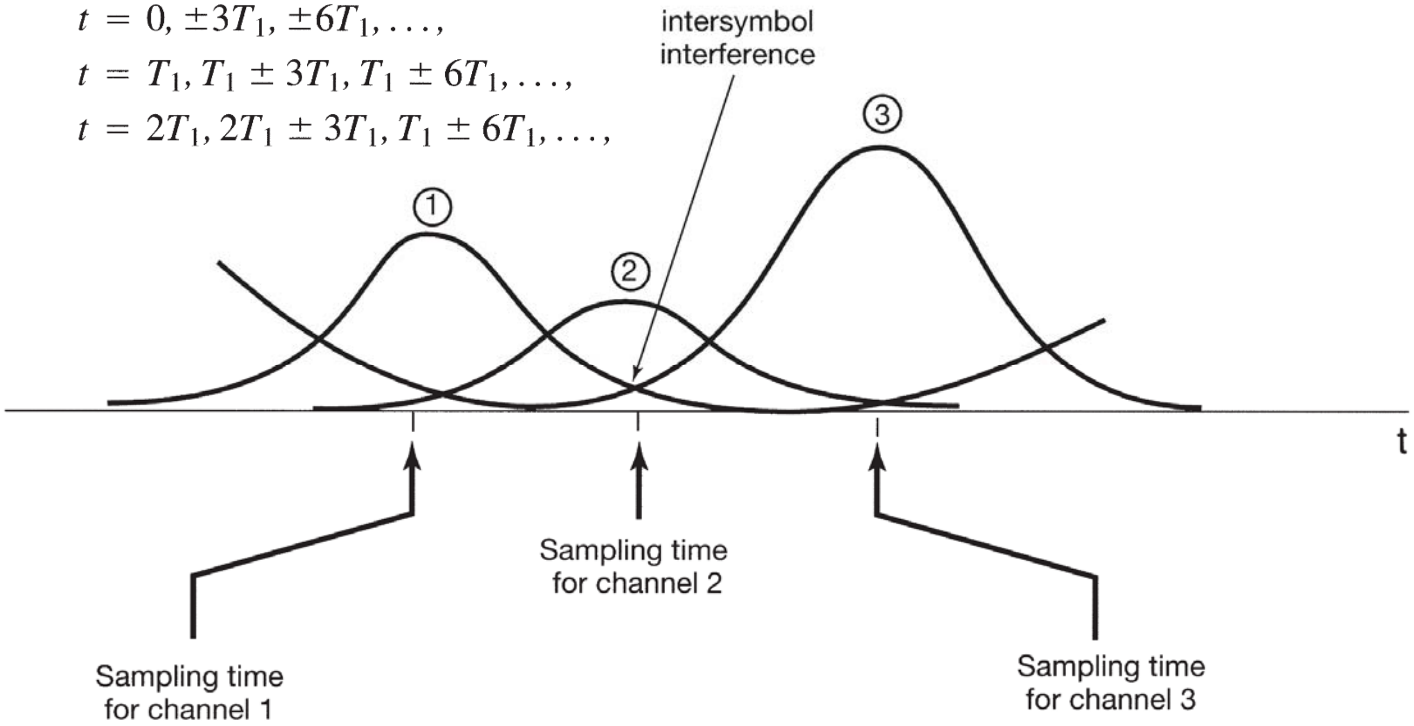


Figure 8.27 Transmitted waveform with three time multiplexed PAM channels. The pulse associated with each channel are distinguished by shading, as well as by the channel number above each pulse. Here, the intemymbol spacing is $T_1 = T/3$.

Sect. 8.6 Pulse Amplitude Modulation

- Intersymbol Interference in PAM Systems

$$
\begin{aligned}
y(t) &= Ax_1(t), \qquad & t &= 0, \pm 3T_1, \pm 6T_1, \ldots, \\
y(t) &= Ax_2(t), \qquad & t &= T_1, T_1 \pm 3T_1, T_1 \pm 6T_1, \ldots, \\
y(t) &= Ax_3(t), \qquad & t &= 2T_1, 2T_1 \pm 3T_1, T_1 \pm 6T_1, \ldots,
\end{aligned}
$$



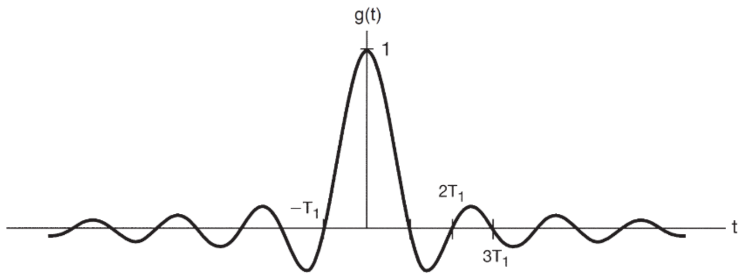
Filtering due to non-ideal frequency response of the channel causes a smearing of the pulses, which can cause the received pulses to overlap in time. This is referred to as intersymbol interference.

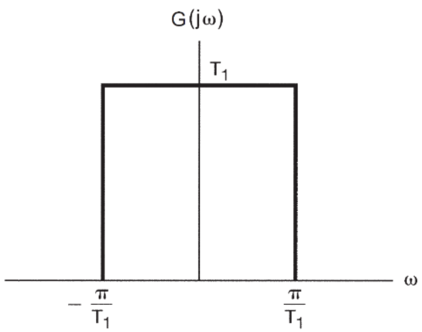
Sect. 8.6 Pulse Amplitude Modulation

- Avoiding Intersymbol Interference in PAM Systems

For example, consider the sinc pulse

$$g(t) = \frac{T_1 \sin(\pi t / T_1)}{\pi t}$$





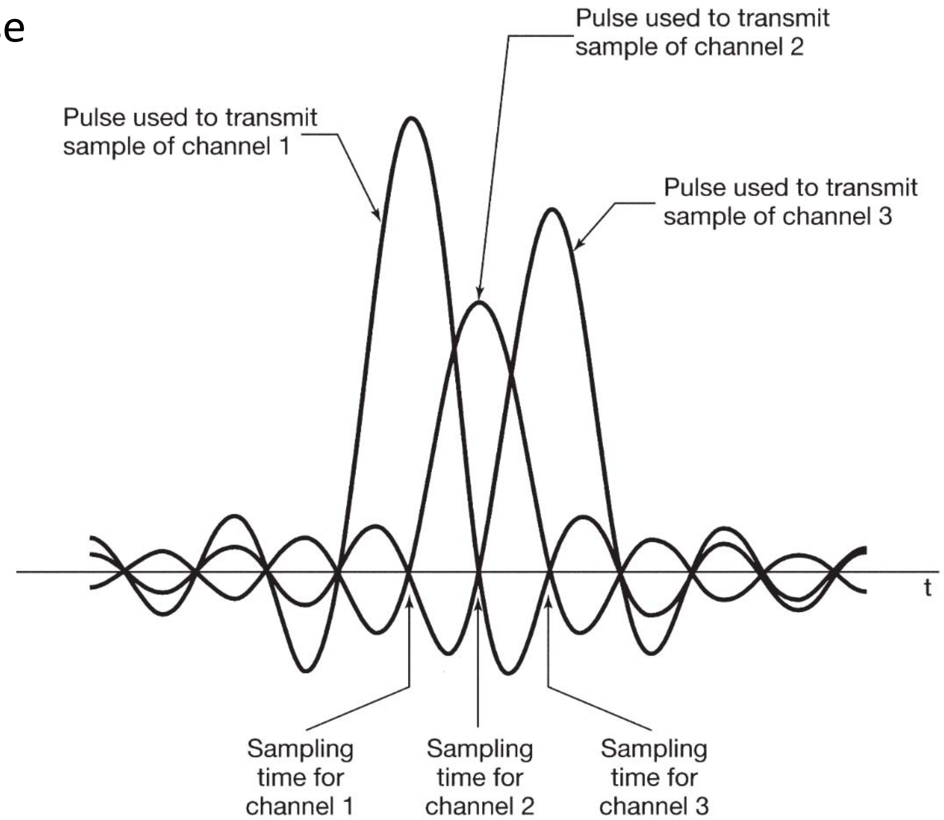


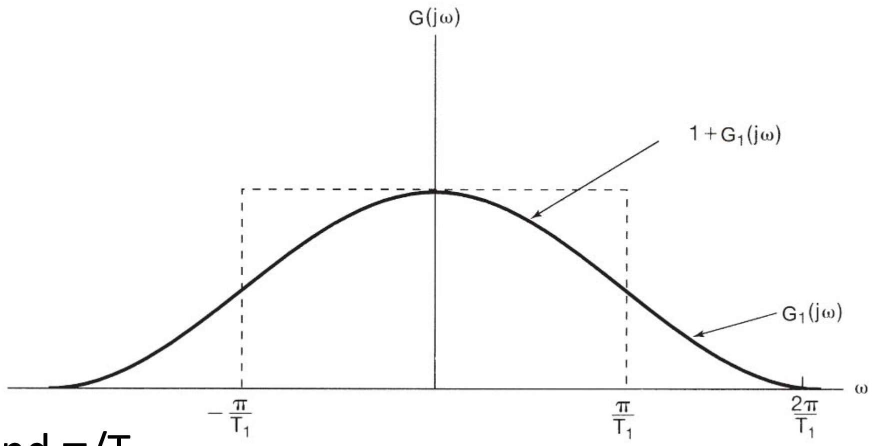
Figure 8.30 Absence of intersymbol interference when sinc pulses with correctly chosen zero-crossings are used.

Sect. 8.6 Pulse Amplitude Modulation

- Generating Band-Limited Pulses with Zero-Crossing at kT_1

More generally, consider a pulse g(t)

$$G(j\omega) = \begin{cases} 1 + G_1(j\omega), & |\omega| \leq \frac{\pi}{T_1}, \\ G_1(j\omega), & \frac{\pi}{T_1} < |\omega| \leq \frac{2\pi}{T_1} \\ 0, & \text{otherwise} \end{cases}$$



and with $G_1(j\omega)$ having odd symmetry around π/T_1, so that

$$G_1\left(-j\omega + j\frac{\pi}{T_1}\right) = -G_1\left(j\omega + j\frac{\pi}{T_1}\right) \quad 0 \leq \omega \leq \frac{\pi}{T_1},$$

Sect. 8.6 Pulse Amplitude Modulation

- Generating Band-Limited Pulses with Zero-Crossing at kT_1 (cont'd)

For the time-multiplexed PAM signal, if $b(t) = \sum_{n} y(nT_1) g(t - nT_1),$

where $g(0) = c,$ and $g(nT_1) = 0$ for nonzero integers n,

if the function $\hat{g}(t)$ obtained by sampling $g(t)$ with a periodic impulse train:

$$\hat{g}(t) = g(t) \sum_{n=-\infty}^{+\infty} \delta(t - nT_1),$$

Taking transforms of both sides, and utilizing the fact that multiplication in the time domain corresponds to convolution in the frequency domain, we obtain

$$\boxed{\hat{G}(j\omega) = c = \frac{1}{T_1} \sum_{n=-\infty}^{+\infty} G\left(j\omega - jn\frac{2\pi}{T_1}\right).}$$

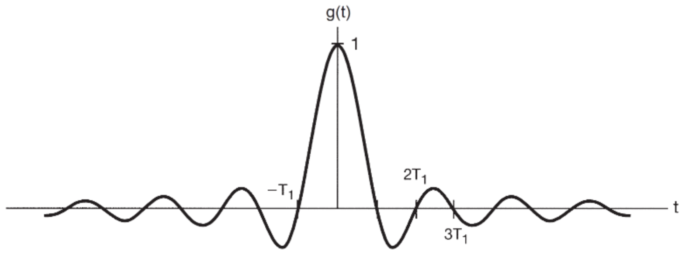
We call equation (8.37) the *Nyquist condition*.

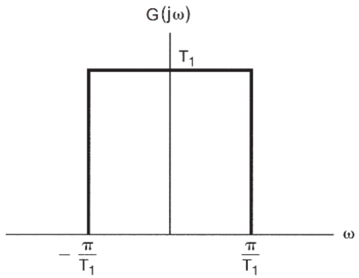
Sect. 8.6 Pulse Amplitude Modulation

- Generating Band-Limited Pulses with Zero-Crossing at kT_1 (cont'd)

Note that, if $g(t)$ is defined as in equation (8.31), i.e., $g(t) = \dfrac{T_1 \sin(\pi t/T_1)}{\pi t}$ then

$$G(j\omega) = 1 \quad \text{when} \quad |\omega| < \pi/T_1,$$
$$G(j\omega) = 0 \quad \text{when} \quad |\omega| > \pi/T_1.$$

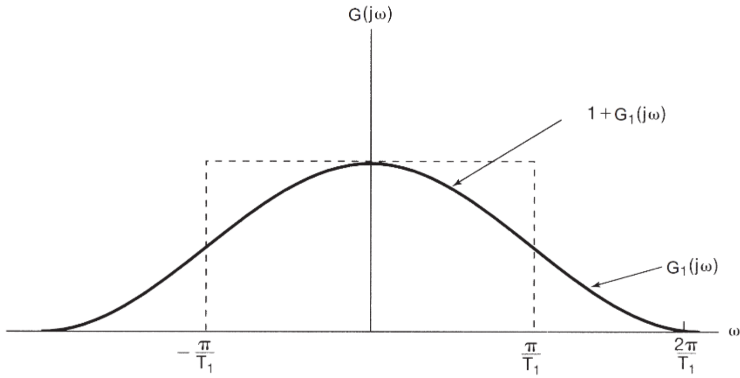




In Figure 8.31, we give another example. Note that,

$$\hat{G}(j\omega) = 1 + G_1(j0) \quad \text{for all } \omega,$$

which also satisfies the Nyquist condition.



Sect. 8.7 Sinusoidal Frequency Modulation

- Frequency Modulation
  - Modulating signal controls the frequency of a sinusoidal carrier.
  - For AM, the peak amplitude of the envelope of the modulated signal can have a large dynamic range.
  - For FM, a constant envelope is generated for the modulated signal. This means that an FM transmitter can always have a better quality than AM reception.
  - But, the price to pay…the bandwidth!

Sect. 8.7 Sinusoidal Frequency Modulation

- **Angle Modulation** $c(t) = A\cos(\omega_c t + \theta_c) = A\cos\theta(t),$
  - **Phase Modulation**

    Use the modulating signal $x(t)$ to vary the phase θ_c

$$y(t) = A\cos\theta(t) = A\cos[\omega_c t + \theta_c(t)]$$

$$\theta_c(t) = \theta_0 + k_p x(t)$$

$$\frac{d\theta(t)}{dt} = \omega_c + k_p \frac{dx(t)}{dt}$$

$$\boxed{\theta(t) = \omega_c t + \theta_0 + k_p x(t),}$$

  - **Frequency Modulation**

    Use the modulating signal $x(t)$ to vary the derivative of the angle

$$y(t) = A\cos\left(\theta(t)\right)$$

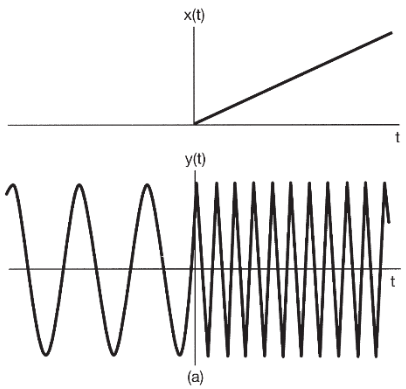
$$\frac{d\theta(t)}{dt} = \omega_c + k_f x(t)$$

$$\boxed{\theta(t) = \omega_c t + \theta_0 + k_f \int x(t)}$$

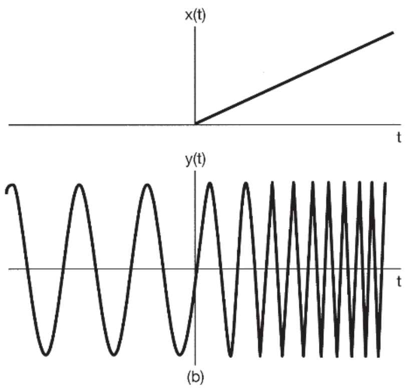
Sect. 8.7 Sinusoidal Frequency Modulation

- **Phase & Frequency Modulation**

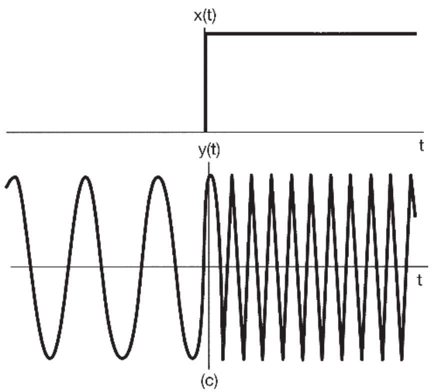
Phase modulation



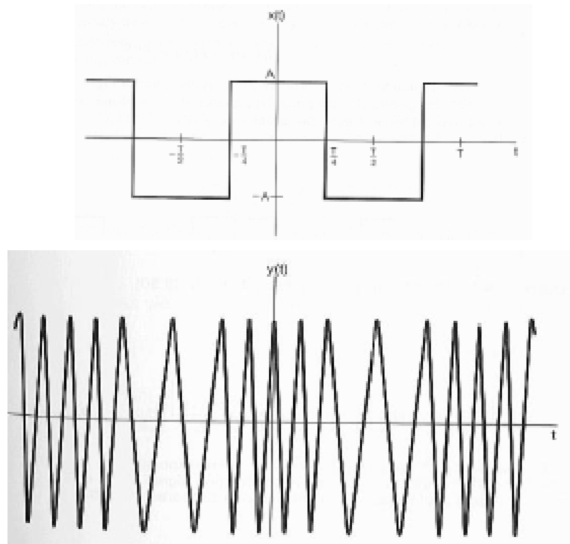
Frequency modulation



Frequency modulation



Frequency modulation



Sect. 8.7 Sinusoidal Frequency Modulation

- Instantaneous Frequency

$$y(t) = A\cos\theta(t), \qquad \omega_i(t) = \frac{d\theta(t)}{dt}.$$

- Phase Modulation:

$$\omega_i = \frac{d\theta(t)}{dt} = \omega_c + k_p \frac{dx(t)}{dt}$$

- Frequency Modulation:

$$\omega_i = \frac{d\theta(t)}{dt} = \omega_c + k_f x(t)$$