Signals & Systems

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Ch. 6 Time & Frequency Characterization of Signals and Systems

- Sec. 6.1 The Magnitude-phase Representation of the Fourier Transform
- Sec. 6.2 The Magnitude-phase Rep. of Frequency Response of LTI Systems
- Sec. 6.3 Time-domain Properties of Ideal Frequency-selective Filters
- Sec. 6.4 Time-domain and Frequency-domain Aspects of Nonideal Filters
- Sec. 6.5* Time and Frequency Characterization for Some Well-known Filters (*: not in the original 2nd edition)
- Sec. 6.6 First-order and Second-order Continuous-time Systems
- Sec. 6.7 First-order and Second-order Discrete-time Systems
- Sec. 6.8 Examples of Time- and Frequency-domain Analysis of Systems

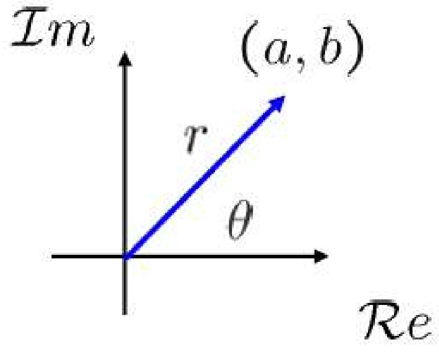
magnitude-phase representation

characters for filters

characters for systems

Magnitude-Phase Representation

$$a + jb \Rightarrow \begin{cases} r = \sqrt{a^2 + b^2} \\ \tan(\theta) = \dfrac{b}{a} \end{cases}$$



$$\Rightarrow a + jb = re^{j\theta}$$

$$X(j\omega) = \mathrm{Re}\{X(j\omega)\} + j\,\mathrm{Im}\{X(j\omega)\} = |X(j\omega)|e^{j\sphericalangle X(j\omega)}$$

$$X(e^{j\omega}) = \mathrm{Re}\{X(e^{j\omega})\} + j\,\mathrm{Im}\{X(e^{j\omega})\} = |X(e^{j\omega})|e^{j\sphericalangle X(e^{j\omega})}$$

$|X(j\omega)|,\ |X(e^{j\omega})|$: magnitude

$\sphericalangle X(j\omega),\ \sphericalangle X(e^{j\omega})$: phase angle

Magnitude-Phase Representation

The *magnitude-phase representation* of the continuous-time Fourier transform $X(j\omega)$ is

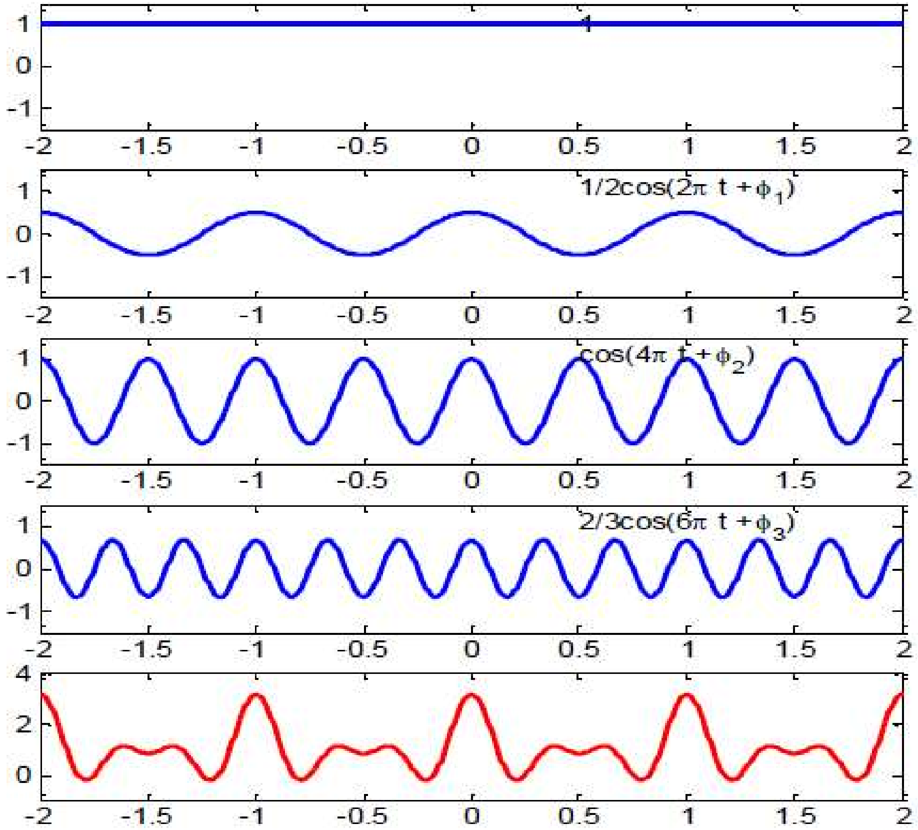
$$\boxed{X(j\omega) = |X(j\omega)|e^{j\sphericalangle X(j\omega)}.} \tag{6.1}$$

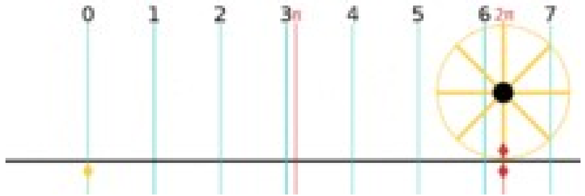
Similarly the magnitude-phase representation of the discrete-time Fourier transform $X(e^{j\omega})$ is

$$\boxed{X(e^{j\omega}) = |X(e^{j\omega})|e^{j\sphericalangle X(e^{j\omega})}.} \tag{6.2}$$

- Fig. 6.1 Impact of Phase on Signals

$$x(t) = 1 + \frac{1}{2}\cos(2\pi t + \phi_1) + \cos(4\pi t + \phi_2) + \frac{2}{3}\cos(6\pi t + \phi_3)$$

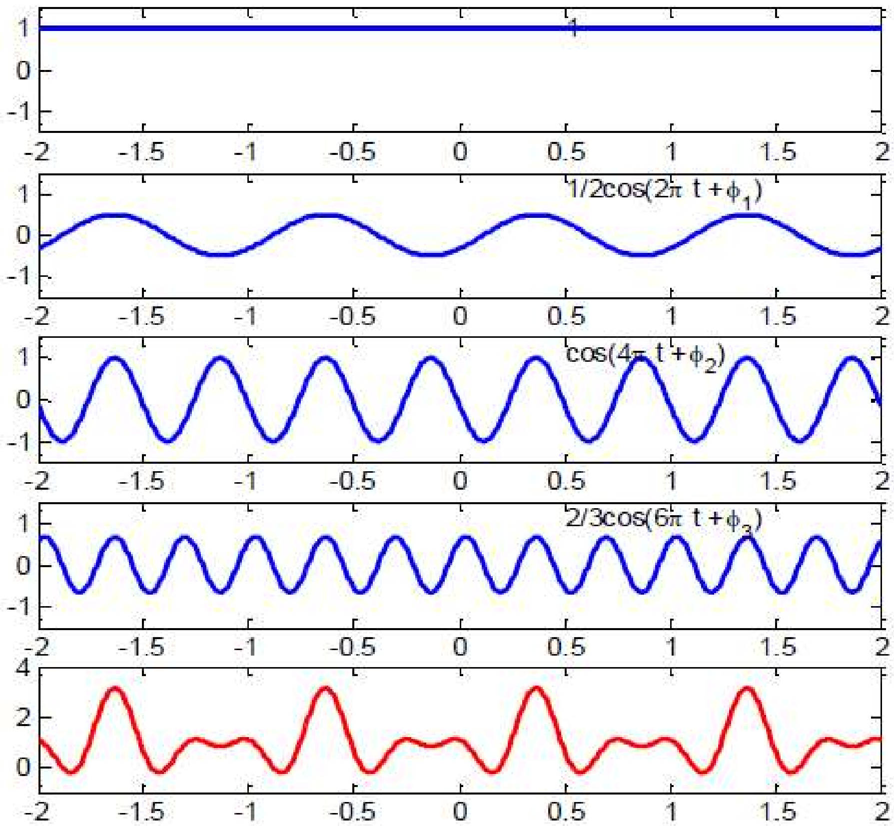


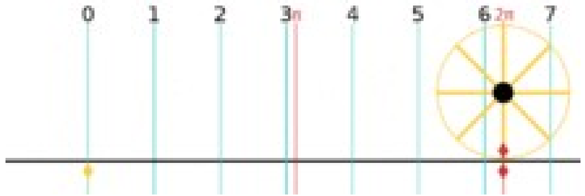


$$\begin{cases} \phi_1 = 0 \quad (rad) \\ \phi_2 = 0 \quad (rad) \\ \phi_3 = 0 \quad (rad) \end{cases}$$

- Fig. 6.1 Impact of Phase on Signals (cont'd)

$$x(t) = 1 + \frac{1}{2}\cos(2\pi t + \phi_1) + \cos(4\pi t + \phi_2) + \frac{2}{3}\cos(6\pi t + \phi_3)$$

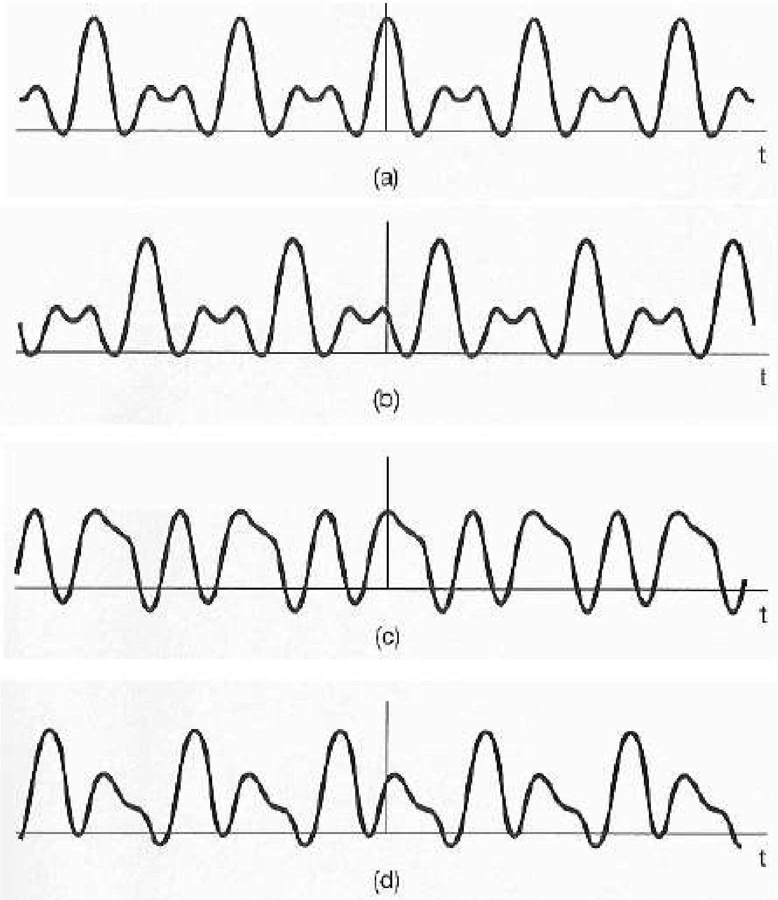




$$\begin{cases} \phi_1 = 4 \quad (rad) \\ \phi_2 = 8 \quad (rad) \\ \phi_3 = 12 \quad (rad) \end{cases}$$

• Fig. 6.1 Impact of Phase on Signals (cont'd)

$$x(t) = 1 + \frac{1}{2}\cos(2\pi t + \phi_1) + \cos(4\pi t + \phi_2) + \frac{2}{3}\cos(6\pi t + \phi_3)$$



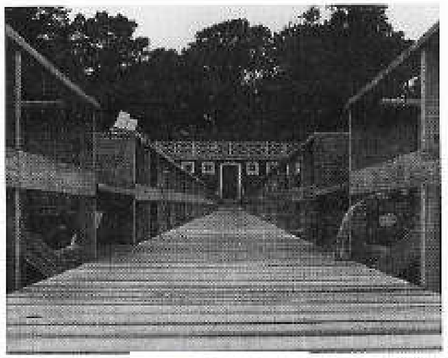
$$\begin{cases} \phi_1 = 0 \quad (rad) \\ \phi_2 = 0 \quad (rad) \\ \phi_3 = 0 \quad (rad) \end{cases}$$

$$\begin{cases} \phi_1 = 4 \quad (rad) \\ \phi_2 = 8 \quad (rad) \\ \phi_3 = 12 \quad (rad) \end{cases}$$

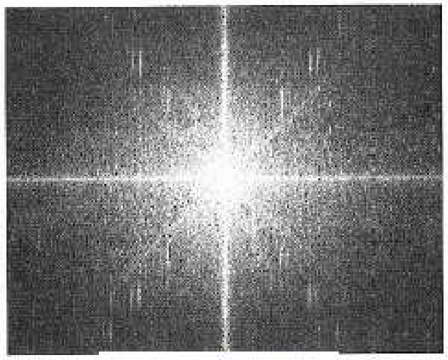
$$\begin{cases} \phi_1 = 6 \quad (rad) \\ \phi_2 = -2.7 \quad (rad) \\ \phi_3 = 0.93 \quad (rad) \end{cases}$$

$$\begin{cases} \phi_1 = 1.2 \quad (rad) \\ \phi_2 = 4.1 \quad (rad) \\ \phi_3 = -7.02 \quad (rad) \end{cases}$$

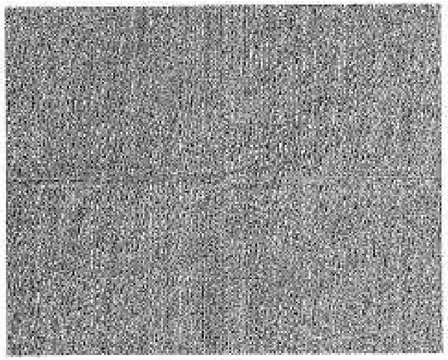
- Example 2 Impact of Phase on *Images*



image

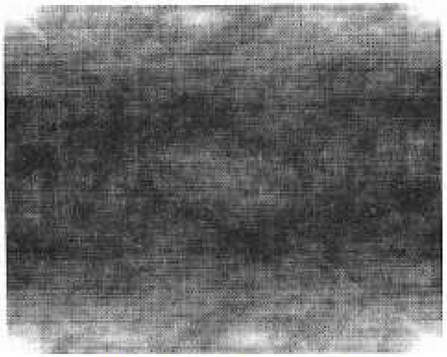


magnitude



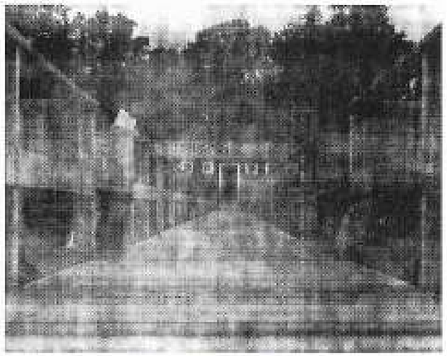
phase

new image



magnitude + zero phase

unit magnitude + phase



magnitude of a new image + phase

- (New) Example 6.1 All-Pass Filters

An *all-pass system* is the system whose frequency-response magnitude is

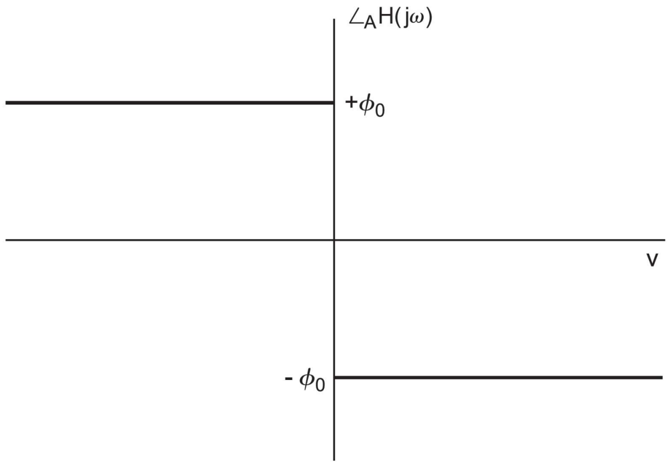
$$|H(j\omega)| = 1 \text{ for all } \omega.$$

- (New) Example 6.1 All-Pass Filters

An *all-pass system* is the system whose frequency-response magnitude is
$|X(j\omega)| = 1$ for all ω.

Suppose that there is an all-pass system whose frequency-response amplitude is
A(jω) = 1 and the unwrapped phase response is

$$\angle_A H(j\omega) = \begin{cases} -\phi_0 & \text{for } \omega > 0 \\ +\phi_0 & \text{for } \omega < 0 \end{cases}, \qquad (6.4)$$



as indicated in the right figure.

In this example, we consider *x(t)* to be of the form

$$x(t) = s(t)\cos(\omega_0 t), \qquad \omega_0 > 0, \qquad (6.5)$$

- (New) Example 6.1 All-Pass Filters

That is, an amplitude-modulated signal at a positive carrier frequency of ω_0. Consequently, X(jω) can be expressed as

$$X(j\omega) = \frac{1}{2} S(j\omega - j\omega_0) + \frac{1}{2} S(j\omega + j\omega_0) \qquad (6.6)$$

where S(jω) denotes the Fourier transform of s(t).

We also assume that S(jω) is bandlimited to $|\omega| < \Delta$, with Δ sufficiently small so that the term $S(j\omega - j\omega_0)$ is zero for $\omega < 0$ and the term $S(j\omega + j\omega_0)$ is zero for $\omega > 0$. That is, we assume that $(\omega_0 - \Delta) > 0$. Thus x(t) is characterized by a slowly varying modulation of its carrier.

With these assumptions on x(t), it is relatively straightforward to determine the output y(t). Specifically, the system frequency response H(jω) is

$$H(j\omega) = \begin{cases} e^{-j\phi_0} & \omega > 0 \\ e^{+j\phi_0} & \omega < 0. \end{cases} \qquad (6.7)$$

- (New) Example 6.1 All-Pass Filters

Since the term S(jω – jω_0) in eq. (6.7) is nonzero only for ω > 0, it is simply multiplied by $e^{-j\phi_0}$, and similarly the term S(jω + jω_0) is multiplied only by $e^{+j\phi_0}$. Consequently, the output Fourier transform Y(jω) is given by

$$\begin{aligned} Y(j\omega) &= X(j\omega)H(j\omega) \\ &= \frac{1}{2}S(j\omega - j\omega_0)e^{-j\phi_0} + \frac{1}{2}S(j\omega + j\omega_0)e^{+j\phi_0}, \end{aligned} \qquad (6.8)$$

which we recognize as a simple phase shift by φ_0 of the carrier in eq. (6.5). Consequently,

$$y(t) = s(t)\cos(\omega_0 t - \phi_0). \qquad (6.9)$$

This change in phase of the carrier can also be expressed in terms of a time delay for the carrier by rewriting eq. (6.9) as

$$\begin{aligned} y(t) &= s(t)\cos\left[\omega_0\left(t - \frac{\phi_0}{\omega_0}\right)\right] \\ &= s(t)\cos[\omega_0(t - \tau_p)] \end{aligned}$$

- (New) Example 6.1 All-Pass Filters

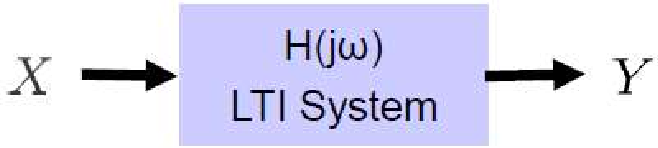
$$
\begin{aligned}
y(t) &= s(t)\cos\left[\omega_0\left(t - \frac{\phi_0}{\omega_0}\right)\right] \\
&= s(t)\cos[\omega_0(t - \tau_p)]
\end{aligned}
$$

where τ_p , the negative of the ratio of the phase at ω_0, *i.e.,* $(-\phi_0)$ to the frequency ω_0, is referred to as the phase delay of the system at frequency ω_0:

$$
\tau_p = -\frac{\angle H(j\omega_0)}{\omega_0} = \frac{\phi_0}{\omega_0}.
$$

6.2 Magnitude-Phase Representation of Frequency Response of LTI Systems

- Effect of an LTI system on an input



$$Y(j\omega) = X(j\omega)H(j\omega)$$

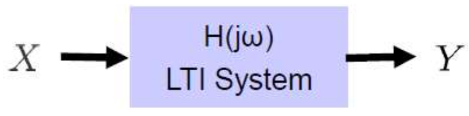
$\Rightarrow$ The system changes the complex amplitude of each frequency component of $x(t)$.

$$\begin{aligned}|Y(j\omega)|e^{j\sphericalangle Y(j\omega)} &= |X(j\omega)|e^{j\sphericalangle X(j\omega)}|H(j\omega)|e^{j\sphericalangle H(j\omega)} \\ &= |X(j\omega)||H(j\omega)|e^{j(\sphericalangle X(j\omega)+\sphericalangle H(j\omega))}\end{aligned}$$

$$\Rightarrow \begin{cases} |H(j\omega)|: & \text{gain of the LTI system} \\ \sphericalangle H(j\omega): & \text{phase shift of the LTI system} \end{cases}$$

If the input is changed in an unwanted manner, the effects are referred to as magnitude and phase *distortions*.

Magnitude-Phase Representation of Frequency Response of LTI Systems



- Linear vs. Non-linear Phases

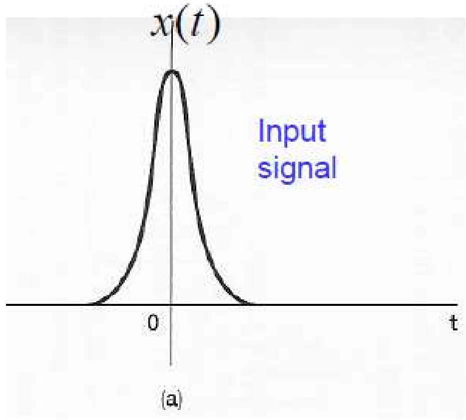
When the phase shift is a linear function of ω, we call it a linear phase shift. For example, consider

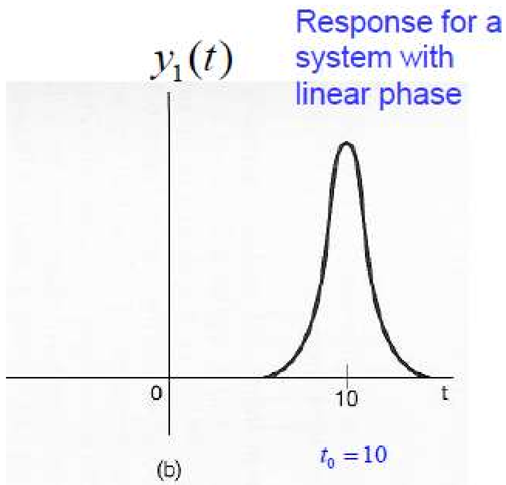
$$H_1(j\omega) = e^{-j\omega t_0}.$$

The system has unit gain and linear phase

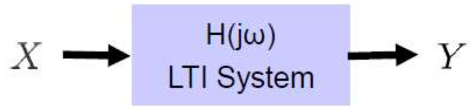
$$|H_1(j\omega)| = 1, \ \measuredangle H_1(j\omega) = -\omega t_0.$$

In the time domain, the system introduces a constant time shift to the signal.



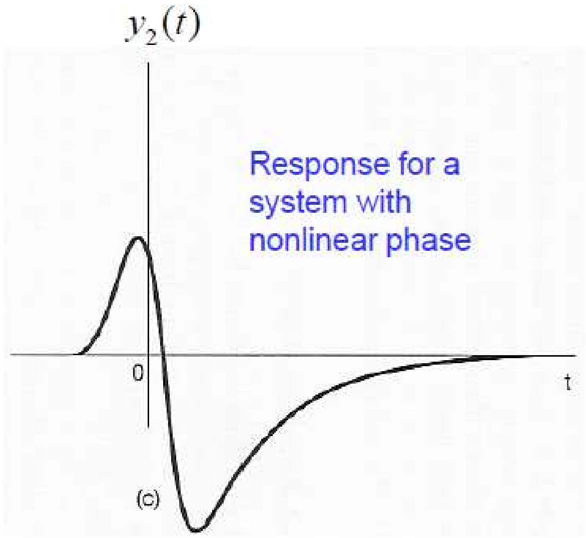


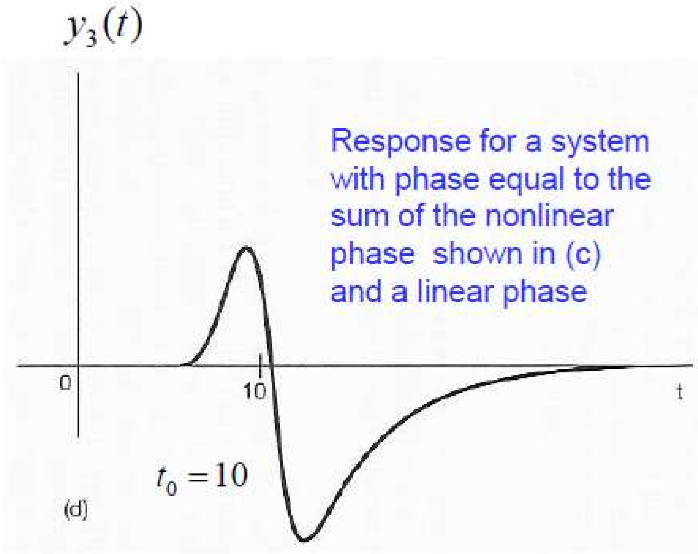
Magnitude-Phase Representation of Frequency Response of LTI Systems



- Linear vs. Non-linear Phases (cont'd)

$$H_2(j\omega) = e^{j\sphericalangle H_2(j\omega)} \qquad\qquad H_3(j\omega) = H_2(j\omega)H_1(j\omega)$$



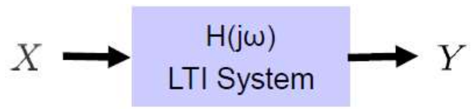


Note:

1) Both systems have unit gain (that is, $|H_2(j\omega)| = |H_1(j\omega)| = 1$) and hence are all-pass filters.

2) The slope of the phase of each filter tells us the size of the time shift (delay).

Magnitude-Phase Representation of Frequency Response of LTI Systems



- Narrowband vs. Broadband Signals

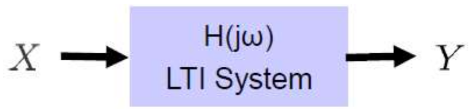
Suppose that $X(j\omega)$ is the Fourier transform of $x(t)$. If $X(j\omega)$ is zero or negligibly small outside a very small band of frequencies centered at $\omega = \pm\omega_0$, then we call $x(t)$ a *narrowband signal*. Otherwise, we call $x(t)$ a *broadband signal*.

By taking the band to be very small, we can accurately approximate the phase of this system in the band with the linear approximation.

$$\sphericalangle H(j\omega) \simeq -\phi - \omega\alpha, \tag{6.20}$$

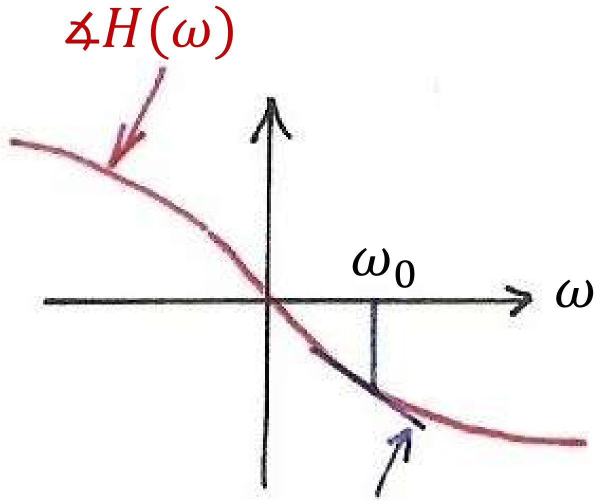
$$Y(j\omega) \simeq X(j\omega)|H(j\omega)|e^{-j\phi}e^{-j\omega\alpha}. \tag{6.21}$$

Magnitude-Phase Representation of Frequency Response of LTI Systems



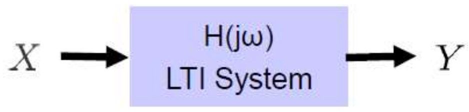
- Group Delay
  - The phase slope tells us the size of the time shift.
  - Such a time shift can be considered as a time delay.
  - E.g., $x(t-t_0) \xleftrightarrow{F} e^{-j\omega t_0} X(j\omega)$

  - The group delay at ω is calculated as the negative slope of the phase at that frequency, i.e., $\tau(\omega) = -\frac{d}{d\omega}[\angle H(j\omega)]$.



$$\frac{d}{d\omega}[\measuredangle H(\omega)]_{\omega=\omega_0}$$

Magnitude-Phase Representation of Frequency Response of LTI Systems



- Effects of LTI Systems on Narrowband Input Signals

- The concept of delay can be extended to include nonlinear phases.
- Consider a narrowband signal x(t) whose Fourier transform is zero or negligibly small outside a small band of frequencies centered at $\omega = \omega_0$.
- We can approximate the phase of the system in the band with a linear approximation centered at $\omega = \omega_0$:

$$\sphericalangle H(j\omega) \simeq -\phi - \omega\alpha$$

where ϕ is a constant, so that

$$Y(j\omega) = X(j\omega)\left|H(j\omega)\right|e^{-j\phi}e^{-j\omega\alpha}$$

This time delay of α seconds is referred to as the group delay at $\omega = \omega_0$.

- Group Delay

$$\boxed{\tau(\omega) = -\frac{d}{d\omega}\{\sphericalangle H(j\omega)\}}$$

- Example 6.1

$$X \longrightarrow \boxed{\text{H1}} \longrightarrow \boxed{\text{H2}} \longrightarrow \boxed{\text{H3}} \longrightarrow Y$$

An all-pass system with non-constant group delay that introduces the so-called dispersion effect where different frequencies in the input are delayed by different amounts

$$H(j\omega) = H_1(j\omega)H_2(j\omega)H_3(j\omega)$$

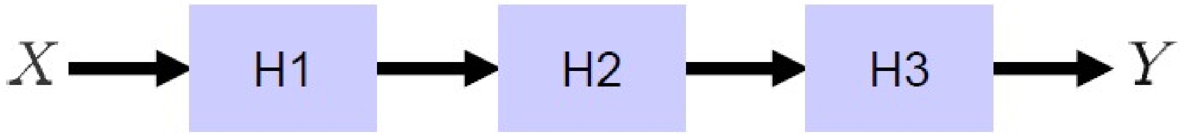
$$H_i(j\omega) = \frac{1 + (j\omega/\omega_i)^2 - 2j\zeta_i(\omega/\omega_i)}{1 + (j\omega/\omega_i)^2 + 2j\zeta_i(\omega/\omega_i)}$$

$$\begin{cases} |H_i(j\omega)| = 1 \\ \measuredangle H_i(j\omega) = -2\arctan\left[\dfrac{2\zeta_i(\omega/\omega_i)}{1-(\omega/\omega_i)^2}\right] \end{cases}$$

$$\Rightarrow \begin{cases} |H(j\omega)| = 1 \\ \measuredangle H(j\omega) = \measuredangle H_1(j\omega) + \measuredangle H_2(j\omega) + \measuredangle H_3(j\omega) \end{cases}$$

$$\tau(\omega) = -\frac{d}{d\omega}\{\measuredangle H(j\omega)\} \qquad \text{Group delay}$$

- Example 6.1 (cont'd)



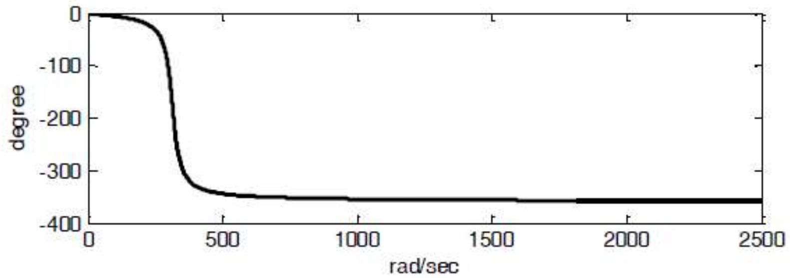
An all-pass system with non-constant group delay that introduces the so-called dispersion effect where different frequencies in the input are delayed by different amounts

$$H_1(j\omega) = \frac{1+(j\omega/\omega_1)^2 - 2j\zeta_1(\omega/\omega_1)}{1+(j\omega/\omega_1)^2 + 2j\zeta_1(\omega/\omega_1)}$$

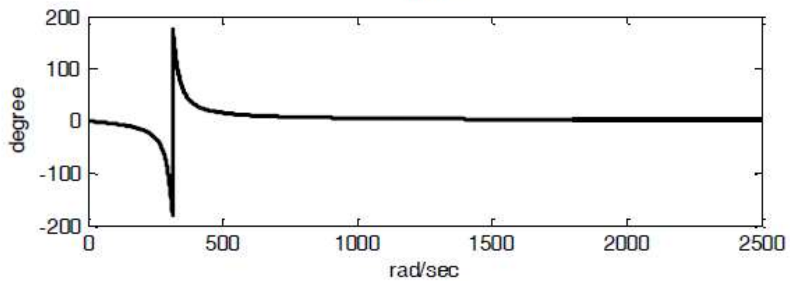
$$\Rightarrow \begin{cases} |H_1(j\omega)| = 1 \\ \sphericalangle H_1(j\omega) = -2\arctan\left[\dfrac{2\zeta_i(\omega/\omega_i)}{1-(\omega/\omega_i)^2}\right] \end{cases}$$

$$\begin{cases} \omega_1 = 315 \text{ rad/sec} \\ \zeta_1 = 0.066 \\ f_1 \approx 50 \text{ Hz} \end{cases}$$

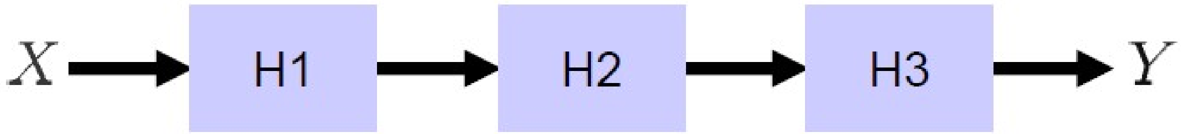
Unwrapped phase



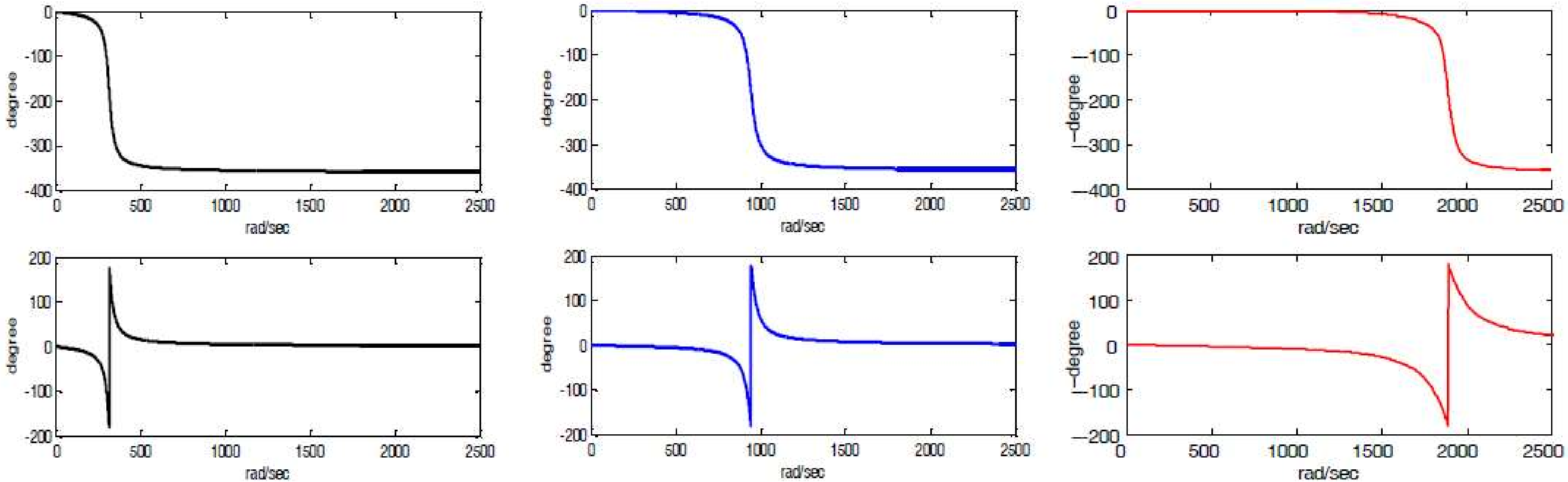
Principal phase



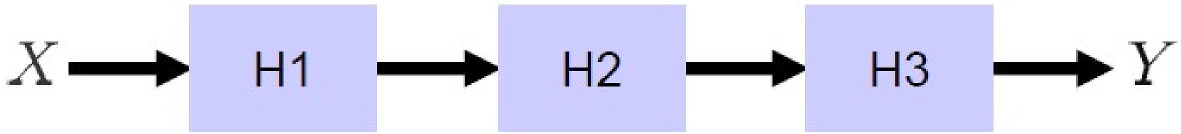
- Example 6.1 (cont'd)



$$\begin{cases} \omega_1 = 315 \quad \text{rad/sec} \\ \omega_2 = 943 \quad \text{rad/sec} \\ \omega_3 = 1888 \quad \text{rad/sec} \end{cases} \qquad \begin{cases} \zeta_1 = 0.066 \\ \zeta_2 = 0.033 \\ \zeta_3 = 0.058 \end{cases}$$

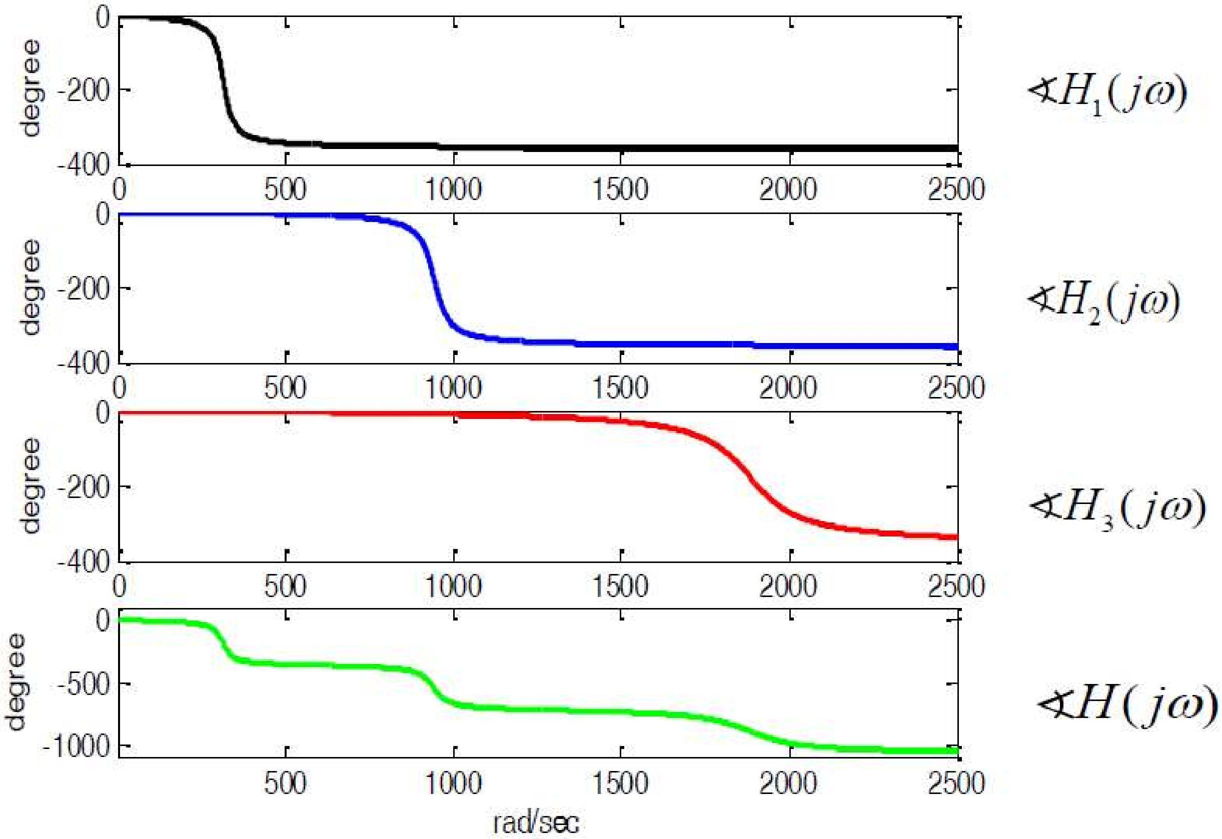


- Example 6.1 (cont'd)



$$\begin{cases} |H(j\omega)| = 1 \\ \measuredangle H(j\omega) = \measuredangle H_1(j\omega) + \measuredangle H_2(j\omega) + \measuredangle H_3(j\omega) \end{cases}$$

Unwrapped phase

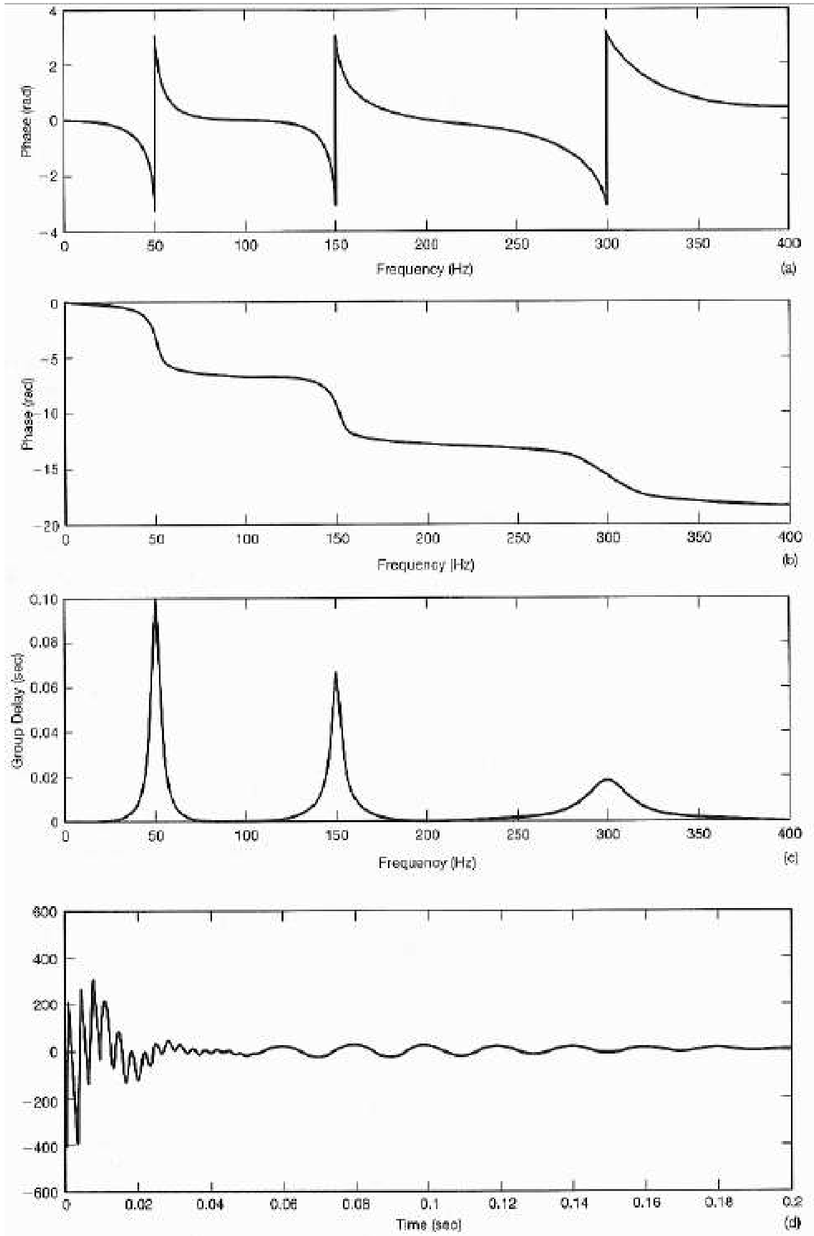


- Example 6.1 (cont'd)

Group delay:

$$\tau(\omega) = -\frac{d}{d\omega}\{\measuredangle H(j\omega)\}$$

Dispersion:
The phenomenon that different frequencies in the input are delayed by different amounts.



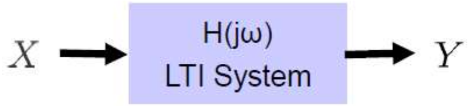
Principal phase

Unwrapped phase

Group delay

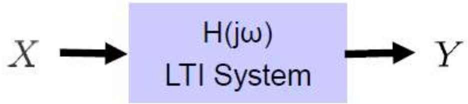
Impulse response

6.2.3 Log-Magnitude & Bode Plots

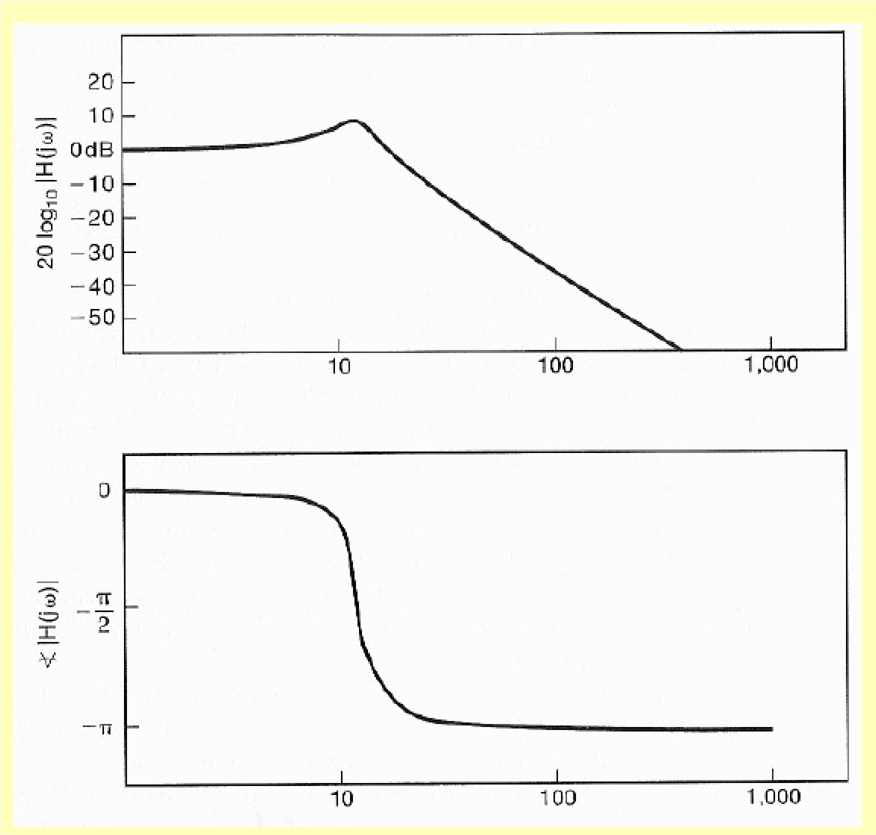


- Decibel: $20\log_{10}$
  - 0dB: a frequency response with magnitude equal to 1
  - 20dB: equivalent to a gain of 10
  - -20dB: corresponds to an attenuation of 0.1
- Bode Plots:
  - plots of $20\log_{10}|H(j\omega)|$ & $\sphericalangle H(j\omega)$ vs. $\log_{10}\omega$

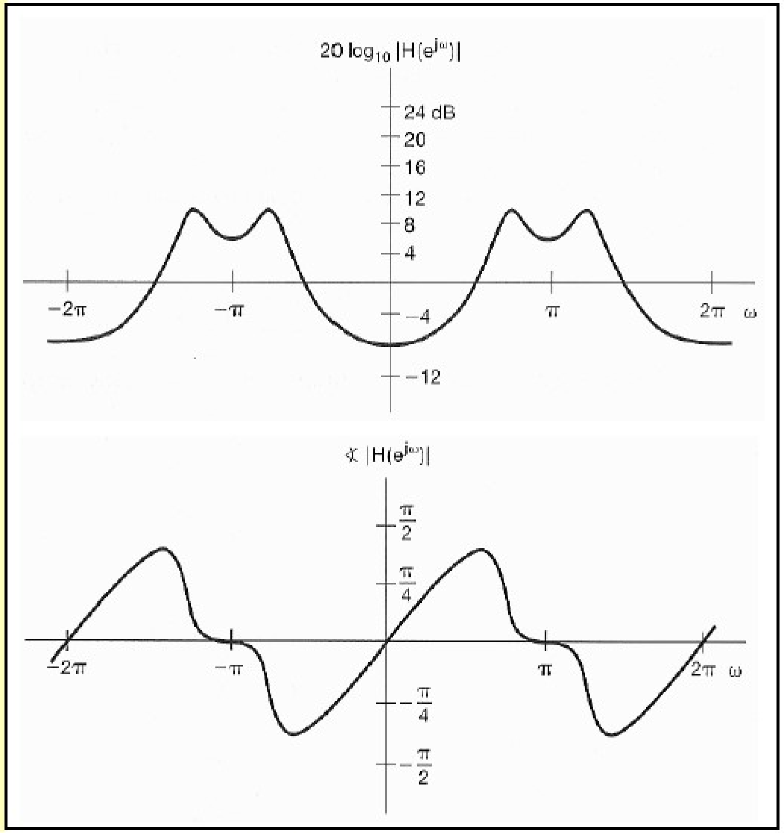
6.2.3 Log-Magnitude & Bode Plots



Continuous-Time Bode plot



Discrete-Time Bode plot



$\log(w), \quad w: 0 \leftrightarrow \infty$

$(w), \quad w: -\pi \leftrightarrow \pi$

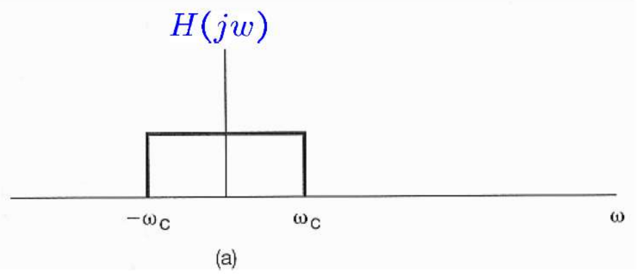
Time-Domain Properties of Ideal Frequency-Selective Filters

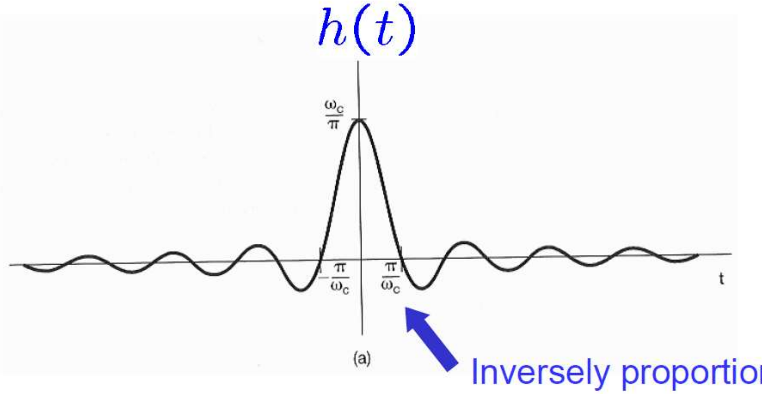
- Ideal LPF

$$H(j\omega) = \begin{cases} 1, & |\omega| \le \omega_c \\ 0, & |\omega| > \omega_c \end{cases}$$

$$\Rightarrow \quad h(t) = \frac{\sin \omega_c t}{\pi t}$$

unit gain, zero phase





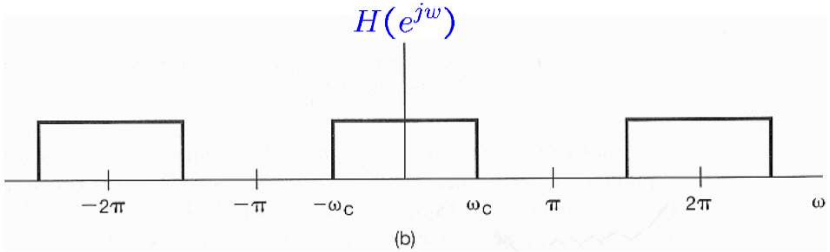
Inversely proportional to ω_c

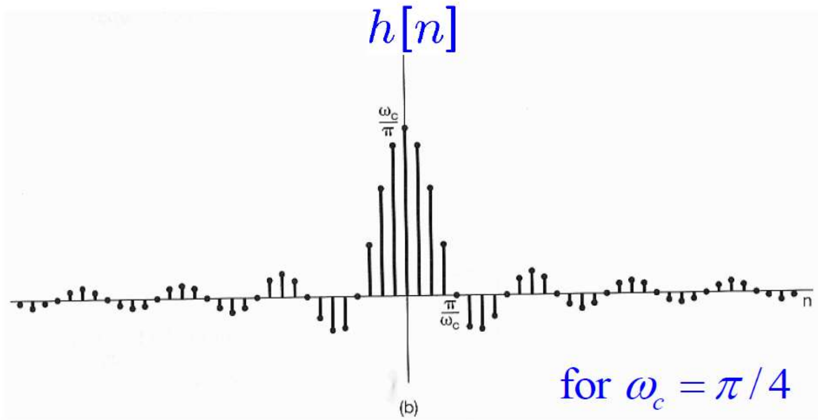
Time-Domain Properties of Ideal Frequency-Selective Filters

- Ideal LPF

$$H(e^{j\omega}) = \begin{cases} 1, & |\omega| \le \omega_c \\ 0, & \omega_c \le |\omega| \le \pi \end{cases}$$

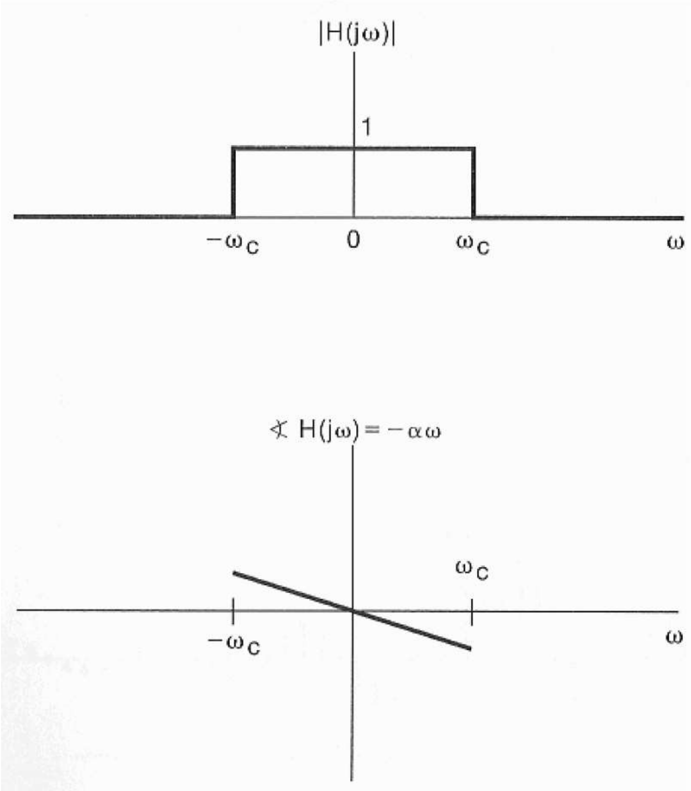
$$\Rightarrow \quad h[n] = \frac{\sin \omega_c n}{\pi n}$$

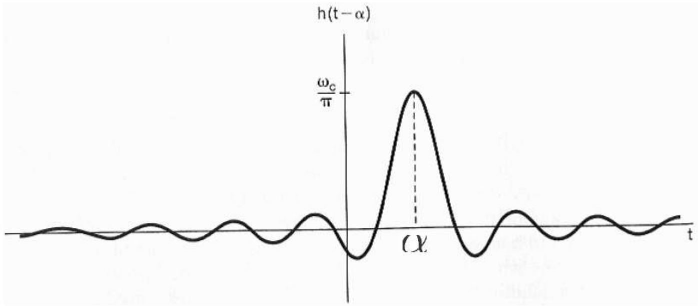




Time-Domain Properties of Ideal Frequency-Selective Filters

- Ideal LPF wit Linear Phase

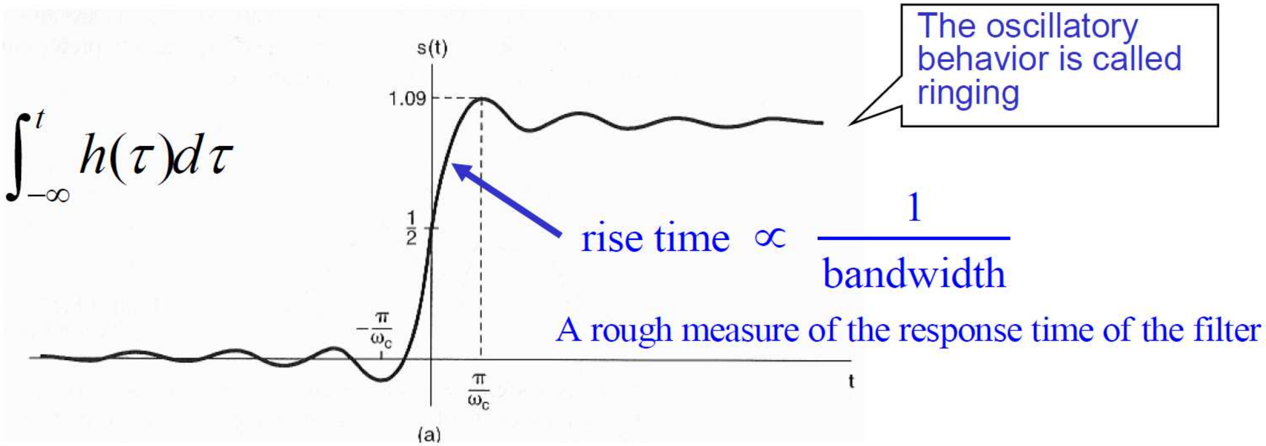




Time-Domain Properties of Ideal Frequency-Selective Filters

- Step Response of Ideal LPF

$$s(t)=\int_{-\infty}^{t} h(\tau)d\tau$$



$$s[n]=\sum_{m=-\infty}^{n} h[m]$$