Signals & Systems

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Ch. 9 Laplace Transform

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definitions, calculation

properties

system analysis

unilateral form

9.1 The Laplace Transform

$$X(\sigma + j\omega) = \int_{-\infty}^{+\infty} x(t)e^{-(\sigma + j\omega)t}\,dt,$$

Fourier Transform vs. Laplace Transform

$$X(\sigma + j\omega) = \int_{-\infty}^{+\infty} [x(t)e^{-\sigma t}]e^{-j\omega t}\,dt.$$

Fourier transform	Laplace transform
$s = j\omega$	$s = \sigma + j\omega$
$X(j\omega) \triangleq \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$	$X(s) \triangleq \int_{-\infty}^{\infty} x(t)e^{-st}dt$
$x(t) \stackrel{\mathcal{F}}{\longleftrightarrow} X(j\omega)$	$x(t) \stackrel{\mathcal{L}}{\longleftrightarrow} X(s)$
$X(j\omega) = \mathcal{F}\{x(t)\}$	$X(s) = \mathcal{L}\{x(t)\}$
$x(t) = \mathcal{F}^{-1}\{X(j\omega)\}$	$x(t) = \mathcal{L}^{-1}\{X(s)\}$

FT is the LT evaluated along the $j\omega$ axis:

$$X(s)\Big|_{s=j\omega} = \mathcal{L}\{x(t)\}\Big|_{s=j\omega} = \mathcal{F}\{x(t)\} = X(j\omega)$$

- Rational Expressions of LT with Poles/Zeros

$$X(s) = \frac{N(s)}{D(s)} \begin{array}{l} \longrightarrow \text{ roots } \longrightarrow \text{ zeros} \\ \longrightarrow \text{ roots } \longrightarrow \text{ poles} \end{array}$$

  – Pole-Zero Plots

  – specifying $X(s)$ except for a scale factor

9.3 The Inverse Laplace Transform

- Inverse Laplace Transform
  - Use partial fraction expansion

In this method, $X(s)$ is expanded into a linear combination of lower order terms so that the inverse Laplace transform of each term can be easily determined.

$$X(s) = \frac{A_1}{s+a_1} + \frac{A_2}{s+a_2} + \ldots + \frac{A_m}{s+a_m}$$

$$x(t) = A_1 e^{-a_1 t} u(t) - A_2 e^{-a_2 t} u(-t) + \ldots + x_m(t)$$

If right-sided　　If left-sided

- Examples 9.9~9.11: Effects of ROC

	$\mathrm{Re}\{s\} < -1$	$-1 < \mathrm{Re}\{s\}$
$\frac{1}{(s+1)}$	$-e^{-t}u(-t)$	$e^{-t}u(t)$

	$\mathrm{Re}\{s\} < -2$	$-2 < \mathrm{Re}\{s\}$
$\frac{1}{(s+2)}$	$-e^{-2t}u(-t)$	$e^{-2t}u(t)$

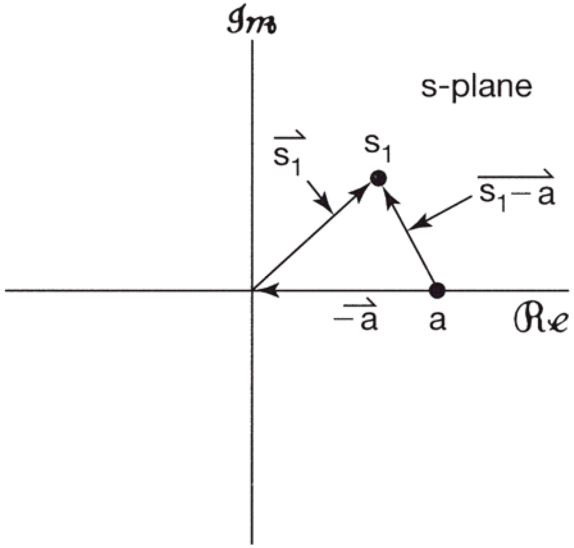
$$\mathrm{Re}\{s\} < -2 \quad \Rightarrow \quad -e^{-t}u(-t) + e^{-2t}u(-t) \xleftrightarrow{\mathcal{L}} \frac{1}{(s+1)} - \frac{1}{(s+2)}$$

$$-2 < \mathrm{Re}\{s\} < -1 \quad \Rightarrow \quad -e^{-t}u(-t) - e^{-2t}u(t) \xleftrightarrow{\mathcal{L}} \frac{1}{(s+1)} - \frac{1}{(s+2)}$$

$$-1 < \mathrm{Re}\{s\} \quad \Rightarrow \quad e^{-t}u(t) - e^{-2t}u(t) \xleftrightarrow{\mathcal{L}} \frac{1}{(s+1)} - \frac{1}{(s+2)}$$

9.4 Geometric Evaluation of the Fourier Transform

- Remarks
  - We discuss a procedure to geometrically evaluate CTFT.
  - More generally, we geometrically evaluate the Laplace transform at any set of values from the pole-zero pattern associated with a rational Laplace transform.
  - A more general rational Laplace transform consists of a product of pole and zero terms as follow:



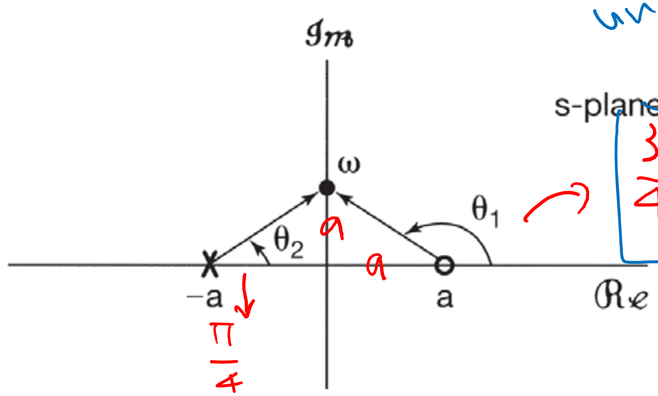
$$X(s) = M \frac{\prod_{i=l}^{R}(s-\beta_i)}{\prod_{j=l}^{P}(s-\alpha_j)}.$$

9.4.3 All-Pass Systems

Such a system is commonly referred to as an *all pass system*, since it passes all frequencies with equal gain (or attenuation).
The phase of frequency response is $\vartheta_1 - \vartheta_2$, or, since $\vartheta_1 = \pi - \vartheta_2$ with $\vartheta_2 = \tan^{-1}(\omega/a)$.

$$H(s) = \frac{s-a}{s+a}$$

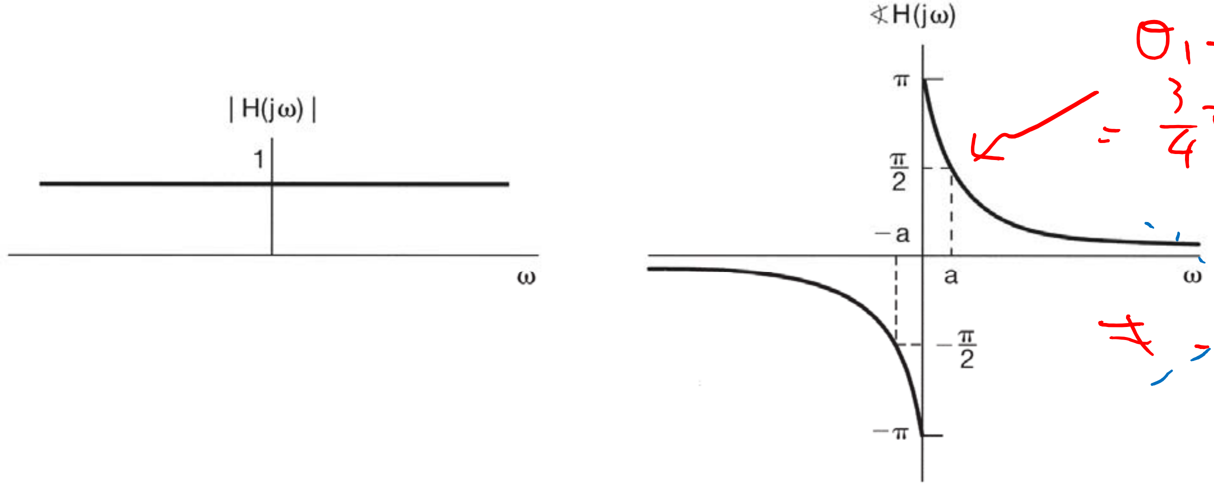
$$H(j\omega) = \frac{j\omega - a}{j\omega + a}$$



unwrapping

$\frac{3}{4}\pi$ or $-\frac{\pi}{4}$?

$\frac{\pi}{4}$



(b)

$\theta_1 - \theta_2 = \frac{3}{4}\pi - \frac{\pi}{4} = \frac{\pi}{2}$

$-\frac{\pi}{4} - \frac{\pi}{4} = -\frac{\pi}{2}$

9.5 Properties of Laplace Transform

- Differentiation in Time and s Domain

$$x(t) \xleftrightarrow{\mathcal{L}} X(s), \quad \text{ROC} = R$$

$$\frac{d}{dt}x(t) \xleftrightarrow{\mathcal{L}} sX(s), \quad \text{ROC contains } R$$

pole-zero cancellation may occur.

$$-tx(t) \xleftrightarrow{\mathcal{L}} \frac{dX(s)}{ds}, \quad \text{ROC} = R$$

Proof :

$$\frac{dx(t)}{dt} = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} X(s)\frac{de^{st}}{dt}ds = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} sX(s)e^{st}ds$$

$$\frac{dX(s)}{ds} = \frac{d}{ds}\int_{-\infty}^{\infty} x(t)e^{-st}dt = \int_{-\infty}^{\infty}(-t)x(t)e^{-st}dt$$

9.5 Properties of Laplace Transform

- Integration in Time

$$x(t) \xleftrightarrow{\mathcal{L}} X(s), \quad \text{ROC=}R$$

$$\int_{-\infty}^{t} x(\tau)d\tau \xleftrightarrow{\mathcal{L}} \frac{1}{s}X(s), \quad \text{ROC contains } R \cap \{\text{Re}\{s\} > 0\}$$

Proof :

$$\int_{-\infty}^{t} x(\tau)d\tau = u(t) * x(t)$$

From Example 9.1, $u(t) \xleftrightarrow{\mathcal{L}} \frac{1}{s}$, Re{s}>0

$\therefore \int_{-\infty}^{t} x(\tau)d\tau \xleftrightarrow{\mathcal{L}} \frac{1}{s}X(s)$, with an ROC containing the intersection of the ROC of $X(s)$ and the ROC of the LT of $u(t)$.

9.5 Properties of Laplace Transform

- The Initial Value Theorem

If $x(t)=0$ for $t<0$ and it contains no impulse or higher order singularities at the origin,

$$x(0^+)=\lim_{s\to\infty} sX(s).$$

- The Final-Value Theorem

If $x(t)=0$ for $t<0$ and $x(t)$ has a finite limit as $t\to\infty$,

$$\lim_{t\to\infty} x(t)=\lim_{s\to 0} sX(s).$$

Note: x(t) has to be causal or the two theorems cannot apply.

- Proof of the Initial Value Theorem

Given $x(t) = 0$ for $t < 0 \Rightarrow x(t) = x(t)u(t)$
By Taylor series expansion at $t = 0+$,

$$x(t) = \left[x(0+) + x^{(1)}(0+)t + \cdots + x^{(n)}(0+)\frac{t^n}{n!} + \cdots \right] u(t) \qquad \text{(Eq. 1)}$$

$$= \sum_0^\infty x^{(n)}(0+)\boxed{\frac{t^n}{n!}}$$

From Example 9.14,

$$e^{-at}\left(\frac{t^n}{n!}\right)u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{(s+a)^{n+1}}, \quad \text{Re}\{s\} > -a$$

Setting $a = 0$ and multiplying both sides by $x^{(n)}(0+)$, we have

$$x^{(n)}(0+)\left(\frac{t^n}{n!}\right)u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{x^{(n)}(0+)}{s^{n+1}}, \quad \text{Re}\{s\} > 0$$

Taking the Laplace transform of Eq. 1, we get

$$X(s) = \sum_0^\infty \frac{x^{(n)}(0+)}{s^{n+1}}$$

$$sX(s) = \boxed{x^{(0)}(0+)} + x^{(1)}(0+)/s + \cdots$$

Therefore,

$$\lim_{s\to\infty} sX(s) = x^{(0)}(0+) = x(0+)$$

- Proof of the Final Value Theorem

Since $x(t)$ is causal, $x(t) = 0$ for $t < 0$.

Since $\dfrac{dx(t)}{dt} \xleftrightarrow{\mathcal{L}} sX(s)$, however by definition,

$$sX(s) = \mathcal{L}\left\{\frac{dx(t)}{dt}\right\} = \int_{-\infty}^{\infty} x'(t)e^{-st}dt = \int_{0^-}^{\infty} x'(t)e^{-st}dt.$$

Thus

$$\lim_{s\to 0} sX(s) = \lim_{s\to 0}\int_{0^-}^{\infty} x'(t)e^{-st}dt = \int_{0^-}^{\infty} x'(t)dt = \lim_{t\to\infty} x(t) - x(0^-).$$

Since $x(0^-) = 0$, we have

$$\lim_{t\to\infty} x(t) = \lim_{s\to 0} sX(s)\ .$$

9.5 Properties of Laplace Transform

TABLE 9.1 PROPERTIES OF THE LAPLACE TRANSFORM

Section	Property	Signal	Laplace Transform	ROC
		$x(t)$ $x_1(t)$ $x_2(t)$	$X(s)$ $X_1(s)$ $X_2(s)$	R R_1 R_2
9.5.1	Linearity	$ax_1(t) + bx_2(t)$	$aX_1(s) + bX_2(s)$	At least $R_1 \cap R_2$
9.5.2	Time shifting	$x(t - t_0)$	$e^{-st_0}X(s)$	R
9.5.3	Shifting in the s-Domain	$e^{s_0 t}x(t)$	$X(s - s_0)$	Shifted version of R (i.e., s is in the ROC if $s - s_0$ is in R)
9.5.4	Time scaling	$x(at)$	$\frac{1}{\lvert a\rvert}X\left(\frac{s}{a}\right)$	Scaled ROC (i.e., s is in the ROC if s/a is in R)
9.5.5	Conjugation	$x^*(t)$	$X^*(s^*)$	R
9.5.6	Convolution	$x_1(t) * x_2(t)$	$X_1(s)X_2(s)$	At least $R_1 \cap R_2$
9.5.7	Differentiation in the Time Domain	$\frac{d}{dt}x(t)$	$sX(s)$	At least R
9.5.8	Differentiation in the s-Domain	$-tx(t)$	$\frac{d}{ds}X(s)$	R
9.5.9	Integration in the Time Domain	$\int_{-\infty}^{t} x(\tau)d(\tau)$	$\frac{1}{s}X(s)$	At least $R \cap \{\mathcal{R}e\{s\} > 0\}$

Initial- and Final-Value Theorems

9.5.10 If $x(t) = 0$ for $t < 0$ and $x(t)$ contains no impulses or higher-order singularities at $t = 0$, then

$$x(0^+) = \lim_{s \to \infty} sX(s)$$

If $x(t) = 0$ for $t < 0$ and $x(t)$ has a finite limit as $t \longrightarrow \infty$, then

$$\lim_{t \to \infty} x(t) = \lim_{s \to 0} sX(s)$$

9.6 Some LT Pairs

- Integration in Time

TABLE 9.2 LAPLACE TRANSFORMS OF ELEMENTARY FUNCTIONS

Transform pair	Signal	Transform	ROC
1	$\delta(t)$	1	All s
2	$u(t)$	$\frac{1}{s}$	$\mathcal{R}e\{s\} > 0$
3	$-u(-t)$	$\frac{1}{s}$	$\mathcal{R}e\{s\} < 0$
4	$\frac{t^{n-1}}{(n-1)!}u(t)$	$\frac{1}{s^n}$	$\mathcal{R}e\{s\} > 0$
5	$-\frac{t^{n-1}}{(n-1)!}u(-t)$	$\frac{1}{s^n}$	$\mathcal{R}e\{s\} < 0$
6	$e^{-\alpha t}u(t)$	$\frac{1}{s+\alpha}$	$\mathcal{R}e\{s\} > -\alpha$
7	$-e^{-\alpha t}u(-t)$	$\frac{1}{s+\alpha}$	$\mathcal{R}e\{s\} < -\alpha$
8	$\frac{t^{n-1}}{(n-1)!}e^{-\alpha t}u(t)$	$\frac{1}{(s+\alpha)^n}$	$\mathcal{R}e\{s\} > -\alpha$

9.6 Some LT Pairs

- Integration in Time

Transform pair	Signal	Transform	ROC
9	$-\frac{t^{n-1}}{(n-1)!}e^{-\alpha t}u(-t)$	$\frac{1}{(s+\alpha)^n}$	$\Re e\{s\} < -\alpha$
10	$\delta(t-T)$	e^{-sT}	All s
11	$[\cos\omega_0 t]u(t)$	$\frac{s}{s^2+\omega_0^2}$	$\Re e\{s\} > 0$
12	$[\sin\omega_0 t]u(t)$	$\frac{\omega_0}{s^2+\omega_0^2}$	$\Re e\{s\} > 0$
13	$[e^{-\alpha t}\cos\omega_0 t]u(t)$	$\frac{s+\alpha}{(s+\alpha)^2+\omega_0^2}$	$\Re e\{s\} > -\alpha$
14	$[e^{-\alpha t}\sin\omega_0 t]u(t)$	$\frac{\omega_0}{(s+\alpha)^2+\omega_0^2}$	$\Re e\{s\} > -\alpha$
15	$u_n(t) = \frac{d^n\delta(t)}{dt^n}$	s^n	All s
16	$u_{-n}(t) = \underbrace{u(t) * \cdots * u(t)}_{n \text{ times}}$	$\frac{1}{s^n}$	$\Re e\{s\} > 0$

9.7 Analysis & Characterization of LTI Systems Using LT

- Stability
  - An LTI system is stable if and only if the ROC of its system function $H(s)$ includes the entire $j\omega$-axis (i.e., $\mathrm{Im}\{s\} = 0$).

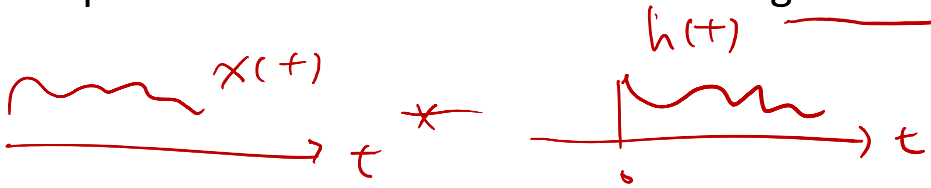
$$H(s) = \mathcal{L}\{h(t)\}$$

$$X(s) \longrightarrow \boxed{\text{LTI System}} \longrightarrow Y(s) = \boxed{H(s)}X(s)$$

CTFT exists.

$H(s)$: system (or transfer) function

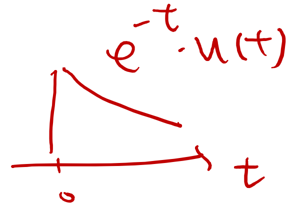
- Causality
  - The ROC associated with the system function for a causal and stable system includes a right half plane.
  - For a system with a rational system function, causality of the system is equivalent to that the ROC is to the right of the rightmost pole.



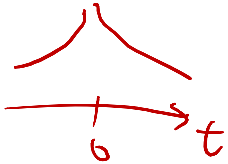
9.7 Analysis & Characterization of LTI Systems Using LT

- Examples 9.17, 9.18, & 9.19

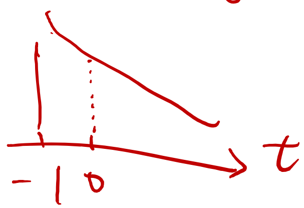
1. $h(t) = e^{-t}u(t) \xleftrightarrow{\mathcal{L}} H(s) = \dfrac{1}{s+1},$ $\quad -1 < \mathrm{Re}\{s\}$



2. $h(t) = e^{-|t|} \xleftrightarrow{\mathcal{L}} H(s) = \dfrac{-2}{s^2-1},$ $\quad -1 < \mathrm{Re}\{s\} < +1$



3. $h(t) = e^{-(t+1)}u(t+1) \xleftrightarrow{\mathcal{L}} H(s) = \dfrac{e^{s}}{s+1},$ $\quad -1 < \mathrm{Re}\{s\}$



1. $h(t)$: causal; $H(s)$: rational; ROC : right-sided

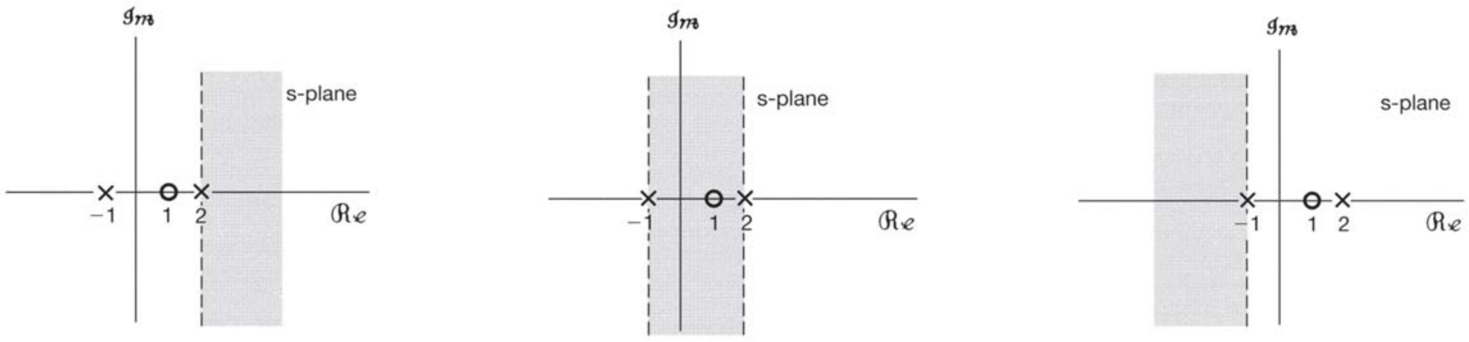
2. $h(t)$: not causal; $H(s)$: rational; ROC : not right-sided

3. $h(t)$: not causal; $H(s)$: not rational; ROC : right-sided

9.7 Analysis & Characterization of LTI Systems Using LT

- Anti-Causality
  - The ROC associated with the system function for a anti-causal system includes a left half plane.
  - For a system with a rational system function, anti-causality of the system is equivalent to the ROC being the left-half plane to the left of the leftmost pole.
- Identifying ROC based on Causality & Stability Information

$$H(s) = \frac{s-1}{(s+1)(s-2)}. = \frac{2/3}{s+1} + \frac{1/3}{s-2}$$



(a) (b) (c)

$h(t) = (\frac{2}{3}e^{-t} + \frac{1}{3}e^{2t})u(t)$ — Causal, unstable

$h(t) = \frac{2}{3}e^{-t}u(t) - \frac{1}{3}e^{2t}u(-t)$ — Non-causal, stable

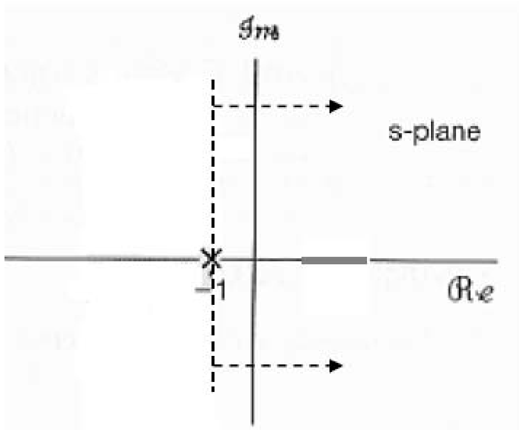
$h(t) = -(\frac{2}{3}e^{-t} + \frac{1}{3}e^{2t})u(-t)$ — Anti-causal, unstable

9.7 Analysis & Characterization of LTI Systems Using LT

- Examples 9.17 & 9.21

$$h(t) = e^{-t}u(t) \xleftrightarrow{\mathcal{L}} H(s) = \frac{1}{s+1}, \quad \mathrm{Re}\{s\} > -1$$

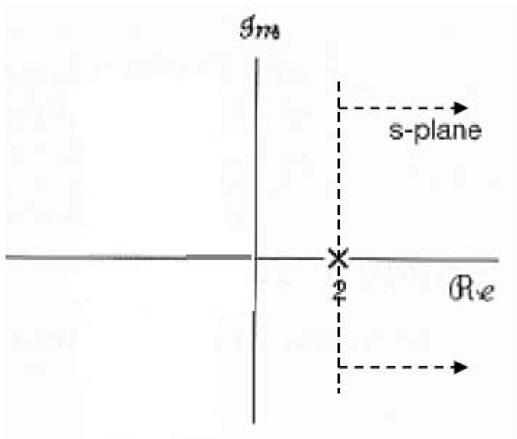
$$\begin{cases} h(t): & \text{causal} \\ H(s): & \text{stable, rational} \end{cases}$$



Consistent with time-domain analysis since h(t) is absolutely integrable and nonzero only if t>0.

$$h(t) = e^{2t}u(t) \xleftrightarrow{\mathcal{L}} H(s) = \frac{1}{s-2}, \quad \mathrm{Re}\{s\} > 2$$

$$\begin{cases} h(t): & \text{causal} \\ H(s): & \text{unstable, rational} \end{cases}$$



9.7.3 LTI Systems by Linear Constant-Coefficient Diff. Equations

- We may directly obtain the system function w/o first calculating the impulse response or time-domain solution.

$$H(s) = \mathcal{L}\{h(t)\}$$

$$X(s) \longrightarrow \boxed{\text{LTI System}} \longrightarrow Y(s) = H(s)X(s)$$

$H(s)$: system (or transfer) function

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}$$

$$\left(\sum_{k=0}^{N} a_k s^k\right) Y(s) = \left(\sum_{k=0}^{M} b_k s^k\right) X(s)$$

$$\boxed{H(s) = \frac{\left\{\sum_{k=0}^{M} b_k s^k\right\}}{\left\{\sum_{k=0}^{N} a_k s^k\right\}}}$$

- The system function for an LTI system described by a differential equation is always rational.

9.7.3 LTI Systems by Linear Constant-Coefficient Diff. Equations

- Example 9.23 Differential Equation

$$\frac{dy(t)}{dt} + 3y(t) = x(t)$$

$$sY(s) + 3Y(s) = X(s) \qquad H(s) = \frac{Y(s)}{X(s)} = \frac{1}{s+3}$$

$h(t) = e^{-3t}u(t)$ when the ROC is Re{s} > -3

$h(t) = -e^{-3t}u(-t)$ when the ROC is Re{s} < -3

9.7.3 LTI Systems by Linear Constant-Coefficient Diff. Equations

- Example 9.25 Relating system behavior to system function

$$x(t)=e^{-3t}u(t) \longrightarrow \boxed{\text{LTI System}} \longrightarrow y(t)=[e^{-t}-e^{-2t}]u(t)$$

$$X(s)=\frac{1}{s+3}, \quad \mathrm{Re}\{s\}>-3$$

$$Y(s)=\frac{1}{(s+1)(s+2)}, \quad \mathrm{Re}\{s\}>-1$$

$$\Rightarrow H(s)=\boxed{\frac{Y(s)}{X(s)}}=\frac{s+3}{(s+1)(s+2)}=\frac{s+3}{s^2+3s+2}$$

Based on the convolution property, the ROC of H(s) must include at least the intersection of the ROCs of X(s) and Y(s).

$$\Rightarrow \text{ROC: } \mathrm{Re}\{s\}>-1$$

$\Rightarrow$ Causal, becasue ROC is to the right of the rightmost pole

$\Rightarrow$ Stable, because both poles have nagative real parts

$$\Rightarrow \frac{d^2y(t)}{dt^2}+3\frac{dy(t)}{dt}+2y(t)=\frac{dx(t)}{dt}+3x(t), \quad \begin{cases} y(0^-)=0 \\ x(0^-)=0 \end{cases}$$

With the condition of initial rest, the differential equation characterizes the system.

9.7.4 Examples Relating System Behavior to System Function

- Example 9.26

Suppose that we are given the following information about an LTI system:

1. The system is causal.
2. The system function is rational and has only two poles, at $s = -2$ and $s = 4$.
3. If $x(t) = 1$, then $y(t) = 0$.
4. The value of the impulse response at $t = 0+$ is 4.

From this info we would like to determine the system function of the system.
From the first two facts, we know that the system is unstable (since it is causal and has a real/positive pole at $s = 4$) and that the system function is of the form

$$H(s) = \frac{p(s)}{(s+2)(s-4)} = \frac{p(s)}{s^2 - 2s - 8},$$

9.7.4 Examples Relating System Behavior to System Function

- Example 9.26

Suppose that we are given the following information about an LTI system:

1. The system is causal.
2. The system function is rational and has only two poles, at $s = -2$ and $s = 4$.
3. If $x(t) = 1$, then $y(t) = 0$.
4. The value of the impulse response at $t = 0+$ is 4.

$$H(s) = \frac{p(s)}{(s+2)(s-4)} = \frac{p(s)}{s^2 - 2s - 8},$$

where $p(s)$ is a polynomial in s.

$s=0 \rightarrow$ $y(t) = H(s)\,e^{st} \rightarrow 0$

Because the response $y(t)$ to the input $x(t) = 1 = e^{0 \cdot t}$ must equal $H(0) \cdot e^{0 \cdot t} = H(0)$. $= y(t)$
We conclude, from fact #3, that $p(0) = 0$, i.e., that $p(s)$ must have a root at $s = 0$ and thus is of the form

$$p(s) = sq(s),$$

where $q(s)$ is another polynomial in s.

9.7.4 Examples Relating System Behavior to System Function

- Example 9.26

Suppose that we are given the following information about an LTI system:

1. The system is causal.
2. The system function is rational and has only two poles, at $s = -2$ and $s = 4$.
3. If $x(t) = 1$, then $y(t) = 0$.
4. The value of the impulse response at $t = 0+$ is 4.

$$H(s) = \frac{p(s)}{(s+2)(s-4)} = \frac{p(s)}{s^2 - 2s - 8}, \qquad p(s) = sq(s). = 4s$$

Finally, from fact 4 and the initial-value theorem in Section 9.5.10, we see that

$$\lim_{s\to\infty} sH(s) = \lim_{s\to\infty} \frac{s^2 \overset{\overset{k}{\shortparallel}}{q(s)}}{s^2 - 2s - 8} = 4. \qquad (9.138)$$

$$\shortparallel$$

$$\lim_{t\to 0^+} h(t)$$

9.7.4 Examples Relating System Behavior to System Function

- Example 9.26

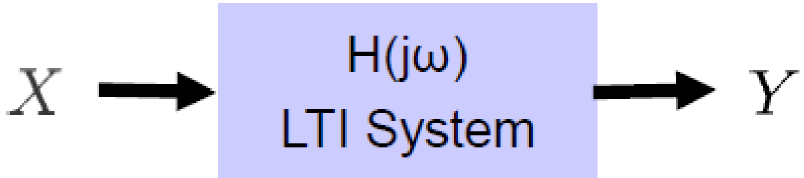
$$H(s) = \frac{p(s)}{(s+2)(s-4)} = \frac{p(s)}{s^2-2s-8}, \qquad p(s) = sq(s),$$

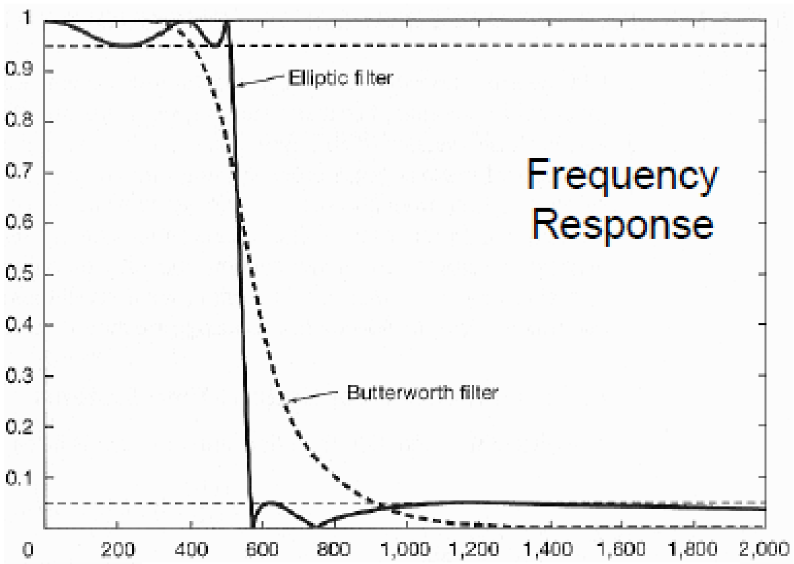
$$\lim_{s\to\infty} sH(s) = \lim_{s\to\infty} \frac{s^2 q(s)}{s^2-2s-8} = 4.$$

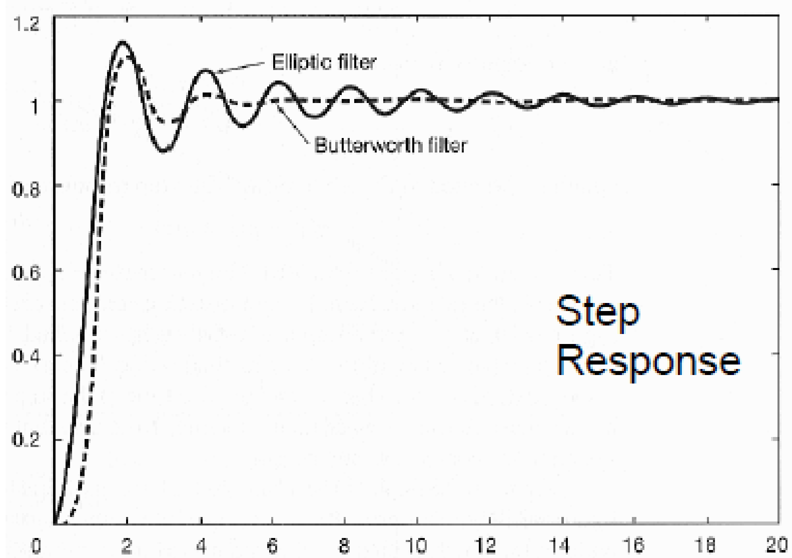
- As $s \to \infty$, the terms of highest power in s in both the numerator and the denominator of $sH(s)$ dominate and thus are the only ones of importance in evaluating eq. (9.138).
- Consequently, we can obtain a finite nonzero value for the limit only if the degree of the numerator *of* $sH(s)$ is the same as the degree of the denominator.
- Since the degree of the denominator is 2, we conclude that, for eq. (9.138) to hold, $q(s)$ must be a constant, i.e., $q(s) = K$. We can evaluate this constant by evaluating

$$\boxed{\lim_{s\to\infty} \frac{Ks^2}{s^2-2s-8}} = \lim_{s\to\infty} \frac{Ks^2}{s^2} = \underset{\substack{\| \\ 4}}{K}. \quad \Rightarrow \quad H(s) = \frac{4s}{(s+2)(s-4)}.$$

9.7.5 Butterworth Filters







9.7.5 Butterworth Filters

Given $|B(j\omega)|^2 = \dfrac{1}{1+(j\omega/j\omega_c)^{2N}}$, where N is the order of a Butterworth filter, determine the system function $B(s)$.

$$|B(j\omega)|^2 = B(j\omega)B^*(j\omega) = B(j\omega)B(-j\omega) = \frac{1}{1+(j\omega/j\omega_c)^{2N}}$$

Suppose the impulse response is real $\Rightarrow B^*(j\omega) = B(-j\omega)$

Becasue $B(j\omega) = B(s)|_{s=j\omega}$, we have

$$B(s)B(-s) = \boxed{\frac{1}{1+(s/j\omega_c)^{2N}}}$$

$\Rightarrow s = +j\omega_c\,(-1)^{1/2N}$

$j \to e^{j\frac{\pi}{2}}$, $\quad (-1) \to \left\{e^{j(\pi+2k\pi)}\right\}^{1/2N}$

which has roots at $s_p = (-1)^{1/2N}(j\omega_c)$.

$= \omega_c \cdot e^{j\left\{\frac{(2k+1)\pi}{2N}+\frac{\pi}{2}\right\}}$

$\Rightarrow |s_p| = \omega_c,\quad \measuredangle s_p = \dfrac{\pi(2k+1)}{2N}+\dfrac{\pi}{2},\quad k$ an integer

$$\Rightarrow s_p = \omega_c e^{j\left[\frac{\pi(2k+1)}{2N}+\frac{\pi}{2}\right]}$$

$\therefore |s_p| = \omega_c$

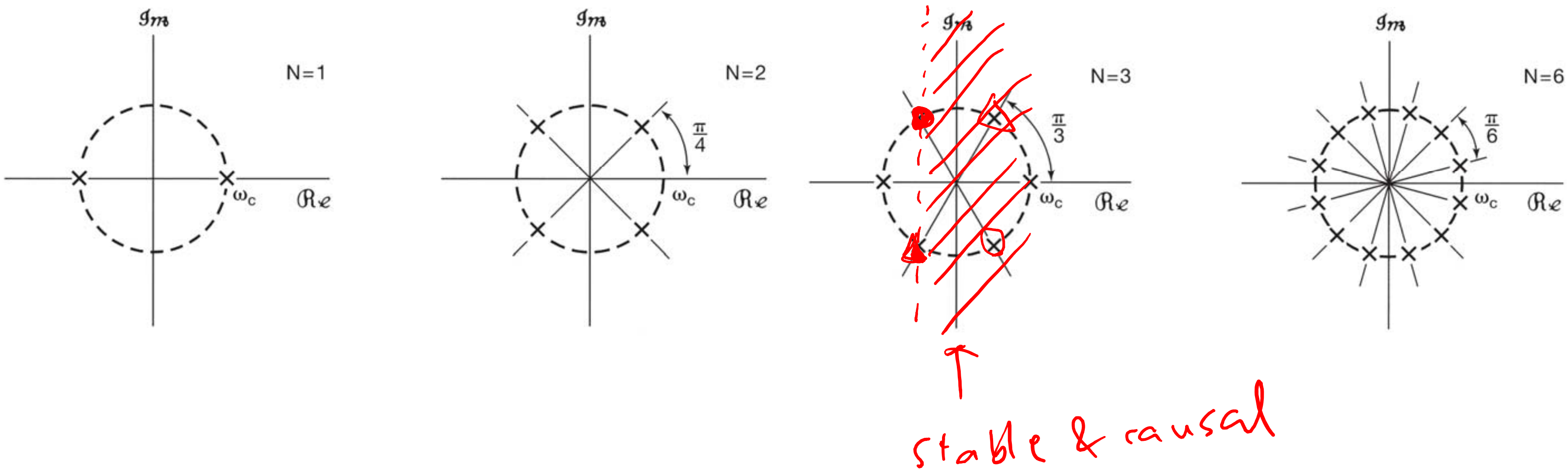
$\measuredangle s_p = \dfrac{(2k+1)\pi}{2N} + \dfrac{\pi}{2}$

9.7.5 Butterworth Filters

$$B(s)B(-s) = \frac{1}{1+(s/j\omega_c)^{2N}}$$

$$s_p = \omega_c e^{j\left[\frac{\pi(2k+1)}{2N}+\frac{\pi}{2}\right]}$$

- Properties
  - There are $2N$ poles equally spaced in angle on a circle of radius ω_c in the s-plane.
  - A pole never lies on $j\omega$-axis and occurs on σ-axis for N odd, but not for N even.
  - The angular spacing between the poles of $B(s)B(-s)$ is π/N radians.



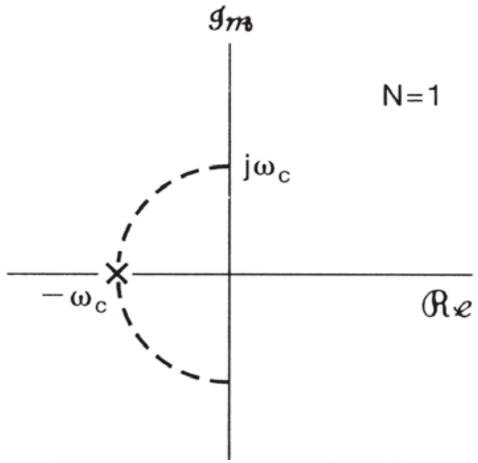
9.7.5 Butterworth Filters

$$B(s)B(-s) = \frac{1}{1+(s/j\omega_c)^{2N}}$$

$$s_p = \omega_c e^{j\left[\frac{\pi(2k+1)}{2N}+\frac{\pi}{2}\right]}$$

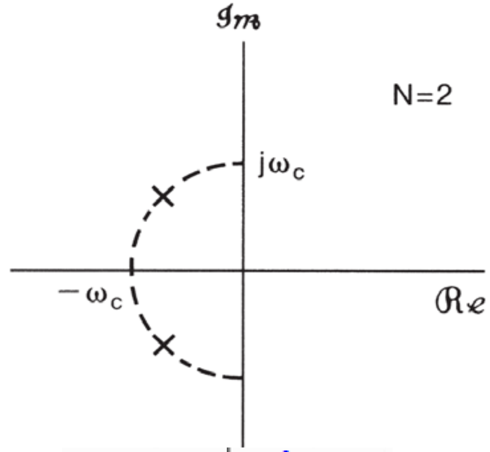
- Constructing B(s)
  - Poles of $B(s)B(-s)$ appear in pairs
    $\Rightarrow$ choose one pole from each pair to construct $B(s)$
  - For the system to be stable and causal,
    $\Rightarrow$ all poles of $B(s)$ should be in the left half plane
  - The pole locations only specify $B(s)$ up to a scale factor.

  Since $B^2(s)|_{s=0} = 1$, we apply it to fix the scale factor.



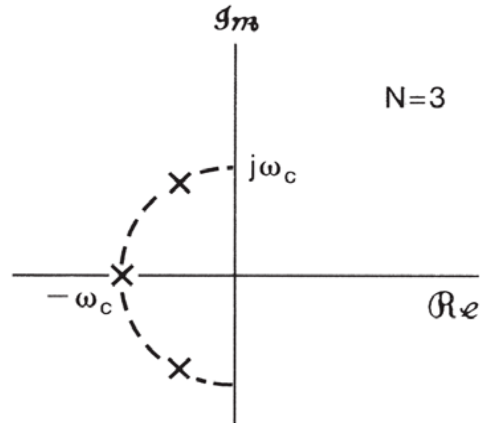
$$B(s) = \frac{\omega_c}{s+\omega_c}$$

$$\frac{dy(t)}{dt} + \omega_c y(t) = \omega_c x(t)$$



$$B(s) = \frac{\omega_c^2}{s^2+\sqrt{2}\omega_c s+\omega_c^2}$$

$$\frac{d^2y(t)}{dt^2} + \sqrt{2}\omega_c \frac{dy(t)}{dt} + \omega_c^2 y(t) = \omega_c^2 x(t)$$



$$B(s) = \frac{\omega_c^3}{s^3+2\omega_c s^2+2\omega_c^2 s+\omega_c^3}$$

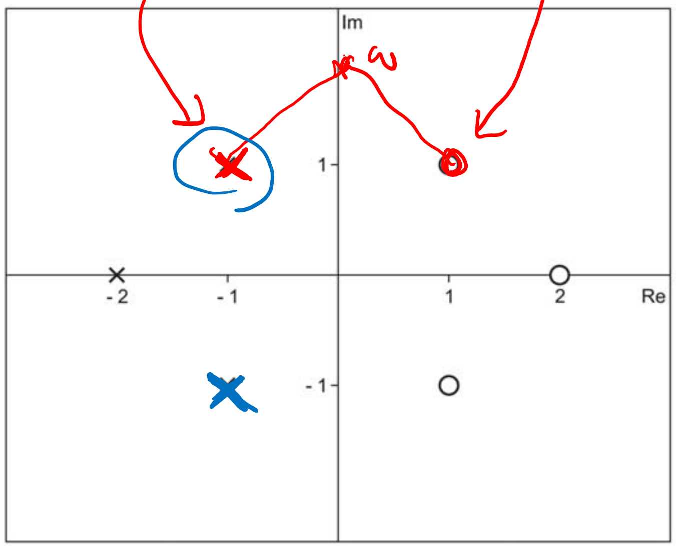
$$\frac{d^3y(t)}{dt^3} + 2\omega_c \frac{d^2y(t)}{dt^2} + 2\omega_c^2 \frac{dy(t)}{dt} + \omega_c^3 y(t) = \omega_c^3 x(t)$$

9.8 All-Pass System, Min. Phase System, and Spectral Factorization*

- Revisit of All-Pass System
  - Assuming that the transfer function H(s) is rational in s, it will have the following form:

$$\boxed{H_{ap}(s) = A\prod_{k=1}^{M}\frac{s+a_k^*}{s-a_k}.} \quad (9.155)$$

  - For an all-pass system, it there is a pole at s = $\sigma_k + j\omega_k$, then there should be a zero at s = $-\sigma_k + j\omega_k$.
  - For a real all-pass system, if there is a pole at s = $\sigma_k + j\omega_k$, then there should also be a pole at s = $\sigma_k - j\omega_k$ and a zero at s = $-\sigma_k + j\omega_k$ and s = $-\sigma_k - j\omega_k$.



9.8 All-Pass System, Min. Phase System, and Spectral Factorization*

- Min. Phase Systems
  - In classical network theory, control systems, and signal processing, a CT LTI system with a rational transfer function is defined as minimum phase if it is stable, causal, and has all its finite zeros strictly within the left-half plane.

- Example 9.28

Consider the causal, stable system with transfer function $H_{cs}(s) = \dfrac{(s-1)(s+2)}{(s+3)(s+4)}$.

Since it has a zero in the right half-plane, specifically at $s = 1$, it is not minimum phase. However, consider the cascade of $H_{cs}(s)$ with an identity factor $\boxed{\dfrac{(s+1)}{(s+1)}}$ to express $H_{cs}(s)$ as

$$H_{cs}(s) = \frac{(s-1)(s+2)}{(s+3)(s+4)} \cdot \frac{(s+1)}{(s+1)}$$

or equivalently,

$$H_{cs}(s) = \frac{(s+1)(s+2)}{(s+3)(s+4)} \cdot \frac{(s-1)}{(s+1)}$$

9.8 All-Pass System, Min. Phase System, and Spectral Factorization*

- Min. Phase Systems
  - In classical network theory, control systems, and signal processing, a CT LTI system with a rational transfer function is defined as minimum phase if it is stable, causal, and has all its finite zeros strictly within the left-half plane.

- Example 9.28

$$H_{cs}(s) = \frac{(s+1)(s+2)}{(s+3)(s+4)} \cdot \frac{(s-1)}{(s+1)}$$

Note that

$$H_{\min}(s) = \frac{(s+1)(s+2)}{(s+3)(s+4)}$$

and

$$H_{ap}(s) = \frac{(s-1)}{(s+1)}.$$

Since all of the poles and the zeros of $H_{min}(s)$ are in the left-half plane and the zero of the pole of $H_{ap}(s)$ are symmetric with respect to the Im(s) axis, $H_{min}(s)$ and $H_{ap}(s)$ are the Laplace transforms of the impulse responses of a minimum-phase filter and an all-pass filter, respectively.

9.8 All-Pass System, Min. Phase System, and Spectral Factorization*

We discuss how to retrieve the transfer function $H(s)$ of a stable, real, and causal system if $|H(j\omega)|$ or $|H(s)|$ is known.

$$|H(j\omega)|^2 = H(j\omega)\, H^*(j\omega)$$

or, since when $h(t)$ is real, $H^*(j\omega) = H(-j\omega)$,

$$|H(j\omega)|^2 = H(j\omega)H(-j\omega).$$

Therefore,

$$|H(j\omega)|^2 = H(\mathrm{s})H(-s)|_{s=j\omega}.$$

9.8 All-Pass System, Min. Phase System, and Spectral Factorization*

- Example 9.29

Consider a frequency-response magnitude that has been measured or approximated as

$$|H(j\omega)|^2 = \frac{\omega^2 + 1}{\omega^4 + 13\omega^2 + 36} = \frac{\omega^2 + 1}{(\omega^2 + 4)(\omega^2 + 9)}.$$

Making the substitution $\omega^2 = -s^2$, we obtain

$$H(s)H(-s) = \frac{-s^2 + 1}{(-s^2 + 4)(-s^2 + 9)}$$

which we further factor it as

$$\boxed{H(s)}H(-s) = \frac{(s + 1)(-s + 1)}{(s + 2)(-s + 2)(s + 3)(-s + 3)}.$$

9.8 All-Pass System, Min. Phase System, and Spectral Factorization*

- Example 9.29 (cont'd)

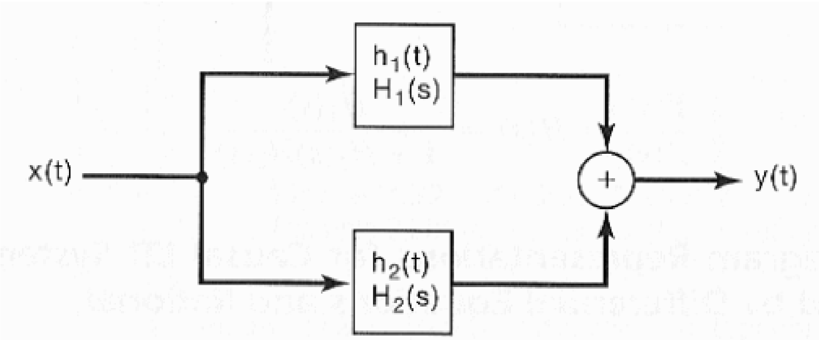
$$H(s)H(-s) = \frac{(s+1)(-s+1)}{(s+2)(-s+2)(s+3)(-s+3)}.$$

It now remains to associate appropriate factors with $H(s)$ and $H(-s)$.
Assuming that the system is causal in addition to being stable, the two left half-plane poles at $s = -2$ and $s = -3$ must be associated with $H(s)$. With no further assumptions, either one of the numerator factors can be associated with $H(s)$ and the other with $H(-s)$. Therefore, we have

$$H(s) = \frac{(s+1)}{(s+2)(s+3)} \quad \text{or} \quad H(s) = \frac{(-s+1)}{(s+2)(s+3)}.$$

9.9 System Function Algebra and Block Diagram Representations

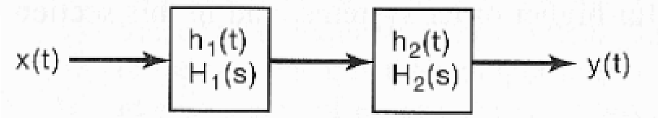
- System Function for Interconnected LTI Systems



Parallel Interconnection

$$h(t) = h_1(t) + h_2(t)$$

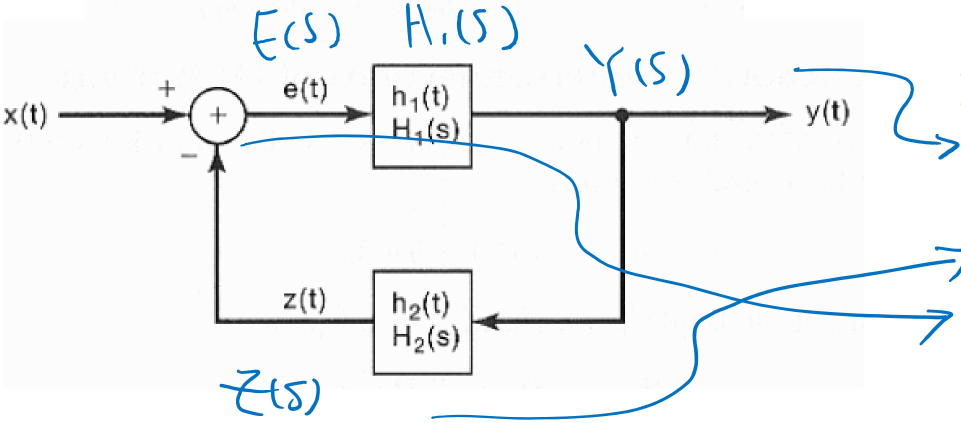
$$H(s) = H_1(s) + H_2(s)$$



Series Interconnection

$$h(t) = h_1(t) * h_2(t)$$

$$H(s) = H_1(s)H_2(s)$$



Feedback Interconnection

$$Y(s) = H_1(s)E(s)$$

$$Z(s) = H_2(s)Y(s)$$

$$\underline{E(s) = X(s) - Z(s)}$$

$$H(s) = \frac{H_1(s)}{1 + H_1(s)H_2(s)}$$

9.9 System Function Algebra and Block Diagram Representations

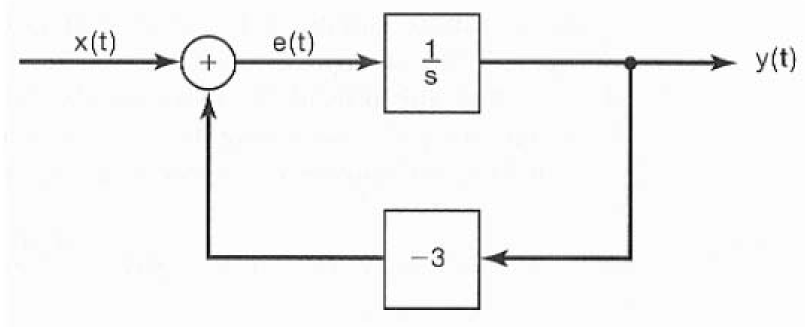
- Example 9.30
  Block diagram construction

$$H(s)=\frac{1}{s+3}$$

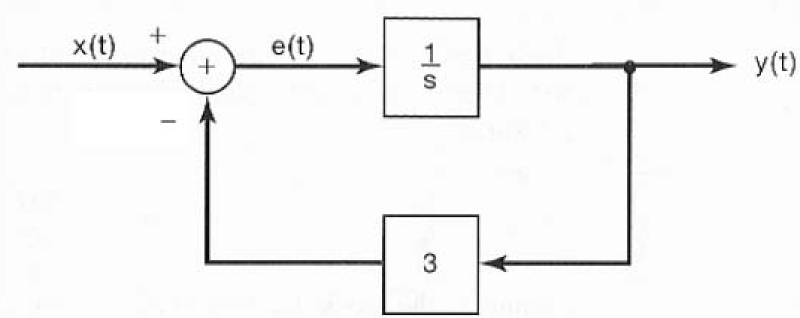
$$Y(s)=\frac{1}{s+3}X(s)$$

$$\frac{d}{dt}y(t)+3y(t)=x(t)$$

$$\frac{d}{dt}y(t)=x(t)-3y(t)$$



(a)



(b)

9.9 System Function Algebra and Block Diagram Representations

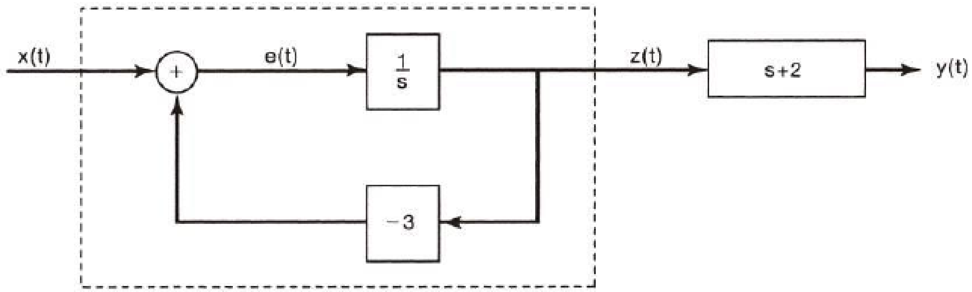
- Example 9.31
  Block diagram representation

$$H(s) = \frac{s+2}{s+3}$$

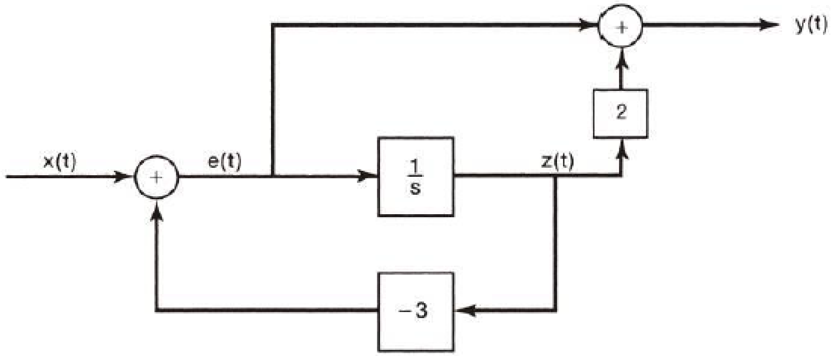
$$= \left(\frac{1}{s+3}\right)(s+2)$$

$$\Rightarrow Z(s) \triangleq \frac{1}{s+3} X(s)$$

$$Y(s) = (s+2)Z(s)$$



(a)



(b)

9.9 System Function Algebra and Block Diagram Representations

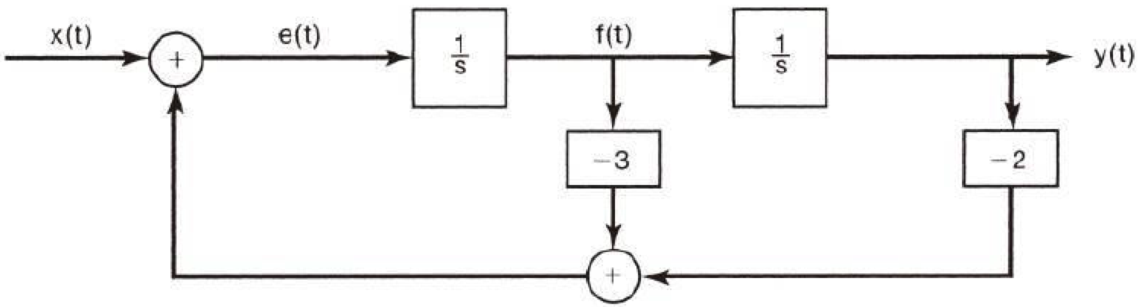
- Example 9.32
  Block diagram representation

$$H(s) = \frac{1}{(s+1)(s+2)} = \frac{1}{s^2+3s+2}$$

$$\Rightarrow s^2Y + 3sY + 2Y = X$$

$$\Rightarrow \begin{cases} F = sY \\ E = sF = s^2Y \end{cases}$$

$$\Rightarrow E = s^2Y = -3F - 2Y + X$$

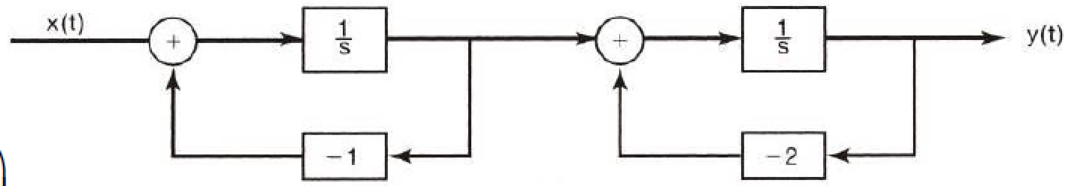


direct form

9.9 System Function Algebra and Block Diagram Representations

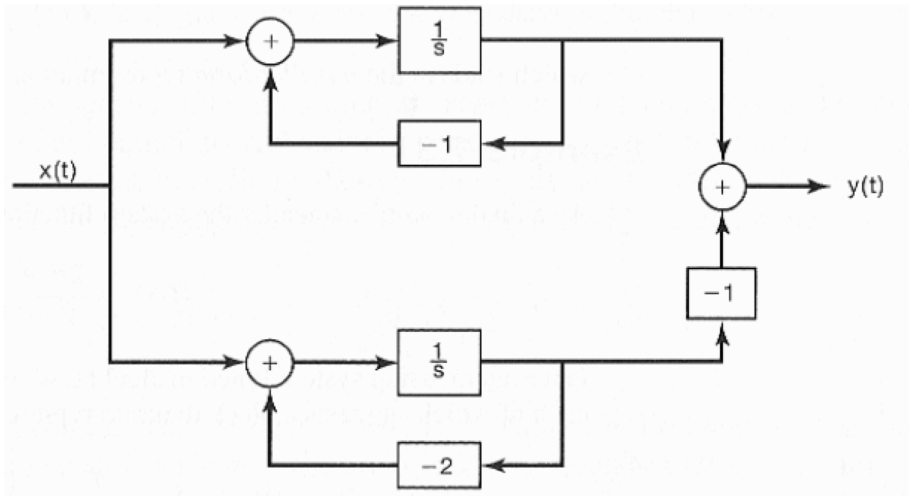
- Example 9.32
  Block diagram representation (cont'd)

$$H(s) = \frac{1}{(s+1)(s+2)} = \left(\frac{1}{s+1}\right)\left(\frac{1}{s+2}\right)$$



cascade form

$$H(s) = \frac{1}{(s+1)(s+2)} = \left(\frac{1}{s+1}\right) + \left(\frac{-1}{s+2}\right)$$



parallel form

9.10 The Unilateral Laplace Transform

- Bilateral vs. Unilateral Laplace Transform
  - The difference between bilateral and unilateral LT is in the lower limit of the integration.

Bilateral LT

$$X(s) \triangleq \int_{-\infty}^{\infty} x(t)e^{-st}\,dt$$

$$x(t) \overset{\mathcal{L}}{\longleftrightarrow} X(s)$$

Unilateral LT

$$\mathcal{X}(s) \triangleq \int_{0^-}^{\infty} x(t)e^{-st}\,dt$$

$$x(t) \overset{\mathcal{UL}}{\longleftrightarrow} \mathcal{X}(s)$$

ROC: always a right half plane

  - Note that the lower limit in unilateral LT signifies that we include in the interval of integration any impulses or higher order singularity functions concentrated at t = 0.

9.10 The Unilateral Laplace Transform

- Example 9.34

$$x(t) = \frac{t^{n-1}}{(n-1)!} e^{-at} u(t). \quad \Longleftrightarrow \quad \mathcal{X}(s) = \frac{1}{(s+a)^n}, \qquad \Re e\{s\} > -a.$$

- Example 9.35

$$x(t) = e^{-a(t+1)} u(t+1). \quad \Longleftrightarrow \quad X(s) = \frac{e^s}{s+a}, \qquad \Re e\{s\} > -a.$$

$$\Rightarrow \quad \mathcal{X}(s) = \int_{0^-}^{\infty} e^{-a(t+1)} u(t+1) e^{-st}\, dt$$

$$= \int_{0^-}^{\infty} e^{-a} e^{-t(s+a)}\, dt$$

$$= e^{-a} \frac{1}{s+a}, \qquad \Re e\{s\} > -a.$$

We should recognize (*s*) as the bilateral transform not of $x(t)$, but of $x(t)u(t)$.

9.10 The Unilateral Laplace Transform

- Example 9.36

$$x(t) = \delta(t) + 2u_1(t) + e^t u(t).$$

$$\Updownarrow$$

$$\mathfrak{X}(s) = X(s) = 1 + 2s + \frac{1}{s-1} = \frac{s(2s-1)}{s-1}, \qquad \Re e\{s\} > 1.$$

9.10 The Unilateral Laplace Transform

- Example 9.38

$$\mathcal{X}(s) = \frac{s^2 - 3}{s + 2}. \Rightarrow \mathcal{X}(s) = A + Bs + \frac{C}{s+2}.$$

$$\Updownarrow$$

$$s^2 - 3 = (A + Bs)(s + 2) + C, \Rightarrow \mathcal{X}(s) = -2 + s + \frac{1}{s+2},$$

$$\Updownarrow$$

$$x(t) = -2\delta(t) + u_1(t) + e^{-2t}u(t) \quad \text{for} \quad t > 0^-.$$

9.10 The Unilateral Laplace Transform

- Properties of Unilateral LT

TABLE 9.3 PROPERTIES OF THE UNILATERAL LAPLACE TRANSFORM

Property	Signal	Unilateral Laplace Transform
	$x(t)$ $x_1(t)$ $x_2(t)$	$\mathcal{X}(s)$ $\mathcal{X}_1(s)$ $\mathcal{X}_2(s)$
Linearity	$ax_1(t) + bx_2(t)$	$a\mathcal{X}_1(s) + b\mathcal{X}_2(s)$
Shifting in the s-domain	$e^{s_0 t}x(t)$	$\mathcal{X}(s - s_0)$
Time scaling	$x(at), \quad a > 0$	$\frac{1}{a}\mathcal{X}\left(\frac{s}{a}\right)$
Conjugation	$x^*(t)$	$X^*(s^*)$
Convolution (assuming that $x_1(t)$ and $x_2(t)$ are identically zero for $t < 0$)	$x_1(t) * x_2(t)$	$\mathcal{X}_1(s)\mathcal{X}_2(s)$

9.10 The Unilateral Laplace Transform

- Properties of Unilateral LT

Differentiation in the time domain	$\frac{d}{dt}x(t)$	$s\mathcal{X}(s) - x(0^-)$
Differentiation in the s-domain	$-tx(t)$	$\frac{d}{ds}\mathcal{X}(s)$
Integration in the time domain	$\int_{0^-}^{t} x(\tau)\,d\tau$	$\frac{1}{s}\mathcal{X}(s)$

Initial- and Final-Value Theorems

If $x(t)$ contains no impulses or higher-order singularities at $t = 0$, then

$$x(0^+) = \lim_{s\to\infty} s\mathcal{X}(s)$$

$$\lim_{t\to\infty} x(t) = \lim_{s\to 0} s\mathcal{X}(s)$$

Recall: 9.5 Properties of Laplace Transform

- Differentiation in Time and s Domain

$$x(t) \xleftrightarrow{\mathcal{L}} X(s), \quad \text{ROC} = R$$

$$\frac{d}{dt}x(t) \xleftrightarrow{\mathcal{L}} sX(s), \quad \text{ROC contains } R$$

pole-zero cancellation may occur.

$$-tx(t) \xleftrightarrow{\mathcal{L}} \frac{dX(s)}{ds}, \quad \text{ROC} = R$$

$$\text{Proof :}$$

$$\frac{dx(t)}{dt} = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} X(s)\frac{de^{st}}{dt}ds = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} sX(s)e^{st}ds$$

$$\frac{d}{ds}\int_{-\infty}^{\infty} x(t)e^{-st}dt = \int_{-\infty}^{\infty}(-t)x(t)e^{-st}dt$$

9.10 The Unilateral Laplace Transform

Integration by parts

$$\int u\frac{dv}{dx}\,dx = u\,v - \int v\frac{du}{dx}dx$$

- Differential Properties

$$\mathcal{UL}\left\{\frac{dx(t)}{dt}\right\} = \int_{0^-}^{\infty}\frac{dx(t)}{dt}e^{-st}dt = \int_{0^-}^{\infty}[\frac{d}{dt}(x(t)e^{-st}) + sx(t)e^{-st}]dt$$

$$y = u\,v$$

$$= x(t)e^{-st}\Big|_{0^-}^{\infty} + s\int_{0^-}^{\infty}x(t)e^{-st}dt$$

$$\frac{dy}{dx} = \frac{d(uv)}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}.$$

$$= 0 - x(0^-) + s\mathcal{X}(s)$$

$$u\frac{dv}{dx} = \frac{d(uv)}{dx} - v\frac{du}{dx}.$$

$$= s\mathcal{X}(s) - x(0^-)$$

$$\int u\frac{dv}{dx}\,dx = \int\frac{d(uv)}{dx}dx - \int v\frac{du}{dx}dx.$$

$$\mathcal{UL}\left\{\frac{d^2x(t)}{dt^2}\right\} = \int_{0^-}^{\infty}\frac{d^2x(t)}{dt^2}e^{-st}dt = s^2\mathcal{X}(s) - sx(0^-) - x'(0^-)$$

9.10 The Unilateral Laplace Transform

- Example 9.39

$$\frac{d^2 y(t)}{dt^2} + 3\frac{dy(t)}{dt} + 2y(t) = x(t)$$

$$x(t) = \alpha u(t); \quad \text{initial conditions: } y(0^-) = \beta, \quad y'(0^-) = \gamma$$

$$\Rightarrow s^2\mathcal{Y}(s) - \beta s - \gamma + 3s\mathcal{Y}(s) - 3\beta + 2\mathcal{Y}(s) = \frac{\alpha}{s}$$

$$\Rightarrow \mathcal{Y}(s) = \underbrace{\frac{\beta(s+3)}{(s+1)(s+2)} + \frac{\gamma}{(s+1)(s+2)}}_{\substack{\text{Zero-input response} \\ (\alpha = 0)}} + \underbrace{\frac{\alpha}{s(s+1)(s+2)}}_{\substack{\text{zero-state response} \\ (\beta = \gamma = 0)}}$$

The overall response is the superposition of the zero-input response and the zero-state response.

$$\Rightarrow \mathcal{Y}(s) = \frac{1}{s} - \frac{1}{s+1} + \frac{3}{s+2}$$

with $\alpha = 2, \beta = 3,$ and $\gamma = -5$

$$\Rightarrow y(t) = [1 - e^{-t} + 3e^{-2t}]\, u(t), \text{ for } t > 0$$

9.10 The Unilateral Laplace Transform

- Initial-Value Theorem for Unilateral LT

$$x(0^+) = \lim_{s \to \infty} s\mathcal{X}(s)$$

Applies only when the order of the numerator polynomial of $X(s)$ is smaller than that of the denominator polynomial.

- Final-Value Theorem

$$\lim_{t \to \infty} x(t) = \lim_{s \to 0} s\mathcal{X}(s)$$

Applies only if all the poles of X(s) are in the left half of the s-plane, with at most a single pole at s=0.