Signals & Systems

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Ch. 9 Laplace Transform

definitions, calculation

properties

system analysis

unilateral form

9.1 The Laplace Transform

$x(t)$
$\|$

Recall that the response of a linear time-invariant system with impulse response $h(t)$ to a complex exponential input of the form e^{st} is

$$y(t) = H(s)e^{st},$$

$y(t) = x(t) * h(t)$

$= \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau$

$e^{s(t-\tau)}$

where

$$H(s) = \int_{-\infty}^{\infty} h(t)e^{-st}\,dt. \quad \text{(i.e., the system function of the system)}$$

For a general signal x(t), a transform like the one above is refereed to as the (bilateral) Laplace transform:

$$X(s) \triangleq \int_{-\infty}^{\infty} x(t)e^{-st}dt$$

where s = σ + jω is a complex variable.
Thus, Laplace transform can be viewed as an extension of CTFT.

9.1 The Laplace Transform

$$X(\sigma + j\omega) = \int_{-\infty}^{+\infty} x(t)e^{-(\sigma + j\omega)t}\, dt,$$

Fourier Transform vs. Laplace Transform

$$X(\sigma + j\omega) = \int_{-\infty}^{+\infty} [x(t)e^{-\sigma t}]e^{-j\omega t}\, dt.$$

Fourier transform	Laplace transform
$s = j\omega$	$s = \sigma + j\omega$
$X(j\omega) \triangleq \int_{-\infty}^{\infty} x(t)e^{-j\omega t} dt$	$X(s) \triangleq \int_{-\infty}^{\infty} x(t)e^{-st} dt$
$x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(j\omega)$	$x(t) \overset{\mathcal{L}}{\longleftrightarrow} X(s)$
$X(j\omega) = \mathcal{F}\{x(t)\}$	$X(s) = \mathcal{L}\{x(t)\}$
$x(t) = \mathcal{F}^{-1}\{X(j\omega)\}$	$x(t) = \mathcal{L}^{-1}\{X(s)\}$

FT is the LT evaluated along the $j\omega$ axis:

$$X(s)\Big|_{s=j\omega} = \mathcal{L}\{x(t)\}\Big|_{s=j\omega} = \mathcal{F}\{x(t)\} = X(j\omega)$$

$\sigma = 0$

$$X(\sigma + j\omega) = \int_{-\infty}^{+\infty} x(t)e^{-(\sigma+j\omega)t}\,dt,$$

9.1 The Laplace Transform

$$X(\sigma + j\omega) = \int_{-\infty}^{+\infty} [x(t)e^{-\sigma t}]e^{-j\omega t}\,dt.$$

Laplace Transform from Fourier Transform

$$s = \sigma + j\omega$$

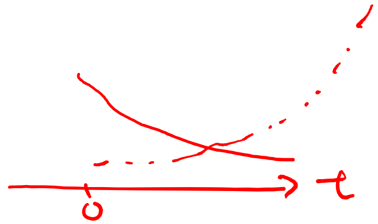
$$\begin{aligned}\mathcal{L}\{x(t)\} &= X(s)\\ &= X(\sigma + j\omega)\\ &= \underline{\int_{-\infty}^{\infty} x(t)e^{-(\sigma+j\omega)t}dt}\\ &= \int_{-\infty}^{\infty} [x(t)e^{-\sigma t}]e^{-j\omega t}dt\\ &= \mathcal{F}\{x(t)e^{-\sigma t}\}\end{aligned}$$

Laplace xform

The Laplace transform of a signal x(t)
is the Fourier transform of the signal $x(t)e^{-\sigma t}$.

Example 9.1

$$x(t) = e^{-at}u(t)$$



$$X(j\omega) = \int_{-\infty}^{\infty} e^{-at}u(t)e^{-j\omega t}dt = \int_{0}^{\infty} e^{-at}e^{-j\omega t}dt = \frac{1}{a + j\omega}$$

Converge if a>0

$$X(s) = \int_{-\infty}^{\infty} e^{-at}u(t)e^{-st}dt = \int_{0}^{\infty} e^{-(a+s)t}dt = -\frac{1}{s+a}e^{-(s+a)t}\Big|_{0}^{\infty} = \frac{1}{s+a}$$

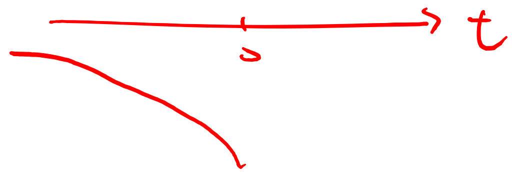
$\boxed{\mathrm{Re}\{s+a\} > 0}$

$$X(\underbrace{\sigma + j\omega}_{= s}) = \int_{0}^{\infty} e^{-(\sigma+a)t}e^{-j\omega t}dt = \frac{1}{(\sigma + a) + j\omega}$$

$\boxed{\sigma + a > 0}$

Example 9.2

$$x(t) = -e^{-at}u(-t)$$



$$X(s) = \int_{-\infty}^{\infty} -e^{-at}u(-t)e^{-st}dt$$

$\mathrm{Re}\{s+a\} < 0$

$$= -\int_{-\infty}^{0} e^{-at}e^{-st}dt$$

$\boxed{\mathrm{Re}\{s\} < -a}$

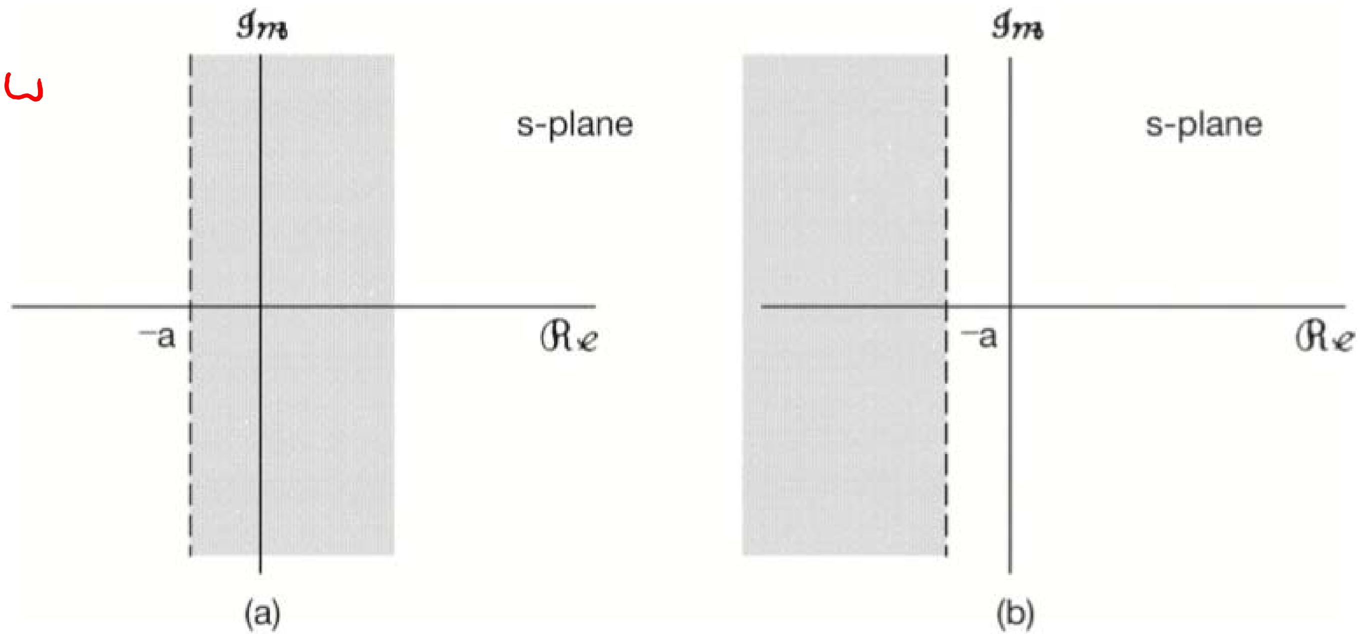
$$= \frac{1}{s+a}$$

Region of Convergence (ROC)

$$e^{-at}u(t) \stackrel{\mathcal{L}}{\longleftrightarrow} \frac{1}{s+a}, \quad \boxed{\mathrm{Re}\{s\} > -a}$$

$$-e^{-at}u(-t) \stackrel{\mathcal{L}}{\longleftrightarrow} \frac{1}{s+a}, \quad \boxed{\mathrm{Re}\{s\} < -a}$$

$s = \sigma + j\omega$



Causal and anti-causal exponential time functions can have the same Laplace transform but different ROCs. Thus in sepecifying the Laplace transform, both the algebraic expression and the ROC are required.

Example 9.3

$$x(t) = 3e^{-2t}u(t) - 2e^{-t}u(t)$$

$$X(s) = \int_{-\infty}^{\infty} [3e^{-2t}u(t) - 2e^{-t}u(t)]e^{-st}\,dt$$

$$= 3\int_{-\infty}^{\infty} e^{-2t}u(t)e^{-st}\,dt - 2\int_{-\infty}^{\infty} e^{-t}u(t)e^{-st}\,dt$$

$$= 3\left(\frac{1}{s+2}\right) - 2\left(\frac{1}{s+1}\right) \quad \#$$

$$e^{-2t}u(t) \xleftrightarrow{\mathcal{L}} \frac{1}{s+2} \qquad \mathrm{Re}\{s\} > -2$$

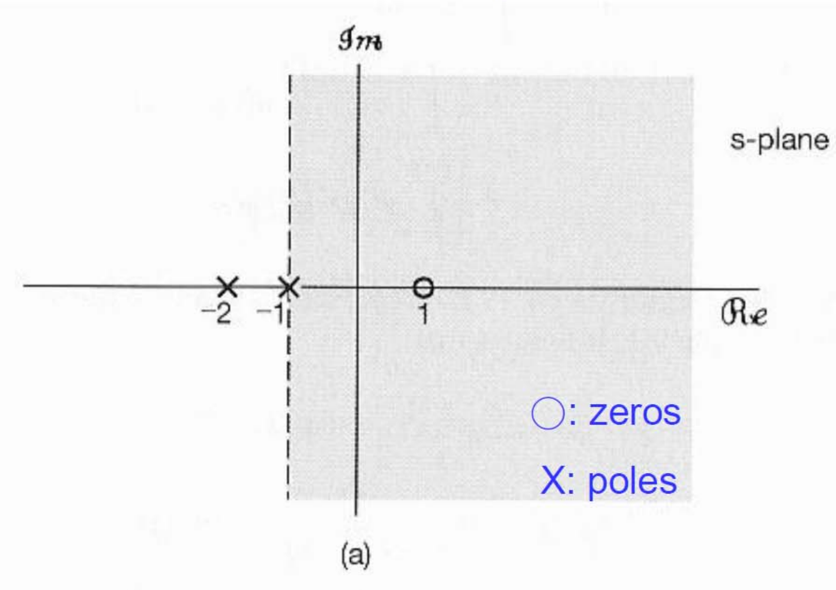
$$e^{-t}u(t) \xleftrightarrow{\mathcal{L}} \frac{1}{s+1} \qquad \mathrm{Re}\{s\} > -1$$

$$3e^{-2t}u(t) - 2e^{-t}u(t) \xleftrightarrow{\mathcal{L}} \frac{3}{s+2} - \frac{2}{s+1} \qquad \mathrm{Re}\{s\} > -1$$

Example 9.3 (cont'd)

$$3e^{-2t}u(t) - 2e^{-t}u(t) \xleftrightarrow{\mathcal{L}} \frac{(s-1)}{(s+2)(s+1)} \qquad \boxed{\mathrm{Re}\{s\} > -1}$$

Pole-zero plot

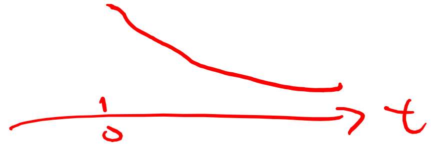


$\sigma = 0,\ s = j\omega$

The $j\omega$-axis is included in the ROC, so Fourier transform of this function exists.

Note: Pole-zero plot and ROC together can specify the Laplace transform of a signal (up to a scale factor).

Example 9.4



$$x(t) = e^{-2t}u(t) + e^{-t}(\cos(3t))u(t)$$

$$= [e^{-2t} + \frac{1}{2}e^{-t}(e^{j3t} + e^{-j3t})]u(t)$$

Re{s+a} > 0

$$e^{-2t}u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s+2} \qquad \mathrm{Re}\{s\} > -2$$

$$e^{-(1-3j)t}u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s+(1-3j)} \qquad \mathrm{Re}\{s\} > -1$$

$$e^{-(1+3j)t}u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s+(1+3j)} \qquad \mathrm{Re}\{s\} > -1$$

$$X(s) = \frac{1}{s+2} + \frac{1}{2}[\frac{1}{s+(1-3j)} + \frac{1}{s+(1+3j)}] = \frac{2s^2+5s+12}{(s^2+2s+10)(s+2)}$$

- Rational Expressions of LR with Poles/Zeros

$$X(s)=\frac{N(s)}{D(s)}\begin{array}{l}\longrightarrow \text{ roots } \longrightarrow \text{ zeros}\\ \longrightarrow \text{ roots } \longrightarrow \text{ poles}\end{array}$$

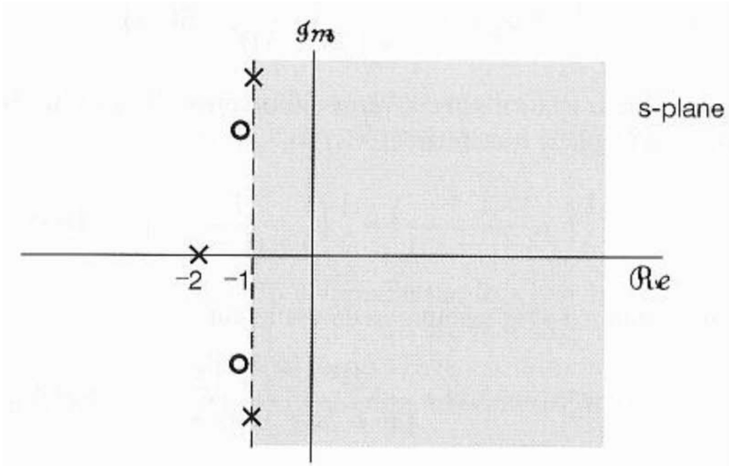
  – Pole-Zero Plots

  – specifying $X(s)$ except for a scale factor

Example 9.4 (cont'd)

$$e^{-2t}u(t)+e^{-t}(\cos(3t))u(t)\xleftrightarrow{\mathcal{L}}\frac{2s^2+5s+12}{(s^2+2s+10)(s+2)} \qquad \mathrm{Re}\{s\}>-1$$

$$\xleftrightarrow{\mathcal{L}}\frac{2(s+1.25-2.11j)(s+1.25+2.11j)}{(s+1-3j)(s+1+3j)(s+2)}$$



The ROC includes the $j\omega$-axis
$\Rightarrow$ Fourier transform exists

Example 9.5

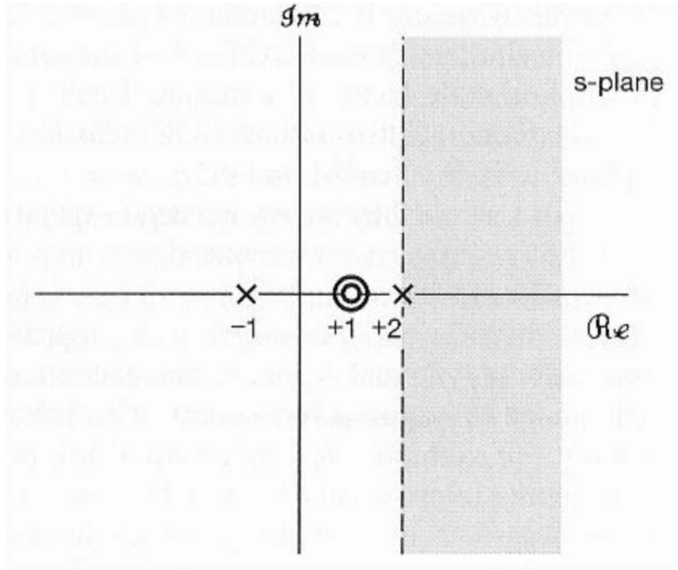
$$x(t)=\delta(t)-\frac{4}{3}e^{-t}u(t)+\frac{1}{3}e^{2t}u(t)$$

$$L\{\delta(t)\}=\int_{-\infty}^{\infty}\delta(t)e^{-st}dt=1,\ \text{entire s-plane}$$

$$X(s)=\boxed{1-\frac{4}{3}\frac{1}{s+1}+\frac{1}{3}\frac{1}{s-2},}\quad \mathrm{Re}\{s\}>2$$

$$\delta(t)-\frac{4}{3}e^{-t}u(t)+\frac{1}{3}e^{2t}u(t)$$

$$\xleftrightarrow{\mathcal{L}}\frac{(s-1)^2}{(s+1)(s-2)},\quad \mathrm{Re}\{s\}>2$$

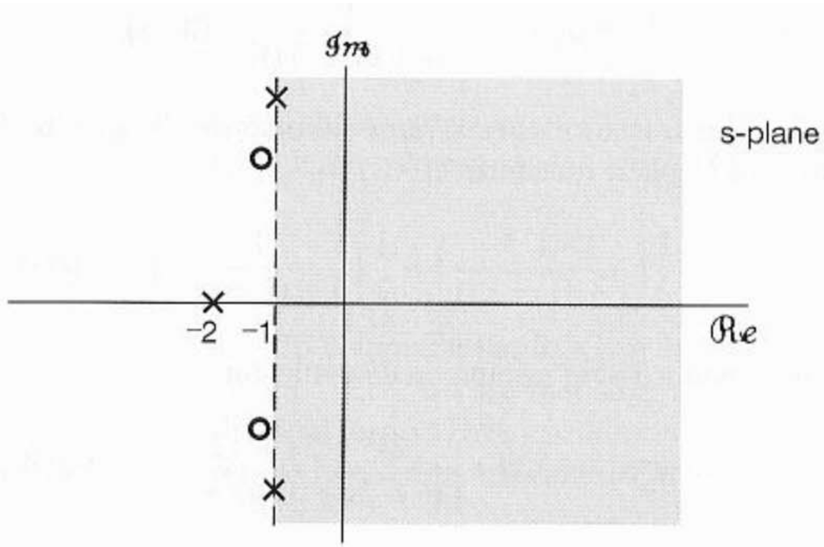


ROC not including the $j\omega$-axis
$\Rightarrow$ FT not existing

- Note that the ROC is the set of values of *s* for which the Laplace transforms of all three terms in *x*(*t*) converge. The pole-zero plot for this example is shown above, together with the ROC.
- Also, since the degrees of the numerator and denominator of *X*(*s*) are equal, *X*(*s*) has neither poles nor zeros at infinity.

9.2 The ROC for Laplace Transform

- Properties of ROC
  1. The ROC of X(s) consists of strips parallel to the jω axis in the s-plane.
  2. For rational Laplace transforms, the ROC does not contain any poles.

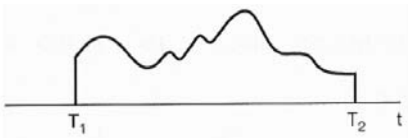


$$\frac{2(s+1.25-2.11j)(s+1.25+2.11j)}{(s+1-3j)(s+1+3j)(s+2)}$$

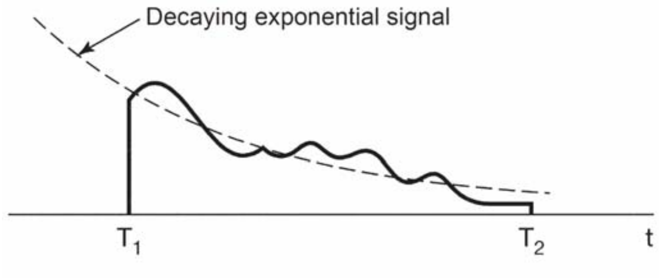
9.2 The ROC for Laplace Transform

- Properties of ROC

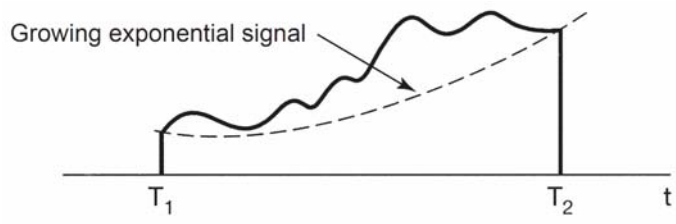
  3. If x(t) is of finite duration and is absolutely integrable, its ROC would be the entire s-plane.



and $\int_{T_1}^{T_2} |x(t)|\,dt < \infty$



(a)



(b)

For $s = \sigma + j\omega$ to be in the ROC,

$x(t)e^{-\sigma t}$ has to be absolutely integrable.

$$\sigma = 0 \;\Rightarrow \int_{T_1}^{T_2} |x(t)| e^{-\sigma t} dt = \int_{T_1}^{T_2} |x(t)|\,dt < \infty$$

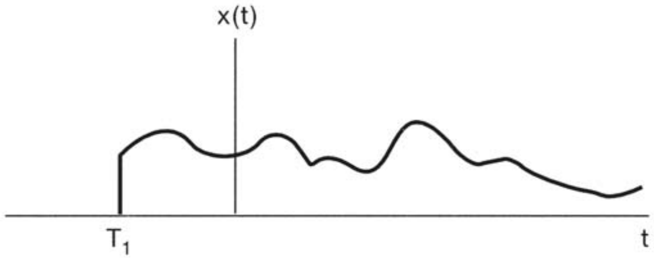
$$\sigma > 0 \;\Rightarrow \int_{T_1}^{T_2} |x(t)| e^{-\sigma t} dt < e^{-\sigma T_1} \int_{T_1}^{T_2} |x(t)|\,dt < \infty$$

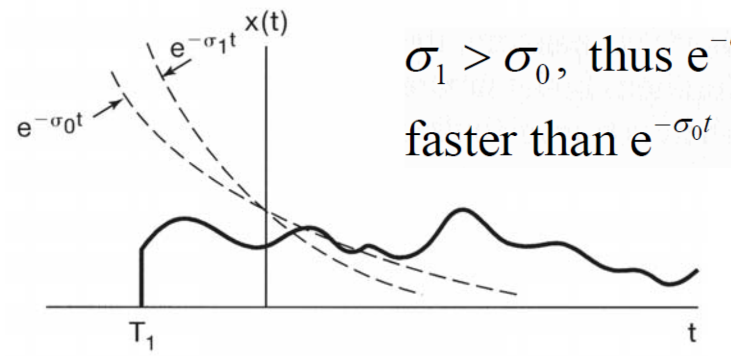
$$\sigma < 0 \;\Rightarrow \int_{T_1}^{T_2} |x(t)| e^{-\sigma t} dt < e^{-\sigma T_2} \int_{T_1}^{T_2} |x(t)|\,dt < \infty$$

9.2 The ROC for Laplace Transform

- Properties of ROC

  4. If x(t) is right-sided, and if the line Re{s} = σ_0 is in the ROC, then all the values of s for which Re{s} > σ_0 will also be in the ROC.

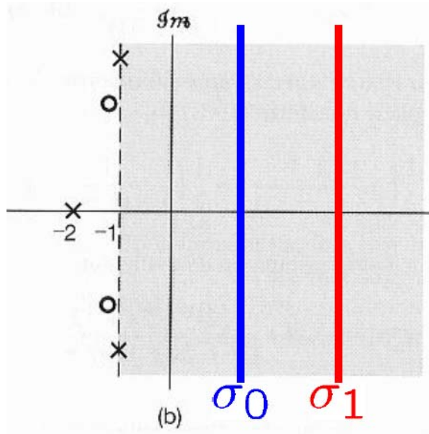




$\sigma_1 > \sigma_0$, thus $e^{-\sigma_1 t}$ decays faster than $e^{-\sigma_0 t}$ as $t \to \infty$.

$$\int_{T_1}^{+\infty} |x(t)| e^{-\sigma_0 t} dt < \infty$$

$$\Rightarrow \int_{T_1}^{+\infty} |x(t)| e^{-\sigma_1 t} dt \le e^{-(\sigma_1 - \sigma_0)T_1} \int_{T_1}^{+\infty} |x(t)| e^{-\sigma_0 t} dt < \infty$$

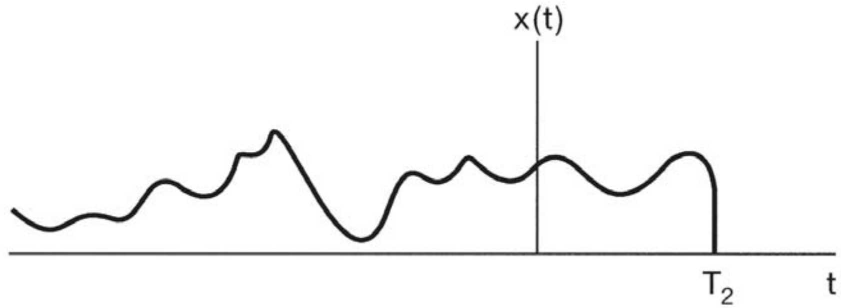


9.2 The ROC for Laplace Transform

- Properties of ROC

  5. If $x(t)$ is left-sided, and if the line $Re\{s\} = \sigma_0$ is in the ROC, then all the values of s for which $Re\{s\} < \sigma_0$ will also be in the ROC.

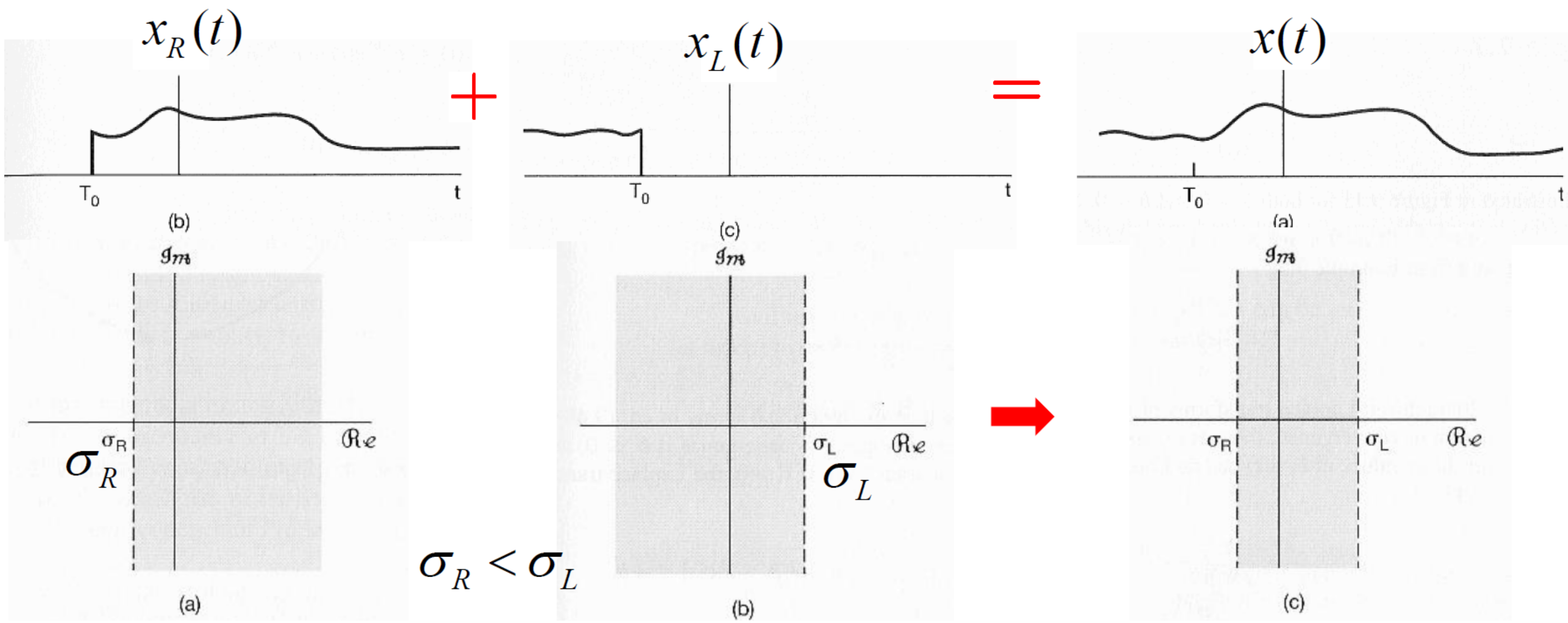
  The proof is similar to that of Property #4.



9.2 The ROC for Laplace Transform

- Properties of ROC

  6. If x(t) is two-sided, and if the line Re{s} = σ_0 is in the ROC, then the ROC will consist of a strip in the s-plane that includes the line Re{s} = σ_0.



Example 9.7

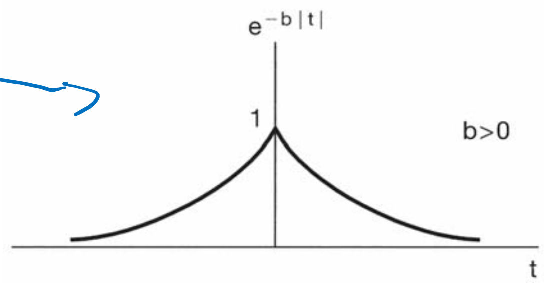
b<0

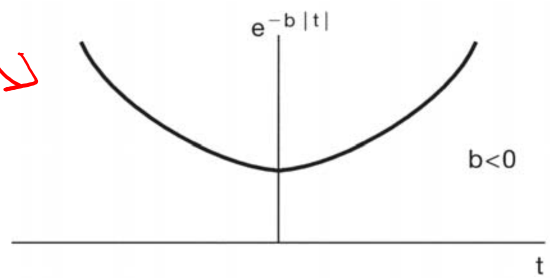
b>0

$$x(t) = e^{-b|t|} = e^{-bt}u(t) + e^{+bt}u(-t)$$

$$e^{-bt}u(t) \stackrel{\mathcal{L}}{\longleftrightarrow} \frac{1}{s+b}, \qquad \mathrm{Re}\{s\} > -b$$

$$e^{+bt}u(-t) \stackrel{\mathcal{L}}{\longleftrightarrow} \frac{-1}{s-b}, \qquad \mathrm{Re}\{s\} < +b$$





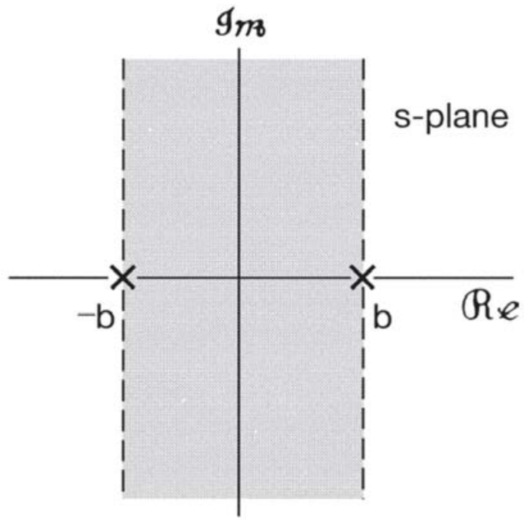
- $b \leq 0$: No common ROC

  $\Rightarrow X(s)$ does not exist

- b>0:

$$e^{-b|t|} \stackrel{\mathcal{L}}{\longleftrightarrow} \frac{1}{s+b} + \frac{-1}{s-b} = \frac{-2b}{(s+b)(s-b)},$$

$$-b < \mathrm{Re}\{s\} < +b$$



9.2 The ROC for Laplace Transform

- A Quick Recap of Properties 3-6

  For any signal with Laplace transform, the ROC must have one of the following four shapes:

  $\delta(t)$ (i) The entire s-plane (if the signal is finite length).

  $u(t)$ (ii) A left-half plane (if the signal is left-sided).

  $u(-t)$ (iii) A right-half plane (if the signal is right-sided).

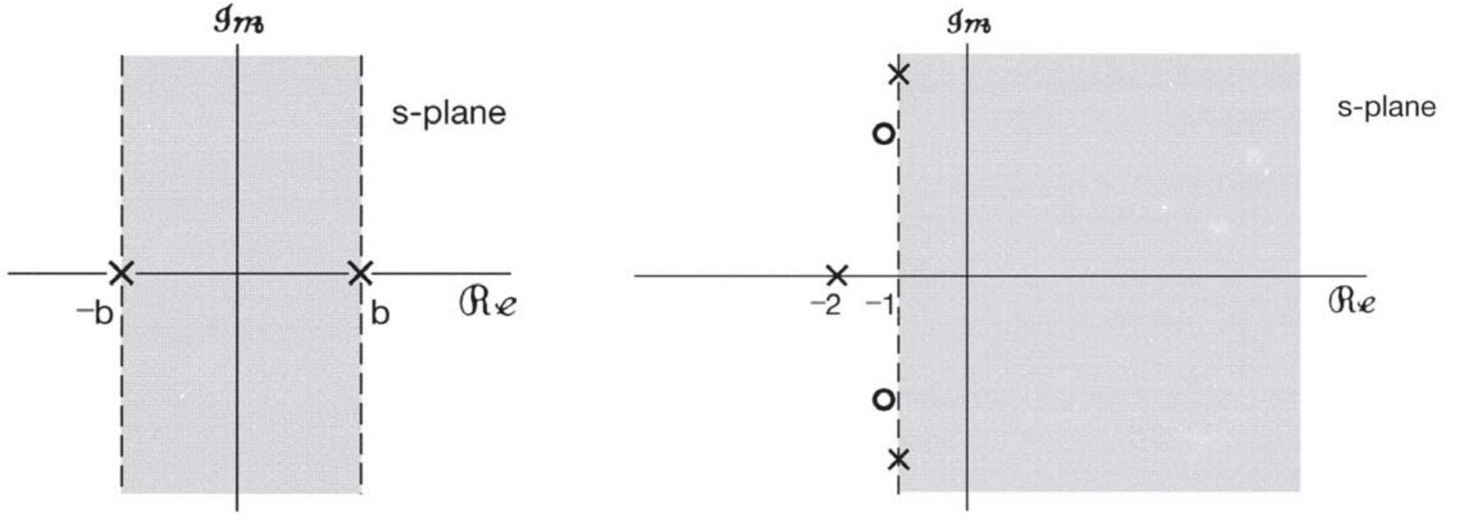
  $e^{-b|t|}$ (iv) A single strip (if the signal is two-sided).

9.2 The ROC for Laplace Transform

- Properties of ROC

  7. If the Laplace transform X(s) of x(t) is rational, then its ROC is bounded by poles or extends to infinity.

  In addition, no poles of X(s) are contained in ROC.



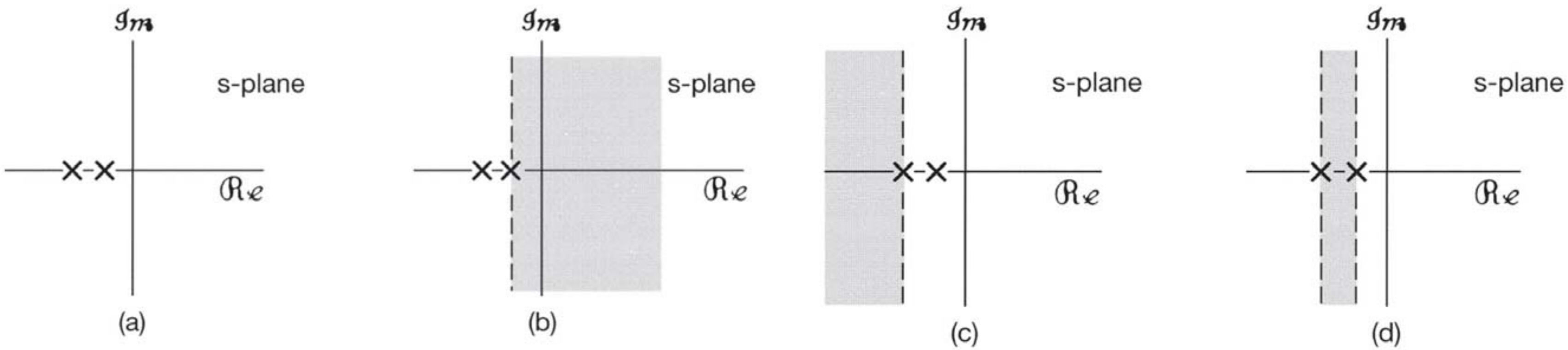
(b)

9.2 The ROC for Laplace Transform

- Properties of ROC

  8. If the Laplace transform X(s) of x(t) is ~~rationl~~ rational

  - If x(t) is right-sided, the ROC is the region in the s-plane to the right of the rightmost pole.
  - If x(t) is left-sided, the ROC is the region in the s-plane to the left of the leftmost pole.



9.3 The Inverse Laplace Transform

- Inverse Laplace Transform

$\sigma + j\omega$

$$X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt$$

$$X(\sigma + j\omega) = \mathcal{F}\{\underline{x(t)e^{-\sigma t}}\} = \int_{-\infty}^{\infty} x(t) e^{-\sigma t} \boxed{e^{-j\omega t}} dt$$

for $s = \sigma + j\omega$ in the ROC

$$x(t)e^{-\sigma t} = \mathcal{F}^{-1}\{X(\sigma + j\omega)\} = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\sigma + j\omega) e^{j\omega t} d\omega$$

$$s = \sigma + j\omega \Rightarrow ds = j d\omega$$

$$\Rightarrow x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\sigma + j\omega) e^{(\sigma + j\omega)t} d\omega = \frac{1}{2\pi j} \int_{\sigma - j\infty}^{\sigma + j\infty} X(s) e^{st} ds$$

$\Rightarrow$ A contour integration along a straight line with $\mathrm{Re}\{s\} = \sigma$.
$\Rightarrow$ We can choose any value of σ such that $X(\sigma + j\omega)$ converges.

9.3 The Inverse Laplace Transform

- Inverse Laplace Transform (cont'd)
  - Use partial fraction expansion

In this method, $X(s)$ is expanded into a linear combination of lower order terms so that the inverse Laplace transform of each term can be easily determined.

$$X(s) = \frac{A_1}{s+a_1} + \frac{A_2}{s+a_2} + \ldots + \frac{A_m}{s+a_m}$$

$$x(t) = \underbrace{A_1 e^{-a_1 t} u(t)}_{\text{If right-sided}} - \underbrace{A_2 e^{-a_2 t} u(-t)}_{\text{If left-sided}} + \ldots + x_m(t)$$

- Example 9.9

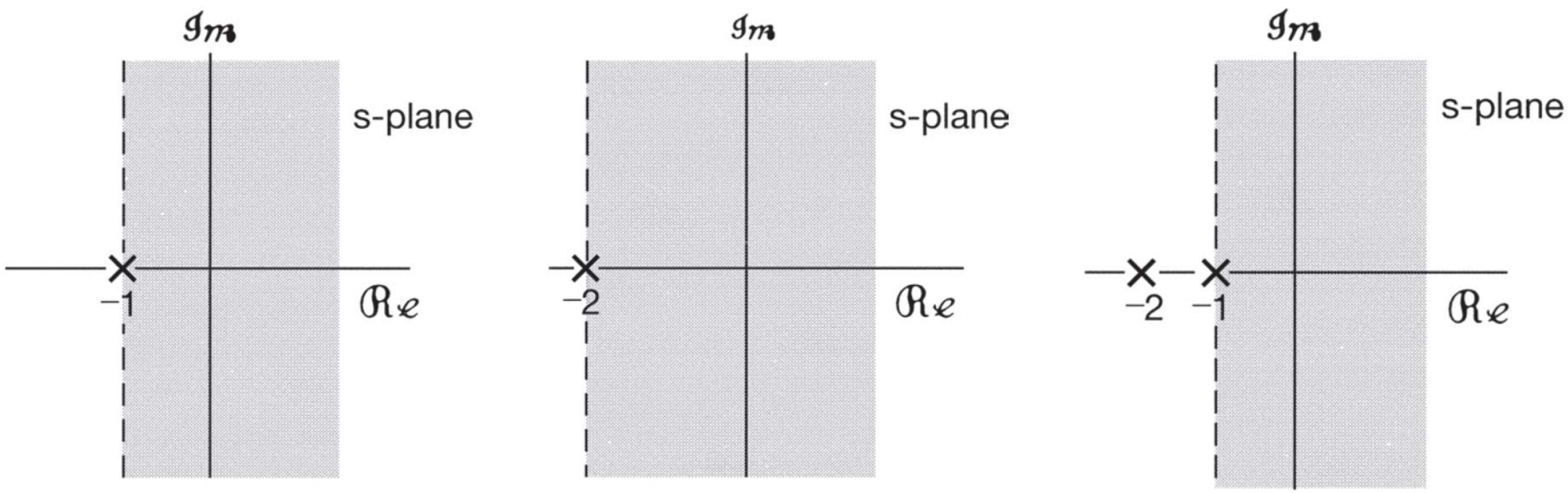
$$\boxed{X(s)=\frac{1}{(s+1)(s+2)},}\quad \mathrm{Re}\{s\}>-1$$

$$=\frac{1}{(s+1)}-\frac{1}{(s+2)}$$

$$e^{-t}u(t)\xleftrightarrow{\mathcal{L}}\frac{1}{(s+1)},\quad \mathrm{Re}\{s\}>-1$$

$$e^{-2t}u(t)\xleftrightarrow{\mathcal{L}}\frac{1}{(s+2)},\quad \mathrm{Re}\{s\}>-2$$

$$[e^{-t}-e^{-2t}]u(t)\xleftrightarrow{\mathcal{L}}\frac{1}{(s+1)(s+2)},\quad \mathrm{Re}\{s\}>-1$$



(b) (c) (a)

- Examples 9.9~9.11: Effects of ROC

	$\mathrm{Re}\{s\} < -1$	$-1 < \mathrm{Re}\{s\}$
$\frac{1}{(s+1)}$	$-e^{-t}u(-t)$	$e^{-t}u(t)$

	$\mathrm{Re}\{s\} < -2$	$-2 < \mathrm{Re}\{s\}$
$\frac{1}{(s+2)}$	$-e^{-2t}u(-t)$	$e^{-2t}u(t)$

$$\mathrm{Re}\{s\} < -2 \quad \Rightarrow \quad -e^{-t}u(-t) + e^{-2t}u(-t) \xleftrightarrow{\mathcal{L}} \frac{1}{(s+1)} - \frac{1}{(s+2)}$$

$$-2 < \mathrm{Re}\{s\} < -1 \quad \Rightarrow \quad -e^{-t}u(-t) - e^{-2t}u(t) \xleftrightarrow{\mathcal{L}} \frac{1}{(s+1)} - \frac{1}{(s+2)}$$

$$-1 < \mathrm{Re}\{s\} \quad \Rightarrow \quad e^{-t}u(t) - e^{-2t}u(t) \xleftrightarrow{\mathcal{L}} \frac{1}{(s+1)} - \frac{1}{(s+2)}$$

Supplement

Note that equation (9.57) is suitable for the case where the order of the denominator polynomial is greater than the numerator polynomial.

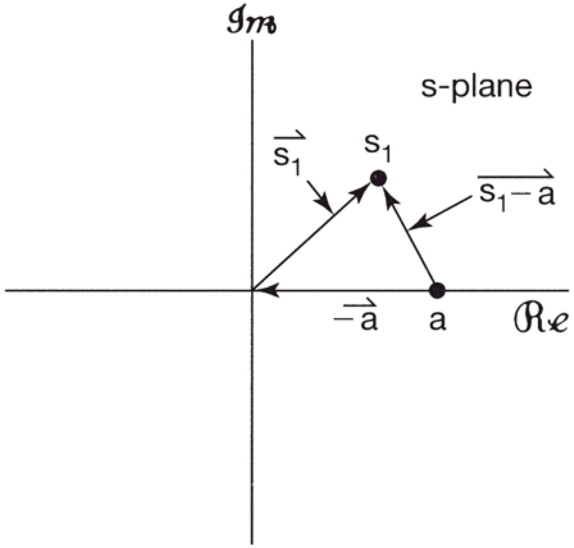
When the order of the denominator polynomial is equal to or smaller than the numerator polynomial, we can use the long division method to expand $X(s)$ as

$$X(s) = \sum_{q=1}^{Q} b_p s^q + b_0 + \sum_{i=1}^{m} \frac{A_i}{s + a_i}. \qquad (9.69a)$$

Will see Sect. 9.6 to determine the inverse Laplace transform of the above X(s).

9.4 Geometric Evaluation of the Fourier Transform

- Remarks
  - We discuss a procedure to geometrically evaluate CTFT.
  - More generally, we geometrically evaluate the Laplace transform at any set of values from the pole-zero pattern associated with a rational Laplace transform.
  - A more general rational Laplace transform consists of a product of pole and zero terms as follow:



$$X(s) = M \frac{\prod_{i=l}^{R}(s - \beta_i)}{\prod_{j=l}^{P}(s - \alpha_j)}.$$

- Example 9.12

$x(t) = e^{-\frac{1}{2}t}u(t)$

Let $$X(s) = \frac{1}{s+\frac{1}{2}}, \qquad \Re e\{s\} > -\frac{1}{2}.$$

The Fourier transform is $X(s)|_{s=j\omega}$.

For this example, then, the Fourier transform is

$$X(j\omega) = \frac{1}{j\omega + 1/2}.$$

with magnitude and phase as:

$$|X(j\omega)|^2 = \frac{1}{\omega^2 + (1/2)^2}$$

$$\measuredangle X(j\omega) = -\tan^{-1} 2\omega.$$

- Example 9.12

Let
$$X(s) = \frac{1}{s+\frac{1}{2}}, \qquad \Re e\{s\} > -\frac{1}{2}.$$ $\rightarrow x(t) = e^{-\frac{1}{2}t}u(t)$

The Fourier transform is $X(s)|_{s = j\omega}$.

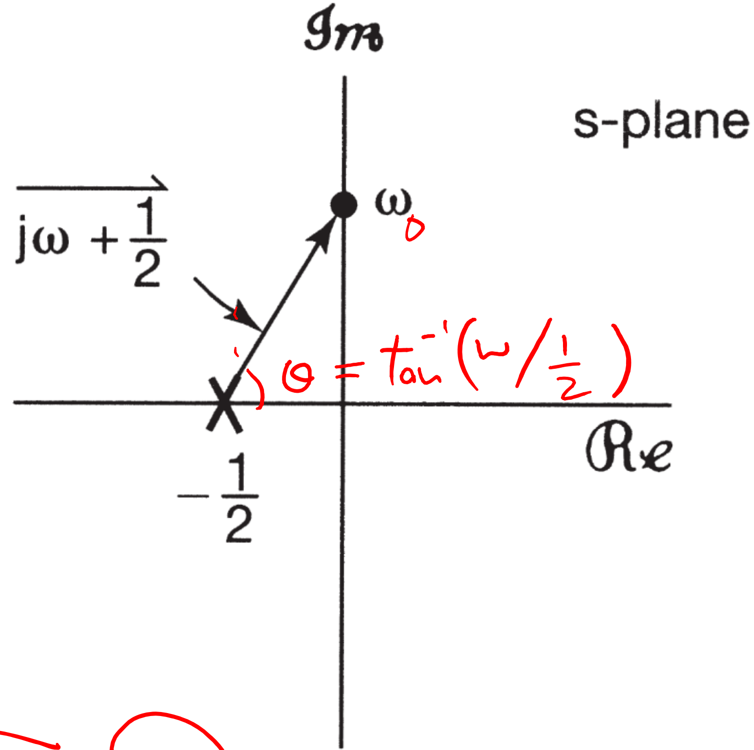
i.e.,
$$X(j\omega) = \frac{1}{j\omega + 1/2}.$$

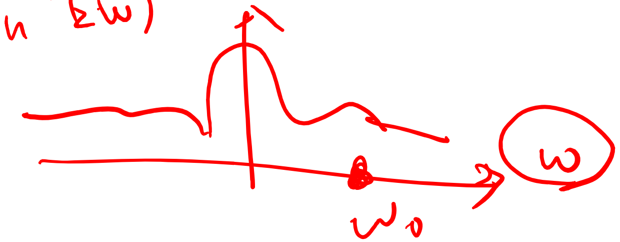
with magnitude and phase as:

$$|X(j\omega)|^2 = \frac{1}{\omega^2 + (1/2)^2}$$

$$\sphericalangle X(j\omega) = 0 - \tan^{-1} 2\omega.$$

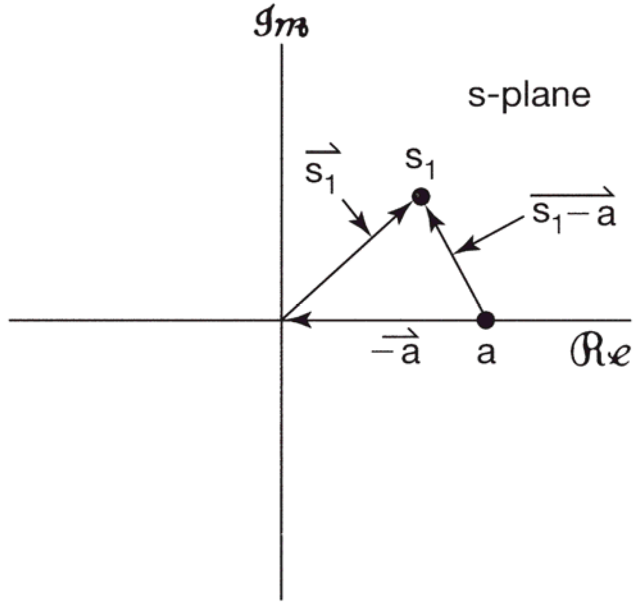
$= -\tan^{-1}(2\omega)$





9.4 Geometric Evaluation of the Fourier Transform

- Remarks
  - We discuss a procedure to geometrically evaluate CTFT.
  - More generally, we geometrically evaluate the Laplace transform at any set of values from the pole-zero pattern associated with a rational Laplace transform.
  - Thus, using the pole-zero plot in the *s*-plane to evaluate the Fourier transform by drawing vectors from zeros (or poles) to the Im{s} axis.



9.4 Geometric Evaluation of the Fourier Transform

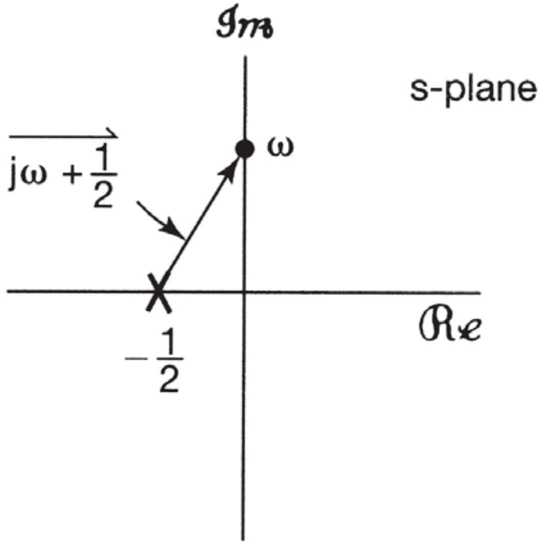
- Example 9.12
  - Let

$$X(s) = \frac{1}{s+\frac{1}{2}}, \qquad \mathcal{R}e\{s\} > -\frac{1}{2}.$$

  - The Fourier transform is $X(s)|_{s\,=\,j\omega}$.
    For this example, then, the Fourier transform is

$$X(j\omega) = \frac{1}{j\omega + 1/2}.$$

$$\left\{\begin{aligned} |X(j\omega)|^2 &= \frac{1}{\omega^2 + (1/2)^2} \\ \sphericalangle X(j\omega) &= -\tan^{-1} 2\omega. \end{aligned}\right.$$



9.4.1 First-Order System

As a generalization of Example 9.12, let us consider the class of first-order systems. The impulse response for such a system is

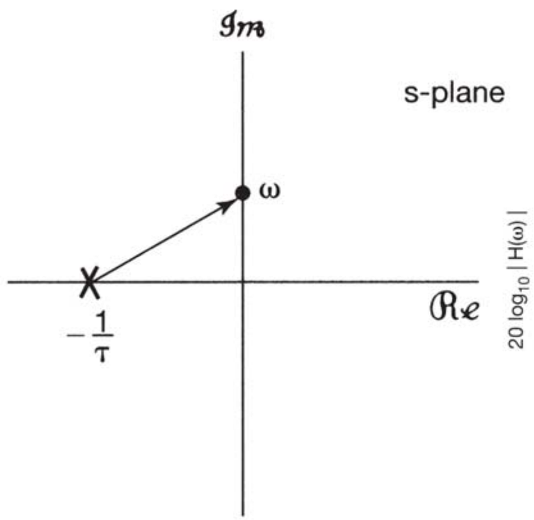
$$h(t) = \frac{1}{\tau} e^{-t/\tau} u(t),$$

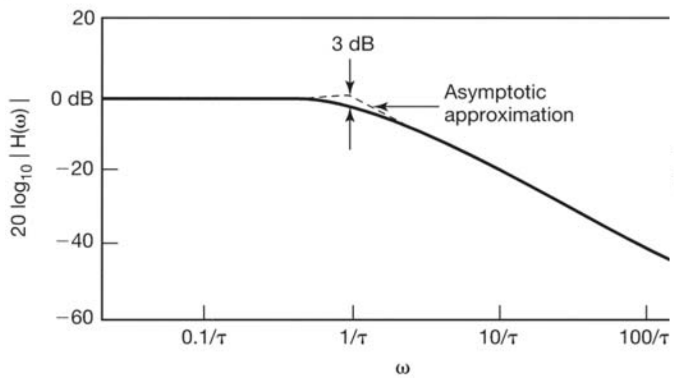
and its Laplace transform is

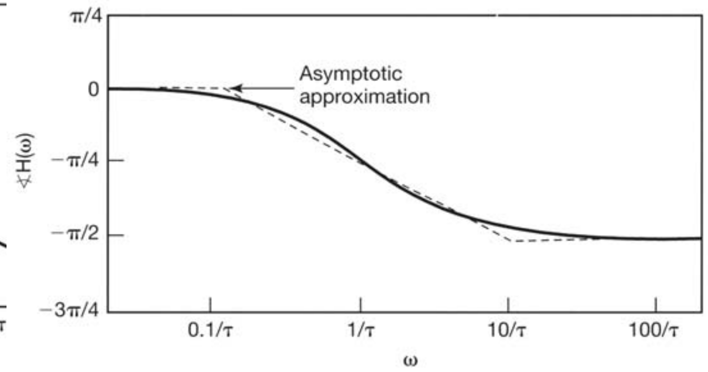
$$H(s) = \frac{1}{s\tau + 1}, \qquad \Re e\{s\} > -\frac{1}{\tau}.$$

$j\omega$

$dB = 20 \log_{10} |H(j\omega)|$







9.4.2 Second-Order System

Consider the class of second-order systems.
The impulse response & frequency responses for such a system are

$$h(t) = M[e^{c_1 t} - e^{c_2 t}]u(t),$$

where

$$c_1 = -\zeta\omega_n + \omega_n\sqrt{\zeta^2 - 1},$$

$$c_2 = -\zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1},$$

$$M = \frac{\omega_n}{2\sqrt{\zeta^2 - 1}},$$

And,

$$H(j\omega) = \frac{\omega_n^2}{(j\omega)^2 + 2\zeta\omega_n(j\omega) + \omega_n^2}.$$

$$H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{\omega_n^2}{(s - c_1)(s - c_2)}.$$

9.4.2 Second-Order System

$$H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{\omega_n^2}{(s - c_1)(s - c_2)}.$$

$$\left.\begin{aligned} c_1 &= -\zeta\omega_n + \omega_n\sqrt{\zeta^2 - 1}, \\ c_2 &= -\zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1}, \end{aligned}\right\}$$

$$M = \frac{\omega_n}{2\sqrt{\zeta^2 - 1}},$$

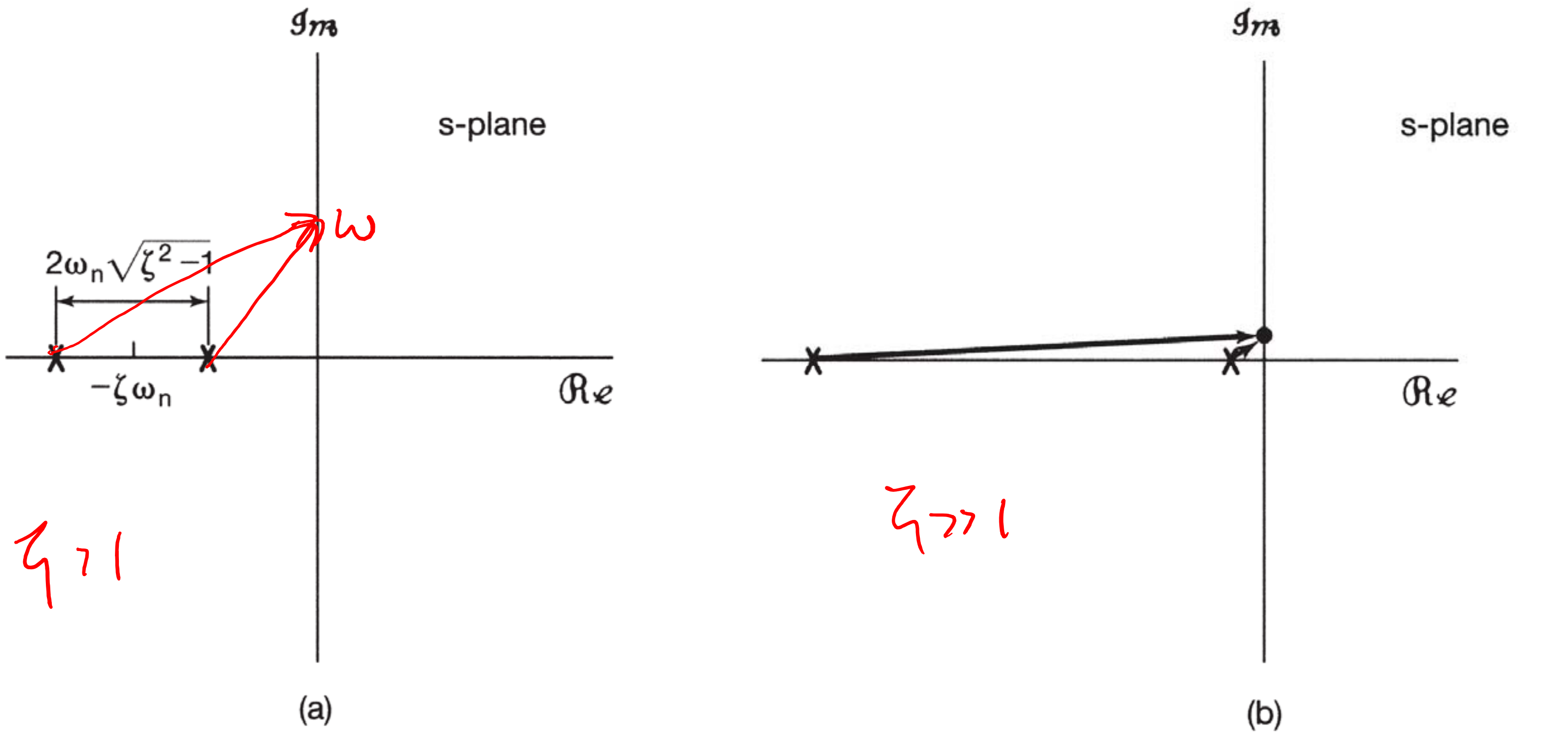


Figure 9.19 (a) Pole-zero plot for a second-order system with $\zeta > 1$; (b) pole vectors for $\zeta \gg 1$; (c) pole-zero plot for a second-order system with $0 < \zeta < 1$; (d) pole vectors for $0 < \zeta < 1$ and for $\omega = \omega_n\sqrt{1-\zeta^2}$ and $\omega = \omega_n\sqrt{1-\zeta^2} \pm \zeta\omega_n$.

9.4.2 Second-Order System

$$H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{\omega_n^2}{(s - c_1)(s - c_2)}.$$

$$\begin{aligned} c_1 &= -\zeta\omega_n + \omega_n\sqrt{\zeta^2 - 1}, \\ c_2 &= -\zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1}, \\ M &= \frac{\omega_n}{2\sqrt{\zeta^2 - 1}}, \end{aligned}$$

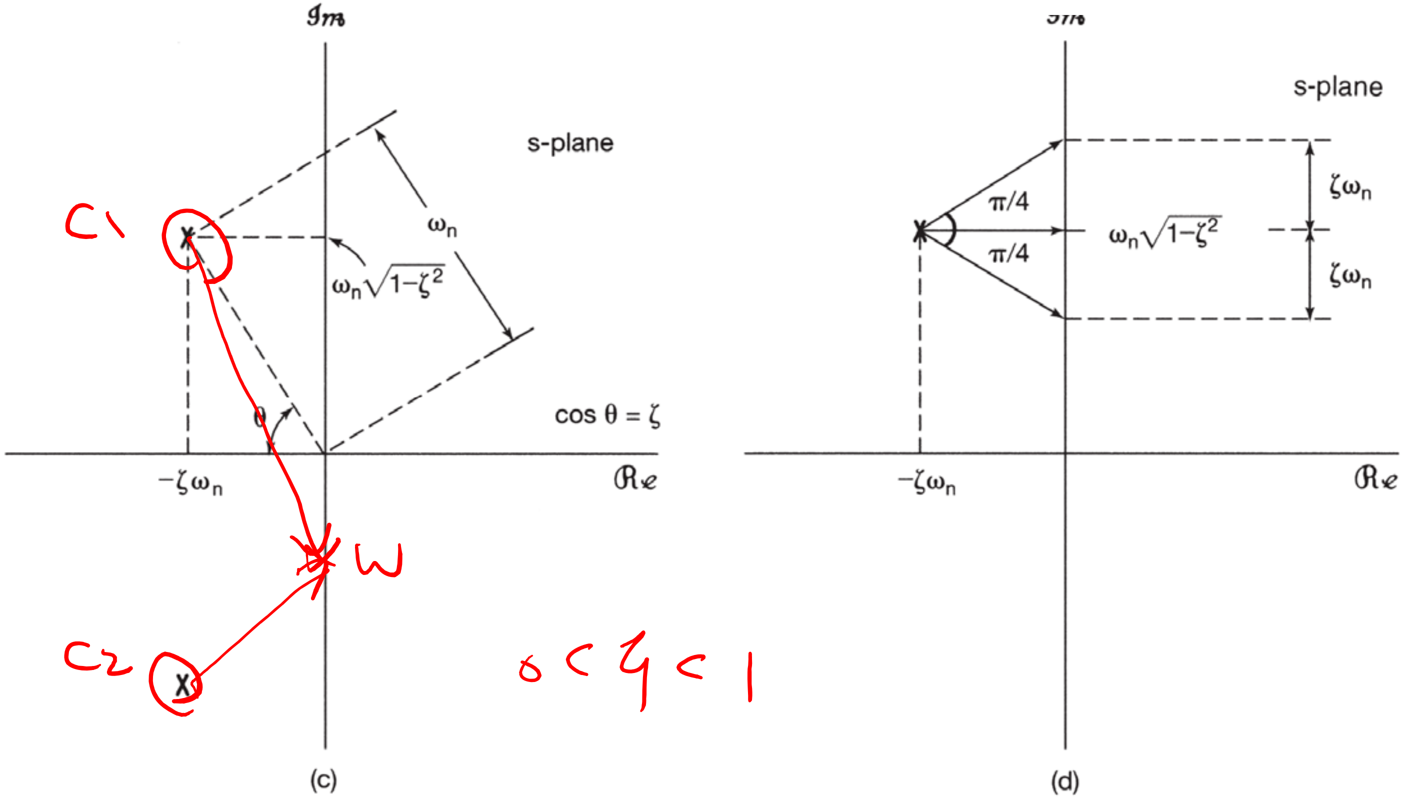


Figure 9.19 (a) Pole-zero plot for a second-order system with $\zeta > 1$; (b) pole vectors for $\zeta \gg 1$; (c) pole-zero plot for a second-order system with $0 < \zeta < 1$; (d) pole vectors for $0 < \zeta < 1$ and for $\omega = \omega_n\sqrt{1-\zeta^2}$ and $\omega = \omega_n\sqrt{1-\zeta^2} \pm \zeta\omega_n$.

9.4.2 Second-Order System

$$H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{\omega_n^2}{(s - c_1)(s - c_2)}.$$

$$c_1 = -\zeta\omega_n + \omega_n\sqrt{\zeta^2 - 1},$$
$$c_2 = -\zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1},$$
$$M = \frac{\omega_n}{2\sqrt{\zeta^2 - 1}},$$

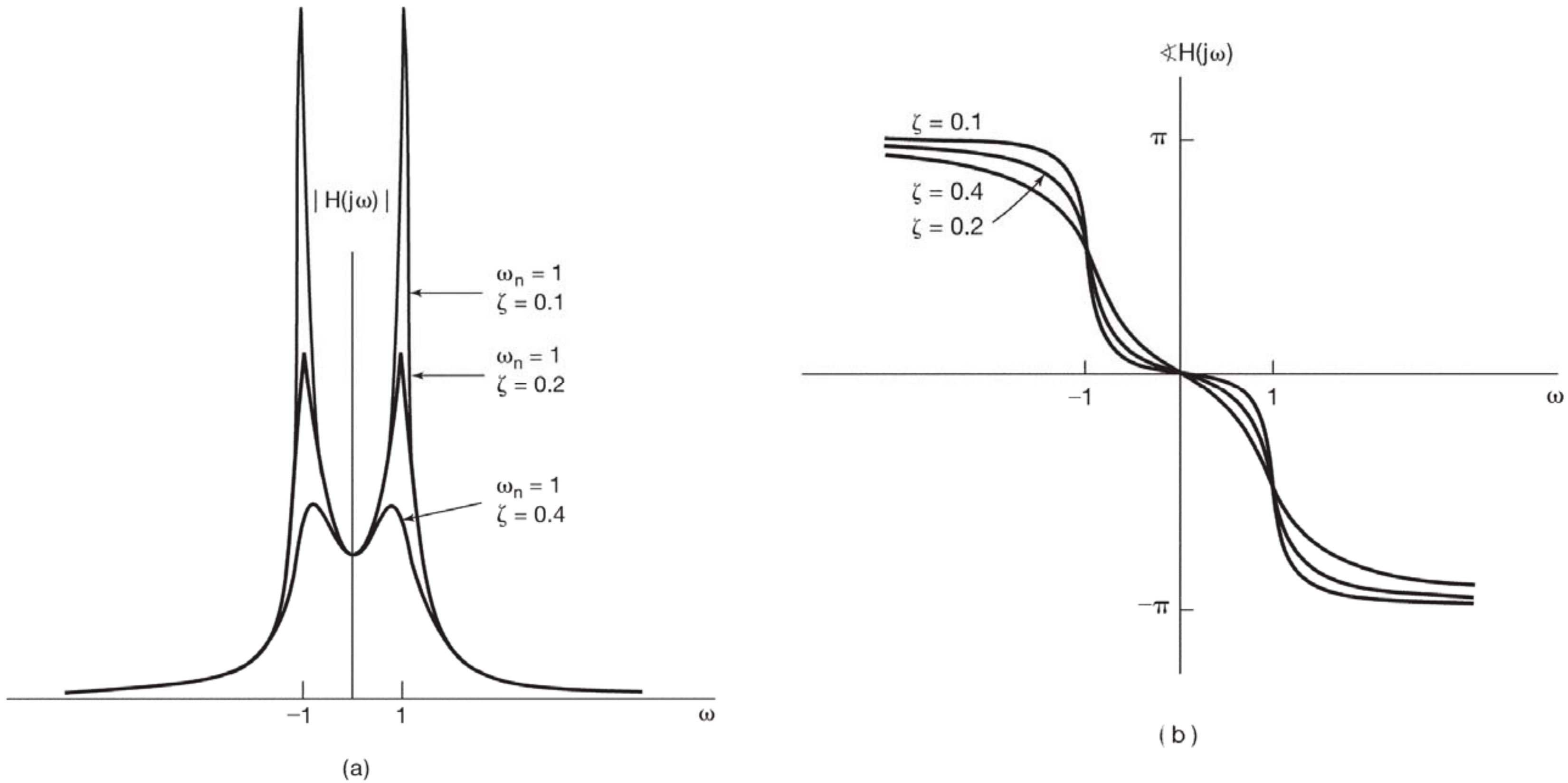


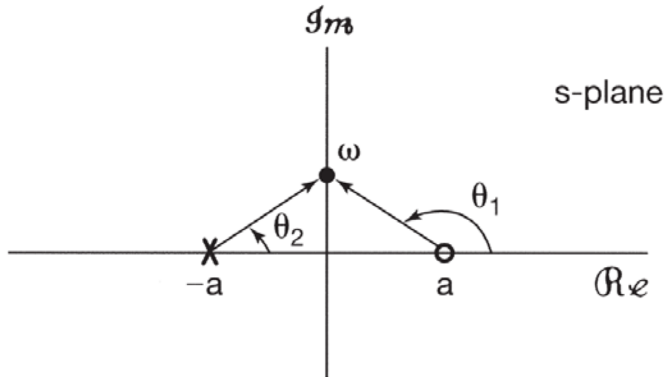
Figure 9.20 (a) Magnitude and (b) phase of the frequency response for a second-order system with $0 < \zeta < 1$.

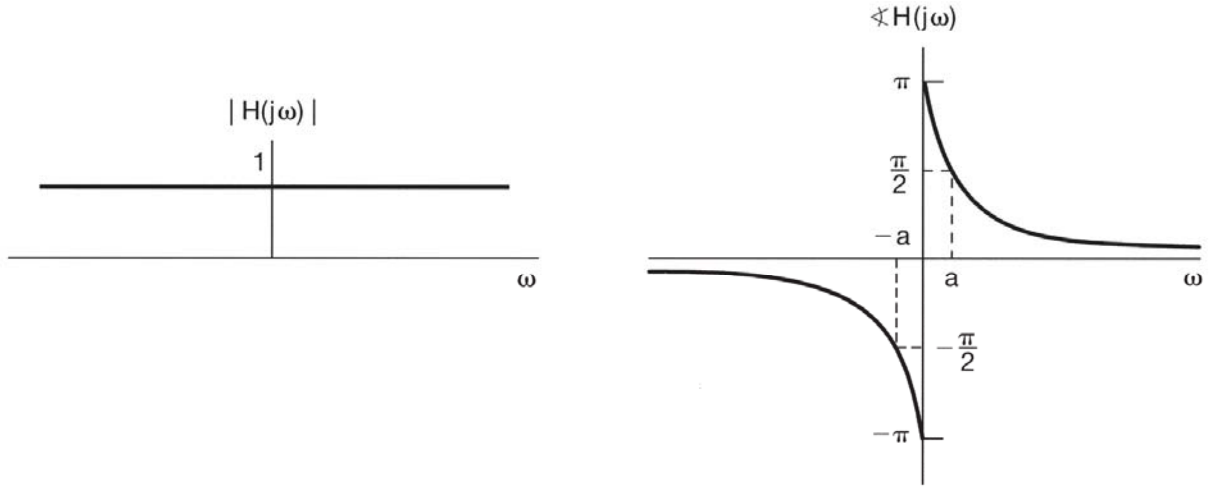
9.4.3 All-Pass Systems

Such a system is commonly referred to as an *all pass system*, since it passes all frequencies with equal gain (or attenuation).
The phase of frequency response is $\vartheta_1 - \vartheta_2$, or, since $\vartheta_1 = \pi - \vartheta_2$ with $\vartheta_2 = \tan^{-1}(\omega/a)$.

$$H(s) = \frac{s-a}{s+a}$$

$$H(j\omega) = \frac{j\omega - a}{j\omega + a}$$





(b)

9.5 Properties of Laplace Transform

- Linearity of Laplace Transform

$$x_1(t) \xleftrightarrow{\mathcal{L}} X_1(s), \quad \text{ROC} = R_1$$

$$x_2(t) \xleftrightarrow{\mathcal{L}} X_2(s), \quad \text{ROC} = R_2$$

$$ax_1(t) + bx_2(t) \xleftrightarrow{\mathcal{L}} aX_1(s) + bX_2(s),$$

ROC contains at least $R_1 \cap R_2$

Proof:

$$X(s) = \int_{-\infty}^{\infty} (ax_1(t) + bx_2(t))e^{-st}dt$$

$$= a\int_{-\infty}^{\infty} x_1(t)e^{-st}dt + b\int_{-\infty}^{\infty} x_2(t)e^{-st}dt$$

$$= aX_1(s) + bX_2(s)$$

The ROC may become larger if pole-zero cancellation occurs.

9.5 Properties of Laplace Transform

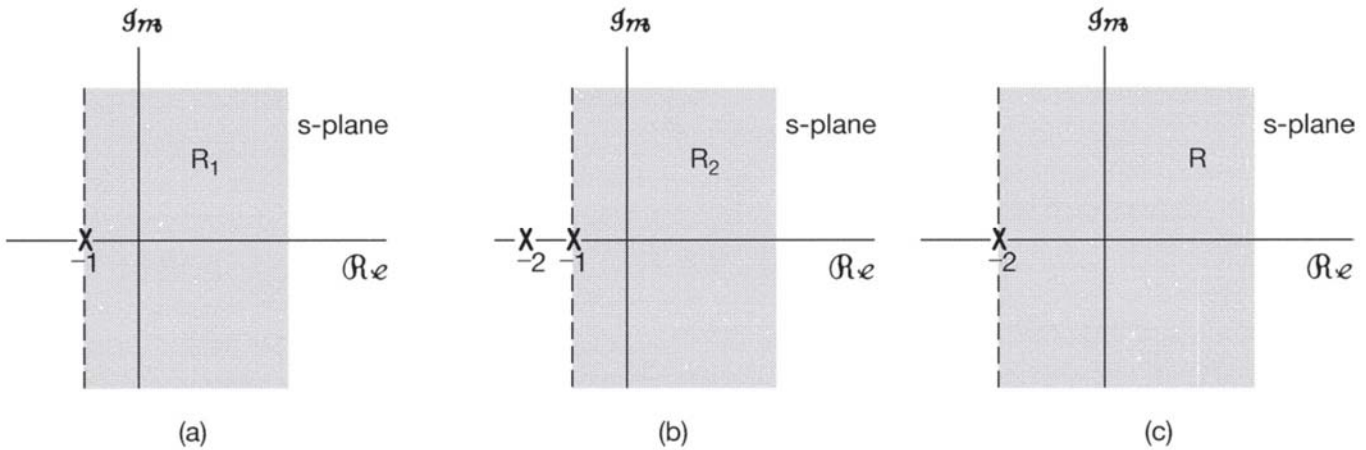
- Example 9.13 ROC may expand with pole-zero cancellation

$$\mathrm{ROC} \supset (R_1 \cap R_2)$$

$$X_1(s) = \frac{1}{(s+1)}, \quad \mathrm{Re}\{s\} > -1$$

$$X_2(s) = \frac{1}{(s+1)(s+2)}, \quad \mathrm{Re}\{s\} > -1$$

$$X(s) = X_1(s) - X_2(s) = \frac{1}{(s+1)} - \frac{1}{(s+1)(s+2)} = \frac{1}{(s+2)}, \quad \mathrm{Re}\{s\} > -2$$



9.5 Properties of Laplace Transform

- Shifting in Time Domain

$$x(t) \xleftrightarrow{\mathcal{L}} X(s), \quad \text{ROC} = R$$

$$x(t-t_0) \xleftrightarrow{\mathcal{L}} e^{-st_0} X(s), \quad \text{ROC} = R$$

$$\int_{-\infty}^{\infty} x(t-t_0) e^{-st} dt$$

$$= \int_{-\infty}^{\infty} x(\tau) e^{-s(\tau+t_0)} d\tau = e^{-st_0} \int_{-\infty}^{\infty} x(\tau) e^{-s\tau} d\tau = e^{-st_0} X(s)$$

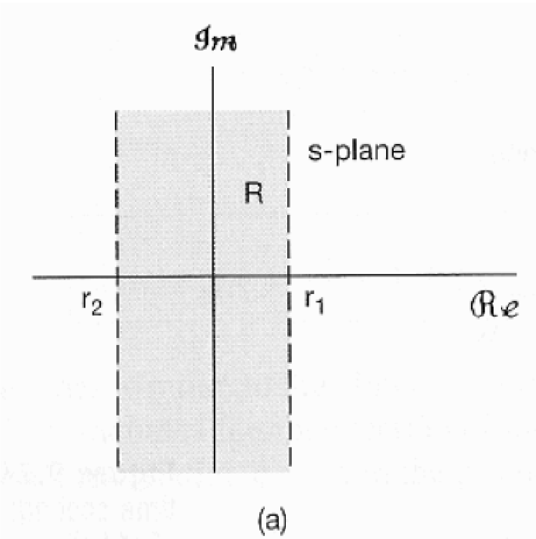
9.5 Properties of Laplace Transform

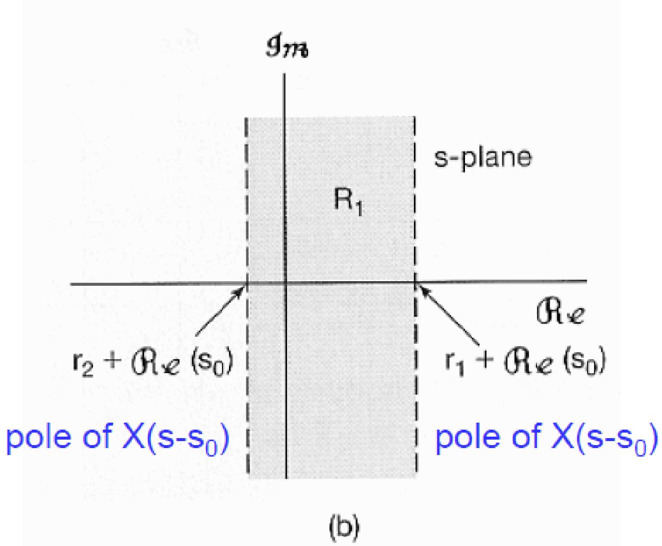
- Shifting in the s-Domain

$$x(t) \xleftrightarrow{\mathcal{L}} X(s), \quad \text{ROC} = R$$

$$e^{s_0 t} x(t) \xleftrightarrow{\mathcal{L}} X(s - s_0), \quad \text{ROC} = R + \mathcal{R}e\{s_0\}$$

The ROC associated with $X(s-s_0)$ is that of $X(s)$, shifted by $\mathcal{R}e(s_0)$.





9.5 Properties of Laplace Transform

- Time Scaling

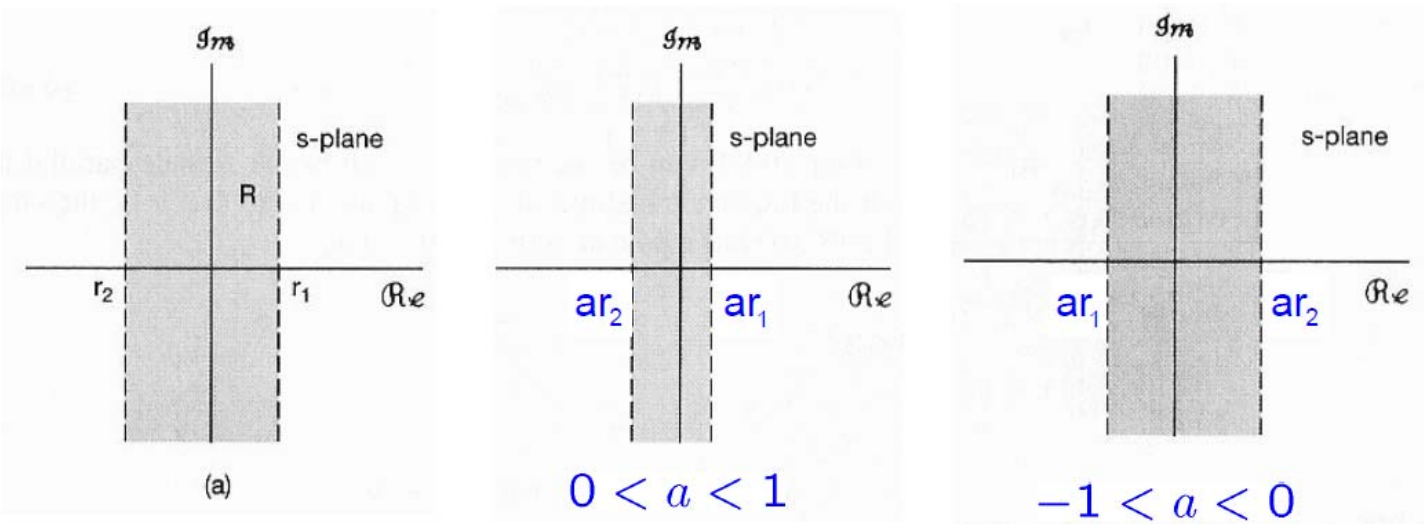
$$x(t) \xleftrightarrow{\mathcal{L}} X(s), \quad \text{ROC} = R$$

$$x(at) \xleftrightarrow{\mathcal{L}} \frac{1}{|a|} X\left(\frac{s}{a}\right), \quad \text{ROC} = aR$$

Scaling of the ROC

$$x(-t) \xleftrightarrow{\mathcal{L}} X(-s), \quad \text{ROC} = -R$$

Reversal of the ROC



9.5 Properties of Laplace Transform

- Conjugation

$$x(t) \overset{\mathcal{L}}{\longleftrightarrow} X(s), \quad \text{ROC} = R$$

$$\Rightarrow x^*(t) \overset{\mathcal{L}}{\longleftrightarrow} X^*(s^*), \quad \text{ROC} = R$$

$$\Rightarrow X(s) = X^*(s^*) \text{ if } x(t) \text{ is real}$$

$\Rightarrow$ Therefore, if X(s) has a pole or zero at $s = s_0$,

it must also have a pole or zero at $s = s_0^*$.

9.5 Properties of Laplace Transform

- Convolution Property

$$x_1(t) \xleftrightarrow{\mathcal{L}} X_1(s), \quad \text{ROC} = R_1$$
$$x_2(t) \xleftrightarrow{\mathcal{L}} X_2(s), \quad \text{ROC} = R_2$$

$$x_1(t) * x_2(t) \xleftrightarrow{\mathcal{L}} X_1(s)X_2(s), \quad \text{ROC contains } R_1 \cap R_2$$

The ROC may become larger if pole-zero cancellation occurs for $X_1(s)X_2(s)$.

9.5 Properties of Laplace Transform

- Differentiation in Time and s Domain

$$x(t) \xleftrightarrow{\mathcal{L}} X(s), \quad \text{ROC} = R$$

$$\frac{d}{dt}x(t) \xleftrightarrow{\mathcal{L}} sX(s), \quad \text{ROC contains } R$$

pole-zero cancellation may occur.

$$-tx(t) \xleftrightarrow{\mathcal{L}} \frac{dX(s)}{ds}, \quad \text{ROC} = R$$

Proof :

$$\frac{dx(t)}{dt} = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} X(s)\frac{de^{st}}{dt}ds = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} sX(s)e^{st}ds$$

$$\frac{d}{ds}\int_{-\infty}^{\infty} x(t)e^{-st}dt = \int_{-\infty}^{\infty} (-t)x(t)e^{-st}dt$$

9.5 Properties of Laplace Transform

- Integration in Time

$$x(t) \xleftrightarrow{\mathcal{L}} X(s), \quad \text{ROC}=R$$

$$\int_{-\infty}^{t} x(\tau)d\tau \xleftrightarrow{\mathcal{L}} \frac{1}{s}X(s), \quad \text{ROC contains } R \cap \{\text{Re}\{s\} > 0\}$$

Proof :

$$\int_{-\infty}^{t} x(\tau)d\tau = u(t) * x(t)$$

From Example 9.1, $u(t) \xleftrightarrow{\mathcal{L}} \frac{1}{s}$, $\text{Re}\{s\}>0$

$\therefore \int_{-\infty}^{t} x(\tau)d\tau \xleftrightarrow{\mathcal{L}} \frac{1}{s}X(s)$, with an ROC containing the intersection of the ROC of $X(s)$ and the ROC of the LT of $u(t)$.

9.5 Properties of Laplace Transform

- The Initial Value Theorem

If $x(t)=0$ for $t<0$ and it contains no impulse or higher order singularities at the origin,

$$x(0^+)=\lim_{s\to\infty} sX(s).$$

- The Final-Value Theorem

If $x(t)=0$ for $t<0$ and $x(t)$ has a finite limit as $t\to\infty$,

$$\lim_{t\to\infty} x(t)=\lim_{s\to 0} sX(s).$$

Note: x(t) has to be causal or the two theorems cannot apply.

- Proof of the Initial Value Theorem

Given $x(t)=0$ for $t<0 \Rightarrow x(t)=x(t)u(t)$
By Taylor series expansion at $t=0+$,

$$x(t)=\left[x(0+)+x^{(1)}(0+)t+\cdots+x^{(n)}(0+)\frac{t^n}{n!}+\cdots\right]u(t) \qquad \text{(Eq. 1)}$$

$$=\sum_{0}^{\infty} x^{(n)}(0+)\frac{t^n}{n!}$$

From Example 9.14,

$$e^{-at}\left(\frac{t^n}{n!}\right)u(t) \xleftrightarrow{\mathcal{L}} \frac{1}{(s+a)^{n+1}}, \quad \mathrm{Re}\{s\}>-a$$

Setting $a=0$ and multiplying both sides by $x^{(n)}(0+)$, we have

$$x^{(n)}(0+)\left(\frac{t^n}{n!}\right)u(t) \xleftrightarrow{\mathcal{L}} \frac{x^{(n)}(0+)}{s^{n+1}}, \quad \mathrm{Re}\{s\}>0$$

Taking the Laplace transform of Eq. 1, we get

$$X(s)=\sum_{0}^{\infty}\frac{x^{(n)}(0+)}{s^{n+1}}$$

$$sX(s)=x^{(0)}(0+)+x^{(1)}(0+)/s+\cdots$$

Therefore,

$$\lim_{s\to\infty} sX(s)=x^{(0)}(0+)=x(0+)$$

- Proof of the Final Value Theorem

Since $x(t)$ is causal, $x(t) = 0$ for $t < 0$.

Since $\dfrac{dx(t)}{dt} \xleftrightarrow{\mathcal{L}} sX(s)$, however by definition,

$$sX(s) = \mathcal{L}\left\{\frac{dx(t)}{dt}\right\} = \int_{-\infty}^{\infty} x'(t)e^{-st}dt = \int_{0^-}^{\infty} x'(t)e^{-st}dt.$$

Thus

$$\lim_{s\to 0} sX(s) = \lim_{s\to 0}\int_{0^-}^{\infty} x'(t)e^{-st}dt = \int_{0^-}^{\infty} x'(t)dt = \lim_{t\to\infty} x(t) - x(0^-).$$

Since $x(0^-) = 0$, we have

$$\lim_{t\to\infty} x(t) = \lim_{s\to 0} sX(s) .$$

9.5 Properties of Laplace Transform

TABLE 9.1 PROPERTIES OF THE LAPLACE TRANSFORM

Section	Property	Signal	Laplace Transform	ROC
		$x(t)$	$X(s)$	R
		$x_1(t)$	$X_1(s)$	R_1
		$x_2(t)$	$X_2(s)$	R_2
9.5.1	Linearity	$ax_1(t) + bx_2(t)$	$aX_1(s) + bX_2(s)$	At least $R_1 \cap R_2$
9.5.2	Time shifting	$x(t - t_0)$	$e^{-st_0}X(s)$	R
9.5.3	Shifting in the s-Domain	$e^{s_0 t}x(t)$	$X(s - s_0)$	Shifted version of R (i.e., s is in the ROC if $s - s_0$ is in R)
9.5.4	Time scaling	$x(at)$	$\frac{1}{\|a\|}X\left(\frac{s}{a}\right)$	Scaled ROC (i.e., s is in the ROC if s/a is in R)
9.5.5	Conjugation	$x^*(t)$	$X^*(s^*)$	R
9.5.6	Convolution	$x_1(t) * x_2(t)$	$X_1(s)X_2(s)$	At least $R_1 \cap R_2$
9.5.7	Differentiation in the Time Domain	$\frac{d}{dt}x(t)$	$sX(s)$	At least R
9.5.8	Differentiation in the s-Domain	$-tx(t)$	$\frac{d}{ds}X(s)$	R
9.5.9	Integration in the Time Domain	$\int_{-\infty}^{t} x(\tau)d(\tau)$	$\frac{1}{s}X(s)$	At least $R \cap \{\mathcal{R}e\{s\} > 0\}$

Initial- and Final-Value Theorems

9.5.10 If $x(t) = 0$ for $t < 0$ and $x(t)$ contains no impulses or higher-order singularities at $t = 0$, then

$$x(0^+) = \lim_{s \to \infty} sX(s)$$

If $x(t) = 0$ for $t < 0$ and $x(t)$ has a finite limit as $t \longrightarrow \infty$, then

$$\lim_{t \to \infty} x(t) = \lim_{s \to 0} sX(s)$$

9.6 Some LT Pairs

- Integration in Time

TABLE 9.2 LAPLACE TRANSFORMS OF ELEMENTARY FUNCTIONS

Transform pair	Signal	Transform	ROC
1	$\delta(t)$	1	All s
2	$u(t)$	$\frac{1}{s}$	$\mathcal{R}e\{s\} > 0$
3	$-u(-t)$	$\frac{1}{s}$	$\mathcal{R}e\{s\} < 0$
4	$\frac{t^{n-1}}{(n-1)!}u(t)$	$\frac{1}{s^n}$	$\mathcal{R}e\{s\} > 0$
5	$-\frac{t^{n-1}}{(n-1)!}u(-t)$	$\frac{1}{s^n}$	$\mathcal{R}e\{s\} < 0$
6	$e^{-\alpha t}u(t)$	$\frac{1}{s+\alpha}$	$\mathcal{R}e\{s\} > -\alpha$
7	$-e^{-\alpha t}u(-t)$	$\frac{1}{s+\alpha}$	$\mathcal{R}e\{s\} < -\alpha$
8	$\frac{t^{n-1}}{(n-1)!}e^{-\alpha t}u(t)$	$\frac{1}{(s+\alpha)^n}$	$\mathcal{R}e\{s\} > -\alpha$

9.6 Some LT Pairs

- Integration in Time

Transform pair	Signal	Transform	ROC
9	$-\frac{t^{n-1}}{(n-1)!}e^{-\alpha t}u(-t)$	$\frac{1}{(s+\alpha)^n}$	$\Re e\{s\} < -\alpha$
10	$\delta(t-T)$	e^{-sT}	All s
11	$[\cos\omega_0 t]u(t)$	$\frac{s}{s^2+\omega_0^2}$	$\Re e\{s\} > 0$
12	$[\sin\omega_0 t]u(t)$	$\frac{\omega_0}{s^2+\omega_0^2}$	$\Re e\{s\} > 0$
13	$[e^{-\alpha t}\cos\omega_0 t]u(t)$	$\frac{s+\alpha}{(s+\alpha)^2+\omega_0^2}$	$\Re e\{s\} > -\alpha$
14	$[e^{-\alpha t}\sin\omega_0 t]u(t)$	$\frac{\omega_0}{(s+\alpha)^2+\omega_0^2}$	$\Re e\{s\} > -\alpha$
15	$u_n(t) = \frac{d^n\delta(t)}{dt^n}$	s^n	All s
16	$u_{-n}(t) = \underbrace{u(t) * \cdots * u(t)}_{n \text{ times}}$	$\frac{1}{s^n}$	$\Re e\{s\} > 0$

9.7 Analysis & Characterization of LTI Systems Using LT

- Stability
  - An LTI system is stable if and only if the ROC of its system function *H*(*s*) includes the entire *j*ω-axis (i.e., Im{*s*} = 0).

$$H(s) = \mathcal{L}\{h(t)\}$$

$$X(s) \longrightarrow \boxed{\text{LTI System}} \longrightarrow Y(s) = H(s)X(s)$$

$$H(s): \text{ system (or transfer) function}$$

- Causality
  - The ROC associated with the system function for a causal and stable system includes a right half plane.
  - For a system with a rational system function, causality of the system is equivalent to that the ROC is to the right of the rightmost pole.

9.7 Analysis & Characterization of LTI Systems Using LT

- Examples 9.17, 9.18, & 9.19

1. $h(t) = e^{-t}u(t) \xleftrightarrow{\mathcal{L}} H(s) = \dfrac{1}{s+1}, \qquad -1 < \mathrm{Re}\{s\}$

2. $h(t) = e^{-|t|} \xleftrightarrow{\mathcal{L}} H(s) = \dfrac{-2}{s^2-1}, \qquad -1 < \mathrm{Re}\{s\} < +1$

3. $h(t) = e^{-(t+1)}u(t+1) \xleftrightarrow{\mathcal{L}} H(s) = \dfrac{e^s}{s+1}, \qquad -1 < \mathrm{Re}\{s\}$

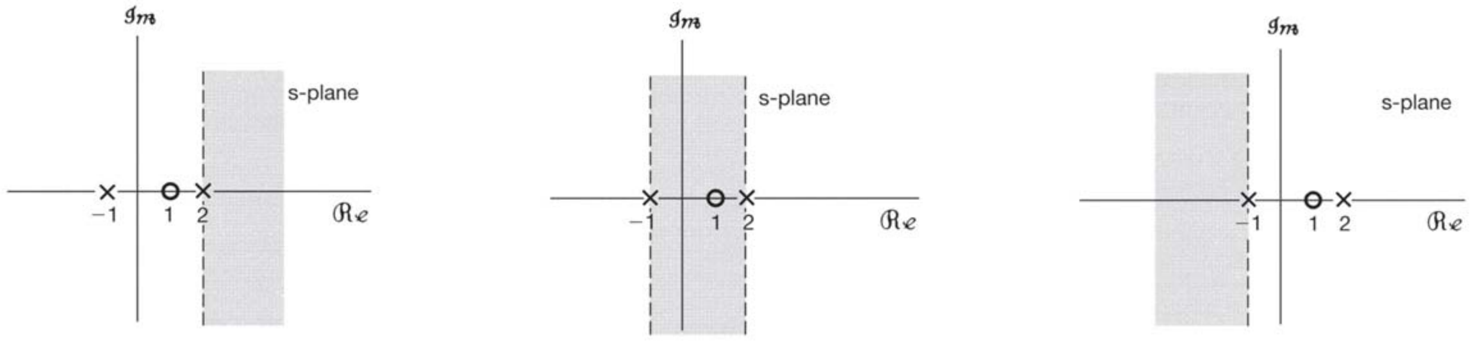
$$1.\begin{cases} h(t): & \text{causal} \\ H(s): & \text{rational} \\ ROC: & \text{right-sided} \end{cases} \qquad 2.\begin{cases} h(t): & \text{not causal} \\ H(s): & \text{rational} \\ ROC: & \text{not right-sided} \end{cases} \qquad 3.\begin{cases} h(t): & \text{not causal} \\ H(s): & \text{not rational} \\ ROC: & \text{right-sided} \end{cases}$$

9.7 Analysis & Characterization of LTI Systems Using LT

- Anti-Causality
  - The ROC associated with the system function for a anti-causal system includes a left half plane.
  - For a system with a rational system function, anti-causality of the system is equivalent to the ROC being the left-half plane to the left of the leftmost pole.
- Identifying ROC based on Causality & Stability Information

$$H(s) = \frac{s-1}{(s+1)(s-2)}.$$

Causal, unstable — Non-causal, stable — Anti-causal, unstable



(a) (b) (c)

$$h(t) = (\frac{2}{3}e^{-t} + \frac{1}{3}e^{2t})u(t) \qquad h(t) = \frac{2}{3}e^{-t}u(t) - \frac{1}{3}e^{2t}u(-t) \qquad h(t) = -(\frac{2}{3}e^{-t} + \frac{1}{3}e^{2t})u(-t)$$

9.7 Analysis & Characterization of LTI Systems Using LT

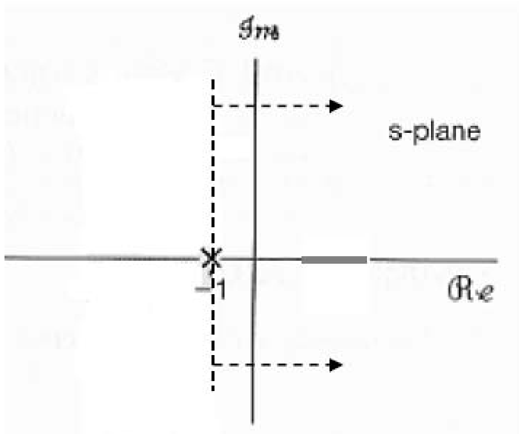
- Examples 9.17 & 9.21

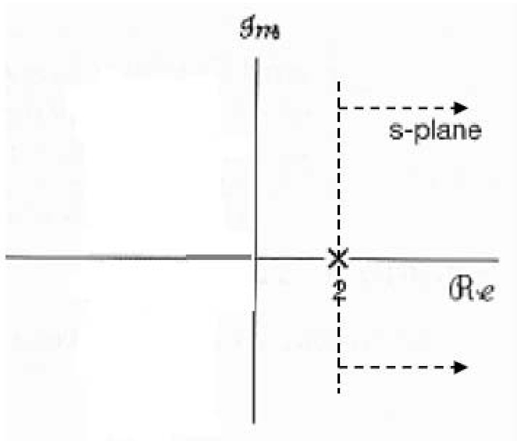
$$h(t) = e^{-t}u(t) \xleftrightarrow{\mathcal{L}} H(s) = \frac{1}{s+1},$$
$$\mathrm{Re}\{s\} > -1$$

$$h(t) = e^{2t}u(t) \xleftrightarrow{\mathcal{L}} H(s) = \frac{1}{s-2},$$
$$\mathrm{Re}\{s\} > 2$$

$$\begin{cases} h(t): & \text{causal} \\ H(s): & \text{stable, rational} \end{cases}$$

$$\begin{cases} h(t): & \text{causal} \\ H(s): & \text{unstable, rational} \end{cases}$$





Consistent with time-domain analysis since h(t) is absolutely integrable and nonzero only if t>0.