Signal and Systems Notes (Midterm Exam)

Ch1 Signals and Systems

1.1 Comparison of CT & DT signals

	CT	DT
complex exponential signal	$x(t) = Ce^{at}$ $a = \sigma + j\omega$	$x[n] = Ce^{bn} = Ca^n$
periodic sinusoidal signal	$x(t) = A\cos(\omega_0 t + \phi)$	$x[n] = A\cos(\omega_0 n + \phi)$
periodic complex exponential signal	$x(t) = e^{j\omega_0 t}$	$x[n] = e^{j\omega_0 n}$
harmonically related periodic exponentials	$\phi_k(t) = e^{jk\omega_0 t}$ $k = 0, \pm 1, \pm 2 \ldots$	$\phi_k[n] = e^{jk\omega_0 n}$ $\omega_0 = 2\pi/N$

1.2 Even and odd signals: $\mathcal{E}v(t) = \frac{1}{2}[x(t) + x(-t)]$; $\mathcal{O}d(t) = \frac{1}{2}[x(t) - x(-t)]$

1.3 Basic system properties

Memoryless: output depends only on the input at the same time.
Invertibility & inverse: distinct inputs lead to distinct outputs.
Causality: output depends only on input at present time & in the past.
Stability: small inputs lead to responses that do not diverge.
Time-invariant: behavior & characteristics are fixed over time.
Linearity: linear system.

Ch2 Linear Time-Invariant System

2.1 Properties of LTI systems

Commutative: $f * g = g * f$
Distributive: $f * (g + h) = f * g + f * h$
Associative: $f * (g * h) = (f * g) * h$
Memoryless: $h(t) = 0, \forall t \neq 0 \Rightarrow h(t) = K\delta(t),\ y(t) = Kx(t)$
Invertibility: $x(t) \xrightarrow{h_1(t)} y(t) \xrightarrow{h_2(t)} w(t) = x(t) \Rightarrow h_1(t) * h_2(t) = \delta(t)$
Causality: $h(t) = 0, \forall t < 0 \Rightarrow y(t) = \int_0^\infty h(\tau)x(t-\tau)d\tau$
Stability: $\int_{-\infty}^{+\infty} |h(\tau)|d\tau < \infty$ (accumulator and $\int$ is not stable)
Delay accumulation: $y[n] = x[n] * h[n] \Rightarrow x[n-j] * h[n-k] = y[n-k-j]$
Unit step response: running sum/integral of its impulse response.

2.2 **IIR** (infinite impulse response) $\Leftrightarrow$ **FIR** (finite impulse response)

2.3 Singularity function (unit doublet)

Differentiator: $u_1(t) \triangleq \frac{d}{dt}\delta(t) \Rightarrow \frac{d}{dt}x(t) = x(t) * u_1(t)$
$u_k(t) = k^{th}$ derivative of $\delta(t) \Rightarrow u_k(t) = u_1(t) * \cdots * u_1(t), k > 0$
Integrator: $u_{-1}(t) \triangleq \int_{-\infty}^{t} \delta(\tau)d\tau = u(t) \Rightarrow x(t) * u(t) = \int_{-\infty}^{t} x(\tau)d\tau$
$u_{-k}(t) = u(t) * \cdots * u(t) = \frac{t^{k-1}}{(k-1)!}u(t)$

Ch3 Fourier Series Representation of Periodic Signals

3.1 Eigenfunctions of LTI systems: e^{st} (CT) and z^n (DT).

CT: $y(t) = \int_{-\infty}^{+\infty} h(\tau)x(t-\tau)d\tau = e^{st}\int_{-\infty}^{+\infty} h(\tau)e^{-s\tau}d\tau = H(s)e^{st}$

DT: $y[n] = \sum_{k=-\infty}^{+\infty} h[k]x[n-k] = z^n \sum_{k=-\infty}^{+\infty} h[k]z^{-k} = H(z)z^n$

3.2 CT & DT periodic signals

	$x(t)/x[n]$	$y(t)/y[n]$	a_k
CT	$\sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$	$\sum_k a_k H(s_k)e^{s_k t}$	$\frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}dt$
DT	$\sum_{k=\langle N\rangle} a_k \phi_k[n]$	$\sum_k a_k H(z_k){z_k}^n$	$\frac{1}{N}\sum_{n=\langle N\rangle} x[n]e^{-jk\omega_0 n}$

$$\phi_k[n] = e^{jk\omega_0 n} = e^{jk(2\pi/N)n},\ k = 0, 1, \cdots, N-1,\ x[n+N] = x[n]$$

3.3 **Euler's relation:** $e^{j\theta} = \cos\theta + j\sin\theta$

$$\cos\theta = \frac{1}{2}(e^{j\theta} + e^{-j\theta}) \quad ; \quad \sin\theta = \frac{1}{2j}(e^{j\theta} - e^{-j\theta})$$

3.4 FS of periodic square wave:

$$x(t) = \begin{cases} 1, & |t| < T_1 \\ 0, & T_1 < |t| < T/2 \end{cases} \Rightarrow a_0 = \frac{2T_1}{T},\ a_k = \frac{2\sin(k\omega_0 T_1)}{k\omega_0 T}$$

3.5 **Dirichlet convergence conditions:**
#1: $x(t)$ is absolutely integrable over any period $\Rightarrow \int_T |x(t)|dt < \infty$.
#2: $x(t)$ is of bounded variation in any finite interval.
#3: $x(t)$ has only finite number of discontinuities in any finite interval.

3.6 Properties of CT Fourier series

Property	Periodic signal	FS coefficients
Linearity	$Ax(t) + By(t)$	$Aa_k + Bb_k$
Time shifting	$x(t - t_0)$	$e^{-jk\omega_0 t_0}a_k$
Frequency shifting	$e^{jM\omega_0 t} = e^{jM(2\pi/T)t}x(t)$	a_{k-M}
Time reversal	$x(-t)$	a_{-k}
Time scaling	$x(\alpha t)$ (period: T/α)	a_k ($\omega_0 \to \alpha\omega_0$)
Periodic convolution	$\int_t x(\tau)y(t-\tau)d\tau$	$Ta_k b_k$
Multiplication	$z(t) = x(t)y(t)$	$c_k = \sum_{l=-\infty}^{+\infty} a_l b_{k-l}$
Differentiation	$\frac{d}{dt}x(t)$	$jk\omega_0 a_k$
Integration	$\int_{-\infty}^{t} x(\tau)d\tau, a_0 = 0$	$\frac{1}{jk\omega_0}a_k$
Conjugate symmetry for real signals	$x^*(t)$	$a_{-k}^* = a_k \quad \|a_k\| = \|a_{-k}\|$ $\mathfrak{Re}\{a_k\} = \mathfrak{Re}\{a_{-k}\}$ $\mathfrak{Im}\{a_k\} = -\mathfrak{Im}\{a_{-k}\}$
Real $\mathcal{E}v(t)$	$x(-t) = x(t)$	$a_{-k} = a_k$ a_k real and even
Real $\mathcal{O}d(t)$	$x(-t) = -x(t)$	$a_{-k} = -a_k$ a_k imaginary and odd

3.7 Properties of DT Fourier series

Property	Periodic signal	FS coefficients
Linearity	$Ax[n] + By[n]$	$Aa_k + Bb_k$
Time shifting	$x[n - n_0]$	$e^{-jk(2\pi/N)n_0}a_k$
Frequency shifting	$e^{jM(2\pi/N)n}x[n]$	a_{k-M}
Time reversal	$x[-n]$	a_{-k}
Time scaling	$x_{(m)}[n] = x[n/m]$ (if n is a multiple of m) $x_{(m)}[n] = 0$ (else)	a_k/m viewed as periodic with period mN
Periodic convolution	$\sum_{r=\langle N\rangle} x[r]y[n-r]$	$Na_k b_k$
Multiplication	$z[n] = x[n]y[n]$	$c_k = \sum_{l=\langle N\rangle} a_l b_{k-l}$
First difference	$x[n] - x[n-1]$	$(1 - e^{-jk(2\pi/N)})a_k$
Running sum	$\sum_{k=-\infty}^{n} x[k],\ a_0 = 0$	$\frac{1}{1 - e^{-jk(2\pi/N)}}a_k$
Conjugate symmetry for real signals	$x^*[n]$	$a_{-k}^* = a_k \quad \|a_k\| = \|a_{-k}\|$ $\mathfrak{Re}\{a_k\} = \mathfrak{Re}\{a_{-k}\}$ $\mathfrak{Im}\{a_k\} = -\mathfrak{Im}\{a_{-k}\}$
Real $\mathcal{E}v(t)$	$x[-n] = x[n]$	$a_{-k} = a_k \Rightarrow even$
Real $\mathcal{O}d(t)$	$x[-n] = -x[n]$	$a_{-k} = -a_k \Rightarrow odd$

3.8 **Parseval's Relation:** The total average power in a periodic signal equals the sum of the average powers in all of its harmonic components.

$$\frac{1}{T}\int_T |x(t)|^2 dt = \sum_{k=-\infty}^{+\infty} |a_k|^2 \qquad \frac{1}{N}\sum_{n=\langle N\rangle} |x[n]|^2 = \sum_{n=\langle N\rangle} |a_k|^2$$

3.9 **Frequency response:** take $s = j\omega$ and $z = e^{j\omega}$.

$$H(j\omega) = \int_{-\infty}^{+\infty} h(t)e^{-j\omega t}dt \quad H(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} h[n]e^{-j\omega n}$$

Ch4 The Continuous-Time Fourier Transform

4.1 CT Fourier transform pair

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)e^{j\omega t}d\omega \overset{\mathcal{F}}{\longleftrightarrow} X(j\omega) = \int_{-\infty}^{+\infty} x(t)e^{-j\omega t}dt$$

$$a_k = \frac{1}{T}X(j\omega)|_{\omega=k\omega_0}$$

4.2 Properties of CT Fourier transform

Property	Aperiodic signal	Fourier transform
Linearity	$ax(t)+by(t)$	$aX(j\omega)+bY(j\omega)$
Time shifting	$x(t-t_0)$	$e^{-j\omega t_0}X(j\omega)$
Frequency shifting	$e^{j\omega_0 t}x(t)$	$X(j(\omega-\omega_0))$
Time reversal	$x(-t)$	$X(-j\omega)$
Time scaling	$x(at)$	$\frac{1}{\lvert a\rvert}X\left(\frac{j\omega}{a}\right)$
Convolution	$x(t)*y(t)$	$X(j\omega)Y(j\omega)$
Integration	$\int_{-\infty}^{t} x(t)dt$	$\frac{1}{j\omega}X(j\omega)+\pi X(0)\delta(\omega)$
Multiplication	$z(t)=x(t)y(t)$	$\frac{1}{2\pi}X(j\omega)*Y(j\omega) =$ $\frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\theta)Y(j(\omega-\theta))d\theta$
Differentiation in time	$\frac{d}{dt}x(t)$	$j\omega X(j\omega)$
Differentiation in frequency	$tx(t)$	$j\frac{d}{d\omega}X(j\omega)$
Conjugate symmetry for real signals	$x^*(t)$	$X(j\omega)=X^*(-j\omega)$ $\lvert X(j\omega)\rvert = \lvert X(-j\omega)\rvert$ $\sphericalangle X(j\omega) = -\sphericalangle X(-j\omega)$ $\mathfrak{Re}\{X(j\omega)\} = \mathfrak{Re}\{X(-j\omega)\}$ $\mathfrak{Im}\{X(j\omega)\} = -\mathfrak{Im}\{X(-j\omega)\}$
Real $\mathcal{E}v(t)$	$x(t)=x(-t)$	$X(j\omega) \Rightarrow$ real & even
Real $\mathcal{O}d(t)$	$x(t)=-x(-t)$	$X(j\omega) \Rightarrow$ imaginary & odd

4.3 Basic Fourier transform pair

Signal	Fourier transform	FS coefficients (if periodic)
$\sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$	$2\pi\sum_{k=-\infty}^{+\infty} a_k\delta(\omega-k\omega_0)$	a_k
$e^{j\omega_0 t}$	$2\pi\delta(\omega-\omega_0)$	$a_1=1$ $a_k=0$, otherwise
$\cos\omega_0 t$	$\pi[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)]$	$a_1=a_{-1}=\frac{1}{2}$ $a_k=0$, otherwise
$\sin\omega_0 t$	$\frac{\pi}{j}[\delta(\omega-\omega_0)-\delta(\omega+\omega_0)]$	$a_1=-a_{-1}=\frac{1}{2j}$ $a_k=0$, otherwise
$x(t)=1$	$2\pi\delta(\omega)$	$a_0=1$ $a_k=0$, otherwise
$\sum_{n=-\infty}^{+\infty}\delta(t-nT)$	$\frac{2\pi}{T}\sum_{k=-\infty}^{+\infty}\delta\left(\omega-\frac{2\pi k}{T}\right)$	$a_k=\frac{1}{T}$ for all k
$x(t)=\begin{cases}1, & \lvert t\rvert<T_1\\ 0, & \lvert t\rvert>T_1\end{cases}$	$\frac{2\sin\omega T_1}{\omega}$	-
$\frac{\sin Wt}{\pi t}$	$X(j\omega)=\begin{cases}1, & \lvert\omega\rvert<W\\ 0, & \lvert\omega\rvert>W\end{cases}$	-
$\delta(t)$	1	-
$u(t)$	$\frac{1}{j\omega}+\pi\delta(\omega)$	-
$e^{-at}u(t)$	$\frac{1}{a+j\omega}$	ps. $\mathfrak{Re}\{a\}>0$
$te^{-at}u(t)$	$\frac{1}{(a+j\omega)^2}$	ps. $\mathfrak{Re}\{a\}>0$
$\frac{t^{n-1}}{(n-1)!}e^{-at}u(t)$	$\frac{1}{(a+j\omega)^n}$	ps. $\mathfrak{Re}\{a\}>0$

4.4 Fourier transform for periodic signals

$$X(j\omega) = \sum_{k=-\infty}^{+\infty} 2\pi a_k\delta(\omega-k\omega_0)$$

4.5 sinc Function: $\mathrm{sinc}(\theta) = \frac{\sin(\pi\theta)}{\pi\theta} \Rightarrow \frac{\sin(Wt)}{\pi t} = \frac{W}{\pi}\mathrm{sinc}\left(\frac{Wt}{\pi}\right)$

4.6 Periodic square wave: $x(t+T)=x(t)$

$$x(t) = \begin{cases}1, & \lvert t\rvert<T_1\\ 0, & T_1<\lvert t\rvert\le T/2\end{cases} \overset{\mathcal{F}}{\longleftrightarrow} \sum_{k=-\infty}^{+\infty}\frac{2\sin k\omega_0 T_1}{k}\delta(\omega-k\omega_0)$$

FS coefficients (if periodic): $\frac{\omega_0 T_1}{\pi}\mathrm{sinc}\left(\frac{k\omega_0 T_1}{\pi}\right) = \frac{\sin k\omega_0 T_1}{k\pi}$

4.7 Parseval's relation for aperiodic signals

$$\int_{-\infty}^{+\infty}\lvert x(t)\rvert^2 dt = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\lvert X(j\omega)\rvert^2 d\omega$$

4.8 A useful class of LTI: $\sum_{k=0}^{N} a_k\frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k\frac{d^k x(t)}{dt^k}$

$$\Rightarrow H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{b_M(j\omega)^M+\cdots+b_1(j\omega)+b_0}{a_N(j\omega)^N+\cdots+a_1(j\omega)+a_0}$$

4.9 Duality Ex. $x(t-t_0)\overset{\mathcal{F}}{\longleftrightarrow}e^{-j\omega t_0}X(j\omega) \Leftrightarrow e^{j\omega_0 t}x(t)\overset{\mathcal{F}}{\longleftrightarrow}X(j(\omega-\omega_0))$

Ch5 The Discrete-Time Fourier Transform

5.1 CT Fourier transform pair

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}d\omega \overset{\mathcal{F}}{\longleftrightarrow} X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n]e^{-j\omega n}$$

$$a_k = \frac{1}{N}X(e^{j\omega})|_{\omega=k\omega_0}$$

5.2 Properties of DT Fourier transform

Property	Aperiodic signal	Fourier transform
Linearity	$ax[n]+by[n]$	$aX(e^{j\omega})+bY(e^{j\omega})$
Time shifting	$x[n-n_0]$	$e^{-j\omega n_0}X(e^{j\omega})$
Frequency shifting	$e^{j\omega_0 n}x[n]$	$X(e^{j(\omega-\omega_0)})$
Time reversal	$x[-n]$	$X(e^{-j\omega})$
Time expansion	$x_{(k)}[n] =$ $\begin{cases}x[n/k], & if\ n=mk\\ 0, & if\ n\neq mk\end{cases}$	$X(e^{jk\omega})$
Convolution	$x[n]*y[n]$	$X(e^{j\omega})Y(e^{j\omega})$
Multiplication	$z[n]=x[n]y[n]$	$\frac{1}{2\pi}X(e^{j\omega})*Y(e^{j\omega}) =$ $\frac{1}{2\pi}\int_{2\pi}X(e^{j\theta})Y(e^{j(\omega-\theta)})d\theta$
Accumulation	$\sum_{k=-\infty}^{+\infty}x[k]$	$\frac{1}{1-e^{-j\omega}}X(e^{j\omega})+$ $\pi X(e^{j0})\sum_{k=-\infty}^{+\infty}\delta(\omega-2\pi k)$
Differencing in time	$x[n]-x[n-1]$	$(1-e^{-j\omega})X(e^{j\omega})$
Differentiation in frequency	$nx[n]$	$j\frac{d}{d\omega}X(e^{j\omega})$
Conjugate symmetry for real signals	$x^*[n]$	$X(e^{j\omega})=X^*(e^{-j\omega})$ $\lvert X(e^{j\omega})\rvert = \lvert X(e^{-j\omega})\rvert$ $\sphericalangle X(e^{j\omega}) = -\sphericalangle X(e^{-j\omega})$ $\mathfrak{Re}\{X(e^{j\omega})\} = \mathfrak{Re}\{X(e^{-j\omega})\}$ $\mathfrak{Im}\{X(e^{j\omega})\} = -\mathfrak{Im}\{X(e^{-j\omega})\}$
Real $\mathcal{E}v[n]$	$x[n]=x[-n]$	$X(e^{j\omega}) \Rightarrow$ real & even
Real $\mathcal{O}d[n]$	$x[n]=-x[-n]$	$X(e^{j\omega}) \Rightarrow$ imaginary & odd

5.3 Fourier transform for periodic signals

$$X(e^{j\omega n}) = \sum_{k=-\infty}^{+\infty}2\pi a_k\delta(\omega-\frac{2\pi k}{N})$$

5.4 Parseval's relation for aperiodic signals

$$\sum_{n=-\infty}^{+\infty}\lvert x[n]\rvert^2 = \frac{1}{2\pi}\int_{2\pi}\lvert X(e^{j\omega})\rvert^2 d\omega$$