Signals & Systems

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Ch. 4 Continuous-Time Fourier Transform

- Sec. 4.1 Representation of Aperiodic Signals:
  The Continuous-Time Fourier Transform
- Sec. 4.2 The Fourier Transform for Periodic Signals
- Sec. 4.3 Properties of the Continuous-Time Fourier Transform
- Sec. 4.4 The Convolution Property
- Sec. 4.5 The Multiplication Property
- Sec. 4.6 Tables of Fourier Properties and of Basic Fourier
  Transform Pairs
- Sec. 4.7 Systems Characterized by Linear Constant Coefficient
  Differential Equations

Sect. 4.3 Properties of CTFT

- Integration

$$x(t) \xleftrightarrow{\mathcal{F}} X(j\omega)$$
$$\Rightarrow \int_{-\infty}^{t} x(\tau)d\tau \xleftrightarrow{\mathcal{F}} \frac{1}{j\omega}X(j\omega) + \boxed{\pi X(j0)\delta(\omega)}$$

?

$$y(t) = \int_{-\infty}^{t} x(\tau)d\tau \quad \Rightarrow \quad \frac{dy(t)}{dt} = x(t)$$

$$Y(j\omega) = \frac{1}{j\omega}X(j\omega)$$

Indeterminate at ω=0 as a result of the differentiation that destroys the dc component of $y(t)$

The value at ω=0 is modified by including an impulse in the transform:

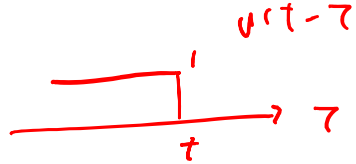
$$Y(j\omega) = \frac{1}{j\omega}X(j\omega) + \pi X(j0)\delta(\omega)$$

Proof of Derivation Property

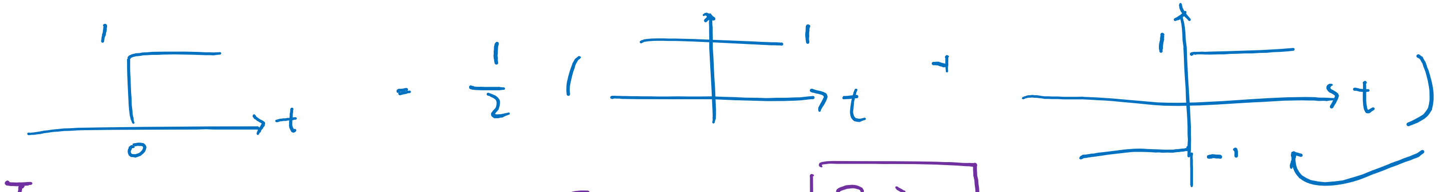
DC term?

$$\int_{-\infty}^{t} x(\tau)\,d\tau \xleftrightarrow{FT} \frac{1}{j\omega}X(j\omega) + \boxed{\pi X(0)\,\delta(\omega)}$$

$$= \int_{-\infty}^{\infty} x(\tau)\,u(t-\tau)\,d\tau$$



$$= x(t) * u(t) \xleftrightarrow{FT} X(j\omega)\,U(j\omega) = X(j\omega)\left[\frac{1}{j\omega} + \pi\,\delta(\omega)\right]$$



"sign of t"

$\mathrm{sgn}(t)$

$$\xleftrightarrow{FT} U(j\omega) = \frac{1}{2}\left(2\pi\,\delta(\omega) + \boxed{S(j\omega)}\right) = \pi\,\delta(\omega) + \frac{1}{j\omega}$$

$$\frac{d}{dt}\,\mathrm{sgn}(t) \xleftrightarrow{FT} j\omega\, S(j\omega)$$

$$2\,\delta(t) \xleftrightarrow{FT} j\omega\, S(j\omega) \quad \& \quad \delta(t) \xleftrightarrow{FT} 1$$

$$\therefore \boxed{S(j\omega) = \frac{2}{j\omega}}$$

Sect. 4.3 Convolution Property

$$\boxed{y(t)=x(t)*h(t)\stackrel{\mathcal{F}}{\longleftrightarrow}Y(j\omega)=X(j\omega)H(j\omega)}$$

$$y(t)=\int_{-\infty}^{+\infty}x(\tau)h(t-\tau)d\tau$$

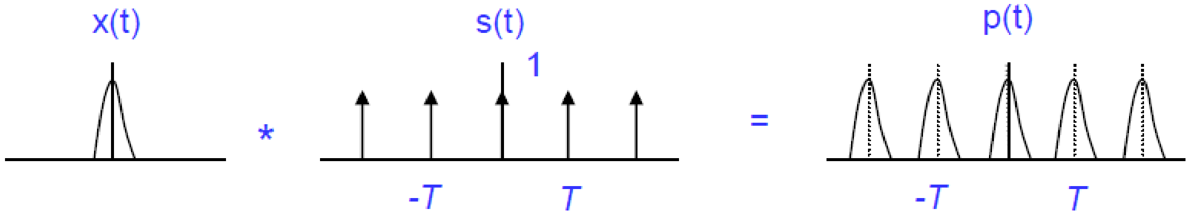
$$\Rightarrow Y(j\omega)=F\{y(t)\}=\int_{-\infty}^{+\infty}\left[\int_{-\infty}^{+\infty}x(\tau)h(t-\tau)d\tau\right]e^{-j\omega t}dt$$

$$=\int_{-\infty}^{+\infty}x(\tau)\left[\int_{-\infty}^{+\infty}h(t-\tau)e^{-j\omega t}dt\right]d\tau$$

$$=\int_{-\infty}^{+\infty}x(\tau)\left[e^{-j\omega\tau}H(j\omega)\right]d\tau$$

$$=H(j\omega)\underbrace{\int_{-\infty}^{+\infty}x(\tau)e^{-j\omega\tau}d\tau}_{X(j\omega)}=H(j\omega)X(j\omega)$$

- Fourier Transform of an Arbitrary Periodic Signal



By convolution,

$$p(t) = x(t) * s(t) = x(t) * \sum_{k=-\infty}^{\infty} \delta(t - kT) = \sum_{k=-\infty}^{\infty} x(t - kT)$$

From Example 3.8 or Example 4.8, we have

$$s(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t} = \boxed{\frac{1}{T}} \sum_{k=-\infty}^{\infty} e^{jk\omega_0 t}$$

That is, we have two expressions for a periodic impulse train.
Express them in terms of FT, we have

$$S(j\omega) = \sum_{k=-\infty}^{\infty} e^{-jk\omega T} = \frac{1}{T} \sum_{k=-\infty}^{\infty} 2\pi\delta(\omega - k\omega_0)$$

Therefore,

$$P(j\omega) = X(j\omega)S(j\omega) = \frac{2\pi}{T} \sum_{k=-\infty}^{\infty} X(j\omega)\delta(\omega - k\omega_0)$$

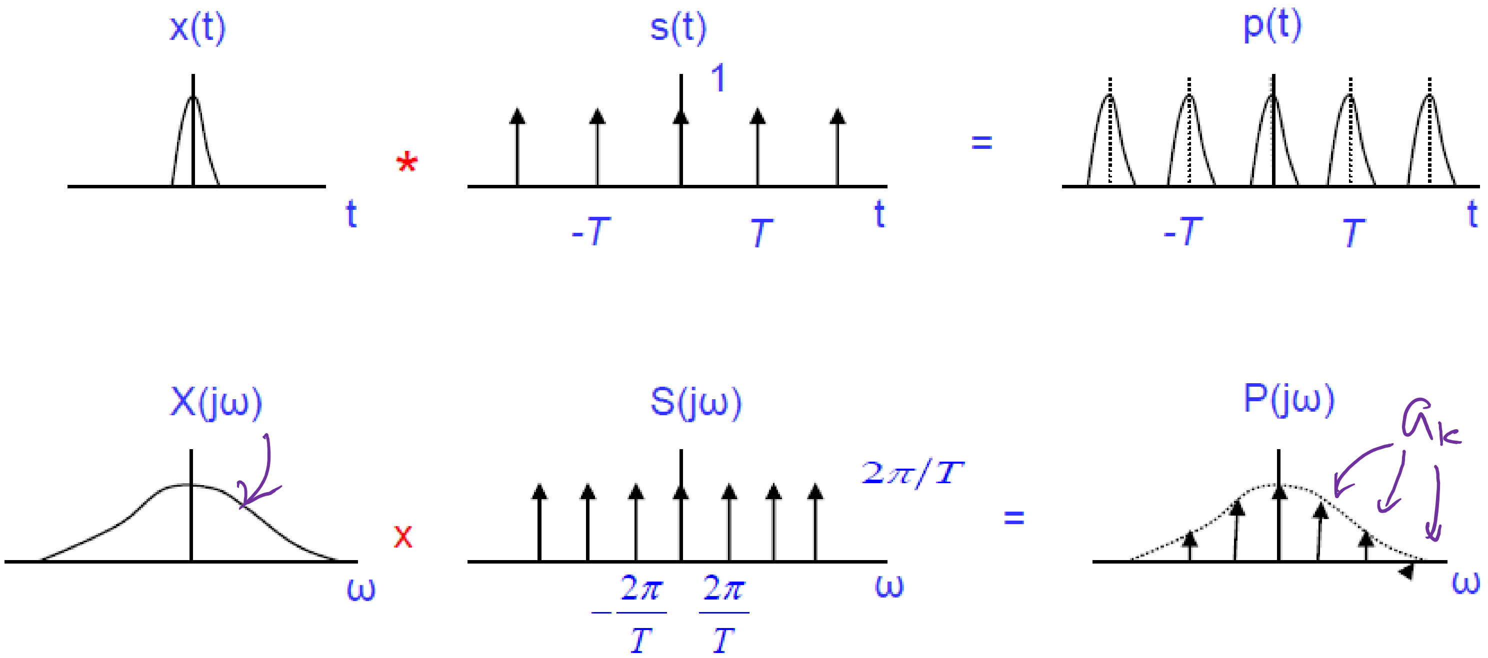
$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\omega)e^{j\omega t} d\omega$$

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t)e^{-j\omega t} dt$$

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$$

$$a_k = \frac{1}{T} \int_{-T/2}^{+T/2} x(t)e^{-jk\omega_0 t} dt$$

- Fourier Transform of an Arbitrary Periodic Signal
- We see that the FT of periodic function consists of impulses in frequency at multiples of the fundamental frequency.
- Thus, CT periodic signals can be represented by a countably infinite number of complex exponentials.



- Fourier Transform of an Arbitrary Periodic Signal
- Recall that $X(j\omega)=\int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt=\int_{-T/2}^{T/2} x(t)e^{-j\omega t}dt$
- Therefore, $\dfrac{X(jk\omega_0)}{T}=\dfrac{1}{T}\int_{-T/2}^{T/2} x(t)e^{-jk\omega_0 t}dt=a_k$ (=Fourier series coefficient)

  which is *exactly the same equation* as Eqn. (4.10). Therefore, for an arbitrary CT periodic signal, it's FT consists of impulses (located at the harmonic frequencies) whose areas are proportional to the FS coeff.

- We can also conclude that the FT of a periodic signal is related to its FS coefficients a_k by

$$P(j\omega)=\sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\omega-k\omega_0),$$

  which is exactly the same as Eqn. (4.22) (i.e., X(jω)).

Sect. 4.4 Multiplication Property

- Multiplication Property

$$\boxed{r(t) = x(t)y(t) \stackrel{\mathcal{F}}{\longleftrightarrow} R(j\omega) = \frac{1}{2\pi} X(j\omega) * Y(j\omega)}$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\phi)e^{j\phi t}d\phi, \qquad y(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} Y(j\eta)e^{j\eta t}d\eta$$

$$r(t) = x(t)y(t) = \frac{1}{(2\pi)^2}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} X(j\phi)Y(j\eta)e^{j(\phi+\eta)t}d\eta d\phi \qquad \leftarrow \ \eta = \omega - \phi$$

$$= \left(\frac{1}{2\pi}\right)\underbrace{\left(\frac{1}{2\pi}\right)\int_{-\infty}^{+\infty}\underbrace{\left[\int_{-\infty}^{+\infty} X(j\phi)Y(j\omega - j\phi)d\phi\right]}_{\text{convolution}} e^{j\omega t}d\omega}_{\text{inverse Fourier transform}}$$

Multiplication in time corresponds to convolution in frequency.

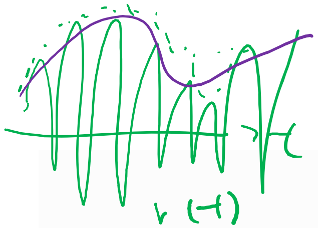
Multiply one signal by another in time is referred to as amplitude modulation.

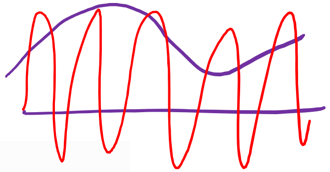
- Example 4.21 Amplitude Modulation (AM)

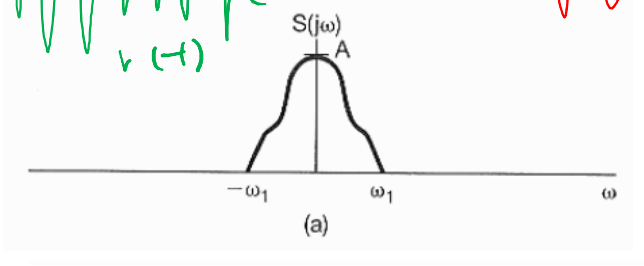
$$s(t) \xleftrightarrow{\mathcal{F}} S(j\omega)$$

$$p(t) \xleftrightarrow{\mathcal{F}} P(j\omega)$$

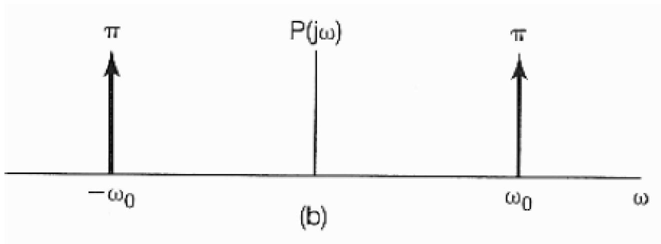
$$\boxed{r(t) = s(t)p(t)}$$





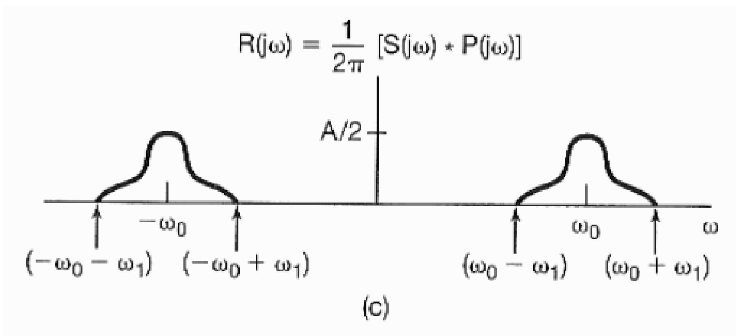


A band-limited signal



$$p(t) = \cos(\omega_0 t) = \frac{1}{2}\left[e^{j\omega_0 t} + e^{-j\omega_0 t}\right]$$

$$P(j\omega) = \pi\delta(\omega-\omega_0) + \pi\delta(\omega+\omega_0)$$



$$R(j\omega) = \frac{1}{2\pi} S(j\omega) * P(j\omega)$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} S(j\theta)P(j(\omega-\theta))d\theta$$

$$= \frac{1}{2}S(j(\omega-\omega_0)) + \frac{1}{2}S(j(\omega+\omega_0))$$

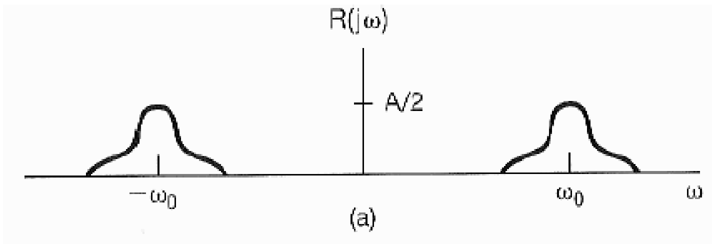
- Example 4.22 Demodulation

$$r(t) \xleftrightarrow{\mathcal{F}} R(j\omega)$$

$$p(t) \xleftrightarrow{\mathcal{F}} P(j\omega)$$

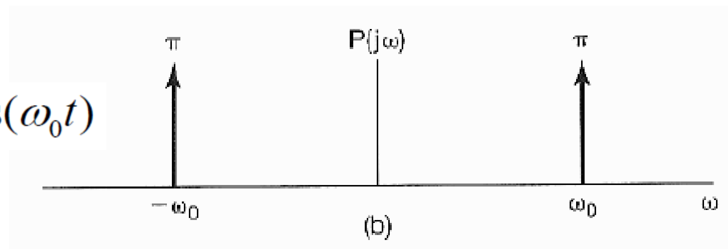
$$g(t) = r(t)\boxed{p(t)}$$

s(t) p(t)



*

$p(t) = \cos(\omega_0 t)$



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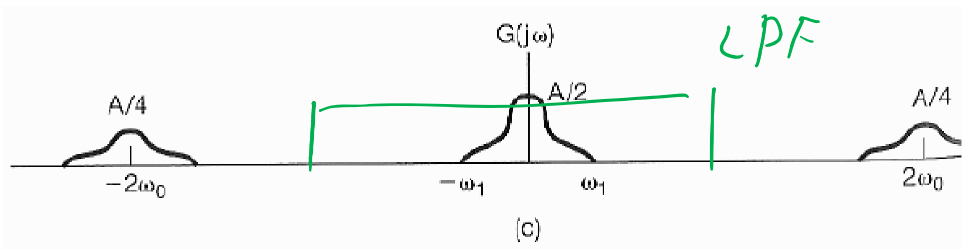

$$G(j\omega) = \frac{1}{2\pi}[R(j\omega) * P(j\omega)]$$

Ch. 4 Continuous-Time Fourier Transform

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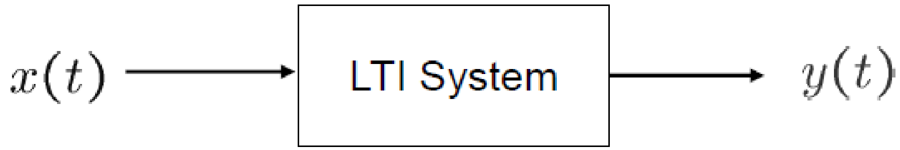
  Differential Equations

Sect. 4.7 Systems Characterized by Linear Constant-Coefficient Differential Equations

- A Useful Class of CT LTI Systems



$$a_N \frac{d^N y(t)}{dt^N} + a_{N-1} \frac{d^{N-1} y(t)}{dt^{N-1}} + \ldots + a_1 \frac{dy(t)}{dt} + a_0 y(t)$$

$$= b_M \frac{d^M x(t)}{dt^M} + b_{M-1} \frac{d^{M-1} x(t)}{dt^{M-1}} + \ldots + b_1 \frac{dx(t)}{dt} + b_0 x(t)$$

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}$$

$$F\left\{ \sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} \right\} = F\left\{ \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k} \right\}$$

Sect. 4.7 Systems Characterized by Linear Constant-Coefficient Differential Equations

- A Useful Class of CT LTI Systems (cont'd)

$$\sum_{k=0}^{N} a_k F\left\{\frac{d^k y(t)}{dt^k}\right\} = \sum_{k=0}^{M} b_k F\left\{\frac{d^k x(t)}{dt^k}\right\}$$

$$\sum_{k=0}^{N} a_k (j\omega)^k Y(j\omega) = \sum_{k=0}^{M} b_k (j\omega)^k X(j\omega)$$

$$Y(j\omega)\left[\sum_{k=0}^{N} a_k (j\omega)^k\right] = X(j\omega)\left[\sum_{k=0}^{M} b_k (j\omega)^k\right]$$

$$\Rightarrow H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{\sum_{k=0}^{M} b_k (j\omega)^k}{\sum_{k=0}^{N} a_k (j\omega)^k} = \frac{b_M (j\omega)^M + \ldots + b_1 (j\omega) + b_0}{a_N (j\omega)^N + \ldots + a_1 (j\omega) + a_0}$$

Sect. 4.7 Systems Characterized by Linear Constant-Coefficient Differential Equations

- Example 4.24

$$\frac{dy(t)}{dt} + ay(t) = x(t)$$

$$j\omega Y(j\omega) + aY(j\omega) = X(j\omega)$$

$$(j\omega + a)Y(j\omega) = X(j\omega)$$

$$\Rightarrow H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{1}{j\omega + a}$$

$$\Rightarrow h(t) = e^{-at}u(t)$$

Sect. 4.7 Systems Characterized by Linear Constant-Coefficient Differential Equations

- Example 4.25

$$\frac{d^2y(t)}{dt^2}+4\frac{dy(t)}{dt}+3y(t)=\frac{dx(t)}{dt}+2x(t)$$

$$\Rightarrow H(j\omega)=\frac{(j\omega)+2}{(j\omega)^2+4(j\omega)+3} \quad =\frac{(j\omega+2)}{(j\omega+1)(j\omega+3)}$$

$$=\frac{1/2}{j\omega+1}+\frac{1/2}{j\omega+3}$$

$$\Rightarrow h(t)=\frac{1}{2}e^{-t}u(t)+\frac{1}{2}e^{-3t}u(t)$$

Ch. 4 Continuous-Time Fourier Transform

- Sec. 4.1 Representation of Aperiodic Signals:
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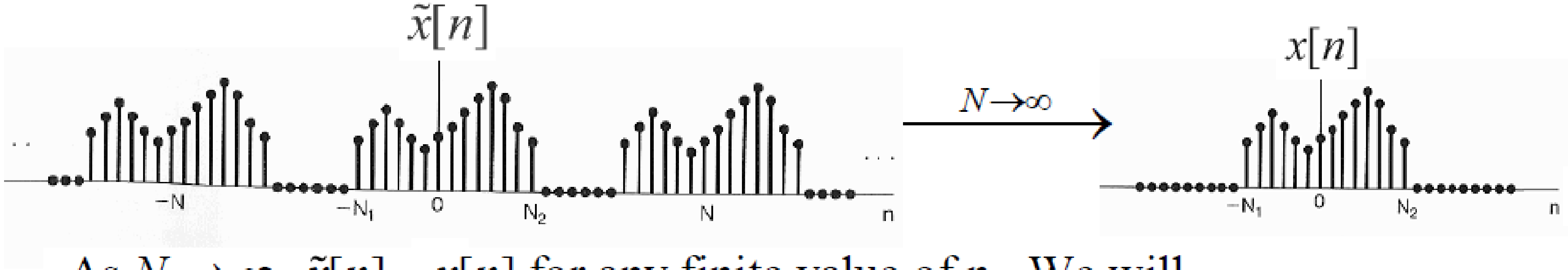
- Sec. 4.3 Properties of the Continuous-Time Fourier Transform
- Sec. 4.4 The Convolution Property
- Sec. 4.5 The Multiplication Property
- Sec. 4.6 Tables of Fourier Properties and of Basic Fourier
  Transform Pairs
- Sec. 4.7 Systems Characterized by Linear Constant Coefficient
  Differential Equations

Ch. 5 Discrete-Time Fourier Transform

- Sec. 5.1 Representation of Aperiodic Signals:
  The Discrete-Time Fourier Transform
- Sec. 5.2 The Fourier Transform for Periodic Signals
- Sec. 5.3 Properties of the Discrete-Time Fourier Transform
- Sec. 5.4 The Convolution Property
- Sec. 5.5 The Multiplication Property
- Sec. 5.6 Tables of FT Properties and Basic FT Pairs
- Sec. 5.7 Duality
- Sec. 5.8 Systems Characterized by Linear Constant Coefficient
  Differential Equations

Sec. 5.1 Representation of Aperiodic Signals: The Discrete-Time Fourier Transform

- Develop DT FT for Aperiodic Signals



As $N \to \infty$, $\tilde{x}[n] = x[n]$ for any finite value of n. We will use this relation to derive the DTFT of aperiodic signals.

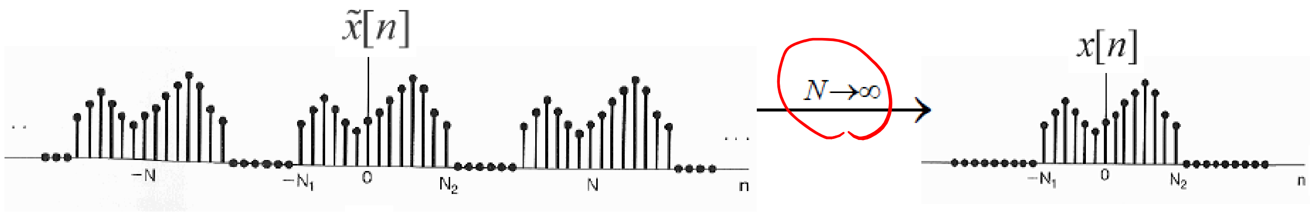
Recall the FS representation of DT signals:

$$\tilde{x}[n] = \sum_{k=<N>} a_k e^{jk\omega_0 n}$$

$$a_k = \frac{1}{N} \sum_{n=<N>} \tilde{x}[n] e^{-jk\omega_0 n}$$

$$\omega_0 = \frac{2\pi}{N}$$

- Develop DT FT for Aperiodic Signals



Since $\tilde{x}[n] = x[n]$ within any period $< N >$, we have

$$a_k = \frac{1}{N} \sum_{n=<N>} \tilde{x}[n]\, e^{-jk\omega_0 n} = \frac{1}{N} \sum_{n=-N_1}^{N_2} x[n]\, e^{-jk\omega_0 n} = \frac{1}{N} \sum_{n=-\infty}^{+\infty} x[n]\, e^{-jk\omega_0 n}$$

$\frac{2\pi}{N}$

$X(e^{jk\omega_0})$

Define

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$ (DTFT)

then we have

$$a_k = \frac{1}{N} X(e^{jk\omega_0}).$$

Substitiuting this a_k to the synthesis equation yields

$$\tilde{x}[n] = \sum_{k=<N>} \frac{1}{N} X(e^{jk\omega_0}) e^{jk\omega_0 n}.$$

Since $\omega_0 = 2\pi / N$, or equivalently, $1/N = \omega_0 / 2\pi$,

$$\tilde{x}[n] = \frac{1}{2\pi} \sum_{k=<N>} X(e^{jk\omega_0}) e^{jk\omega_0 n} \omega_0.$$

- Develop DT FT for Aperiodic Signals (cont'd)

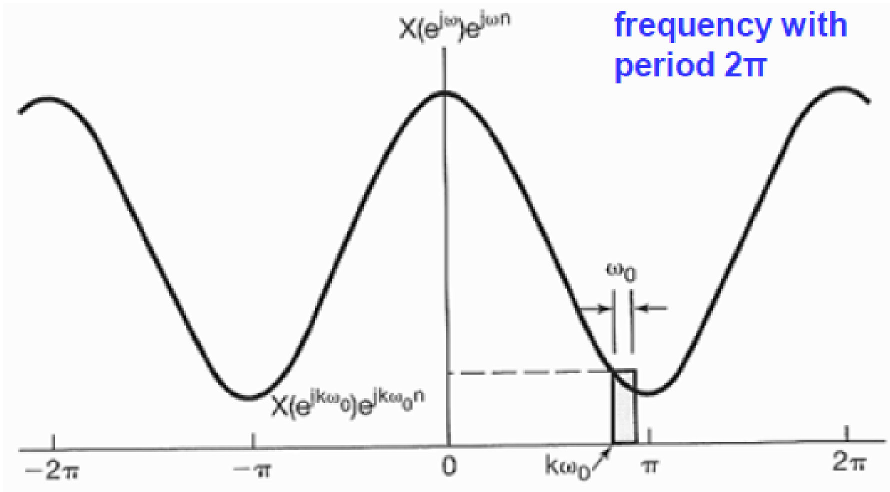
Note: 1) Both $\boxed{X(e^{j\omega})}$ and $\boxed{e^{j\omega n}}$ are periodic in ω with period 2π, so is the product $X(e^{j\omega})e^{j\omega n}$.

2) The total interval of integration will become 2π since the summation is carried over N consecutive of intervals of width $\omega_0 = 2\pi / N$.

Therefore, as $N \to \infty$, $\tilde{x}[n] \to x[n]$, $\omega_0 \to 0$, we have

$$\tilde{x}[n] = \frac{1}{2\pi} \sum_{k=<N>} X(e^{jk\omega_0}) e^{jk\omega_0 n} \omega_0 \qquad \to \qquad \boxed{x[n] = \frac{1}{2\pi} \int_{2\pi} X(e^{j\omega}) e^{j\omega n} d\omega.}$$

Periodic in frequency with period 2π



Sec. 5.1 Representation of Aperiodic Signals: The Discrete-Time Fourier Transform

- DTFT

$$\boxed{x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}\,d\omega}$$

Inverse Fourier transform
Synthesis equation

$$\boxed{X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n]e^{-j\omega n}}$$

Fourier transform
Analysis equation

Recall the CTFT:

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}\,d\omega$$

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt$$

Sec. 5.1 Representation of Aperiodic Signals: The Discrete-Time Fourier Transform

- Periodicity

$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n]e^{-j\omega n}$$

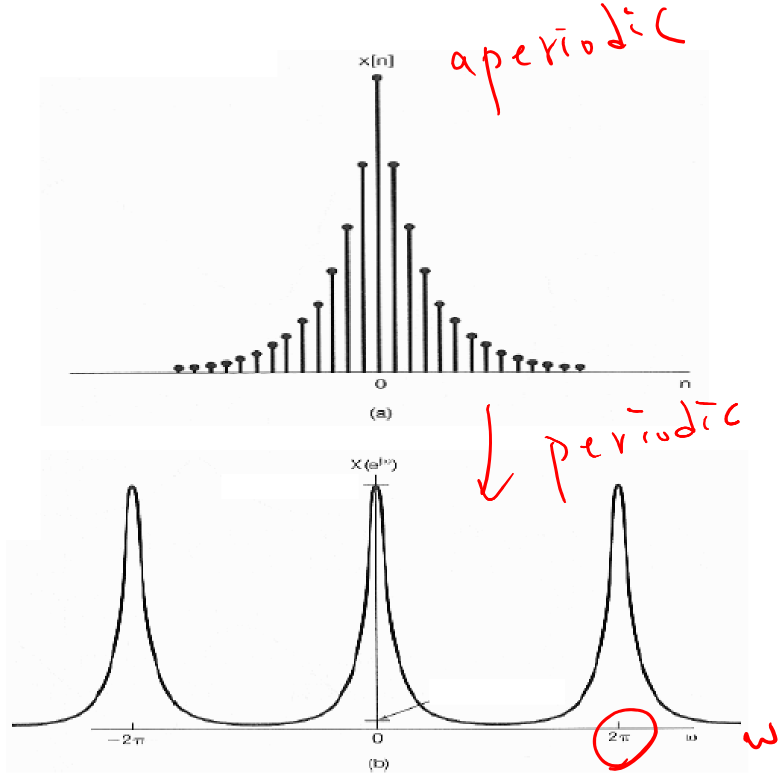
$$X(e^{j(\omega+2\pi)}) = \sum_{n=-\infty}^{+\infty} x[n]e^{-j(\omega+2\pi)n}$$

$$= \sum_{n=-\infty}^{+\infty} x[n]e^{-j\omega n}e^{-j2\pi n}$$

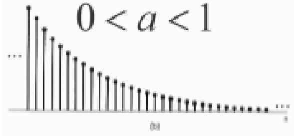
$$= \sum_{n=-\infty}^{+\infty} x[n]e^{-j\omega n}$$

$$\therefore X(e^{j(\omega+2\pi)}) = X(e^{j\omega})$$

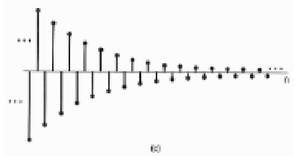
$$\Rightarrow X(e^{j\omega}) \text{ is periodic with period } 2\pi$$



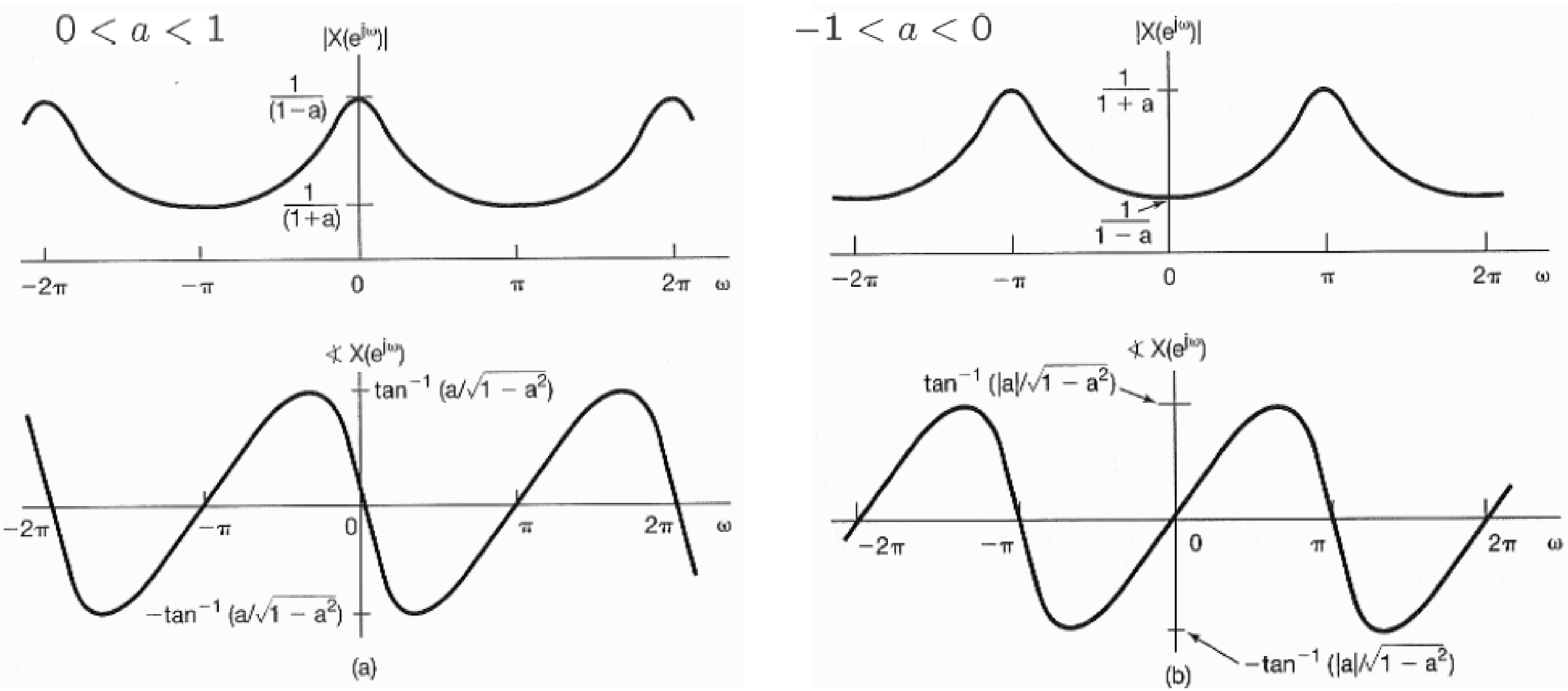
- Example 5.1 $x[n] = a^n u[n], \quad |a| < 1$



$-1 < a < 0$



$$\Rightarrow X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} a^n u[n] e^{-j\omega n} = \sum_{n=0}^{\infty} (ae^{-j\omega})^n = \frac{1}{1 - ae^{-j\omega}}$$



- Example 5.2 $\quad x[n] = a^{|n|}, \quad |a| < 1$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} a^{|n|} e^{-j\omega n}$$

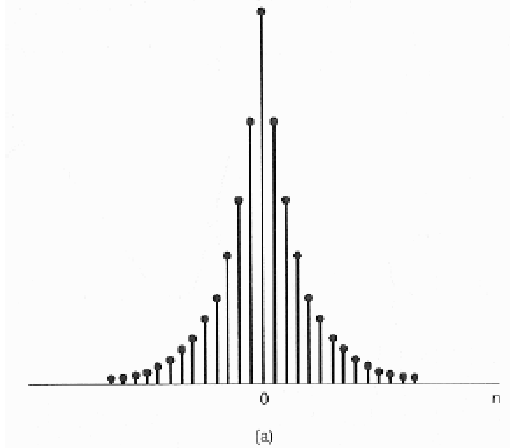
$$= \sum_{n=0}^{+\infty} a^{n} e^{-j\omega n} + \sum_{n=-\infty}^{-1} a^{-n} e^{-j\omega n}$$

$$= \sum_{n=0}^{+\infty} (a e^{-j\omega})^{n} + \sum_{m=1}^{+\infty} (a e^{j\omega})^{m}$$

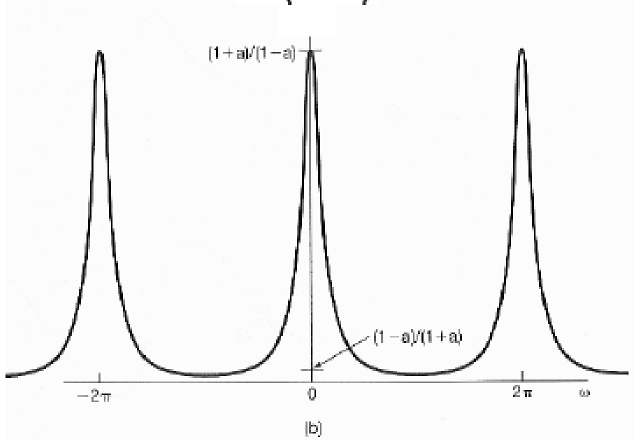
$$= \frac{1}{1 - a e^{-j\omega}} + \frac{a e^{j\omega}}{1 - a e^{j\omega}}$$

$$= \frac{1 - a^2}{1 - 2a\cos\omega + a^2}$$

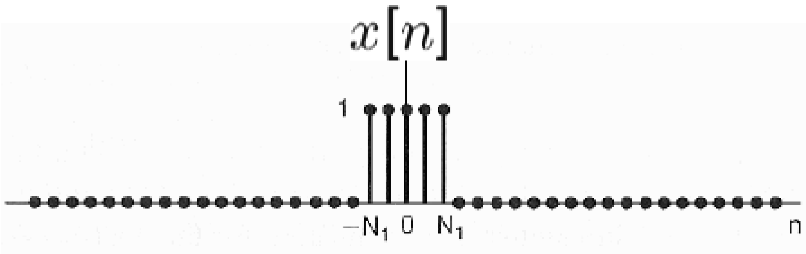
$x[n]$



$X(e^{jw})$



- Example 5.3 $x[n]=\begin{cases}1, & |n|\le N_1\\ 0, & |n|>N_1\end{cases}$



(a)

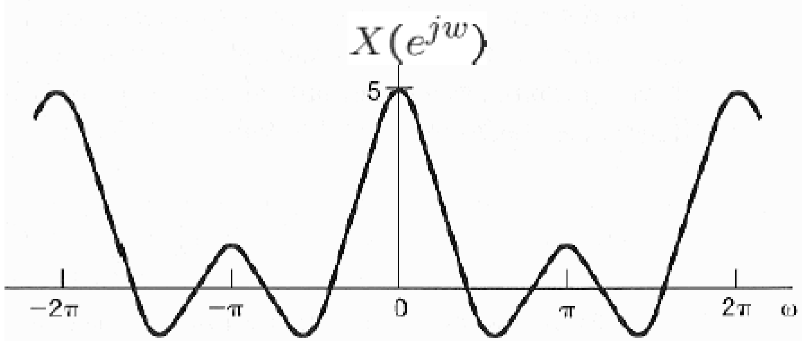
$$X(e^{j\omega})=\sum_{n=-N_1}^{N_1} e^{-j\omega n}$$

$$=e^{-j\omega(-N_1)}+\ldots+e^{-j\omega(N_1)}$$

$$=e^{-j\omega(-N_1)}\left(\frac{1-(e^{-j\omega})^{2N_1+1}}{1-(e^{-j\omega})}\right)$$

$$=e^{j\omega(N_1)}\left(\frac{(e^{-j\omega})^{N_1+\frac{1}{2}}((e^{j\omega})^{N_1+\frac{1}{2}}-(e^{-j\omega})^{N_1+\frac{1}{2}})}{(e^{-j\omega/2})((e^{j\omega/2})-(e^{-j\omega/2}))}\right)$$

$$=\frac{\sin(\omega(N_1+\frac{1}{2}))}{\sin(\omega/2)}$$



(b)

- **Convergence of DTFT**

- Sufficient Conditions for the Convergence of FT
- Derivation of FT suggests the same convergence condition as that of FS.

Define $\hat{x}(t) = \dfrac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)e^{j\omega t}d\omega$ and $e(t) = \hat{x}(t) - x(t)$.

If $x(t)$ has finite energy (that is, it is square integrable),

$$\int_{-\infty}^{+\infty} |x(t)|^2\, dt < \infty$$

Then we are guaranteed that $X(j\omega)$ is finite and that

$$\int_{-\infty}^{+\infty} |e(t)|^2\, dt = 0.$$

That is, there is no energy in their difference, even if $\hat{x}(t)$ may differ significantly at individual values of t.

Sec. 5.1 Representation of Aperiodic Signals: The Discrete-Time Fourier Transform

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}d\omega$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n]e^{-j\omega n}$$

- Convergence of DTFT

The analysis equation

$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n]e^{-j\omega n}$$

will converge if $x[n]$ is absolutely summable,

$$\sum_{n=-\infty}^{+\infty} |x[n]| < \infty,$$

or if $x[n]$ has finite energy,

$$\sum_{n=-\infty}^{+\infty} |x[n]|^2 < \infty,$$

very much like its counterpart in the CT case.

Sec. 5.1 Representation of Aperiodic Signals: The Discrete-Time Fourier Transform

- Convergence of DTFT

But the synthesis equation

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}d\omega$$

has no convergence issue associated with it because the integral is over a finite interval of integration.

Also, in contrast to the CT case, the Gibbs phenomenon does not exist if we approximate $x[n]$ by

$$\hat{x}[n] = \frac{1}{2\pi}\int_{-W}^{W} X(e^{j\omega})e^{j\omega n}d\omega,$$

where $W \leq \pi$. The amplitude of the oscillations exhibited in $\hat{x}[n]$ relative to the magnitude of $\hat{x}[0]$ decreases as W is increased.

Revisit of Sect. 3.6 FS Representation of DT Periodic Signals

- Partial Sum

$N = 9,\ 2N_1 + 1 = 5$

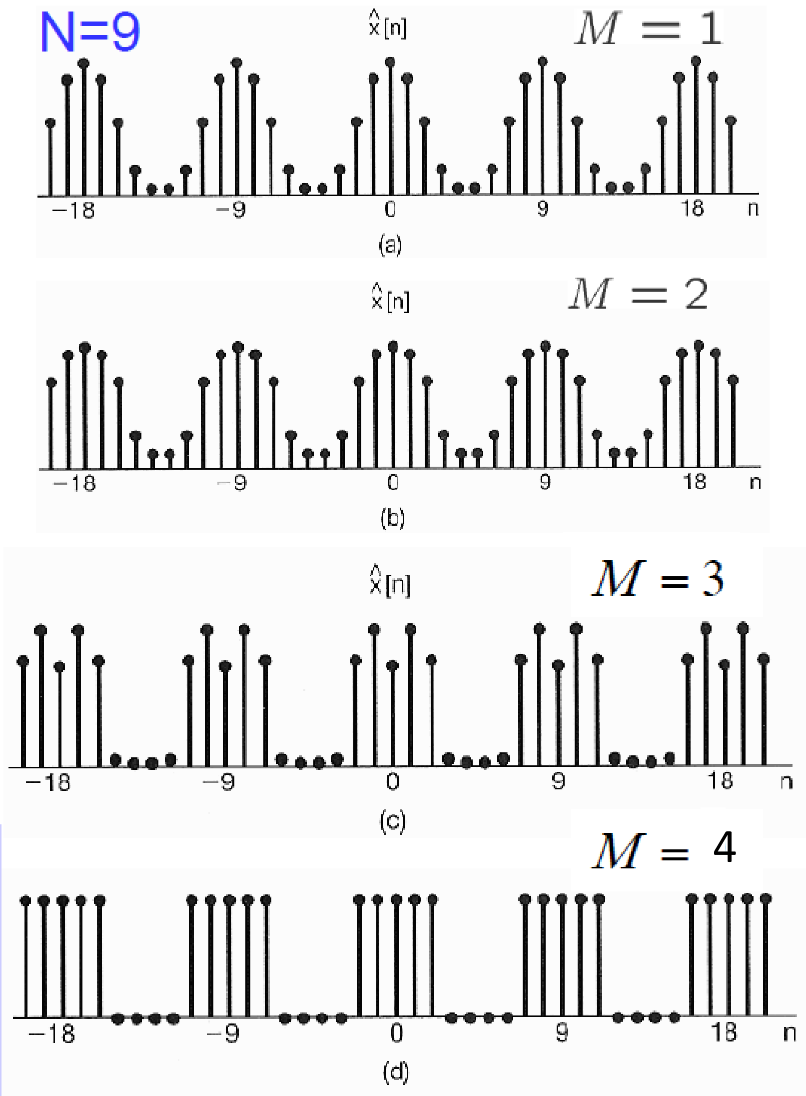
$$x[n] = \sum_{k=<N>} a_k e^{jk(\frac{2\pi}{N})n}$$

If N is odd

$$\hat{x}[n] = \sum_{k=-M}^{M} a_k e^{jk(\frac{2\pi}{N})n}$$

If N is even

$$\hat{x}[n] = \sum_{k=-M+1}^{M} a_k e^{jk(\frac{2\pi}{N})n}$$

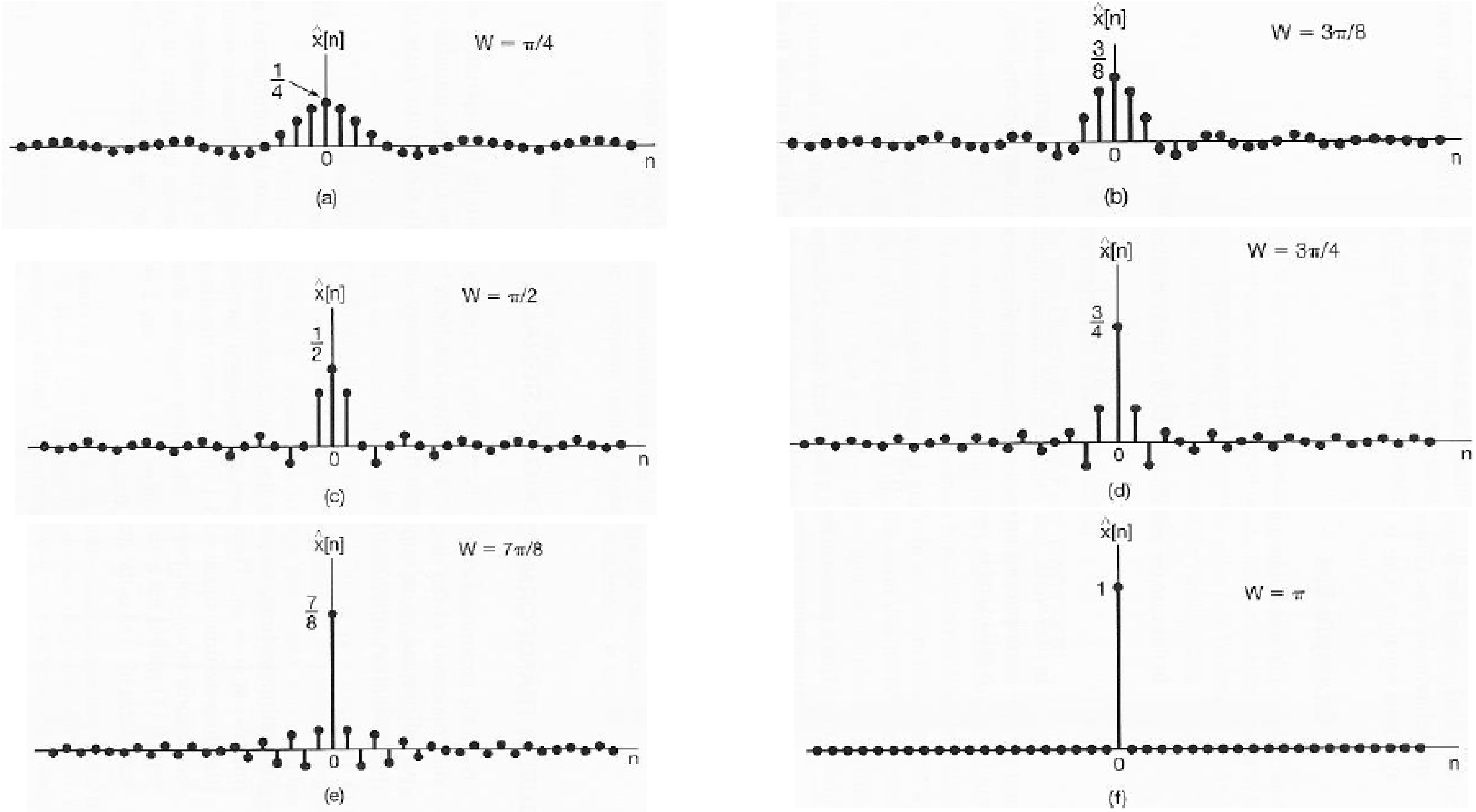


Gibbs phenomenon does not exist for DT signals because DT signals are represented by a finite number of FS coefficients. For the same reason, there is no convergence issue with DTFS.

- Convergence of DTFT

$$x[n]=\delta[n] \xleftrightarrow{F} X(e^{j\omega})=1$$

$$\hat{x}[n]=\frac{1}{2\pi}\int_{-W}^{W} X(e^{j\omega})e^{j\omega n}d\omega=\frac{\sin Wn}{\pi n}$$



Sect. 5.2 FT for *Periodic* Signals

Reall:

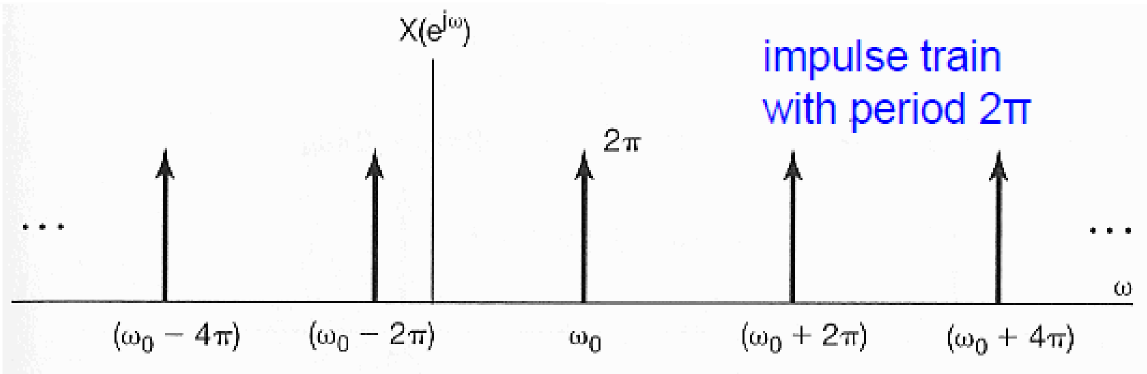
$$e^{j\omega_0 t} \xleftrightarrow{F} 2\pi\delta(\omega-\omega_0)$$

in the CT domain.

- FT from FS

$$x[n] = e^{j\omega_0 n}, \quad \omega_0 = \frac{2\pi}{N}$$

$$\Rightarrow X(e^{j\omega}) = \sum_{l=-\infty}^{+\infty} 2\pi\delta(\omega-\omega_0-2\pi l)$$



Proof:

$$\frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}d\omega = \frac{1}{2\pi}\int_{2\pi}\sum_{l=-\infty}^{+\infty} 2\pi\delta(\omega-\omega_0-2\pi l)e^{j\omega n}d\omega$$

$$= e^{j(\omega_0+2\pi r)n} = e^{j\omega_0 n}$$

Sect. 5.2 FT for *Periodic* Signals

- FT from FS (cont'd)

Thus, for a periodic sequence $x[n]$ with period N and with the FS representation

$$x[n] = \sum_{k=<N>} a_k e^{jk\omega_0 n},$$

$a_{k+N} = a_k$

its FT is related to its Foureir coefficient by

$$\boxed{X(e^{j\omega}) = \sum_{k=-\infty}^{+\infty} 2\pi a_k \delta(\omega - k\omega_0).}$$

The FT of a periodic signal can be directly constructed from its Fourier coefficients.

We can verify this equation graphically by expressing $x[n]$ as

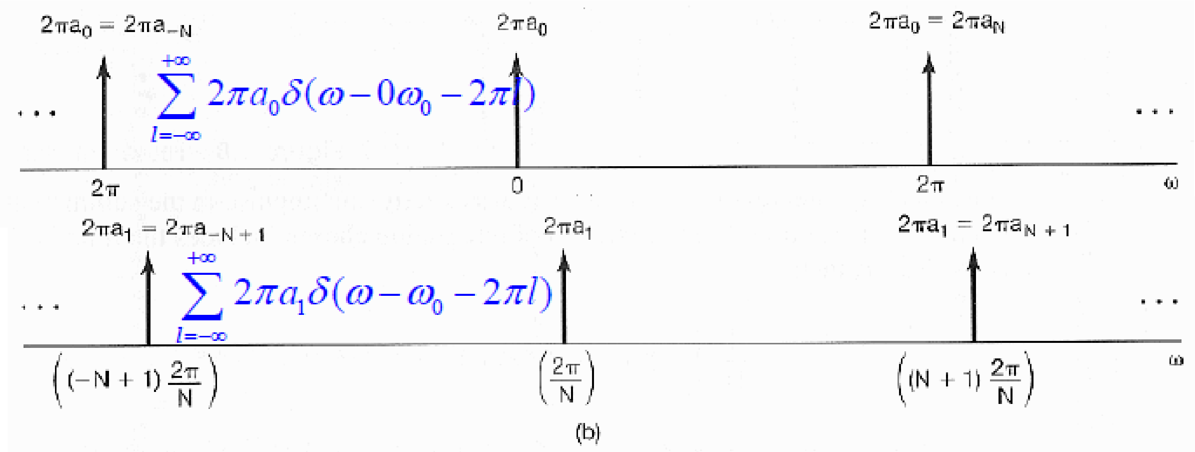
$$x[n] = a_0 + a_1 e^{j\omega_0 n} + a_2 e^{j2\omega_0 n} + \ldots + a_{N-1} e^{j(N-1)\omega_0 n},$$

plot the FT of each term, and then superimpose them.

- FT from FS (cont'd)

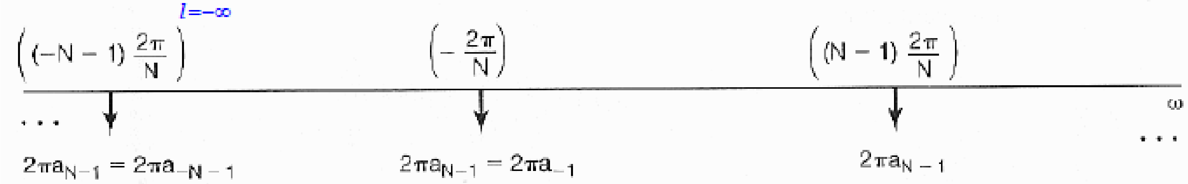
$$a_0 \overset{F}{\longleftrightarrow} \sum_{l=-\infty}^{+\infty} 2\pi a_0 \delta(\omega - 0\omega_0 - 2\pi l)$$

$$a_1 e^{j\omega_0 n} \overset{F}{\longleftrightarrow} \sum_{l=-\infty}^{+\infty} 2\pi a_1 \delta(\omega - \omega_0 - 2\pi l)$$

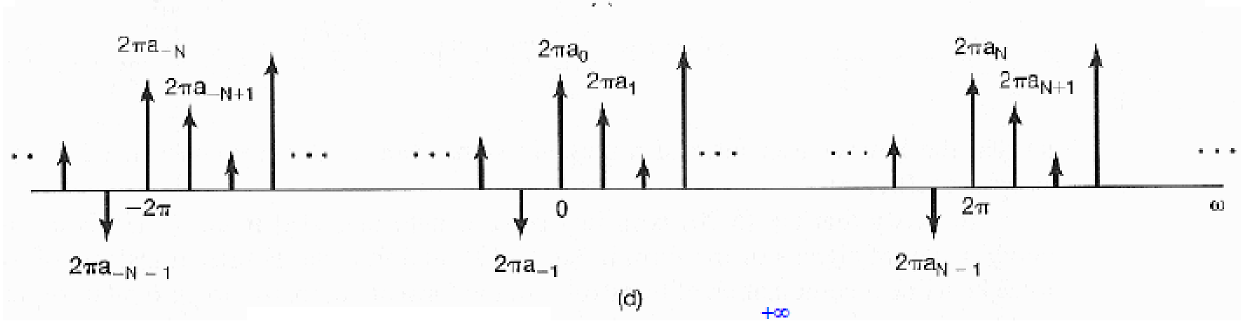


…

$$a_{N-1} e^{j(N-1)\omega_0 n} \overset{F}{\longleftrightarrow} \sum_{l=-\infty}^{+\infty} 2\pi \delta(\omega - (N-1)\omega_0 - 2\pi l)$$



$$\sum_{k=<N>} a_k e^{jk\omega_0 n} \overset{F}{\longleftrightarrow}$$

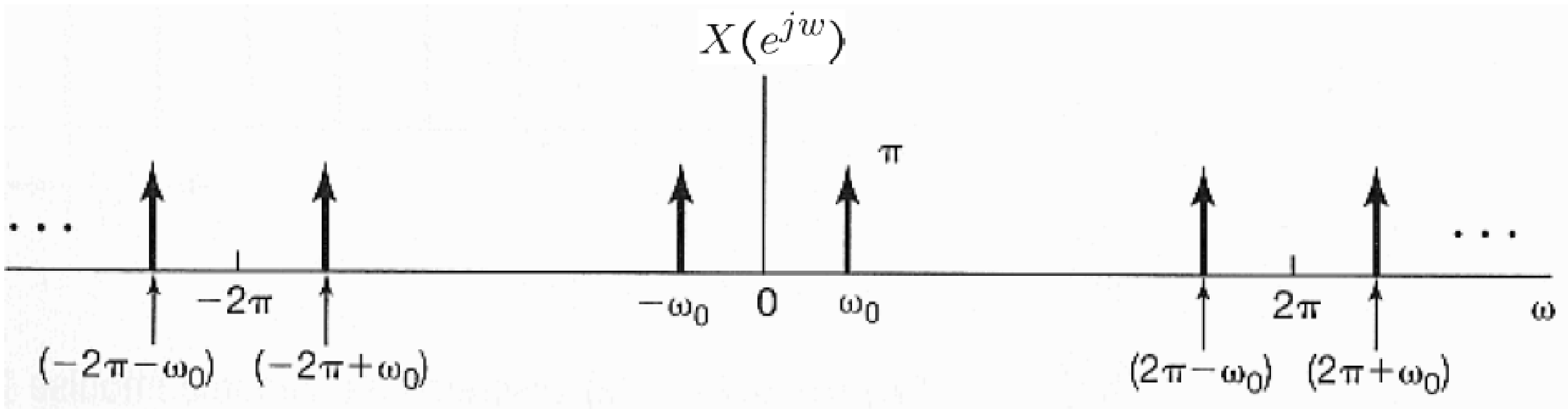


$$a_{k+N} = a_k \quad \Rightarrow \quad X(e^{j\omega}) = \sum_{k=-\infty}^{+\infty} 2\pi a_k \delta(\omega - k\omega_0)$$

- Example 5.5

$$x[n]=\cos(\omega_0 n)=\frac{e^{j\omega_0 n}+e^{-j\omega_0 n}}{2}, \qquad \omega_0=\frac{2\pi}{5}$$

$$X(e^{j\omega})=\sum_{l=-\infty}^{+\infty}\pi\delta(\omega-\frac{2\pi}{5}-2\pi l)+\sum_{l=-\infty}^{+\infty}\pi\delta(\omega+\frac{2\pi}{5}-2\pi l)$$



- Example 5.6 DTFT of Impulse Trains

$$x[n]=\sum_{k=-\infty}^{+\infty}\delta[n-kN]$$

$$\Rightarrow a_k=\frac{1}{N}\sum_{n=<N>}x[n]e^{-jk(2\pi/N)n}$$

$$=\frac{1}{N}\sum_{n=0}^{N-1}x[n]e^{-jk(2\pi/N)n}$$

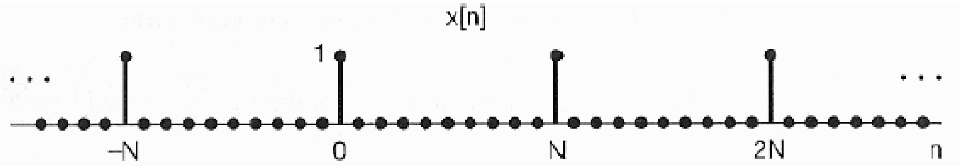
$$=\frac{1}{N}$$

$$\Rightarrow X(e^{j\omega})=\frac{2\pi}{N}\sum_{k=-\infty}^{+\infty}\delta(\omega-k\frac{2\pi}{N})$$

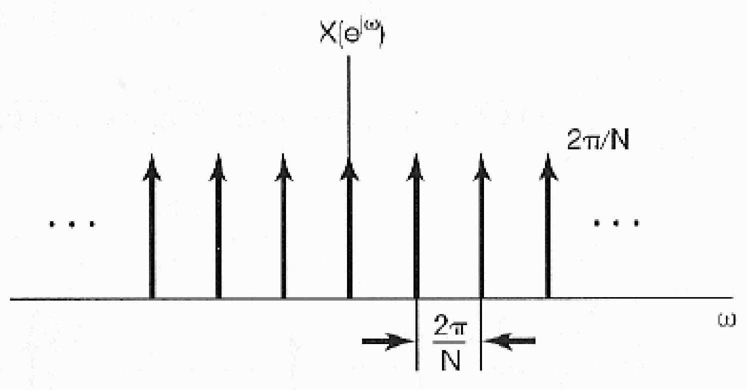
$$x[n]=\sum_{k=<N>}a_k e^{jk(2\pi/N)n}$$

$$a_k=\frac{1}{N}\sum_{n=<N>}x[n]e^{-jk(2\pi/N)n}$$

$$X(e^{j\omega})=\sum_{k=-\infty}^{+\infty}2\pi a_k\delta(\omega-\frac{2\pi k}{N})$$



(a)



(b)

Sect. 5.3 Properties of DTFT

- Recall that…

Synthesis equation

IFT $$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}\,d\omega$$

Analysis equation

DTFT $$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n]e^{-j\omega n}$$

$$X(e^{j\omega}) = F\{x[n]\}$$

$$x[n] = F^{-1}\{X(e^{j\omega})\}$$

$$x[n] \overset{F}{\longleftrightarrow} X(e^{j\omega})$$

$$\frac{1}{1-ae^{j\omega}} = F\{a^n u[n]\}, \quad |a|<1$$

$$a^n u[n] = F^{-1}\{\frac{1}{1-ae^{j\omega}}\}$$

$$a^n u[n] \overset{F}{\longleftrightarrow} \frac{1}{1-ae^{j\omega}}$$

Sect. 5.3 Properties of DTFT (cont'd)

- Periodicity of DT Fourier Transform:

$$X(e^{j(\omega+2\pi)}) = X(e^{j\omega})$$

- Linearity:

$$x[n] \xleftrightarrow{F} X(e^{j\omega})$$

$$y[n] \xleftrightarrow{F} Y(e^{j\omega})$$

$$\Rightarrow ax[n] + by[n] \xleftrightarrow{F} aX(e^{j\omega}) + bY(e^{j\omega})$$

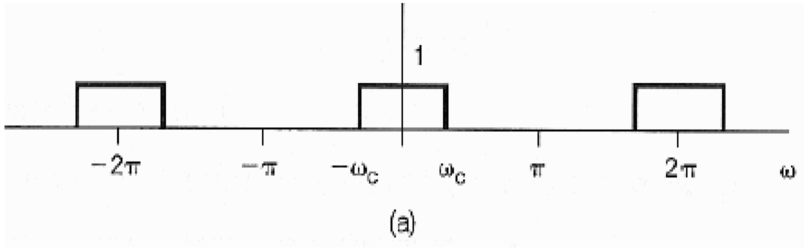
- Time & Frequency Shifting:

$$x[n-n_0] \xleftrightarrow{F} e^{-j\omega n_0} X(e^{j\omega})$$

$$e^{j\omega_0 n} x[n] \xleftrightarrow{F} X(e^{j(\omega-\omega_0)}) \rightsquigarrow \text{AM}$$

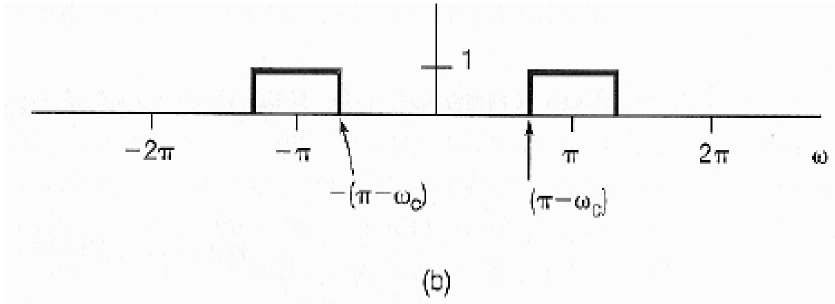
- Example 5.7 Relationship between LPF & HPF

$$H_{lp}(e^{j\omega})$$



(a)

$$H_{hp}(e^{j(\omega-\pi)})$$



(b)

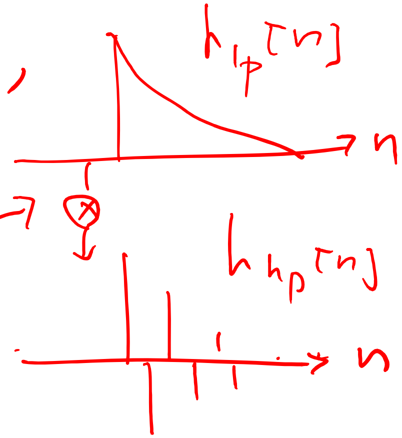
$$H_{hp}(e^{j\omega}) = H_{lp}(e^{j(\omega-\pi)})$$

$$\Rightarrow h_{hp}[n] = e^{j\pi n} h_{lp}[n]$$

$$= \underline{\underline{(-1)^n}}\, h_{lp}[n]$$

$$e^{j\pi n} = \cos(\pi n) + j\sin(\pi n)$$

E.g.,



Sect. 5.3 Properties of DTFT (cont'd)

- Conjugation & Conjugate Symmetry

$$x[n] \xleftrightarrow{F} X(e^{j\omega}) \quad \Rightarrow \quad x^*[n] \xleftrightarrow{F} X^*(e^{-j\omega})$$

If $x[n]$ is real, then $x[n] = x^*[n]$ and $X(e^{-j\omega}) = X^*(e^{j\omega})$.

That is, $X(e^{j\omega})$ is conjugate symmetric and

$$Ev\{x[n]\} \xleftrightarrow{F} Re\{X(e^{j\omega})\}$$

$$Od\{x[n]\} \xleftrightarrow{F} jIm\{X(e^{j\omega})\}$$

Let $X(e^{j\omega}) = Re\{X(e^{j\omega})\} + jIm\{X(e^{j\omega})\}$

$$\Rightarrow \quad Re\{X(e^{j\omega})\} = Re\{X(e^{-j\omega})\}$$

$$\Rightarrow \quad Im\{X(e^{j\omega})\} = -Im\{X(e^{-j\omega})\}$$

Real part is an even function
Imaginary part is an odd function

Let $X(e^{j\omega}) = \left|X(e^{j\omega})\right| e^{\angle X(e^{j\omega})}$

$$\Rightarrow \quad \left|X(e^{j\omega})\right| \text{ even}, \quad \angle X(e^{j\omega}) \text{ odd}$$

Magnitude: an even function
Phase: an odd function

Sect. 5.3 Properties of DTFT (cont'd)

- Conjugation & Conjugate Symmetry

$$x[n] \xleftrightarrow{F} X(e^{j\omega}) \quad \Rightarrow \quad x^*[n] \xleftrightarrow{F} X^*(e^{-j\omega})$$

- If $x[n] = x^*[n]$ and $x[-n] = x[n]$

$\Rightarrow X(e^{-j\omega}) = X^*(e^{j\omega})$ and $X(e^{-j\omega}) = X(e^{j\omega})$

$\Rightarrow X(e^{j\omega}) = X^*(e^{j\omega})$

$\Rightarrow$ If $x[n]$ is real and even, then $X(e^{j\omega})$ is real and even.

- If $x[n]$ is real and odd, then $X(e^{j\omega})$ is pure imaginary and odd.

Sect. 5.3 Properties of DTFT (cont'd)

- Differencing & Accumulation

$$x[n] \xleftrightarrow{F} X(e^{j\omega})$$

$$x[n]-x[n-1] \xleftrightarrow{F} (1-e^{-j\omega})X(e^{j\omega})$$

$$y[n]=\sum_{m=-\infty}^{n} x[m] \xleftrightarrow{F} \frac{1}{(1-e^{-j\omega})}X(e^{j\omega}) + \underbrace{\pi X(e^{j0})\sum_{k=-\infty}^{+\infty}\delta(\omega-2\pi k)}_{\text{dc value}}$$

Sect. 5.3 Properties of DTFT (cont'd)

- Differentiation in Frequency

$$x[n] \xleftrightarrow{F} X(e^{j\omega}) \Rightarrow nx[n] \xleftrightarrow{F} j\frac{d}{d\omega}X(e^{j\omega})$$

Proof:

$$\frac{d}{d\omega}X(e^{j\omega}) = \frac{d}{d\omega}\sum_{n=-\infty}^{+\infty} x[n]e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{+\infty}(-jn)x[n]e^{-j\omega n} = (-j)\sum_{n=-\infty}^{+\infty}\left(nx[n]\right)e^{-j\omega n}$$

- Time Reversal

$$x[n] \xleftrightarrow{F} X(e^{j\omega}) \Rightarrow x[-n] \xleftrightarrow{F} X(e^{-j\omega})$$

Proof:

$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n]e^{-j\omega n}, \quad X(e^{j(-\omega)}) = \sum_{n=-\infty}^{+\infty} x[n]e^{-j(-\omega)n}$$

Sect. 5.3 Properties of DTFT (cont'd)

- Time Expansion

$$x[n] \Rightarrow x[an] = ?$$

If a is an integer and a>1, $x[an]$ is a time-compressed version of $x[n]$. For example, $x[2n]$ is the even samples of $x[n]$.

However, if a is not an integer, the value of $x[an]$ is unknown because discrete-time signals are defined over integer intervals. Consequently, we cannot slow down the signal by making $a < 1$.

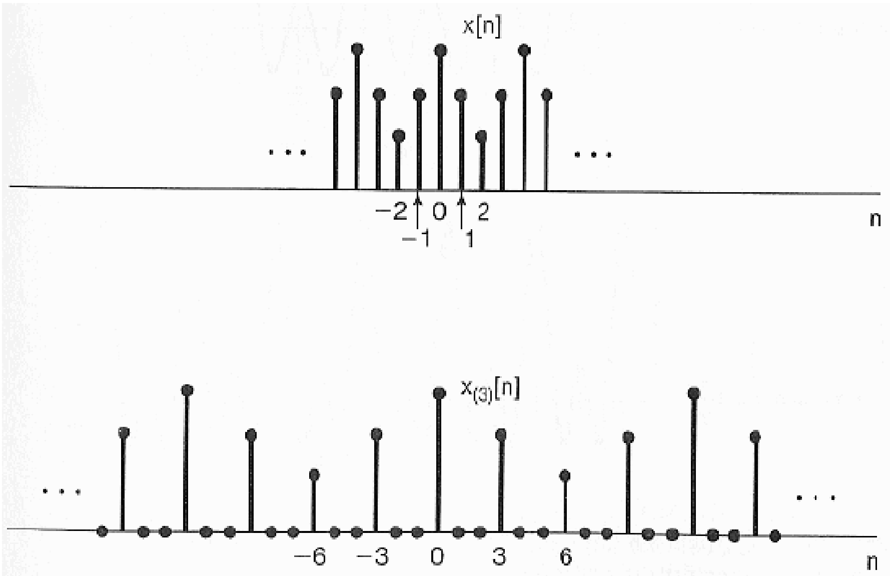
We resort to an alternative method (on next page).

Sect. 5.3 Properties of DTFT (cont'd)

- Time Expansion

Define

$$x_{(k)}[n] = \begin{cases} x[n/k], & \text{if n is a multiple of } k \\ 0, & \text{otherwise.} \end{cases}$$



$x_{(k)}[n]$ is obtained by placing k-1 zeros between successive samples of the original signal.

Sect. 5.3 Properties of DTFT (cont'd)

- Time Expansion

$$X_{(k)}(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x_{(k)}[n]e^{-j\omega n}$$

$$= \sum_{r=-\infty}^{+\infty} x_{(k)}[rk]e^{-j\omega rk}$$

$x_{(k)}[rk] = x[r]$

$$= \sum_{r=-\infty}^{+\infty} x[r]e^{-jk\omega r}$$

$$= X(e^{jk\omega})$$

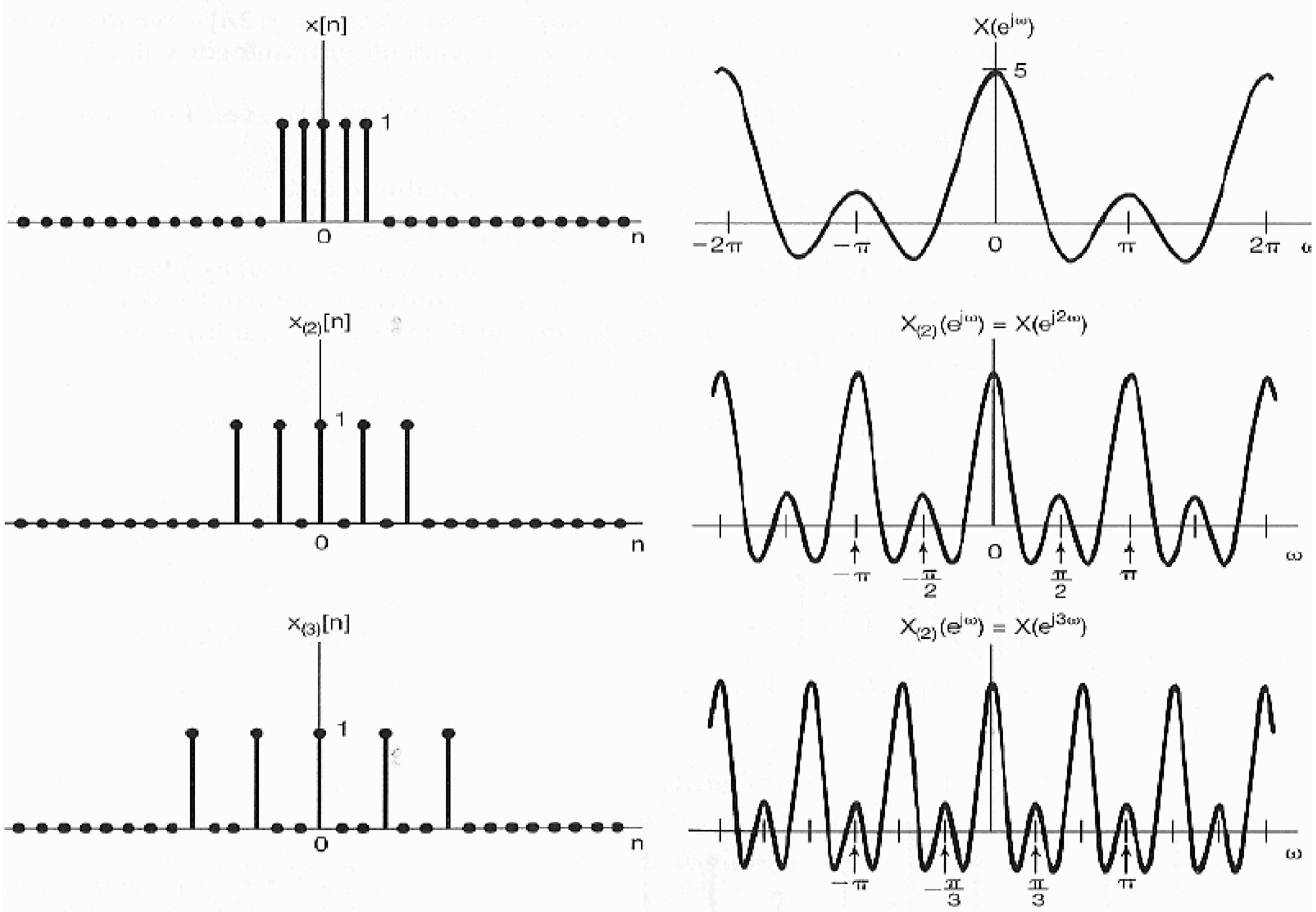
$$x_{(k)}[n] \xleftrightarrow{F} X(e^{jk\omega})$$

As a signal is spread out and slowed down in time, its FT is compressed.

Sect. 5.3 Properties of DTFT (cont'd)

- Time Expansion

Inverse relationship between the time and frequency domains

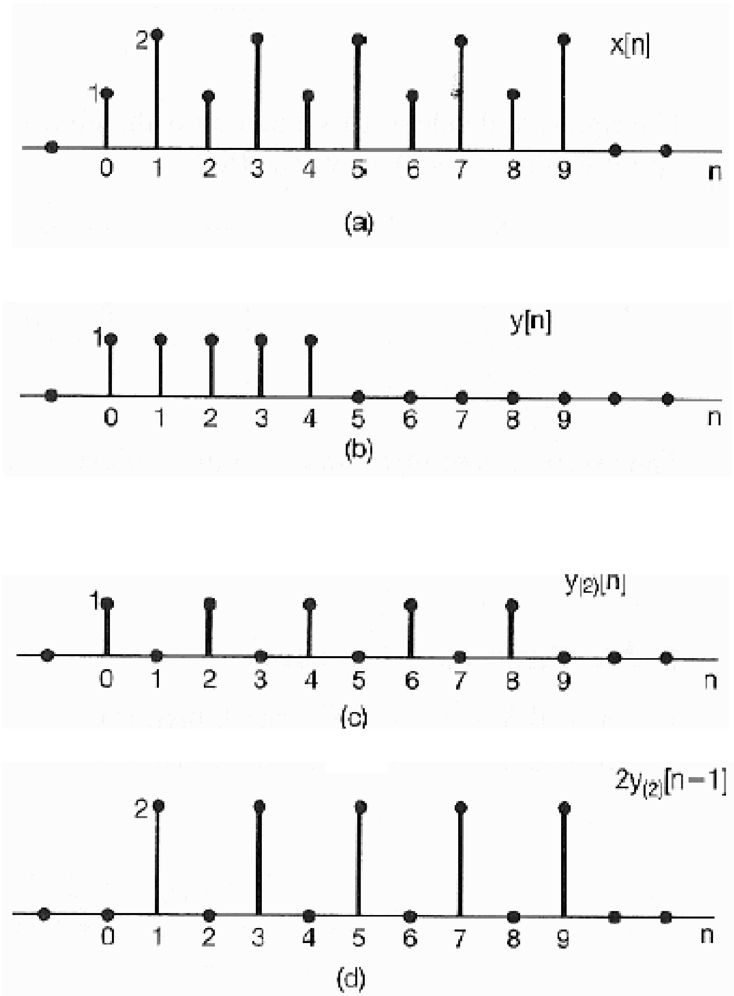


$X(e^{j\omega})$

$X(e^{j2\omega})$

$X(e^{j3\omega})$

- Example 5.9



$$x[n] = y_{(2)}[n] + 2y_{(2)}[n-1]$$

$$Y(e^{j\omega}) = e^{-j2\omega}\frac{\sin(5\omega/2)}{\sin(\omega/2)}$$

$$y_{(2)}[n] = \begin{cases} 2y[n/2], & \text{if } n \text{ is even} \\ 0, & \text{if } n \text{ is odd} \end{cases}$$

$$y_{(2)}[n] \xleftrightarrow{F} e^{-j4\omega}\frac{\sin(5\omega)}{\sin(\omega)}$$

$$2y_{(2)}[n-1] \xleftrightarrow{F} 2e^{-j\omega}e^{-j4\omega}\frac{\sin(5\omega)}{\sin(\omega)}$$

$$X(e^{j\omega}) = (1+2e^{-j\omega})\cdot e^{-j4\omega}\cdot\frac{\sin(5\omega)}{\sin(\omega)}$$

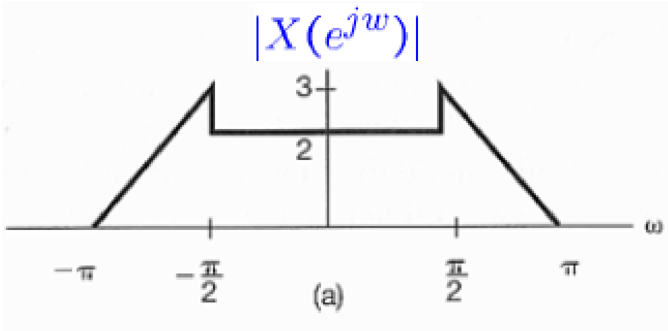
Sect. 5.3 Properties of DTFT (cont'd)

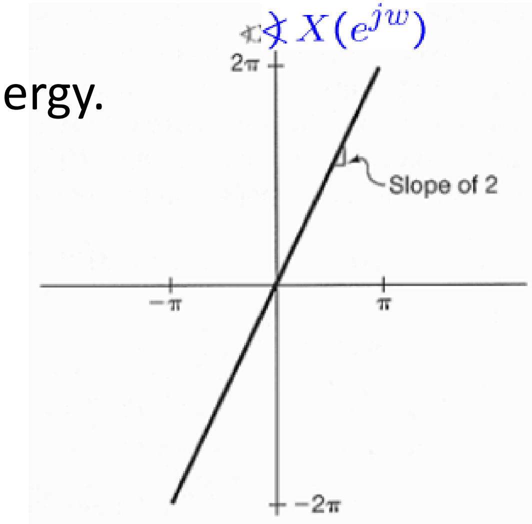
- Parseval's relation

$$x[n] \xleftrightarrow{F} X(e^{j\omega})$$

$$\underbrace{\sum_{n=-\infty}^{+\infty} |x[n]|^2}_{\text{Total energy}} = \frac{1}{2\pi}\int_{2\pi} \underbrace{|X(e^{j\omega})|^2}_{\text{Energy density spectrum}} d\omega$$

- Example 5.10
  Determine if x[n] is periodic/real/even/finite energy.





$X(e^{j\omega}) \neq$ impulse train $\Rightarrow$ $x[n]$ is NOT periodic

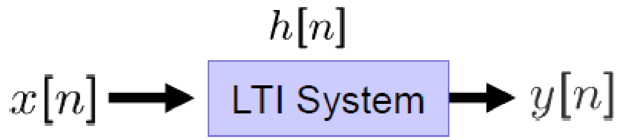
Even magnitude odd phase $\Rightarrow$ $x[n]$ is real

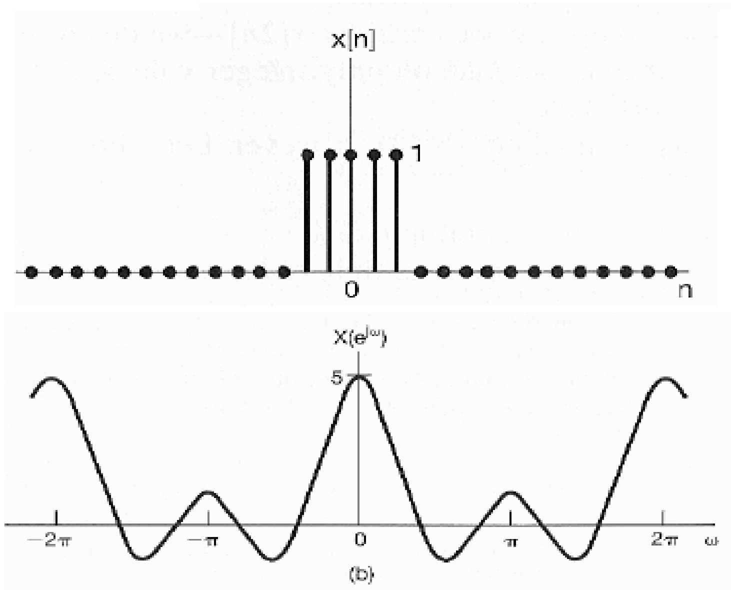
$X(e^{j\omega})$ is not real $\Rightarrow$ $x[n]$ is NOT even

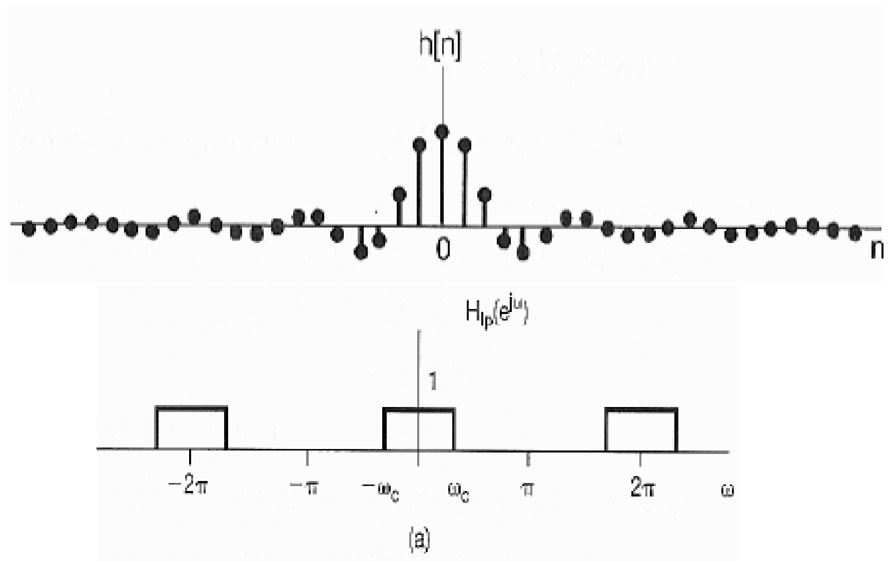
$X(e^{j\omega})$ has finite energy $\Rightarrow$ $x[n]$ is finite

Sect. 5.4 & 5.5
Convolution vs. Multiplication Property

- Convolution Property



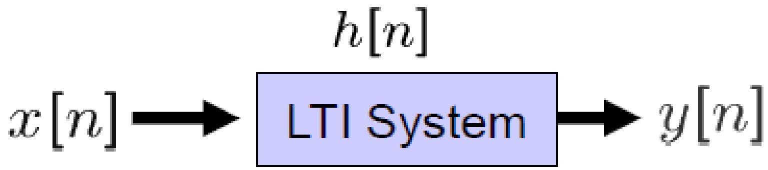




$$\boxed{y[n] = x[n] * h[n] \quad \overset{F}{\longleftrightarrow} \quad Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega})}$$

$$= \sum_{n=-\infty}^{+\infty} x[k]h[n-k]$$

- Example 5.11 Time shifting property



$$h[n] = \delta[n - n_0]$$

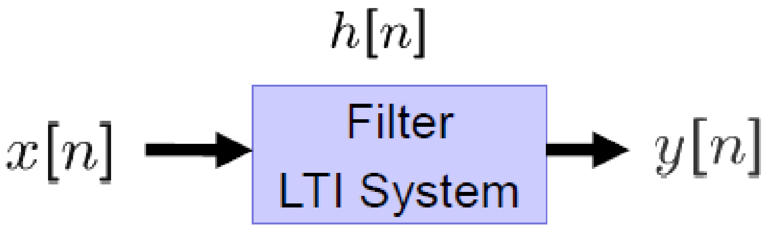
$$\Rightarrow H(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} \delta[n - n_0] e^{-j\omega n} = e^{-j\omega n_0}$$

$$\Rightarrow Y(e^{j\omega}) = H(e^{j\omega}) X(e^{j\omega})$$

$$= e^{-j\omega n_0} X(e^{j\omega})$$

$$\Rightarrow y[n] = x[n - n_0]$$

- Example 5.13 Determine y[n]



$$h[n] = a^n u[n], \quad |a| < 1 \qquad \Rightarrow H(e^{j\omega}) = \frac{1}{1 - ae^{-j\omega}}$$

$$x[n] = b^n u[n], \quad |b| < 1 \qquad \Rightarrow X(e^{j\omega}) = \frac{1}{1 - be^{-j\omega}}$$

$$\Rightarrow Y(e^{j\omega}) = H(e^{j\omega}) X(e^{j\omega})$$

$$= \frac{1}{1 - ae^{-j\omega}} \frac{1}{1 - be^{-j\omega}}$$

- Example 5.13 (cont'd)

if $a \neq b$ $$Y(e^{j\omega}) = [(\frac{a}{a-b})\frac{1}{1-ae^{-j\omega}} + (\frac{-b}{a-b})\frac{1}{1-be^{-j\omega}}]$$

$$\Rightarrow y[n] = (\frac{a}{a-b})a^n u[n] - (\frac{b}{a-b})b^n u[n]$$

if $a = b$ $$Y(e^{j\omega}) = (\frac{1}{1-ae^{-j\omega}})^2 = \frac{j}{a}e^{j\omega}\frac{d}{d\omega}(\frac{1}{1-ae^{-j\omega}})$$

since $$a^n u[n] \overset{F}{\longleftrightarrow} \frac{1}{1-ae^{-j\omega}}$$

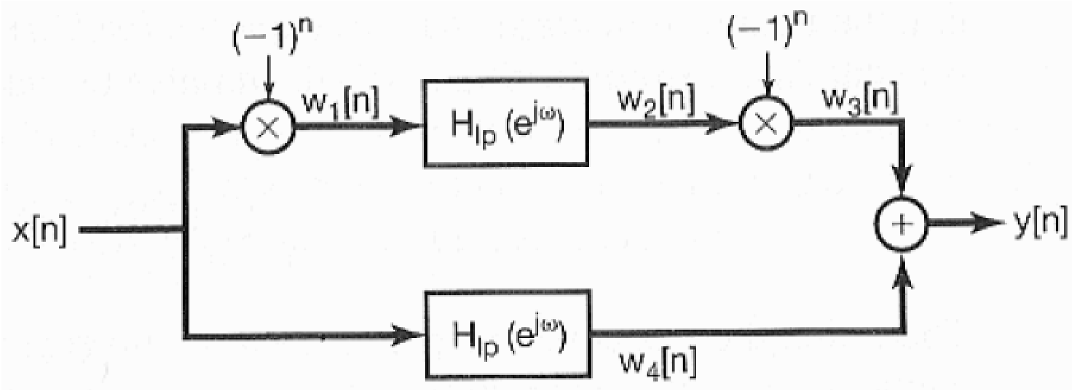
$$na^n u[n] \overset{F}{\longleftrightarrow} j\frac{d}{d\omega}(\frac{1}{1-ae^{-j\omega}})$$

$$(n+1)a^{n+1}u[n+1] \overset{F}{\longleftrightarrow} je^{j\omega}\frac{d}{d\omega}(\frac{1}{1-ae^{-j\omega}})$$

$$\Rightarrow y[n] = (n+1)a^n u[n+1] \quad = (n+1)a^n u[n]$$

- Example 5.14

$H_{lp}(e^{j\omega})$: Low-pass filter with $\omega_c = \pi/4$



$$(-1)^n = e^{j\pi n}$$

$$w_1[n] = e^{j\pi n}x[n] = (-1)^n x[n]$$

$$\Rightarrow W_1(e^{j\omega}) = X(e^{j(\omega-\pi)})$$

$$W_2(e^{j\omega}) = H_{lp}(e^{j\omega})X(e^{j(\omega-\pi)})$$

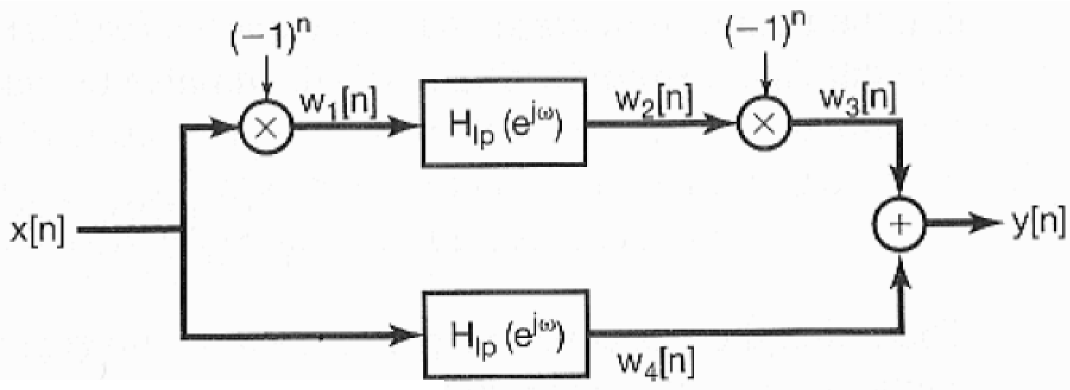
$$w_3[n] = e^{j\pi n}w_2[n] = (-1)^n w_2[n]$$

$$\Rightarrow W_3(e^{j\omega}) = W_2(e^{j(\omega-\pi)}) = H_{lp}(e^{j(\omega-\pi)})X(e^{j(\omega-2\pi)})$$

$$= H_{lp}(e^{j(\omega-\pi)})X(e^{j\omega})$$

- Example 5.14

$H_{lp}(e^{j\omega})$: Low-pass filter with $\omega_c = \pi/4$



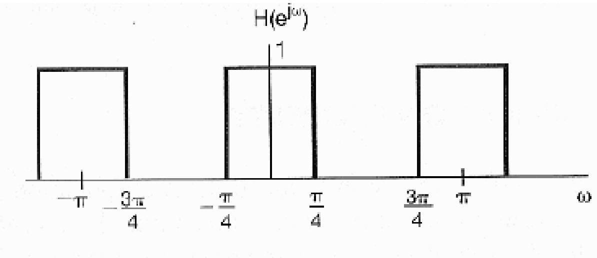
$$W_4(e^{j\omega}) = H_{lp}(e^{j\omega})X(e^{j\omega})$$

$$Y(e^{j\omega}) = W_3(e^{j\omega}) + W_4(e^{j\omega})$$

$$= H_{lp}(e^{j(\omega-\pi)})X(e^{j\omega}) + H_{lp}(e^{j\omega})X(e^{j\omega})$$

$$= [H_{lp}(e^{j(\omega-\pi)}) + H_{lp}(e^{j\omega})]X(e^{j\omega})$$

$$H(e^{j\omega}) = \underbrace{\underbrace{H_{lp}(e^{j(\omega-\pi)})}_{\text{high-pass filter}} + \underbrace{H_{lp}(e^{j\omega})}_{\text{low-pass filter}}}_{\text{bandstop filter}}$$



(b)

Sect. 5.4 & 5.5
Convolution vs. Multiplication Property

- Multiplication Property

$$\boxed{r[n] = s[n]p[n] \xleftrightarrow{F} R(e^{j\omega}) = \frac{1}{2\pi}\int_{2\pi} S(e^{j\theta})P(e^{j(\omega-\theta)})d\theta}$$

$$R(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} r[n]e^{-j\omega n} = \sum_{n=-\infty}^{+\infty} s[n]p[n]e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{+\infty} s[n]\{\frac{1}{2\pi}\int_{2\pi} P(e^{j\theta})e^{j\theta n}d\theta\}e^{-j\omega n}$$

$$= \frac{1}{2\pi}\int_{2\pi} P(e^{j\theta})[\sum_{n=-\infty}^{+\infty} s[n]e^{-j(\omega-\theta)n}]d\theta$$

$$= \frac{1}{2\pi}\int_{2\pi} P(e^{j\theta})S(e^{j(\omega-\theta)})d\theta$$

$$= \frac{1}{2\pi}\int_{2\pi} P(e^{j(\omega-\theta)})S(e^{j\theta})d\theta$$

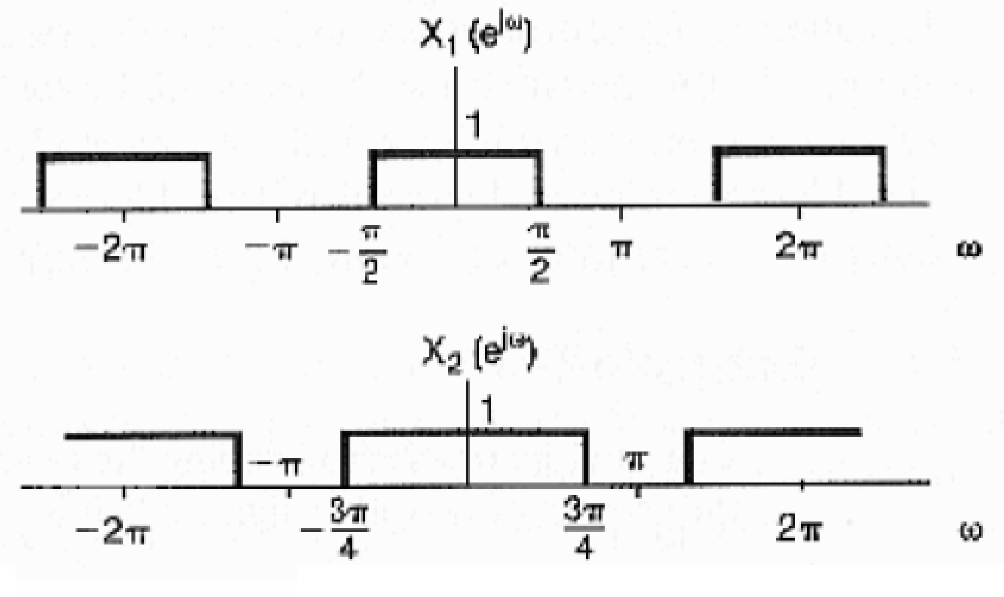
Periodic convolution

- Example 5.15
  Converting periodic convolution into ordinary convolution

$$x[n] = x_1[n]x_2[n]$$

$$x_1[n] = \frac{\sin(\frac{\pi}{2}n)}{\pi n}$$

$$x_2[n] = \frac{\sin(\frac{3\pi}{4}n)}{\pi n}$$



$$X(e^{j\omega}) = \frac{1}{2\pi}\int_{2\pi} X_1(e^{j\theta})X_2(e^{j(\omega-\theta)})d\theta$$

We can convert this equation to an ordinary convolution. Define

$$\hat{X}_1(e^{j\theta}) = \begin{cases} X_1(e^{j\theta}), & \text{for } -\pi < \theta \le \pi \\ 0, & \text{otherwise.} \end{cases}$$

- Example 5.15
  Converting periodic convolution into ordinary convolution

$$X(e^{j\omega}) = \frac{1}{2\pi}\int_{-\pi}^{+\pi} \hat{X}_1(e^{j\theta})X_2(e^{j(\omega-\theta)})d\theta$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} \hat{X}_1(e^{j\theta})X_2(e^{j(\omega-\theta)})d\theta$$