Signals & Systems

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Ch. 5 Discrete-Time Fourier Transform

- Sec. 5.1 Representation of Aperiodic Signals:
  The Discrete-Time Fourier Transform
- Sec. 5.2 The Fourier Transform for Periodic Signals
- Sec. 5.3 Properties of the Discrete-Time Fourier Transform
- Sec. 5.4 The Convolution Property
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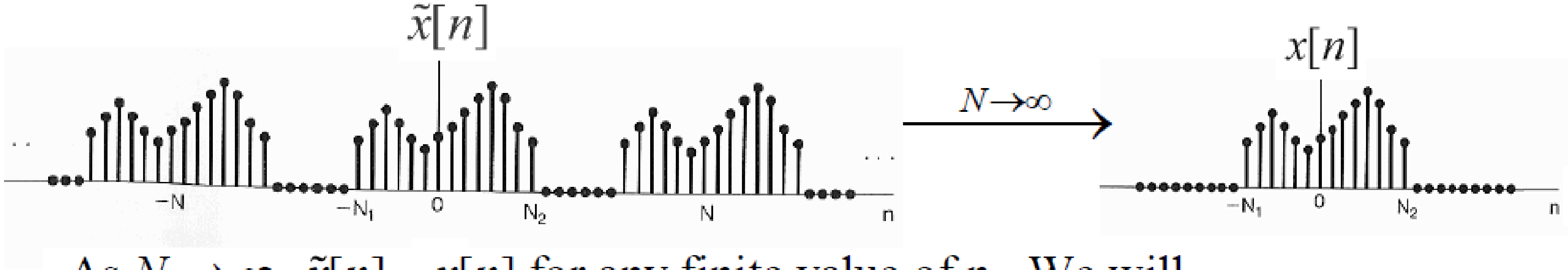
- Sec. 5.8 Systems Characterized by Linear Constant Coefficient
  Differential Equations

Sec. 5.1 Representation of Aperiodic Signals: The Discrete-Time Fourier Transform

- Develop DT FT for Aperiodic Signals



As $N \to \infty$, $\tilde{x}[n] = x[n]$ for any finite value of n. We will use this relation to derive the DTFT of aperiodic signals.

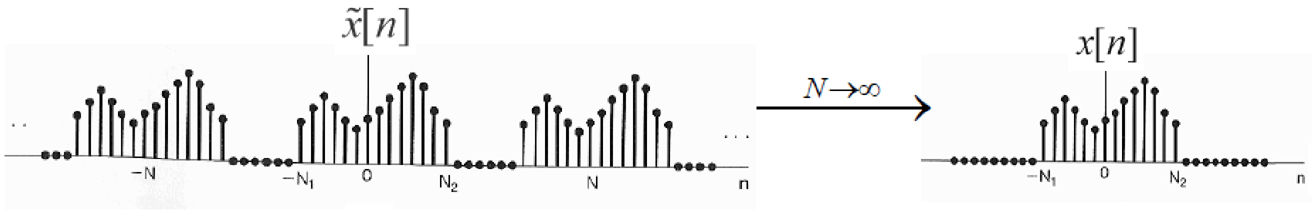
Recall the FS representation of DT signals:

$$\tilde{x}[n] = \sum_{k=<N>} a_k e^{jk\omega_0 n}$$

$$a_k = \frac{1}{N} \sum_{n=<N>} \tilde{x}[n] e^{-jk\omega_0 n}$$

$$\omega_0 = \frac{2\pi}{N}$$

- Develop DT FT for Aperiodic Signals



Since $\tilde{x}[n] = x[n]$ within any period $<N>$, we have

$$a_k = \frac{1}{N}\sum_{n=<N>} \tilde{x}[n]\, e^{-jk\omega_0 n} = \frac{1}{N}\sum_{n=-N_1}^{N_2} x[n]\, e^{-jk\omega_0 n} = \frac{1}{N}\sum_{n=-\infty}^{+\infty} x[n]\, e^{-jk\omega_0 n}$$

Define

DTFT $$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

then we have

$$a_k = \frac{1}{N} X(e^{jk\omega_0}).$$

Substitiuting this a_k to the synthesis equation yields

$$\tilde{x}[n] = \sum_{k=<N>} \boxed{\frac{1}{N} X(e^{jk\omega_0})} e^{jk\omega_0 n}.$$

Since $\omega_0 = 2\pi / N$, or equivalently, $1/N = \omega_0 / 2\pi$,

$$\tilde{x}[n] = \frac{1}{2\pi}\sum_{k=<N>} X(e^{jk\omega_0}) e^{jk\omega_0 n} \omega_0.$$

Sec. 5.1 Representation of Aperiodic Signals: The Discrete-Time Fourier Transform

- DTFT

$$\boxed{x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}\,d\omega}$$

Inverse Fourier transform
Synthesis equation

$$\boxed{X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n]e^{-j\omega n}}$$

Fourier transform
Analysis equation

Recall the CTFT:

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}\,d\omega$$

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt$$

Sec. 5.1 Representation of Aperiodic Signals: The Discrete-Time Fourier Transform

- Periodicity

$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n]e^{-j\omega n}$$

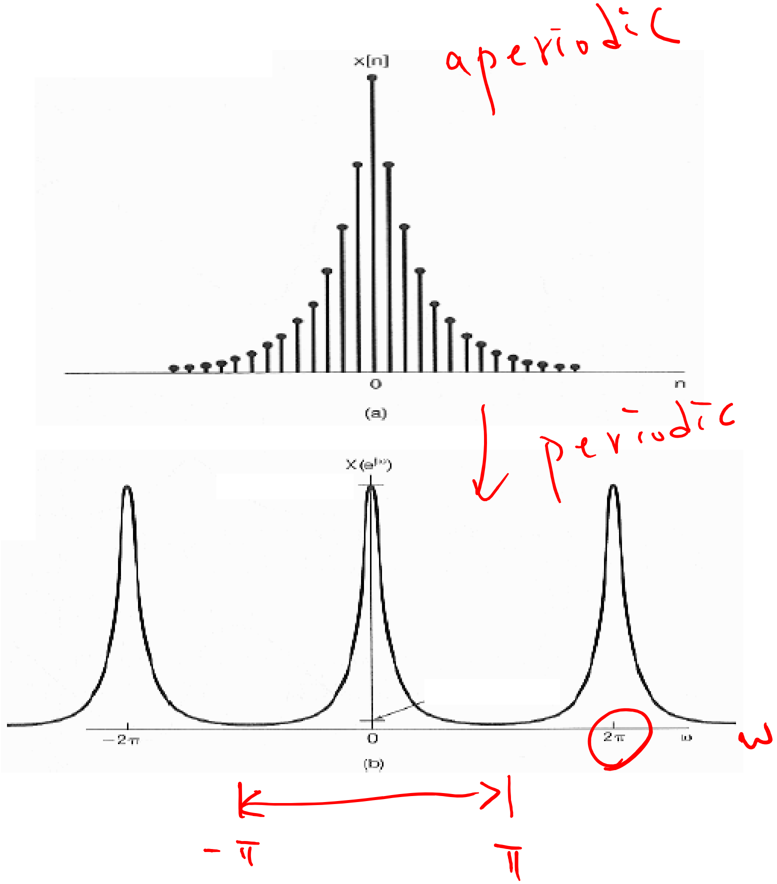
$$X(e^{j(\omega+2\pi)}) = \sum_{n=-\infty}^{+\infty} x[n]e^{-j(\omega+2\pi)n}$$

$$= \sum_{n=-\infty}^{+\infty} x[n]e^{-j\omega n}e^{-j2\pi n}$$

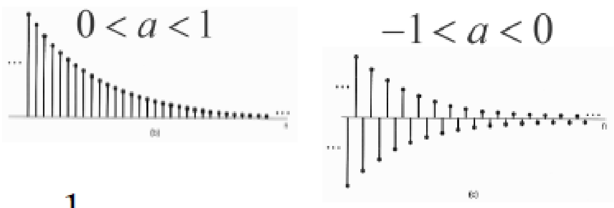
$$= \sum_{n=-\infty}^{+\infty} x[n]e^{-j\omega n}$$

$$\therefore X(e^{j(\omega+2\pi)}) = X(e^{j\omega})$$

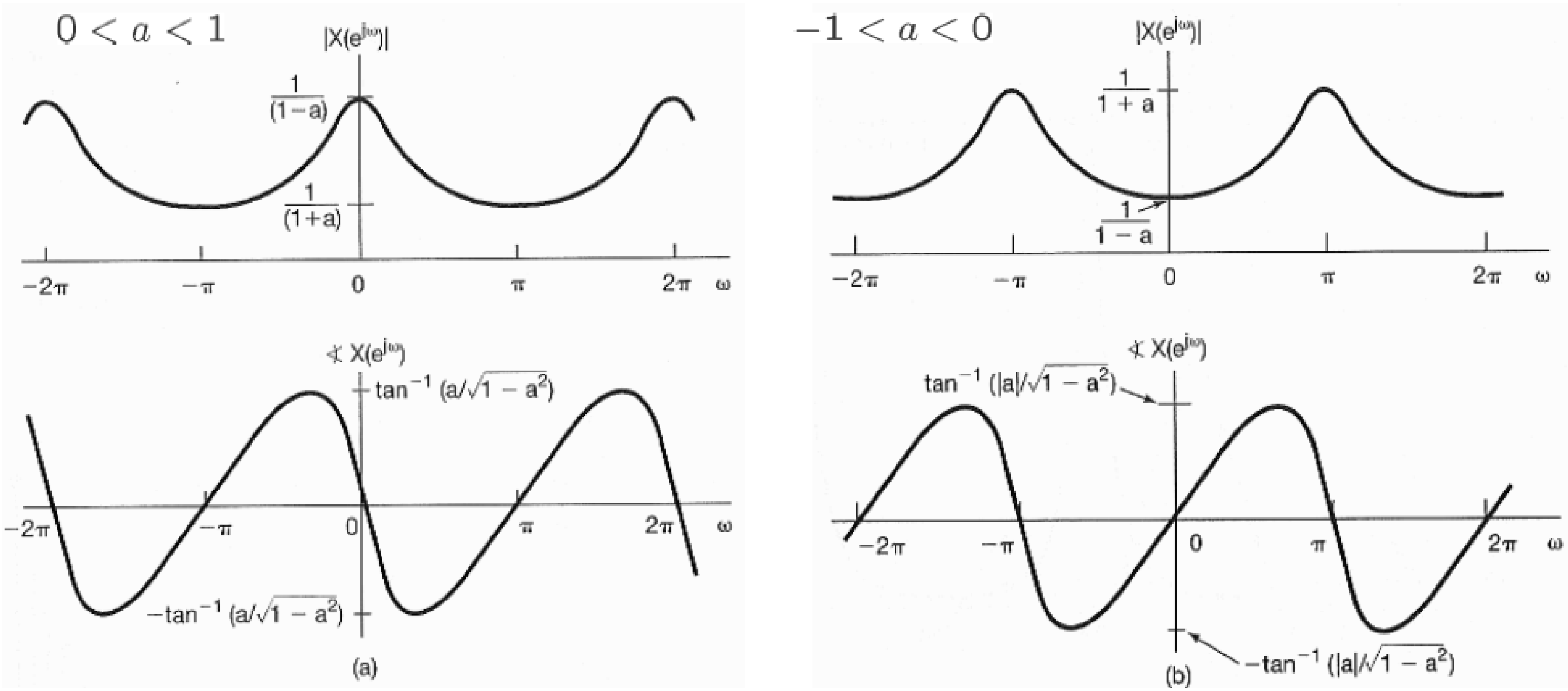
$\Rightarrow X(e^{j\omega})$ is periodic with period 2π



- Example 5.1 $\quad x[n] = a^n u[n], \quad |a| < 1$



$$\Rightarrow X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} a^n u[n] e^{-j\omega n} = \sum_{n=0}^{\infty} (ae^{-j\omega})^n = \frac{1}{1 - ae^{-j\omega}}$$



Sect. 5.2 FT for *Periodic* Signals

Reall:

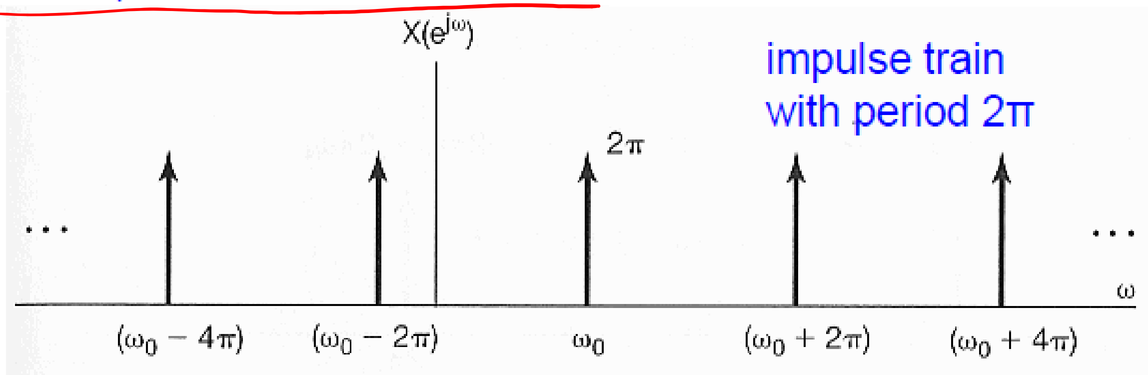
$$e^{j\omega_0 t} \xleftrightarrow{F} 2\pi\delta(\omega-\omega_0)$$

in the CT domain.

- FT from FS

$$x[n]=e^{j\omega_0 n}, \quad \omega_0=\frac{2\pi}{N}$$

$$\Rightarrow X(e^{j\omega})=\sum_{l=-\infty}^{+\infty} 2\pi\delta(\omega-\omega_0-2\pi l)$$



Proof:

$$\frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}d\omega=\frac{1}{2\pi}\int_{2\pi}\sum_{l=-\infty}^{+\infty} 2\pi\delta(\omega-\omega_0-2\pi l)e^{j\omega n}d\omega$$

$$=e^{j(\omega_0+2\pi r)n}=e^{j\omega_0 n}$$

Sect. 5.2 FT for *Periodic* Signals

- FT from FS (cont'd)

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}\,d\omega$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n]e^{-j\omega n}$$

Thus, for a periodic sequence $x[n]$ with period N and with the FS representation

$$x[n] = \sum_{k=\langle N\rangle} a_k e^{jk\omega_0 n},$$

$$a_{k+N} = a_k$$

its FT is related to its Foureir coefficient by

$$\boxed{X(e^{j\omega}) = \sum_{k=-\infty}^{+\infty} 2\pi a_k \delta(\omega - k\omega_0).}$$

$\frac{2\pi}{N}$

The FT of a periodic signal can be directly constructed from its Fourier coefficients.

We can verify this equation graphically by expressing $x[n]$ as

$$x[n] = a_0 + a_1 e^{j\omega_0 n} + a_2 e^{j2\omega_0 n} + \ldots + a_{N-1}e^{j(N-1)\omega_0 n},$$

plot the FT of each term, and then superimpose them.

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}d\omega$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n]e^{-j\omega n}$$

Sect. 5.2 FT for *Periodic* Signals

- FT from FS (cont'd)

To determine the discrete-time Fourier transform for a periodic discrete-time signal $x[n]$ (i.e., $x[n] = x[n+N]$)

(1) First, use the discrete-time Fourier series (Sec. 3.6) to express $x[n]$ by

$$x[n] = \sum_{n=\langle N\rangle} a_k e^{jk(2\pi/N)n}$$ where $$a_k = \frac{1}{N}\sum_{n=\langle N\rangle} x[n]e^{-jk(2\pi/N)n}$$

(2) Then

$$X(e^{j\omega}) = \sum_{k=-\infty}^{+\infty} 2\pi a_k \delta\left(\omega - \frac{2\pi k}{N}\right)$$ for $-\pi \leq \omega < \pi$

$$X(e^{j\omega}) = X(e^{j(\omega+2\pi)})$$

- FT from FS (cont'd)

$x[n]$

$=$

$$a_0 \overset{F}{\longleftrightarrow} \sum_{l=-\infty}^{+\infty} 2\pi a_0 \delta(\omega - 0\omega_0 - 2\pi l)$$

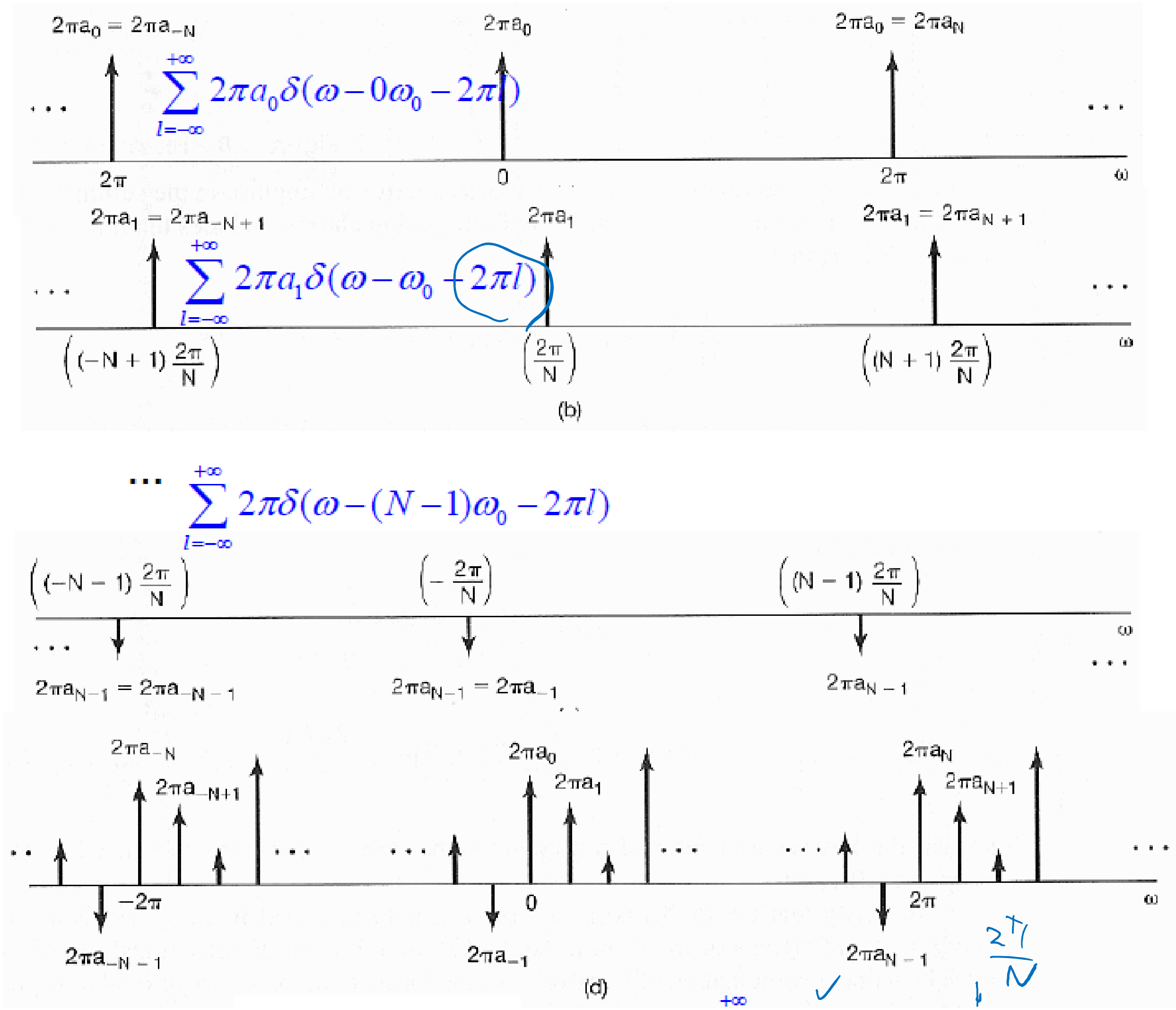
$$a_1 e^{j\omega_0 n} \overset{F}{\longleftrightarrow} \sum_{l=-\infty}^{+\infty} 2\pi a_1 \delta(\omega - \omega_0 - 2\pi l)$$

$$\cdots \quad \sum_{l=-\infty}^{+\infty} 2\pi \delta(\omega - (N-1)\omega_0 - 2\pi l)$$

$$a_{N-1} e^{j(N-1)\omega_0 n} \overset{F}{\longleftrightarrow}$$

$=$

$$\sum_{k=\langle N \rangle} a_k e^{jk\omega_0 n} \overset{F}{\longleftrightarrow}$$



$$a_{k+N} = a_k \quad \Rightarrow \quad X(e^{j\omega}) = \sum_{k=-\infty}^{+\infty} 2\pi a_k \delta(\omega - k\omega_0)$$

$\frac{2\pi}{N}$

$= a_{k+N}$

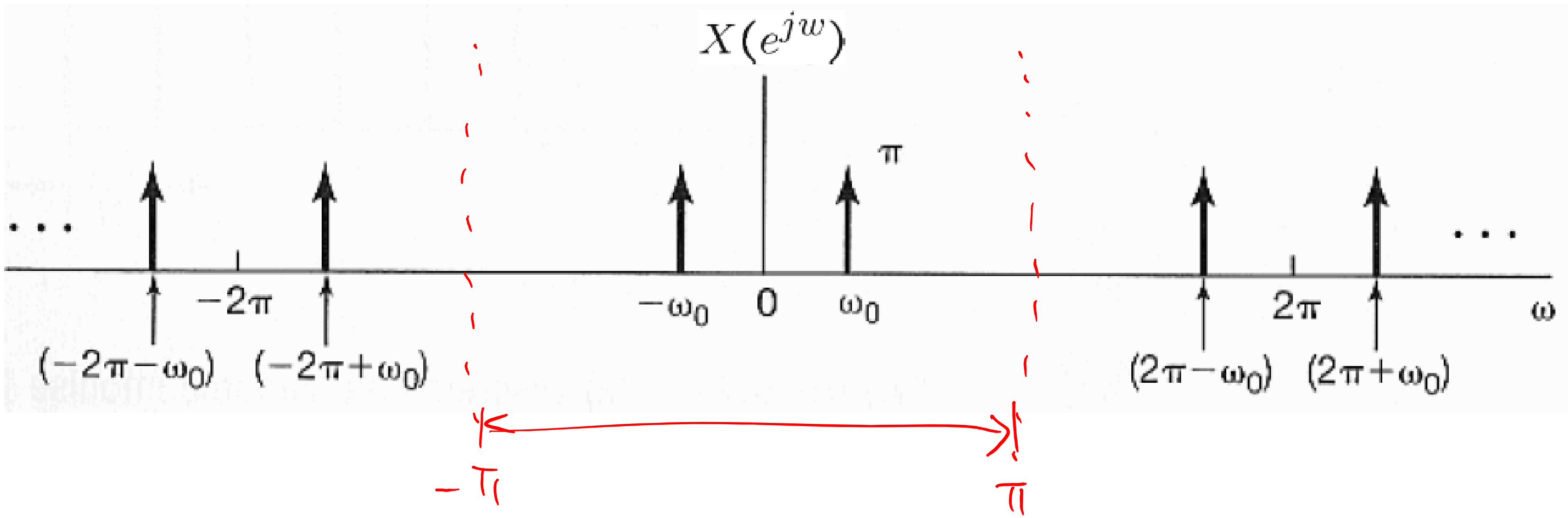
- Example 5.5

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}\,d\omega$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n]e^{-j\omega n}$$

$$x[n] = \cos(\omega_0 n) = \frac{e^{j\omega_0 n} + e^{-j\omega_0 n}}{2}, \qquad \omega_0 = \frac{2\pi}{5}$$

$$X(e^{j\omega}) = \sum_{l=-\infty}^{+\infty} \pi\delta(\omega - \frac{2\pi}{5} - 2\pi l) + \sum_{l=-\infty}^{+\infty} \pi\delta(\omega + \frac{2\pi}{5} - 2\pi l)$$



- Example 5.6 DTFT of Impulse Trains

$$x[n] = \sum_{k=-\infty}^{+\infty} \delta[n-kN]$$

$$\Rightarrow a_k = \frac{1}{N} \sum_{n=<N>} x[n] \boxed{e^{-jk(2\pi/N)n}}$$

$$= \frac{1}{N} \sum_{n=0}^{N-1} x[n] e^{-jk(2\pi/N)n}$$

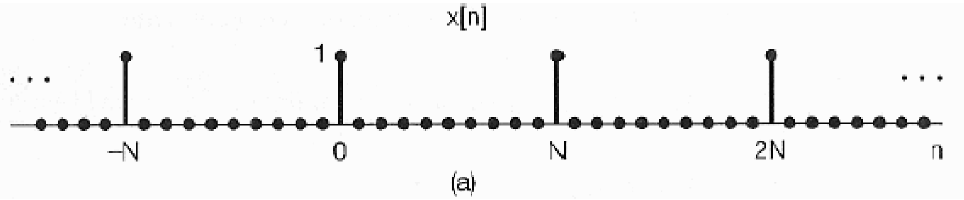
$$= \frac{1}{N}$$

$$\Rightarrow X(e^{j\omega}) = \frac{2\pi}{N} \sum_{k=-\infty}^{+\infty} \delta(\omega - k\frac{2\pi}{N})$$

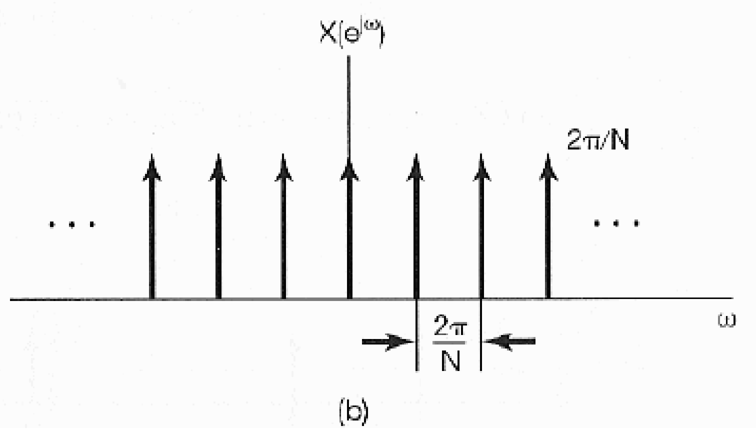
$$x[n] = \sum_{k=<N>} a_k e^{jk(2\pi/N)n}$$

$$a_k = \frac{1}{N} \sum_{n=<N>} x[n] e^{-jk(2\pi/N)n}$$

$$X(e^{j\omega}) = \sum_{k=-\infty}^{+\infty} 2\pi a_k \delta(\omega - \frac{2\pi k}{N})$$



(a)



(b)

Sect. 5.3 Properties of DTFT

- Recall that…

Synthesis equation

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}d\omega$$

Analysis equation

$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n]e^{-j\omega n}$$

$$X(e^{j\omega}) = F\{x[n]\}$$

$$x[n] = F^{-1}\{X(e^{j\omega})\}$$

$$x[n] \overset{F}{\longleftrightarrow} X(e^{j\omega})$$

$$\frac{1}{1-ae^{j\omega}} = F\{a^n u[n]\}, \quad |a|<1$$

$$a^n u[n] = F^{-1}\{\frac{1}{1-ae^{j\omega}}\}$$

$$a^n u[n] \overset{F}{\longleftrightarrow} \frac{1}{1-ae^{j\omega}}$$

Sect. 5.3 Properties of DTFT (cont'd)

- Periodicity of DT Fourier Transform:

$$X(e^{j(\omega+2\pi)}) = X(e^{j\omega})$$

- Linearity:

$$x[n] \xleftrightarrow{F} X(e^{j\omega})$$

$$y[n] \xleftrightarrow{F} Y(e^{j\omega})$$

$$\Rightarrow ax[n] + by[n] \xleftrightarrow{F} aX(e^{j\omega}) + bY(e^{j\omega})$$

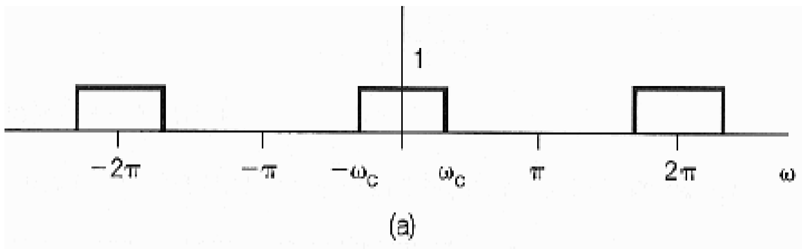
- Time & Frequency Shifting:

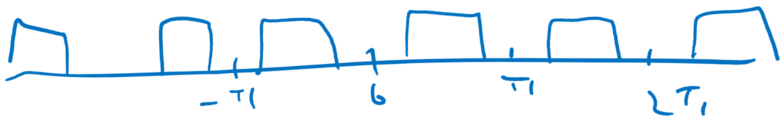
$$x[n-n_0] \xleftrightarrow{F} e^{-j\omega n_0} X(e^{j\omega})$$

$$e^{j\omega_0 n} x[n] \xleftrightarrow{F} X(e^{j(\omega-\omega_0)})$$

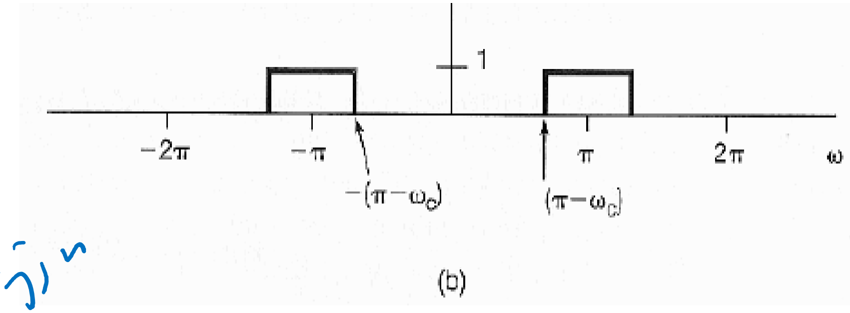
- Example 5.7 Relationship between LPF & HPF

$$H_{lp}(e^{j\omega})$$





$$H_{hp}(e^{j(\omega-\pi)})$$



$(-1)^n$

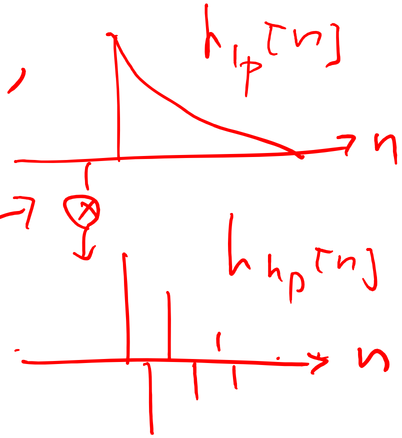
$$H_{hp}(e^{j\omega}) = H_{lp}(e^{j(\omega-\pi)})$$

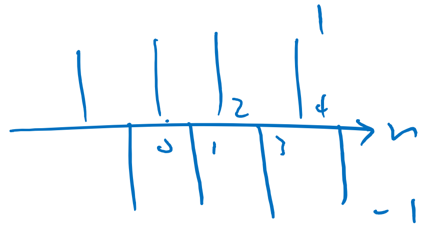
$$\Rightarrow h_{hp}[n] = e^{j\pi n} h_{lp}[n]$$

$$= (-1)^n h_{lp}[n]$$

$$e^{j\pi n} = \cos(\pi n) + j\sin(\pi n)$$

E.g.,





Simple LP/HP/BP Filter Conversion Techniques

$$\text{lowpass filter} \xrightarrow{\text{multiplied by } (-1)^n} \text{highpass filter}$$

$$\text{highpass filter} \xrightarrow{\text{multiplied by } (-1)^n} \text{lowpass filter}$$

$$\text{lowpass filter} \xrightarrow{\text{multiplied by } j^n} \text{bandpass filter}$$

$$\text{highpass filter} \xrightarrow{\text{multiplied by } j^n} \text{bandpass filter}$$

Sect. 5.3 Properties of DTFT (cont'd)

- Conjugation & Conjugate Symmetry

$$x[n] \stackrel{F}{\longleftrightarrow} X(e^{j\omega}) \quad \Rightarrow \quad x^*[n] \stackrel{F}{\longleftrightarrow} X^*(e^{-j\omega})$$

If $x[n]$ is real, then $x[n] = x^*[n]$ and $X(e^{-j\omega}) = X^*(e^{j\omega})$.

That is, $X(e^{j\omega})$ is conjugate symmetric and

$$Ev\{x[n]\} \stackrel{F}{\longleftrightarrow} Re\{X(e^{j\omega})\}$$

$$Od\{x[n]\} \stackrel{F}{\longleftrightarrow} jIm\{X(e^{j\omega})\}$$

Let $X(e^{j\omega}) = Re\{X(e^{j\omega})\} + jIm\{X(e^{j\omega})\}$

$$\Rightarrow \quad Re\{X(e^{j\omega})\} = Re\{X(e^{-j\omega})\}$$

$$\Rightarrow \quad Im\{X(e^{j\omega})\} = -Im\{X(e^{-j\omega})\}$$

Real part is an even function
Imaginary part is an odd function

Let $X(e^{j\omega}) = \left|X(e^{j\omega})\right| e^{\angle X(e^{j\omega})}$

$$\Rightarrow \quad \left|X(e^{j\omega})\right| \text{ even}, \quad \angle X(e^{j\omega}) \text{ odd}$$

Magnitude: an even function
Phase: an odd function

Sect. 5.3 Properties of DTFT (cont'd)

- Conjugation & Conjugate Symmetry

$$x[n] \xleftrightarrow{F} X(e^{j\omega}) \quad \Rightarrow \quad x^*[n] \xleftrightarrow{F} X^*(e^{-j\omega})$$

- If $x[n] = x^*[n]$ and $x[-n] = x[n]$

$\Rightarrow X(e^{-j\omega}) = X^*(e^{j\omega})$ and $X(e^{-j\omega}) = X(e^{j\omega})$

$\Rightarrow X(e^{j\omega}) = X^*(e^{j\omega})$

$\Rightarrow$ If $x[n]$ is real and even, then $X(e^{j\omega})$ is real and even.

- If $x[n]$ is real and odd, then $X(e^{j\omega})$ is pure imaginary and odd.

Sect. 5.3 Properties of DTFT (cont'd)

- Differencing & Accumulation

$$x[n] \xleftrightarrow{F} X(e^{j\omega})$$

$$x[n]-x[n-1] \xleftrightarrow{F} (1-e^{-j\omega})X(e^{j\omega})$$

$$y[n]=\sum_{m=-\infty}^{n} x[m] \xleftrightarrow{F} \frac{1}{(1-e^{-j\omega})}X(e^{j\omega})+\underbrace{\pi X(e^{j0})\sum_{k=-\infty}^{+\infty}\delta(\omega-2\pi k)}_{\text{dc value}}$$

Sect. 5.3 Properties of DTFT (cont'd)

- Differentiation in Frequency

$$x[n] \xleftrightarrow{F} X(e^{j\omega}) \Rightarrow nx[n] \xleftrightarrow{F} j\frac{d}{d\omega}X(e^{j\omega})$$

Proof:

$$\frac{d}{d\omega}X(e^{j\omega}) = \frac{d}{d\omega}\sum_{n=-\infty}^{+\infty} x[n]e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{+\infty} (-jn)x[n]e^{-j\omega n} = (-j)\sum_{n=-\infty}^{+\infty} (nx[n])e^{-j\omega n}$$

- Time Reversal

$$x[n] \xleftrightarrow{F} X(e^{j\omega}) \Rightarrow x[-n] \xleftrightarrow{F} X(e^{-j\omega})$$

Proof:

$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n]e^{-j\omega n}, \quad X(e^{j(-\omega)}) = \sum_{n=-\infty}^{+\infty} x[n]e^{-j(-\omega)n}$$

Sect. 5.3 Properties of DTFT (cont'd)

- Time Expansion

$$x[n] \Rightarrow x[an] = ?$$

If a is an integer and a>1, $x[an]$ is a time-compressed version of $x[n]$. For example, $x[2n]$ is the even samples of $x[n]$.

However, if a is not an integer, the value of $x[an]$ is unknown because discrete-time signals are defined over integer intervals. Consequently, we cannot slow down the signal by making $a < 1$.

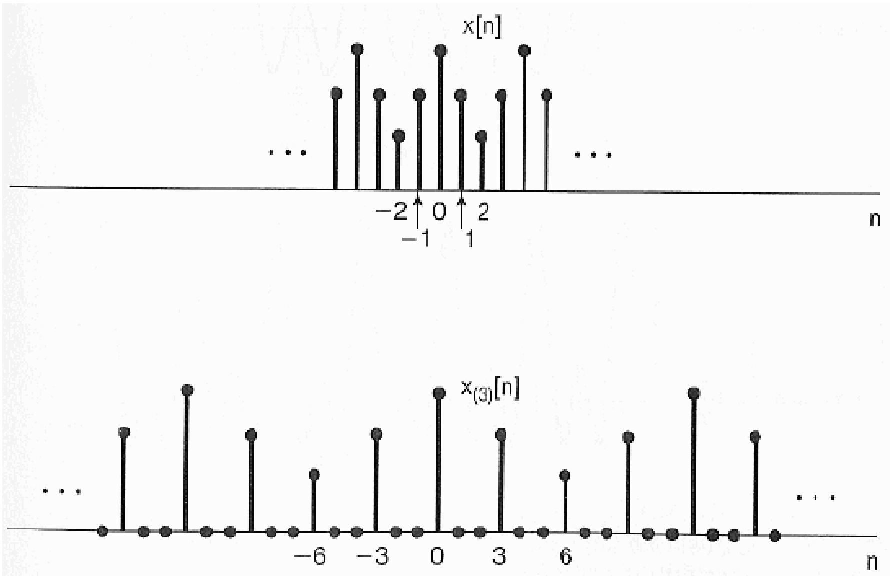
We resort to an alternative method (on next page).

Sect. 5.3 Properties of DTFT (cont'd)

- Time Expansion

Define

$$x_{(k)}[n] = \begin{cases} x[n/k], & \text{if n is a multiple of } k \\ 0, & \text{otherwise.} \end{cases}$$



$x_{(k)}[n]$ is obtained by placing k-1 zeros between successive samples of the original signal.

Sect. 5.3 Properties of DTFT (cont'd)

- Time Expansion

$$X_{(k)}(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x_{(k)}[n]e^{-j\omega n}$$

$$= \sum_{r=-\infty}^{+\infty} x_{(k)}[rk]e^{-j\omega rk}$$

$x_{(k)}[rk] = x[r]$

$$= \sum_{r=-\infty}^{+\infty} x[r]e^{-jk\omega r}$$

$$= X(e^{jk\omega})$$

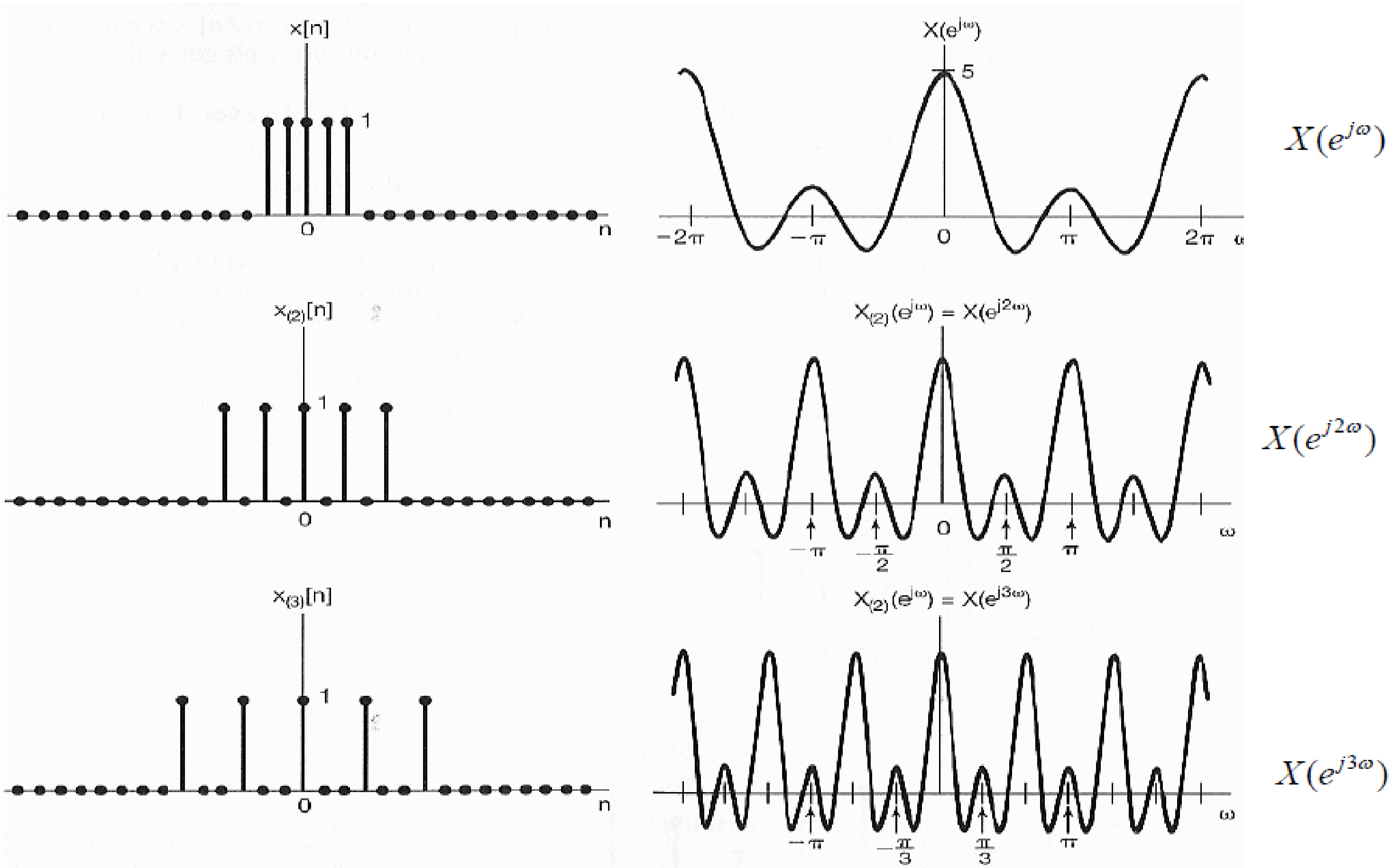
$$x_{(k)}[n] \xleftrightarrow{F} X(e^{jk\omega})$$

As a signal is spread out and slowed down in time, its FT is compressed.

Sect. 5.3 Properties of DTFT (cont'd)

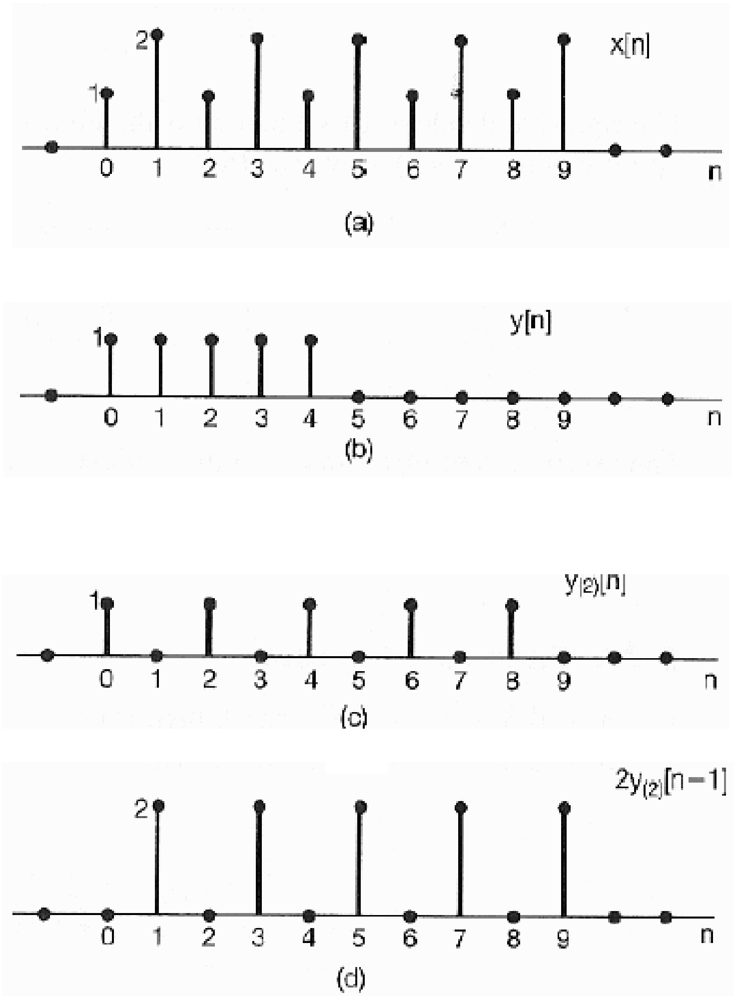
- Time Expansion

Inverse relationship between the time and frequency domains



- Example 5.9

$$x[n] = y_{(2)}[n] + 2y_{(2)}[n-1]$$



$$y_{(2)}[n] = \begin{cases} y[n/2], & \text{if } n \text{ is even} \\ 0, & \text{if } n \text{ is odd} \end{cases}$$

$$Y(e^{j\omega}) = e^{-j2\omega} \frac{\sin(5\omega/2)}{\sin(\omega/2)}$$

$\omega \to 2\omega$

$$y_{(2)}[n] \stackrel{\mathcal{F}}{\longleftrightarrow} e^{-j4\omega} \frac{\sin(5\omega)}{\sin(\omega)}$$

$$2y_{(2)}[n-1] \stackrel{\mathcal{F}}{\longleftrightarrow} 2e^{-j5\omega} \frac{\sin(5\omega)}{\sin(\omega)}$$

$$X(e^{j\omega}) = e^{-j4\omega}(1 + 2e^{-j\omega}) \left(\frac{\sin(5\omega)}{\sin(\omega)} \right)$$

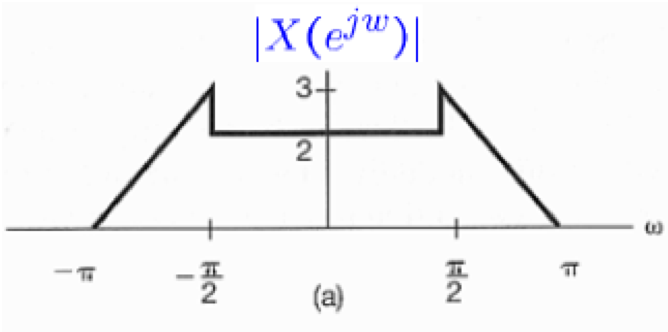
Sect. 5.3 Properties of DTFT (cont'd)

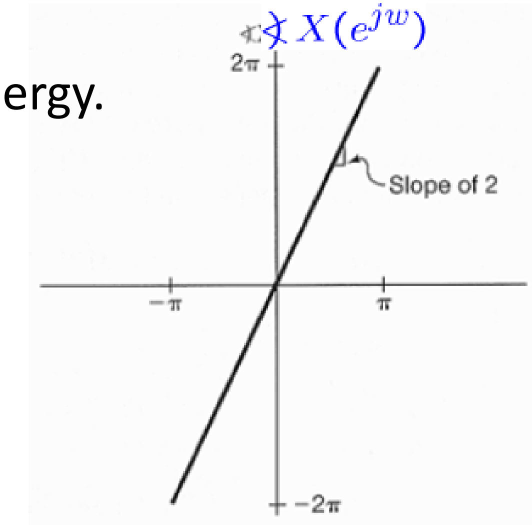
- Parseval's relation

$$x[n] \xleftrightarrow{F} X(e^{j\omega})$$

$$\underbrace{\sum_{n=-\infty}^{+\infty} |x[n]|^2}_{\text{Total energy}} = \frac{1}{2\pi}\int_{2\pi} \underbrace{|X(e^{j\omega})|^2}_{\text{Energy density spectrum}} d\omega$$

- Example 5.10
  Determine if x[n] is periodic/real/even/finite energy.





$X(e^{j\omega}) \neq$ impulse train $\Rightarrow$ $x[n]$ is NOT periodic

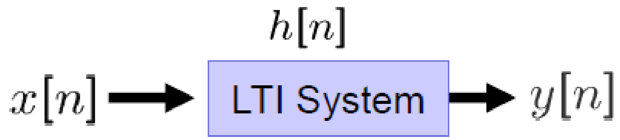
Even magnitude odd phase $\Rightarrow$ $x[n]$ is real

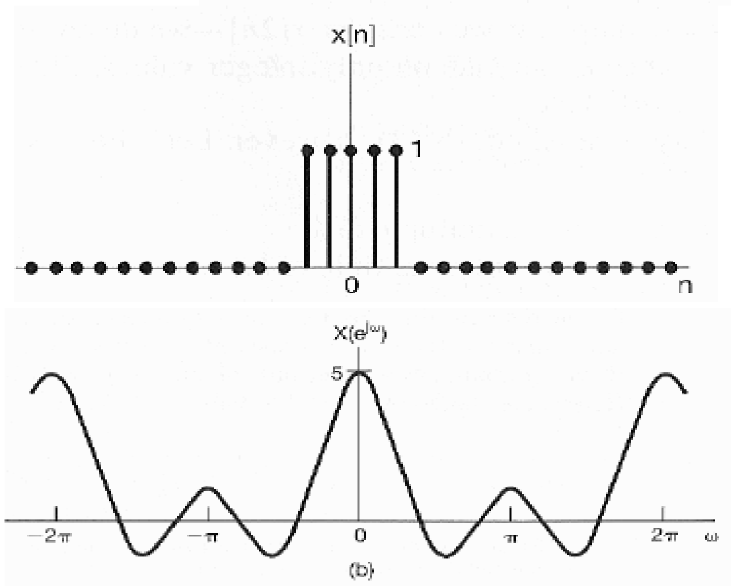
$X(e^{j\omega})$ is not real $\Rightarrow$ $x[n]$ is NOT even

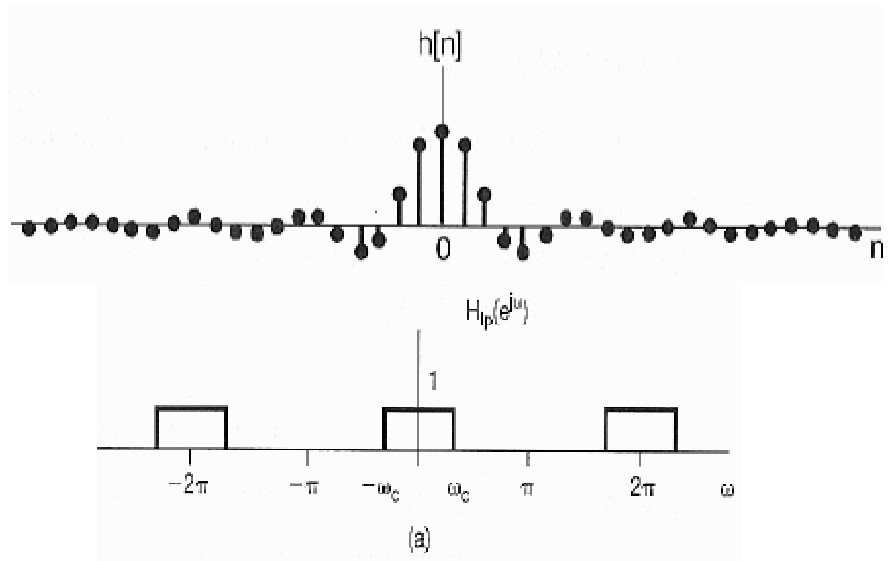
$X(e^{j\omega})$ has finite energy $\Rightarrow$ $x[n]$ is finite

Sect. 5.4 & 5.5
Convolution vs. Multiplication Property

- Convolution Property







$$\boxed{y[n] = x[n] * h[n] \quad \xleftrightarrow{F} \quad Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega})}$$

$$= \sum_{n=-\infty}^{+\infty} x[k]h[n-k]$$

- Example 5.11 Time shifting property

$$x[n] \longrightarrow \underset{}{\overset{h[n]}{\boxed{\text{LTI System}}}} \longrightarrow y[n]$$

$$h[n] = \delta[n - n_0]$$

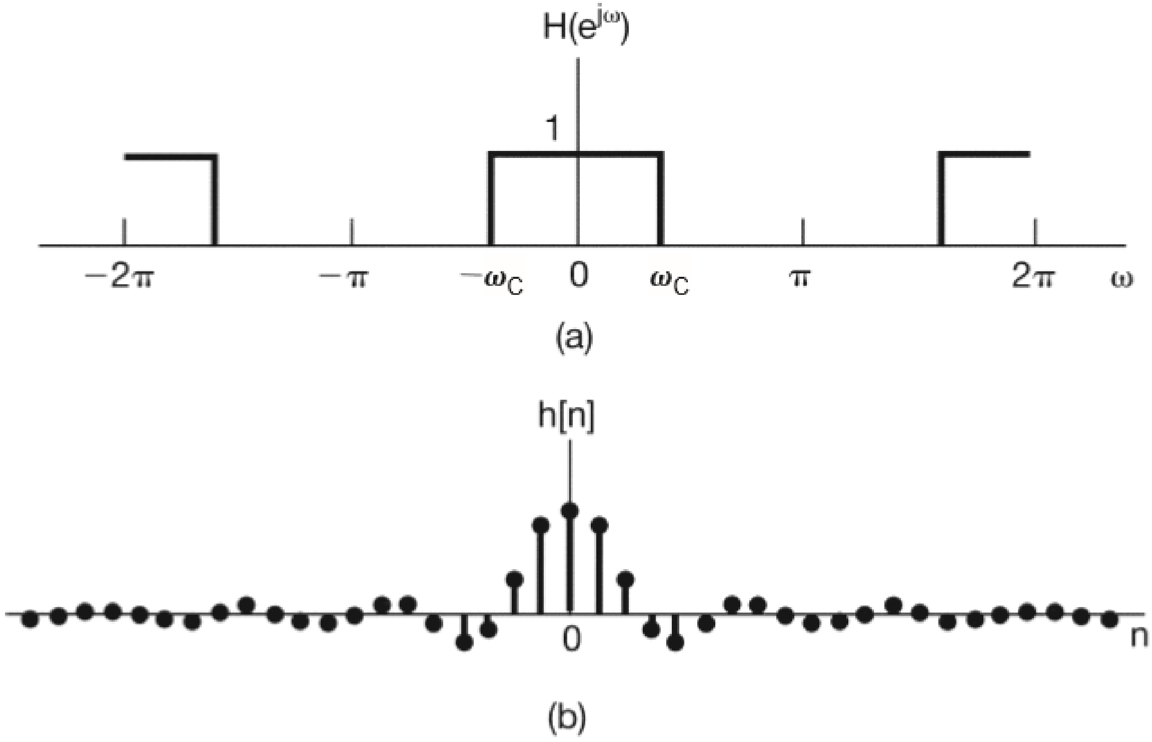
$$\Rightarrow H(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} \delta[n - n_0] e^{-j\omega n} = e^{-j\omega n_0}$$

$$\Rightarrow Y(e^{j\omega}) = H(e^{j\omega}) X(e^{j\omega})$$

$$= e^{-j\omega n_0} X(e^{j\omega})$$

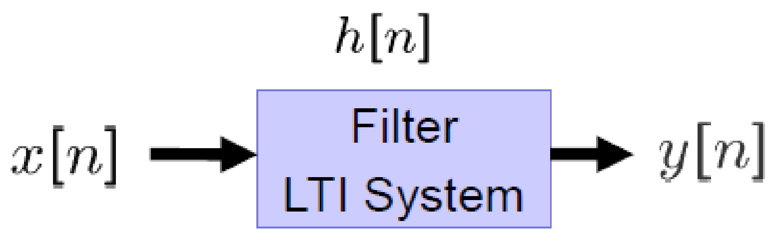
$$\Rightarrow y[n] = x[n - n_0]$$

- Example 5.12 Ideal LPF



$$h[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(e^{j\omega})e^{j\omega n}d\omega = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{j\omega n}d\omega = \frac{\sin \omega_c n}{\pi n}$$

- Example 5.13 Determine y[n]



$$h[n] = a^n u[n], \quad |a| < 1 \qquad \Rightarrow H(e^{j\omega}) = \frac{1}{1 - ae^{-j\omega}}$$

$$x[n] = b^n u[n], \quad |b| < 1 \qquad \Rightarrow X(e^{j\omega}) = \frac{1}{1 - be^{-j\omega}}$$

$$\Rightarrow Y(e^{j\omega}) = H(e^{j\omega}) X(e^{j\omega})$$

$$= \frac{1}{1 - ae^{-j\omega}} \frac{1}{1 - be^{-j\omega}}$$

- Example 5.13 (cont'd)

if $a \neq b$ $$Y(e^{j\omega}) = [(\frac{a}{a-b})\frac{1}{1-ae^{-j\omega}} + (\frac{-b}{a-b})\frac{1}{1-be^{-j\omega}}]$$

$$\Rightarrow y[n] = (\frac{a}{a-b})a^n u[n] - (\frac{b}{a-b})b^n u[n]$$

if $a = b$ $$Y(e^{j\omega}) = (\frac{1}{1-ae^{-j\omega}})^2 = \frac{j}{a}e^{j\omega}\frac{d}{d\omega}(\frac{1}{1-ae^{-j\omega}})$$

since $$a^n u[n] \overset{F}{\longleftrightarrow} \frac{1}{1-ae^{-j\omega}}$$

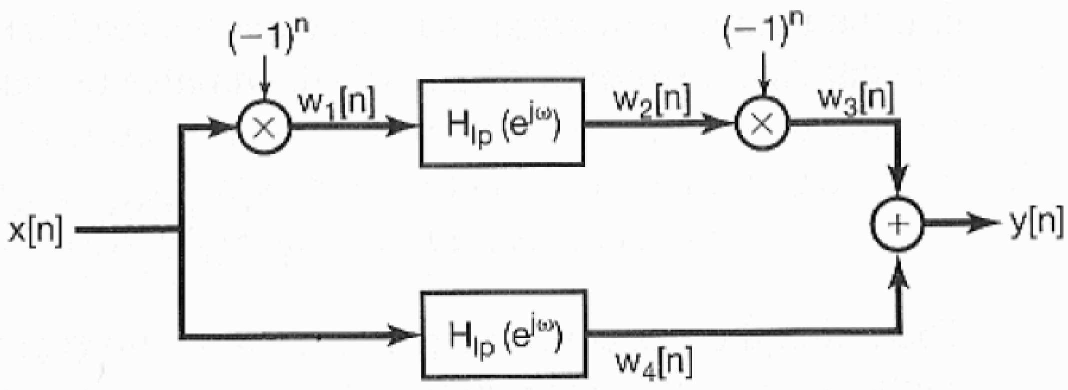
$$na^n u[n] \overset{F}{\longleftrightarrow} j\frac{d}{d\omega}(\frac{1}{1-ae^{-j\omega}})$$

$$(n+1)a^{n+1}u[n+1] \overset{F}{\longleftrightarrow} je^{j\omega}\frac{d}{d\omega}(\frac{1}{1-ae^{-j\omega}})$$

$$\Rightarrow y[n] = (n+1)a^n u[n+1] \quad = (n+1)a^n u[n]$$

- Example 5.14

$H_{lp}(e^{j\omega})$: Low-pass filter with $\omega_c = \pi/4$



$$(-1)^n = e^{j\pi n}$$

$$w_1[n] = e^{j\pi n} x[n] = (-1)^n x[n]$$

$$\Rightarrow W_1(e^{j\omega}) = X(e^{j(\omega-\pi)})$$

$$W_2(e^{j\omega}) = H_{lp}(e^{j\omega}) X(e^{j(\omega-\pi)})$$

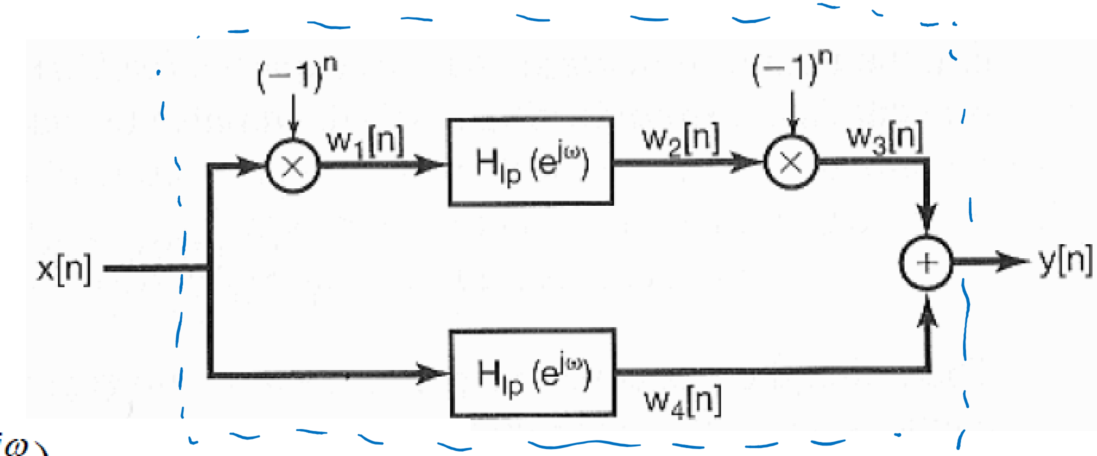
$$w_3[n] = e^{j\pi n} w_2[n] = (-1)^n w_2[n]$$

$$\Rightarrow W_3(e^{j\omega}) = W_2(e^{j(\omega-\pi)}) = H_{lp}(e^{j(\omega-\pi)}) X(e^{j(\omega-2\pi)})$$

$$= H_{lp}(e^{j(\omega-\pi)}) X(e^{j\omega})$$

- Example 5.14

$$H_{lp}(e^{j\omega}):\ \text{Low-pass filter with } \omega_c = \pi/4$$



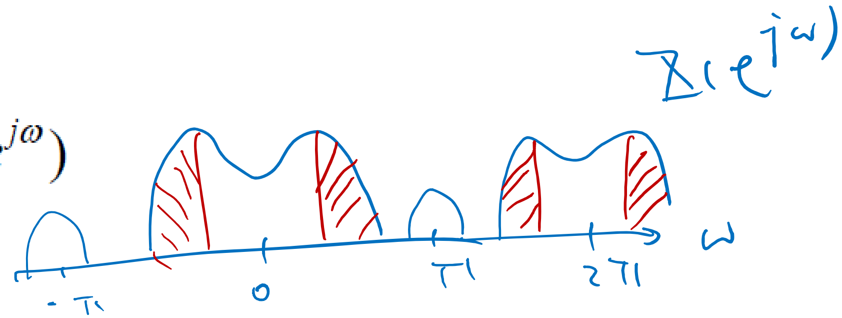
$H(e^{j\omega})$

$$W_4(e^{j\omega}) = H_{lp}(e^{j\omega})X(e^{j\omega})$$

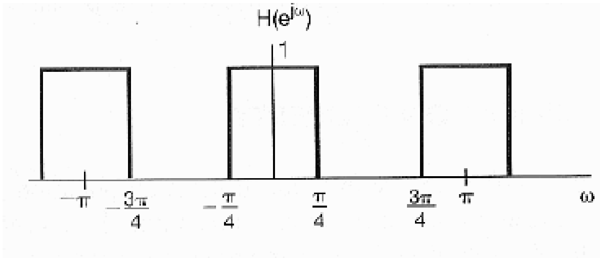
$$Y(e^{j\omega}) = W_3(e^{j\omega}) + W_4(e^{j\omega})$$

$$= H_{lp}(e^{j(\omega-\pi)})X(e^{j\omega}) + H_{lp}(e^{j\omega})X(e^{j\omega})$$

$$= [H_{lp}(e^{j(\omega-\pi)}) + H_{lp}(e^{j\omega})]X(e^{j\omega})$$



$$H(e^{j\omega}) = \underbrace{\underbrace{H_{lp}(e^{j(\omega-\pi)})}_{\text{high-pass filter}} + \underbrace{H_{lp}(e^{j\omega})}_{\text{low-pass filter}}}_{\text{bandstop filter}}$$



(b)

Sect. 5.4 & 5.5
Convolution vs. Multiplication Property

■ Multiplication Property

$$y[n] = x_1[n]x_2[n] \quad \Longleftrightarrow \quad \boxed{Y(e^{j\omega}) = \frac{1}{2\pi}\int_{2\pi} X_1(e^{j\theta})X_2(e^{-j(\omega-\theta)})d\theta}$$

(Proof): $$Y(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} y[n]e^{-j\omega n} = \sum_{n=-\infty}^{+\infty} x_1[n]x_2[n]e^{-j\omega n}$$

$$Y(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x_2[n]\left\{\frac{1}{2\pi}\int_{2\pi} X_1(e^{j\theta})e^{j\theta n}d\theta\right\}e^{-j\omega n}$$

$$Y(e^{j\omega}) = \frac{1}{2\pi}\int_{2\pi} X_1(e^{j\theta})\left[\sum_{n=-\infty}^{+\infty} x_2[n]e^{-j(\omega-\theta)n}\right]d\theta$$

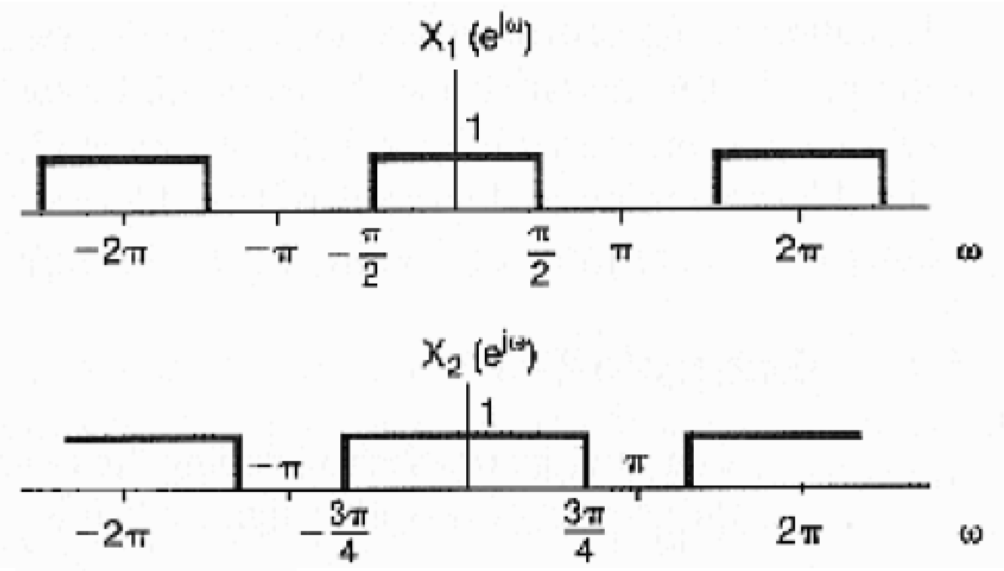
$$= \frac{1}{2\pi}\int_{2\pi} X_1(e^{j\theta})X_2(e^{-j(\omega-\theta)})d\theta$$

- Example 5.15
  Converting periodic convolution into ordinary convolution

$$x[n] = x_1[n]x_2[n]$$

$$x_1[n] = \frac{\sin(\frac{\pi}{2}n)}{\pi n}$$

$$x_2[n] = \frac{\sin(\frac{3\pi}{4}n)}{\pi n}$$



$$X(e^{j\omega}) = \frac{1}{2\pi}\int_{2\pi} X_1(e^{j\theta})X_2(e^{j(\omega-\theta)})d\theta$$

We can convert this equation to an ordinary convolution. Define

$$\hat{X}_1(e^{j\theta}) = \begin{cases} X_1(e^{j\theta}), & \text{for } -\pi < \theta \le \pi \\ 0, & \text{otherwise.} \end{cases}$$

- Example 5.15
  Converting periodic convolution
  into ordinary convolution

$$X(e^{j\omega}) = \frac{1}{2\pi}\int_{-\pi}^{+\pi} \hat{X}_1(e^{j\theta})X_2(e^{j(\omega-\theta)})d\theta$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} \hat{X}_1(e^{j\theta})X_2(e^{j(\omega-\theta)})d\theta$$

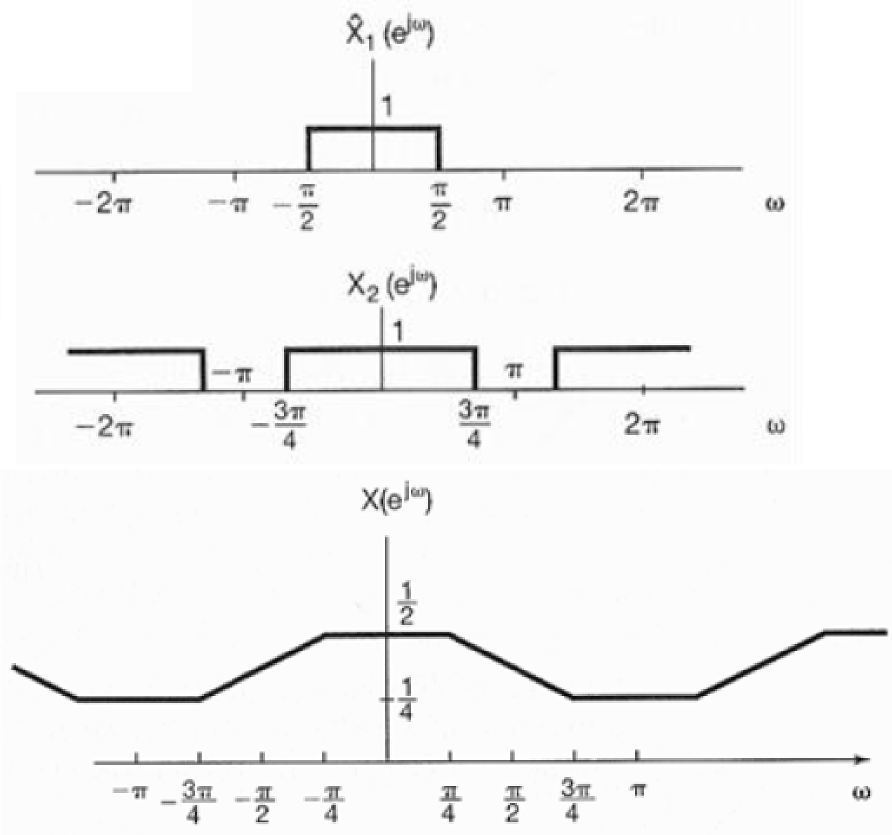


TABLE 5.1 PROPERTIES OF THE DISCRETE-TIME FOURIER TRANSFORM

Section	Property	Aperiodic Signal	Fourier Transform
		$x[n]$	$X(e^{j\omega})$ periodic with
		$y[n]$	$Y(e^{j\omega})$ period 2π
5.3.2	Linearity	$ax[n] + by[n]$	$aX(e^{j\omega}) + bY(e^{j\omega})$
5.3.3	Time Shifting	$x[n - n_0]$	$e^{-j\omega n_0} X(e^{j\omega})$
5.3.3	Frequency Shifting	$e^{j\omega_0 n} x[n]$	$X(e^{j(\omega-\omega_0)})$
5.3.4	Conjugation	$x^*[n]$	$X^*(e^{-j\omega})$
5.3.6	Time Reversal	$x[-n]$	$X(e^{-j\omega})$
5.3.7	Time Expansion	$x_{(k)}[n] = \begin{cases} x[n/k], & \text{if } n = \text{multiple of } k \\ 0, & \text{if } n \neq \text{multiple of } k \end{cases}$	$X(e^{jk\omega})$
5.4	Convolution	$x[n] * y[n]$	$X(e^{j\omega})Y(e^{j\omega})$
5.5	Multiplication	$x[n]y[n]$	$\frac{1}{2\pi} \int_{2\pi} X(e^{j\theta})Y(e^{j(\omega-\theta)})d\theta$
5.3.5	Differencing in Time	$x[n] - x[n-1]$	$(1 - e^{-j\omega})X(e^{j\omega})$
5.3.5	Accumulation	$\sum_{k=-\infty}^{n} x[k]$	$\frac{1}{1 - e^{-j\omega}} X(e^{j\omega})$ $+\pi X(e^{j0}) \sum_{k=-\infty}^{+\infty} \delta(\omega - 2\pi k)$
5.3.8	Differentiation in Frequency	$nx[n]$	$j\frac{dX(e^{j\omega})}{d\omega}$

5.3.4	Conjugate Symmetry for Real Signals	$x[n]$ real	$\begin{cases} X(e^{j\omega}) = X^*(e^{-j\omega}) \\ \Re e\{X(e^{j\omega})\} = \Re e\{X(e^{-j\omega})\} \\ \Im m\{X(e^{j\omega})\} = -\Im m\{X(e^{-j\omega})\} \\ \lvert X(e^{j\omega})\rvert = \lvert X(e^{-j\omega})\rvert \\ \sphericalangle X(e^{j\omega}) = -\sphericalangle X(e^{-j\omega}) \end{cases}$
5.3.4	Symmetry for Real, Even Signals	$x[n]$ real an even	$X(e^{j\omega})$ real and even
5.3.4	Symmetry for Real, Odd Signals	$x[n]$ real and odd	$X(e^{j\omega})$ purely imaginary and odd
5.3.4	Even-odd Decomposition of Real Signals	$x_e[n] = \mathcal{E}v\{x[n]\}$ $[x[n]$ real$]$ $x_o[n] = \mathcal{O}d\{x[n]\}$ $[x[n]$ real$]$	$\Re e\{X(e^{j\omega})\}$ $j\Im m\{X(e^{j\omega})\}$
5.3.9	Parseval's Relation for Aperiodic Signals $\sum_{n=-\infty}^{+\infty} \lvert x[n]\rvert^2 = \frac{1}{2\pi}\int_{2\pi} \lvert X(e^{j\omega})\rvert^2 d\omega$		

TABLE 5.2 BASIC DISCRETE-TIME FOURIER TRANSFORM PAIRS

Signal	Fourier Transform	Fourier Series Coefficients (if periodic)
$\sum_{k=\langle N\rangle} a_k e^{jk(2\pi/N)n}$	$2\pi \sum_{k=-\infty}^{+\infty} a_k \delta\left(\omega - \frac{2\pi k}{N}\right)$	a_k
$e^{j\omega_0 n}$	$2\pi \sum_{l=-\infty}^{+\infty} \delta(\omega - \omega_0 - 2\pi l)$	(a) $\omega_0 = \frac{2\pi m}{N}$ $a_k = \begin{cases} 1, & k = m, m \pm N, m \pm 2N, \ldots \\ 0, & \text{otherwise} \end{cases}$ (b) $\frac{\omega_0}{2\pi}$ irrational $\Rightarrow$ The signal is aperiodic
$\cos \omega_0 n$	$\pi \sum_{l=-\infty}^{+\infty} \{\delta(\omega - \omega_0 - 2\pi l) + \delta(\omega + \omega_0 - 2\pi l)\}$	(a) $\omega_0 = \frac{2\pi m}{N}$ $a_k = \begin{cases} \frac{1}{2}, & k = \pm m, \pm m \pm N, \pm m \pm 2N, \ldots \\ 0, & \text{otherwise} \end{cases}$ (b) $\frac{\omega_0}{2\pi}$ irrational $\Rightarrow$ The signal is aperiodic
$\sin \omega_0 n$	$\frac{\pi}{j} \sum_{l=-\infty}^{+\infty} \{\delta(\omega - \omega_0 - 2\pi l) - \delta(\omega + \omega_0 - 2\pi l)\}$	(a) $\omega_0 = \frac{2\pi r}{N}$ $a_k = \begin{cases} \frac{1}{2j}, & k = r, r \pm N, r \pm 2N, \ldots \\ -\frac{1}{2j}, & k = -r, -r \pm N, -r \pm 2N, \ldots \\ 0, & \text{otherwise} \end{cases}$ (b) $\frac{\omega_0}{2\pi}$ irrational $\Rightarrow$ The signal is aperiodic
$x[n] = 1$	$2\pi \sum_{l=-\infty}^{+\infty} \delta(\omega - 2\pi l)$	$a_k = \begin{cases} 1, & k = 0, \pm N, \pm 2N, \ldots \\ 0, & \text{otherwise} \end{cases}$

Periodic square wave $x[n] = \begin{cases} 1, & \lvert n\rvert \leq N_1 \\ 0, & N_1 < \lvert n\rvert \leq N/2 \end{cases}$ and $x[n+N] = x[n]$	$2\pi \sum_{k=-\infty}^{+\infty} a_k \delta\left(\omega - \frac{2\pi k}{N}\right)$	$a_k = \frac{\sin[(2\pi k/N)(N_1 + \frac{1}{2})]}{N \sin[2\pi k/2N]}, \ k \neq 0, \pm N, \pm 2N, \ldots$ $a_k = \frac{2N_1 + 1}{N}, \ k = 0, \pm N, \pm 2N, \ldots$
$\sum_{k=-\infty}^{+\infty} \delta[n - kN]$	$\frac{2\pi}{N} \sum_{k=-\infty}^{+\infty} \delta\left(\omega - \frac{2\pi k}{N}\right)$	$a_k = \frac{1}{N}$ for all k
$a^n u[n], \quad \lvert a\rvert < 1$	$\frac{1}{1 - ae^{-j\omega}}$	—
$x[n] = \begin{cases} 1, & \lvert n\rvert \leq N_1 \\ 0, & \lvert n\rvert > N_1 \end{cases}$	$\frac{\sin[\omega(N_1 + \frac{1}{2})]}{\sin(\omega/2)}$	—
$\frac{\sin Wn}{\pi n} = \frac{W}{\pi} \operatorname{sinc}\left(\frac{Wn}{\pi}\right)$ $0 < W < \pi$	$X(\omega) = \begin{cases} 1, & 0 \leq \lvert\omega\rvert \leq W \\ 0, & W < \lvert\omega\rvert \leq \pi \end{cases}$ $X(\omega)$ periodic with period 2π	—
$\delta[n]$	1	—
$u[n]$	$\frac{1}{1 - e^{-j\omega}} + \sum_{k=-\infty}^{+\infty} \pi\delta(\omega - 2\pi k)$	—
$\delta[n - n_0]$	$e^{-j\omega n_0}$	—
$(n+1)a^n u[n], \quad \lvert a\rvert < 1$	$\frac{1}{(1 - ae^{-j\omega})^2}$	—
$\frac{(n + r - 1)!}{n!(r-1)!} a^n u[n], \quad \lvert a\rvert < 1$	$\frac{1}{(1 - ae^{-j\omega})^r}$	—

Sect. 5.2 FT for *Periodic* Signals

Reall:

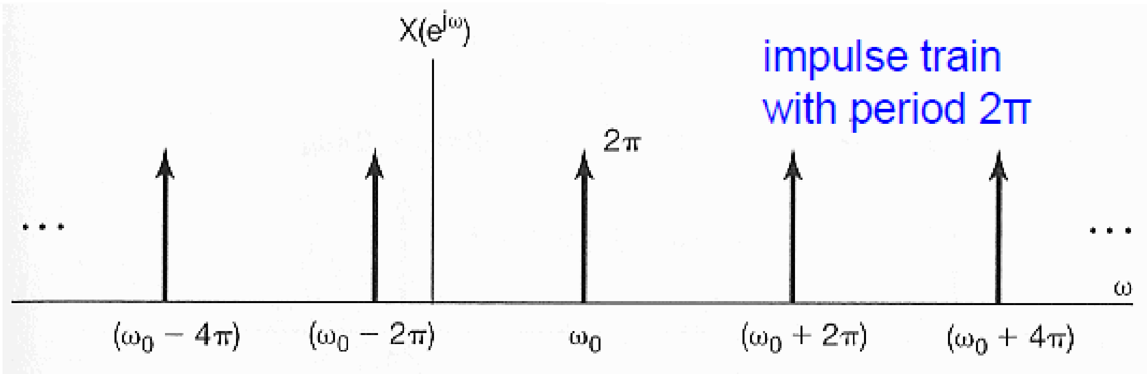
$$e^{j\omega_0 t} \xleftrightarrow{F} 2\pi\delta(\omega-\omega_0)$$

in the CT domain.

- FT from FS

$$x[n] = e^{j\omega_0 n}, \quad \omega_0 = \frac{2\pi}{N}$$

$$\Rightarrow X(e^{j\omega}) = \sum_{l=-\infty}^{+\infty} 2\pi\delta(\omega-\omega_0-2\pi l)$$



Proof:

$$\frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}d\omega = \frac{1}{2\pi}\int_{2\pi} \sum_{l=-\infty}^{+\infty} 2\pi\delta(\omega-\omega_0-2\pi l)e^{j\omega n}d\omega$$

$$= e^{j(\omega_0+2\pi r)n} = e^{j\omega_0 n}$$

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}d\omega$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n]e^{-j\omega n}$$

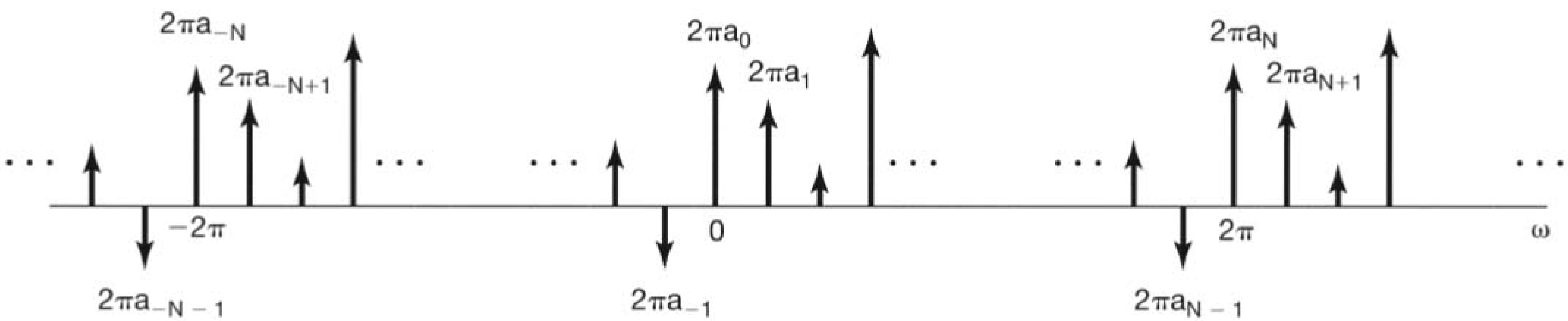
Sect. 5.2 FT for *Periodic* Signals

- FT from FS (cont'd)

If $x[n] = \sum_{k=\langle N\rangle} a_k e^{jk(2\pi/N)n}$

then $X(e^{j\omega}) = \sum_{k=-\infty}^{+\infty} 2\pi a_k \delta\left(\omega - \frac{2\pi k}{N}\right)$ for $-\pi \le \omega < \pi$

$$X(e^{j\omega}) = X(e^{j(\omega+2\pi)})$$



Sect. 5.2 FT for *Periodic* Signals

- FT from FS (cont'd)

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}d\omega$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n]e^{-j\omega n}$$

Thus, for a periodic sequence $x[n]$ with period N and with the FS representation

$$x[n] = \sum_{k=<N>} a_k e^{jk\omega_0 n},$$

$$a_{k+N} = a_k$$

its FT is related to its Foureir coefficient by

$$\boxed{X(e^{j\omega}) = \sum_{k=-\infty}^{+\infty} 2\pi a_k \delta(\omega - k\omega_0).}$$

The FT of a periodic signal can be directly constructed from its Fourier coefficients.

We can verify this equation graphically by expressing $x[n]$ as

$$x[n] = a_0 + a_1 e^{j\omega_0 n} + a_2 e^{j2\omega_0 n} + \ldots + a_{N-1}e^{j(N-1)\omega_0 n},$$

plot the FT of each term, and then superimpose them.

Sect. 5.2 FT for *Periodic* Signals

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}d\omega$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n]e^{-j\omega n}$$

- FT from FS (cont'd)

To determine the DTFT for a periodic discrete-time signal $x[n]$ (i.e., $x[n] = x[n+N]$)

(1) First, use the DTFS (Sec. 3.6) to express $x[n]$ by

$$x[n] = \sum_{n=\langle N\rangle} a_k e^{jk(2\pi/N)n}$$ where $$a_k = \frac{1}{N}\sum_{n=\langle N\rangle} x[n]e^{-jk(2\pi/N)n}$$

(2) Then

$$X(e^{j\omega}) = \sum_{k=-\infty}^{+\infty} 2\pi a_k \delta\left(\omega - \frac{2\pi k}{N}\right)$$ for $-\pi \leq \omega < \pi$

$$X(e^{j\omega}) = X(e^{j(\omega+2\pi)})$$

- FT from FS (cont'd)

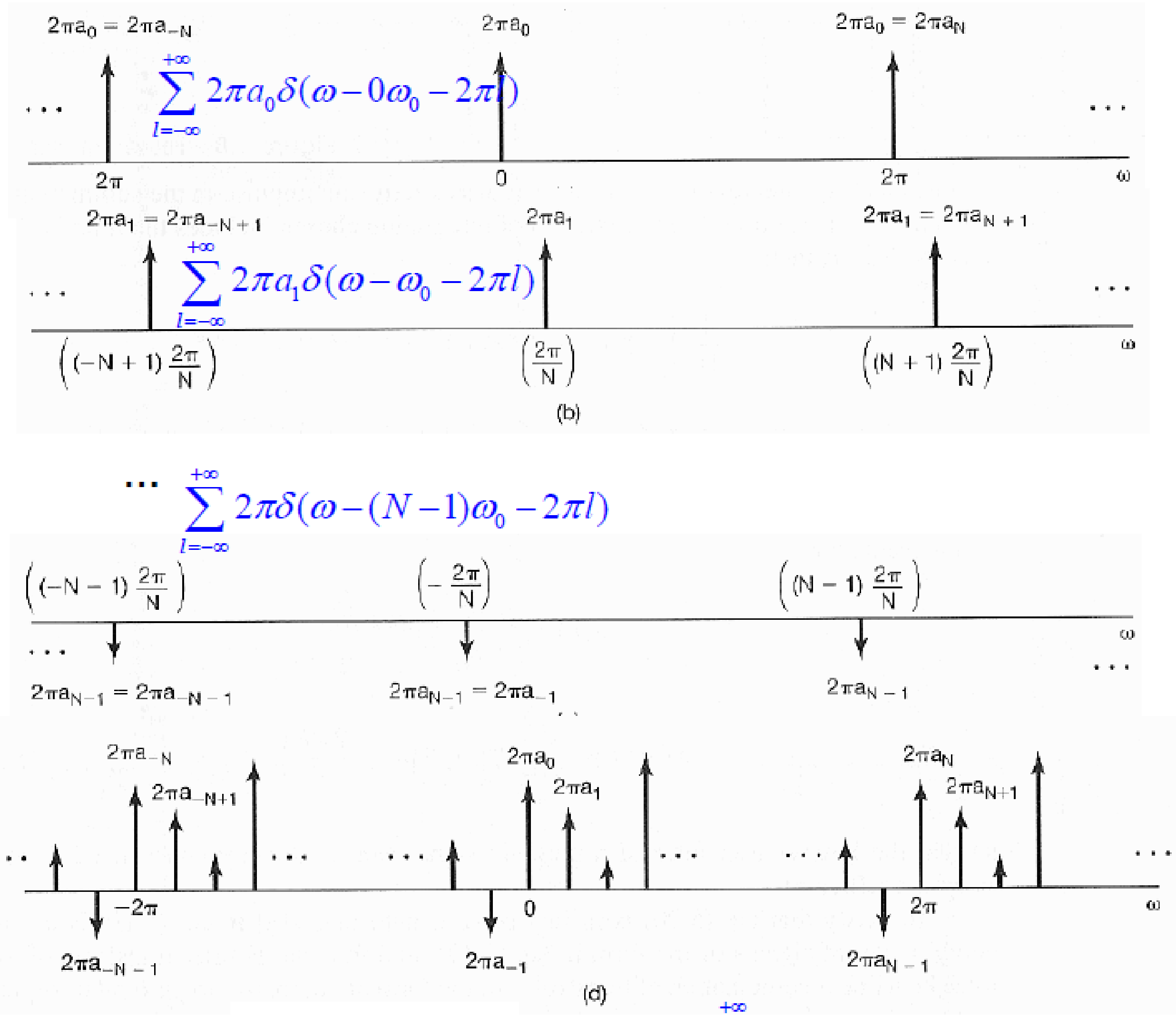
$$a_0 \overset{F}{\longleftrightarrow} \sum_{l=-\infty}^{+\infty} 2\pi a_0 \delta(\omega - 0\omega_0 - 2\pi l)$$

$$a_1 e^{j\omega_0 n} \overset{F}{\longleftrightarrow} \sum_{l=-\infty}^{+\infty} 2\pi a_1 \delta(\omega - \omega_0 - 2\pi l)$$

$$\cdots \sum_{l=-\infty}^{+\infty} 2\pi \delta(\omega - (N-1)\omega_0 - 2\pi l)$$

$$a_{N-1} e^{j(N-1)\omega_0 n} \overset{F}{\longleftrightarrow}$$

$$\sum_{k=<N>} a_k e^{jk\omega_0 n} \overset{F}{\longleftrightarrow}$$



$$a_{k+N} = a_k \quad \Rightarrow \quad X(e^{j\omega}) = \sum_{k=-\infty}^{+\infty} 2\pi a_k \delta(\omega - k\omega_0)$$

- Example 5.6 DTFT of Impulse Trains

$$x[n]=\sum_{k=-\infty}^{+\infty}\delta[n-kN]$$

$$\Rightarrow a_k=\frac{1}{N}\sum_{n=<N>}x[n]e^{-jk(2\pi/N)n}$$

$$=\frac{1}{N}\sum_{n=0}^{N-1}x[n]e^{-jk(2\pi/N)n}$$

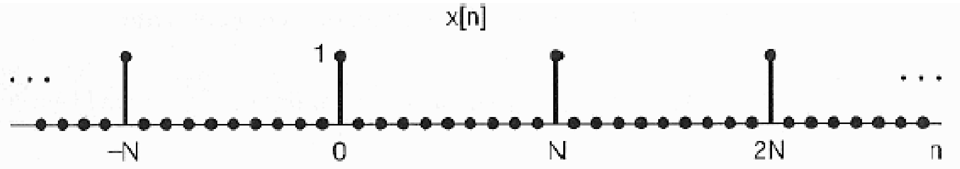
$$=\frac{1}{N}$$

$$\Rightarrow X(e^{j\omega})=\frac{2\pi}{N}\sum_{k=-\infty}^{+\infty}\delta(\omega-k\frac{2\pi}{N})$$

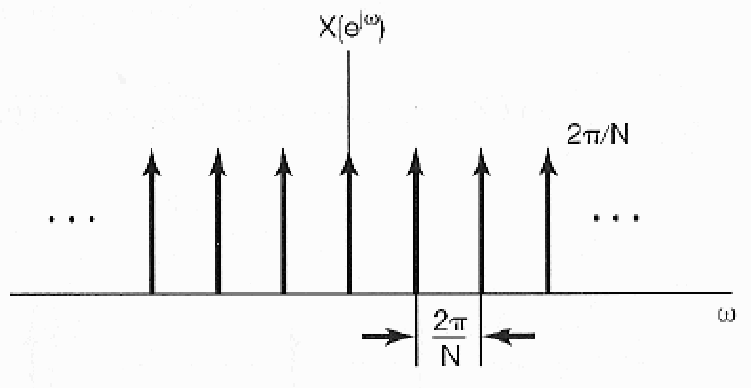
$$x[n]=\sum_{k=<N>}a_k e^{jk(2\pi/N)n}$$

$$a_k=\frac{1}{N}\sum_{n=<N>}x[n]e^{-jk(2\pi/N)n}$$

$$X(e^{j\omega})=\sum_{k=-\infty}^{+\infty}2\pi a_k\delta(\omega-\frac{2\pi k}{N})$$



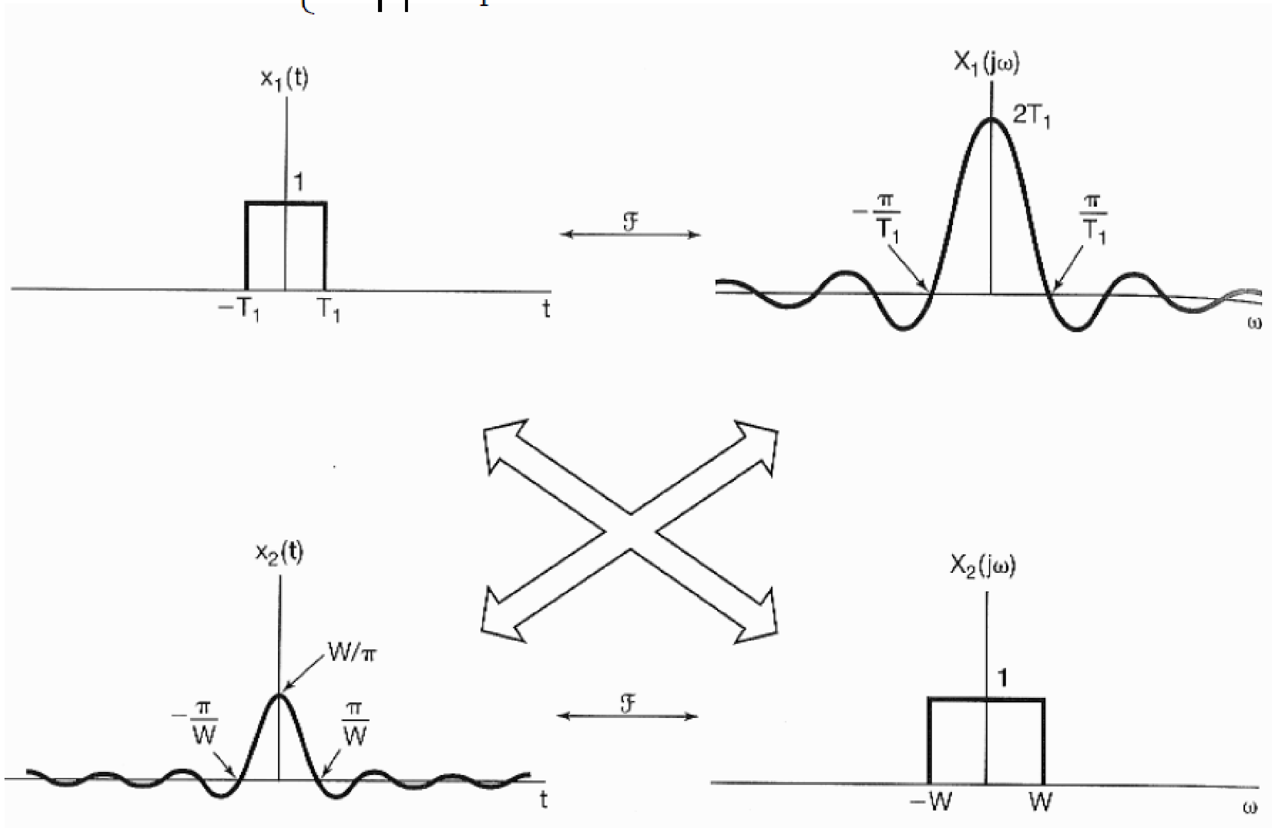
(a)



(b)

Sect. 4.3.6 Duality in CTFT

- Duality $$x_1(t) = \begin{cases} 1, & |t| < T_1 \\ 0, & |t| > T_1 \end{cases} \xleftrightarrow{\mathcal{F}} X_1(j\omega) = \frac{2\sin(\omega T_1)}{\omega}$$



$$x_2(t) = \frac{\sin(Wt)}{\pi t} \xleftrightarrow{\mathcal{F}} X_2(j\omega) = \begin{cases} 1, & |\omega| < W \\ 0, & |\omega| > W \end{cases}$$

Example 4.13

From Example 4.2 $e^{-|t|} \longleftrightarrow \dfrac{2}{1+\omega^2}$

Therefore, $\mathcal{F}\left(\dfrac{2}{1+t^2}\right) = 2\pi e^{-|\omega|}$

(Proof):

$$e^{-|t|} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\frac{2}{1+\omega^2}\right)e^{j\omega t}d\omega$$

$$2\pi e^{-|t|} = \int_{-\infty}^{\infty}\left(\frac{2}{1+\omega^2}\right)e^{-j\omega t}d\omega$$

$$2\pi e^{-|\omega|} = \int_{-\infty}^{\infty}\left(\frac{2}{1+t^2}\right)e^{-j\omega t}dt$$

(2) Duality for Properties

Duality

$$\frac{dx(t)}{dt} \xleftrightarrow{\mathcal{F}} j\omega X(j\omega)$$

$$-jtx(t) \xleftrightarrow{\mathcal{F}} \frac{dX(j\omega)}{d\omega}$$

$$x(t-t_0) \xleftrightarrow{\mathcal{F}} e^{-j\omega t_0} X(j\omega)$$

$$e^{j\omega_0 t} x(t) \xleftrightarrow{\mathcal{F}} X(j(\omega-\omega_0))$$

$$\int_{-\infty}^{t} x(\tau)d\tau \xleftrightarrow{\mathcal{F}} \frac{1}{j\omega} X(j\omega) + \pi X(0)\delta(\omega)$$

$$-\frac{1}{jt} x(t) + \pi x(0)\delta(t) \xleftrightarrow{\mathcal{F}} \int_{-\infty}^{\omega} X(j\eta)d\eta$$

Sect. 5.7 Duality

- Duality in DTFS

The Fourier coefficient a_k of a periodic sequence x[n] is also periodic. So we can apply DTFS again to a_k.

$$\underbrace{x[n] = \sum_{k=<N>} a_k e^{jk(2\pi/N)n}}_{\substack{\text{discrete time} \\ \text{periodic in time}}} \xleftrightarrow{FS} \underbrace{a_k = \sum_{n=<N>} \frac{1}{N} x[n]\, e^{-jk(2\pi/N)n}}_{\substack{\text{discrete frequency} \\ \text{periodic in frquency}}}$$

Sect. 5.7 Duality

- Duality in DTFS (cont'd)

Consider two periodic sequences related by

$$f[m] = \frac{1}{N} \sum_{r=<N>} g[r] e^{-jr(2\pi/N)m}$$

$$\Rightarrow \text{Set } m = k, r = n \Rightarrow f[k] = \frac{1}{N} \sum_{n=<N>} g[n] e^{-jk(2\pi/N)n} \Rightarrow g[n] \xleftrightarrow{FS} f[k]$$

$$\Rightarrow \text{Set } m = n, r = -k \Rightarrow f[n] = \frac{1}{N} \sum_{k=<N>} g[-k] e^{jk(2\pi/N)n} \Rightarrow f[n] \xleftrightarrow{FS} \frac{1}{N} g[-k]$$

For the DT Fourier series pair $x[n] \xleftrightarrow{FS} a_k$

$$x[n] = \sum_{k=<N>} a_k e^{jk\omega_0 n} = \sum_{k=<N>} a_k e^{jk(2\pi/N)n}$$

$$a_k = \frac{1}{N} \sum_{k=<N>} x[n] e^{-jk\omega_0 n} = \frac{1}{N} \sum_{k=<N>} x[n] e^{-jk(2\pi/N)n}$$

The duality implies that the Fourier coefficients for a_k are $\frac{1}{N} x[-n]$

Sect. 5.7 Duality (cont'd)

- Duality in DT Fourier Series (cont'd)

The duality implies that every property of the DT FS has a dual.

For example,

$$\left.\begin{array}{l} x[n-n_0] \xleftrightarrow{FS} e^{-jk(2\pi/N)n_0} a_k \\ e^{+jm(2\pi/N)n} x[n] \xleftrightarrow{FS} a_{k-m} \end{array}\right\} \text{dual}$$

$$\left.\begin{array}{l} \sum\limits_{r=<N>} x[n]y[n-r] \xleftrightarrow{FS} N a_k b_k \\ x[n]y[n] \xleftrightarrow{FS} \sum\limits_{l=<N>} a_l b_{k-l} \end{array}\right\} \text{dual}$$

See Table 3.2 on p. 221.

Sect. 5.7 Duality (cont'd)

5.7.2 Duality between DTFT and CTFS

CTFS

$$a_k = \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}dt$$

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$$

$$\omega_0 = 2\pi / T$$

$$t \to \omega$$
$$k \to -n$$
$$T = 2\pi \quad \omega_0 = 1$$

$$k \to -n$$
$$\omega_0 t \to \omega$$

DTFT

$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n]e^{-j\omega n}$$

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}d\omega$$

We can interpret the DTFT pair as the FS representation of $X(e^{j\omega})$

$\Rightarrow$ The nth Fourier coefficient is $x[-n]$

$\Rightarrow$ The period of $X(e^{j\omega})$ is 2π

- Example 5.17 Determine DTFT by duality

$$x[n] = \frac{\sin(\pi n/2)}{\pi n} \xleftrightarrow{DTFT} X(e^{j\omega}) = ?$$

Which CT signal has the Fourier coefficient $a_k = x[k]$ and $T=2\pi$?

From Example 3.5,

$$g(t) = \begin{cases} 1, & |t| \le T_1 \\ 0, & T_1 < |t| \le \pi \end{cases} \xleftrightarrow{FS} a_k = \frac{\sin(kT_1)}{k\pi}$$

Let $T_1 = \pi/2 \Rightarrow a_k = x[k]$ and

$$a_k = \frac{\sin(\pi k/2)}{\pi k} = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(t)e^{-jkt}dt = \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2} e^{-jkt}dt$$

Replacing k by $-n$ and t by ω yields

$$\frac{\sin(\pi n/2)}{\pi n} = \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2} e^{jn\omega}d\omega$$

Thus

$$X(e^{j\omega}) = \begin{cases} 1, & |\omega| \le \pi/2 \\ 0, & \pi/2 < |\omega| \le \pi \end{cases}$$

TABLE 5.3 SUMMARY OF FOURIER SERIES AND TRANSFORM EXPRESSIONS

	Continuous time		Discrete time	
	Time domain	**Frequency domain**	**Time domain**	**Frequency domain**
Fourier Series	$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$ continuous time periodic in time	Chap. 3 $a_k = \frac{1}{T_0}\int_{T_0} x(t)e^{-jk\omega_0 t}$ discrete frequency aperiodic in frequency	$x[n] = \sum_{k=\langle N\rangle} a_k e^{jk(2\pi/N)n}$ discrete time periodic in time	Sec. 3.6 $a_k = \frac{1}{N}\sum_{k=\langle N\rangle} x[n]e^{-jk(2\pi/N)n}$ duality discrete frequency periodic in frequency
Fourier Transform	$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)e^{j\omega t}d\omega$ continuous time aperiodic in time	Chap. 4 $X(j\omega) = \int_{-\infty}^{+\infty} x(t)e^{-j\omega t}dt$ duality continuous frequency aperiodic in frequency	duality $x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}d\omega,$ discrete time aperiodic in time	Chap. 5 $X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n]e^{-j\omega n}$ continuous frequency periodic in frequency

Sect. 5.8 Systems Characterized by Linear Constant-Coefficient Difference Equations

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

$$x[n] \longrightarrow \boxed{\text{LTI System}\ \overset{h[n]}{}} \longrightarrow y[n]$$

$\Rightarrow$ Determine the frequency response $H(e^{j\omega})$ of the system

Approach 1: Use eigenfunctions

Let $x[n]=e^{j\omega n} \quad \Rightarrow \quad y[n]=H(e^{j\omega})e^{j\omega n}$

Approach 2: Use DTFT

$$Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega}) \Rightarrow H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{\sum_{k=0}^{M} b_k e^{-jk\omega}}{\sum_{k=0}^{N} a_k e^{-jk\omega}}$$

Sect. 5.8 Systems Characterized by Linear Constant-Coefficient Difference Equations

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

$$x[n] \longrightarrow \overset{h[n]}{\boxed{\text{LTI System}}} \longrightarrow y[n]$$

$$F\left\{\sum_{k=0}^{N} a_k y[n-k]\right\} = F\left\{\sum_{k=0}^{M} b_k x[n-k]\right\}$$

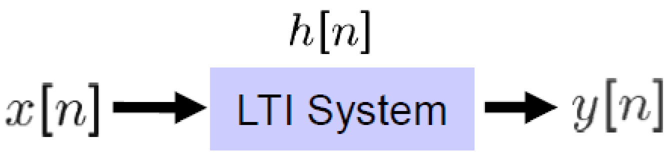
$$\sum_{k=0}^{N} a_k F\{y[n-k]\} = \sum_{k=0}^{M} b_k F\{x[n-k]\}$$

$$x[n-n_0] \stackrel{F}{\longleftrightarrow} e^{-j\omega n_0} X(e^{j\omega})$$

$$\sum_{k=0}^{N} a_k e^{-jk\omega} Y(e^{j\omega}) = \sum_{k=0}^{M} b_k e^{-jk\omega} X(e^{j\omega})$$

$$\Rightarrow H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{\sum_{k=0}^{M} b_k e^{-jk\omega}}{\sum_{k=0}^{N} a_k e^{-jk\omega}}$$

- Example 5.18



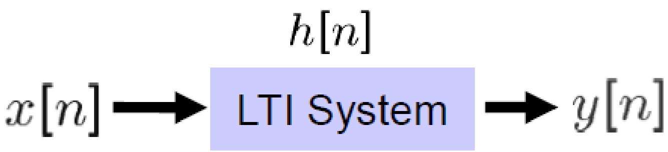
$$y[n]-ay[n-1]=x[n], \quad |a|<1$$

$$\Rightarrow H(e^{j\omega})=\frac{1}{1-ae^{-j\omega}}$$

From Example 5.1, we immediately find that

$$h[n]=a^n u[n]$$

- Example 5.19



$$y[n]-\frac{3}{4}y[n-1]+\frac{1}{8}y[n-2]=2x[n]$$

$$\Rightarrow H(e^{j\omega})=\frac{2}{1-\frac{3}{4}e^{-j\omega}+\frac{1}{8}e^{-j2\omega}}=\frac{2}{(1-\frac{1}{2}e^{-j\omega})(1-\frac{1}{4}e^{-j\omega})}$$

$$=\frac{4}{(1-\frac{1}{2}e^{-j\omega})}-\frac{2}{(1-\frac{1}{4}e^{-j\omega})}$$

$$\Rightarrow h[n]=4(\frac{1}{2})^{n}u[n]-2(\frac{1}{4})^{n}u[n]$$

- Example 5.20

Given $x[n] = (\frac{1}{4})^n u[n]$ and $H(e^{j\omega}) = \dfrac{2}{(1-\frac{1}{2}e^{-j\omega})(1-\frac{1}{4}e^{-j\omega})}$,

find $y[n]$.

$$
\begin{aligned}
Y(e^{j\omega}) &= X(e^{j\omega})H(e^{j\omega}) \\
&= \frac{1}{1-\frac{1}{4}e^{-j\omega}} \cdot \frac{2}{(1-\frac{1}{2}e^{-j\omega})(1-\frac{1}{4}e^{-j\omega})} \\
&= \frac{2}{(1-\frac{1}{2}e^{-j\omega})(1-\frac{1}{4}e^{-j\omega})^2} \\
&= \frac{8}{(1-\frac{1}{2}e^{-j\omega})} - \frac{4}{(1-\frac{1}{4}e^{-j\omega})} - \frac{2}{(1-\frac{1}{4}e^{-j\omega})^2}
\end{aligned}
$$

$$\Rightarrow y[n] = \{8(\tfrac{1}{2})^n - 4(\tfrac{1}{4})^n - 2(n+1)(\tfrac{1}{4})^n\}u[n]$$

- We have learned FS or FT of infinite-duration signals

	Aperiodic	Periodic
Continuous-Time	FT	FS FT
Discrete-Time	FT	FS FT

- Discrete Fourier Transform for DT signals of finite duration

Recall DT FS pair $\tilde{x}[n] \xleftrightarrow{FS} a_k$:

$$\tilde{x}[n] = \sum_{k=<N>} a_k e^{jk\omega_0 n}$$

$$a_k = \frac{1}{N} \sum_{n=<N>} \tilde{x}[n] e^{-jk\omega_0 n}$$

$$a_k = a_{k+N}$$

$$\omega_0 = \frac{2\pi}{N}$$

DFT of $x[n],\ 0 \le n \le N-1$

$$x[n] = \sum_{0}^{N-1} \mathcal{X}[k] e^{jk(2\pi/N)n}$$

$$\boxed{\begin{aligned}\mathcal{X}[k] &= \frac{1}{N} \sum_{0}^{N-1} x[n] e^{-jk(2\pi/N)n} \\ &= \frac{1}{N} \sum_{0}^{N-1} x[n] W_N^{nk}\end{aligned}}$$

$$W_N = e^{-j2\pi/N}$$