Signals & Systems

Spring 2019

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2019/03/04 & 07

Chapter 2 Linear Time Invariant Systems

- Sec. 2.1 Discrete-time LTI Systems: The Convolution Sum
- Sec. 2.2 Continuous-time LTI Systems: The Convolution Integral
- Sec. 2.3 Properties of Linear Time-invariant Systems
- Sec. 2.4 Causal LTI Systems Described by Differential and Difference Equations
- Sec. 2.5 Singularity Functions

2.1 DT LTI Systems: The Convolution Sum

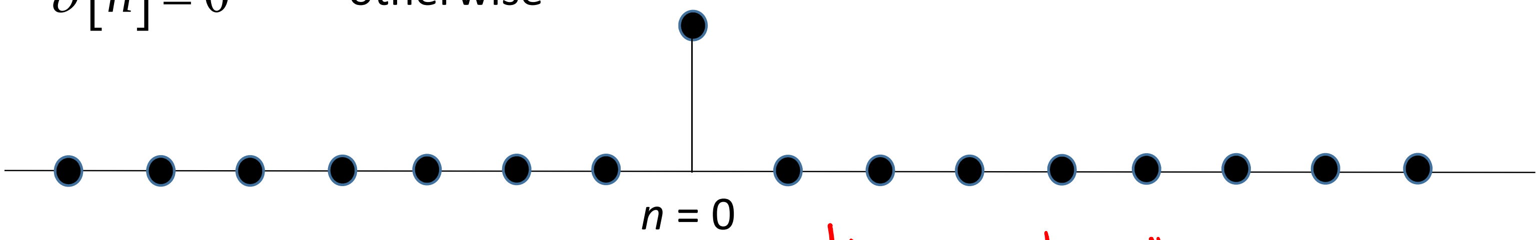
- Highlights
  - Definition of DT convolution
  - Unit impulse response
  - Any DT LTI system can be modeled by a DT convolution operation.
  - Any DT signals can be represented by a sum of impulses.

2.1.1 Representation of DT Signals in term of Impulses

- Recall that, unit impulses are…

$\delta[n] = 1$ when n = 0,

$\delta[n] = 0$ otherwise



linear combination

- Any DT signals can be represented by a sum of impulses.

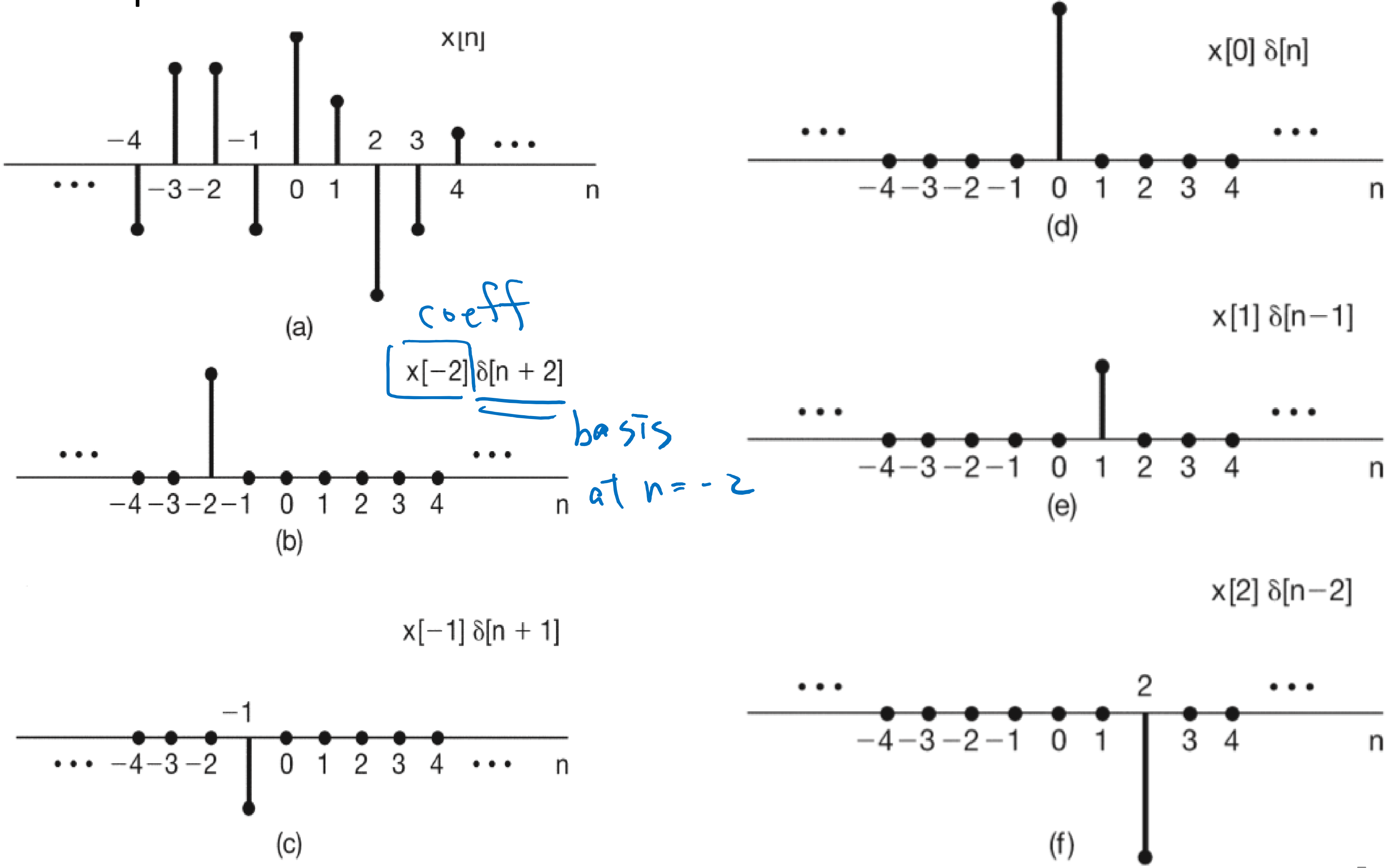
coeff

$$x[n] = \ldots + x[-3]\delta[n+3] + x[-2]\delta[n+2] + x[-1]\delta[n+1] + x[0]\delta[n] \\ + x[1]\delta[n-1] + x[2]\delta[n-2] + x[3]\delta[n-3] + \ldots$$

$$x[n] = \sum_{k=-\infty}^{+\infty} \underbrace{x[k]}_{\text{coeff}}\underbrace{\delta[n-k]}_{\text{basis}}.$$

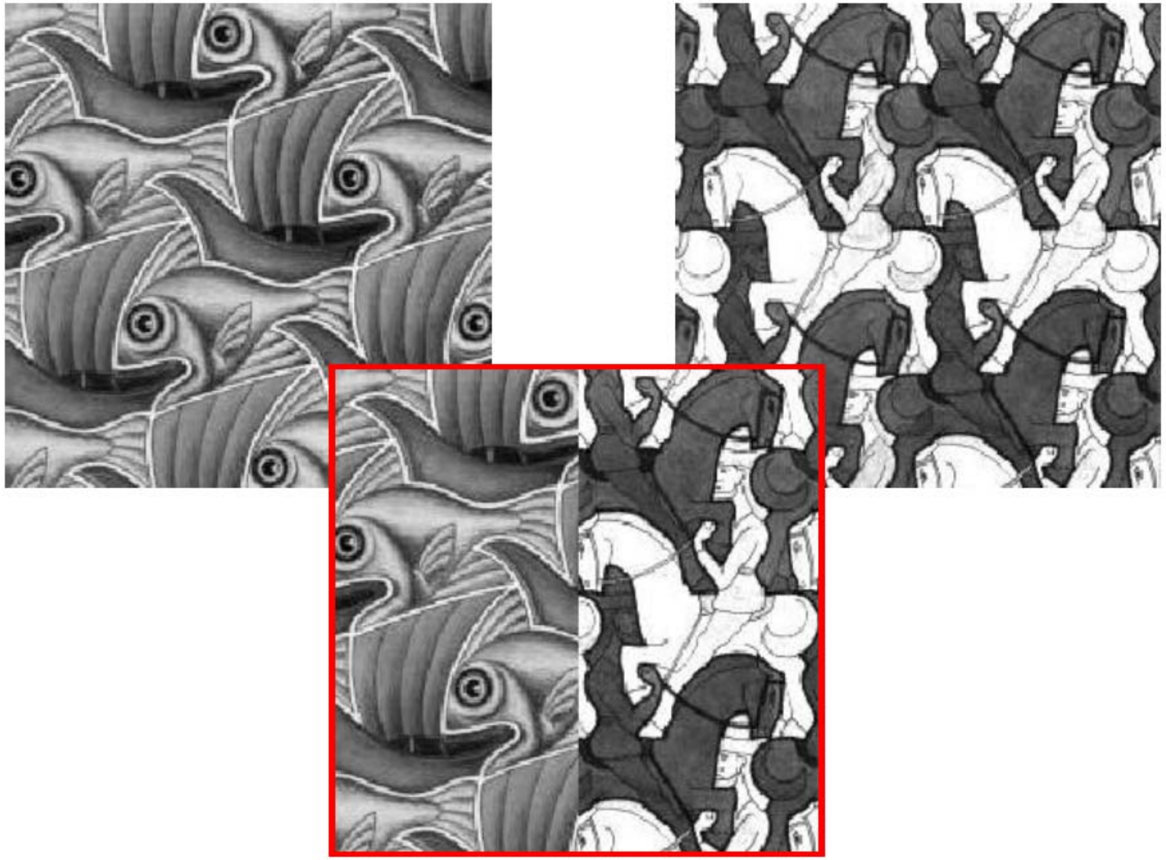
2.1.1 Representation of DT Signals in term of Impulses

- Examples



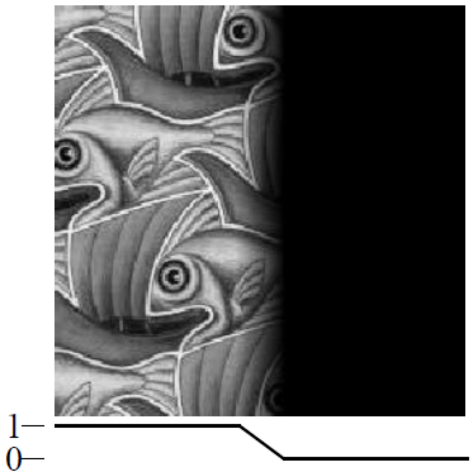
Extension:
Representation of Signals in terms of Basis Functions

- An image-based example: image blending



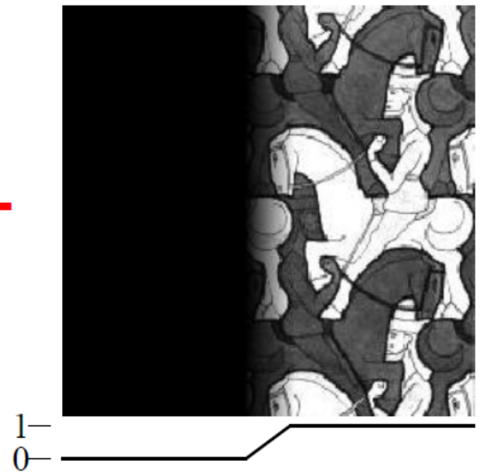
Extension:
Representation of Signals in terms of Basis Functions

- An image-based example: alpha image blending/feathering



$\alpha =$

+



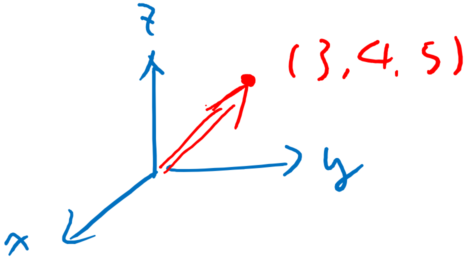
$= 1-\alpha$

=

$$I_{blend} = \alpha \underbrace{I_{left}}_{\text{basis}} + (1\text{-}\alpha)\underbrace{I_{right}}_{\text{basis}}$$

Extension:
Representation of Signals in terms of Basis Functions

- Detailed Remarks:



coeff

$$\begin{bmatrix}3\\4\\5\end{bmatrix} = 3\underbrace{\begin{bmatrix}1\\0\\0\end{bmatrix}}_{= x\text{-axis}} + 4\underbrace{\begin{bmatrix}0\\1\\0\end{bmatrix}}_{= y} + 5\underbrace{\begin{bmatrix}0\\0\\1\end{bmatrix}}_{= z}$$

basis functions

$$\underline{a} = \sum_{i=1}^{d} \alpha_i \underline{u}_i , \quad \underline{a}, \underline{u}_i \in \mathbb{R}^d$$

- Will see more in Ch. 3 Fourier Series, etc.

2.1.2 The DT Unit Impulse Response and the Convolution Sum Representation of LTI Systems

- For a DT signal, we can represent it as:

$$x[n] = \sum_{k=-\infty}^{+\infty} x[k]\delta[n-k].$$

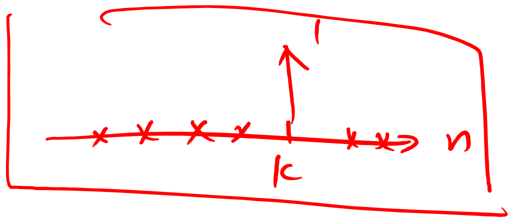
- If a system is linear, then its output corresponding to $x[n]$ can be expressed as:

$$y[n] = \sum_{k=-\infty}^{+\infty} x[k]\boxed{h_k[n]}$$

where $h_k[n]$ is the system output $y[n]$ with $\delta[n-k]$ as input.

output

- Why?



Input

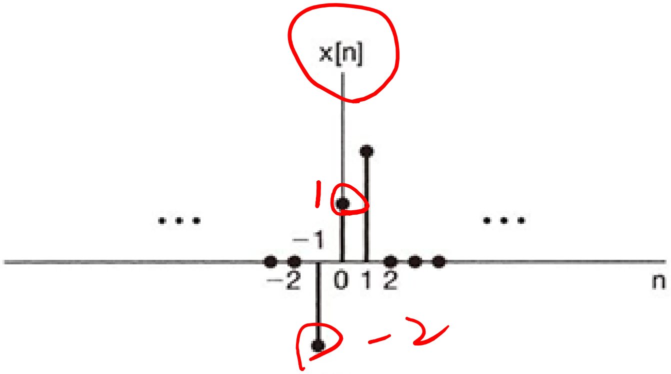
For a system that is linear but not time-invariant:
(will discuss LTI systems in a min)

Input

$$x[n] = \sum_{k=-\infty}^{+\infty} x[k]\delta[n-k]$$

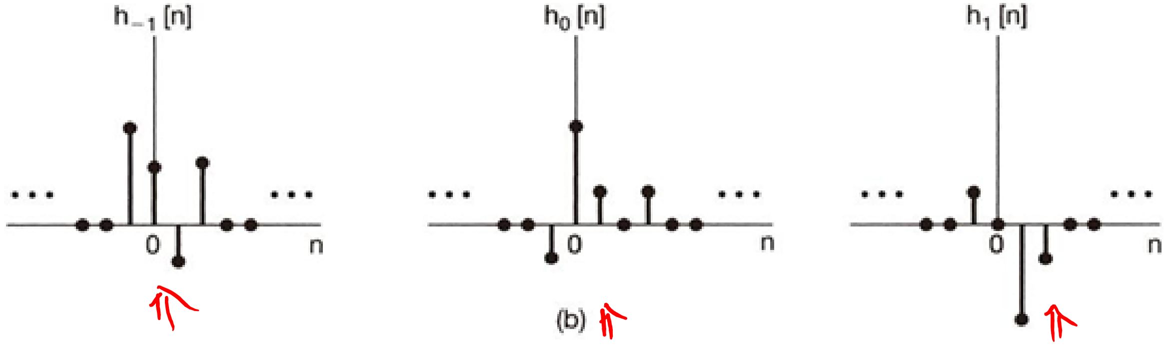
Output

$$y[n] = \sum_{k=-\infty}^{+\infty} x[k]h_k[n]$$



(a)

impulse response at k= ....



(b)

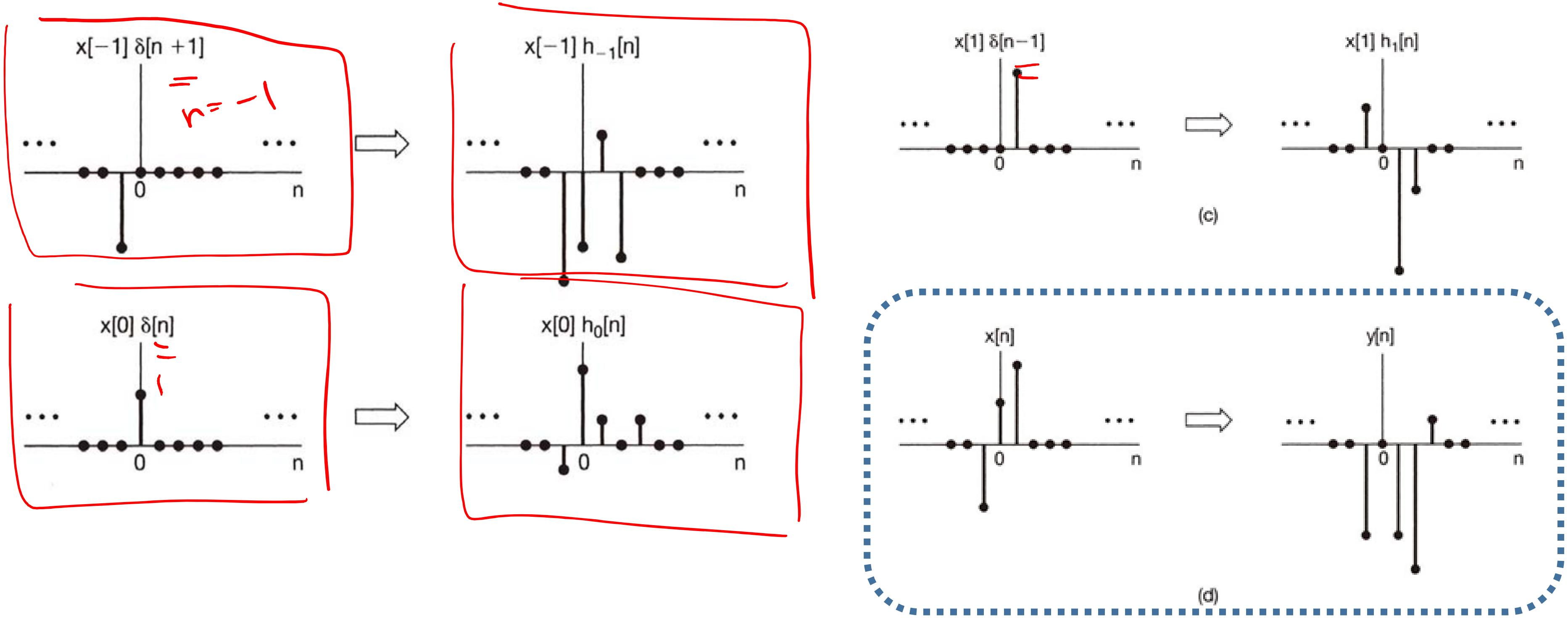
Input : $\delta[n+1]$ $\delta[n]$ $\delta[n-1]$

Input

$$x[n]=\sum_{k=-\infty}^{+\infty} x[k]\delta[n-k]$$

Output

$$y[n]=\sum_{k=-\infty}^{+\infty} x[k]h_k[n]$$



- If the aforementioned system is time-invariant, then

Input

$$x[n] = \sum_{k=-\infty}^{+\infty} x[k]\delta[n-k]$$

Output

$$y[n] = \sum_{k=-\infty}^{+\infty} x[k]h_k[n]$$

we denote the impulse response at time k as $h_k[n] = h_0[n-k]$.

- For simplicity, we drop the subscript and denote $h_0[n]$ by $h[n]$:

LTI system: $$y[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k]$$

This is the Discrete-Time Convolution.

- The convolution operator is typically denoted by *.

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k]$$

2.1 DT LTI Systems: The Convolution Sum

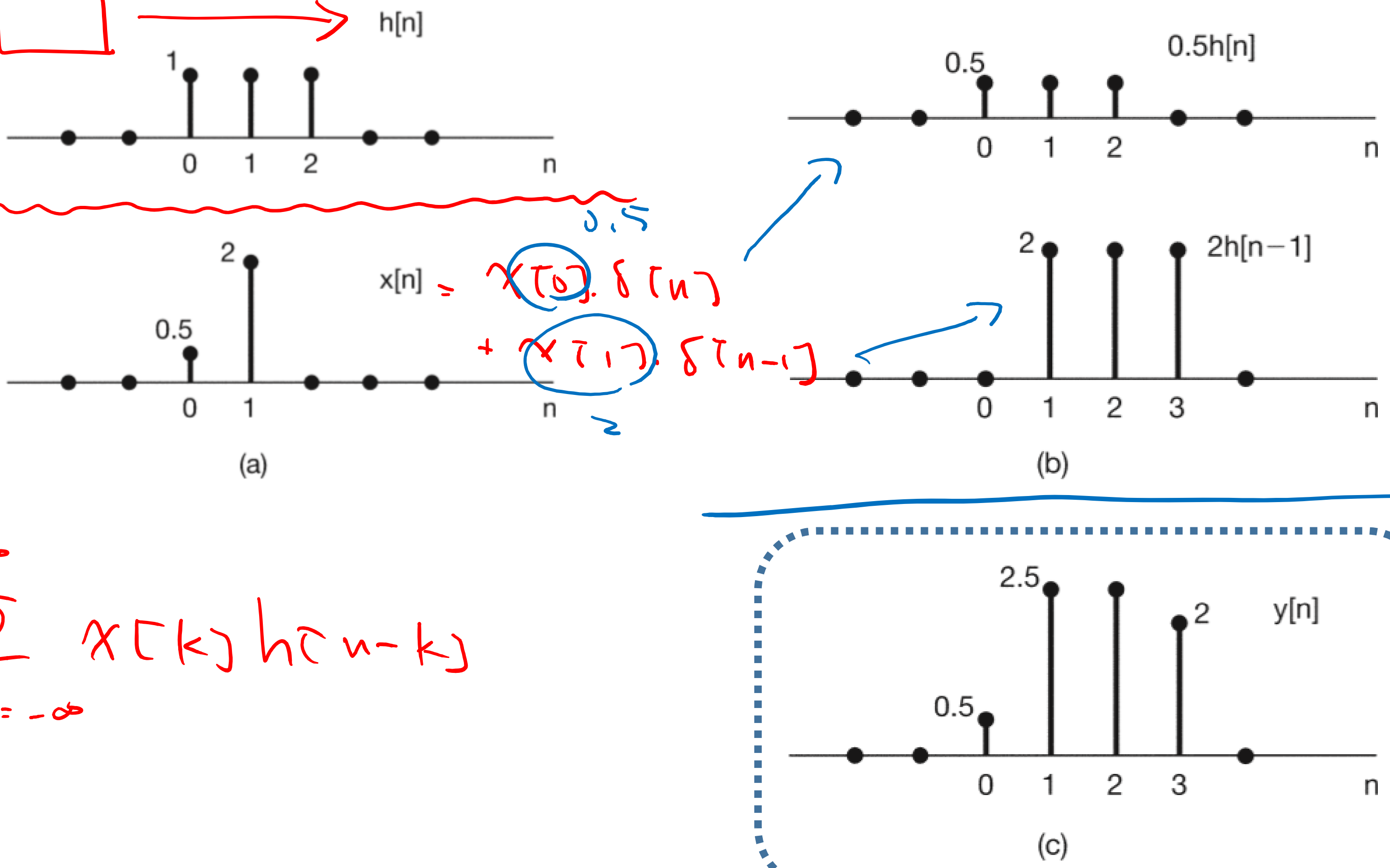
- Highlights
  - Definition of DT convolution
  - Unit impulse response
  - Any DT LTI system can be modeled by a DT convolution operation.
  - Any DT signals can be represented by a sum of impulses.

Example 2.1

Consider an LTI system with impulse response $h[n]$ and input $x[n]$.
We have

$$y[n] = x[n] * h[n] = x[0]h[n-0] + x[1]h[n-1] = 0.5h[n] + 2h[n-1].$$

$\delta[n] \rightarrow \boxed{} \rightarrow$



(a) (b) (c)

$x[n] = \underset{0.5}{x[0]} \cdot \delta[n] + \underset{2}{x[1]} \cdot \delta[n-1]$

$+$

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

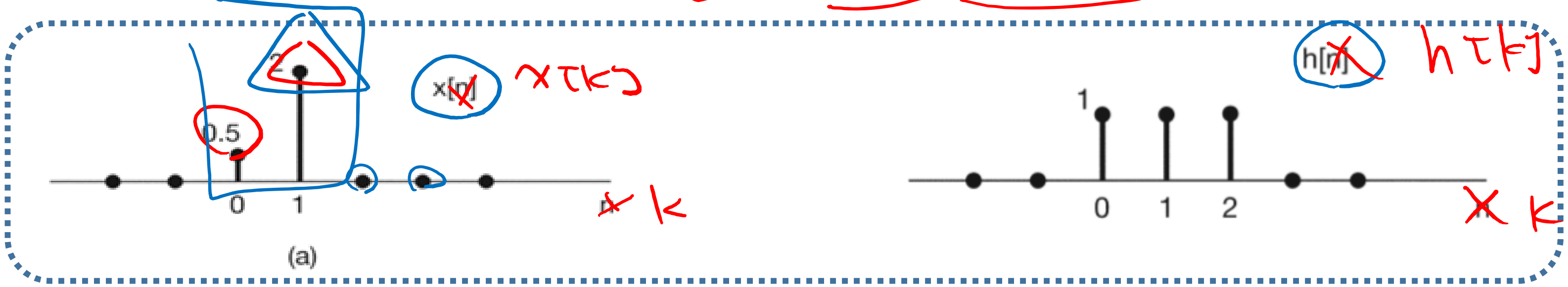
Example 2.2

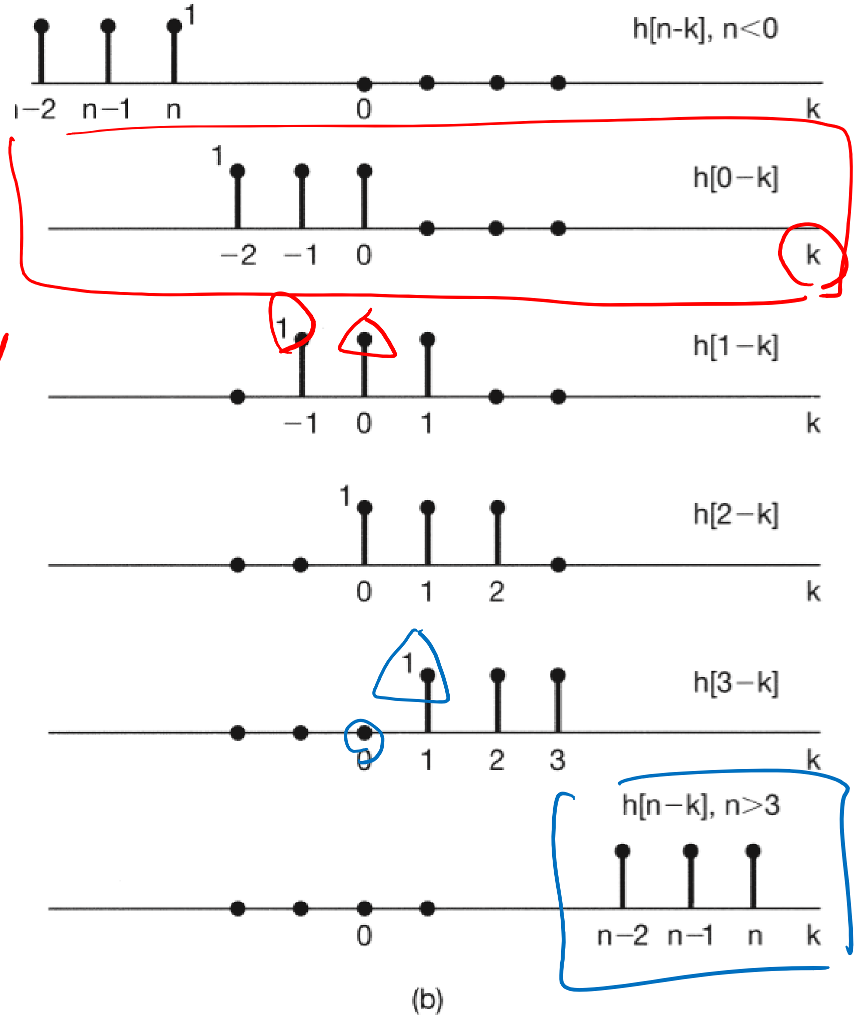
Same as Example 2.1 but from a different point of view…

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

That is, $y[n]$ is the sum of the products of $x[k]$ and $h[n-k]$ over k.

Example 2.2 $y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$





$$y[0] = \sum_{k=-\infty}^{\infty} x[k]h[0-k] = 0.5.$$

$$y[1] = \sum_{k=-\infty}^{\infty} x[k]h[1-k] = 0.5 + 2.0 = 2.5.$$

$$y[2] = \sum_{k=-\infty}^{\infty} x[k]h[2-k] = 0.5 + 2.0 = 2.5.$$

n

$$y[3] = \sum_{k=-\infty}^{\infty} x[k]h[3-k] = 2.0.$$

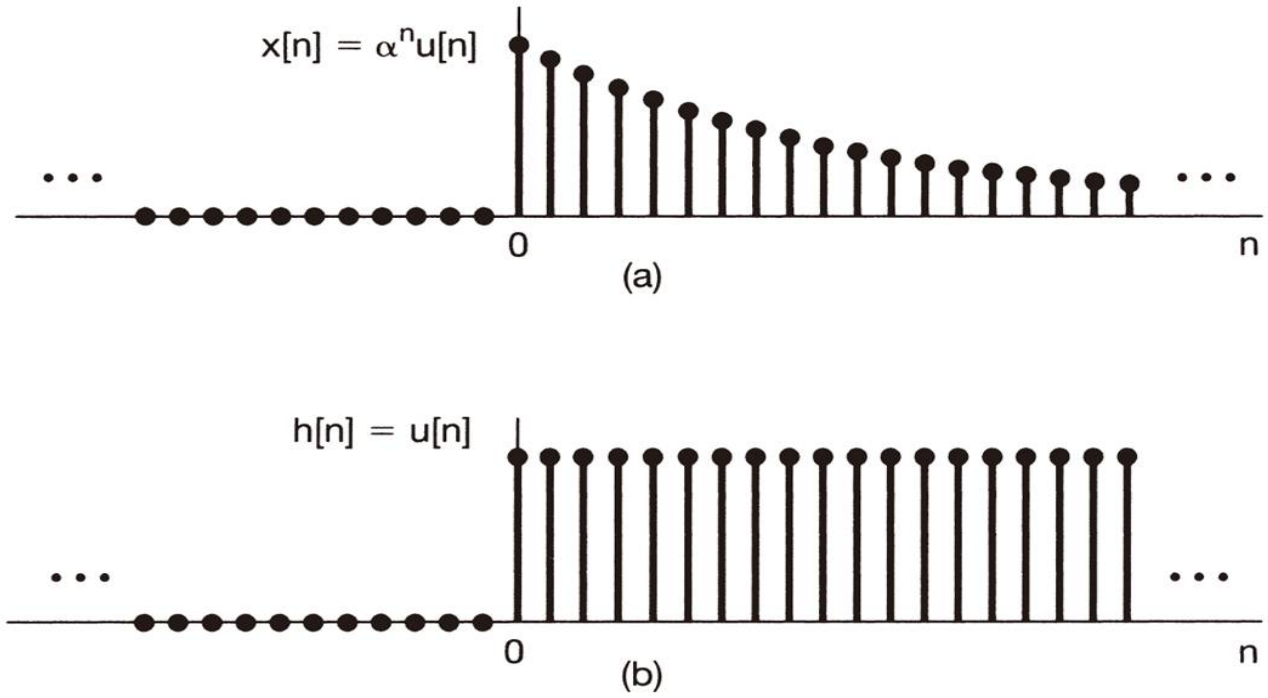
$y[n] = 0$ otherwise

Example 2.3

LTI

$$x[n]=\alpha^n u[n],$$
$$h[n]=u[n],$$

$$y[n]=\sum_{k=-\infty}^{\infty} x[k]\,h[n-k]$$



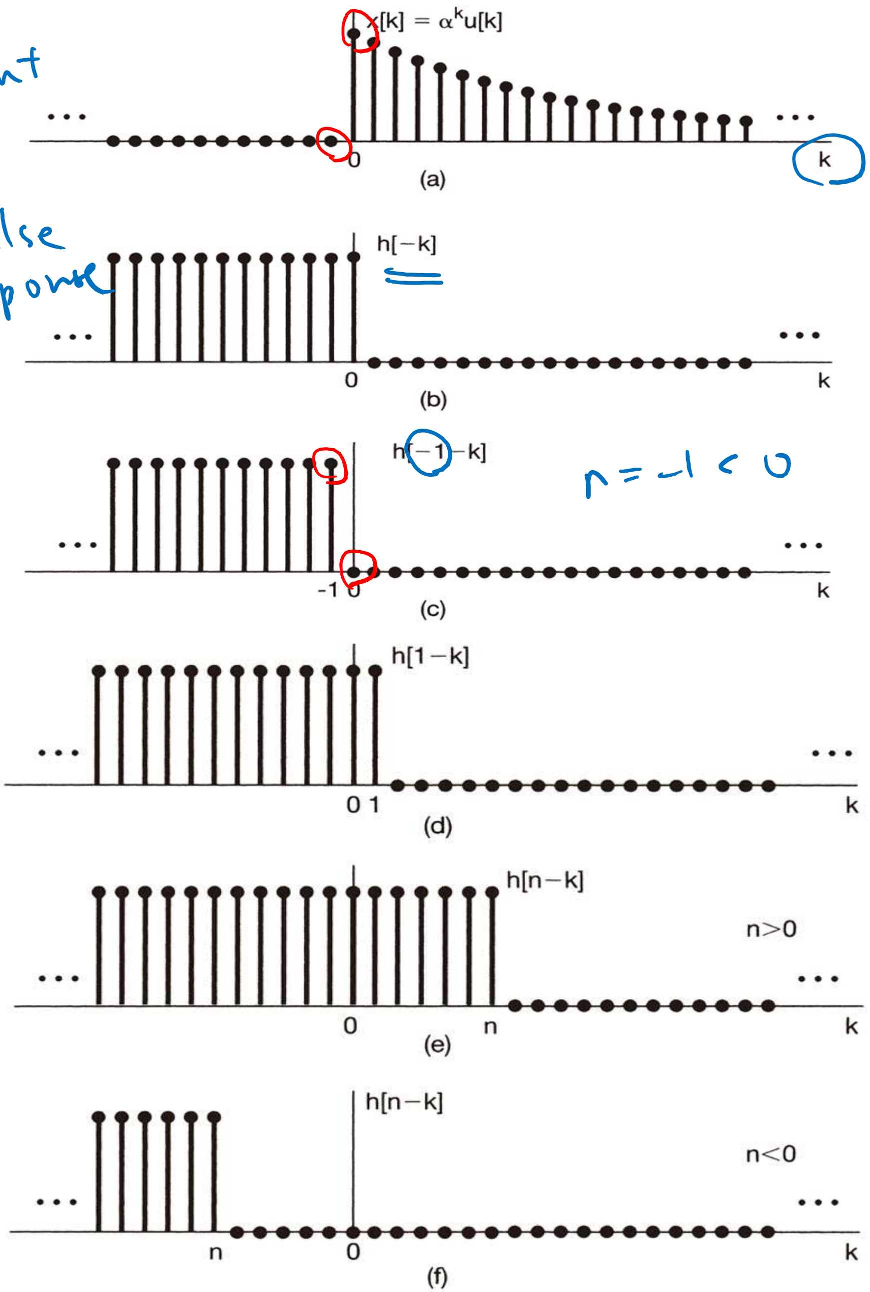
Example 2.3 (cont'd) $y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$

$$x[n] = \alpha^n u[n],$$

$$h[n] = u[n],$$

Input

Impulse response

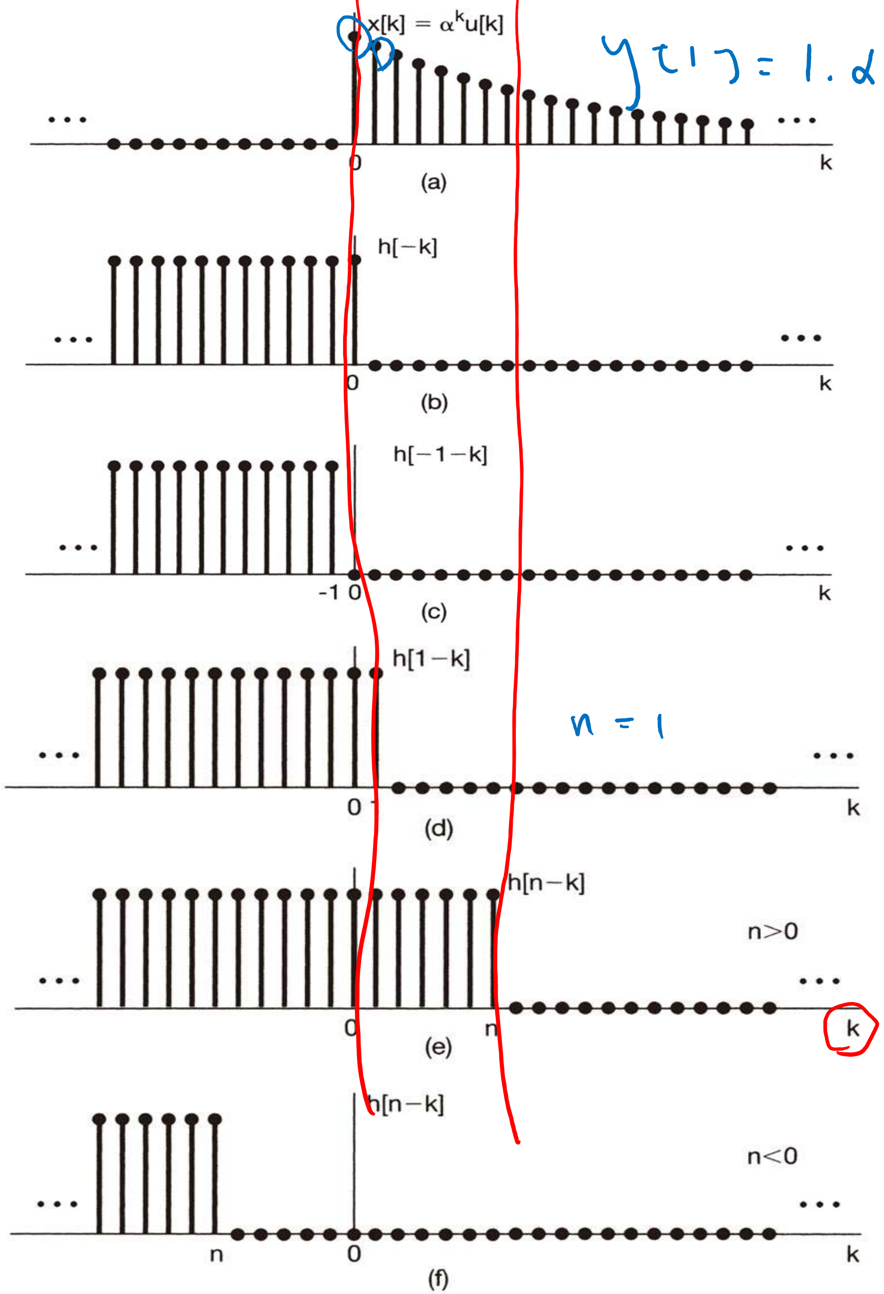


$n = -1 < 0$

Example 2.3 (cont'd)

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

$$y[1] = 1\cdot\alpha^0 + 1\cdot\alpha^1 = \sum_{k=0}^{1} \alpha^k \cdot 1$$



$$x[n] = \alpha^n u[n],$$

$$h[n] = u[n],$$

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = \sum_{k=0}^{n} \alpha^k,$$

$\boxed{n \geq 0}$

$u[k]$

$$\boxed{y[n] = \left(\frac{1-\alpha^{n+1}}{1-\alpha}\right)u[n].}$$

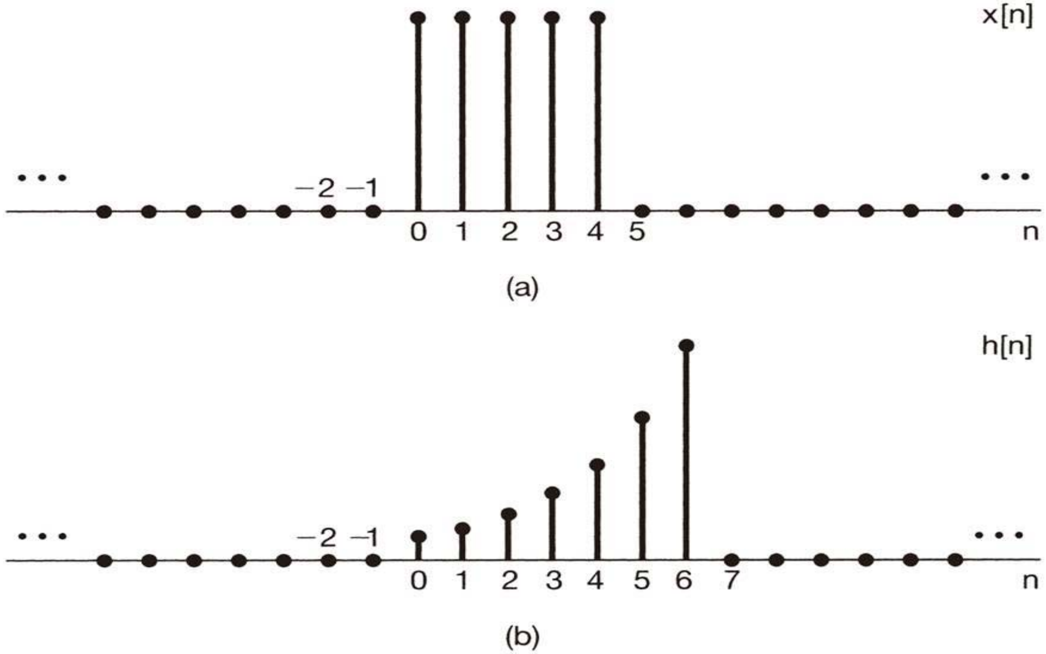
general form

0, n

$$y[n] = \begin{cases} \dfrac{1-\alpha^{n+1}}{1-\alpha}, & n \geq 0. \\ 0, & n < 0. \end{cases}$$

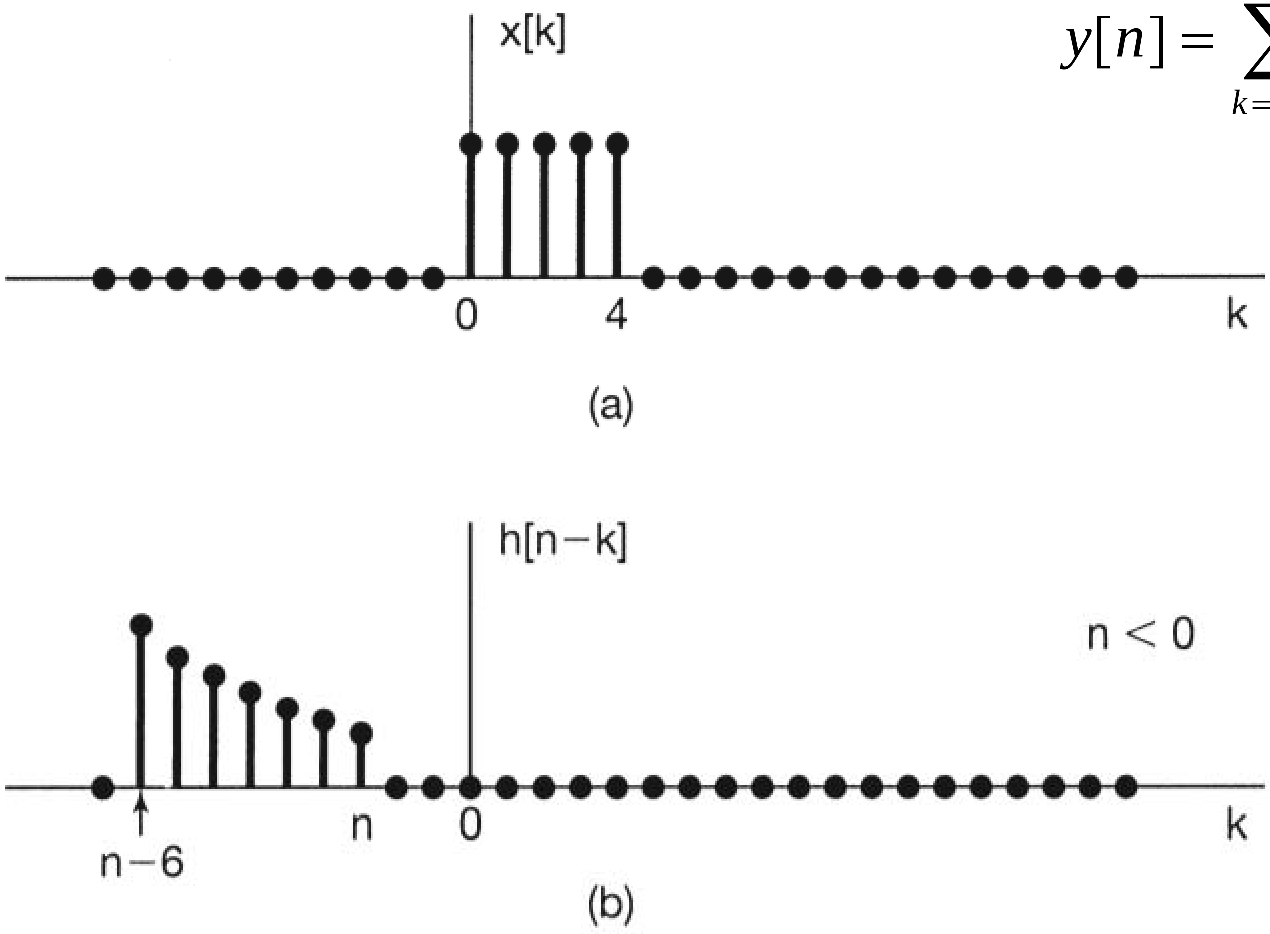
Example 2.4 $x[n]=\begin{cases}1, & 0\le n\le 4\\ 0, & otherwise\end{cases}$ $h[n]=\begin{cases}\alpha^n, & 0\le n\le 6\\ 0, & otherwise\end{cases}.$

$$y[n]=x[n]*h[n]=\sum_{k=-\infty}^{\infty}x[k]h[n-k]$$



Example 2.4 (cont'd)

Interval 1: $n < 0$



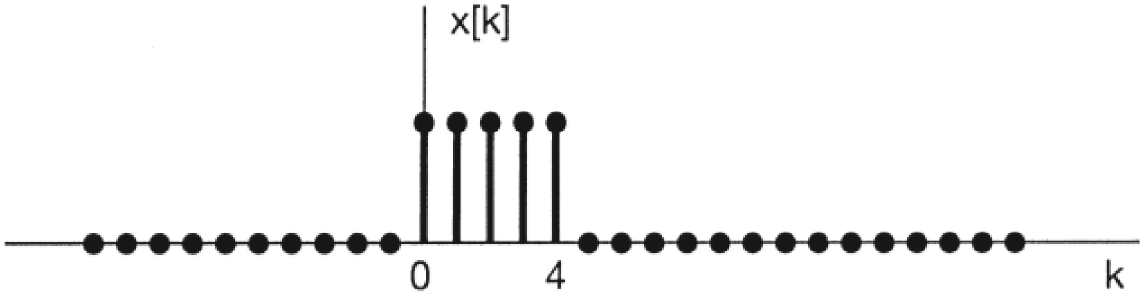
$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = 0$$

Example 2.4 (cont'd)

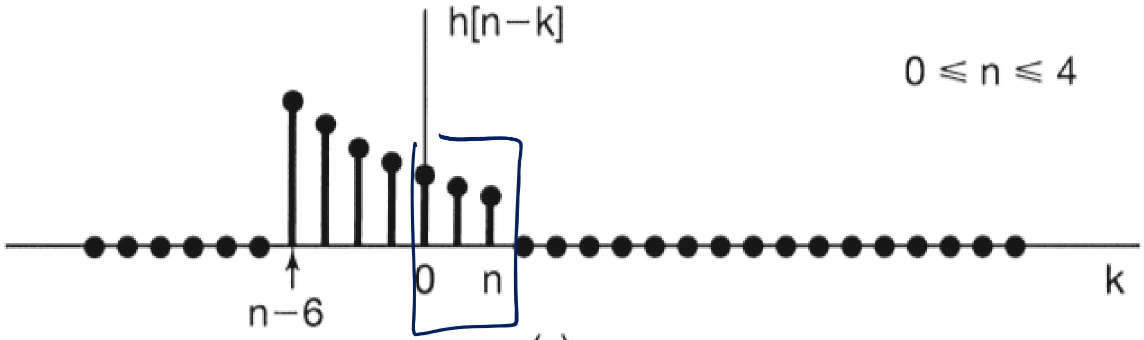
Interval 2: $0 \le n \le 4$

$$y[n] = \qquad = \frac{1-\alpha^{n+1}}{1-\alpha}. = \sum_{k=0}^{n} \underbrace{x[k]}_{1}\, \underbrace{h[n-k]}_{\alpha^{n-k}}$$

$$= \alpha^{n} \boxed{\sum_{k=0}^{n} \alpha^{-k}}$$



(a)

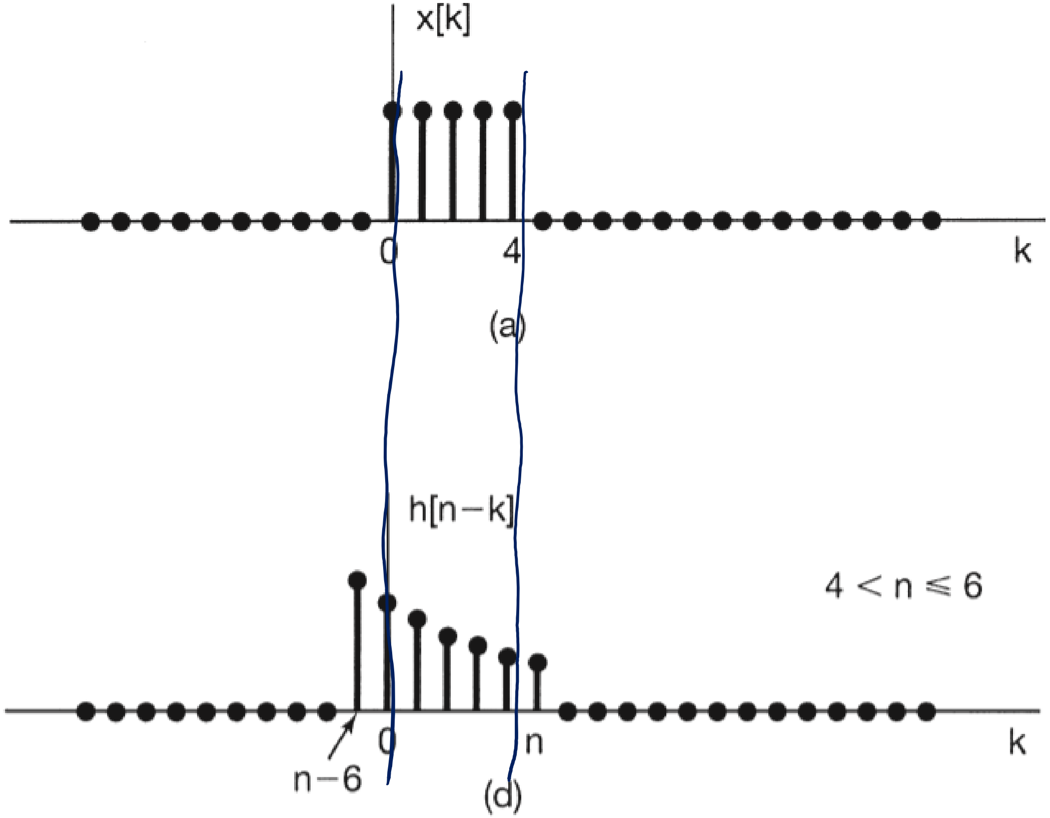


(c)

Example 2.4 (cont'd)

Interval 3: $4 < n \leq 6$

$$y[n] = \boxed{} = \frac{\alpha^{n-4} - \alpha^{n+1}}{1-\alpha}.$$



Example 2.4 (cont'd)

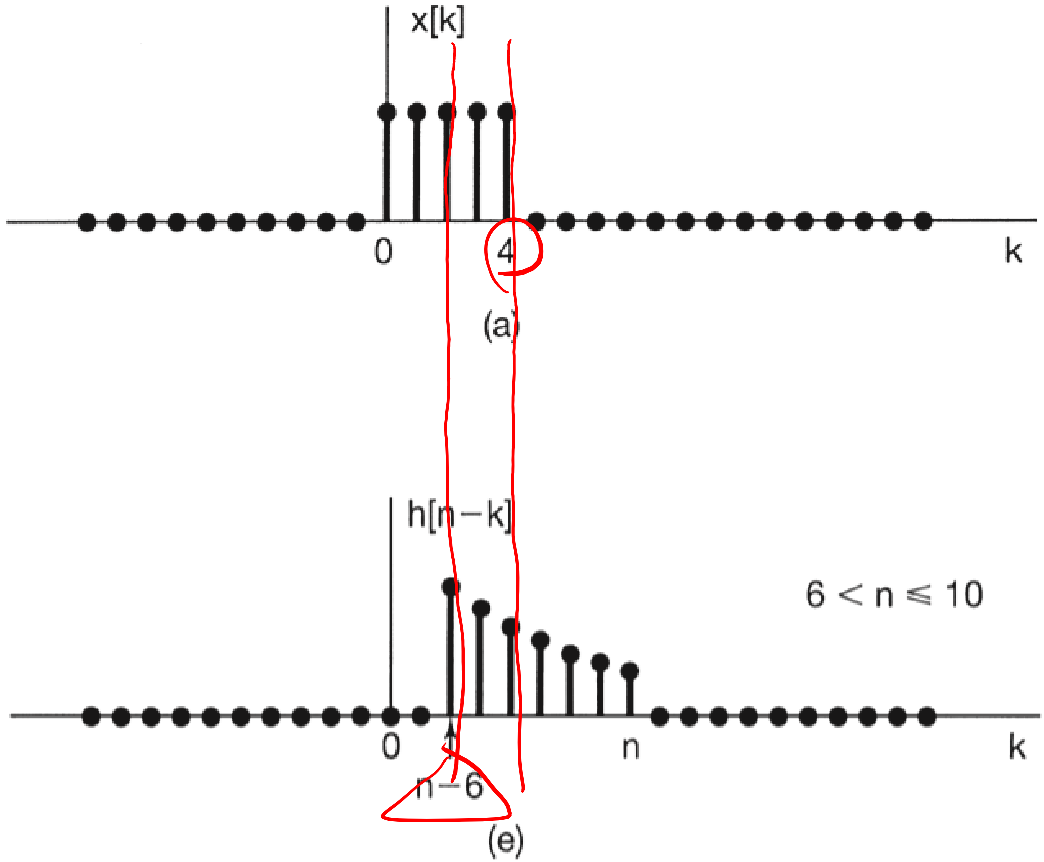
$$y[n]=\sum_{k=-\infty}^{\infty}x[k]h[n-k]=\boxed{\sum_{k=n-6}^{4}\alpha^{n-k}}.$$

Interval 4: $\boxed{6<n\leq 10}$

$$y[n]=\sum_{r=0}^{10-n}\alpha^{6-r}=\frac{\alpha^{n-4}-\alpha^{7}}{1-\alpha}.$$

let $r=k-(n-6)$

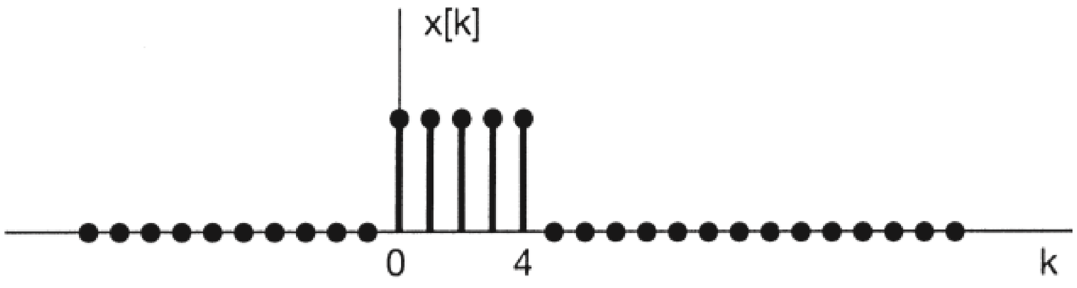
$\boxed{k=r+n-6}$



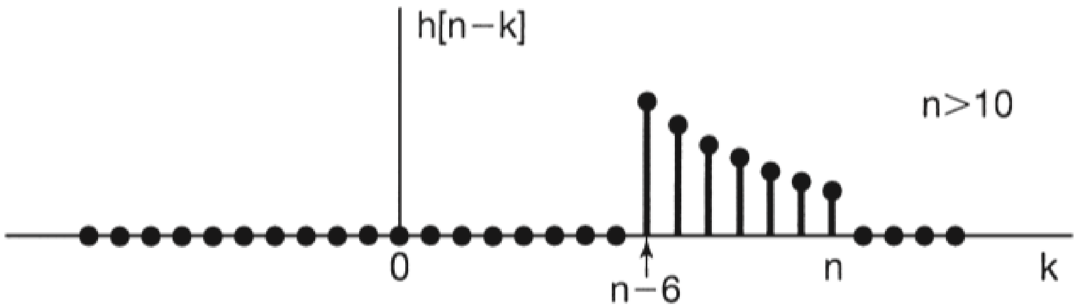
Example 2.4 (cont'd)

Interval 5: $n > 10$

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = 0.$$



(a)



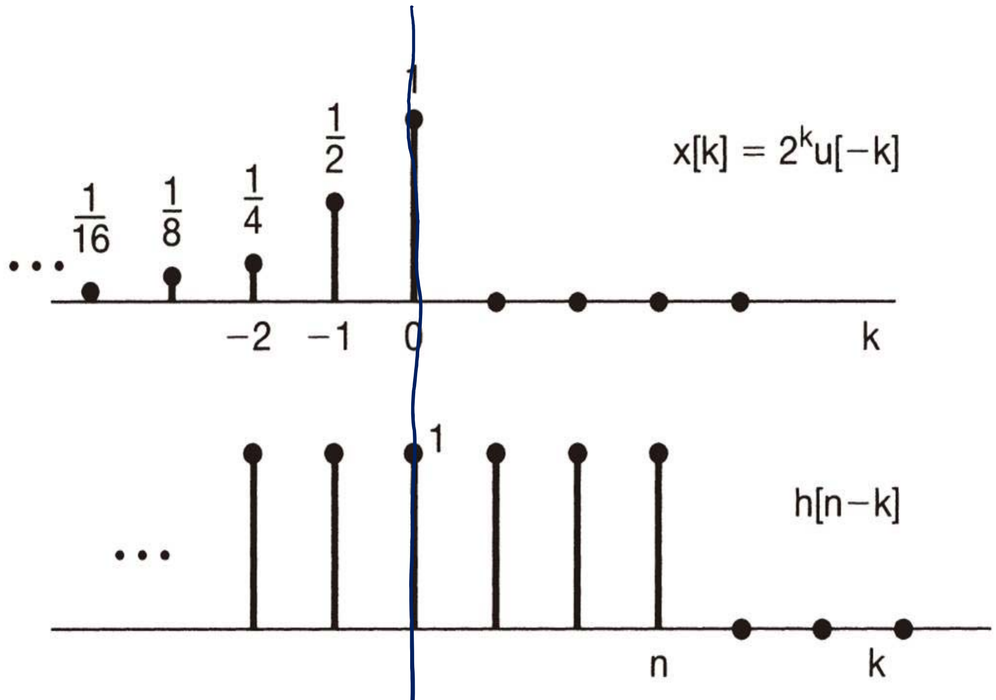
(f)

$$y[n] = \begin{cases} \_\_ & n<0 \\ \_\_ & 0 \le n < 4 \\ \vdots & \vdots \\ \_\_ & n > 10 \end{cases}$$

Example 2.5

$$x[n] = 2^n u[-n], \qquad h[n] = u[n].$$

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$



(a)

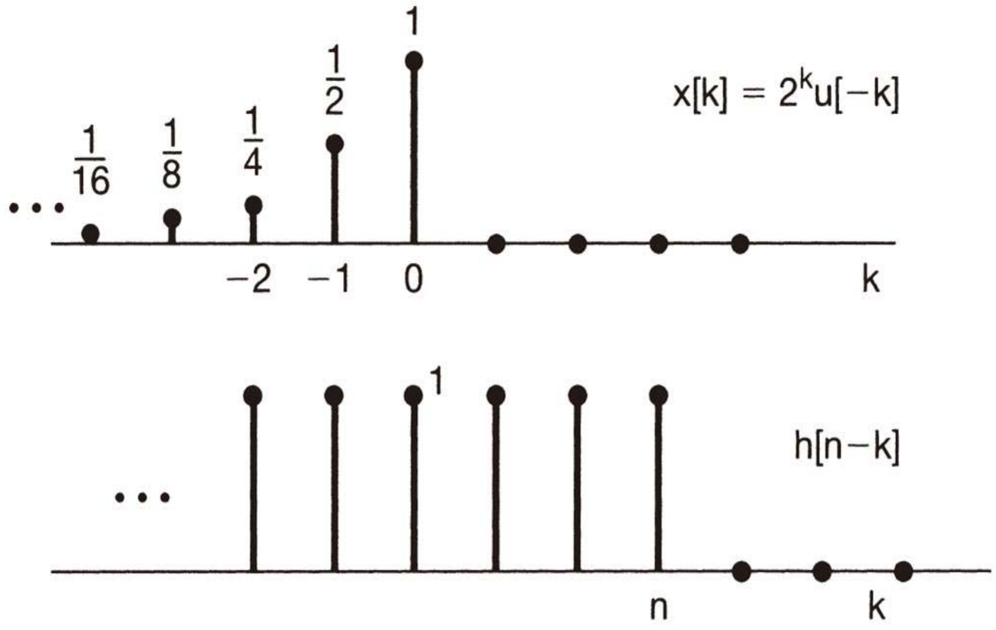
Interval 1: $n \geq 0$

$$y[n] = \sum_{k=-\infty}^{0} x[k]h[n-k] = \sum_{k=-\infty}^{0} 2^k = 2$$

Example 2.5

$$x[n] = 2^n u[-n], \qquad h[n] = u[n].$$

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]\boxed{h[n-k]}$$



(a)

Interval 2: $n < 0$

$$y[n] = \sum_{k=-\infty}^{n} 2^k = \sum_{l=-n}^{\infty} (\frac{1}{2})^l = \sum_{m=0}^{\infty} (\frac{1}{2})^{m-n}$$

$$= \left(\frac{1}{2}\right)^{-n} \sum_{m=0}^{\infty} (\frac{1}{2})^m = 2^n \cdot 2 = 2^{n+1}.$$

Recall that in Sect. 2.1
DT LTI Systems: The Convolution Sum

- Summary
  - Definition of DT convolution
  - Unit impulse response
  - Any DT LTI system can be modeled by a DT convolution operation.
  - Any DT signals can be represented by a sum of impulses.

Chapter 2 Linear Time Invariant Systems

- Sec. 2.1 Discrete-time LTI Systems: The Convolution Sum
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- Sec. 2.5 Singularity Functions

Sect. 2.2 CT LTI Systems: The Convolution Integral

- Highlights
  - Definition of the convolution integral for CT signals
  - Any CT LTI system can be modeled by a CT convolution integral operation

2.2.1 Representation of CT Signals in term of Impulses

- Recall that, unit impulse in CT is defined as

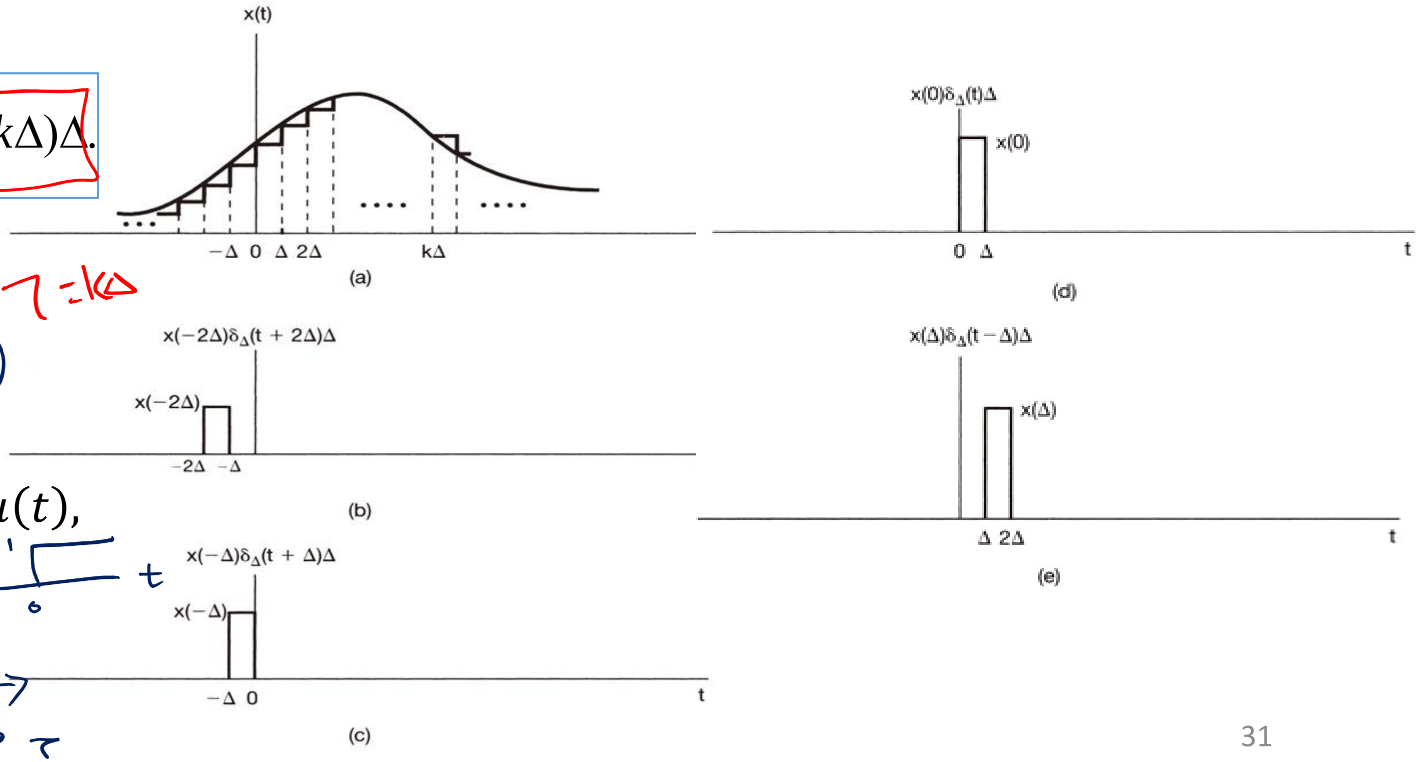
$$\delta_\Delta(t) = \begin{cases} \frac{1}{\Delta}, & 0 \le t \le \Delta \\ 0, & otherwise \end{cases},$$

- Any CT signals can be represented by a sum of impulses.

$$\boxed{x(t) = \lim_{\Delta \to 0} \sum_{k=-\infty}^{+\infty} x(k\Delta)\delta_\Delta(t-k\Delta)\Delta.}$$

$\tau = k\Delta$

$$\boxed{x(t) = \int_{-\infty}^{+\infty} x(\tau)\delta(t-\tau)d\tau.}$$



- When $x(t) = u(t)$,

$$u(t) = \int_0^\infty \delta(t-\tau)d\tau.$$

t

τ

2.2.2 The CT Unit Impulse Response and the Convolution Integral Representation of LTI Systems

- For a CT signal, we can represent it as:

$$x(t) = \lim_{\Delta \to 0} \sum_{k=-\infty}^{+\infty} x(k\Delta)\delta_{\Delta}(t - k\Delta)\Delta.$$

- If a system is linear, then its output $y(t)$ corresponding to $x(t)$ can be expressed as:

$$y(t) = \lim_{\Delta \to 0} \sum_{k=-\infty}^{+\infty} x(k\Delta)h_{k\Delta}(t)\Delta.$$

where $\boxed{h_{k\Delta}(t)}$ is the system output with $\delta_{\Delta}(t - k\Delta)$ as input.

- Here we go again…

2.2.2 The CT Unit Impulse Response and the Convolution Integral Representation of LTI Systems

- For a CT linear system…

$$x(t) = \lim_{\Delta \to 0} \sum_{k=-\infty}^{+\infty} x(k\Delta)\delta_{\Delta}(t - k\Delta)\Delta. \quad \Rightarrow \quad y(t) = \lim_{\Delta \to 0} \sum_{k=-\infty}^{+\infty} x(k\Delta)h_{k\Delta}(t)\Delta$$

$$= \int_{-\infty}^{+\infty} x(\tau)\boxed{h_{\tau}(t)}d\tau.$$

by setting $\tau = k\Delta$

- Moreover, if such a system is time-invariant, we have…

$$h_{\tau}(t) = h_0(t - \tau)$$

- For notation convenience, we use $h(t)$ to denote $h_0(t)$:

$$\boxed{y(t) = \int_{-\infty}^{+\infty} x(\tau)h(t - \tau)d\tau.}$$

2.2.2 The CT Unit Impulse Response and the Convolution Integral Representation of LTI Systems

- In other words, for a CT LTI system, we observe…

$$x(t) \Rightarrow \boxed{\text{LTI system}} \Rightarrow y(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau.$$

- CT Convolution:

$$y(t) = x(t) * h(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau.$$

$h(t)$: unit impulse response (impulse response)

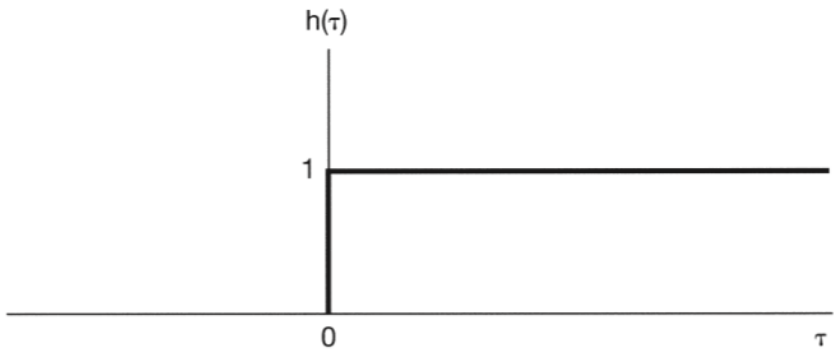
i.e., the output of the system when the input is $\delta(t)$

(see slide #10 for the DT version)

Example 2.6 $x(t) = e^{-at}u(t), \quad a > 0$

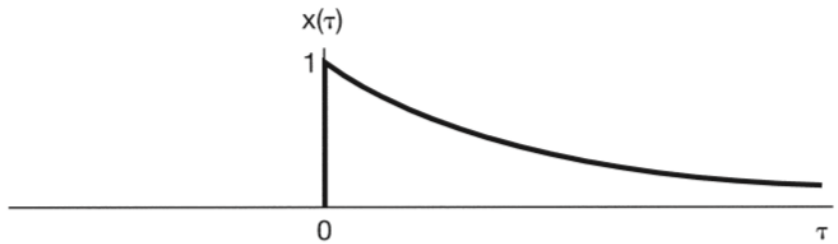
$$h(t) = u(t).$$

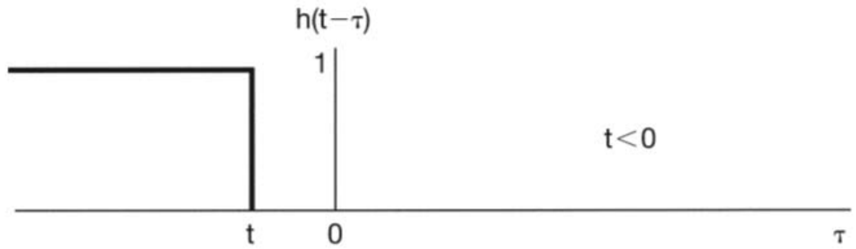
$$y(t) = x(t) * h(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau.$$



Interval 1: $t < 0$

$y(t) = 0$

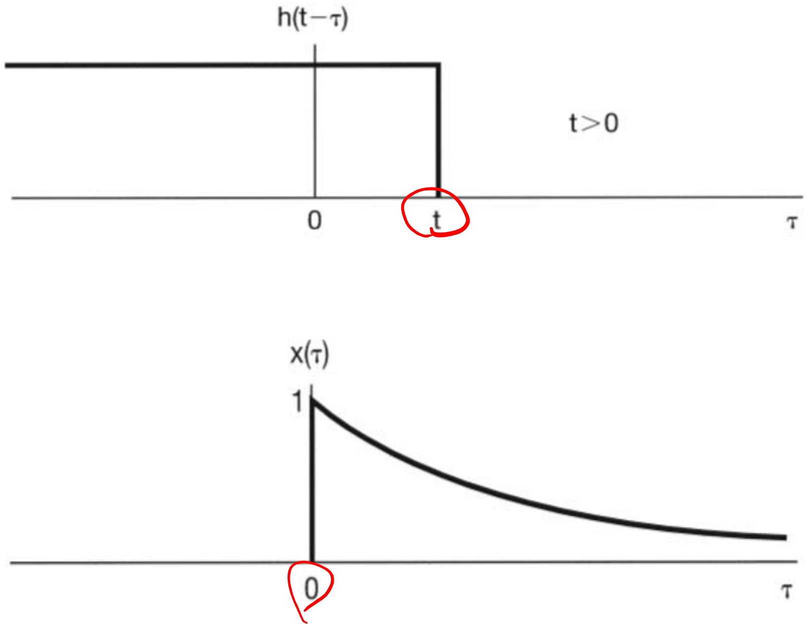




Example 2.6 $x(t) = e^{-a\tau}u(t), \quad a > 0$

$h(t) = u(t).$

$$y(t) = x(t) * h(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau.$$



Interval 2: $t > 0$

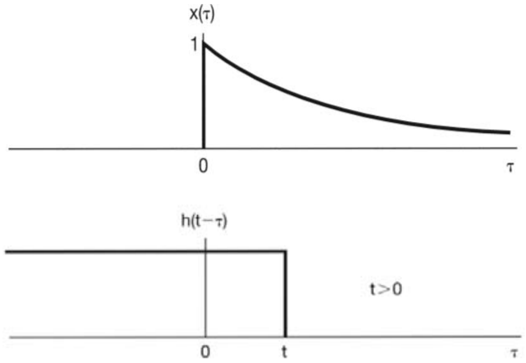
$$y(t) = \int_{-\infty}^{\infty} x(\tau)\,h(t-\tau)\,d\tau$$

$$= \int_{0}^{t} e^{-a\tau}\,d\tau$$

Example 2.6 $x(t) = e^{-a\tau}u(t), \quad a > 0$

$h(t) = u(t).$

$y(t) = x(t) * h(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau.$



$y(t) = \frac{1}{a}(1 - e^{-at})$ for $t > 0$

$y(t) = 0$ for $t < 0$

$y(t) = \frac{1}{a}(1 - e^{-at})u(t)$

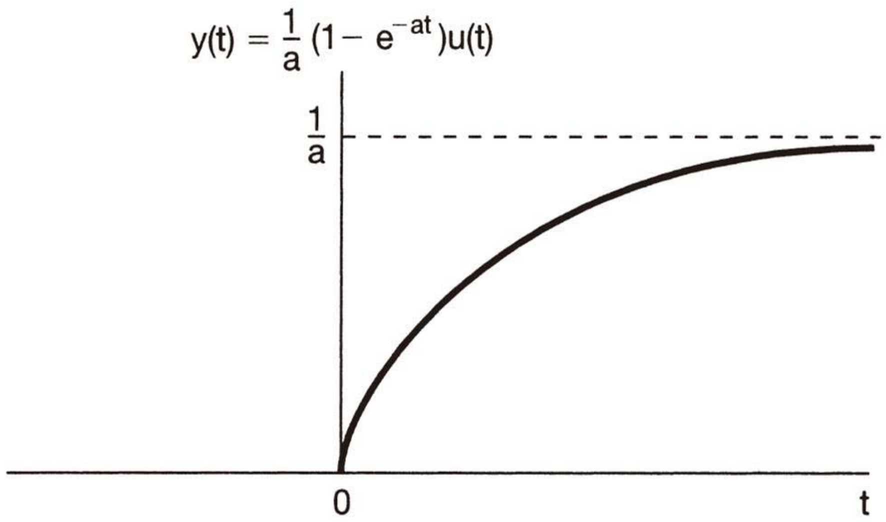


Figure 2.18 Response of the system in Example 2.6 with impulse response $h(t) = u(t)$ to the input $x(t) = e^{-at}u(t)$.

Chapter 2 Linear Time Invariant Systems

- Sec. 2.1 Discrete-time LTI Systems: The Convolution Sum
- Sec. 2.2 Continuous-time LTI Systems: The Convolution Integral
- Sec. 2.3 Properties of Linear Time-invariant Systems
- Sec. 2.4 Causal LTI Systems Described by Differential and Difference Equations
- Sec. 2.5 Singularity Functions

Sect. 2.3 Properties of LTI Systems

- Highlights
  - All of the LTI systems have the following properties:
    (a) linearity, (b) time invariance, (c) commutative property,
    (d) distributive property, and (e) the associative property.
  - Moreover, some of the LTI systems have the properties of
    (a) memory (or memoryless), (b) invertibility, (c) causality, and (d) stability.
  - Learn the definitions of
    (a) absolutely summable, (b) absolutely integrable, and (c) the unit step response

Revisit of DT/CT LTI Systems

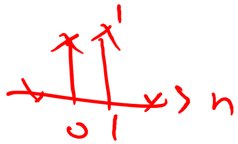
- DT/CT LTI systems are determined by their impulse responses:

$$y[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k] = x[n]*h[n]$$

$$y(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau = x(t)*h(t)$$

- **[Example 2.9] counterexamples of systems of unit impulse responses**

Consider a DT system with unit impulse response as: $h[n] = \begin{cases} 1, & n = 0,1 \\ 0, & otherwise \end{cases}$.



If the system is LTI, we have $y[n] = x[n]*h[n] = x[n] + x[n-1]$.

However, the following systems also have the same impulse responses, but such systems are not LTI (they are actually nonlinear ones):

$$y[n] = (x[n] + x[n-1])^2,$$

$$y[n] = \max(x[n], x[n-1]).$$

2.3.1 The Commutative Property

- DT

$$x[n] * h[n] = h[n] * x[n]$$

$$\sum_{k=-\infty}^{+\infty} x[k]h[n-k], = \sum_{k=-\infty}^{+\infty} h[k]x[n-k],$$

- CT

$$x(t) * h(t) = h(t) * x(t)$$

$$\int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau = \int_{-\infty}^{+\infty} h(\tau)x(t-\tau)d\tau.$$

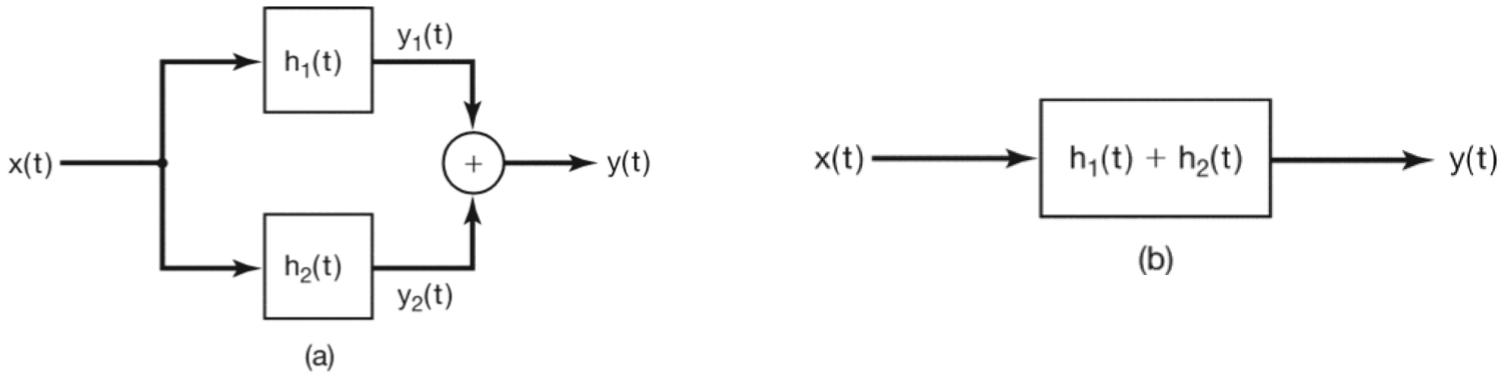
2.3.2 The Distributive Property

- Recall that the LTI systems are of interest here.
- DT

$$x[n]*(h_1[n]+h_2[n]) = x[n]*h_1[n]+x[n]*h_2[n],$$

- CT

$$x(t)*(h_1(t)+h_2(t)) = x(t)*h_1(t)+x(t)*h_2(t)$$



2.3.2 The Distributive Property (cont'd)

- Example 2.10

$$x[n] = \left(\frac{1}{2}\right)^n u[n] + 2^n u[-n], \qquad h[n] = u[n].$$

$$y[n] = x[n] * h[n] = y_1[n] + y_2[n]$$

$$y_1[n] = \left(\frac{1}{2}\right)^n u[n] * h[n] \qquad y_2[n] = 2^n u[-n] * h[n]$$

2.3.3 The Associative Property

- Recall that the LTI systems are of interest here.
- DT

$$x[n]*(h_1[n]*h_2[n])=(x[n]*h_1[n])*h_2[n]$$

- CT

$$x(t)*[h_1(t)*h_2(t)]=[x(t)*h_1(t)]*h_2(t)$$

2.3.4 LTI Systems w/ and w/o Memories

- Recall that the LTI systems are of interest here.
- DT

memoryless $y[n] = Kx[n]$

i.e., $h[n] = 0$ when $n \neq 0$

otherwise, the system has memory.

- CT

memoryless $y(t) = Kx(t)$

i.e., $h(t) = 0$ when $t \neq 0$

otherwise, the system has memory.

2.3.4 Invertibility of LTI Systems

- DT:
  If $h[n]$ is the impulse response of a discrete LTI system, then the system has the reversibility property if and only if there exists an $h_1[n]$ such that

$$h[n] * h_1[n] = \delta[n]$$

- CT:
  If $h(t)$ is the impulse response of a discrete LTI system, then the system has the reversibility property if and only if there exists an $h_1(t)$ such that

$$h(t) * h_1(t) = \delta(t)$$

2.3.4 Invertibility of LTI Systems (cont'd)

- Example 2.11

$$y(t) = x(t) * h(t) = x(t - t_0)$$

$$h(t) = \delta(t - t_0)$$

If $$\boxed{h_1(t) = \delta(t + t_0)}$$

$$y(t) * h_1(t) = y(t + t_0) = x(t)$$

$$h(t) * h_1(t) = \delta(t)$$

2.3.4 Invertibility of LTI Systems (cont'd)

- Example 2.12

If $h[n] = u[n]$

$$y[n] = x[n] * h[n] = \sum_{m=-\infty}^{\infty} x[m]u[n-m] = \sum_{m=-\infty}^{n} x[m]$$

When $h_1[n] = \delta[n] - \delta[n-1]$

$$y[n] * h_1[n] = y[n] - y[n-1] = x[n]$$

$$h[n] * h_1[n] = \delta[n]$$

2.3.6 Causality of LTI Systems

- DT $\quad h[n]=0 \quad$ for $n<0$

$$y[n]=\sum_{k=-\infty}^{n} x[k]h[n-k]=\sum_{k=0}^{\infty} h[k]x[n-k],$$

- CT $\quad h(t)=0 \quad$ for $t<0$

$$y(t)=\int_{-\infty}^{t} x(\tau)h(t-\tau)d\tau=\int_{0}^{\infty} h(\tau)x(t-\tau)d\tau$$

2.3.7 Stability of LTI Systems

- CT
  - If $|x(t)|$ is bounded, then $|y(t)|$ is also bounded (for all t).
  - *Sufficient condition for a continuous-time LTI system to be stable:*

$$\int_{-\infty}^{+\infty} |\, h(\tau) \,|\, d\tau < \infty$$

  - For a CT LTI stable system:

$$|x(t)| < B \text{ for all } t$$

$$|y(t)| = \left| \int_{-\infty}^{+\infty} h(\tau) x(t-\tau) d\tau \right|$$

$$\Rightarrow |y(t)| \leq \int_{-\infty}^{+\infty} |h(\tau)| |x(t-\tau)| d\tau$$

$$\Rightarrow |y(t)| \leq B \left(\int_{-\infty}^{+\infty} |h(\tau)| d\tau \right)$$

$$\text{if } \int_{-\infty}^{+\infty} |h(\tau)| d\tau < \infty \ \text{ then } y(t) \text{ is bounded}$$

2.3.7 Stability of LTI Systems

- DT
  - If $|x[n]|$ is bounded, then $|y[n]|$ is also bounded (for all n).
  - *Sufficient condition for a discrete-time LTI system to be stable*:

$$\sum_{k=-\infty}^{+\infty} \left| h[k] \right| < \infty$$

- CT
  - If $|x(t)|$ is bounded, then $|y(t)|$ is also bounded (for all t).
  - *Sufficient condition for a continuous-time LTI system to be stable:*

$$\int_{-\infty}^{+\infty} | h(\tau) | \, d\tau < \infty$$

2.3.7 Stability of LTI Systems (cont'd)

- Example 2.13 Stable or Not?

$$h[n] = \delta[n]$$

$$h(t) = \delta(t)$$

$\Rightarrow$

$$h[n] = u[n]$$

$$h(t) = u(t)$$

$\Rightarrow$

2.3.8 The Unit Step Response of LTI Systems

- Unit Step Response
  - The response (i.e., system output) when the input is $u[n]$ or $u(t)$.
- DT

The unit step response $s[n]$ is

$$s[n] = u[n] * h[n] = \sum_{k=-\infty}^{n} h[k]$$

Therefore, $h[n] = s[n] - s[n-1]$

- CT

$$s(t) = u(t) * h(t) = \int_{-\infty}^{t} h(\tau) d\tau,$$

Therefore, $h(t) = \dfrac{ds(t)}{dt} = s'(t)$

2.3.9* Variation of Support and Length After Convolution

- Support: A set of points where a function has nonzero values.
- CT case:

If $x(t) = 0$ for $t < t_1$ and $t > t_2$, $t_2 > t_1$,

$x(t) \neq 0$ for $t_1 < t < t_2$,

support: $t \in (t_1, t_2)$

length: $t_2 - t_1$.

2.3.9 Variation of Support and Length After Convolution

- Support: A set of points where a function has nonzero values.
- CT case (cont'd):

If the support of $x(t)$ is $t \in (t_1, t_2)$

the support of $h(t)$ is $t \in (t_3, t_4)$

$$y(t) = x(t) * h(t)$$

then the support of $y(t)$ is equal to (or within) $t \in (t_1 + t_3,\ t_2 + t_4)$

the length of $y(t)$ is $L_y = t_2 + t_4 - t_3 - t_1 = L_x + L_h$.

2.3.9 Variation of Support and Length After Convolution

- Support: A set of points where a function is nonzero.
- DT case:

If $x[n] = 0$ for $t < n_1$ and $t > n_2$, $n_2 > n_1$,

$x[n] \neq 0$ for $n_1 < t < n_2$,

support: $n \in [n_1, n_2]$

length: $n_2 - n_1 + 1$

2.3.9 Variation of Support and Length After Convolution

- Support: A set of points where a function is nonzero.
- DT case:

If the support of $x[n]$ is $n \in [n_1, n_2]$

the support of $h[n]$ is $n \in [n_3, n_4]$

$$y[n] = x[n] * h[n]$$

then the support of $y[n]$ is equal to (or within) $n \in [n_1 + n_3, \; n_2 + n_4]$

the length of $y[n]$ is $L_y = n_2 + n_4 - n_3 - n_1 + 1 = L_x + L_h - 1$.

Chapter 2 Linear Time Invariant Systems

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2.4 Systems Described by Differential/Difference Equations

- **CT**
  - Differential equation specification for input/output relationships

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}$$

  - Derived by physical phenomena and relationships (e.g., circuits)
  - Auxiliary conditions are often needed to completely specify the system of interest.

2.4 Systems Described by Differential/Difference Equations

- **CT (cont'd)**
  - Differential equation specification for input/output relationships

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}$$

  - The response $y(t)$ to an input $x(t)$ generally consists of two parts:
    - Homogeneous solution (natural response):

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = 0$$

    - Particular solution (to the complete differential equation):

2.4 Systems Described by Differential/Difference Equations

- **CT (cont'd)**
  - Differential equation specification for input/output relationships

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}$$

  - Initial rest condition for causal systems

$$x(t) = 0,\, t \le t_0 \;\rightarrow\; \quad y(t) = 0,\, t \le t_0$$

  - Initial conditions

$$y(t_0) = \frac{dy(t_0)}{dt} = \frac{d^2 y(t_0)}{dt^2} = \ldots\ldots = \frac{d^N y(t_0)}{dt^N} = 0$$

2.4 Systems Described by Differential/Difference Equations

- **DT**
  - Difference equation specification

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

  - Derived by sequential behavior of difference processes
  - Auxiliary conditions might be needed.
  - Response $y[n]$ generally consists of
    - Homogeneous solution for $$\sum_{k=0}^{N} a_k y[n-k] = 0$$
    - Particular solution

2.4 Systems Described by Differential/Difference Equations

- **DT (cont'd)**
  - Difference equation specification

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

  - Initial rest conditions for causal systems

$$x[n] = 0,\ n \le n_0 \ \rightarrow \quad y[n] = 0,\ n \le n_0$$

  - Recursive equation

$$y[n] = \frac{1}{a_0}\left\{\sum_{k=0}^{M} b_k x[n-k] - \sum_{k=1}^{N} a_k y[n-k]\right\}$$

i.e., output at time n expressed in terms of inputs/outputs in previous times

2.4 Systems Described by Differential/Difference Equations

- **DT (cont'd)**
  - Recursive equation $$y[n] = \frac{1}{a_0}\left\{\sum_{k=0}^{M} b_k x[n-k] - \sum_{k=1}^{N} a_k y[n-k]\right\}$$
  - When N=0, reduced to a convolution sum

$$y(n) = \sum_{k=0}^{M} \left(\frac{b_k}{a_0}\right) x[n-k]$$

  - Note that $h[n] = b_n / a_0$ for $0 \leq n \leq M$; otherwise $h[n] = 0$
  - Finite impulse response (FIR) vs. infinite impulse response (IIR) systems

2.4.1 Linear Constant-Coefficient Differential Equations

- Example 2.14 $$\frac{dy(t)}{dt} + 2y(t) = x(t) \quad \text{where} \quad x(t) = Ke^{3t}u(t)$$

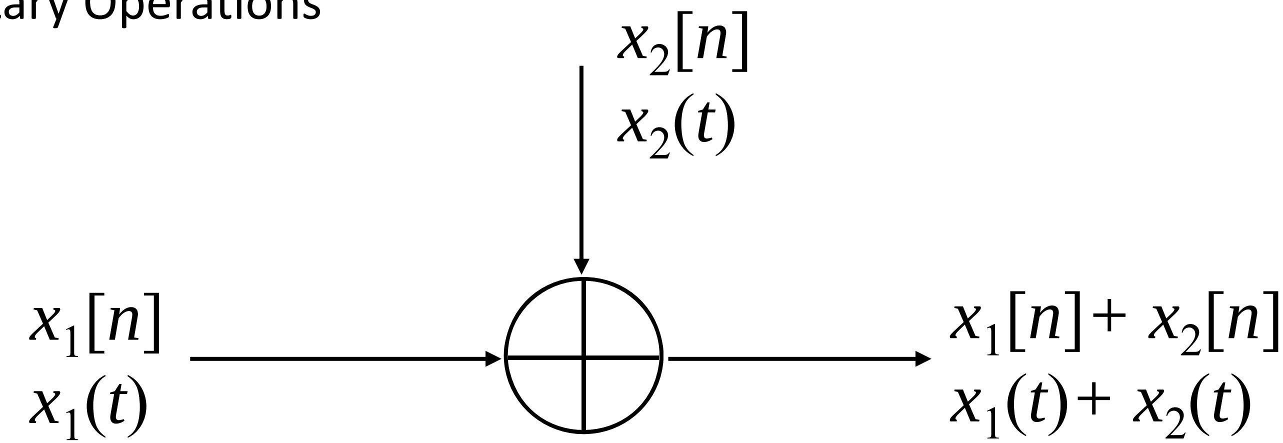
Solution: $$y(t) = y_p(t) + y_h(t)$$

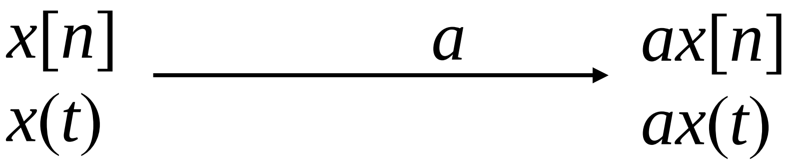
$y_h(t)$ is the solution of $\frac{dy(t)}{dt} + 2y(t) = 0$ $\quad y_h(t) = Ae^{st}$

$y_p(t)$ is any the original solution $\quad y_p(t) = \frac{K}{5}e^{3t}, \quad t > 0$

Block Diagram Representation

- Elementary Operations





Block Diagram Representation

- Elementary Operations (cont'd)

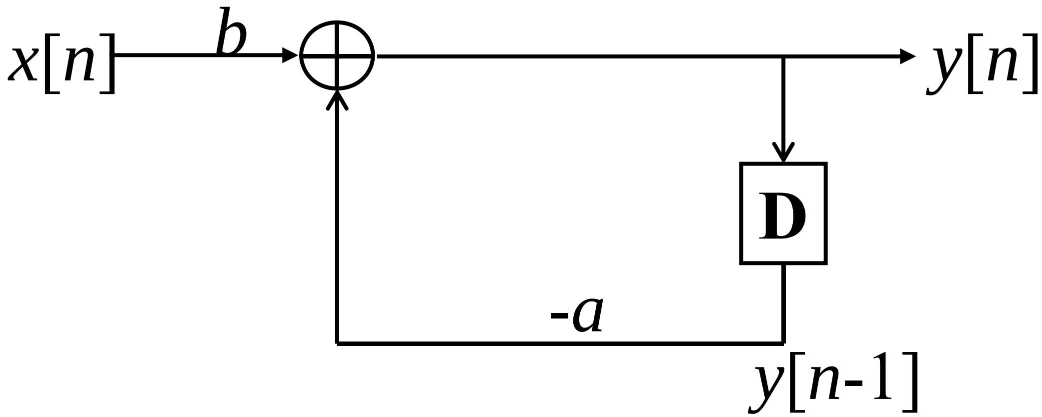
$$x[n] \longrightarrow \boxed{\mathbf{D}} \longrightarrow x[n\text{-}1]$$

$$x(t) \longrightarrow \boxed{\frac{d}{dt}} \longrightarrow \frac{dx(t)}{dt}$$

$$x(t) \longrightarrow \boxed{\int} \longrightarrow \int_{-\infty}^{t} x(\tau)\, d\tau$$

Block Diagram Representation

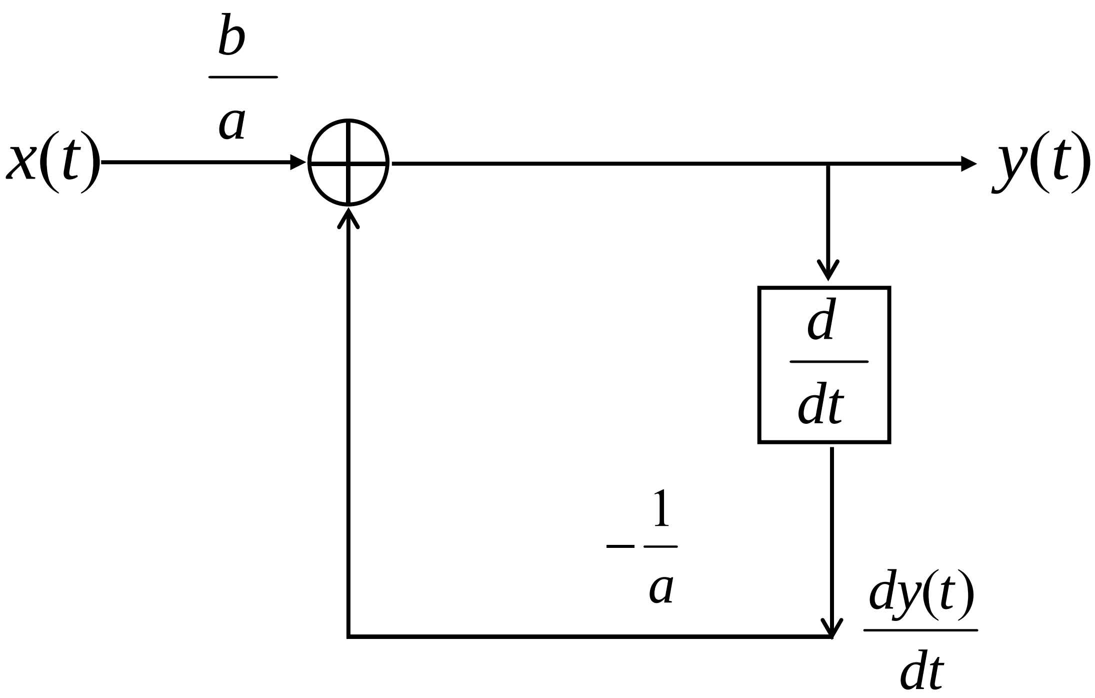
- Example: $y[n]+ay[n-1]=bx[n]$



- Note that, this systems observes feedback (i.e., with memory).
  Initial value of the memory element = initial condition of the systems
- Initial rest condition: initial value in the memory element is zero.

Block Diagram Representation

- CT Example: $$\frac{dy(t)}{dt} + ay(t) = bx(t)$$

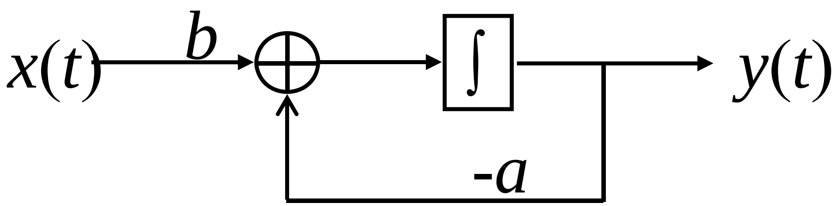


Block Diagram Representation

- CT Example: $$\frac{dy(t)}{dt} + ay(t) = bx(t)$$
- Expressed by integrator, assuming initially at rest

$$y(t) = \int_{-\infty}^{t} [bx(\tau) - ay(\tau)]d\tau$$

$$y(t) = y(t_0) + \int_{t_0}^{t} [bx(\tau) - ay(\tau)]d\tau$$



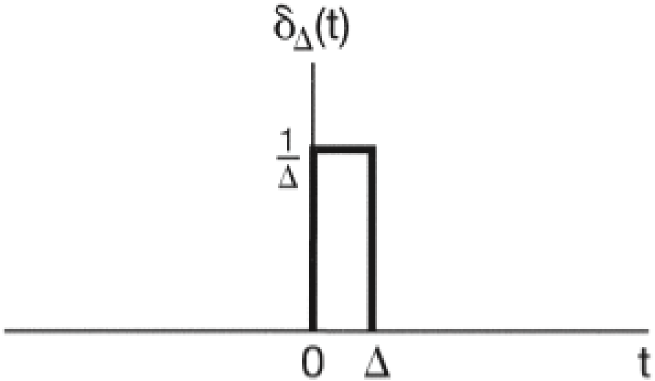
- The integrator represents the memory element.

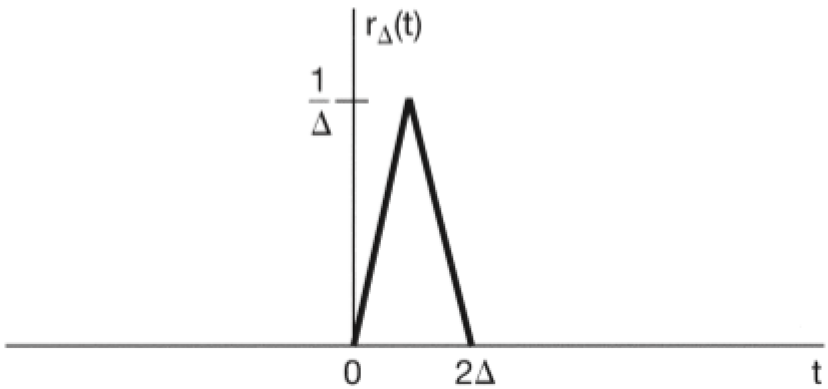
2.5 Singularity Functions: Properties of CT unit impulse

- **2.5.1 The Unit Impulse as an Idealized Short Pulse**
  - There is no explicit form of a unit impulse.
  - Instead, we can say some "functions" behave like a unit impulse.

$$x(t) = x(t) * \delta(t)$$ when $x(t) = \delta(t)$ $$\delta(t) = \delta(t) * \delta(t)$$

$$r_\Delta(t) = \delta_\Delta(t) * \delta_\Delta(t)$$





$\delta_\Delta(t)\big|_{\Delta\to 0}$ and $r_\Delta(t)\big|_{\Delta\to 0}$ can all be viewed as a unit impulse.

2.5 Singularity Functions: Properties of CT unit impulse

- **2.5.2 Defining the Unit Impulse through Convolution**
  - We define $\delta(t)$ as the signal for which

$$x(t) = x(t) * \delta(t)$$

is satisfied.

2.5 Singularity Functions: Properties of CT unit impulse

- **2.5.3 Unit Doublets and Other Singularity Functions**
  - Define

$$u_1(t) = \frac{d}{dt}\delta(t) \quad \Longrightarrow \quad \frac{d}{dt}x(t) = x(t) * u_1(t)$$

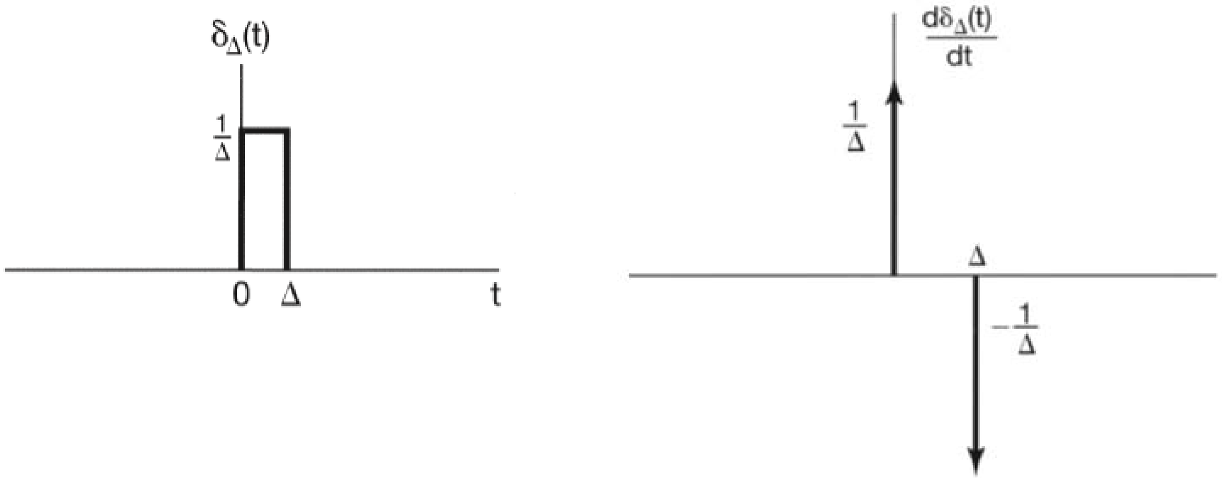
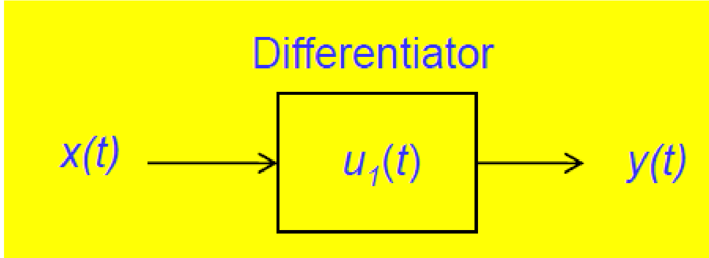


Figure 2.36 The derivative $d\delta_\Delta(t)/dt$ of the short rectangular pulse $\delta_\Delta(t)$ of Figure 1.34.

2.5 Singularity Functions: Properties of CT unit impulse



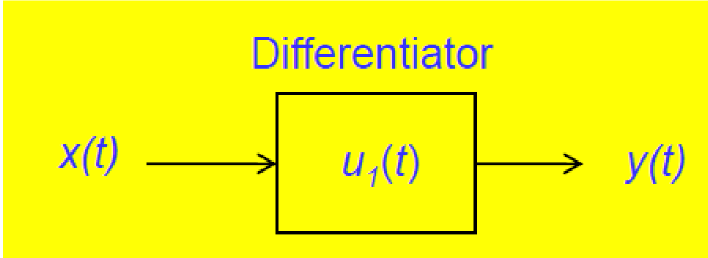
- **2.5.3 Unit Doublets and Other Singularity Functions**
  - Consider the system $y(t) = \frac{d}{dt}x(t)$.
  - The unit impulse response of the system is the derivative of the unit impulse, which is called the unit doublet $u_1(t)$, which is defined as:

$$u_1(t) = \frac{d}{dt}\delta(t).$$

  - From the convolution representation of LTI systems, we have

$$\frac{d}{dt}x(t) = x(t) * u_1(t).$$

2.5 Singularity Functions: Properties of CT unit impulse



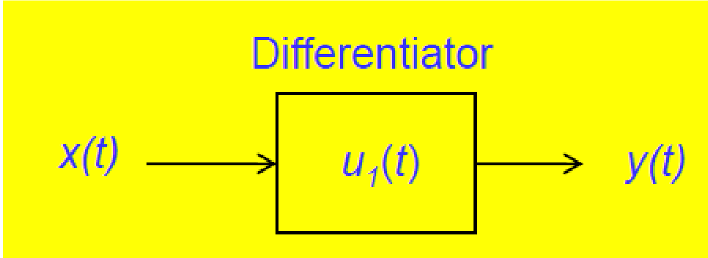
- **2.5.3 Unit Doublets and Other Singularity Functions**
  - Similarly, we may define $\frac{d^2}{dt^2}x(t) = x(t) * u_2(t)$.
  - We have $\frac{d^2}{dt^2}x(t) = \frac{d}{dt}\left(\frac{d}{dt}x(t)\right) = (x(t) * u_1(t)) * u_1(t)$.
  - Therefore, we observe

$$u_2(t) = u_1(t) * u_1(t).$$

  - In general, for the kth derivative of $\delta(t)$, we have

$$u_k(t) = \underbrace{u_1(t) * \cdots * u_1(t)}_{\text{k times}}, \mathrm{k} > 0.$$

2.5 Singularity Functions: Properties of CT unit impulse



- **2.5.3 Unit Doublets and Other Singularity Functions**
  - Consider $x(t) = 1$, we have

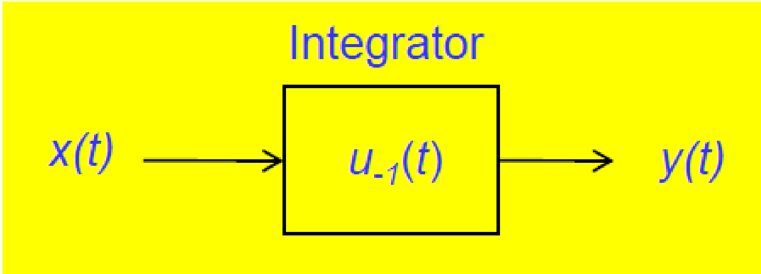
$$0 = \frac{dx(t)}{dt} = x(t) * u_1(t)$$

$$= \int_{-\infty}^{\infty} u_1(\tau) x(t-\tau) d\tau$$

$$= \int_{-\infty}^{\infty} u_1(\tau) d\tau$$

  - That is, the unit doublet has zero area.

2.5.3 Unit Doublets and Other Singularity Functions



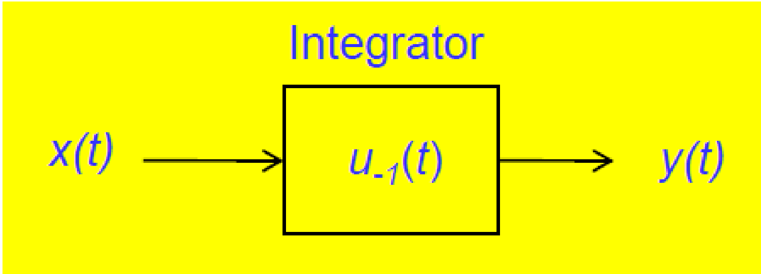
- **Integral of Unit Impulse**
  - Consider an integrator: $y(t) = \int_{-\infty}^{t} x(\tau) d\tau$
  - The impulse response of an integrator is the unit step. Why?

$$u_{-1}(t) \triangleq \int_{-\infty}^{t} \delta(\tau) d\tau = u(t)$$

  - Thus, we have the following operational definition of $u(t)$.

$$x(t) * u(t) = \int_{-\infty}^{t} x(\tau) d\tau$$

2.5.3 Unit Doublets and Other Singularity Functions

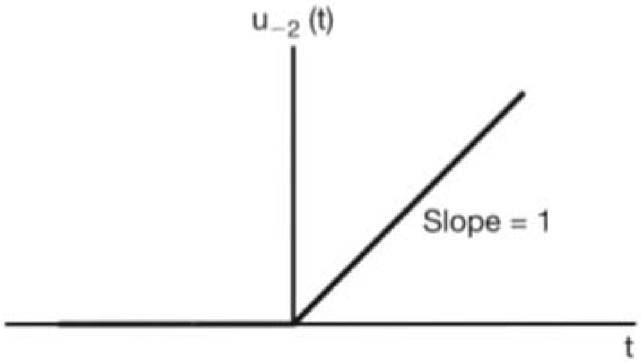


- **Integral of Unit Impulse**
  - Similarly, we observe

$$u_{-2}(t) = u(t) * u(t) = \int_{-\infty}^{t} u(\tau) d\tau$$

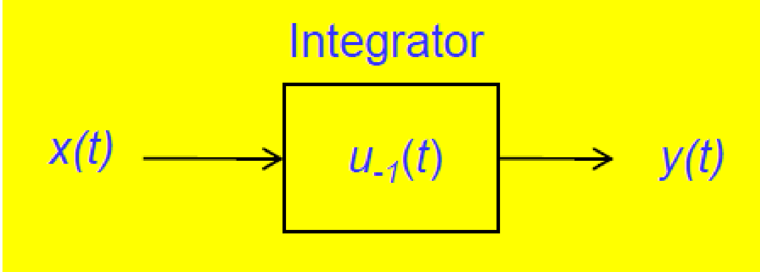
  - Since $u(t)$ equals 0 for $t < 0$ and 1 for $t \geq 0$, it follows that

$$u_{-2}(t) = tu(t)$$



unit ramp function

2.5.3 Unit Doublets and Other Singularity Functions



- **Integral of Unit Impulse**
  - Moreover

$$x(t) * u_{-2}(t) = x(t) * u(t) * u(t)$$

$$= \left(\int_{-\infty}^{t} x(\sigma) d\sigma \right) * u(t)$$

$$= \int_{-\infty}^{t} \left(\int_{-\infty}^{\tau} x(\sigma) d\sigma \right) d\tau$$

  - In general,

$$u_{-k} = u(t) * \cdots * u(t) = \int_{-\infty}^{t} u_{-(k-1)}(\tau) d\tau$$

$$u_{-k}(t) = \frac{t^{k-1}}{(k-1)!} u(t)$$

2.5 Singularity Functions: Properties of CT unit impulse

- **Summary**

$$\delta(t) = u_0(t)$$

$$u(t) = u_{-1}(t)$$

$$u_k(t)\begin{cases} k > 0, & \text{Impulse response of a cascade of } k \text{ differentiators} \\ k < 0, & \text{Impulse response of a cascade of } |k| \text{ integrators} \end{cases}$$

$$u(t) * u_1(t) = \delta(t) \qquad \text{or} \quad u_{-1}(t) * u_1(t) = u_0(t)$$

$$\Rightarrow u_k(t) * u_r(t) = u_{k+r}(t)$$

2.5 Singularity Functions: Properties of CT unit impulse

Property or Definition	Formula
(1) Integration	$\int_{-\infty}^{\infty} \delta(t)dt = 1$
(2) Relation with the unit step function	$\int_{-\infty}^{t} \delta(\tau)d\tau = u(t), \qquad \frac{d}{dt}u(t) = \delta(\tau)$
(3) Convolution	$x(t) * \delta(t) = x(t)$
(4) Auto convolution	$\delta(t) * \delta(t) = \delta(t), \qquad \delta(t) * \delta(t) * \ldots. * \delta(t) = \delta(t)$
(5) Sifting (I)	$\int_{a}^{b} f(t)\delta(t-t_0)dt = f(t_0)$ if $a < t_0 < b$
(6) Sifting (II)	$f(t)\delta(t-t_0) = f(t_0)\delta(t-t_0)$
(7) Unit doublet $u_1(t)$	$u_1(t) = \frac{d}{dt}\delta(t)$ $x(t) * u_1(t) = \frac{d}{dt}x(t)$
(8) $u_k(t)$ (k is a positive integer)	$u_k(t) = \underbrace{u_1(t) * \cdots * u_1(t)}_{k \text{ times}} = \frac{d^k}{dt^k}\delta(t)$ $x(t) * u_k(t) = \frac{d^k}{dt^k}x(t)$
(9) $u_{-1}(t)$	$u_{-1}(t) = u(t),$
(10) $u_{-k}(t)$ (k is a positive integer)	$u_{-k}(t) = \underbrace{u(t) * \cdots * u(t)}_{k \text{ times}} = \frac{t^{k-1}}{(k-1)!}u(t),$ $x(t) * u_{-k}(t) = \int_{-\infty}^{t}\int_{-\infty}^{\tau_{k-1}} \ldots \int_{-\infty}^{\tau_2}\left(\int_{-\infty}^{\tau_1} x(\sigma)d\sigma\right)d\tau_1 d\tau_2 \ldots d\tau_{k-1}.$ (k times of integration)
When $k = 2$, it is called a unit ramp function	