Signals & Systems

Spring 2019

https://sites.google.com/site/ntusands/

https://ceiba.ntu.edu.tw/1072EE2011_04

Yu-Chiang Frank Wang 王鈺強, Associate Professor

Dept. Electrical Engineering, National Taiwan University

2019/02/18 & 21

信號與系統 Signals & Systems

- Time: Mondays 2, 3 (09:10-11:10) & Thursday 7 (14:20-15:10)
- Location: EE2-106 [UPDATED!]
- Website:

  https://sites.google.com/site/ntusands/ (重要事項，包含作業考試等資訊)

  https://ceiba.ntu.edu.tw/1072EE2011_04 (個人成績等相關資訊)

  Please make sure that your NTU email account is working!
- Required Knowledge & Skills
  - Linear algebra, calculus, and probability
  - Basic programming skills

Disclaimer

- I'm teaching this course for the very 1st time. ☺
- Other classes (highly recommended!!) taught by
  - 李琳山教授 @ EE2-143
  - 李枝宏教授 @ EE2-229
  - ~~陳宏銘教授~~ @ EE2-106 (course suspended due to sick leave)

What to Expect from this Course?

- What are Signals & Systems?
- Why study Signals & Systems?
- Lots of stuff to learn, but hopefully would be helpful and with lots of fun!

Course Information

- **加簽原則** (How to sign up if not already in?)
- 講師/助教群介紹 Teaching Team & Office Hours
- 課程大綱與精神
- 成績計算方式

加簽原則

- Capacity
  - 教室容量**62**人，目前已選上60人
  - 加簽上限以教室容量為準 (有可能換教室?)
- Priority
  - 電機系同學 (必修)
  - 他系同學依年級排序

Teaching Team & Office Hours

- Instructor: Yu-Chiang Frank Wang (王鈺強)
- Research Areas
  - Computer Vision, Machine Learning, Deep Learning, & Artificial Intelligence
- Education
  - PhD, ECE, Carnegie Mellon University, 2009
  - MS, ECE, Carnegie Mellon University, 2004
  - BS, EE, National Taiwan University, 2001

- Contact Info
  - Email: ycwang@ntu.edu.tw
- Office Hour for Signals & Systems
  - Mon 11:10am-12pm
  - Alternative time can be arranged by email appointments.

TAs & Office Hours:

(本班助教 #1)
蘇峯廣 Wed 3-5pm @ 系K (BL-B1)

(本班助教 #2)
林志皓 Tue 1-2pm @ 系K (BL-B1)

TAs & Office Hours:

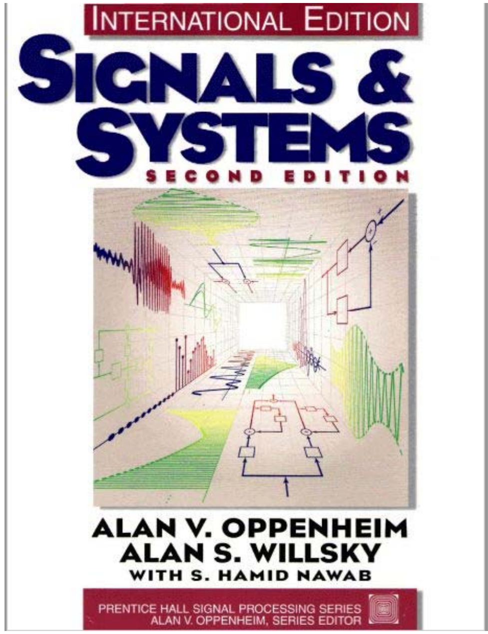
(統籌助教: 課程協調/課後輔導)
陳皇志

(Tight) Schedule

Week	Date	HW	HW Content	Remark
1	2/18 - 2/22	-		
2	2/25 - 3/01	HW1	Periodic signal, inverse function (Ch. 1)	Matlab Tutorial; 2/28 (Thu.) 放假
3	3/04 - 3/08	HW2	LTI, causality, stable properties, convolution (Ch. 2-1~2-3)	
4	3/11 - 3/15	Matlab1	Convolution (Ch. 2-1~2-2)	
5	3/18 - 3/22	HW3	Fourier series (Ch. 3)	
6	3/25 - 3/29	HW4	Fourier transform (Ch. 4-1~4-3)	
7	4/01 - 4/05	Matlab2	DFT, FFT (Ch. 5-1)	4/4 (Thu.) 放假
8	4/08 - 4/12	HW5	Fourier transform (Ch. 5-1~5-3)	
9	4/15 - 4/19	-		4/15 (Mon.) Midterm Exam
10	4/22 - 4/26	-		
11	4/29 - 5/03	HW6	Filter, frequency response (Ch. 6)	
12	5/06 - 5/10	HW7	Sampling theorem (Ch. 7)	
13	5/13 - 5/17	Matlab3	Digital filter (Ch. 6)	
14	5/20 - 5/24	HW8	Laplace transform (Ch. 9)	
15	5/27 - 5/31	HW9	z-transform (Ch. 10)	
16	6/03 - 6/07	Matlab4	z-transform (Ch. 10)	
17	6/10 - 6/14	HW10	Modulation, demodulation (Ch. 8)	
18	6/17 - 6/21	-		6/17 (Mon.) Final Exam

Textbook & Lecture Slides/Notes

- Signals and Systems
  - Alan V. Oppenheim, et al.
  - Pearson; 2nd edition (August 16, 1996)



- Lecture slides/notes
  - Available on Ceiba before the day of class

About Grading & Academic Integrity

- **HWs 30%**
  - Handwritten HWs x 10 (1% each)
  - Matlab HWs x 4 (5% each)
  - 每周的HW於周五出，手寫作業deadline為一星期後的周五。MATLAB作業deadline為兩周後的周五。
- **Exams 70%**
  - Midterm 70% x 0.45
  - Final 70% x 0.55
- Participation (optional bonus; not guaranteed)
  - Not necessarily 點名
  - Can be 課堂互動、Q&A等
- Can discuss HW with peers, but DO NOT copy and/or share code
  - 任一次作業抄襲/被抄襲者，按校規論且本課程學期成績為F
  - This is university policy and not negotiable.

Final Grade

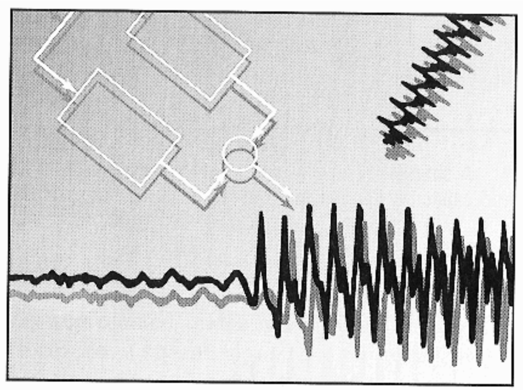
Letter Grading System	Definition	Grade Points	Conversion Scale
A+	All goals achieved beyond expectation	4.3	90-100
A	All goals achieved	4.0	85-89
A-	All goals achieved, but need some polish	3.7	80-84
B+	Some goals well achieved	3.3	77-79
B	Some goals adequately achieved	3.0	73-76
B- (passing grade for graduate students)	Some goals achieved with minor flaws	2.7	70-72
C+	Minimum goals achieved	2.3	67-69
C	Minimum goals achieved with minor flaws	2.0	63-66
C- (passing grade for undergraduate students)	Minimum goals achieved with major flaws	1.7	60-62
F	Minimum goals not achieved	0	59 and below
X	Not graded due to unexcused absences or other reasons	0	0
W	Withdrawal		
NG	No grade reported		
IP	In progress		
TR	Transfer credit		
EX	Exempted		

http://www.oia.ntu.edu.tw/upload/files/20150616223619.pdf

Ch. 1 Signals & Systems

What Are Signals?

- Definition
  - A signal is a function that "conveys information about the behavior or attributes of some phenomenon". (Wikipedia)
  - Signals: any variables that carry information
  - Commonly observed and processed (e.g. calculated, stored, transmitted, recovered, predicted, and so on) in communication systems, signal processing, etc. areas.
  - Ever heard of data (or "big data")?
    They are observed/represented/processed as signals too!



What Are Signals? (cont’d)

- Examples
  - Voice/sound/music $s = f(t)$ as signals

What Are Signals? (cont'd)

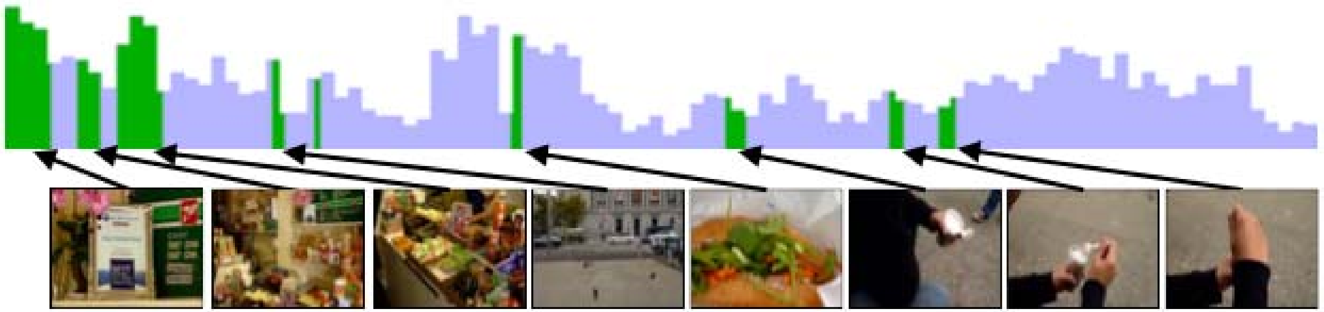
- Examples
  - Image $\mathrm{I} = f(x, y)$ as signals

$f(1,1) = 157$

157	153	174	168	150	152	129	151	172	161	155	156
155	182	163	74	75	62	33	17	110	210	180	154
180	180	50	14	34	6	10	33	48	106	159	181
206	109	5	124	131	111	120	204	166	15	56	180
194	68	137	251	237	239	239	228	227	87	71	201
172	105	207	233	233	214	220	239	228	98	74	206
188	88	179	209	185	215	211	158	139	75	20	169
189	97	165	84	10	168	134	11	31	62	22	148
199	168	191	193	158	227	178	143	182	106	36	190
205	174	155	252	236	231	149	178	228	43	95	234
190	216	116	149	236	187	86	150	79	38	218	241
190	224	147	108	227	210	127	102	36	101	255	224
190	214	173	66	103	143	96	50	2	109	249	215
187	196	235	75	1	81	47	0	6	217	255	211
183	202	237	145	0	0	12	108	200	138	243	236
195	206	123	207	177	121	123	200	175	13	96	218

What Are Signals? (cont'd)

- Examples
  - Video $\mathrm{V} = f(x, y, t)$ as signals



$v_1 = f(x, y, t_1)$ $v_2 = f(x, y, t_2)$

What Are Signals? (cont'd)

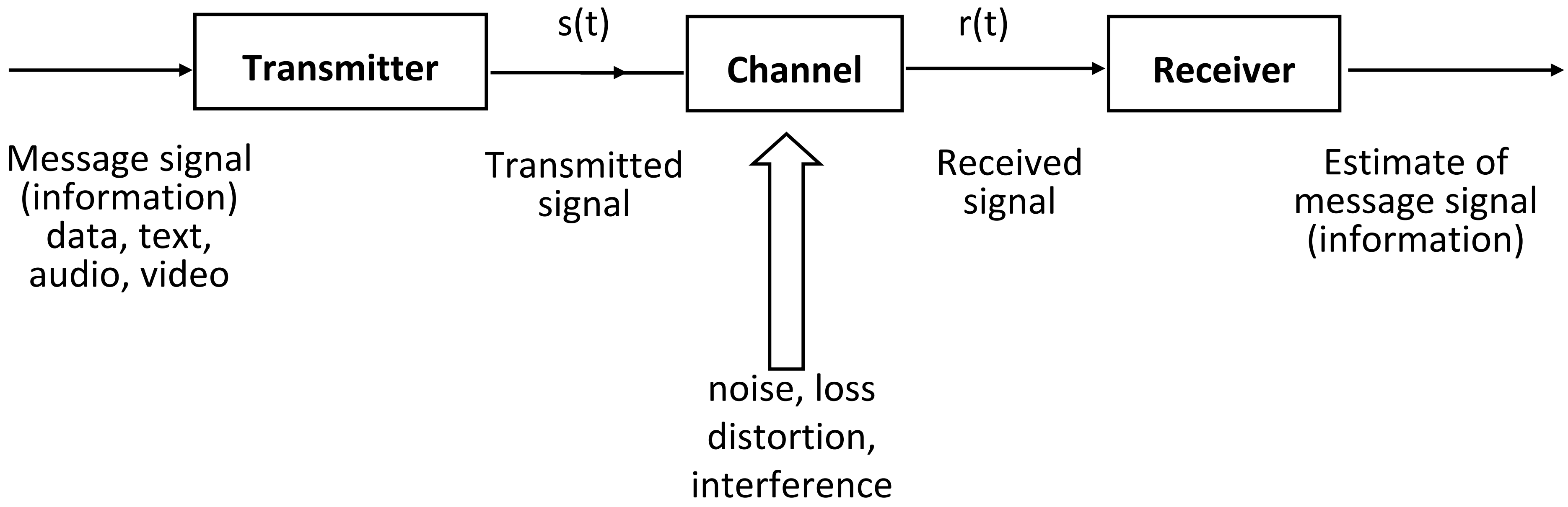
- Properties
  - Signals can be 1D, 2D, 3D, or beyond.
  - Signals can be consisted of 1s and 0s (i.e., binary signals).
  - Signals can be of real or complex values.
  - Signals can be continuous-time or discrete-time signals. (e.g., signals processed in analog or digital circuits)

What Are Systems?

- Definition
  - A system is a group of interacting or interrelated entities that form a unified whole.
  - A system is delineated by its spatial and temporal boundaries, surrounded and influenced by its environment, described by its structure and purpose and expressed in its functioning.
  - A system processes <u>input signals</u> to produce <u>output signals</u>.
  - Sometimes referred to as “models”.

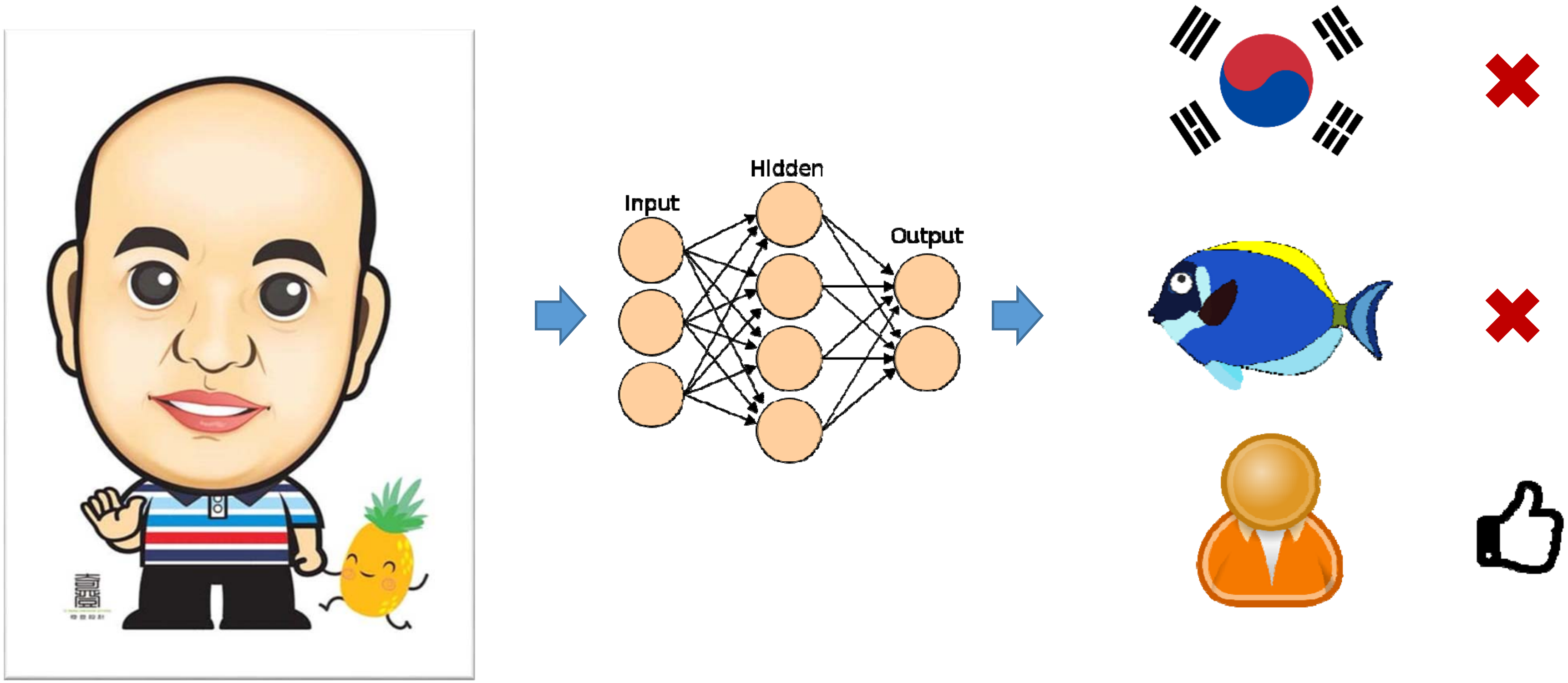
What Are Systems? (cont'd)

- Examples
  - Communication systems



What Are Systems? (cont'd)

- Examples
  - Image recognition systems

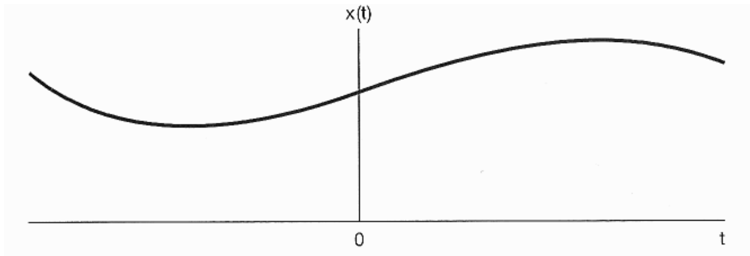


Sect. 1.1 Continuous-Time and Discrete-Time Signals

Sect. 1.1 Continuous/Discrete-Time Signals

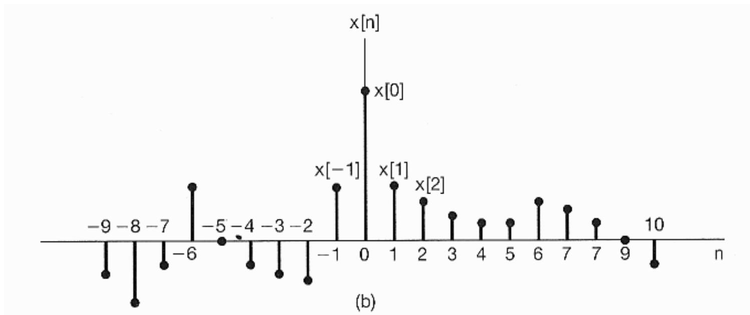
- Mathematical Representation
  - For a continuous-time (CT) signal, the independent variable is always enclosed by a parenthesis $(\cdot)$, e.g.,

$$x(t), y(t), z(t), I(x,y), f(x,y,t), etc.$$



  - For a discrete-time (DT) signal, the independent variable is always enclosed by a brackets $[\cdot]$, e.g.,

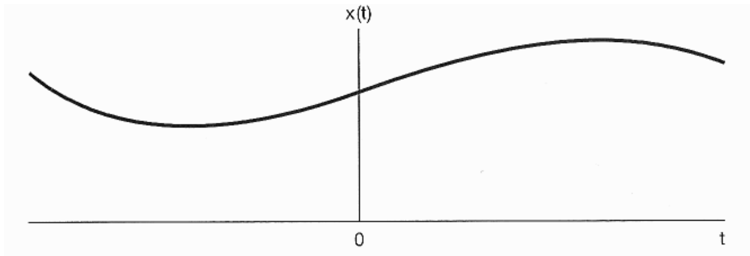
$$x[n], y[n], z[n], I[m,n], F[u,v,n], etc.$$



Sect. 1.1 Continuous/Discrete-Time Signals

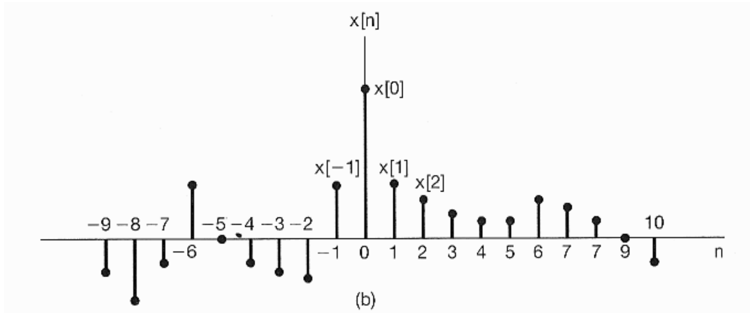
- Mathematical Representation
  - For a continuous-time (CT) signal, the independent variable is always enclosed by a parenthesis $(\cdot)$, e.g.,

$$x(t), y(t), z(t), I(x, y), f(x, y, t), etc.$$



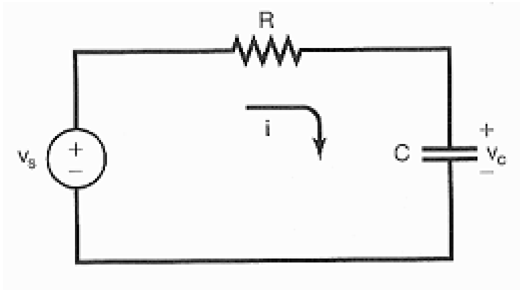
  - For a discrete-time (CT) signal, the independent variable is always enclosed by a brackets $[\cdot]$, e.g.,

$$x[n], y[n], z[n], I[m, n], F[u, v, n], etc.$$

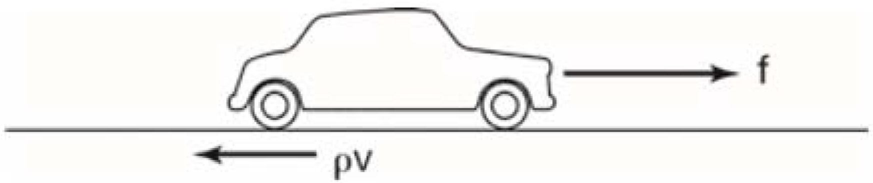


Examples of Continuous Signals

- Circuits (e.g., voltage, current, electric charge)

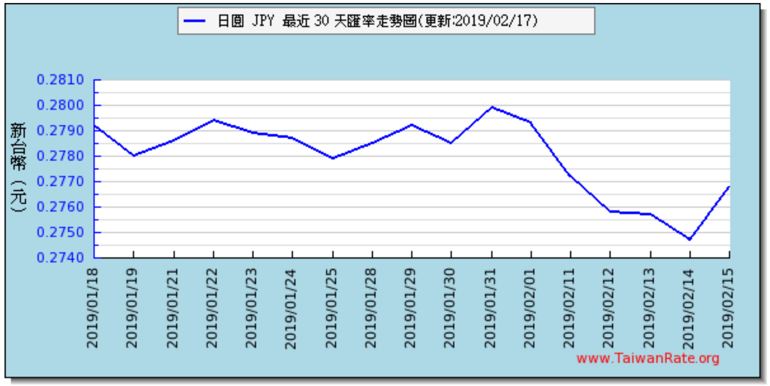


- Motion (e.g., location, velocity, acceleration)

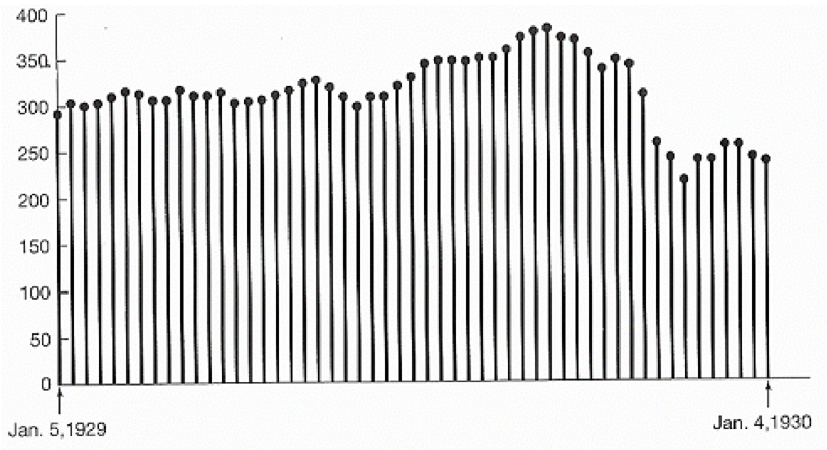


Examples of Discrete Signals

- Foreign exchange rate (e.g., JPY to NTD)



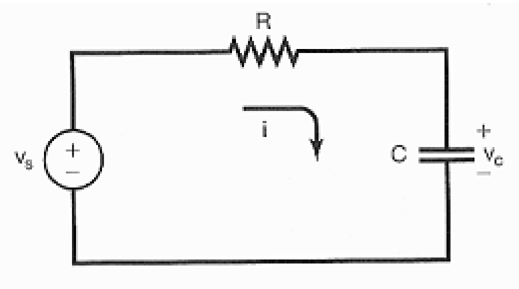
- Stock market index (e.g., weekly Dow-Jones stock market index)



Sect. 1.1 Continuous/Discrete-Time Signals

- Signal Energy & Power
  - Instantaneous Power $p(t) = v(t)i(t) = \frac{1}{R}v^2(t)$
  - Total energy over a finite time interval $\int_{t_1}^{t_2} p(t)dt = \int_{t_1}^{t_2} \frac{1}{R}v^2(t)dt$
  - Average power over a finite time interval

$$\frac{1}{t_2 - t_1}\int_{t_1}^{t_2} p(t)dt = \frac{1}{t_2 - t_1}\int_{t_1}^{t_2} \frac{1}{R}v^2(t)dt$$



Sect. 1.1 Continuous/Discrete-Time Signals

- Signal Energy & Power (cont'd)
  - Total energy over a finite time interval

$$E \triangleq \int_{t_1}^{t_2} |x(t)|^2 dt \qquad \text{continuous-time}$$

$$E \triangleq \sum_{n=n_1}^{n_2} |x[n]|^2 \qquad \text{discrete-time}$$

  - Time-averaged power over a finite time interval

$$P \triangleq \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} |x(t)|^2 dt \qquad \text{continuous-time}$$

$$P \triangleq \frac{1}{n_2 - n_1 + 1} \sum_{n=n_1}^{n_2} |x[n]|^2 \qquad \text{discrete-time}$$

Sect. 1.1 Continuous/Discrete-Time Signals

- Signal Energy & Power (cont'd)
  - Total energy over an infinite time interval

$$E_\infty \triangleq \lim_{T\to\infty} \int_{-T}^{T} |x(t)|^2 dt = \int_{-\infty}^{+\infty} |x(t)|^2 dt$$

$$E_\infty \triangleq \lim_{N\to\infty} \sum_{n=-N}^{+N} |x[n]|^2 = \sum_{n=-\infty}^{+\infty} |x[n]|^2$$

  - Time-averaged power over an infinite time interval

$$P_\infty \triangleq \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 dt$$

$$P_\infty \triangleq \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{+N} |x[n]|^2$$

Sect. 1.2 Transformations of the Independent Variable

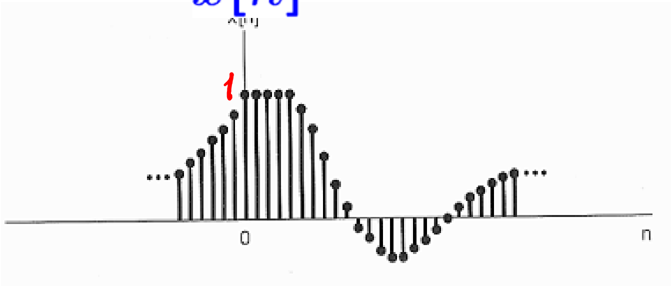
Key Concepts about the Transformations

- Properties
  - Time shift
  - Time reversal
  - Time scaling
  - Periodic signal and its fundamental period
  - Even and odd signals

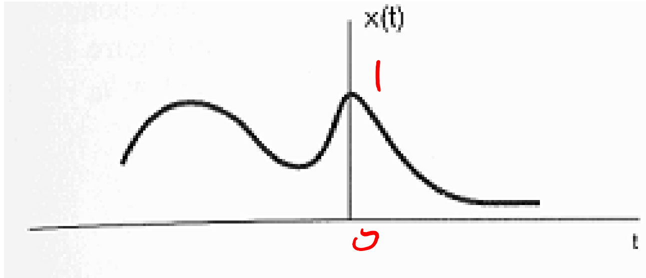
Shifting, Reversal, and Scaling

- Time shift: $\begin{cases} n_0,\ t_0 > 0 : & \text{delay} \checkmark \\ n_0,\ t_0 < 0 : & \text{advance} \end{cases}$

$x[n]$

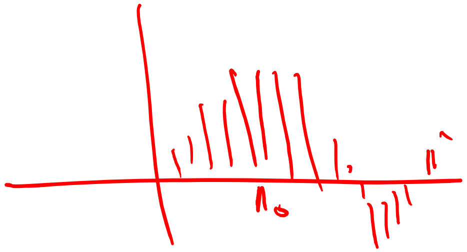


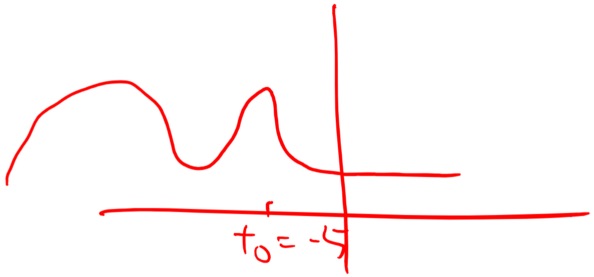
$x(t)$



$x[n - n_0]$

$x(t - t_0)$





$8 = n_0 > 0$

$x[n - 8]$

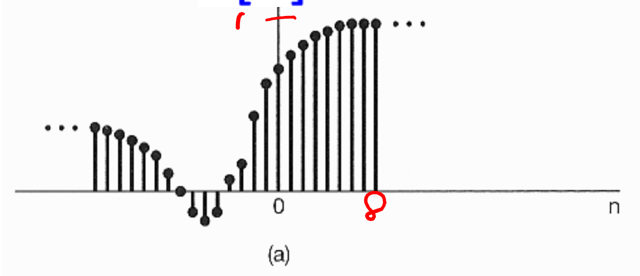
$-5 = t_0 < 0$

$x(t + 5)$

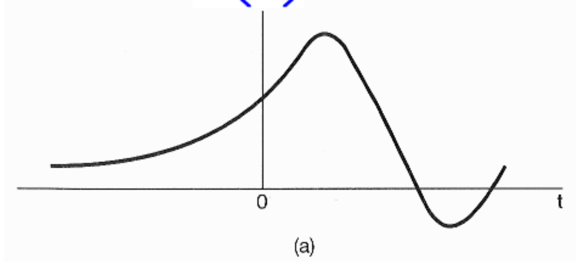
Shifting, Reversal, and Scaling

- Time reversal:

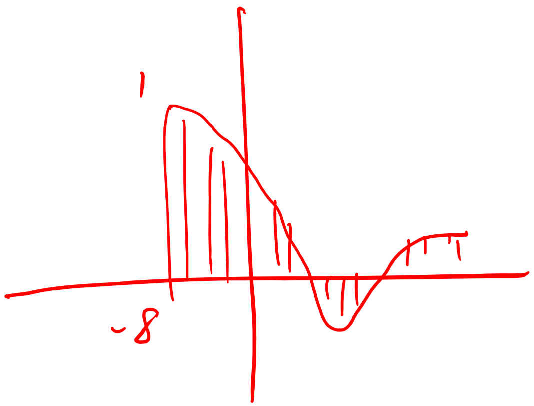
$x[n]$



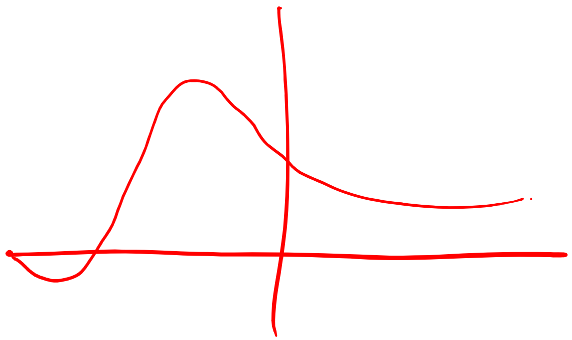
$x(t)$



$x[-n]$

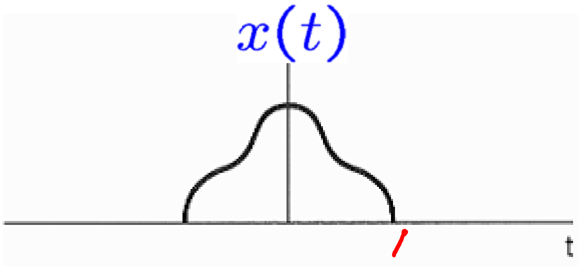


$x(-t)$

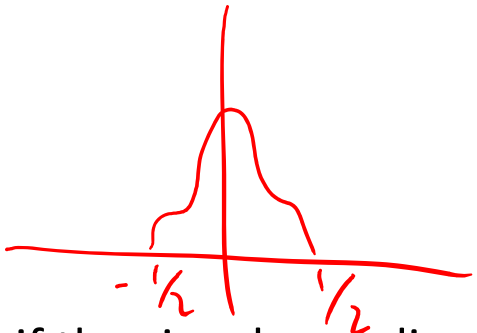


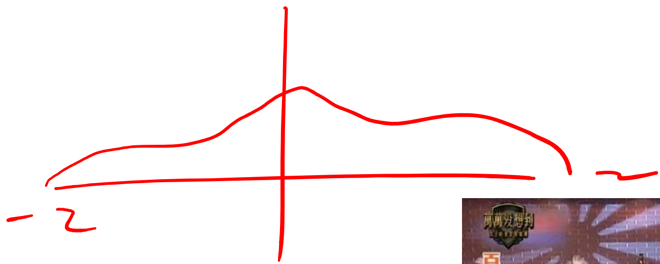
Shifting, Reversal, and Scaling

- Time scaling:



$t \rightarrow 2t$ $\quad$ $x(2t)$ $\qquad$ $t \rightarrow t/2$ $\quad$ $x(t/2)$



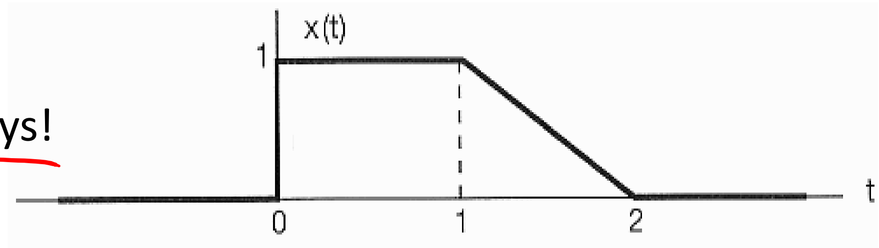


- What if the signals are discrete? What would $x[kn]$ be?

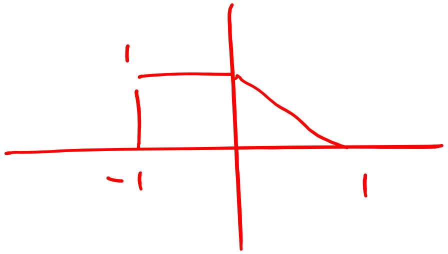
Shifting, Reversal, and Scaling (cont'd)

- Example 1.1
  - Determine $x(-t+1)$
  - You can solve this in two different ways!

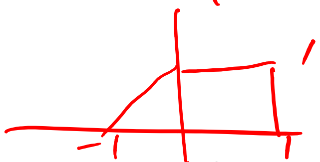
$x(t)$



(i) $x(t) \to x(t+1)$

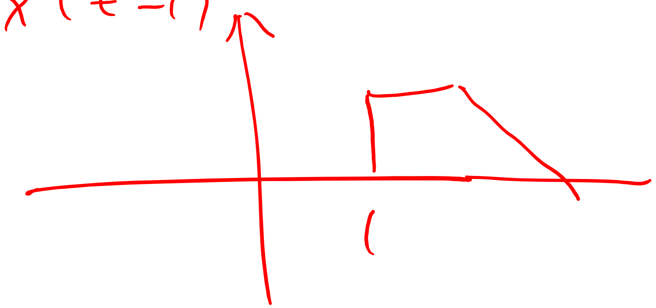


$x(t+1) \to x(-t+1)$

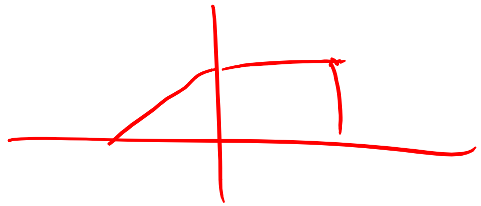


(ii) $x(-t+1) \to x(-(t-1))$

$x(t-1)$



$x(-(t-1))$



- Try Example 1.2 by yourself!

Shifting, Reversal, and Scaling (cont'd)

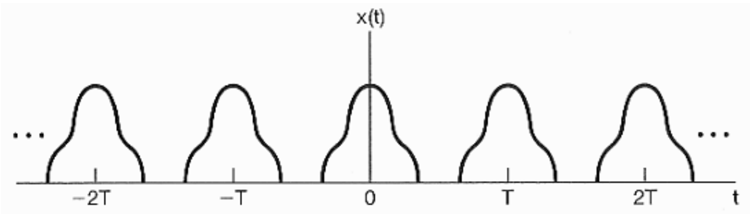
- Example 1.3
  - $x(t) \rightarrow x(\alpha t + \beta)$
  - $|\alpha| < 1$
  - $|\alpha| > 1$
  - $\alpha < 0$
  - $\beta > 0$
  - $\beta < 0$

- Linearly stretched
- Linearly compressed
- Time reversal
- Advanced time shift
- Delayed time shift

1.2.2 Periodic Signals

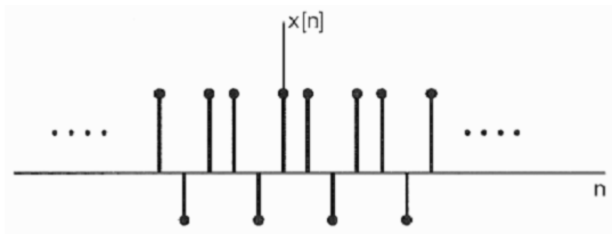
- What are periodic signals?
  - Signals which are periodic…
  - Seriously, periodic signals are the signals which would be unchanged by a time shift of T (and obviously 2T, 3T, etc.)
  - Thus, T (or N if discrete) is called the fundamental period, denoted as T_0 or N_0.
  - If a signal is *not* periodic, we call it an aperiodic signal.

A CT Periodic Signal



$x(t) = x(t+T)$ for $T > 0$ and all values of t

A DT Periodic Signal

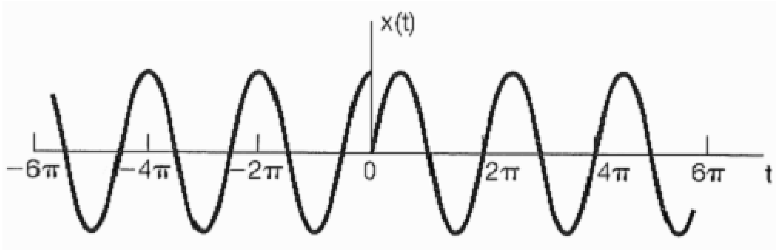


$N = 3$

$x[n] = x[n+N]$ for $N > 0$ and all values of n

1.2.2 Periodic Signals (cont'd)

- Example 1.4
  - Is $x(t) = \begin{cases} \cos(t), & \text{if } t < 0 \\ \sin(t), & \text{if } t \geq 0 \end{cases}$ periodic? If so, what is its periodicity?



  - Let's first consider the cases when t<0 and t>0…

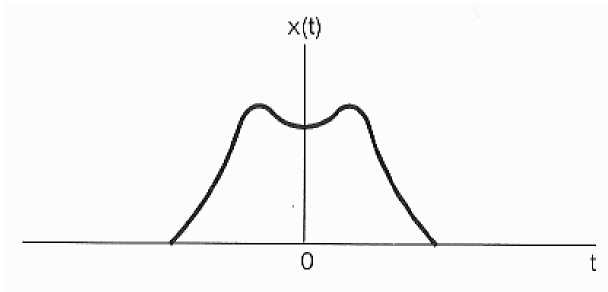
    $t<0, \; x(t) = \cos(t) \Rightarrow T_0 = 2\pi$

    $t \geq 0, \; x(t) = \sin(t) \Rightarrow T_0 = 2\pi$

  - What about t=0? Discontinuity occurs, and it does NOT recur at any other time.
  - Since every feature in the shape of a periodic signal must recur periodically, we conclude that x(t) is… aperiodic signal.

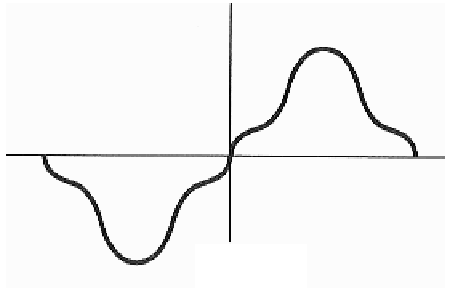
1.2.3 Even and Odd Signals

even



(a)

odd



- What are even and odd signals?
  - A signal is even if $x(-t) = x(t)$ or $x[-n] = x[n]$.
  - A signal is odd if $x(-t) = -x(t)$ or $x[-n] = -x[n]$.
- Why even/odd signals are important?
  - ANY signals can be decomposed into a sum of an even signal and an odd signal!
  - BUT, how to get the even/odd parts of a signal?

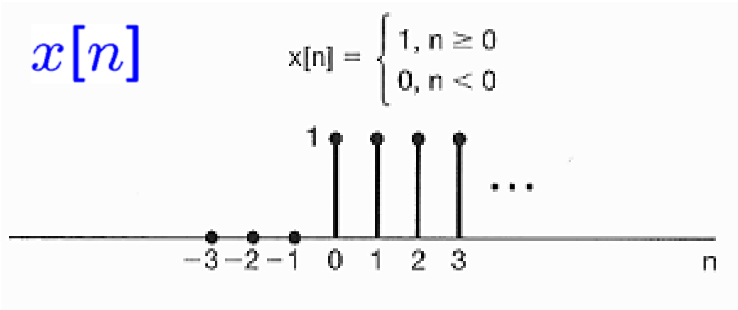
$$Even(x(t)) = \frac{1}{2}\{x(t) + x(-t)\}$$

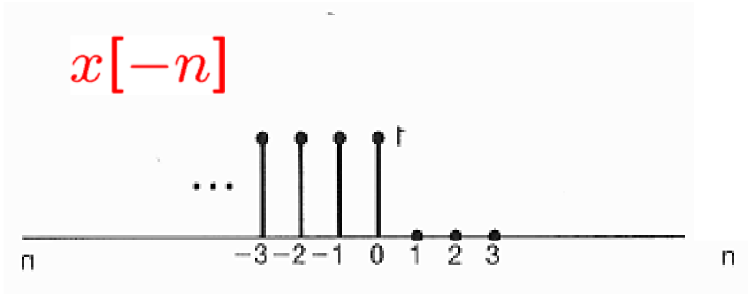
$$Odd\{x(t)\} = \frac{1}{2}\{x(t) - x(-t)\}$$

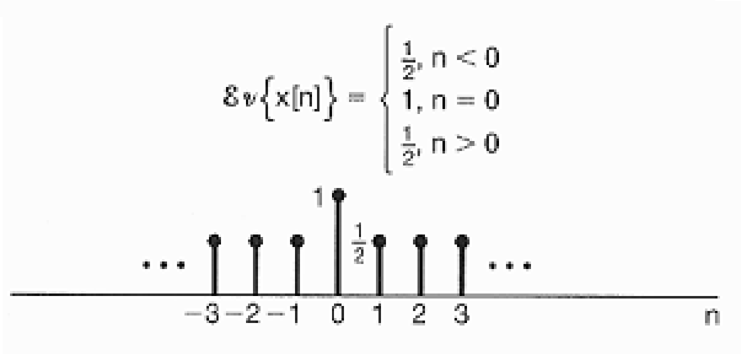
$$\Rightarrow x(t) = \mathcal{E}v\left\{x(t)\right\} + \mathcal{O}d\left\{x(t)\right\}$$

1.2.3 Even and Odd Signals (cont'd)

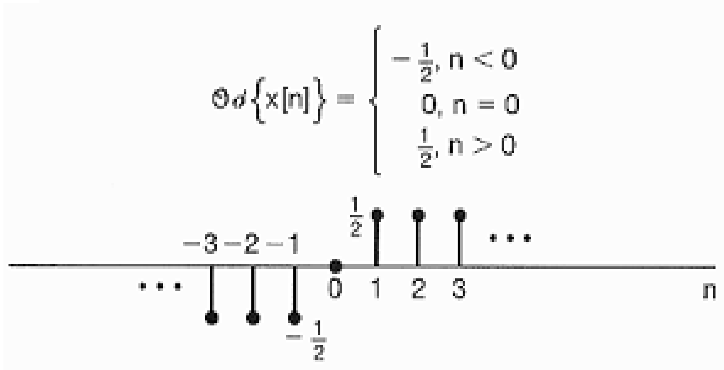
- Even-odd decomposition of a DT signal







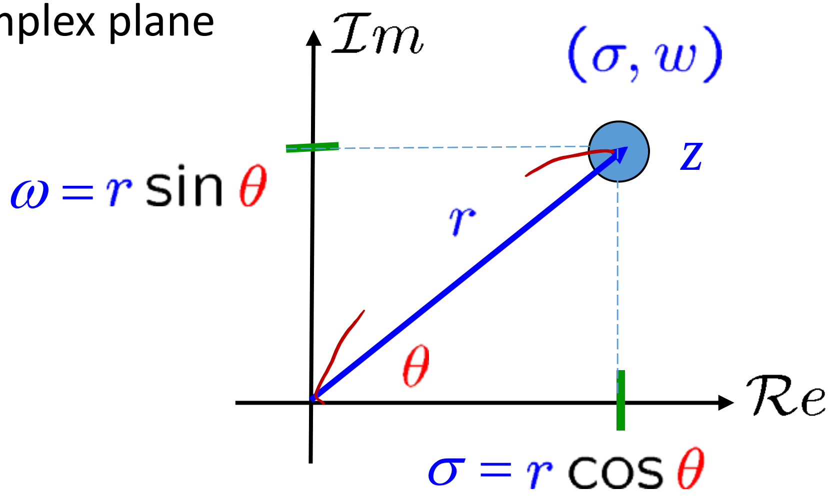
$$\mathcal{E}v\left\{x[n]\right\} = \frac{1}{2}\left[x[n] + x[-n]\right]$$



$$\mathcal{O}d\left\{x[n]\right\} = \frac{1}{2}\left[x[n] - x[-n]\right]$$

1.3 Exponential and Sinusoidal Signals

- Mathematical Review: Complex number & complex plane
  - Complex plane



$$z = \sigma + j\omega$$

$$j = \sqrt{-1}$$

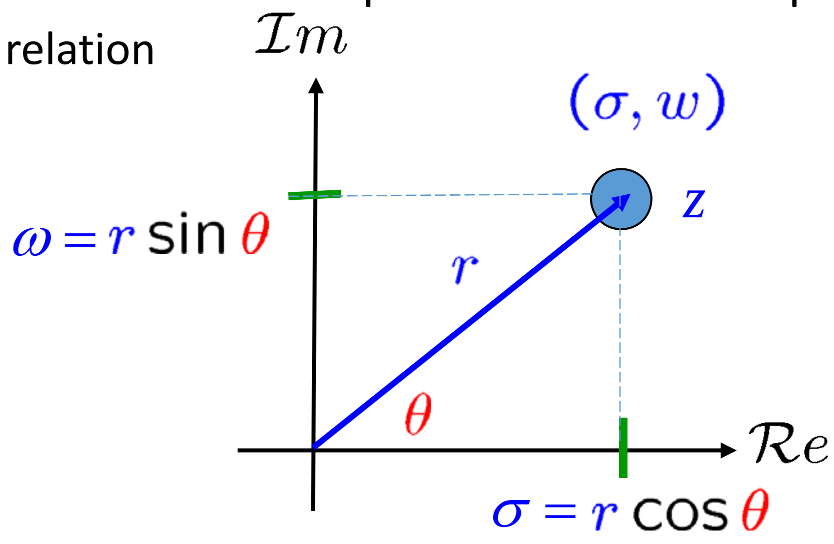
  - Magnitude and phase representation

$$\sigma + jw \quad \Rightarrow \quad \begin{cases} r = \sqrt{\sigma^2 + w^2} \\ \tan(\theta) = \frac{w}{\sigma} \end{cases} \quad \Rightarrow \quad \sigma + jw \; = \; r\, e^{j\theta}$$

$= r \cdot e^{j\theta}$

1.3 Exponential and Sinusoidal Signals

- Mathematical Review: Complex number & complex plane
  - Euler's relation



$$e^{j\theta} = \cos\theta + j\sin\theta$$

$$\Rightarrow z = \sigma + j\omega = r\left(\cos\theta + j\sin\theta\right)$$

$$= (r\cos\theta) + j(r\sin\theta)$$

1.3.1 Continuous-Time Complex Exponential and Sinusoidal Signals

- Continuous-time (CT) complex exponential signals

$$x(t) = Ce^{at}$$

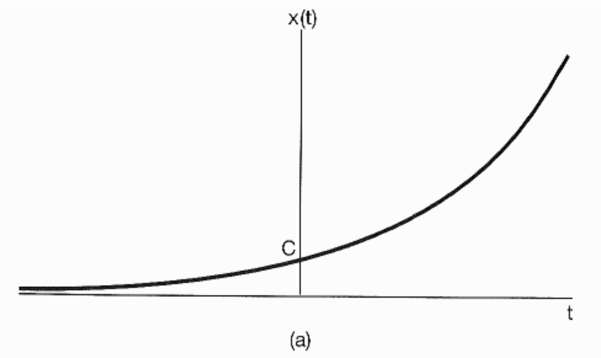
where C and a are, in general, complex numbers.

$$a = \sigma + jw$$

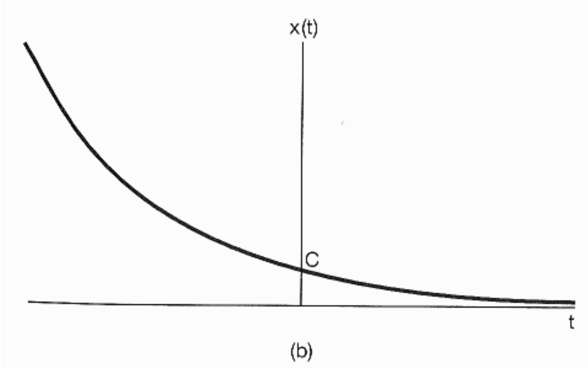
$$C = |C|\, e^{j\theta}$$

1.3.1 Continuous-Time Complex Exponential and Sinusoidal Signals (cont'd)

- Continuous-time (CT) complex exponential signals: $x(t) = Ce^{at}$
- Special case 1:
  if C and a are real.



$a > 0$



$a < 0$

1.3.1 Continuous-Time Complex Exponential and Sinusoidal Signals (cont'd)

- Continuous-time (CT) complex exponential signals: $x(t) = Ce^{at}$
- Special case 2:
  if a is pure imaginary, i.e., $a = j\omega_0$ (where ω_0 is real)
  and C is complex $C = Ae^{j\phi}$, then

$$x(t) = Ae^{j(\omega_0 t + \phi)}$$

- It is periodic and the fundamental period $T_0 = \frac{2\pi}{\omega_0}$. Why?
  Proof:

1.3.1 Continuous-Time Complex Exponential and Sinusoidal Signals (cont'd)

- Continuous-time (CT) complex exponential signals: $x(t) = Ce^{at}$
- Energy of CT complex periodic signals

$$E_{period} = \int_0^{T_0} \left| Ae^{j(\omega_0 t+\phi)} \right|^2 dt \qquad P_{period} = \frac{1}{T_0} E_{period} = A^2$$

$$= \int_0^{T_0} A^2 dt = A^2 T_0$$

$$E_\infty = \infty \qquad P_\infty = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \left| Ae^{j(\omega_0 t+\phi)} \right|^2 dt = A^2$$

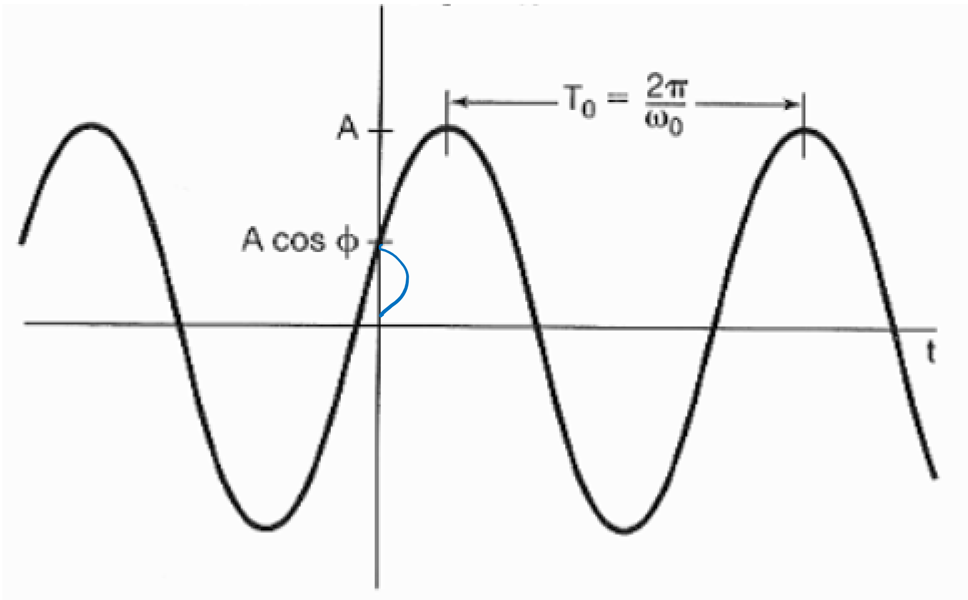
- For ANY non-zero periodic signals, the total energy E must be infinite.

1.3.1 Continuous-Time Complex Exponential and Sinusoidal Signals (cont'd)

- For $x(t) = Ae^{j(\omega_0 t+\phi)}$, its real part is a sinusoidal signal with fundamental period $T_0 = \frac{2\pi}{\omega_0}$.

$$y(t) = \mathcal{Re}\{x(t)\} = A\cos(\omega_0 t + \phi)$$

$$A\cos(\omega_0 t + \phi)$$



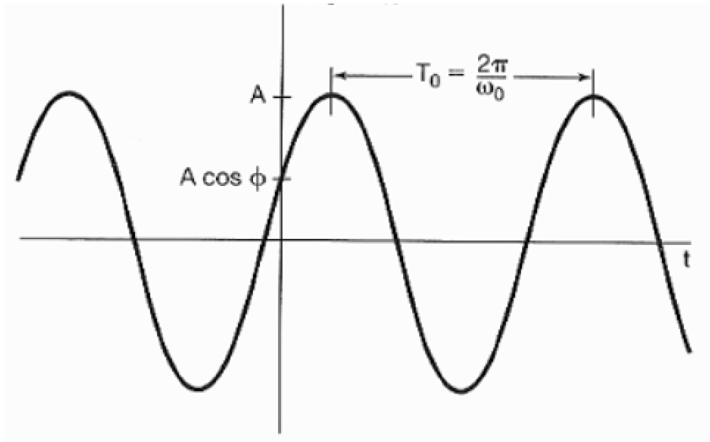
1.3.1 Continuous-Time Complex Exponential and Sinusoidal Signals (cont'd)

- Fundamental period vs. fundamental frequency

  fundamental frequency = 2π / fundamental period

  fundamental period = 2π / fundamental frequency

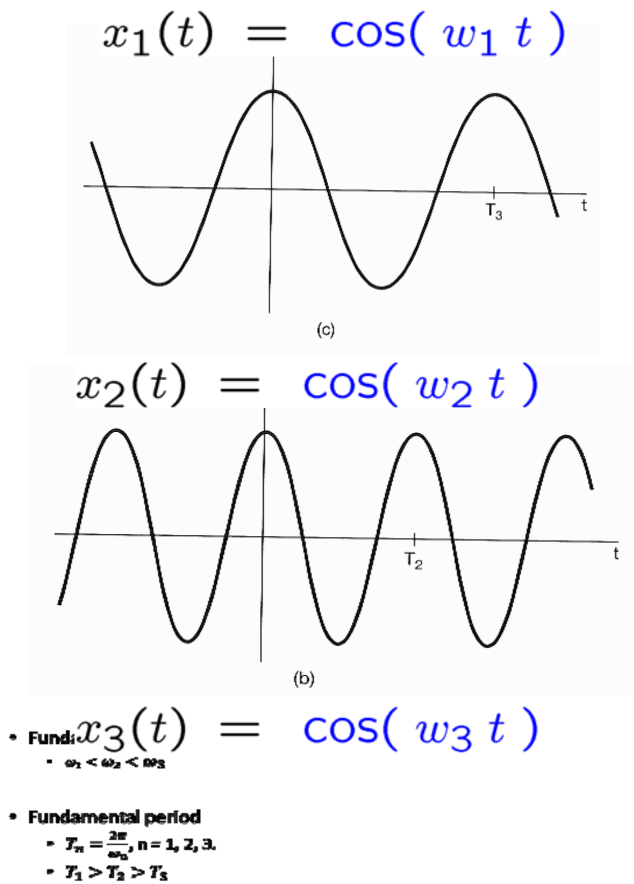
- For example, we have $Acos(\omega_0 t + \phi)$, whose fundamental period is $T_0 = \frac{2\pi}{\omega_0}$, and fundamental frequency is $\frac{2\pi}{T_0} = \omega_0$.



1.3.1 Continuous-Time Complex Exponential and Sinusoidal Signals (cont'd)

- Fundamental frequency
  - $\omega_1 < \omega_2 < \omega_3$

- Fundamental period
  - $T_n = \dfrac{2\pi}{\omega_n}$, n = 1, 2, 3.
  - $T_1 > T_2 > T_3$



1.3.1 Continuous-Time Complex Exponential and Sinusoidal Signals (cont'd)

- Example 1.5
  What is the fundamental period of $x(t) = e^{j2t} + e^{j3t}$?
- Let's check each complex signal first…

e^{j2t} : fundamental period $= \pi$

e^{j3t} " $= \frac{2}{3}\pi$

$\Rightarrow$ "2π"

1.3.1 Continuous-Time Complex Exponential and Sinusoidal Signals (cont'd)

- CT Complex Exponential Signals: $x(t) = Ce^{at}$
- General case:
  Both C and a are complex.

  C is expressed in a polar form: $C = |C|e^{j\phi}$

  a is expressed in a rectangular form: $a = r + j\omega_0$

$$Ce^{at} = (|C|e^{j\theta})(e^{(r+jw_0)t})$$
$$= |C|\ e^{rt}\ e^{j(w_0 t+\theta)}$$
$$= |C|e^{rt}\ \cos(w_0 t+\theta) + j|C|e^{rt}\ \sin(w_0 t+\theta)$$

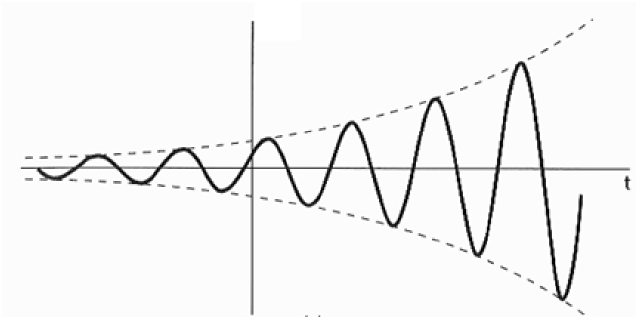
Note: The real part is a growing sinusoid signal:

$$|C|e^{rt}\cos(\omega_0 t + \theta)$$

1.3.1 Continuous-Time Complex Exponential and Sinusoidal Signals (cont'd)

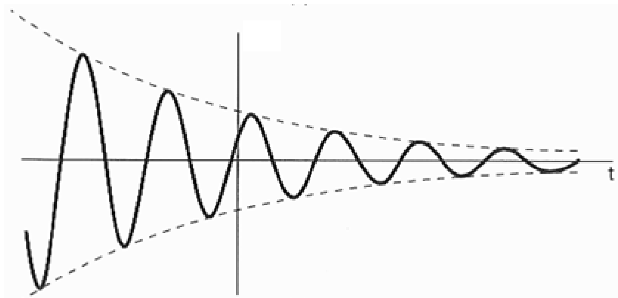
- CT Complex Exponential Signals: $x(t) = Ce^{at}$
- General case: Both C and a are complex.
- The real part is a growing sinusoidal signal: $|C|e^{rt}\cos(\omega_0 t + \theta)$

$r > 0$



(a)

$r < 0$



(b)

1.3.2 Discrete-Time Complex Exponential and Sinusoidal Signals

- CT complex exponential signal: $x(t) = Ce^{at}$
- DT complex exponential signal:

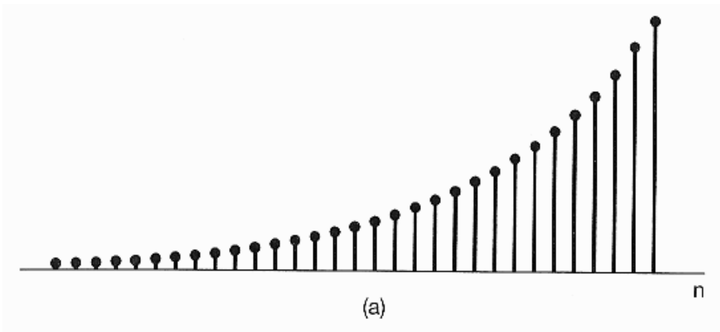
$$\begin{aligned} x[n] &= Ce^{\beta n} \\ &= C(e^{\beta})^n \\ &= C\alpha^n \end{aligned}$$

where C and α are, in general, complex numbers.

1.3.2 Discrete-Time Complex Exponential and Sinusoidal Signals (cont'd)

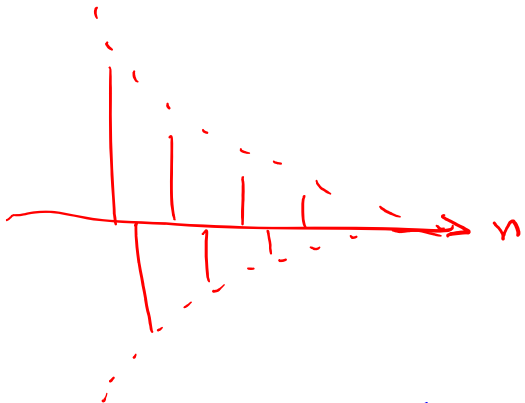
- DT complex exponential signals: $x[n] = C\alpha^n$
- <u>Special case 1</u>:
  if C and α are both real numbers.

$\alpha > 1$

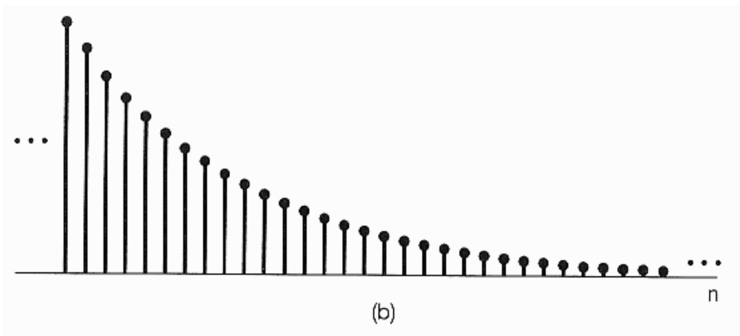


(a)

$-1 < \alpha < 0$

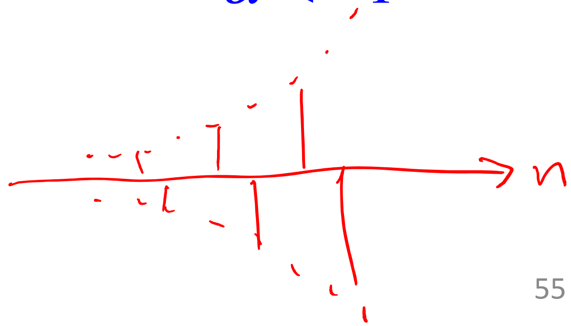


$0 < \alpha < 1$



(b)

$\alpha < -1$



1.3.2 Discrete-Time Complex Exponential and Sinusoidal Signals (cont'd)

- DT complex exponential signals: $x[n] = C\alpha^n$ where $\alpha = e^{\beta}$
- Special case 2:
  β is purely imaginary and can be expressed as $\beta = j\omega_0$ (ω_0 is real), i.e., $\alpha = e^{j\omega_0}$ and $|\alpha| = 1$.

$$x[n] = Ce^{j\omega_0 n}$$

Moreover, if $C = Ae^{j\phi}$, we have

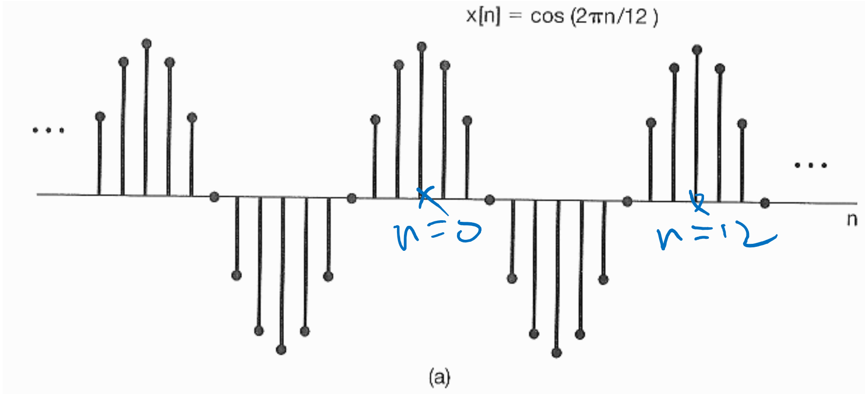
$$x[n] = A \cdot e^{j\phi} \cdot e^{j\omega_0 \cdot n}$$
$$= A \cdot e^{j(\omega_0 n + \phi)} = \boxed{A \cdot \cos(\omega_0 n + \phi)} + jA\sin(\omega_0 n + \phi)$$

The real part is a sinusoid signal. Why?

1.3.2 Discrete-Time Complex Exponential and Sinusoidal Signals (cont'd)

- DT sinusoid signal: $Acos[\omega_0 n + \phi]$

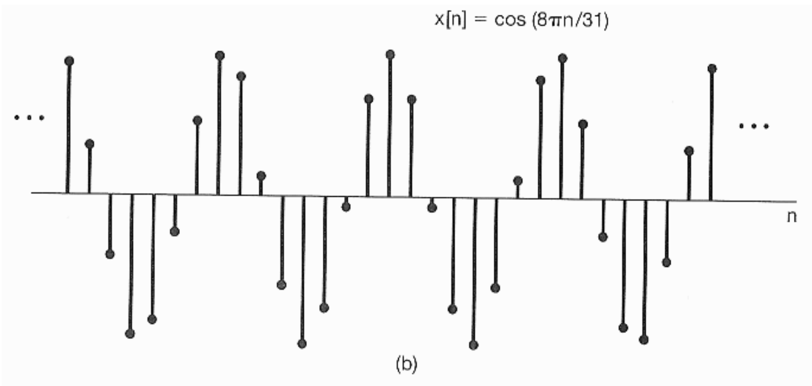
$$x[n] = \cos\left(\frac{2\pi}{12}n\right)$$



(a)

$$x[n] = \cos\left(\frac{n}{6}\right)$$

$$x[n] = \cos\left(\frac{4\cdot 2\pi}{31}n\right)$$



(b)

- DT sinusoid signal: $Acos(\omega_0 n + \phi)$

1.3.2 Discrete-Time Complex Exponential and Sinusoidal Signals (cont'd)

- DT complex exponential signals: $x[n] = C\alpha^n$
- General case:
  if C and α are both complex numbers & their amplitude may not be 1.
  $C = |C|e^{j\theta}$, $\alpha = |\alpha|e^{j\omega_0}$

$$x[n] = C\alpha^n =$$

$$x[n] = |C|\cdot|\alpha|^n\cdot e^{j(\omega_0 n+\theta)}$$

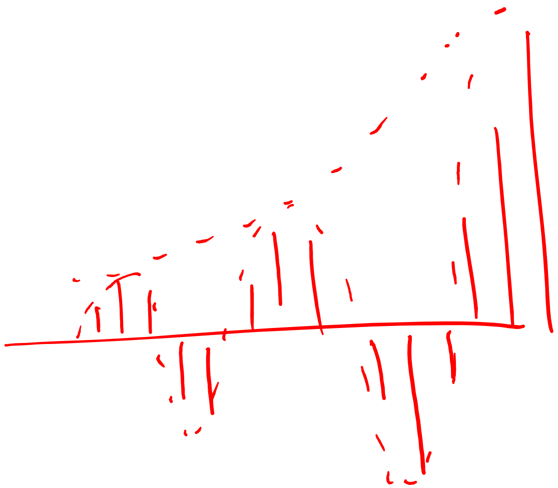
$$= \boxed{|C|\cdot|\alpha|^n\cdot \cos(\omega_0 n+\theta)} + j\{\quad\}\sin(\quad)$$

Its real part is a growing DT sinusoid signal:

1.3.2 Discrete-Time Complex Exponential and Sinusoidal Signals (cont'd)

- Growing DT sinusoid signals: $|C||\alpha|^n \cos(\omega_0 n + \theta)$

$|\alpha| > 1$



$|\alpha| < 1$

1.3.3 Periodicity Properties of DT Complex Exponential Signals

- Recall that for CT exponential signal $x(t) = e^{j\omega_0 t}$, the fundamental frequency is ω_0 . $T_0 = \frac{2\pi}{\omega_0}$
- A larger ω corresponds to a larger fundamental frequency.
- For DT exponential ones $x[n] = e^{j\omega_0 n}$, is the fundamental frequency? ω_0

- Let's take a closer look…
  - For $x[n] = e^{j\omega_0 n}$, the fund. frequency does NOT always increase with ω_0. This is because $e^{j(\omega_0+2\pi)n} = \overset{1}{\boxed{e^{j2\pi n}}} e^{j\omega_0 n} = e^{j\omega_0 n}$
- When ω_0 is changed into $\omega_0 + 2\pi k$ and k is an integer, then the fundamental frequency is unchanged!

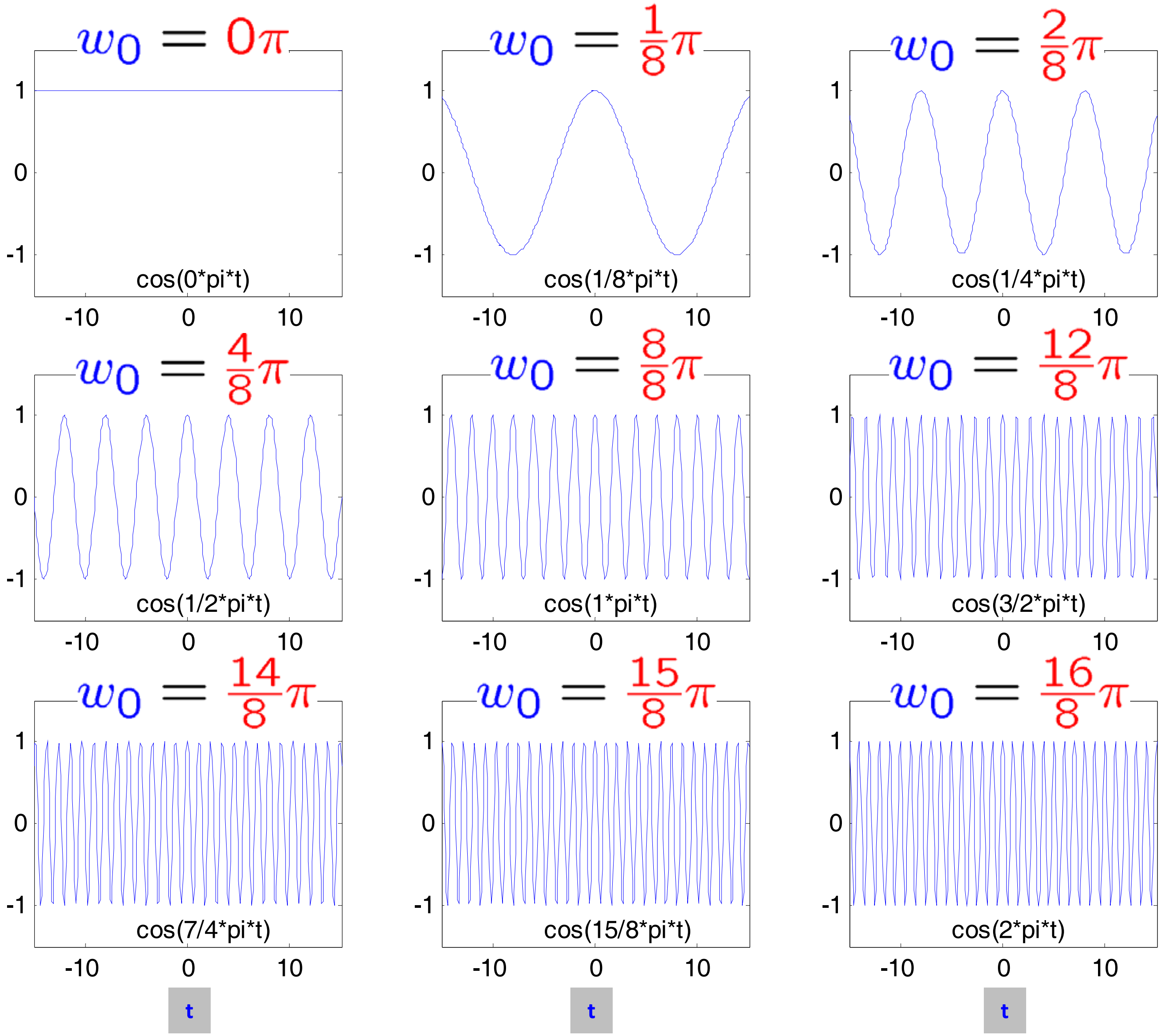
1.3.3 Periodicity of DT Complex Exponential Signals (cont'd)

- Now we have $e^{j(\omega_0+2\pi)n}$=$e^{j2\pi n}e^{j\omega_0 n} = e^{j\omega_0 n}$.
- Therefore, only a frequency interval of length 2π is considered.
- Typically we have $0 \leq \omega_0 < 2\pi$ or $-\pi \leq \omega_0 < \pi$.
- Given the above observation, we have the
  low frequencies located at $\omega_0 =$ 0 , $\pm 2\pi$, …
  high frequencies located at $\omega_0 =$ $\pm\pi$, $\pm 3\pi$, …

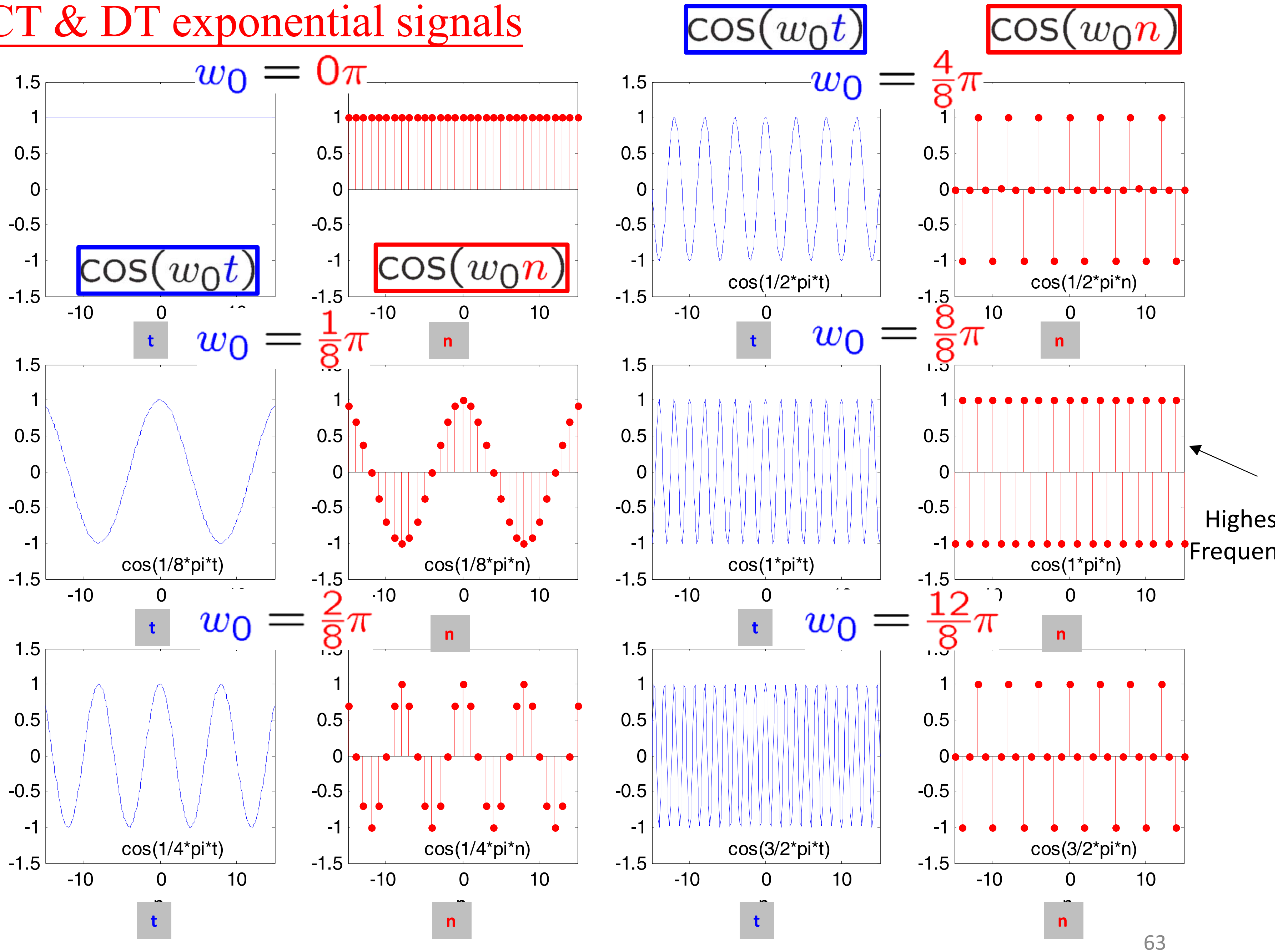
- Why? Think about $e^{j\omega_0 n} =$ 1 when $\omega_0 =$ 0, $\pm 2\pi$, ...
  and $e^{j\omega_0 n} =$ $(-1)^n$ when $\omega_0 =$ π, $\pm 3\pi$, ...

CT exponential signals

$\boxed{\cos(w_0 t)}$

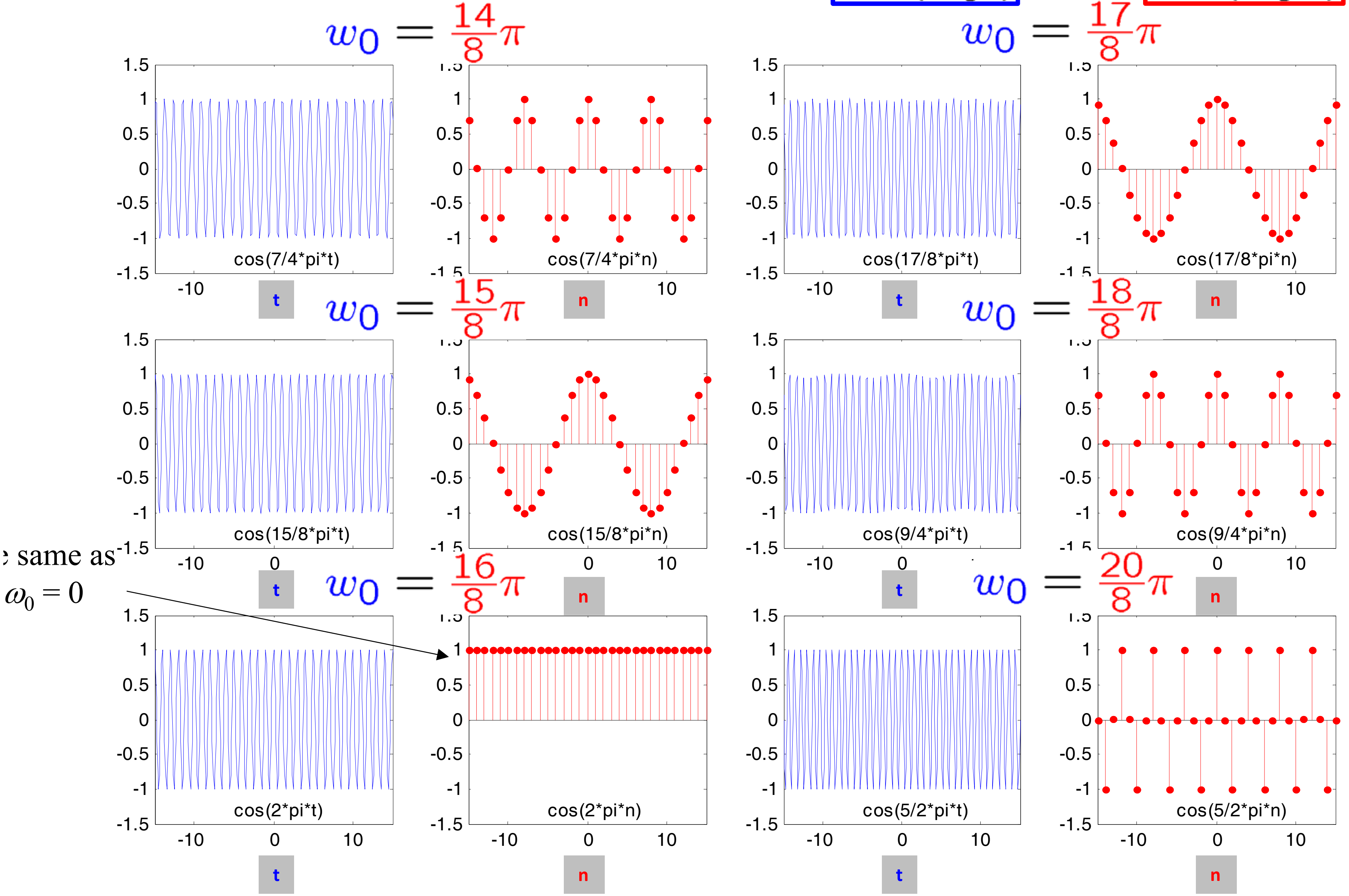


- <u>CT & DT exponential signals</u>

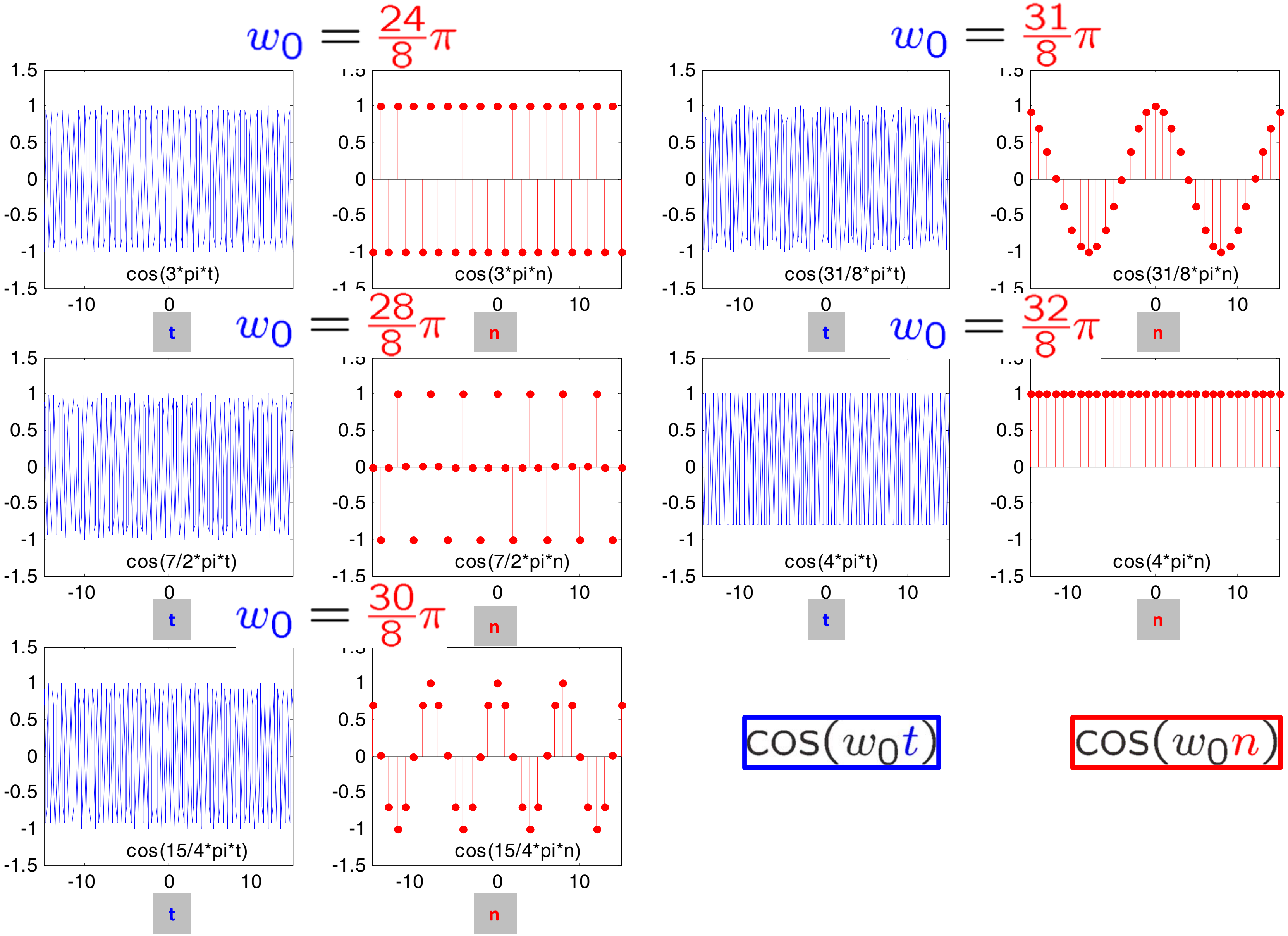


• CT & DT exponential signals

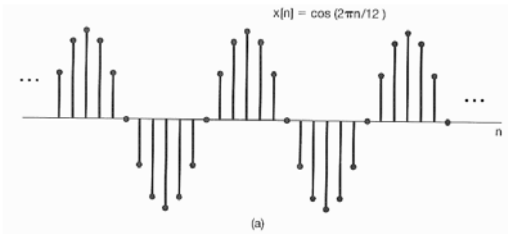
$\boxed{\cos(w_0 t)}$ $\boxed{\cos(w_0 n)}$



- <u>CT & DT exponential signals</u>

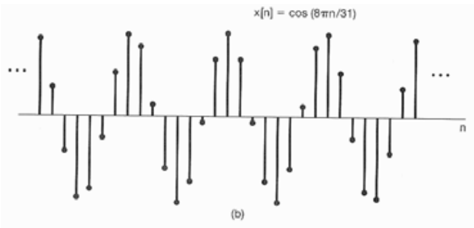


Comparisons of the Periods in the CT and DT Signals



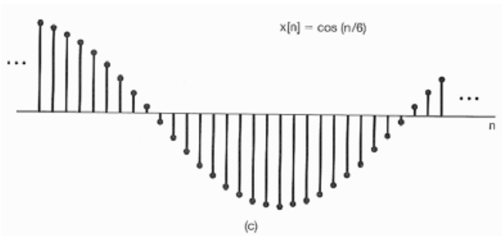
$$x(t) = \cos\left(\frac{2\pi}{12}t\right) \qquad T = 12$$

$$x[n] = \cos\left(\frac{2\pi}{12}n\right) \qquad N = 12$$



$$x(t) = \cos\left(\frac{4\cdot 2\pi}{31}t\right) \qquad T = \frac{31}{4}$$

$$x[n] = \cos\left(\frac{4\cdot 2\pi}{31}n\right) \qquad N = \frac{31}{4}?$$



$$x(t) = \cos\left(\frac{1}{6}t\right) \qquad T = 12\pi$$

$$x[n] = \cos\left(\frac{1}{6}n\right) \qquad N = 12\pi?$$

1.3.3 Periodicity of DT Complex Exponential Signals (cont'd)

- In the CT case, the period can be ANY positive real number.
- What about the DT case? The period should be a positive **integer**.
- For example, we have $e^{j\omega_0 n}$ or $\cos(\omega_0 n)$.
- If $\omega_0 = 2\pi \frac{m}{N}$, m and N are some integers and m is a prime to N,
  the fundamental period = N.
  the fundamental frequency = $\frac{2\pi}{N}$.

  If ω_0 does not have the form $2\pi \frac{m}{N}$, then the DT signal is aperiodic.

1.3.3 Periodicity of DT Complex Exponential Signals (cont'd)

TABLE 1.1 Comparison of the signals $e^{j\omega_0 t}$ and $e^{j\omega_0 n}$.

$e^{j\omega_0 t}$	$e^{j\omega_0 n}$
Distinct signals for distinct values of ω_0	Identical signals for values of ω_0 separated by multiples of 2π
Periodic for any choice of ω_0	Periodic only if $\omega_0 = 2\pi m/N$ for some integers $N > 0$ and m.
Fundamental frequency ω_0	Fundamental frequency* ω_0/m
Fundamental period $\omega_0 = 0$: undefined $\omega_0 \neq 0$: $\frac{2\pi}{\omega_0}$	Fundamental period* $\omega_0 = 0$: undefined $\omega_0 \neq 0$: $m\left(\frac{2\pi}{\omega_0}\right)$

*Assumes that m and N do not have any factor in common.

1.3.3 Periodicity of DT Complex Exponential Signals (cont'd)

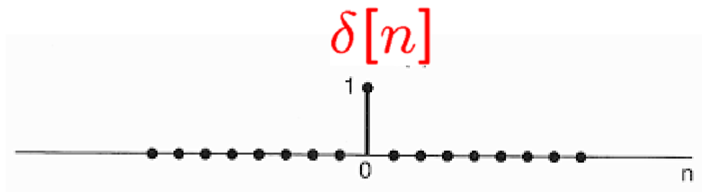
- Example 1.6
  What is the fundamental period of $x[n] = e^{j(2\pi/3)n} + e^{j(3\pi/4)n}$?

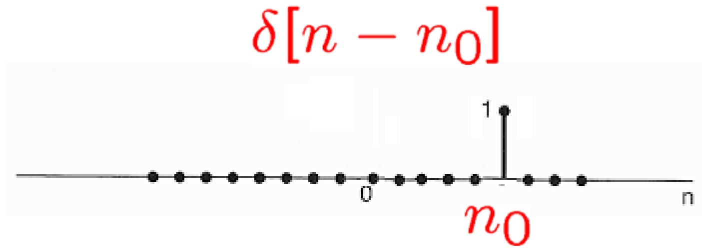
1.4 The Unit Impulse and Unit Step Functions

- 1.4.1 The DT Unit Impulse and Unit Step Sequences
- Definitions
  - Unit impulse (or unit sample)

$$\delta[n] = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases}$$

$$\delta[n-n_0] = \begin{cases} 1, & n = n_0 \\ 0, & n \neq n_0 \end{cases}$$

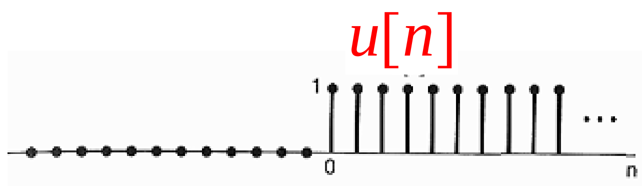


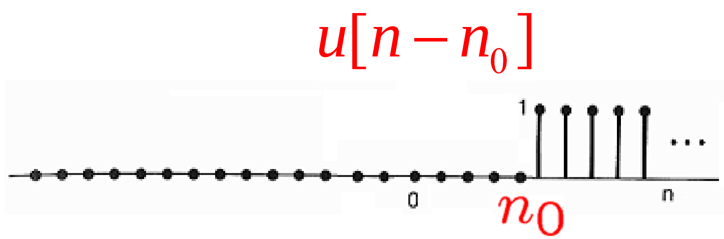


  - Unit step

$$u[n] = \begin{cases} 0, & n < 0 \\ 1, & n \geq 0 \end{cases}$$

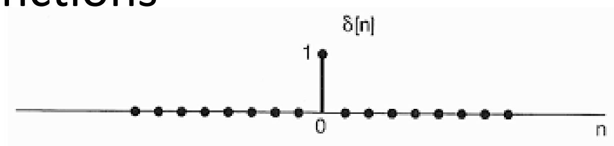
$$u[n-n_0] = \begin{cases} 0, & n < n_0 \\ 1, & n \geq n_0 \end{cases}$$





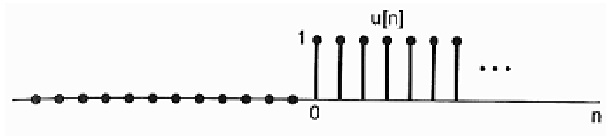
1.4 The Unit Impulse and Unit Step Functions (cont'd)

- Relations between Impulse and Step Functions
- First difference



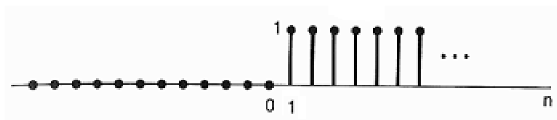
$\delta[n]$

$$\delta[n] = u[n] - u[n-1]$$



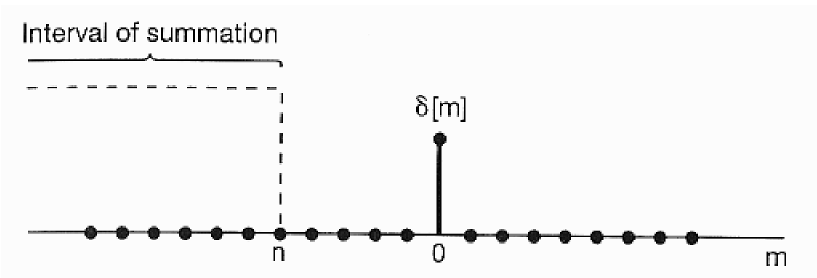
$u[n]$

- Running sum



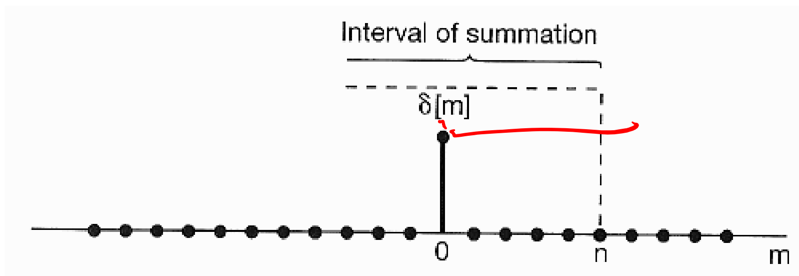
$u[n-1]$

$$u[n] = \sum_{m=-\infty}^{n} \delta[m] = \begin{cases} 0, & n < 0 \\ 1, & n \geq 0 \end{cases}$$



(a)

$n < 0$



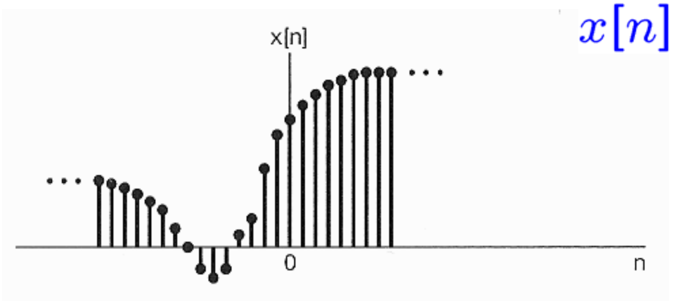
(b)

$n \geq 0$

1.4 The Unit Impulse and Unit Step Functions (cont'd)

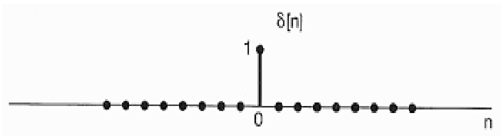
- Sampling (sifting) property
- For $x[n]$

$$x[n]\delta[n] = x[0]\delta[n]$$

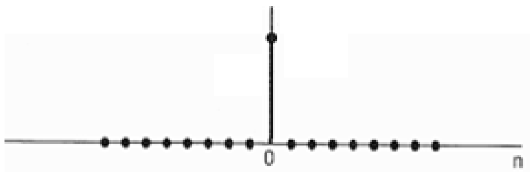


(a)

$x[n]$



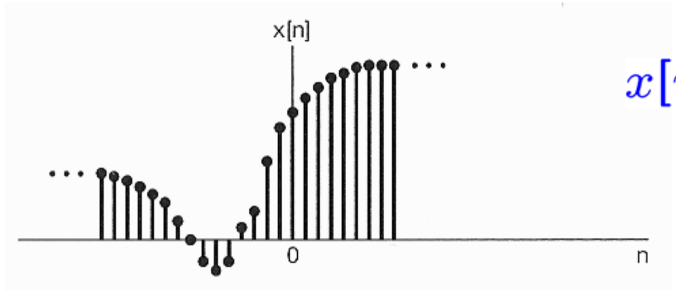
$\delta[n]$



$x[n]\delta[n]$

- More generally,

$$x[n]\delta[n-n_0] = x[n_0]\delta[n-n_0]$$



(a)

$x[n]$

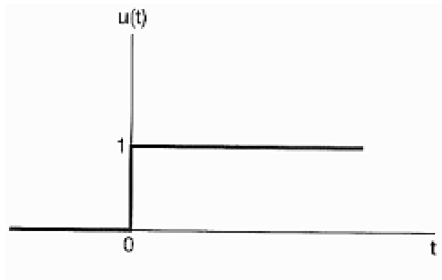
$\delta[n-5]$

$x[5]\delta[n-5]$

1.4 The Unit Impulse and Unit Step Functions

- 1.4.2 The CT Unit Impulse and Unit Step Sequences
- Definitions
  - Unit step function

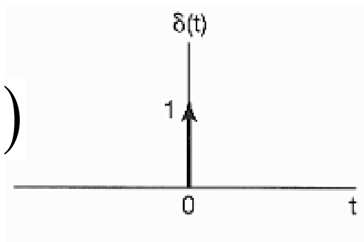
$$u(t) = \begin{cases} 0, & t < 0 \\ 1, & t > 0 \end{cases}$$



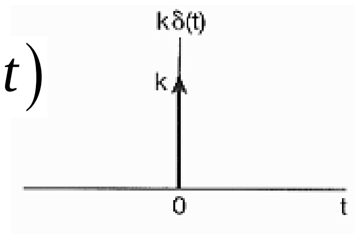
  - Unit impulse function

$$\delta(t)$$

$$\delta(t)$$

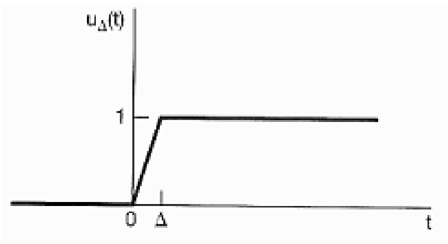


$$k\delta(t)$$

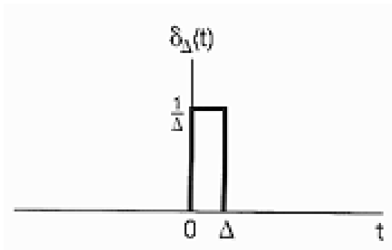


1.4 The Unit Impulse and Unit Step Functions

- 1.4.2 The CT Unit Impulse and Unit Step Sequences
- Approximation

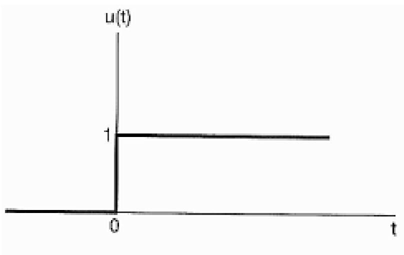


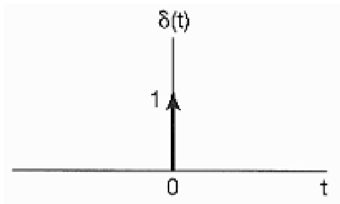
$$\delta_\Delta(t) = \frac{du_\Delta(t)}{dt}$$



$$u(t) = \lim_{\Delta \to 0} u_\Delta(t)$$

$$\delta(t) = \lim_{\Delta \to 0} \delta_\Delta(t)$$





1.4.2 The CT Unit Impulse and Unit Step Sequences (cont'd)

- Relations between Impulse and Step Functions
- First derivative

$$\delta(t) = \frac{du(t)}{dt}$$

- Running integral

$$u(t) = \int_{-\infty}^{t} \delta(\tau) d\tau \quad = \begin{cases} 0, & t < 0 \\ 1, & t > 0 \end{cases}$$

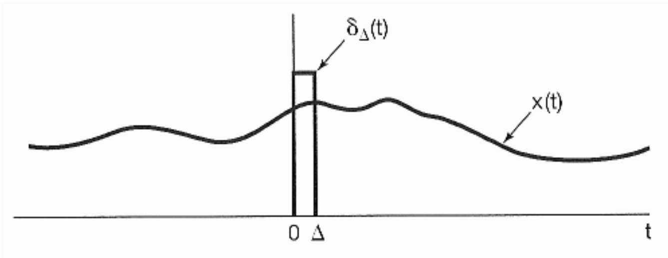
1.4.2 The CT Unit Impulse and Unit Step Sequences (cont'd)

- Sampling (sifting) property
- For *x*(t)

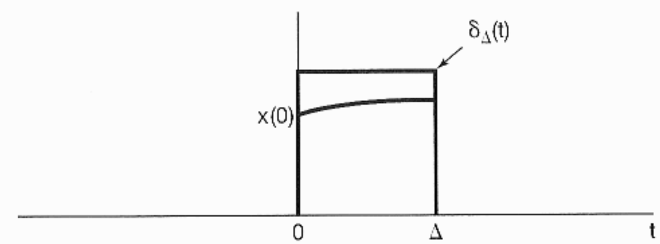
$$x(t)\delta(t) = x(0)\delta(t)$$

- More generally,

$$x(t)\delta(t - t_0) = x(t_0)\delta(t - t_0)$$



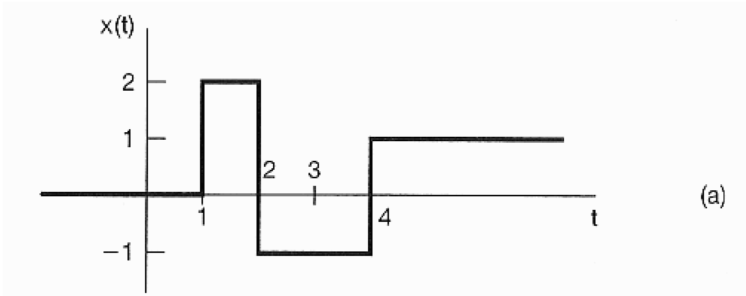
(a)



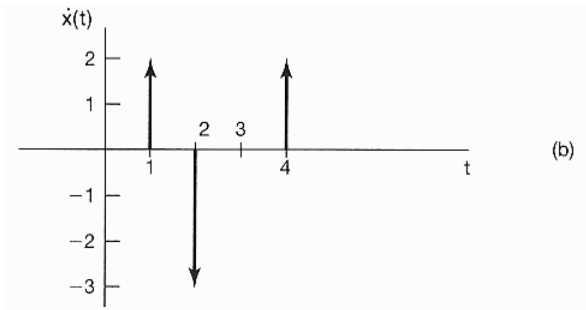
(b)

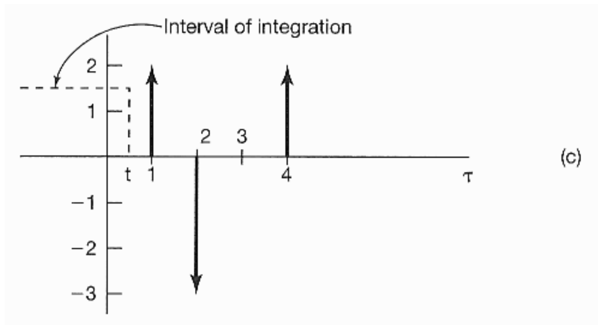
1.4.2 The CT Unit Impulse and Unit Step Sequences (cont'd)

- Example 1.7
  Express x(t) and x'(t) in terms of CT unit impulse/step functions.



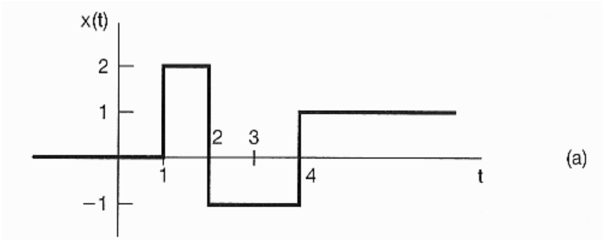
$$x(t) = \int_0^t \dot{x}(\tau)d\tau$$





[Example 1.7]

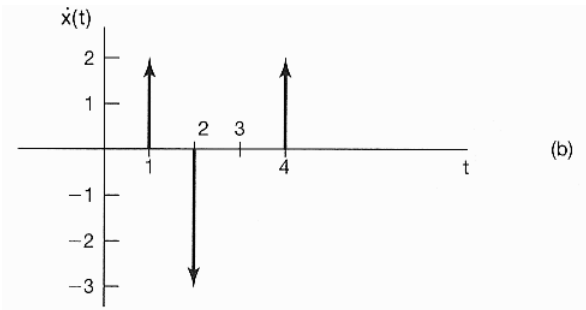
Suppose that $x(t)$ is

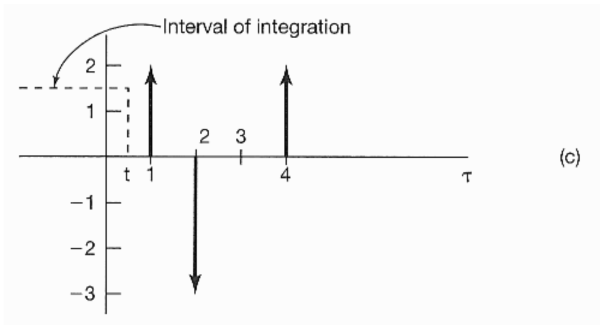


then

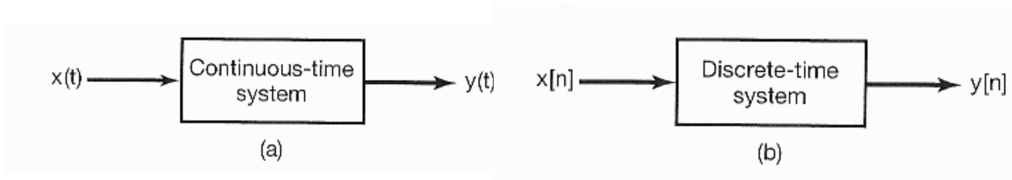
$$x(t) = 2u(t-1) - 3u(t-2) + 2u(t-1)$$

$$\dot{x}(t) = 2\delta(t-1) - 3\delta(t-2) + 2\delta(t-1) \qquad x(t) = \int_0^t \dot{x}(\tau)d\tau$$



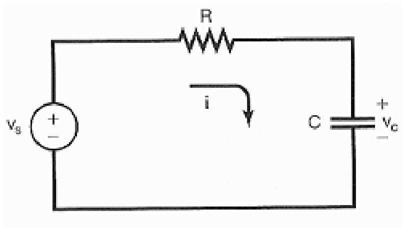


1.5 CT and DT Systems



- A system can be viewed as a process in which input signals are transformed into other signals (outputs).
- Example 1.8 RC circuit (a CT system)

Input signal: $v_s(t)$



Output signal: $v_C(t)$

$$i(t) = \frac{v_s(t) - v_C(t)}{R}$$

$$i(t) = C\frac{dv_C(t)}{dt}$$

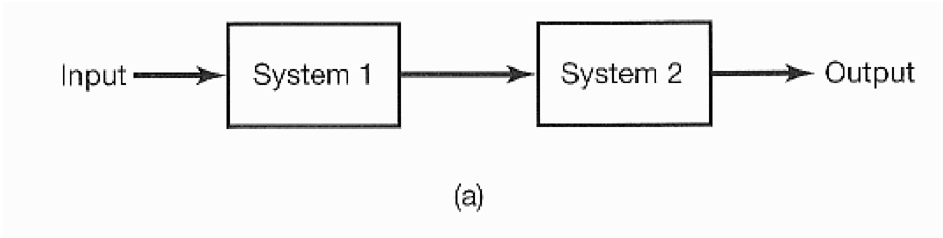
$$\Rightarrow \frac{v_s(t) - v_c(t)}{R} = C\frac{d}{dt}v_c(t)$$

$$\Rightarrow \frac{dv_C(t)}{dt} + \frac{1}{RC}v_C(t) = \frac{1}{RC}v_s(t)$$

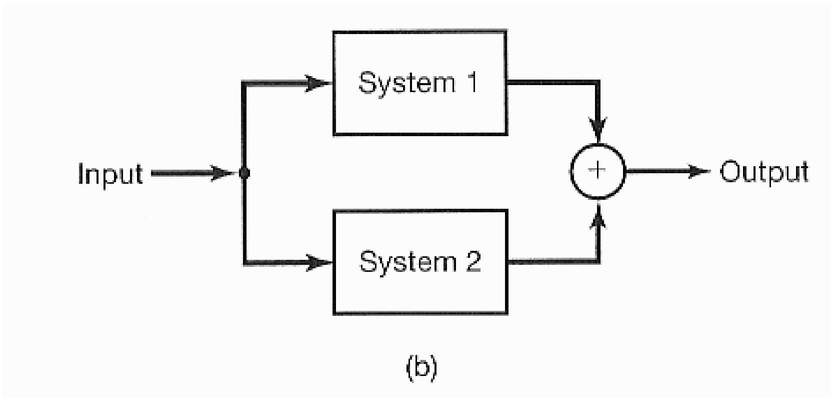
$$\Rightarrow \frac{dy(t)}{dt} + ay(t) = bx(t) \qquad a = b = \frac{1}{RC}$$

1.5.2 Interconnections of Systems

- Series or cascade interconnection of 2 systems (e.g., receiver/amplifier)

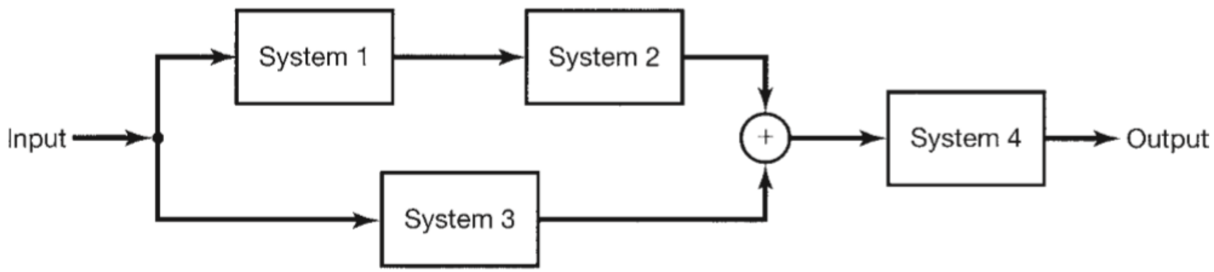


- Parallel interconnection of 2 systems
  (e.g., audio systems with multi-microphones/speakers)

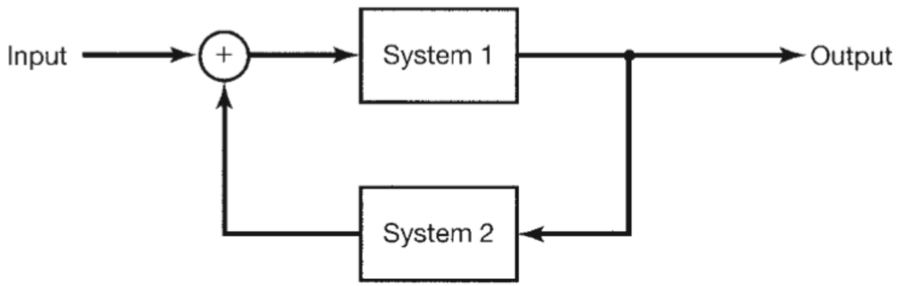


1.5.2 Interconnections of Systems

- Hybrid of series and cascade interconnections

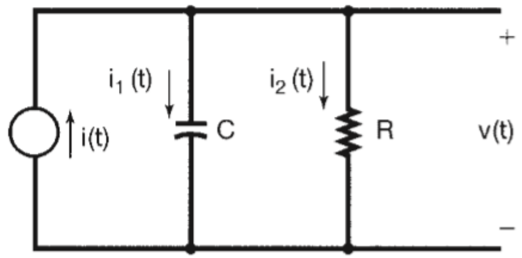


- Feedback interconnections (e.g., control systems, circuits, etc.)

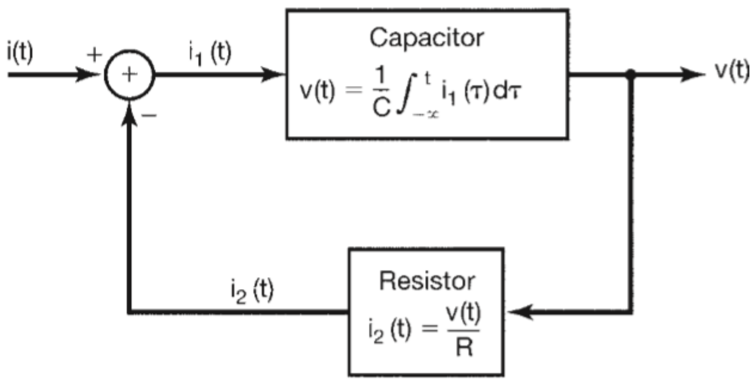


1.5.2 Interconnections of Systems

- An example of a feedback system in circuits
  - (a) Simple electrical circuit



  - (b) Block diagram in which the circuit is depicted as the feedback interconnection of two circuit elements



1.6 Basic System Properties

- Key Concepts
    - Memory and memoryless (1.6.1)
    - Invertibility (1.6.2)
    - Causaility (1.6.3)
    - Stability and BIBO stable (1.6.4)
    - Time invariance (1.6.5)
    - Linearity (additivity property for a linear system) (1.6.6)
    - Superposition property for a linear system
    - Incrementally linear system and zero-input response

1.6.1 Systems with and without Memory

- CT
  - If $y(t)$ is independent of $x(t+\tau)$ where $\tau \neq 0$, the systems is memoryless.
- DT
  - If $y[n]$ is independent of $x[n+k]$ where $k \neq 0$, the systems is memoryless.
- Example
  - Memoryless systems $y[n] = (2x[n] - x[n]^2)^2$

    $y[n] = x[n]$ (identity)

    $y(t) = x(t)$ (identity)
  - Systems with memory

    $y[n] = \sum_{k=-\infty}^{n} x[k]$ (accumulator)

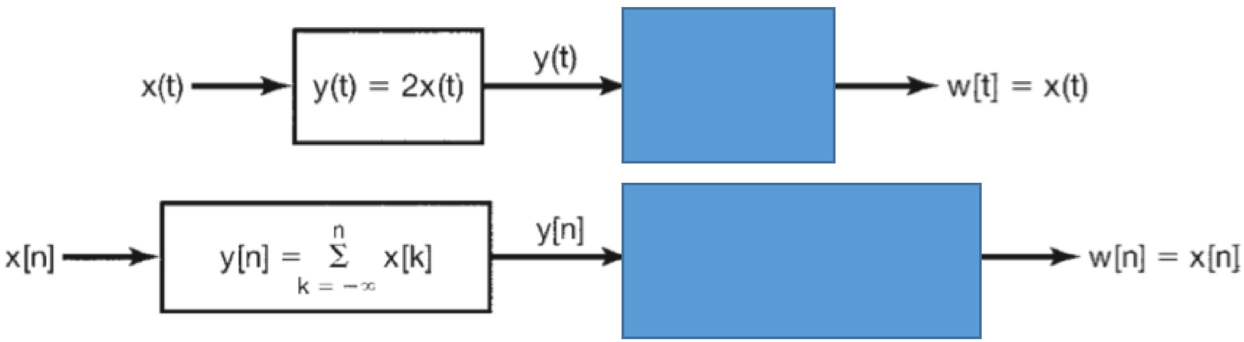
    $y(t) = \int_{-\infty}^{t} x(\tau)\,d\tau$ (integral)

    $y[n] = x[n-1]$ (delay)

1.6.2 Invertibility and Inverse Systems

- For a system, if there exits another system that can retrieve the input from the output, then the system is invertible.

- Examples
  - Is $y(t) = 2x(t)$ invertible?
  - Is the summation operation reversible?
  - Is $y(t) = x(t)^2$ invertible?



1.6.3 Causality

- Causal systems
  - The present output ($y(t)$ or $y[n]$) depends only on the input at the present time ($x(t)$ or $x[n]$) & those in the past.
  - Future inputs do NOT affect the present output.

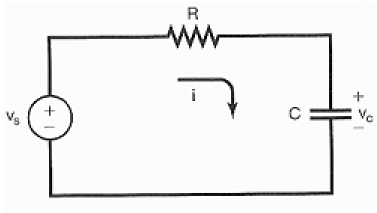
  - In a causal CT system
    $y(t)$ is independent of $x(t+\tau)$ where $\tau > 0$.

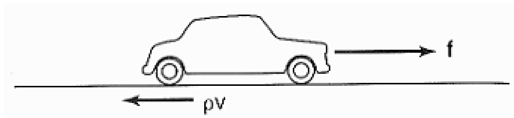
  - In a causal DT system
    $y[n]$ is independent of $x[n+k]$ where $k > 0$.

1.6.3 Causality (cont'd)

- Causal systems $y(t)=\int_{-\infty}^{t} x(\tau)d\tau$ $y[n]=x[n]-x[n-1]$

Circuit System and motion system are also causal systems,
since the future input is impossible to affect the present output.





- Non-causal systems

$$y[n]=x[n]-x[n+1]$$

$$y(t)=x(t+1)$$

$$y[n]=\frac{1}{2M+1}\sum_{k=-M}^{+M} x[n-k].$$

1.6.3 Causality (cont'd)

- Example 1.12
  Are the following two systems causal and why?
  (i) $y[n] = x[-n]$
  (ii) $y(t) = x(t)\cos(t + 1)$

1.6.4 Stability

- Causal systems