Signals & Systems

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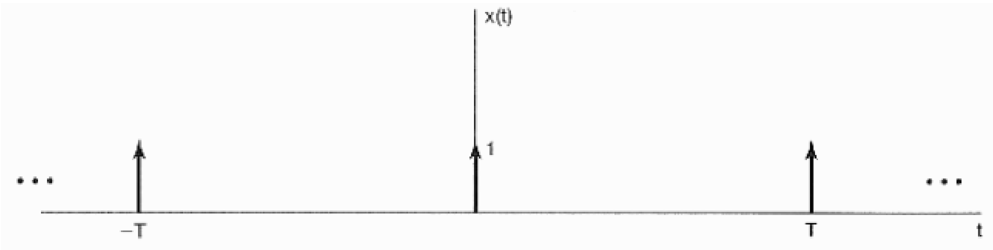
Ch. 7 Sampling

- Sec. 7.1 Representation of a CT Signal by Its Samples: The Sampling Theorem
- Sec. 7.2 Reconstruction of a Signal from Its Samples Using Interpolation
- Sec. 7.3 The Effect of Undersampling: Aliasing
- Sec. 7.4 DT Processing of CT Signals
- Sec. 7.5 Sampling of DT Signals — sampling for DT signal
- Sec. 7.6* Computing CTFT by DTFS — DFT computation

sampling for CT signals

Recall: FS & FT of an Impulse Train

$$FS\begin{cases} x(t)=\sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t} \\ a_k=\frac{1}{T}\int_{-T/2}^{T/2} x(t)e^{-jk\omega_0 t}dt \end{cases} \qquad FT\begin{cases} x(t)=\frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)e^{j\omega t}d\omega \\ X(j\omega)=\int_{-\infty}^{+\infty} x(t)e^{-j\omega t}dt \end{cases}$$



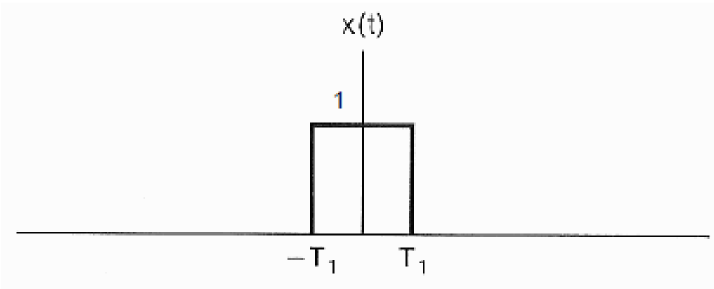
(a)

$$x(t)=\sum_{k=-\infty}^{\infty}\delta(t-kT) \xleftrightarrow{FS} a_k=\frac{1}{T}\int_{-T/2}^{T/2}\delta(t)e^{-jk2\pi t/T}dt=\frac{1}{T}$$

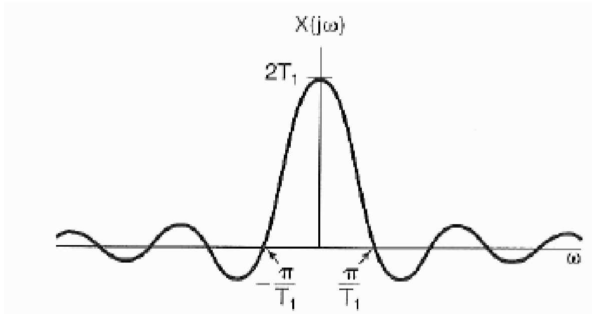
$$x(t)=\sum_{k=-\infty}^{\infty}\delta(t-kT) \xleftrightarrow{\mathcal{F}} X(j\omega)=\sum_{k=-\infty}^{+\infty}2\pi a_k\delta(\omega-k\omega_0)=\frac{2\pi}{T}\sum_{k=-\infty}^{+\infty}\delta(\omega-k\omega_0)$$

Recall: FT of a Square Wave

$$FT\begin{cases} x(t)=\dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{+\infty}X(j\omega)e^{j\omega t}d\omega \\ X(j\omega)=\displaystyle\int_{-\infty}^{+\infty}x(t)e^{-j\omega t}dt \end{cases}$$



(a)



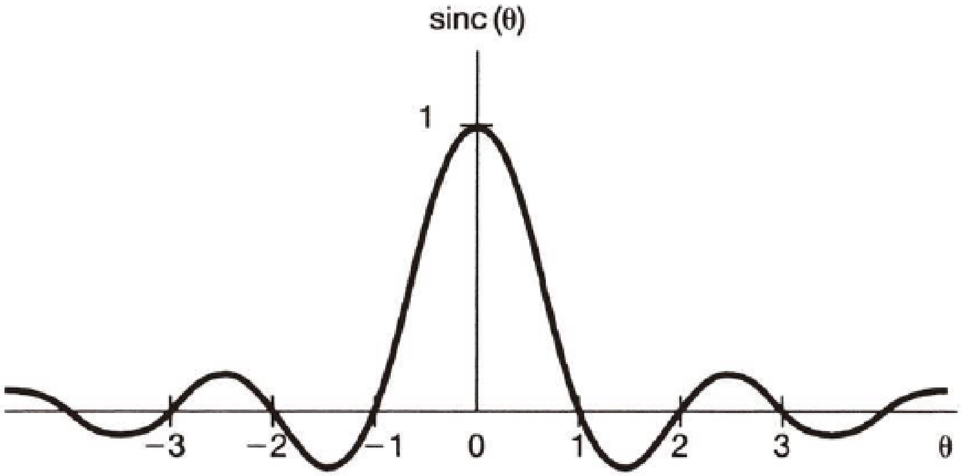
(b)

$$x(t)=\begin{cases}1, & |t|<T_1\\ 0, & |t|>T_1\end{cases} \xleftrightarrow{\mathcal{F}} X(j\omega)=\int_{-T_1}^{T_1}e^{-j\omega t}dt=\frac{1}{-j\omega}e^{-j\omega t}\Big|_{-T_1}^{T_1}$$

$$=\frac{1}{j\omega}(e^{j\omega T_1}-e^{-j\omega T_1})=2\frac{\sin(\omega T_1)}{\omega}$$

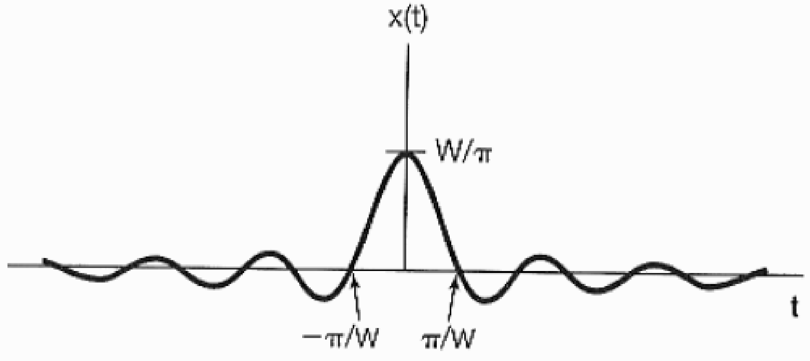
Recall: Sinc Signal

$$\mathrm{sinc}(\theta) = \frac{\sin(\pi\theta)}{\pi\theta}$$

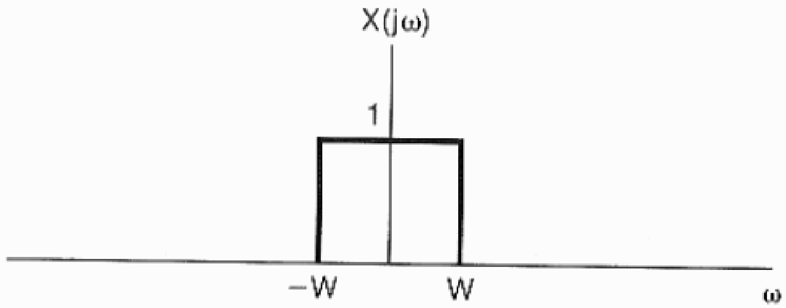


Recall: FT of a Sinc Signal

$$FT\begin{cases} x(t)=\dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{+\infty}X(j\omega)e^{j\omega t}d\omega \\ X(j\omega)=\displaystyle\int_{-\infty}^{+\infty}x(t)e^{-j\omega t}dt \end{cases}$$



(b)



(a)

$$x(t)=\frac{\sin(Wt)}{\pi t}=\frac{W}{\pi}\mathrm{sinc}\left(\frac{Wt}{\pi}\right)\xleftrightarrow{\mathcal{F}}X(j\omega)=\begin{cases}1, & |\omega|<W\\ 0, & |\omega|>W\end{cases}$$

7.1 Representation of a CT Signal by Its Samples: The Sampling Theorem

In Figure 7.1 we illustrate three different continuous-time signals, all of which have identical values at integer multiples of *T*; that is,

$$x_1(kT) = x_2(kT) = x_3(kT).$$

Sampling theorem:
for band-limited CT signals, the samples are taken uniquely and sufficiently close to each other, so that the CT signal can be reconstructed accordingly.

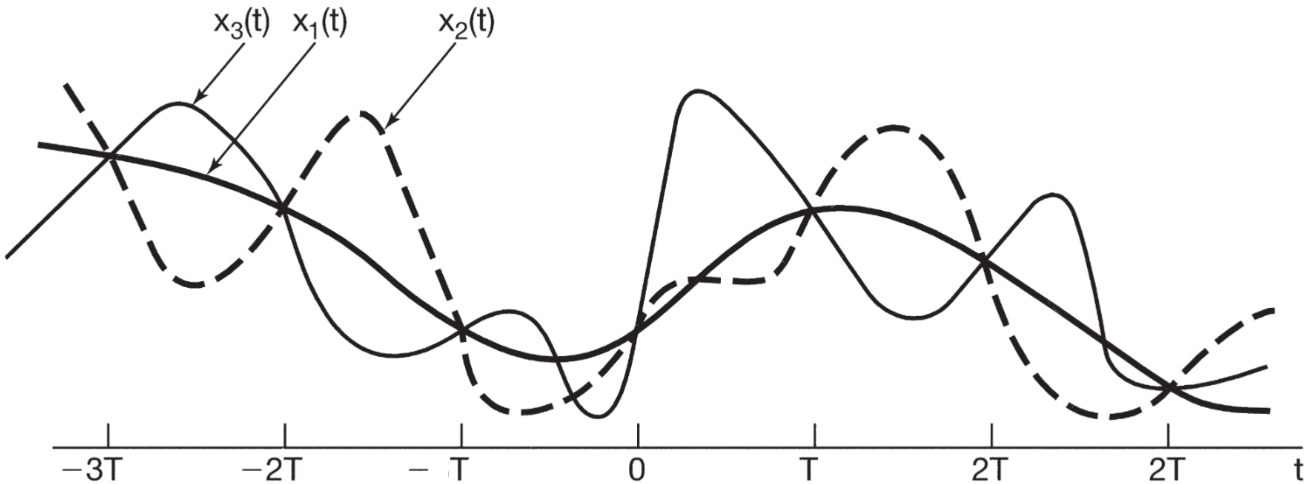


Figure 7.1 Three continuous-time signals with identical values at integer multiples of *T*.

7.1 Representation of a CT Signal by Its Samples: The Sampling Theorem

Model of Sampling: Impulse Modulation

$p(t)$: sampling function

T: sampling period

$\omega_s = \dfrac{2\pi}{T}$: sampling frequency

$$x_p(t) = x(t)p(t)$$

$$p(t) = \sum_{n=-\infty}^{+\infty} \delta(t - nT)$$

$$x_p(t) = \sum_{n=-\infty}^{+\infty} x(nT)\delta(t - nT)$$

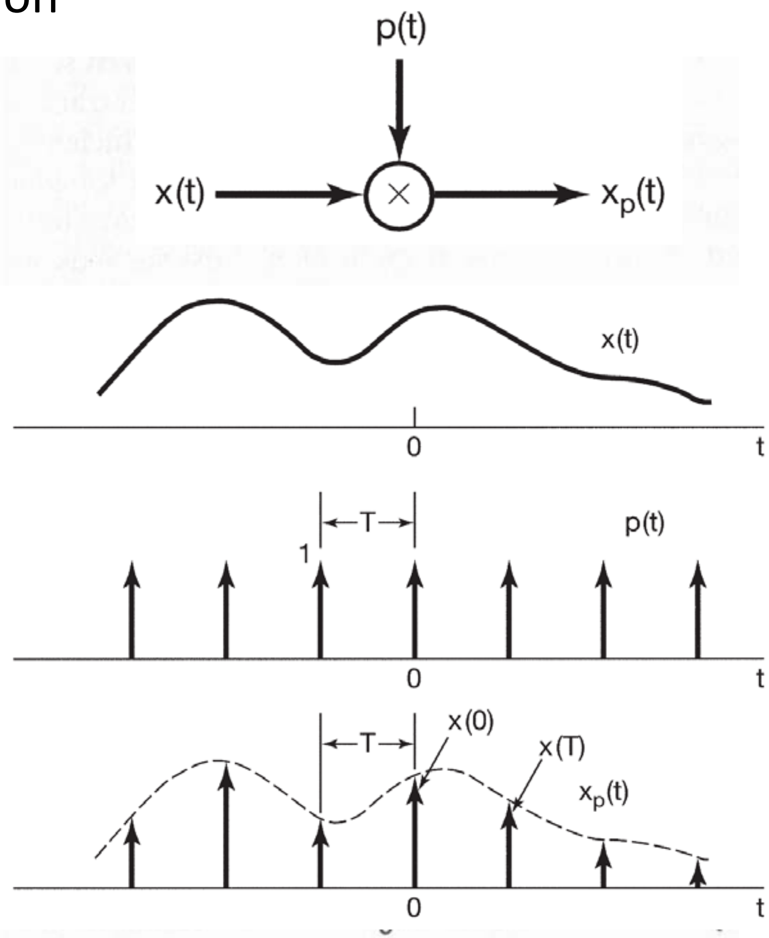


Figure 7.2 Impulse-train sampling.

7.1 Representation of a CT Signal by Its Samples: The Sampling Theorem

Impulse-Train Sampling

$$x(t) \overset{F}{\longleftrightarrow} X(j\omega)$$

$$P(j\omega) = \frac{2\pi}{T} \sum_{k=-\infty}^{+\infty} \delta(\omega - k\omega_s)$$

From the multiplication property:

$$X_p(j\omega) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\theta) P(j(\omega-\theta)) d\theta$$

$$= \frac{1}{T} \sum_{k=-\infty}^{+\infty} X(j(\omega - k\omega_s))$$

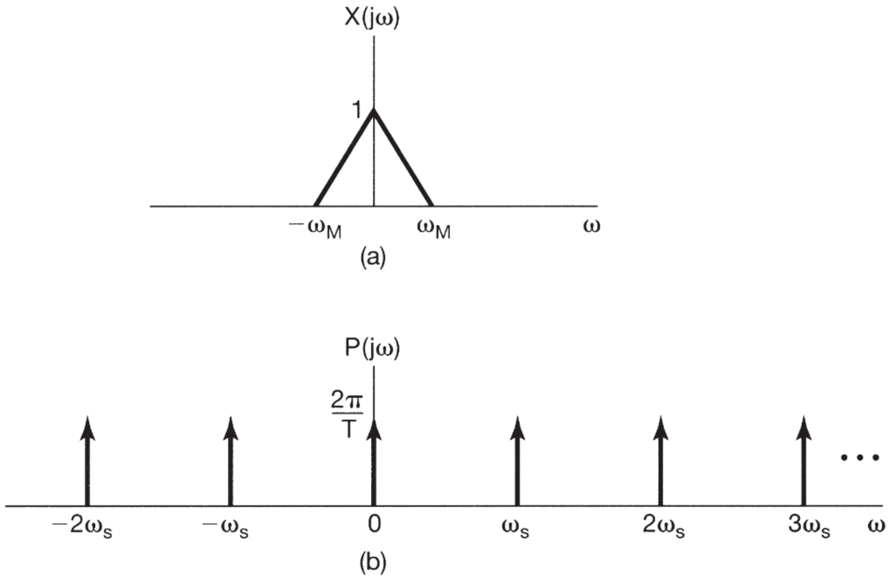


Figure 7.3 Effect in the frequency domain of sampling in the time domain: (a) spectrum of original signal; (b) spectrum of sampling function;

Impulse-Train Sampling

$$x(t) \overset{F}{\longleftrightarrow} X(j\omega)$$

$$P(j\omega) = \frac{2\pi}{T} \sum_{k=-\infty}^{+\infty} \delta(\omega - k\omega_s)$$

$$X_p(j\omega) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\theta) P(j(\omega - \theta)) d\theta$$

$$= \frac{1}{T} \sum_{k=-\infty}^{+\infty} X(j(\omega - k\omega_s))$$

$\omega_s - \omega_M > \omega_M$

$\omega_s > 2\omega_M$

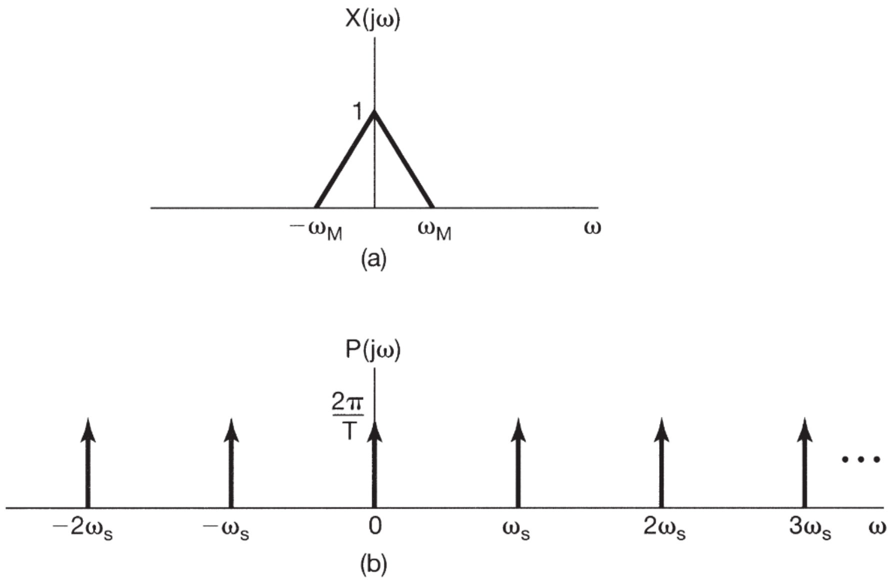


Figure 7.3 Effect in the frequency domain of sampling in the time domain: (a) spectrum of original signal; (b) spectrum of sampling function;

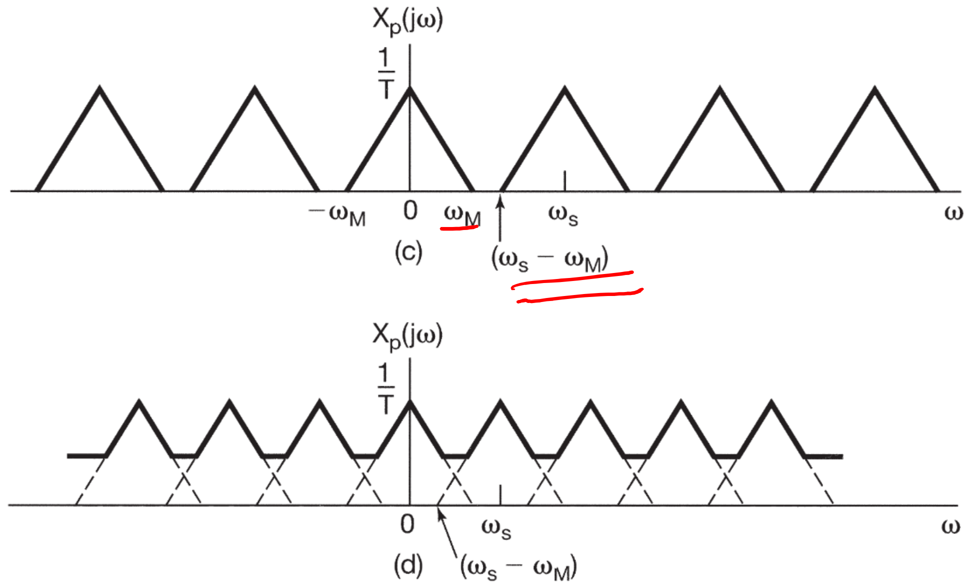


Figure 7.3 *Continued* (c) spectrum of sampled signal with $\omega_s > 2\omega_M$; (d) spectrum of sampled signal with $\omega_s < 2\,\omega_M$.

7.1 Representation of a CT Signal by Its Samples: The Sampling Theorem

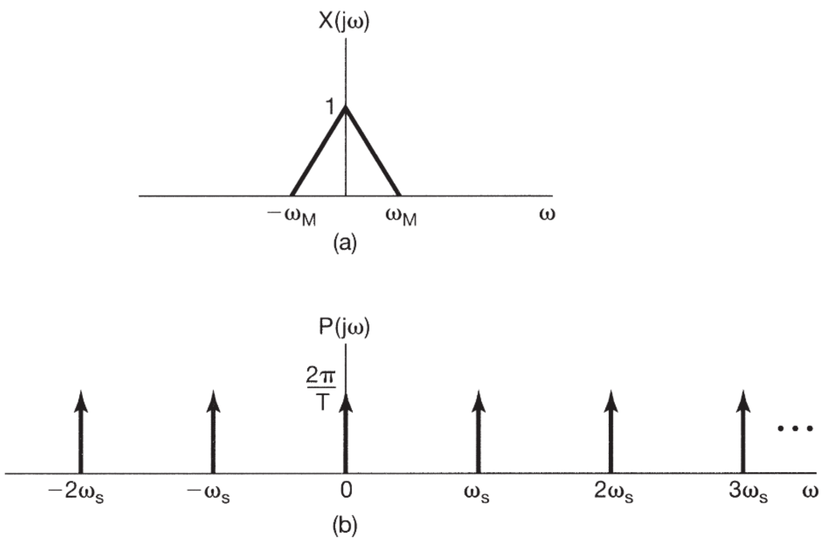
- Sampling Theorem

$x(t)$: a band-limited signal with $X(j\omega) = 0$ for $|\omega| > \omega_M$

$\Rightarrow x(t)$ is uniquely determined by $x(nT)$ if $\omega_s > 2\omega_M$,
where $n = 0,\ \pm 1,\ \pm 2,\ \ldots,$ and $\omega_s = 2\pi/T$.

$\Rightarrow 2\omega_M$: Nyquist rate

ω_M : Nyquist frequency



7.1 Representation of a CT Signal by Its Samples: The Sampling Theorem

- Sampling Theorem

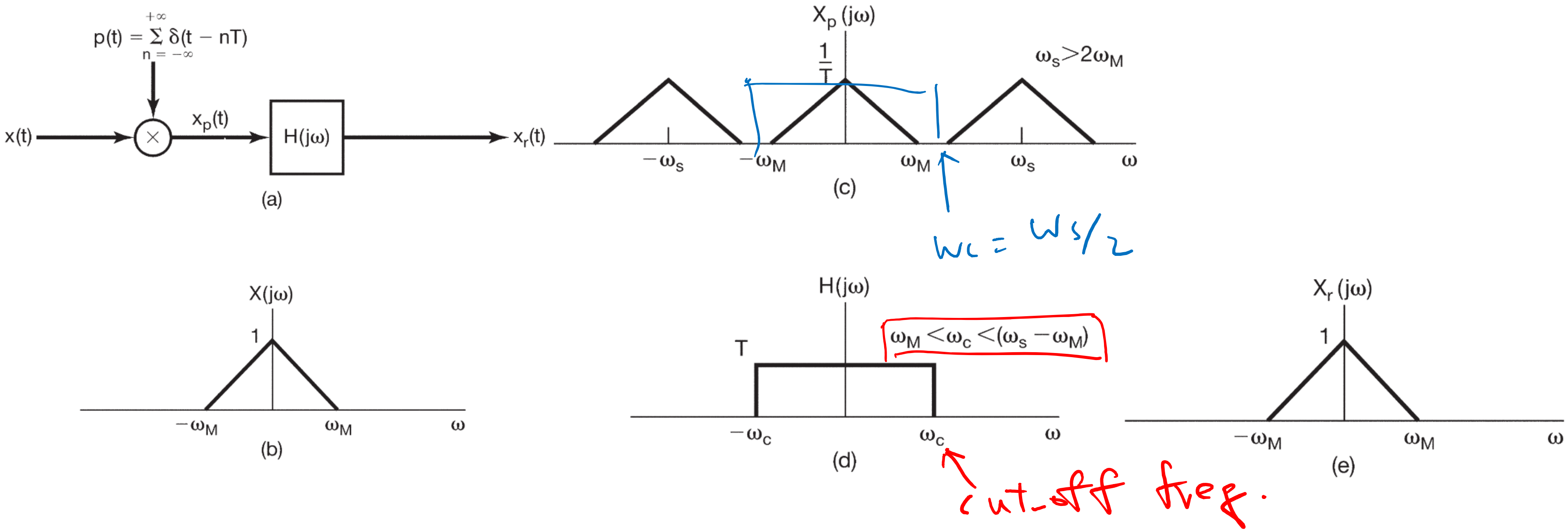
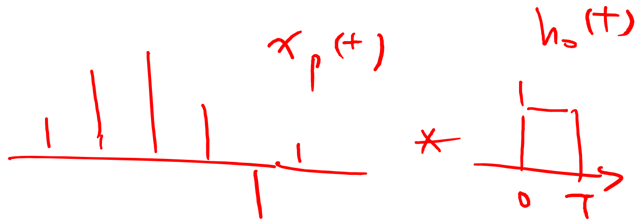


Figure 7.4 Exact recovery of a continuous-time signal from its samples using an ideal lowpass filter: (a) system for sampling and reconstruction; (b) representative spectrum for $x(t)$; (c) corresponding spectrum for $x_p(t)$; (d) ideal lowpass filter to recover $X(j\omega)$ from $X_p(j\omega)$; (e) spectrum of $X_r(t)$.

7.1 Representation of a CT Signal by Its Samples: The Sampling Theorem

$x_p(t)$ $h_0(t)$



- Sampling with a Zero-Order Hold
  - In practice, it's difficult to generate impulses.
  - It's typically more convenient to generate the sampled signal by a zero-order hold, which samples a signal at a given instant and holds the value until the next instant at which another sample is taken.
  - The zero-order hold (ZOH) in Figures 7.5 and 7.6 can be expressed as the convolution of $x_0(t) = x_p(t) * h_0(t)$, where $x_p(t)$ is defined as in eq. (7.1), and

$$h_0(t) = 1 \qquad \text{for } 0 \le t \le T, \qquad h_0(t) = 0 \qquad \text{otherwise.} \tag{7.6}$$

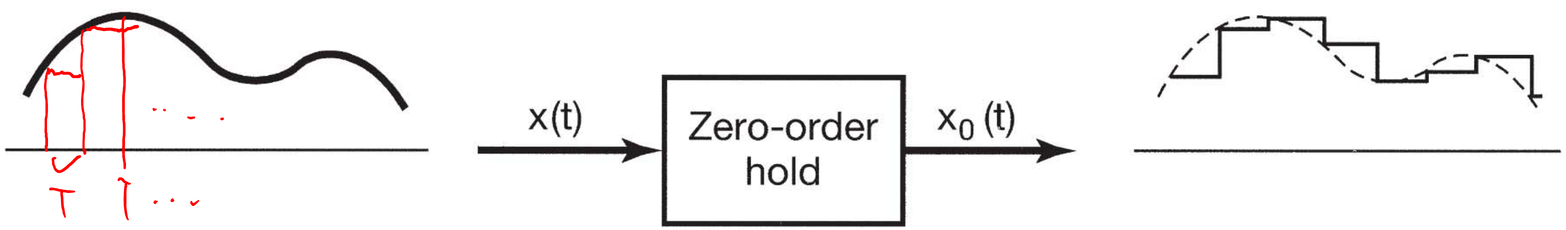
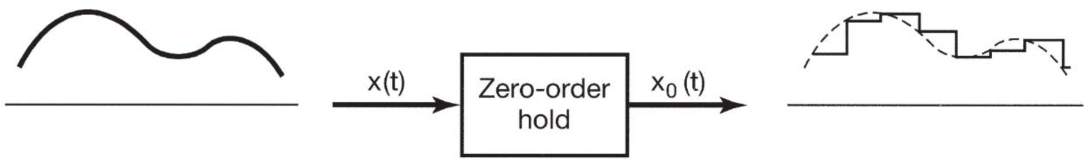


Figure 7.5 Sampling utilizing a zero-order hold.



- Sampling with a Zero-Order Hold
  - The reconstruction of x(t) from the output of a ZOH can be carried by a LPF, but the required filter no longer has constant gain in the pass band.

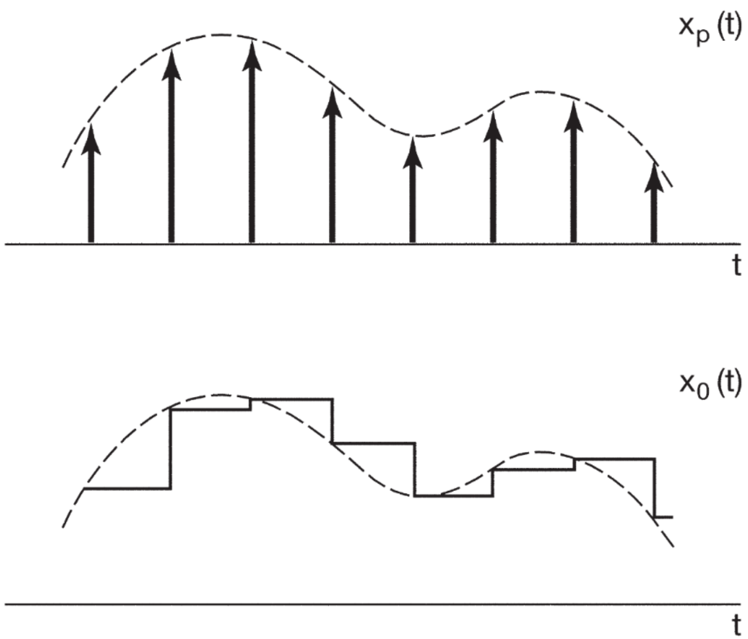
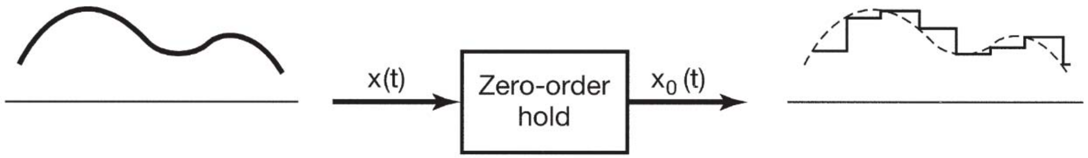


Figure 7.6 Zero-order hold as impulse-train sampling followed by an LTI system with a rectangular impulse response.



- Sampling with a Zero-Order Hold
  - The reconstruction of x(t) from the output of a ZOH can be carried by a LPF, but the required filter no longer has constant gain in the pass band.

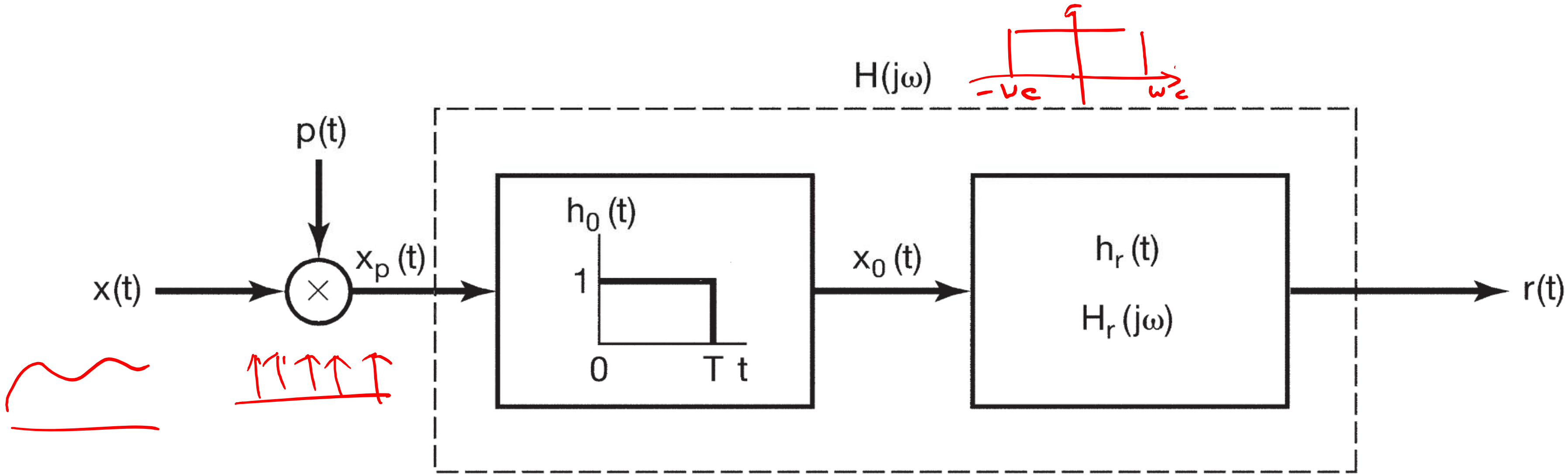
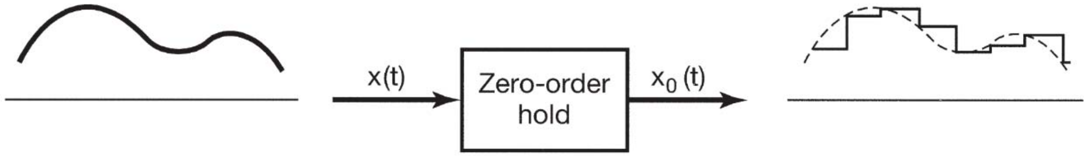
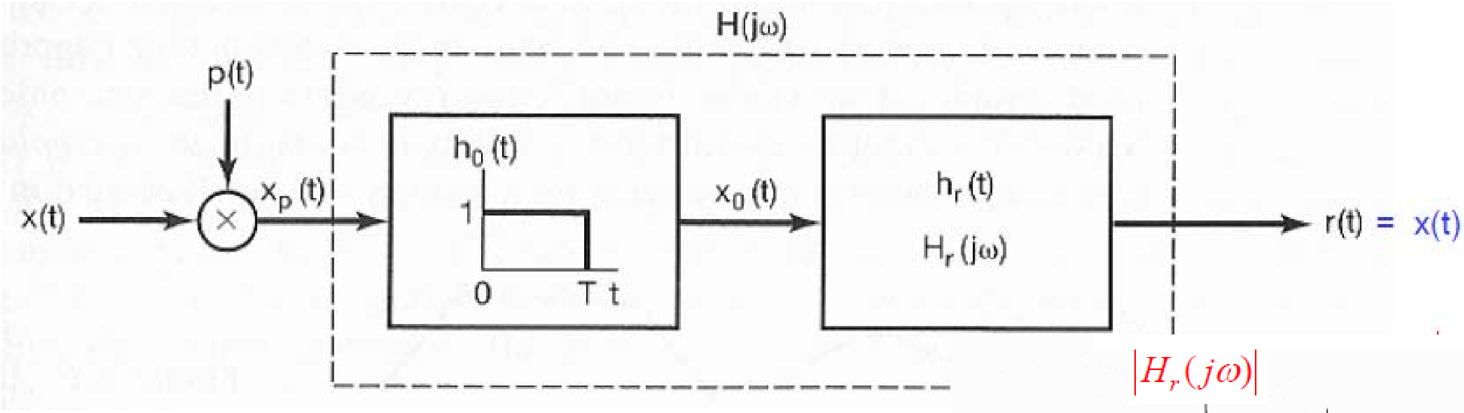


Figure 7.7 Cascade of the representation of a zero-order hold (Figure 7.6) with a reconstruction filter.



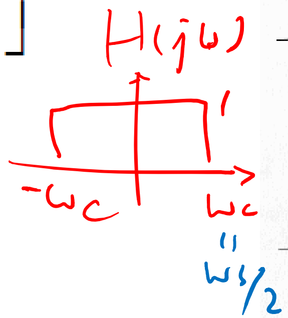
- Sampling with a Zero-Order Hold
  - The reconstruction of x(t) from the output of a ZOH can be carried by a LPF, but the required filter no longer has constant gain in the pass band.



$$H_0(j\omega) = e^{-j\omega T/2}\left[\frac{2\sin(\omega T/2)}{\omega}\right]$$

$$H(j\omega) = H_0(j\omega)H_r(j\omega)$$

$$\Rightarrow H_r(j\omega) = \frac{e^{j\omega T/2}H(j\omega)}{\dfrac{2\sin(\omega T/2)}{\omega}}$$



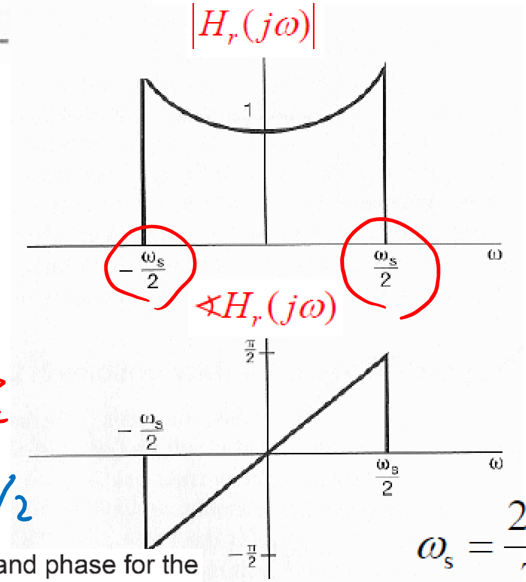


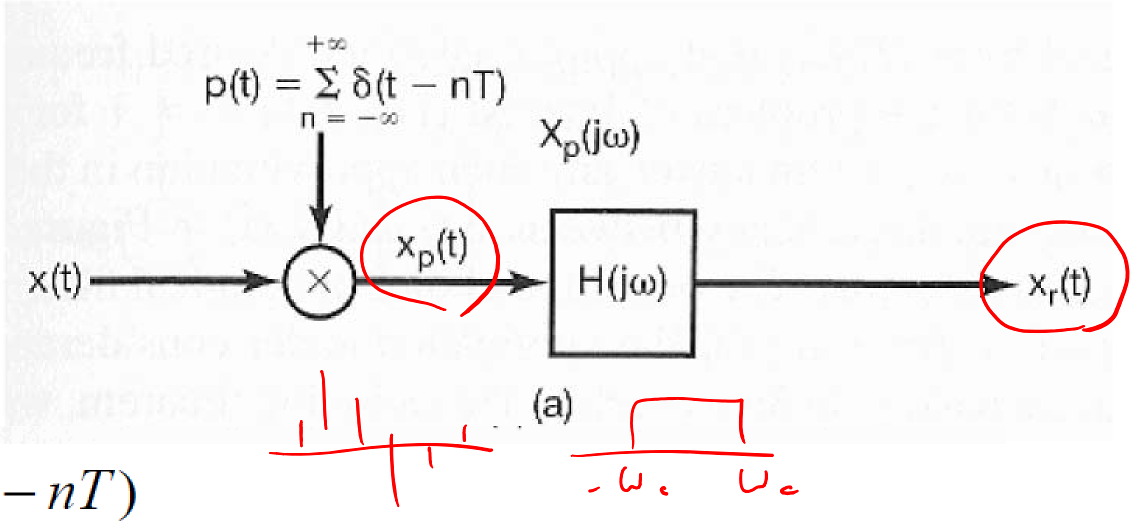
Figure 7.8 Magnitude and phase for the reconstruction filter for a zero-order hold.

$$\omega_s = \frac{2\pi}{T}$$

7.2 Reconstruction of a Signal from Its Samples Using Interpolation

- Reconstruction Using Interpolation

- Ideal low pass filter
- Zero-order interpolation
- 1st-order interpolation (linear interpolation)
- Higher-order interpolation



$$x_p(t) = \sum_{n=-\infty}^{+\infty} x(nT)\delta(t - nT)$$

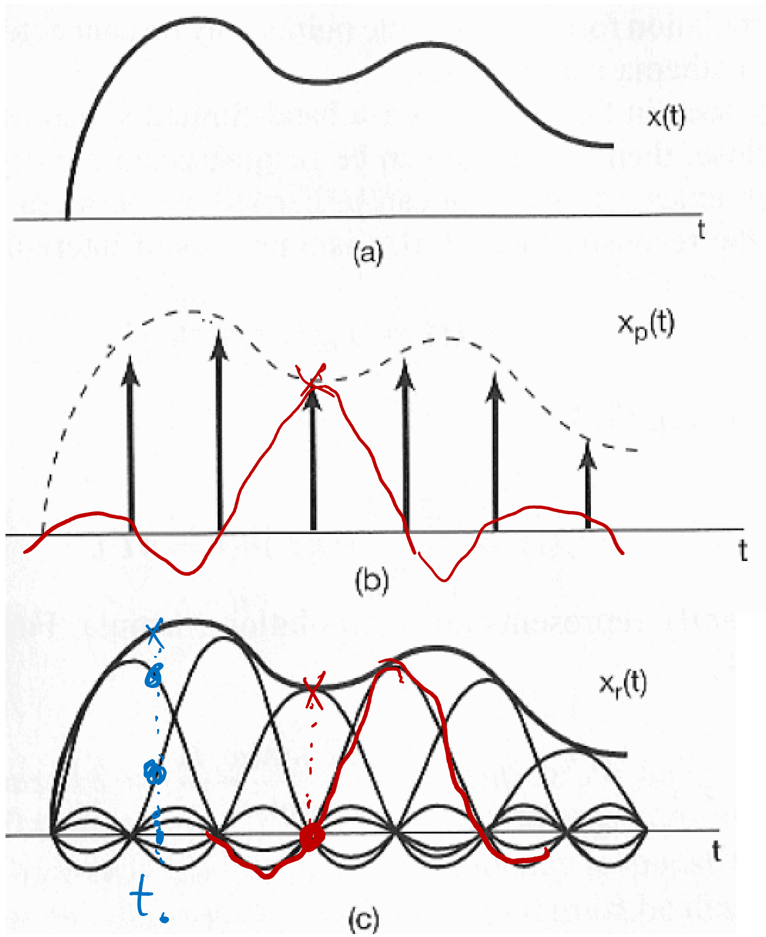
$$x_r(t) = x_p(t) * h(t)$$

$$x_r(t) = \sum_{n=-\infty}^{+\infty} x(nT)h(t - nT)$$

⇦ **This equation describes how to fit a continuous curve between sample points and hence represents an interpolation formula.**

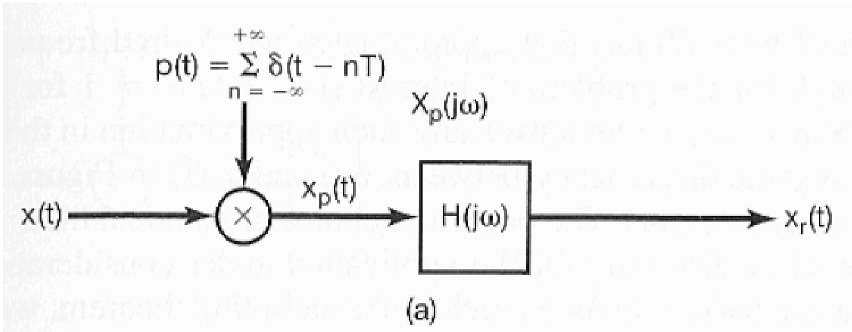
7.2 Reconstruction of a Signal from Its Samples Using Interpolation

- Reconstruction by LPF

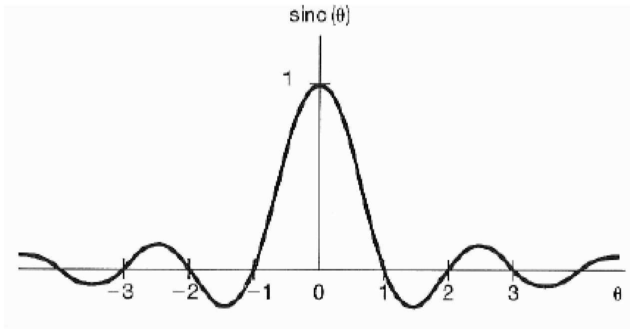


7.2 Reconstruction of a Signal from Its Samples Using Interpolation

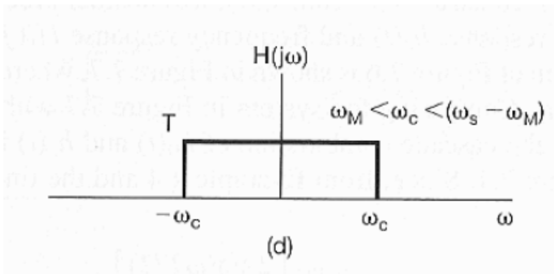
- Interpolation by Ideal LPF



(a)



$$h(t) = \frac{\omega_c T \sin(\omega_c t)}{\pi \omega_c t}$$



(d)

$$x_r(t) = \sum_{n=-\infty}^{+\infty} x(nT) h(t - nT)$$

$$= \sum_{n=-\infty}^{+\infty} x(nT) \frac{\omega_c T}{\pi} \frac{\sin(\omega_c (t - nT))}{\omega_c (t - nT)}$$

Exact reconstruction can be obtained if x(t) is band limited and if the sampling frequency is greater than the Nyquist rate.

7.2 Reconstruction of a Signal from Its Samples Using Interpolation

- Interpolation by first-order hold (FOH)

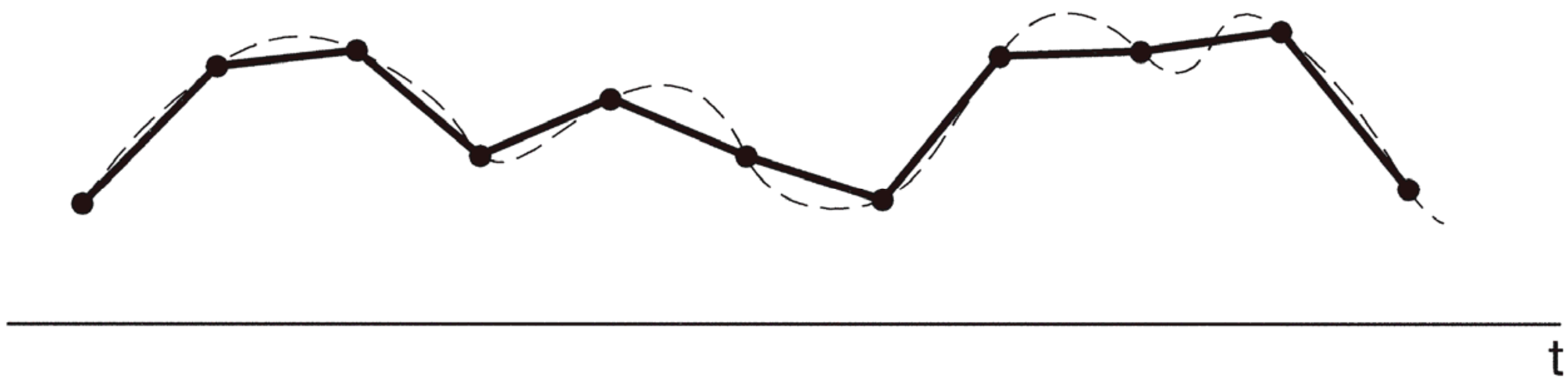
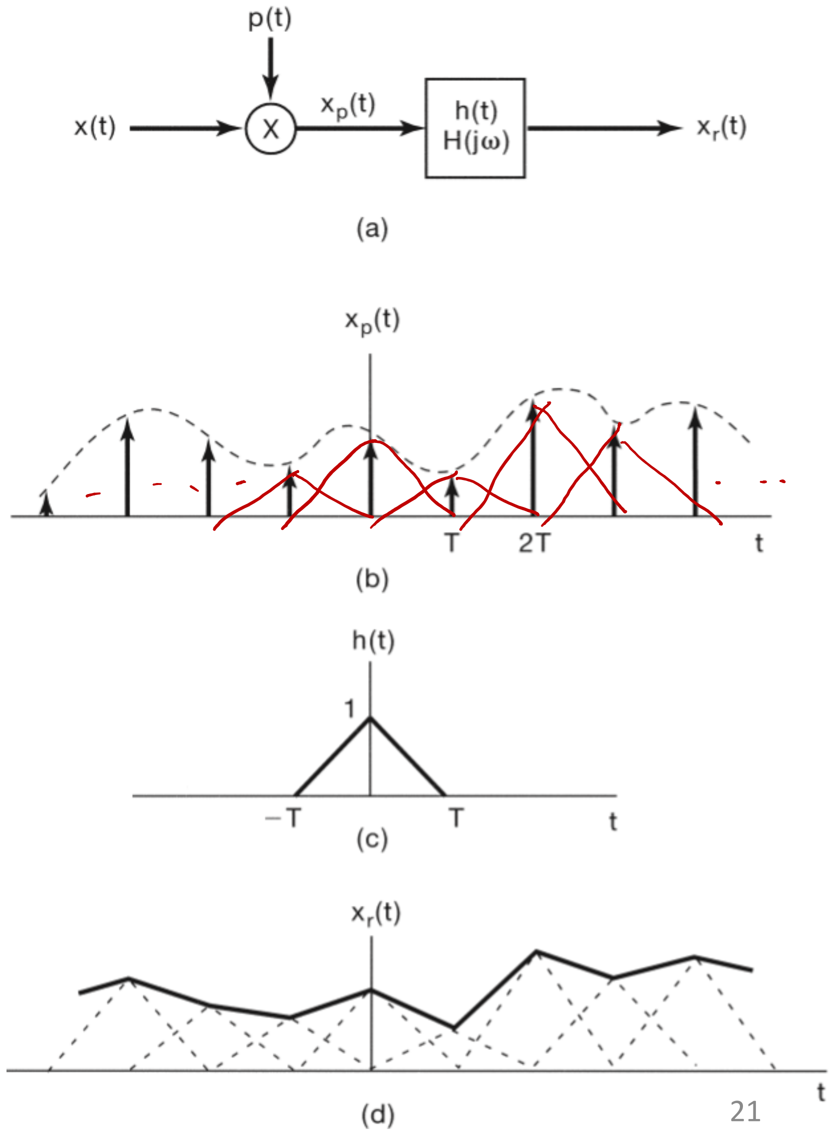


Figure 7.9 Linear interpolation between sample points. The dashed curve represents the original signal and the solid curve the linear interpolation.

7.2 Reconstruction of a Signal from Its Samples Using Interpolation

- Interpolation by First-Order Hold

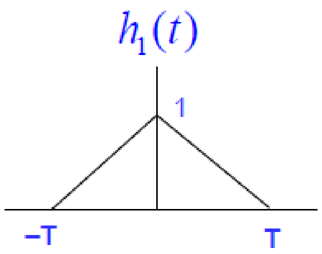


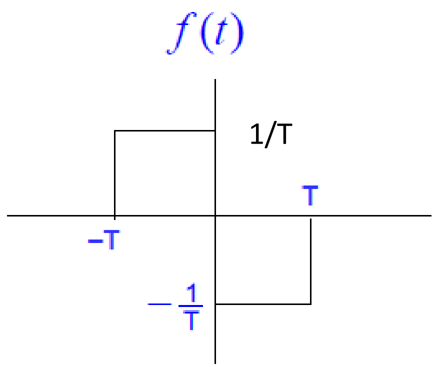
$h(t)$ triangular $\quad H(j\omega) = \dfrac{1}{T}\left[\dfrac{\sin(\omega T/2)}{\omega/2}\right]^2.$

$$\text{i.e.,}\quad H(j\omega) = T\operatorname{sinc}^2\left(\frac{\omega T}{2\pi}\right)$$

See derivation in the next slide.

Exercise: Determine $H_1(j\omega)$



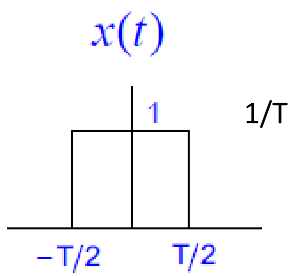


$$X(j\omega)=\frac{\sin(\omega T/2)}{\omega/2}$$

$$f(t)=\frac{1}{T}\left(x(t+\frac{T}{2})-x(t-\frac{T}{2})\right)$$

$$F(j\omega)=\frac{1}{T}X(j\omega)(e^{j\omega T/2}-e^{-j\omega T/2})$$

$$=\frac{j2}{T}X(j\omega)\sin(\omega T/2)$$

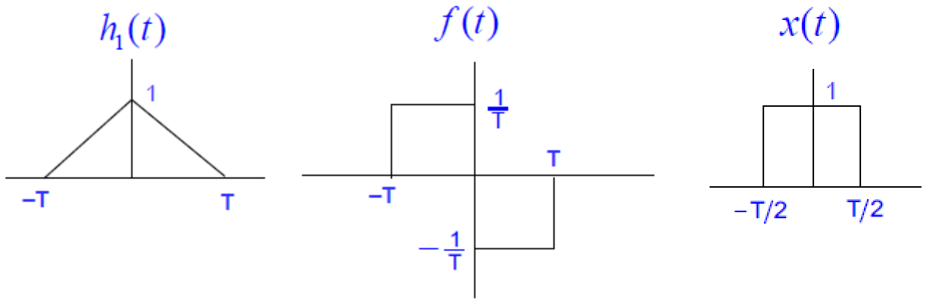


$$h_1(t)=\int_{-\infty}^{t}f(\tau)d\tau,\ \text{so } H(j\omega)=\frac{1}{j\omega}F(j\omega)+\pi F(0)\delta(\omega)$$

$$f(t)\text{ real and odd }\Rightarrow\ F(j\omega)\text{ pure imaginary and odd }\Rightarrow F(0)=0$$

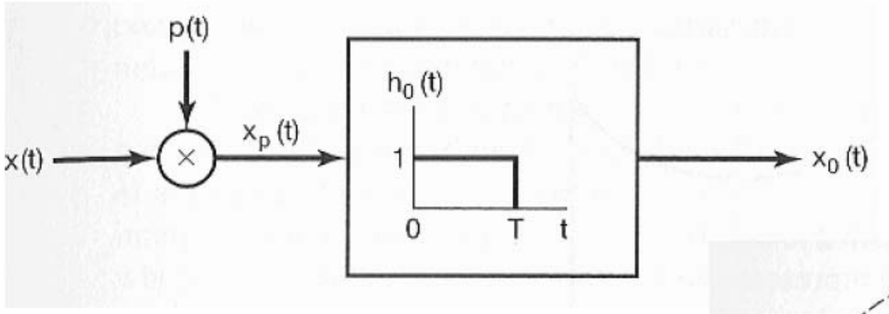
$$\therefore H_1(j\omega)=\frac{1}{T}\left[\frac{\sin(\omega T/2)}{\omega/2}\right]^2$$

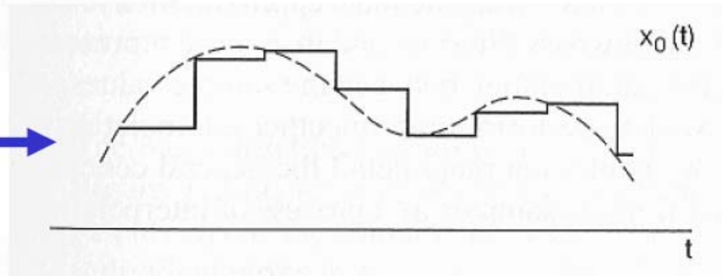
Alternatively, $h_1(t)=x(t)*x(t)$.
Thus, $H_1(j\omega)=X^2(j\omega)$



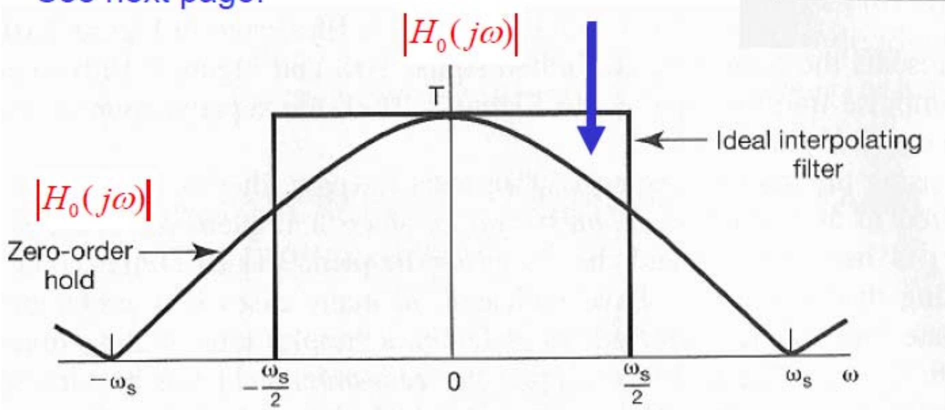
7.2 Reconstruction of a Signal from Its Samples Using Interpolation

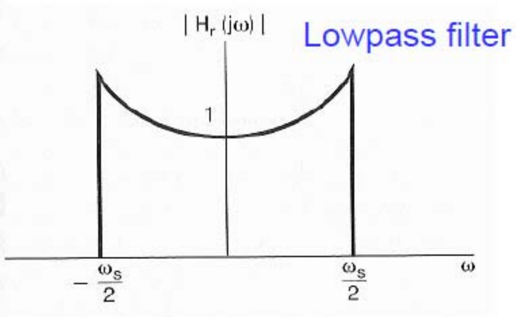
- Revisit of ZOH (Interpolation by Zero-Order Hold Filter)





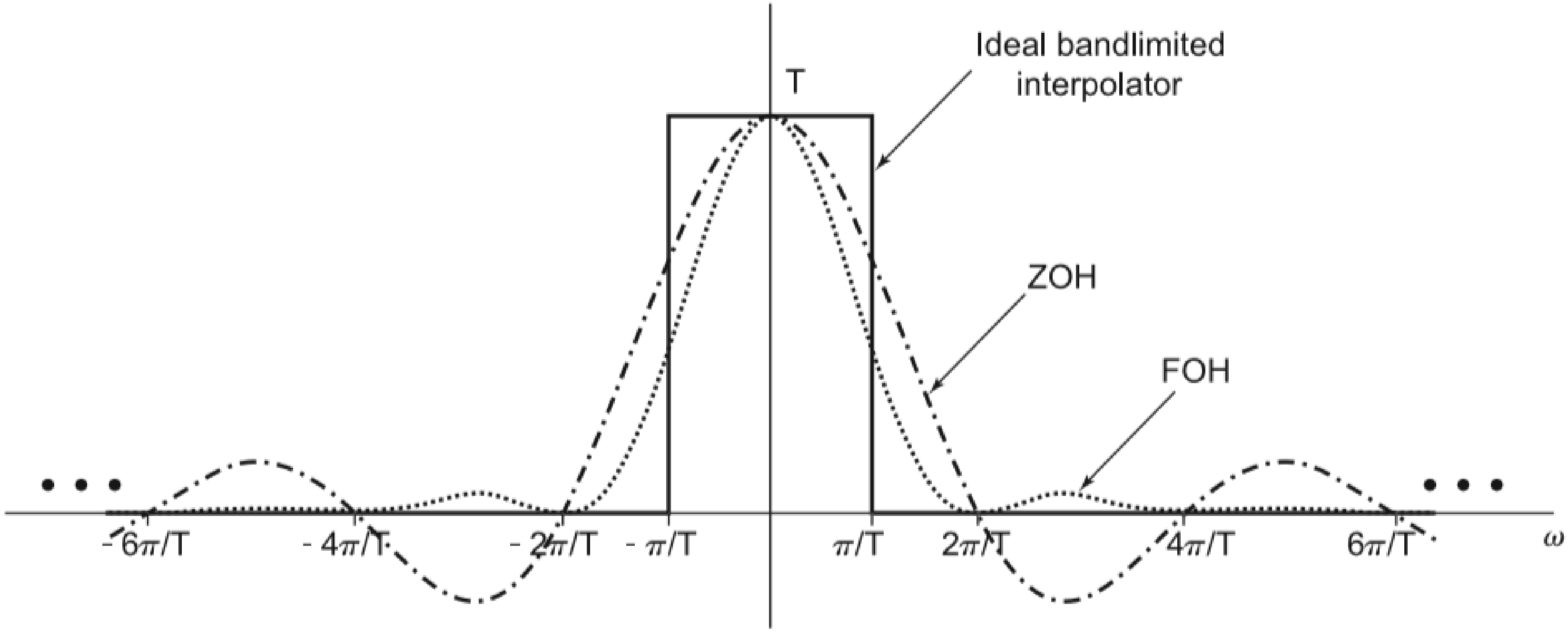
We can see that ZOH is a very rough approximation. It can often be improved by adding additional lowpass filtering. See next page.





7.2 Reconstruction of a Signal from Its Samples Using Interpolation

- Comparisons of transfer functions of ideal interpolation filter, ZOH & FOH

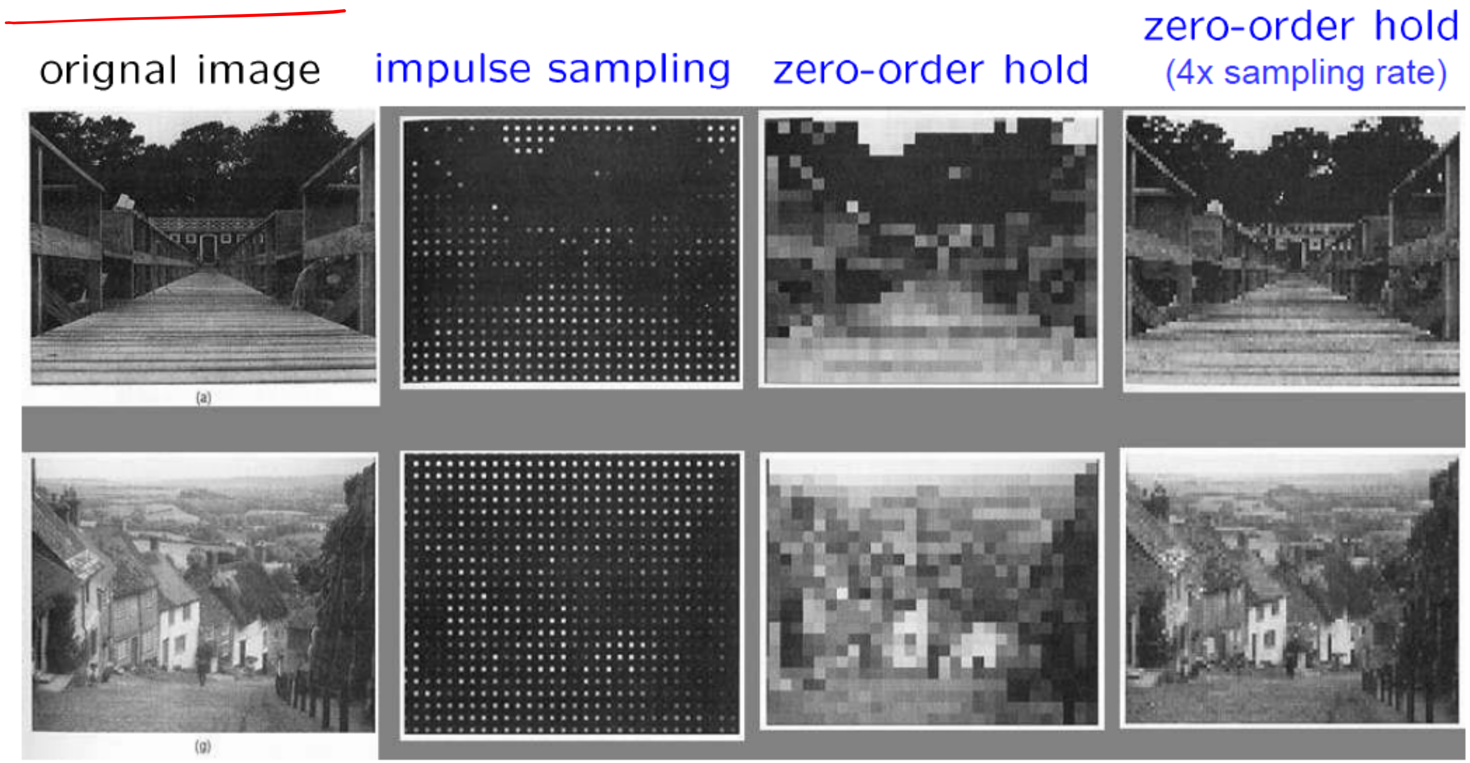


(e)

The (FOH) has a smaller magnitude than that of the zero-order hold (ZOH), especially when $|\omega|$ is larger (remember that $|\mathrm{sinc}(a)| \le 1$ and $\mathrm{sinc}^2(a) \le |\mathrm{sinc}(a)|$). Therefore, using the FOH can obtain a smoother interpolation result than using the ZOH.

7.2 Reconstruction of a Signal from Its Samples Using Interpolation

- Illustration of Sampling & Interpolation



Note: Additional lowpass filtering introduced by human eyes makes the image look smoother.
For example, the mosaic effect is smoothed when the image is viewed from distance.

Results

7.3 The Effect of Undersampling: Aliasing

$\omega_s - \omega_M \ngtr \omega_M$

$\omega_s \ngtr 2\omega_M$

- Revisit of Sampling Theorem:

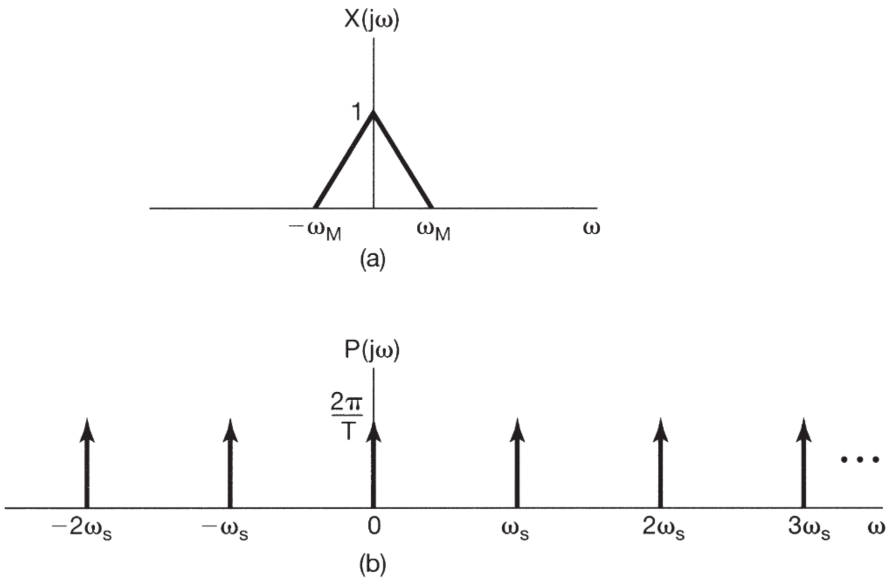


Figure 7.3 Effect in the frequency domain of sampling in the time domain: (a) spectrum of original signal; (b) spectrum of sampling function;

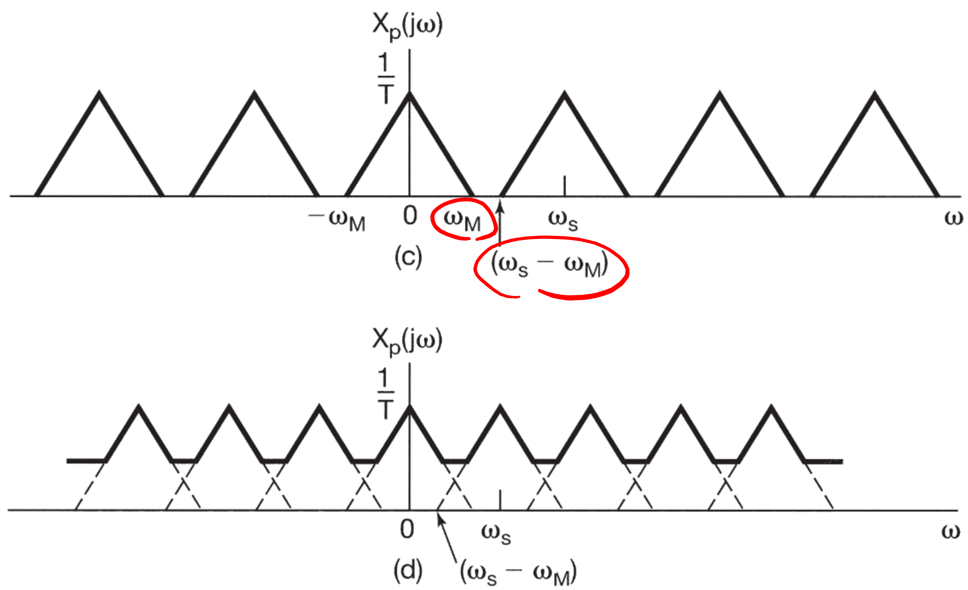
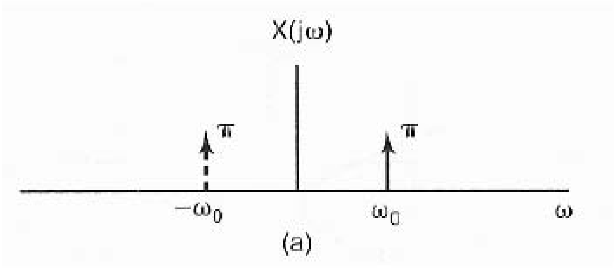


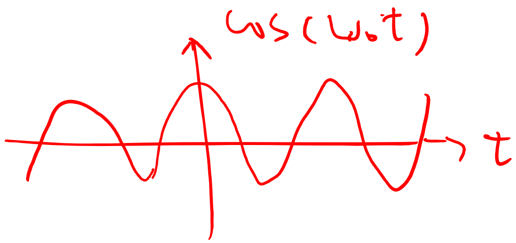
Figure 7.3 *Continued* (c) spectrum of sampled signal with $\omega_s > 2\omega_M$; (d) spectrum of sampled signal with $\omega_s < 2\,\omega_M$.

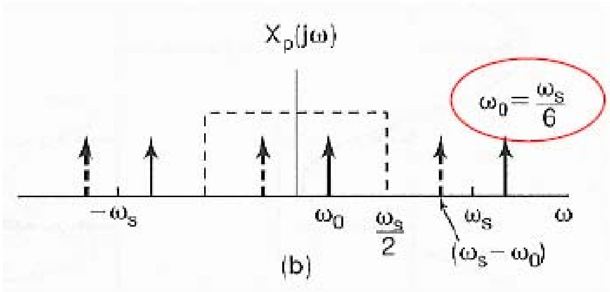
7.3 The Effect of Undersampling: Aliasing

- Effect of Aliasing in Frequency Domain

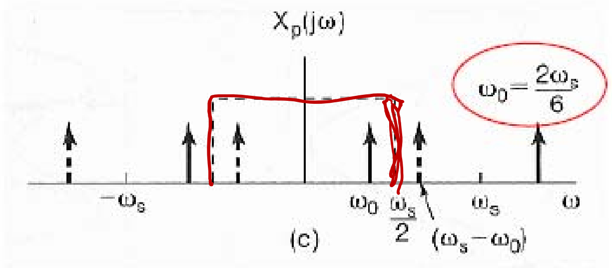
$$x(t) = \cos(\omega_0 t)$$



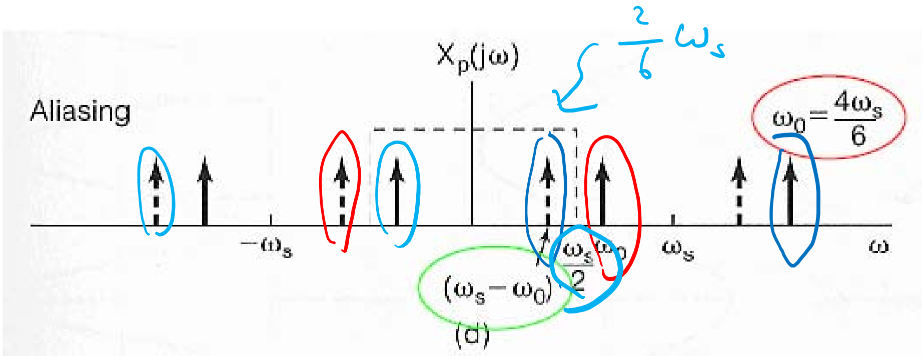




$\omega_s > 2\omega_0$



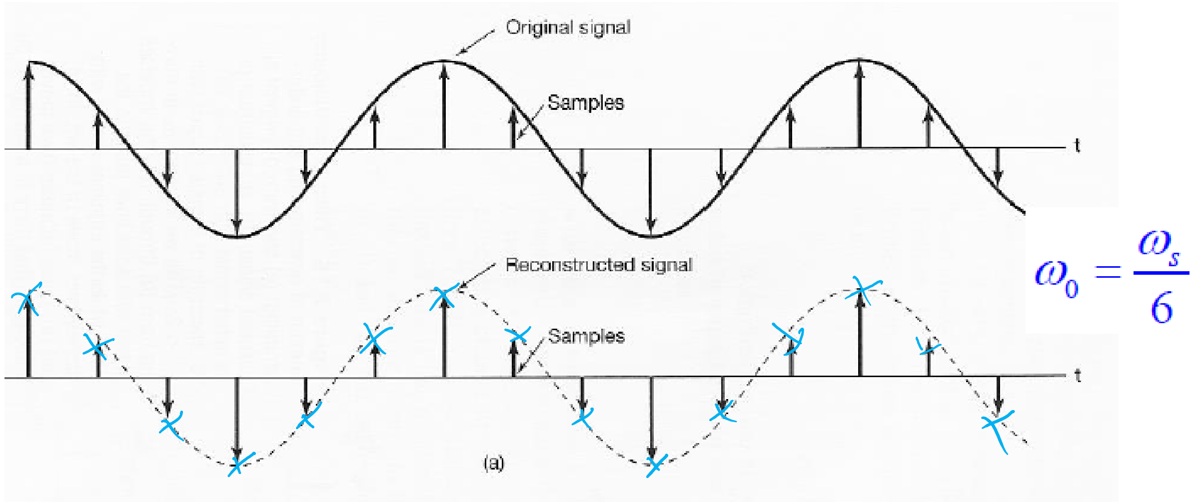
$\omega_s > 2\omega_0$

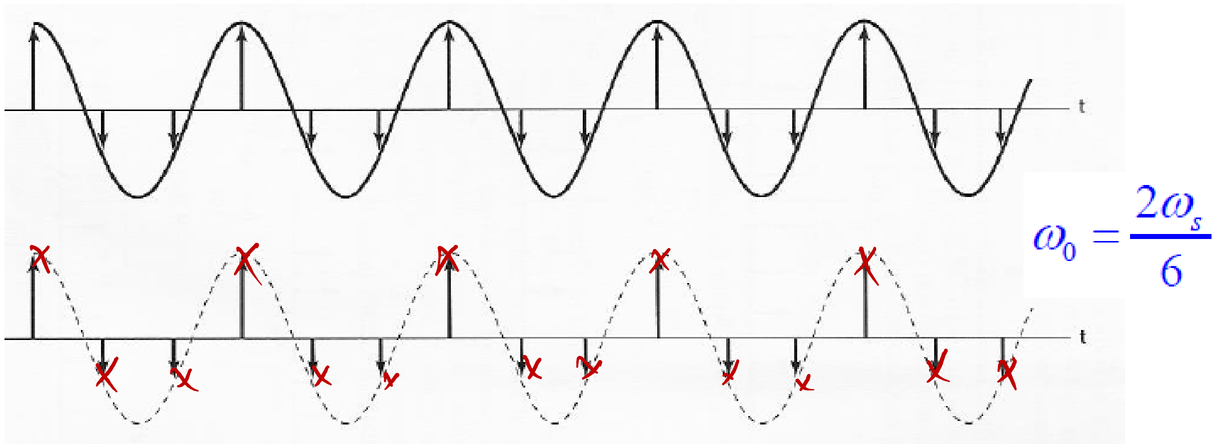


$\omega_s < 2\omega_0$

7.3 The Effect of Undersampling: Aliasing

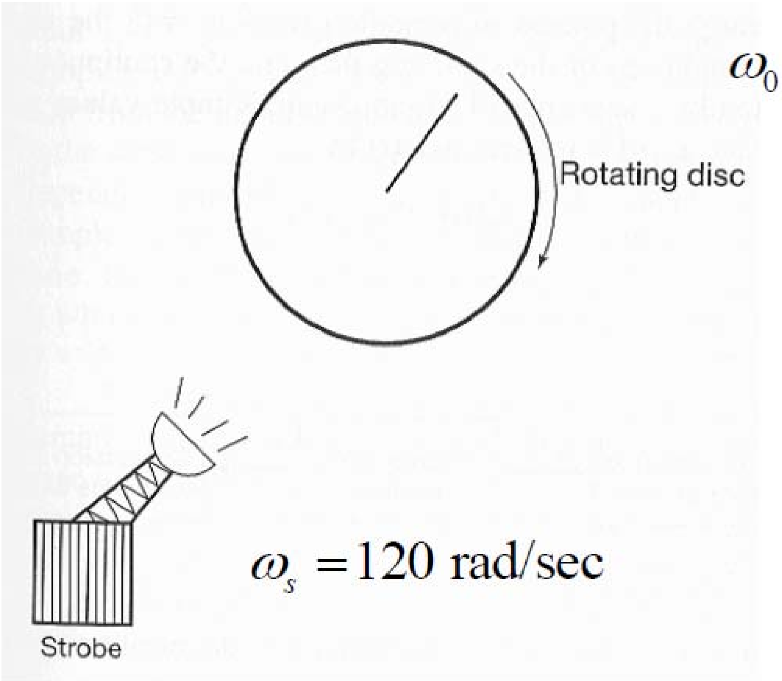
- Effect of Aliasing in Time Domain





7.3 The Effect of Undersampling: Aliasing

- Strobe Effect

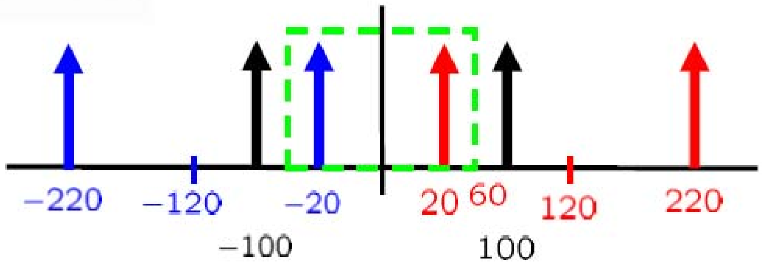


$$\omega_0 = 100 \text{ rad/sec}$$

$$\omega_s = 120 \text{ rad/sec}$$

$$\Rightarrow \omega = \pm\omega_s \pm \omega_0$$
$$= +20, \ -20$$

$w_c = \frac{w_s}{2} = 60$



7.3 The Effect of Undersampling: Aliasing

- Have you noticed this effect in practice?

Photo Credit: Frank ☺

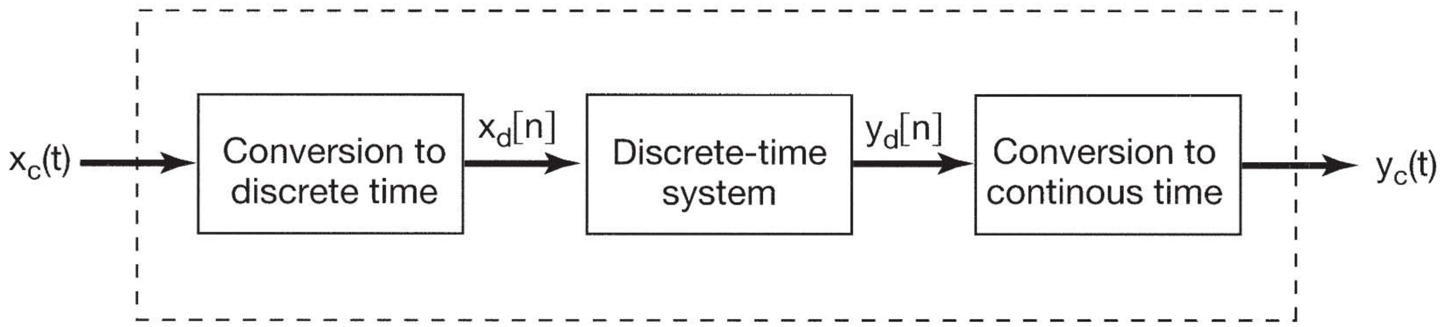
7.3 The Effect of Undersampling: Aliasing

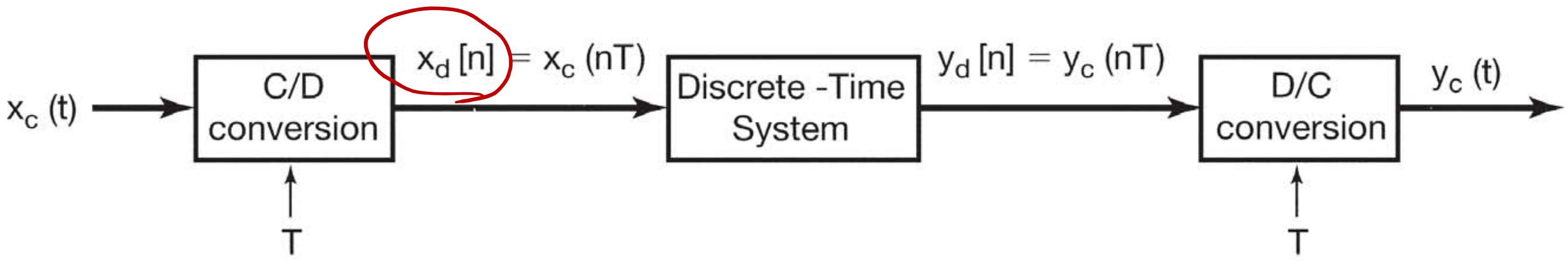
- Wagon-Wheel Effect

https://www.youtube.com/watch?v=QOwzkND_ooU

7.4 DT Processing of CT Signals

- The system can be represented as the cascade of three operations:

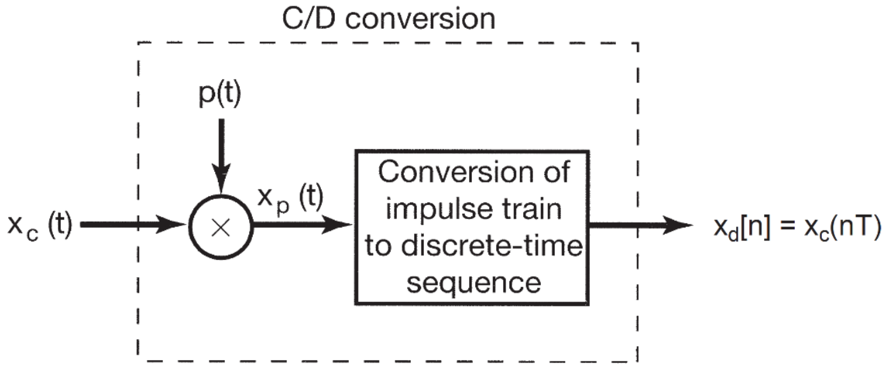




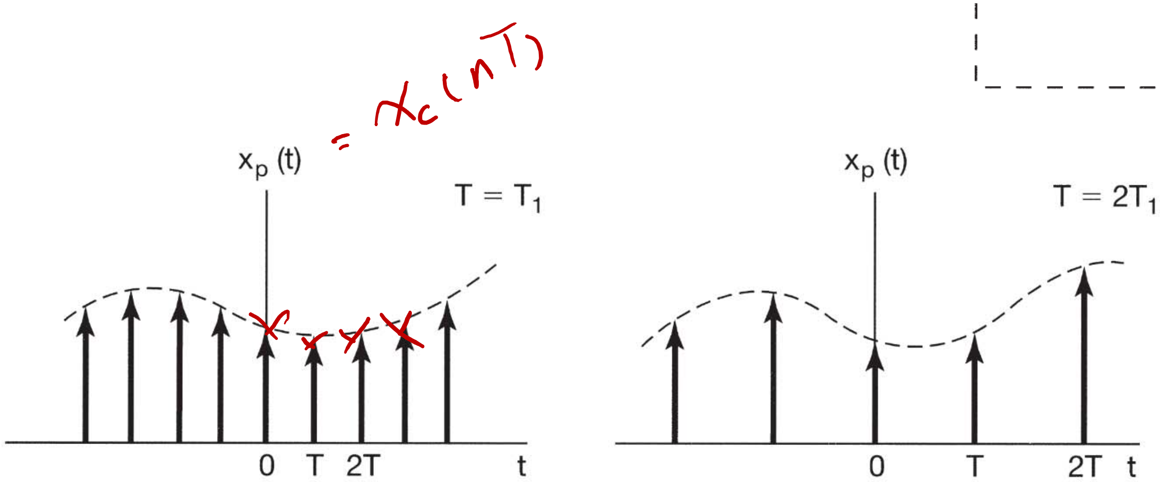
Example: Digital video player with analog display

7.4 DT Processing of CT Signals

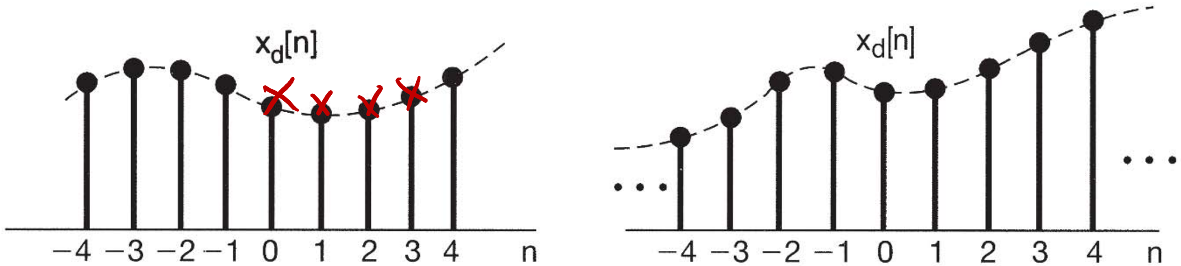
- C/D conversion



(a)



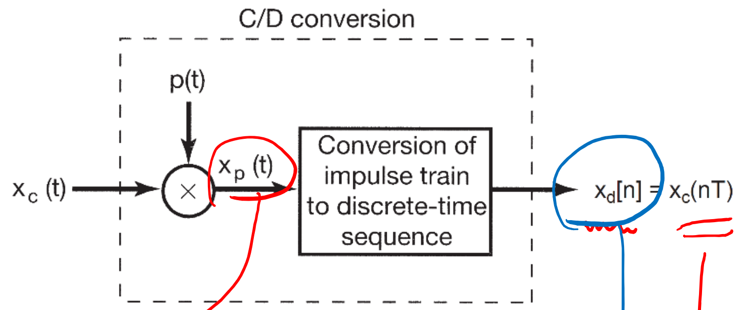
(b)



(c)

Which one has higher frequency components?

7.4 DT Processing of CT Signals



- Let's look into the details…

Now, examine the relation between $x_c(t)$ and $x_d[n]$ in the frequency domain.

$$x_p(t) = \sum_{n=-\infty}^{\infty} x_c(nT)\delta(t-nT)$$ CT

$$X_p(j\omega) = \int_{-\infty}^{\infty} x_p(t)e^{-j\omega t}dt = \sum_{n=-\infty}^{\infty} x_c(nT)\int_{-\infty}^{\infty} \delta(t-nT)e^{-j\omega t}dt = \sum_{n=-\infty}^{\infty} x_c(nT)e^{-j\omega nT}$$

$$X_d(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} x_d[n]e^{-j\Omega n} = \sum_{n=-\infty}^{\infty} x_c(nT)e^{-j\Omega n}$$

Comparing the above two equations, we have

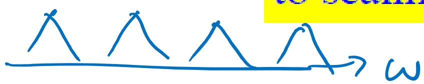
$$X_d(e^{j\Omega}) = X_p(j\Omega / T) \qquad \boxed{\Omega = \omega T} \Rightarrow$$

$X_d(e^{j\Omega})$ is a frequency-scaled version of $X_p(j\omega)$.

Scaling the frequency by T is eq. to scaling the time by $1/T$.

From Eq. (7.6),

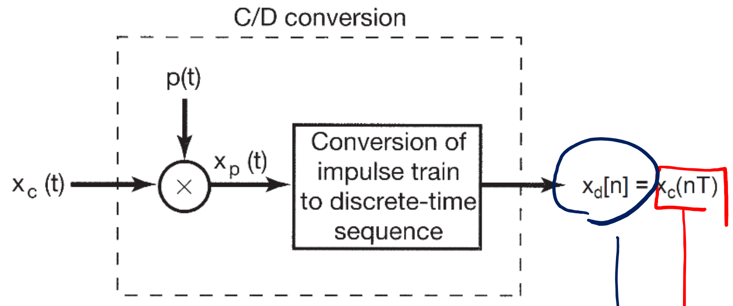
$$X_p(j\omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c(j(\omega - k\omega_s)).$$



$$\Rightarrow \boxed{X_d(e^{j\Omega}) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c(j(\Omega - 2\pi k)/T)} \Rightarrow$$

The spectrum of $x_d[n]$ is a periodic replica of $x_c(t)$, followed by a linear frequency scaling.

7.4 DT Processing of CT Signals



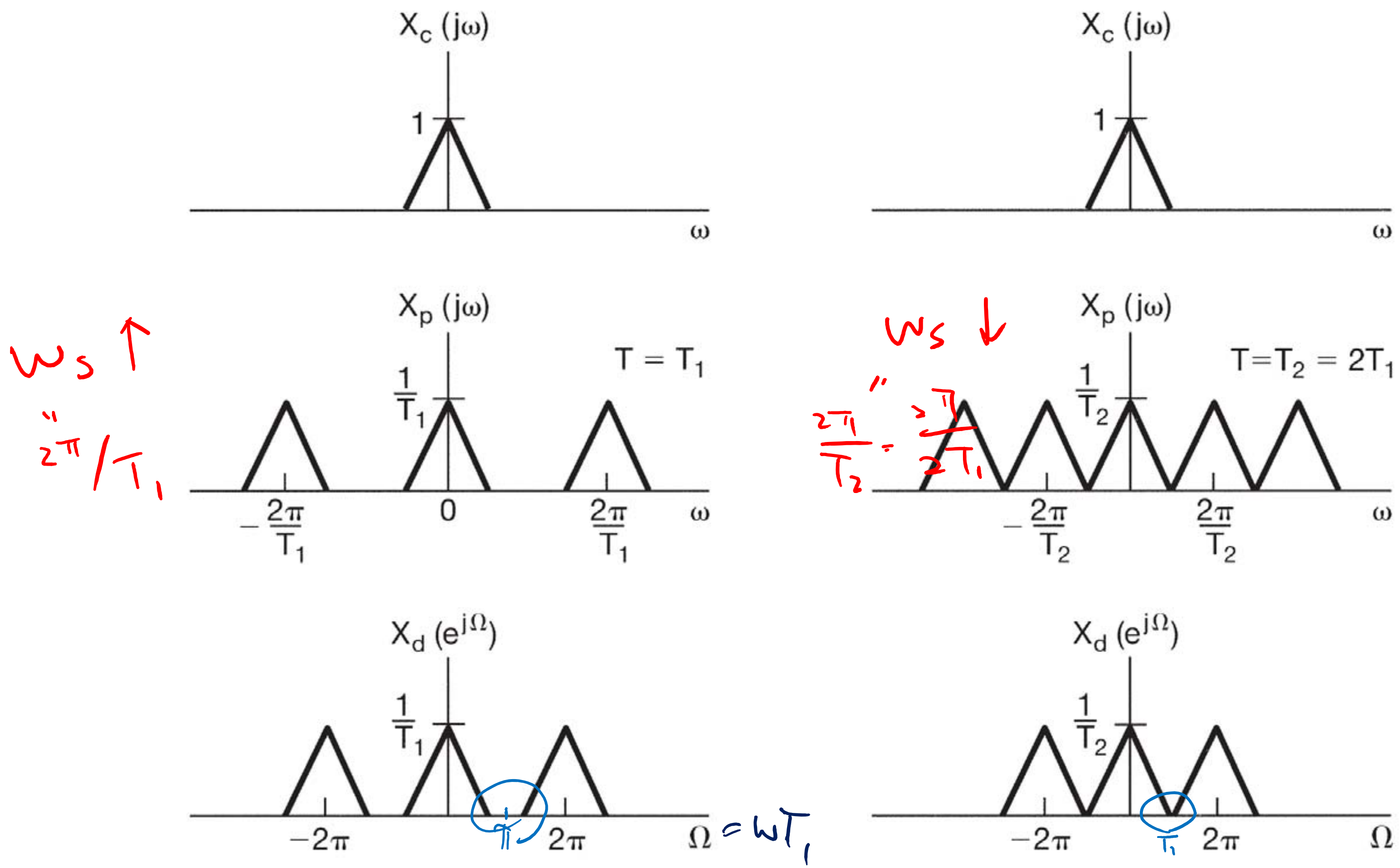
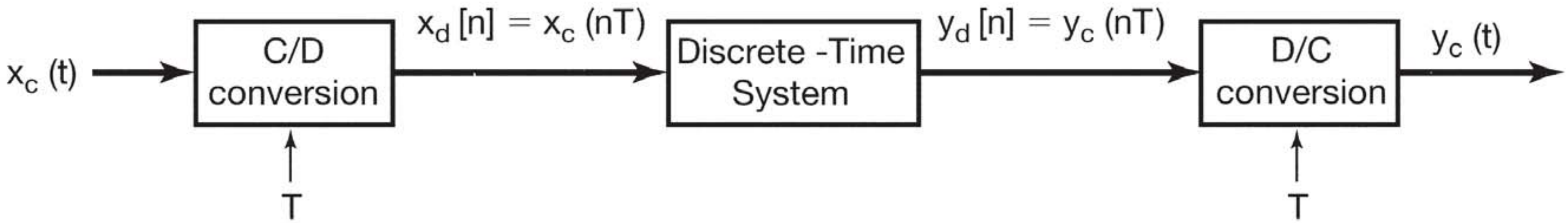


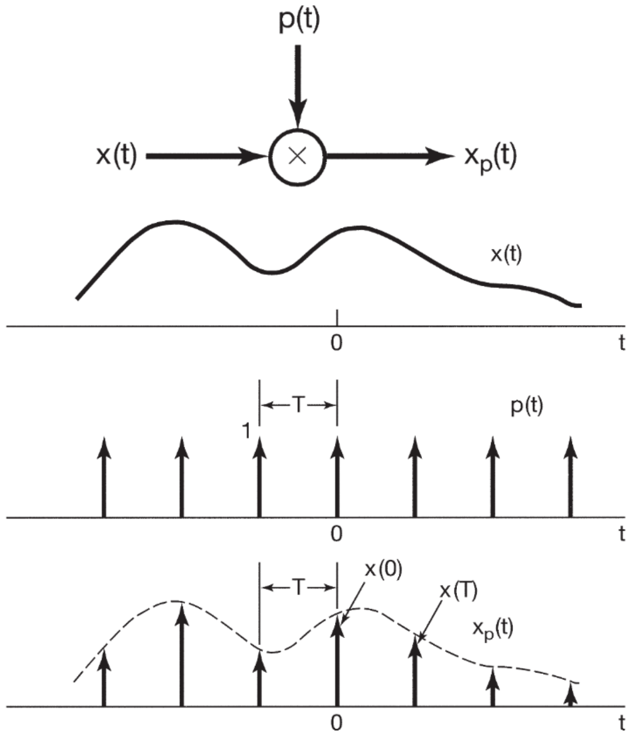
Figure 7.22 Relationship between $X_c(j\omega)$, $X_p(j\omega)$, and $X_d(e^{j\Omega})$ for two different sampling rates.

Which one has higher frequency components?

7.4 DT Processing of CT Signals

- D/C conversion





$$x_p(t) = x(t)p(t)$$

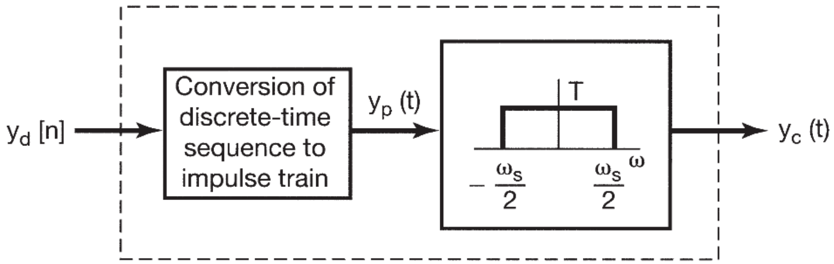
$$p(t) = \sum_{n=-\infty}^{+\infty} \delta(t - nT)$$

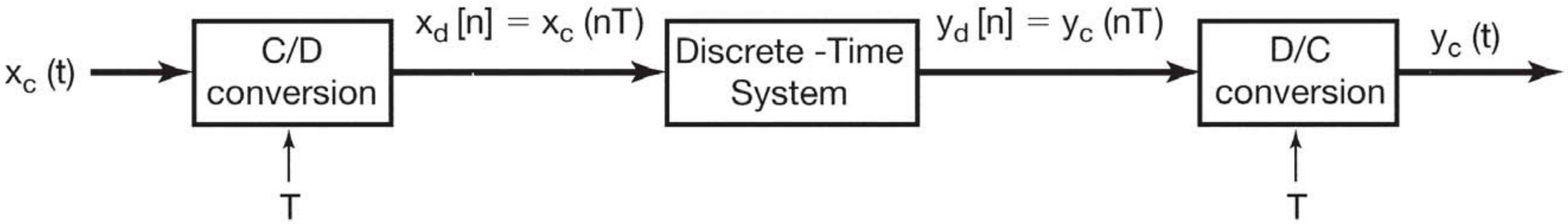
$$x_p(t) = \sum_{n=-\infty}^{+\infty} x(nT)\delta(t - nT)$$

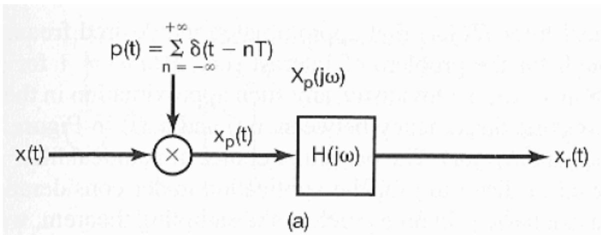
7.4 DT Processing of CT Signals

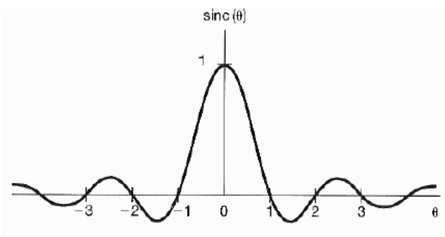
- D/C conversion

Figure 7.23 Conversion of a discrete-time sequence to a continuous-time signal.

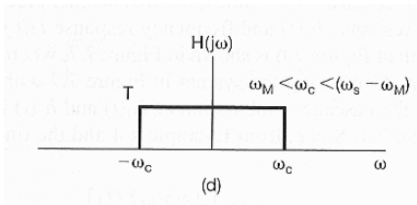








$$h(t) = \frac{\omega_c T \sin(\omega_c t)}{\pi \omega_c t}$$

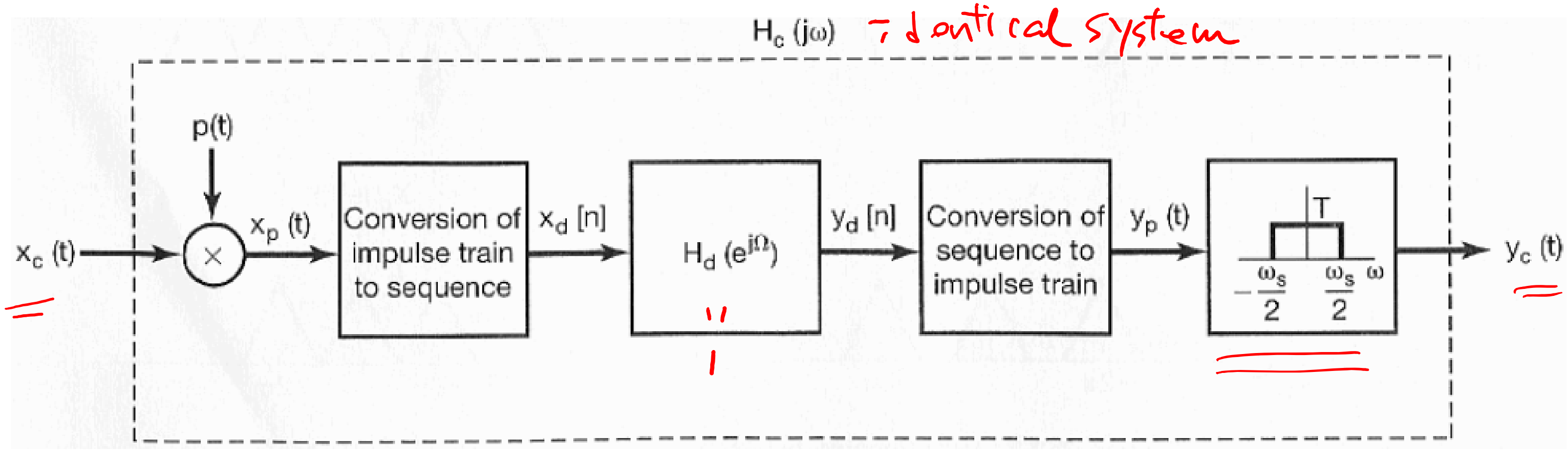


$$x_r(t) = \sum_{n=-\infty}^{+\infty} x(nT)h(t-nT)$$

$$= \sum_{n=-\infty}^{+\infty} x(nT)\frac{\omega_c T}{\pi}\frac{\sin(\omega_c(t-nT))}{\omega_c(t-nT)}$$

Exact reconstruction can be obtained if x(t) is band limited and if the sampling frequency is greater than the Nyquist rate.

7.4 DT Processing of CT Signals



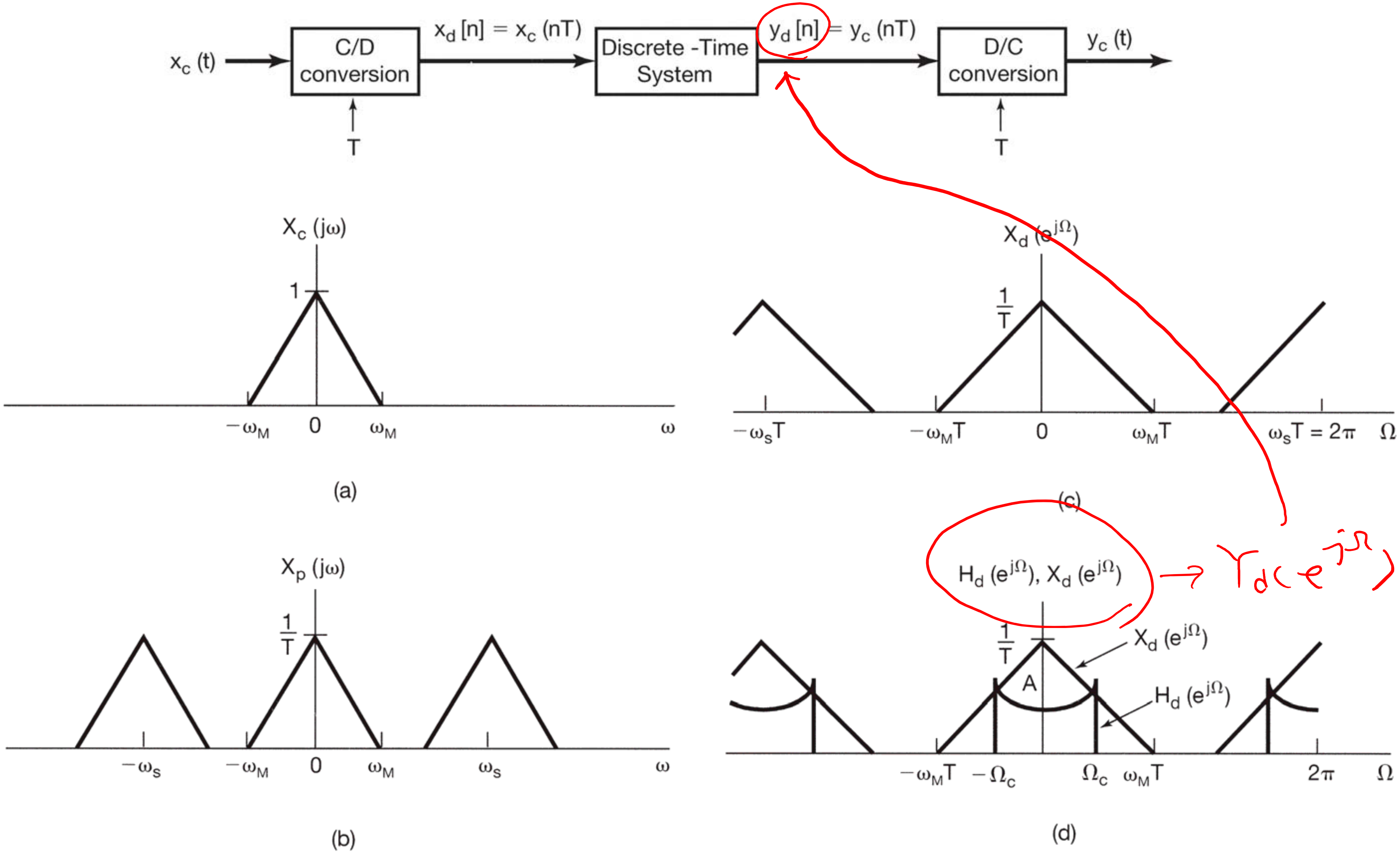
$$Y_c(j\omega) = X_c(j\omega)H_d(e^{j\omega T}).$$

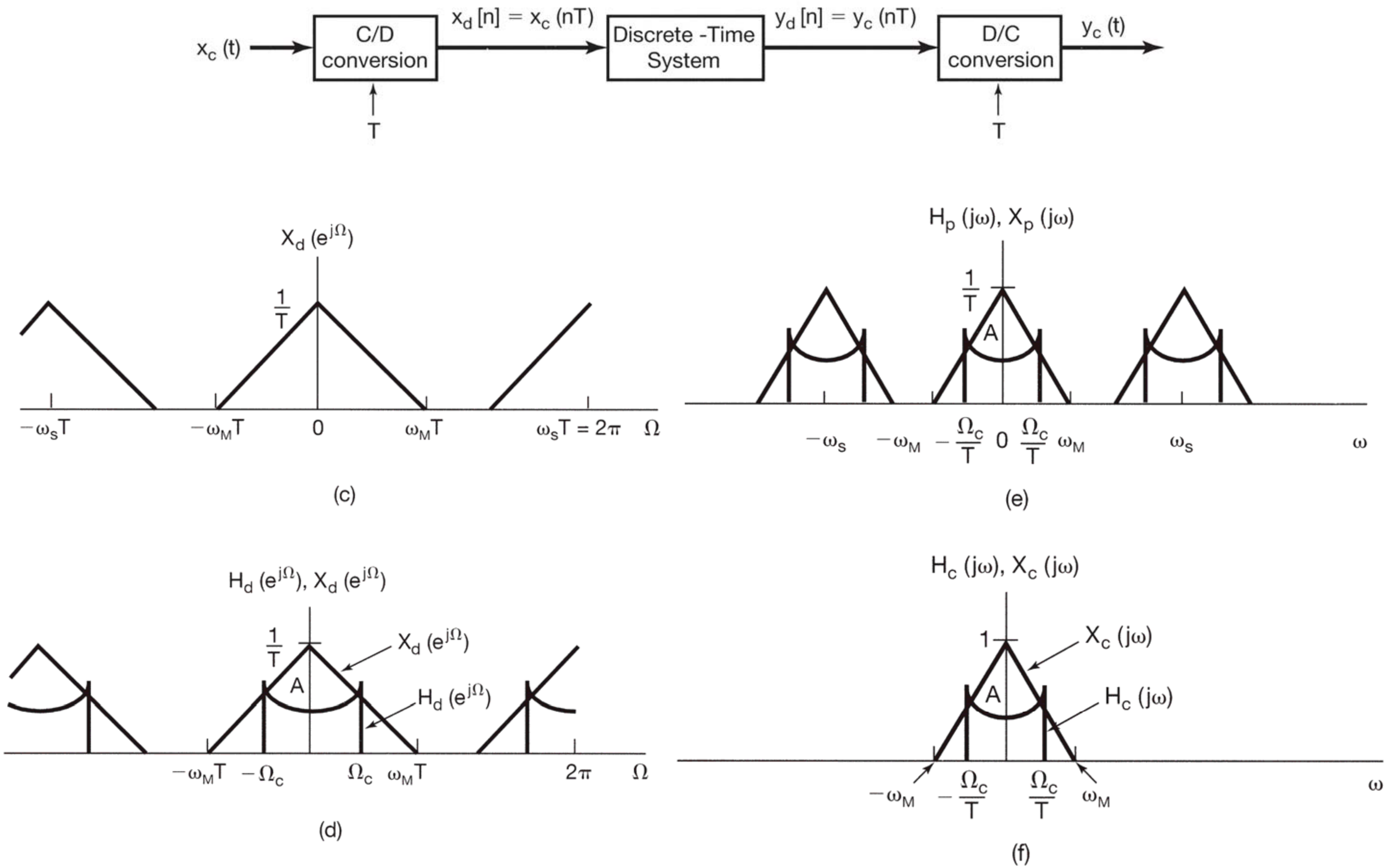
$$H_c(j\omega) = \begin{cases} H_d(e^{j\omega T}), & |\omega| < \omega_s/2 \\ 0, & |\omega| > \omega_s/2 \end{cases}.$$

In summary, the frequency response $H_d(e^{j\Omega})$ in Figure 7.24 can be derived from:

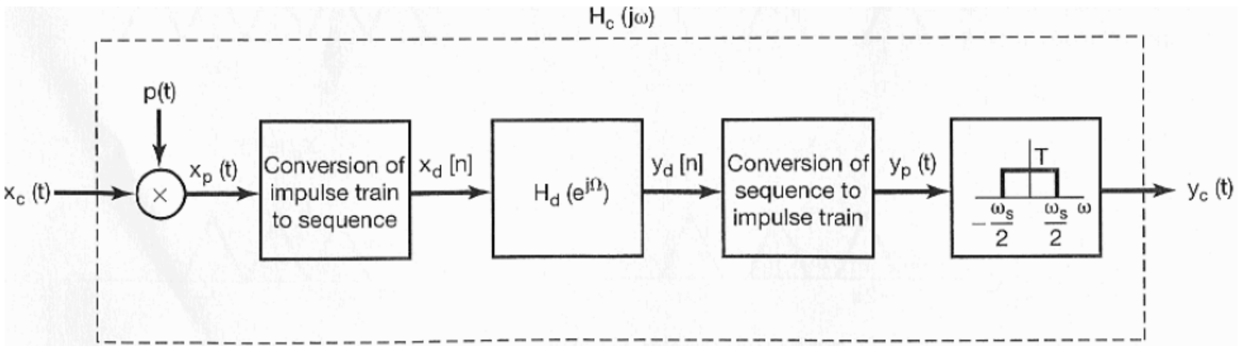
$$H_d(e^{j\Omega}) = H_c(j\Omega/T) \quad \text{for } |\Omega| < \pi,$$

$$H_d(e^{j\Omega}) = H_d(e^{j(\Omega+2\pi)}).$$


$x_c(t)$
C/D conversion
T
$x_d[n] = x_c(nT)$
Discrete -Time System
$y_d[n] = y_c(nT)$
D/C conversion
T
$y_c(t)$
$X_c(j\omega)$
1
$-\omega_M$
0
ω_M
ω
(a)
$X_d(e^{j\Omega})$
$\frac{1}{T}$
$-\omega_s T$
$-\omega_M T$
0
$\omega_M T$
$\omega_s T = 2\pi$
Ω
(c)
$X_p(j\omega)$
$\frac{1}{T}$
$-\omega_s$
$-\omega_M$
0
ω_M
ω_s
ω
(b)
$H_d(e^{j\Omega}), X_d(e^{j\Omega})$
$\rightarrow Y_d(e^{j\Omega})$
$\frac{1}{T}$
$X_d(e^{j\Omega})$
A
$H_d(e^{j\Omega})$
$-\omega_M T$
$-\Omega_c$
Ω_c
$\omega_M T$
2π
Ω
(d)



$x_c(t)$
C/D conversion
$x_d[n] = x_c(nT)$
Discrete -Time System
$y_d[n] = y_c(nT)$
D/C conversion
$y_c(t)$
T
T
$X_d(e^{j\Omega})$
$\frac{1}{T}$
$-\omega_s T$
$-\omega_M T$
0
$\omega_M T$
$\omega_s T = 2\pi$
Ω
(c)
$H_p(j\omega), X_p(j\omega)$
$\frac{1}{T}$
A
$-\omega_s$
$-\omega_M$
$-\frac{\Omega_c}{T}$
0
$\frac{\Omega_c}{T}$
ω_M
ω_s
ω
(e)
$H_d(e^{j\Omega}), X_d(e^{j\Omega})$
$\frac{1}{T}$
$X_d(e^{j\Omega})$
A
$H_d(e^{j\Omega})$
$-\omega_M T$
$-\Omega_c$
Ω_c
$\omega_M T$
2π
Ω
(d)
$H_c(j\omega), X_c(j\omega)$
1
$X_c(j\omega)$
A
$H_c(j\omega)$
$-\omega_M$
$-\frac{\Omega_c}{T}$
$\frac{\Omega_c}{T}$
ω_M
ω
(f)



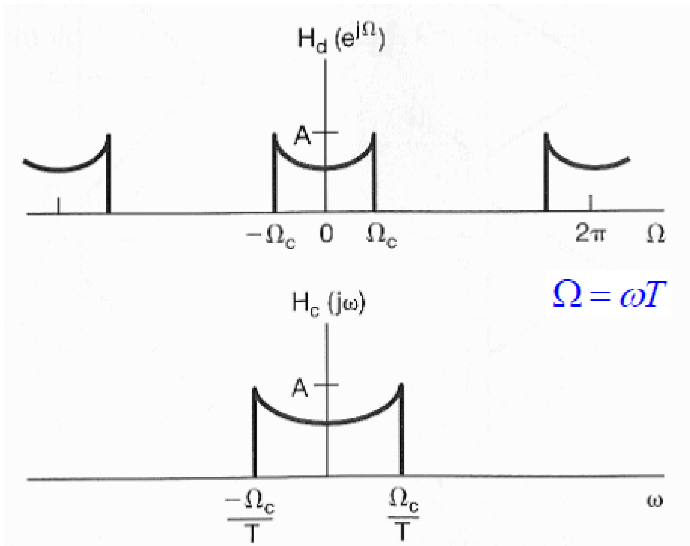
- Relation between CT & DT responses

Comparing Figs. (a) and (f) yileds

$$Y_c(j\omega) = X_c(j\omega)H_d(e^{j\omega T})$$

$$H_c(j\omega) = \begin{cases} H_d(e^{j\omega T}), & |\omega| < \omega_s/2 \\ 0, & |\omega| > \omega_s/2 \end{cases}$$

$H_c(j\omega)$ is one period of $H_d(e^{j\Omega})$ with a linear scale change applied to the frequency axis.

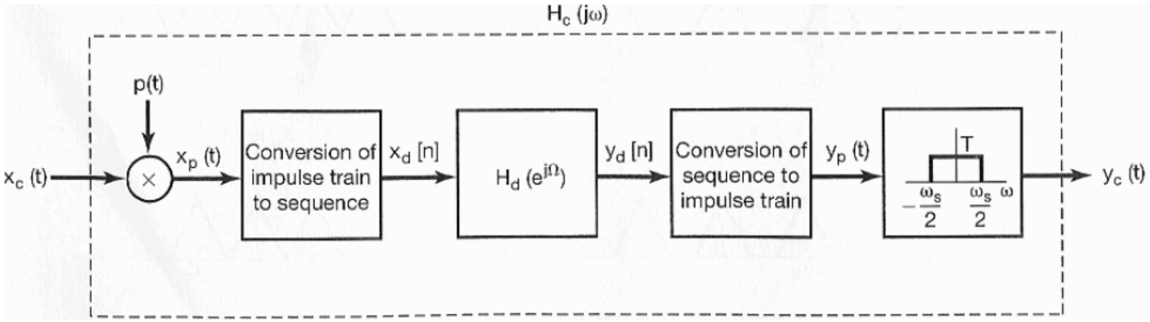


$x(t) \longleftrightarrow X(j\omega)$

$\frac{d\,x(t)}{dt} \longleftrightarrow j\omega\, X(j\omega)$

- Digital Differentiator

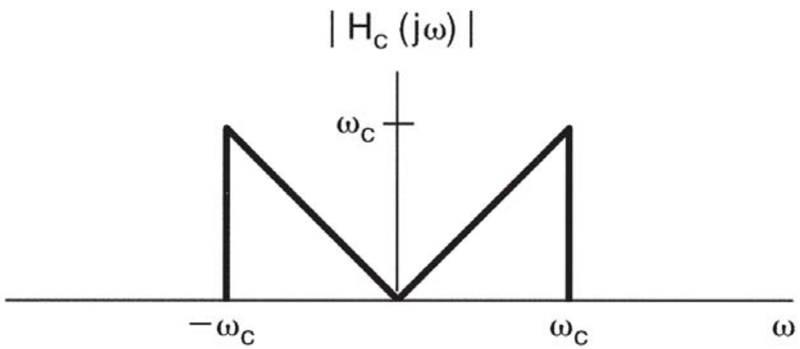
$H(j\omega)\, X(j\omega)$

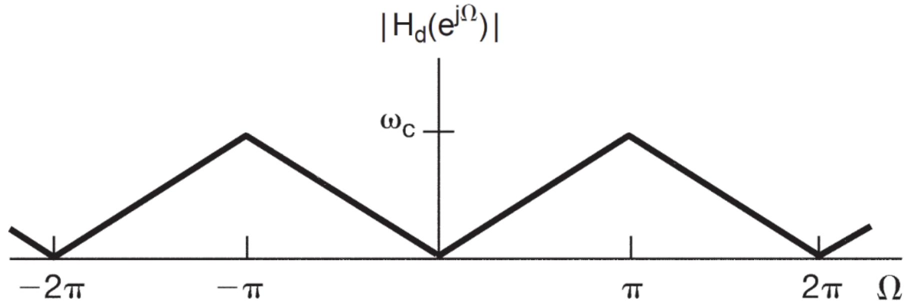


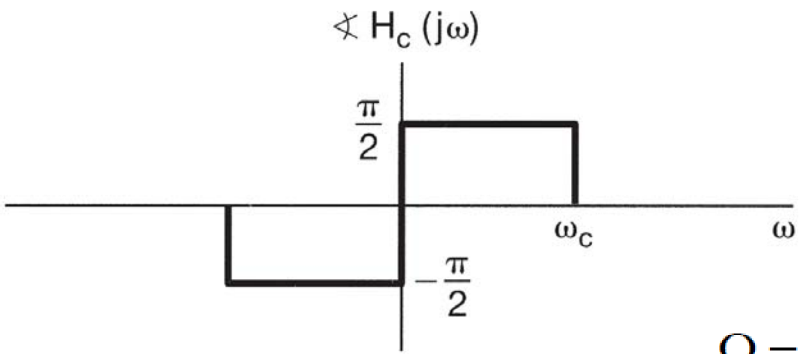
$$H_c(j\omega) = \begin{cases} j\omega, & |\omega| < \omega_c \\ 0, & |\omega| > \omega_c \end{cases}$$

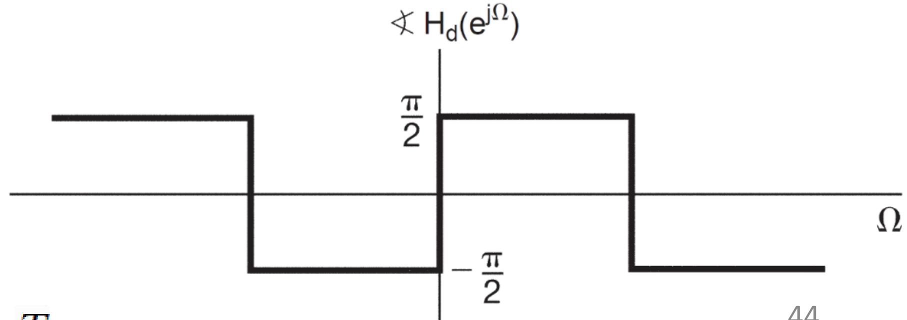
Choosing $\omega_s = 2\omega_c$, we have

$$H_d(e^{j\Omega}) = j\left(\frac{\Omega}{T}\right), \quad |\Omega| < \pi$$

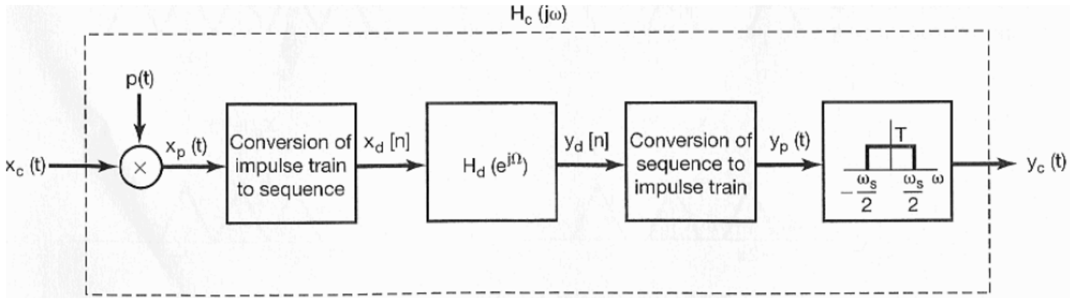








$$\Omega = \omega T$$



- Digital Differentiator $h_d[n]$
  - To determine the impulse response of the DT filter H_d, we may set

$$x_c(t) = \frac{\sin(\pi t/T)}{\pi t}, \quad T\text{: sampling period}$$

Then

$$y_c(t) = \frac{d}{dt}x_c(t) = \frac{\cos(\pi t/T)}{Tt} - \frac{\sin(\pi t/T)}{\pi t^2}$$

$$\boxed{x_d[n]} = x_c(nT) = \frac{\sin(\pi n)}{\pi nT} = \boxed{\frac{1}{T}\delta[n]}$$

$$\boxed{y_d[n]} = y_c(nT) = \begin{cases} \dfrac{\cos(\pi n)}{nT^2} = \boxed{\dfrac{(-1)^n}{nT^2}}, & n \neq 0 \\ 0, & n = 0 \end{cases}$$

$$\Rightarrow \quad h_d[n] = \begin{cases} \dfrac{(-1)^n}{nT}, & n \neq 0 \\ 0, & n = 0 \end{cases}$$

$\frac{j\Omega}{T}$

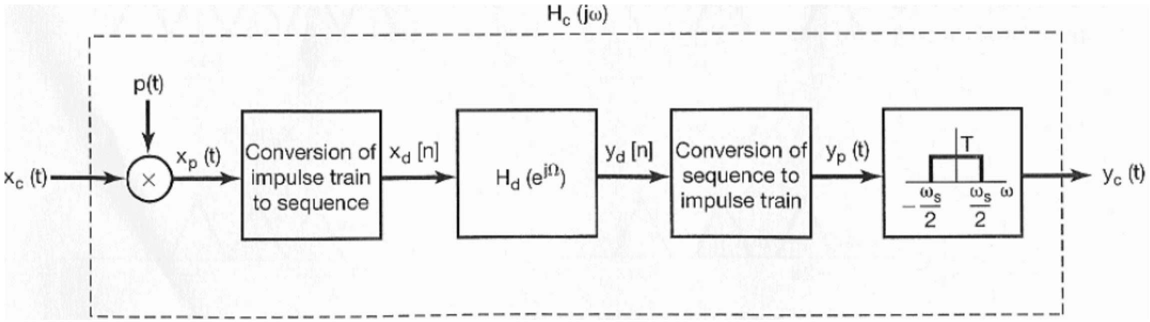
$$h_d[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} \boxed{H_d(e^{j\Omega})} e^{j\Omega n} d\Omega = \frac{j}{2\pi T}\int_{-\pi}^{\pi} \Omega e^{j\Omega n} d\Omega$$

$$= \frac{\Omega}{2\pi Tn} e^{j\Omega n}\Big|_{-\pi}^{\pi} - \frac{1}{2\pi Tn}\int_{-\pi}^{\pi} e^{j\Omega n} d\Omega = \frac{(-1)^n}{nT}$$

Output of the discrete-time filter when the input is δ(n)/T

Output of the discrete-time filter when the input is δ(n)

- Half-Sample Delay

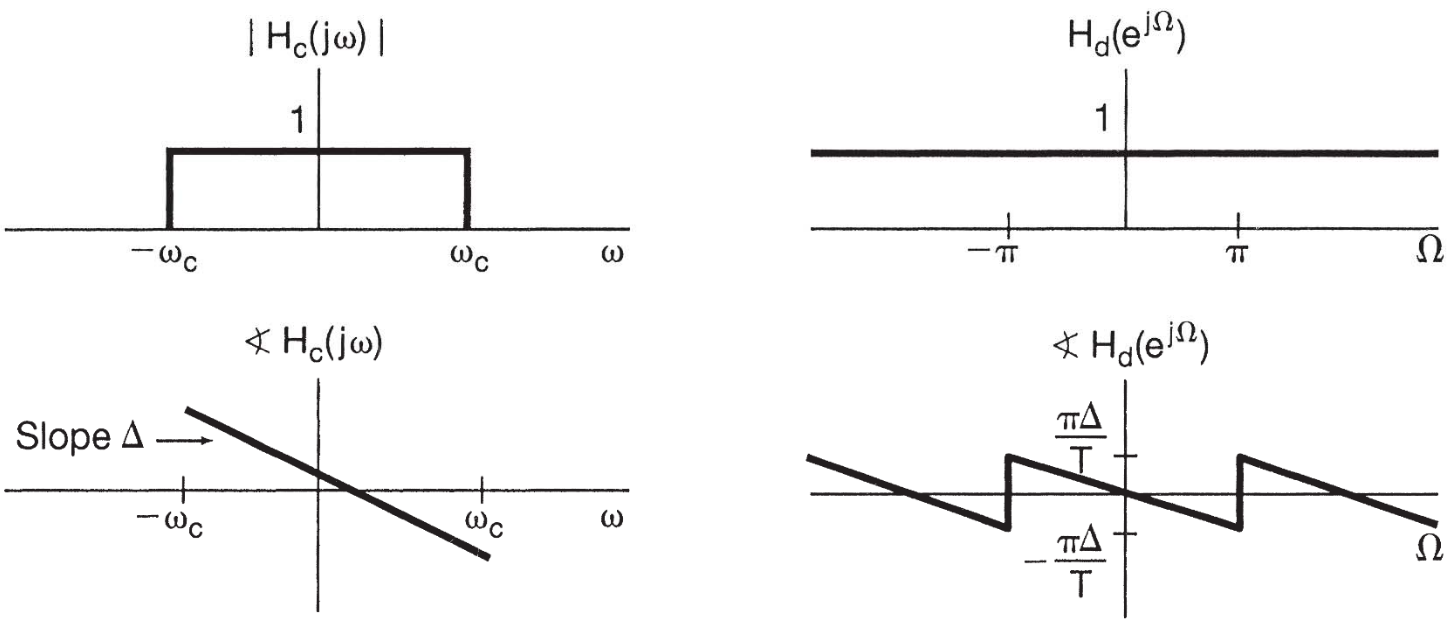


$$y_c(t) = x_c(t-\Delta)$$

$$H_c(j\omega) = \begin{cases} e^{-j\omega\Delta}, & |\omega| < \omega_c \\ 0, & |\omega| > \omega_c \end{cases}$$

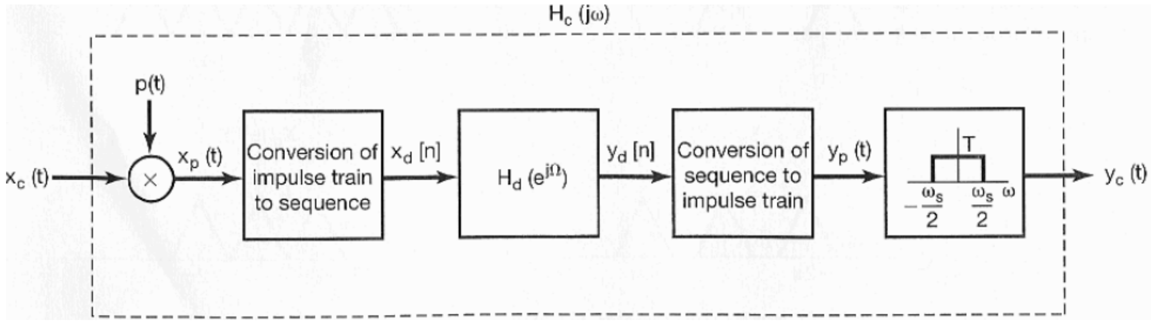
$$H_d(e^{j\Omega}) = e^{-j\Omega\boxed{\Delta/T}}, \quad |\Omega| < \pi$$

$$y_d[n] = x_d[n - \underline{\Delta/T}]$$



(a) (b)

- Half-Sample Delay



$$y_c(t) = x_c(t-\Delta)$$

$$H_c(j\omega) = \begin{cases} e^{-j\omega\Delta}, & |\omega| < \omega_c \\ 0, & |\omega| > \omega_c \end{cases}$$

$$H_d(e^{j\Omega}) = e^{-j\Omega\Delta/T}, \quad |\Omega| < \pi$$

$$y_d[n] = x_d[n - \Delta / T]$$

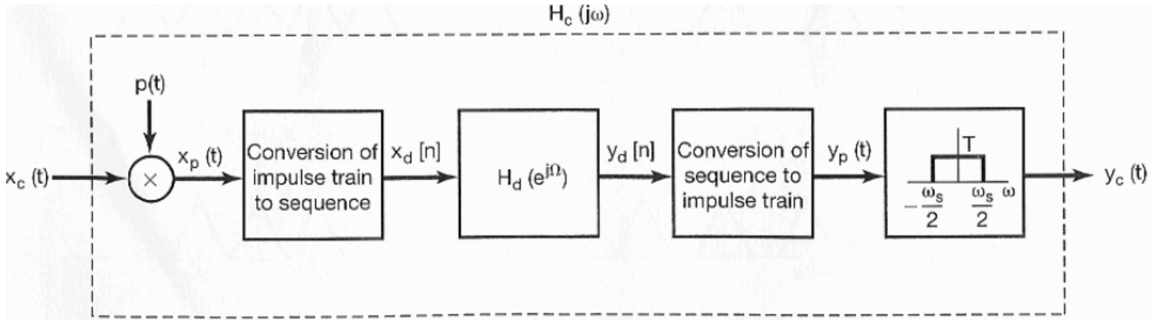
$$h_d[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\Omega})e^{j\Omega n}d\Omega = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{j\Omega(n-\Delta/T)}d\Omega$$
$$= \frac{\sin(\pi(n-\Delta/T))}{\pi(n-\Delta/T)}.$$

For Δ/T an integer, the sequence $y_d[n]$ is a delayed replica of $x_d[n]$; that is,

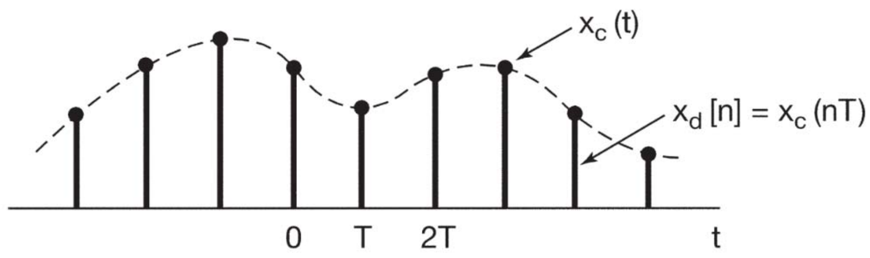
$$y_d[n] = x_d\left[n - \frac{\Delta}{T}\right].$$

- Half-Sample Delay

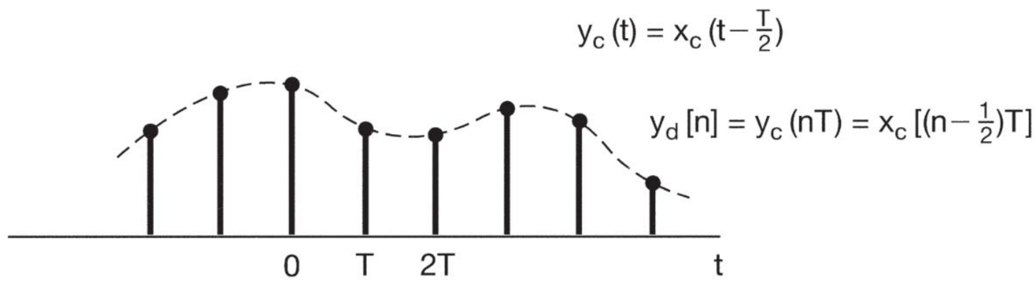
$$y_d[n] = x_d\left[n - \frac{\Delta}{T}\right].$$



For Δ/*T* a non-integer, we can interpret the relationship between $x_d[n]$ and $y_d[n]$ in terms of band-limited interpolation. The sequence $y_d[n]$ is equal to samples of a shifted version of the band-limited interpolation of $x_d[n]$ as:



(a)



(b)

[Example 7.3]

The approach in Example 7.2 is also applicable to determining the impulse response $h_d[n]$ of the discrete-time filter in the half-sample delay system.
With reference to Figure 7.24, let

$$x_c(t) = \frac{\sin(\pi t/T)}{\pi t}. \tag{7.37}$$

It follows from Example 7.2 that

$$x_d[n] = x_c(nT) = \frac{1}{T}\delta[n].$$

Also, since there is no aliasing for the band-limited input in eq. (7.37), the output of the half-sample delay system is

$$\underline{y_c(t) = x_c(t - T/2)} = \underline{\frac{\sin(\pi(t - T/2)/T)}{\pi(t - T/2)}},$$

and the sequence $y_d[n]$ in Figure 7.24 is $y_d[n] = y_c(nT) = \underline{\frac{\sin(\pi(n - \frac{1}{2}))}{T\pi(n - \frac{1}{2})}}.$

We conclude that

$$h_d[n] = \frac{\sin(\pi(n - \frac{1}{2}))}{\pi(n - \frac{1}{2})}.$$

$x(t) * h(t) = y(t)$

$\delta(t) * h(t) = h(t)$

7.5 Sampling of DT Signals

- Modeling by Impulse-Train Sampling

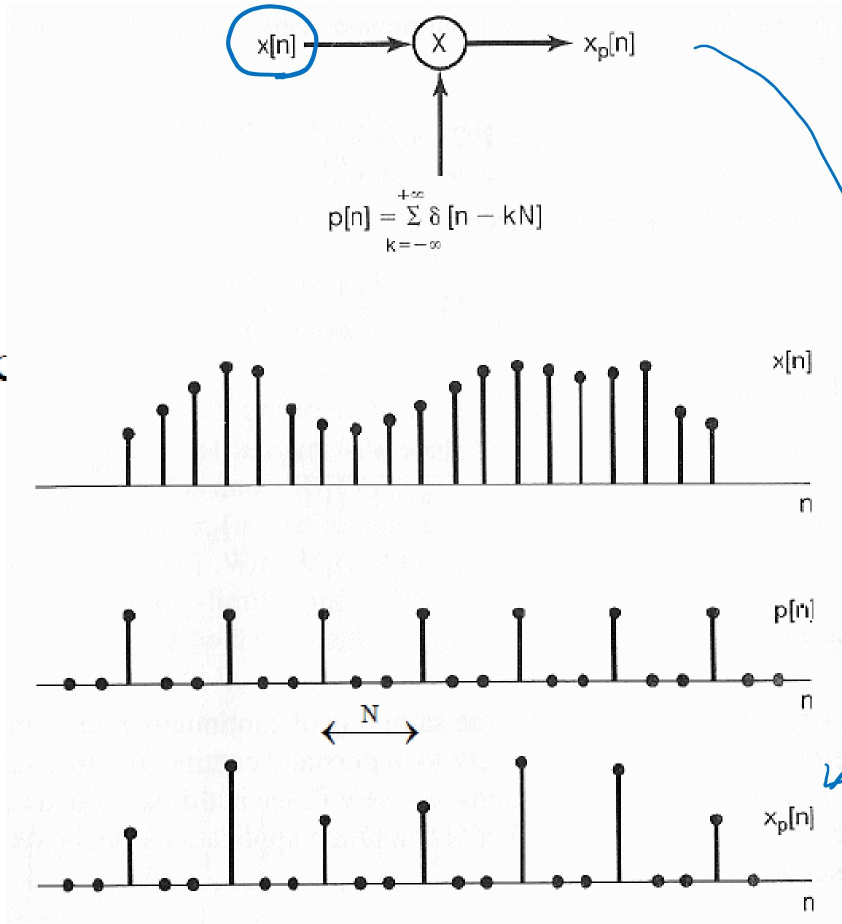
$$x_p[n] = \begin{cases} x[n], & \text{if } n = kN \\ 0, & \text{otherwise} \end{cases}$$

N : sampling period

$$\omega_s = \frac{2\pi}{N} : \text{ sampling frequenc}$$

$$x_p[n] = x[n]p[n]$$

$$= \sum_{k=-\infty}^{+\infty} x[kN]\delta[n-kN]$$



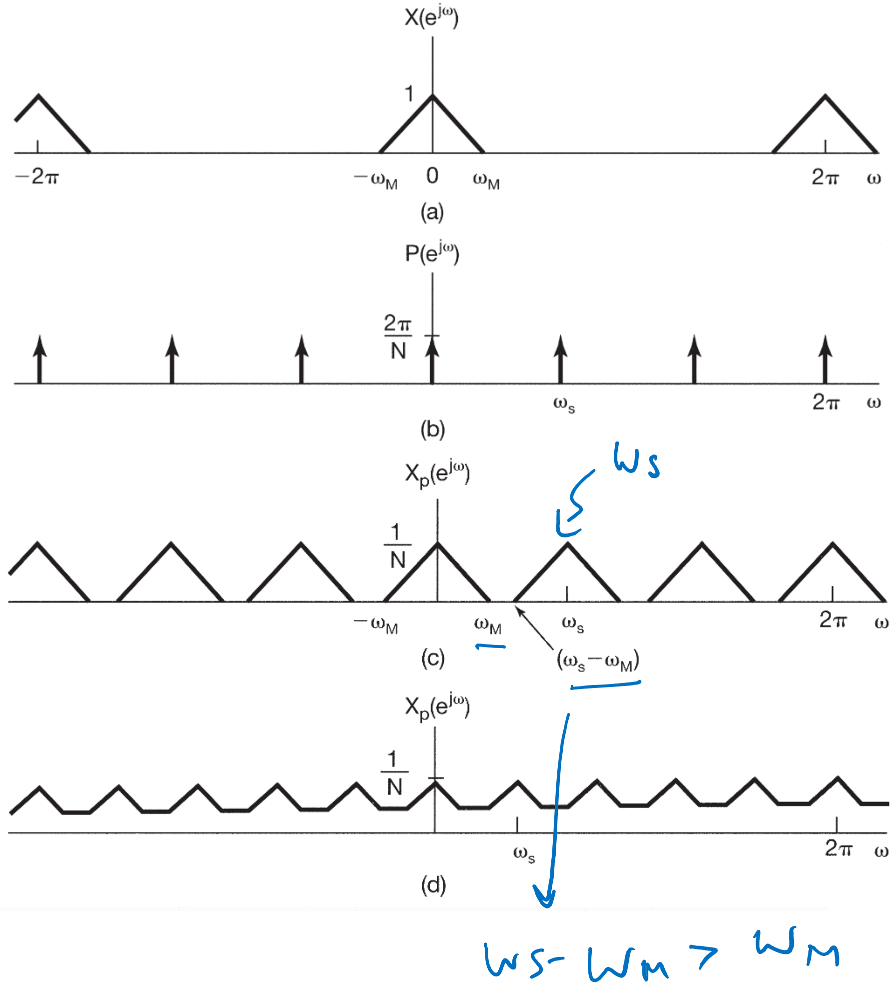
7.5 Sampling of DT Signals

- Frequency-domain analysis of DT sampling

$$X_p(e^{j\omega}) = \frac{1}{2\pi}\int_{2\pi} P(e^{j\theta})X(e^{j(\omega-\theta)})d\theta$$

$$P(e^{j\omega}) = \frac{2\pi}{N}\sum_{k=-\infty}^{+\infty}\delta(\omega-k\omega_s)$$

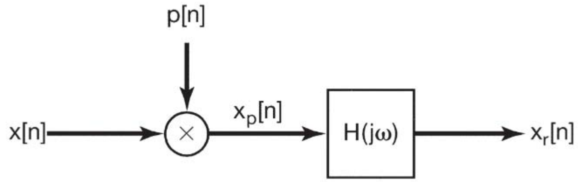
$$\Rightarrow X_p(e^{j\omega}) = \frac{1}{N}\sum_{k=0}^{N-1}X(e^{j(\omega-k\omega_s)})$$



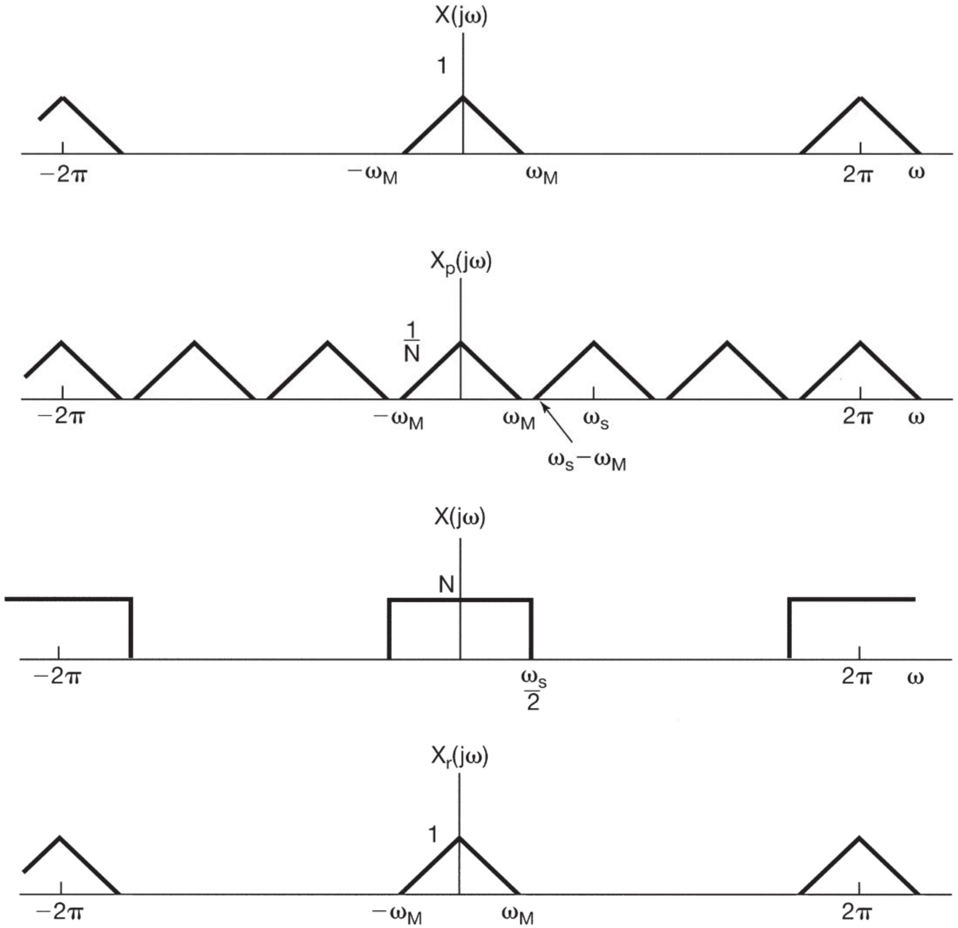
7.5 Sampling of DT Signals

- Exactly Recovery Using LPF

$x[n]$ can be recovered from $x_p[n]$ by means of a lowpass filter with gain N and a cutoff frequency ω_c, $\omega_M < \omega_c < \omega_s - \omega_M$.



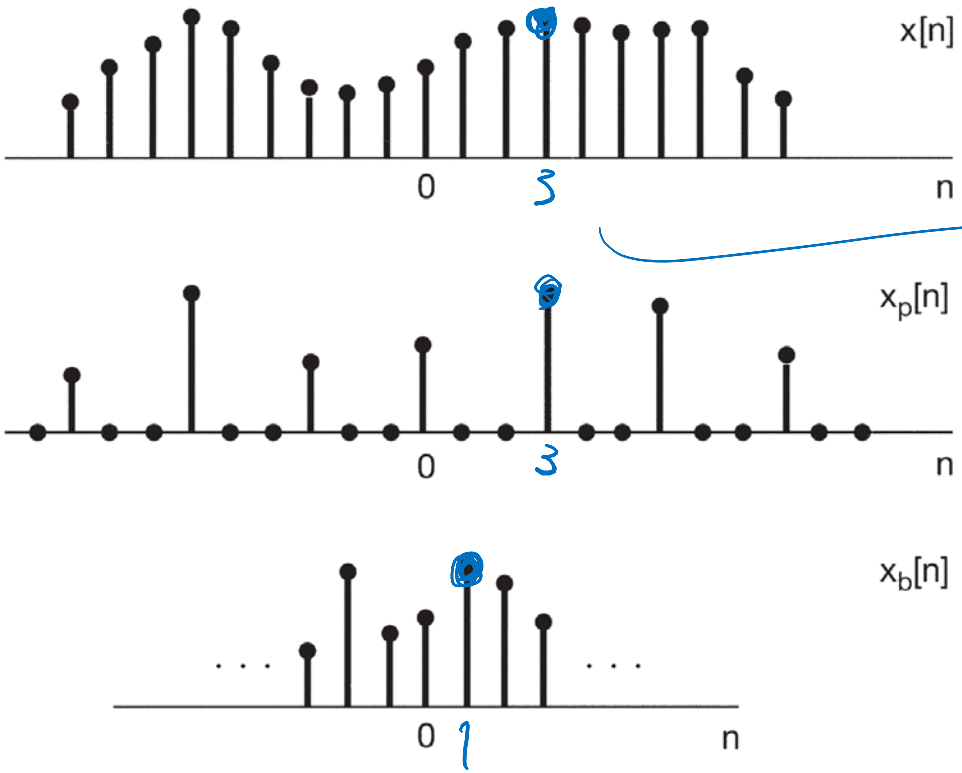
(a)



(b)

7.5 Sampling of DT Signals

- DT Decimation: Down-Sampling
  - The operation of extracting every Nth sample is called decimation. The process of decimation is often called downsampling.



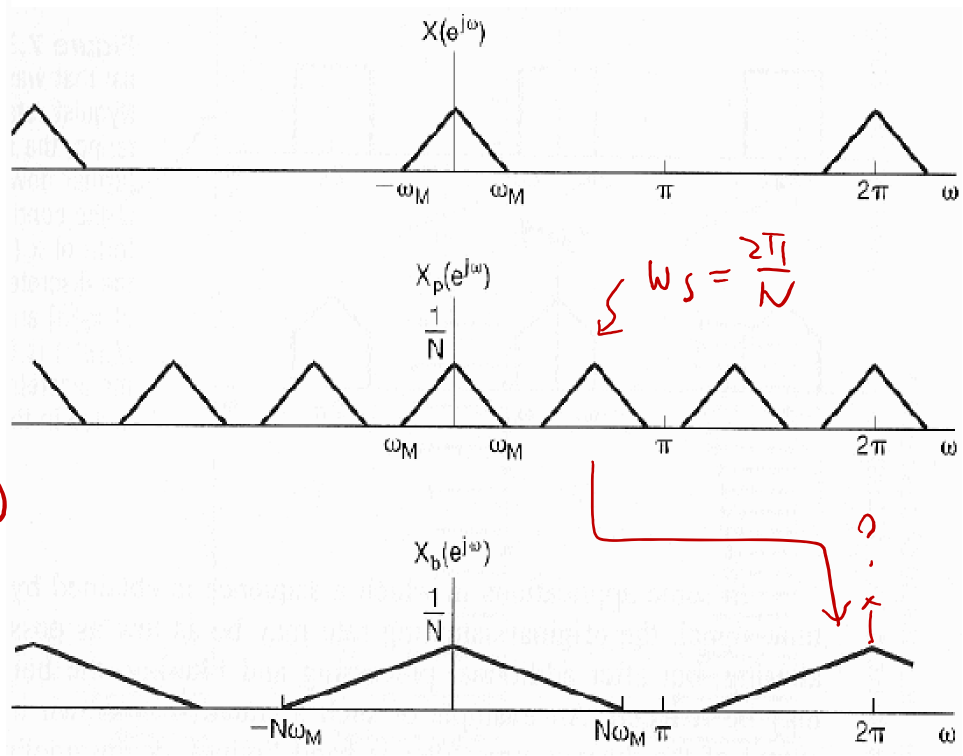
n=1

3

$$x_b[n] = x_p[nN].$$

$$x_b[n] = x[nN],$$

- DT Decimation: Down-Sampling



$$x_b[n] = x_p[nN] = x[nN]$$

$$X_b(e^{j\omega}) = \sum_{k=-\infty}^{\infty} x_b[k]e^{-j\omega k} = \sum_{k=-\infty}^{\infty} x_p[kN]e^{-j\omega k}$$

$$= \sum_{\substack{n=\text{integer}\\ \text{multiples of N}}} x_p[n]e^{-j\omega n/N}$$

Since $x_p[n] = 0$ when n is not an integer multiple of N,

$$X_b(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x_p[n]e^{-j\omega n/N} = X_p(e^{j\omega/N})$$

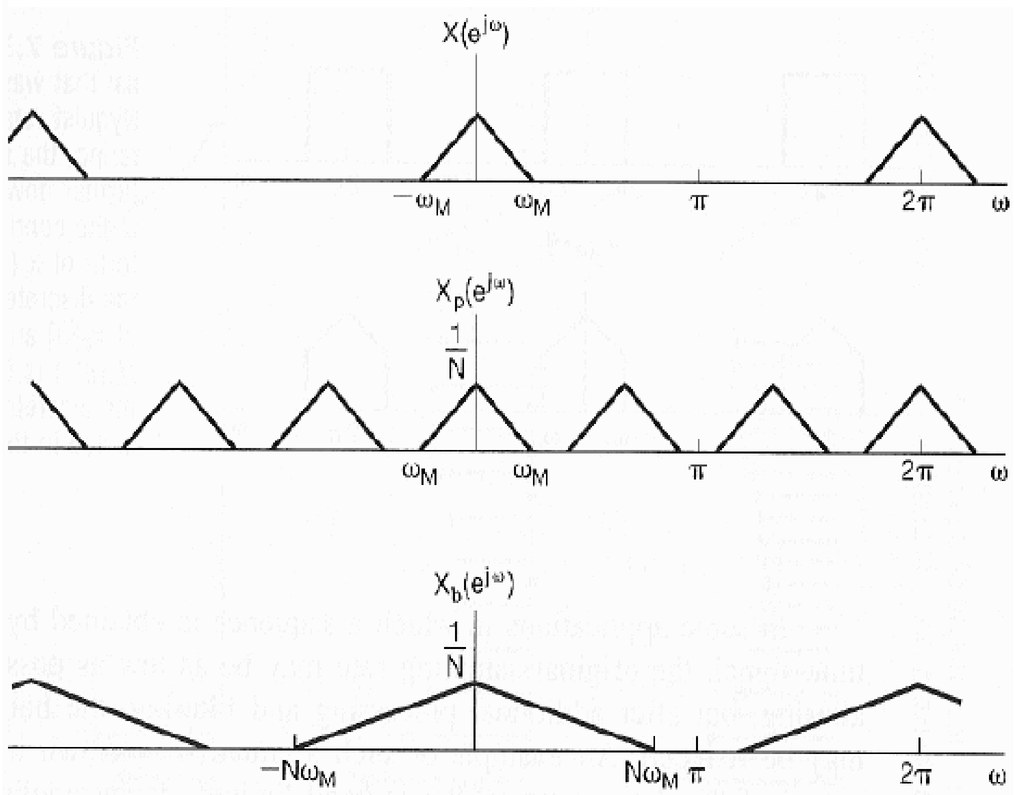
Scale the time axis by N $\Rightarrow$ Frequency is scaled by $\frac{1}{N}$.

$$\boxed{X_b(e^{j\omega}) = X_p(e^{j\omega/N}).}$$

$$\boxed{X_b(e^{j\omega}) = \frac{1}{N}\sum_{k=0}^{N-1} X(e^{j((\omega-2\pi k)/N)}).}$$

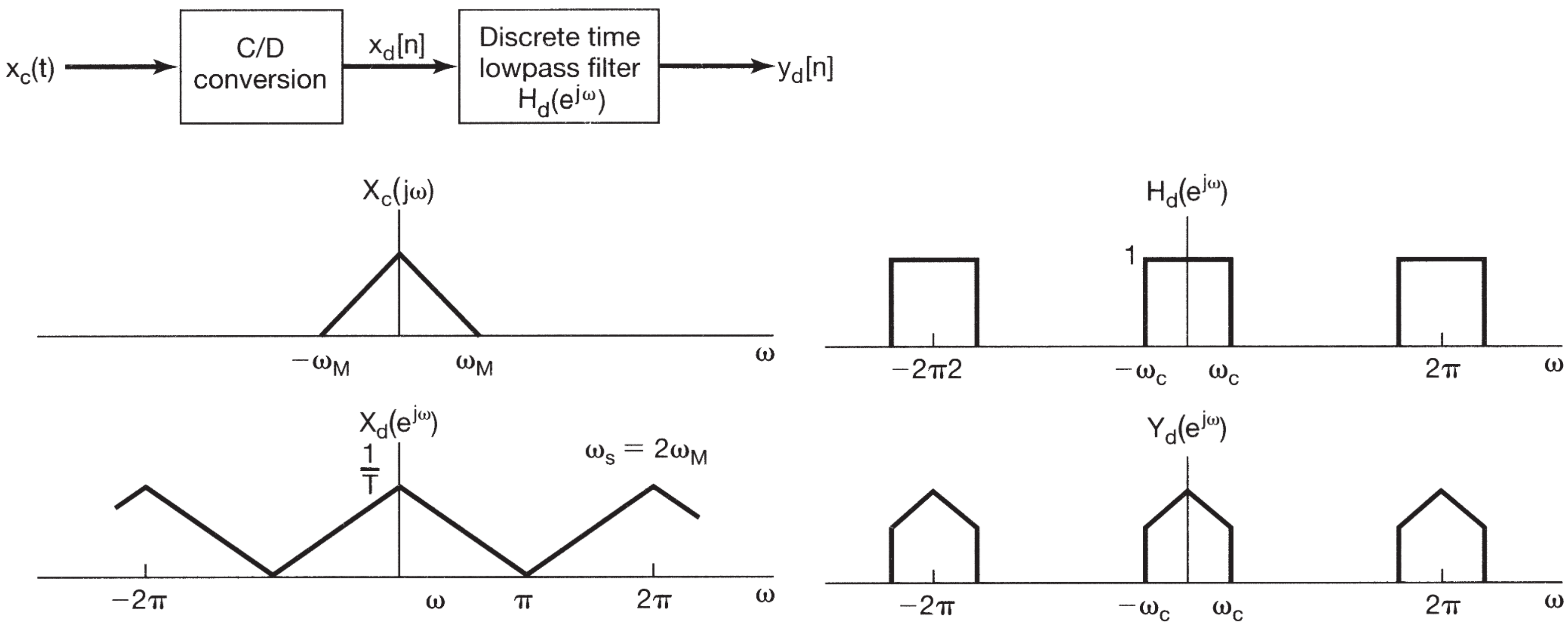
7.5 Sampling of DT Signals

- Remark #1 for DT Decimation (i.e., down-sampling)
  - If the signal is to be decimated without introducing aliasing, the original CT signal must be oversampled so that the sampling rate can be reduced without aliasing.
  - That is, $X(e^{j\omega})$ cannot occupy the full frequency band.



7.5 Sampling of DT Signals

- Remark #2 for DT Decimation (i.e., down-sampling)
  - In some applications, the original sampling rate for CT signals can be as low as possible to avoid aliasing.
  - With additional digital filtering/processing, the bandwidth of the sampled DT signals can be reduced, allowing further down-sampling.

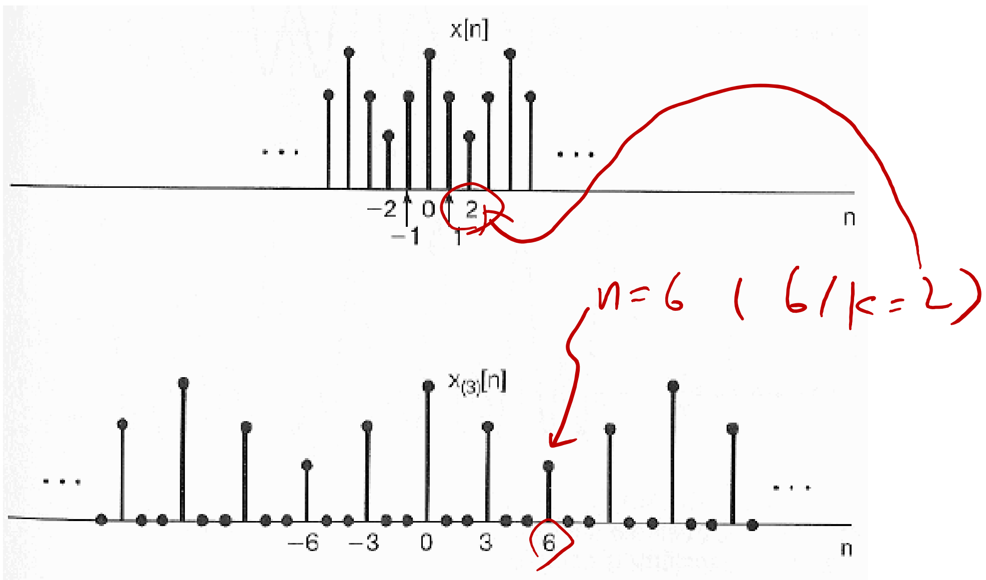


7.5 Sampling of DT Signals

- DT Interpolation: Up-Sampling
  - Recall that time expansion property in Sect. 5.3.7

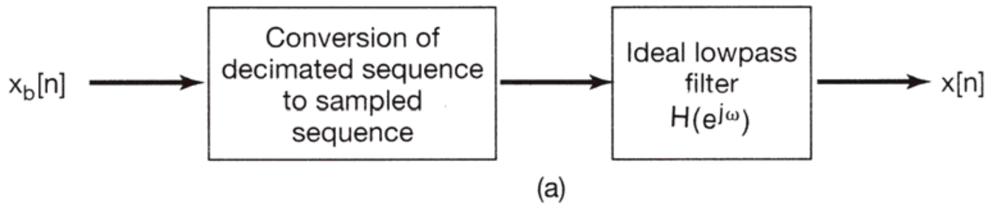
$$x_{(k)}[n] = \begin{cases} x[n/k], & \text{if n is a multiple of } k \\ 0, & \text{otherwise.} \end{cases}$$

$$x_{(k)}[n] \xleftrightarrow{F} X(e^{jk\omega})$$

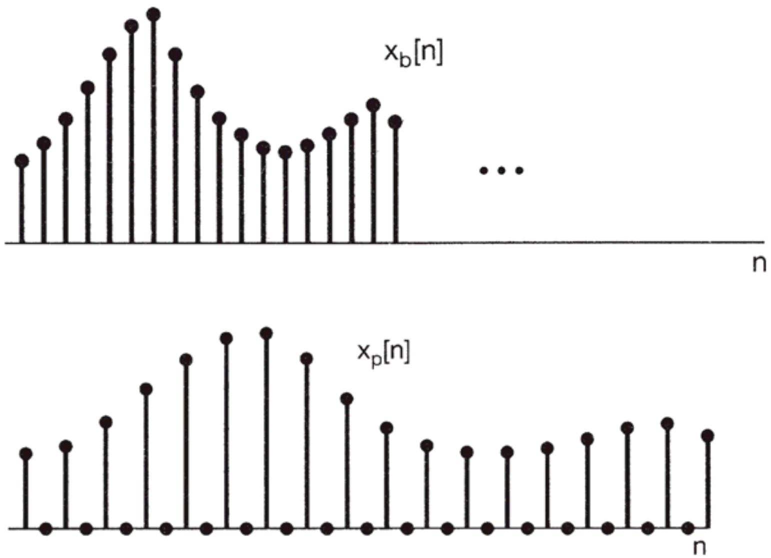


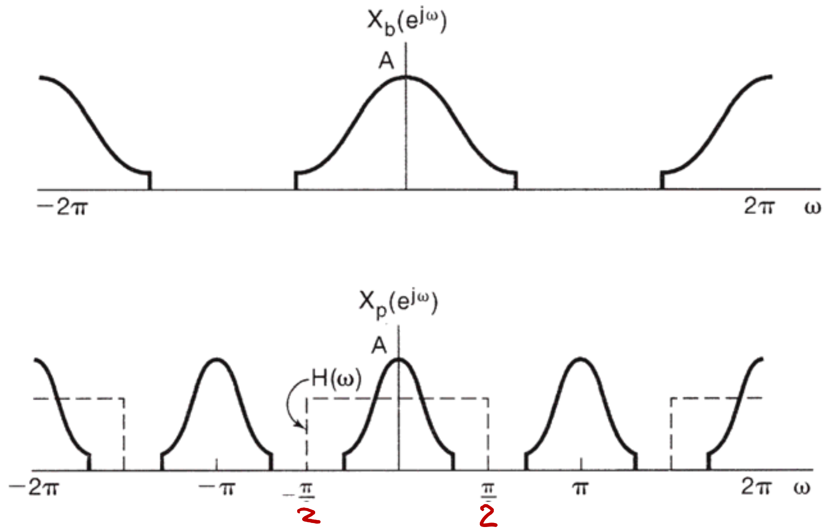
7.5 Sampling of DT Signals

- DT Interpolation: Up-Sampling
  - The operation of converting a signal to higher equivalent sampling rate is called interpolation or up-sampling.

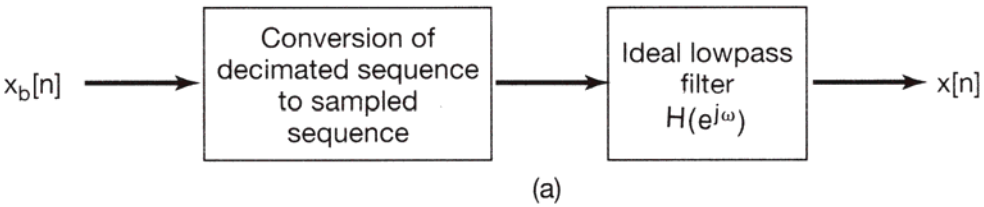


(a)

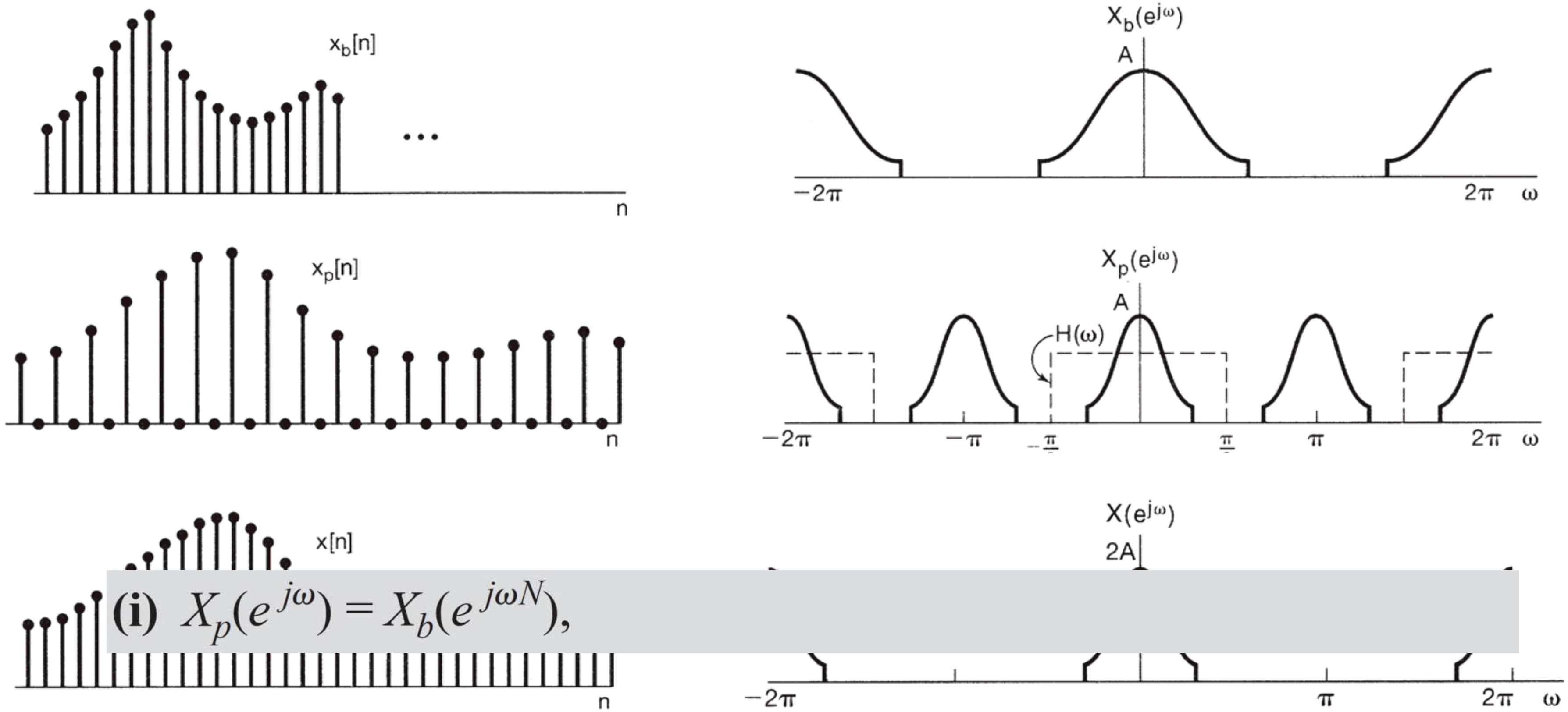




7.5 Sampling of DT Signals

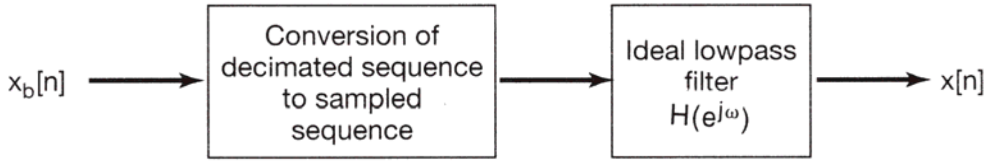


(i) $x_p[Nn] = x_b[n]$,
$x_p[n] = 0$ if n is not a multiple of N,



(i) $X_p(e^{j\omega}) = X_b(e^{j\omega N})$,

7.5 Sampling of DT Signals



(a)

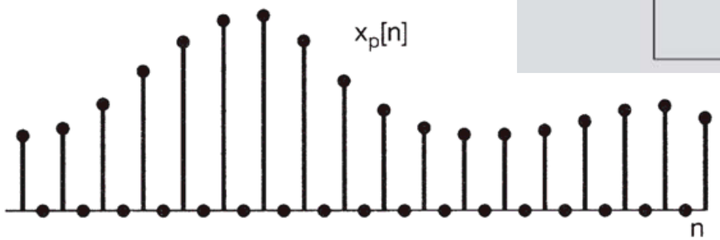
(ii) $X(e^{j\omega}) = X_p(e^{j\omega})H(e^{j\omega})$

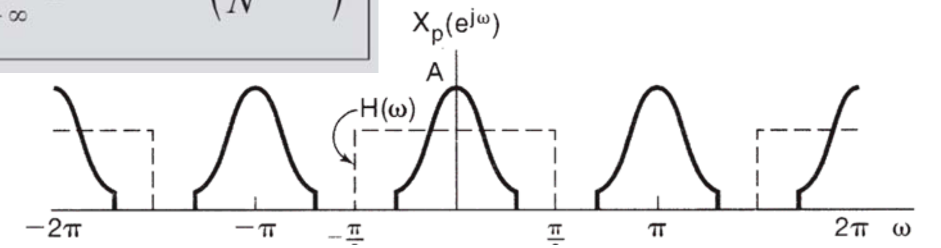
$$\text{where} \quad H(e^{j\omega}) = N \text{ when } 0 \le |\omega| < \pi/N,$$

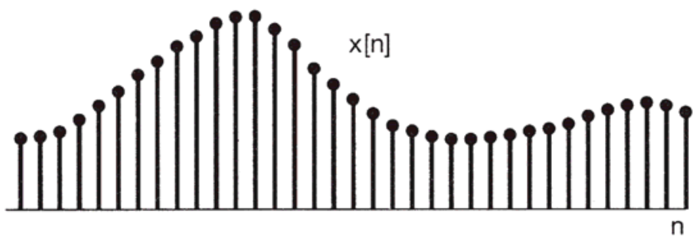
$$H(e^{j\omega}) = 0 \quad \text{when } \pi/N < |\omega| \le \pi, \qquad H(e^{j\omega}) = H(e^{j(\omega+2\pi)}).$$

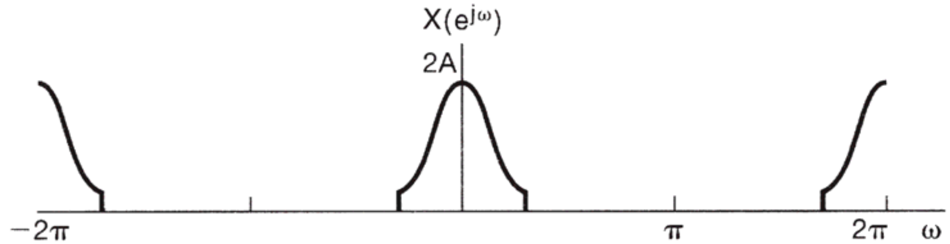
$$\textbf{(ii)}\ x[n] = \sum_{m=-\infty}^{\infty} x_p[mN] \operatorname{sinc}\left(\frac{n}{N} - m\right),$$

$$\text{i.e.,} \quad \boxed{x[n] = \sum_{m=-\infty}^{\infty} x_b[m] \operatorname{sinc}\left(\frac{n}{N} - m\right)}.$$

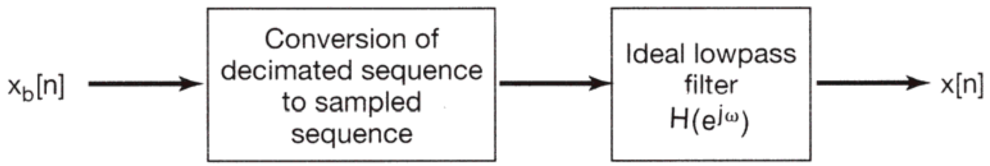








7.5 Sampling of DT Signals



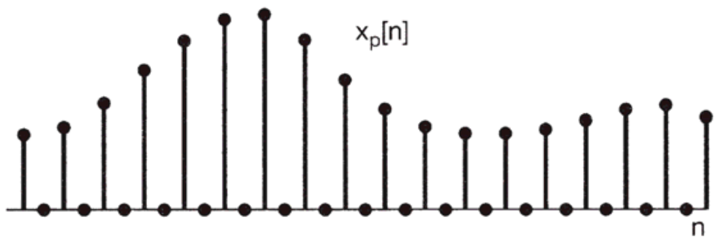
(a)

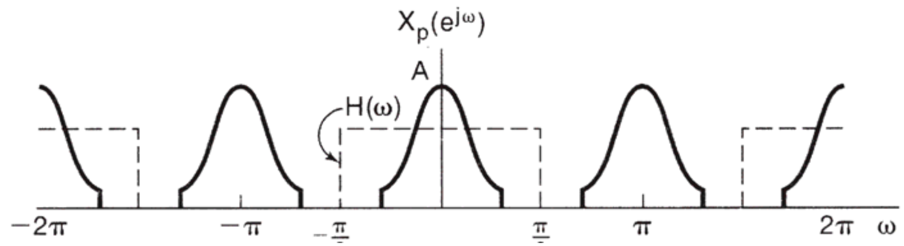
(ii) $x[n] = \sum_{m=-\infty}^{\infty} x_p[mN] \operatorname{sinc}\left(\frac{n}{N} - m\right)$,

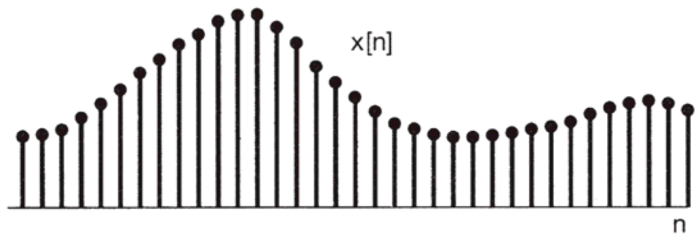
i.e., $\boxed{x[n] = \sum_{m=-\infty}^{\infty} x_b[m] \operatorname{sinc}\left(\frac{n}{N} - m\right)}$.

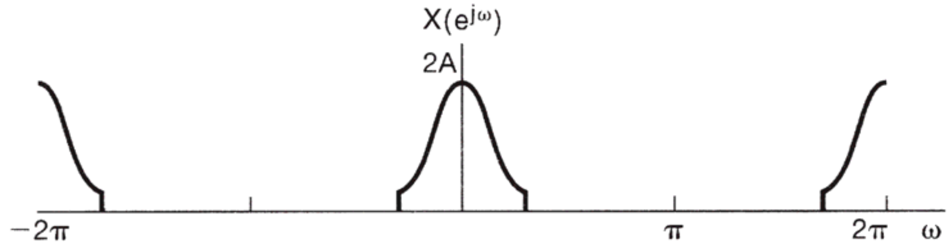
$$\boxed{X(e^{j\omega}) = NX_b(e^{j\omega N}) \quad \text{when } 0 \le |\omega| < \pi/N},$$

$$\boxed{X(e^{j\omega}) = 0 \quad \text{when } \pi/N < |\omega| \le \pi,} \qquad \boxed{X(e^{j\omega}) = X(e^{j(\omega + 2\pi)})}.$$

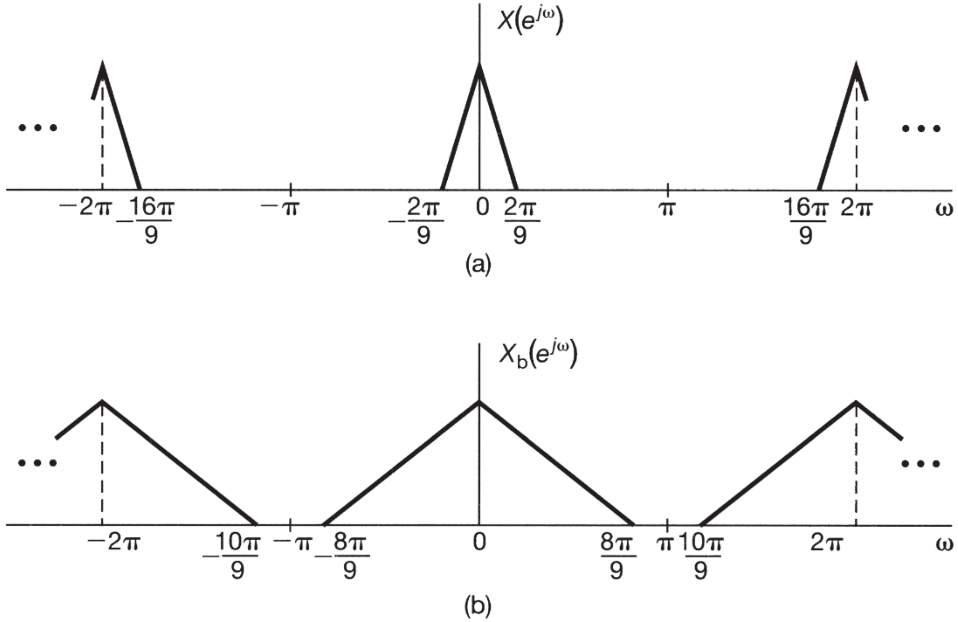








- Downsampling + Upsampling:
  How to deal with non-integer up/down-sampling?
  - The lowest sampling rate is…



  - Can we do better?

- Downsampling + Upsampling:
  How to deal with non-integer up/down-sampling?
  - For example, we cannot directly scale the frequency by 9/2, since it is not an integer.
  - We need to do so in 2 steps.