Signals & Systems

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Yu-Chiang Frank Wang 王鈺強, Associate Professor

Dept. Electrical Engineering, National Taiwan University

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Ch. 9 Laplace Transform

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definitions, calculation

properties

system analysis

unilateral form

9.1 The Laplace Transform

CTFT vs. Laplace Transform

Fourier transform	Laplace transform
$s = j\omega$	$s = \sigma + j\omega$
$X(j\omega) \triangleq \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$	$X(s) \triangleq \int_{-\infty}^{\infty} x(t) e^{-st} dt$
$x(t) \xleftrightarrow{\mathcal{F}} X(j\omega)$	$x(t) \xleftrightarrow{\mathcal{L}} X(s)$
$X(j\omega) = \mathcal{F}\{x(t)\}$	$X(s) = \mathcal{L}\{x(t)\}$
$x(t) = \mathcal{F}^{-1}\{X(j\omega)\}$	$x(t) = \mathcal{L}^{-1}\{X(s)\}$

FT is the LT evaluated along the $j\omega$ axis:

$$X(s)\Big|_{s=j\omega} = \mathcal{L}\{x(t)\}\Big|_{s=j\omega} = \mathcal{F}\{x(t)\} = X(j\omega)$$

9.5 Properties of Laplace Transform

- Differentiation in Time and s Domain

$$x(t) \xleftrightarrow{\mathcal{L}} X(s), \quad \text{ROC} = R$$

$$\boxed{\frac{d}{dt}x(t) \xleftrightarrow{\mathcal{L}} sX(s),} \text{ ROC contains } R$$

pole-zero cancellation may occur.

$$-tx(t) \xleftrightarrow{\mathcal{L}} \frac{dX(s)}{ds}, \quad \text{ROC} = R$$

Proof :

$$\frac{dx(t)}{dt} = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} X(s)\frac{de^{st}}{dt}ds = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} sX(s)e^{st}ds$$

$$\frac{d}{ds}\int_{-\infty}^{\infty} x(t)e^{-st}dt = \int_{-\infty}^{\infty} (-t)x(t)e^{-st}dt$$

9.5 Properties of Laplace Transform

- Integration in Time

$$x(t) \xleftrightarrow{\mathcal{L}} X(s), \quad \text{ROC}=R$$

$$\int_{-\infty}^{t} x(\tau)d\tau \xleftrightarrow{\mathcal{L}} \frac{1}{s}X(s), \quad \text{ROC contains } R \cap \{\mathrm{Re}\{s\} > 0\}$$

Proof :

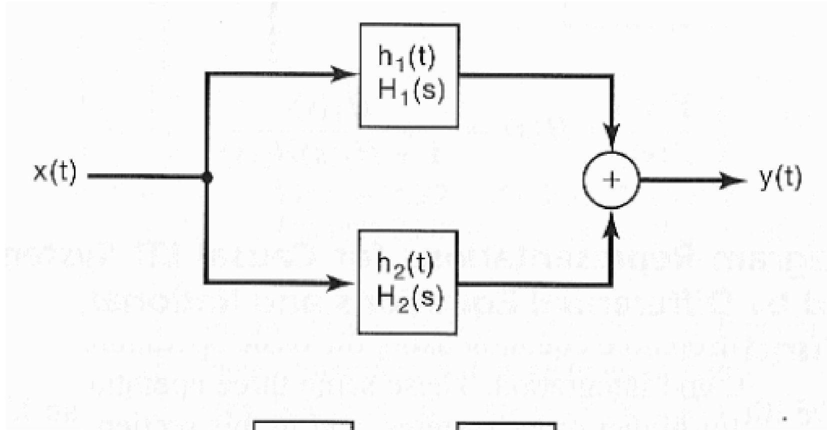
$$\int_{-\infty}^{t} x(\tau)d\tau = u(t) * x(t)$$

From Example 9.1, $u(t) \xleftrightarrow{\mathcal{L}} \frac{1}{s}$, $\mathrm{Re}\{s\}>0$

$\therefore \int_{-\infty}^{t} x(\tau)d\tau \xleftrightarrow{\mathcal{L}} \frac{1}{s}X(s)$, with an ROC containing the intersection of the ROC of $X(s)$ and the ROC of the LT of $u(t)$.

9.9 System Function Algebra and Block Diagram Representations

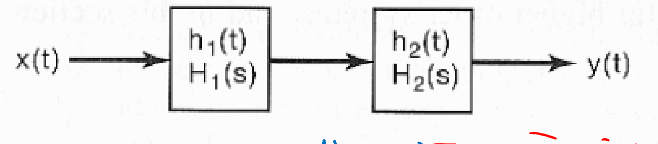
- System Function for Interconnected LTI Systems



Parallel Interconnection

$$h(t) = h_1(t) + h_2(t)$$

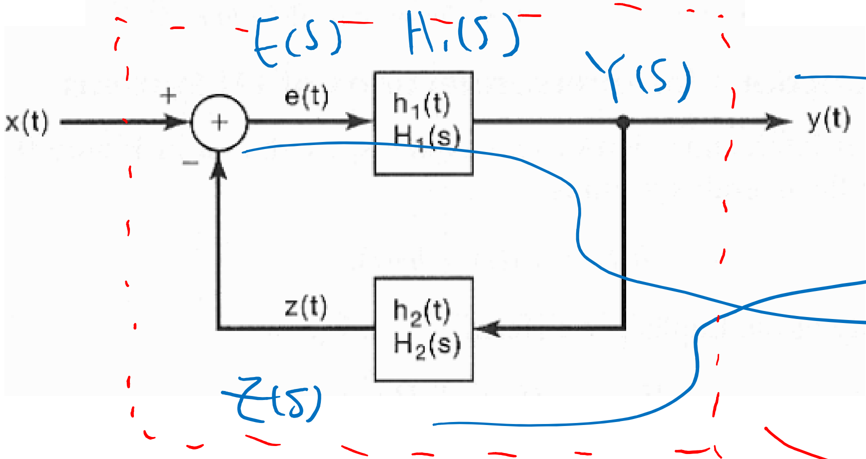
$$H(s) = H_1(s) + H_2(s)$$



Series Interconnection

$$h(t) = h_1(t) * h_2(t)$$

$$H(s) = H_1(s)H_2(s)$$



Feedback Interconnection

$$Y(s) = H_1(s)E(s)$$

$$Z(s) = H_2(s)Y(s)$$

$$E(s) = X(s) - Z(s)$$

$$H(s) = \boxed{\frac{H_1(s)}{1 + H_1(s)H_2(s)}} = \frac{Y(s)}{X(s)}$$

9.9 System Function Algebra and Block Diagram Representations

- Example 9.30
  Block diagram construction

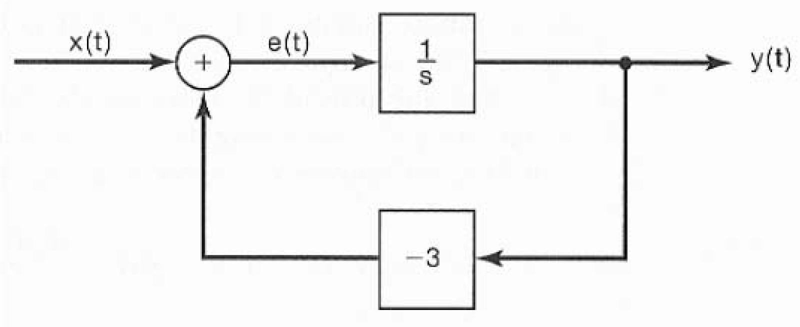
$$H(s)=\frac{1}{s+3} = \frac{Y(s)}{X(s)}$$

$$Y(s)=\frac{1}{\boxed{s+3}}X(s)$$

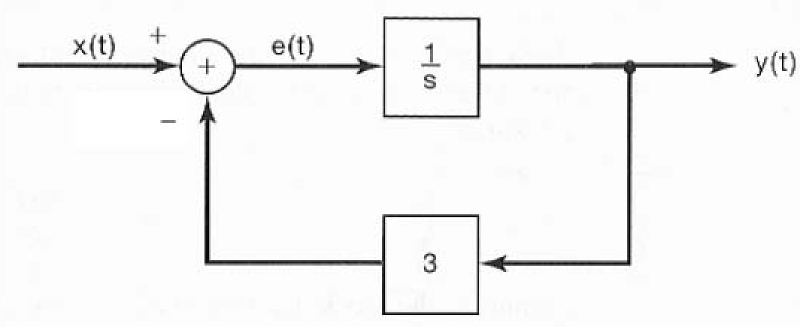
$$\frac{d}{dt}y(t)+3y(t)=x(t)$$

$$\frac{d}{dt}y(t)=x(t)-3y(t)$$

$$\underline{sY(s)}+3Y(s)=X(s)$$



(a)



(b)

9.9 System Function Algebra and Block Diagram Representations

- Example 9.31
  Block diagram representation

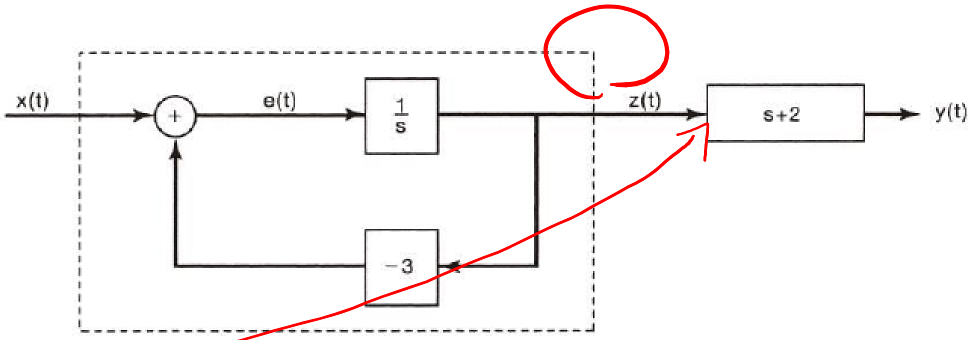
$$H(s)=\frac{s+2}{s+3}$$

$$=\left(\frac{1}{s+3}\right)(s+2)$$

$$\Rightarrow Z(s)\triangleq\frac{1}{s+3}X(s)$$

$$Y(s)=(s+2)Z(s)$$

$$=\boxed{s\,Z(s)}+2\,Z(s)$$



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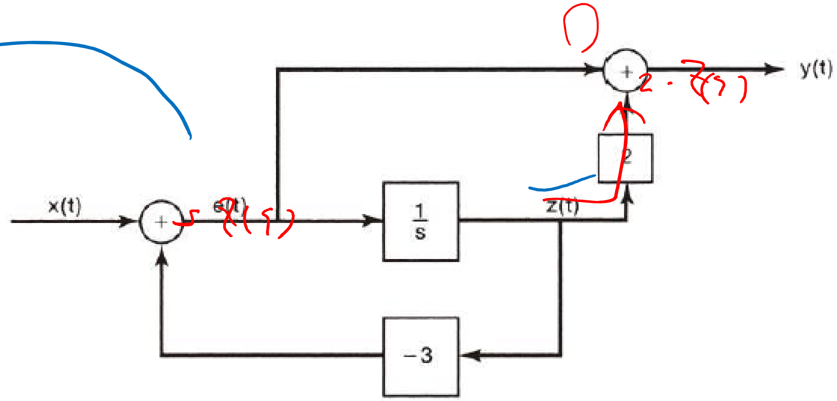


(b)

9.9 System Function Algebra and Block Diagram Representations

- Example 9.32
  Block diagram representation

$$H(s) = \frac{1}{(s+1)(s+2)}$$

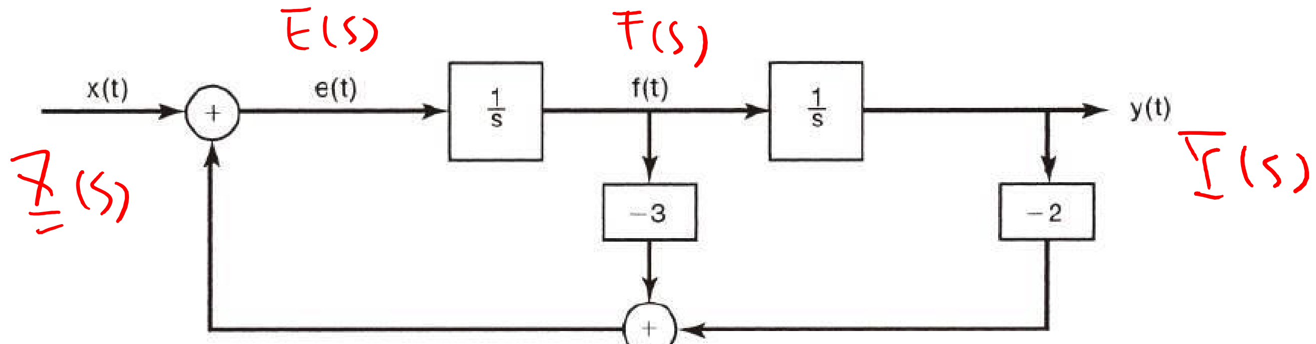
$$\frac{Y(s)}{X(s)} = \frac{1}{s^2 + 3s + 2}$$

$E + 3F + 2Y = X$

$$\Rightarrow s^2Y + 3sY + 2Y = X$$

$$\Rightarrow \begin{cases} F = \boxed{sY} \\ E = sF = \boxed{s^2Y} \end{cases}$$

$$\Rightarrow E = s^2Y = -3F - 2Y + X$$

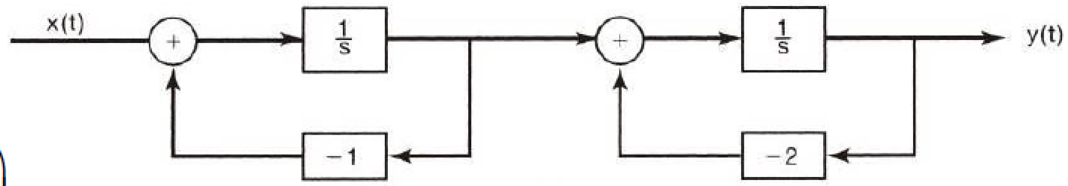


direct form

9.9 System Function Algebra and Block Diagram Representations

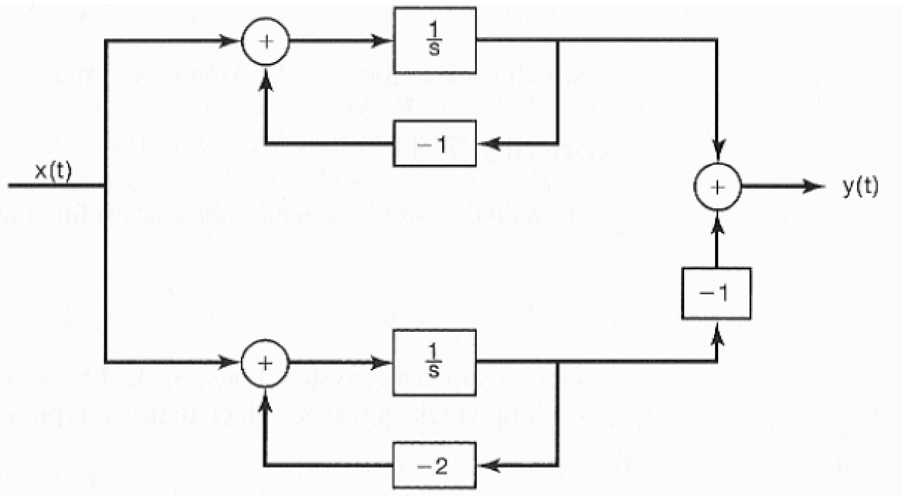
- Example 9.32
  Block diagram representation (cont'd)

$$H(s)=\frac{1}{(s+1)(s+2)}$$
$$=\left(\frac{1}{s+1}\right)\left(\frac{1}{s+2}\right)$$



cascade form

$$H(s)=\frac{1}{(s+1)(s+2)}$$
$$=\left(\frac{1}{s+1}\right)+\left(\frac{-1}{s+2}\right)$$



parallel form

9.10 The Unilateral Laplace Transform

- Bilateral vs. Unilateral Laplace Transform
  - The difference between bilateral and unilateral LT is in the lower limit of the integration.

Bilateral LT

$$X(s) \triangleq \int_{-\infty}^{\infty} x(t)e^{-st}dt$$

$$x(t) \xleftrightarrow{\mathcal{L}} X(s)$$

Unilateral LT

$$\mathcal{X}(s) \triangleq \int_{0^-}^{\infty} x(t)e^{-st}dt$$

$$x(t) \xleftrightarrow{\mathcal{UL}} \mathcal{X}(s)$$

ROC: always a right half plane

- Note that the lower limit in unilateral LT signifies that we include in the interval of integration any impulses or higher order singularity functions concentrated at t = 0.

9.10 The Unilateral Laplace Transform

- Example 9.34

$$x(t) = \frac{t^{n-1}}{(n-1)!}e^{-at}u(t). \Longleftrightarrow \mathcal{X}(s) = \frac{1}{(s+a)^n}, \qquad \mathcal{Re}\{s\} > -a.$$

- Example 9.35

Bilateral LT (by time-shifting property)

$$\boxed{x(t) = e^{-a(t+1)}u(t+1). \Longleftrightarrow X(s) = \frac{e^s}{s+a}, \qquad \mathcal{Re}\{s\} > -a.}$$

Unilateral LT ⇨

$$\begin{aligned}\mathcal{X}(s) &= \int_{0^-}^{\infty} e^{-a(t+1)}u(t+1)e^{-st}\,dt \\ &= \int_{0^-}^{\infty} e^{-a}e^{-t(s+a)}\,dt \\ &= e^{-a}\frac{1}{s+a}, \qquad \mathcal{Re}\{s\} > -a.\end{aligned}$$

We should recognize X(s) as the bilateral transform not of $x(t)$, but of $x(t)u(t)$.

- Example 9.36

Transform pair	Signal	Transform	ROC
9	$-\frac{t^{n-1}}{(n-1)!}e^{-\alpha t}u(-t)$	$\frac{1}{(s+\alpha)^n}$	$\Re e\{s\} < -\alpha$
10	$\delta(t-T)$	e^{-sT}	All s
11	$[\cos\omega_0 t]u(t)$	$\frac{s}{s^2+\omega_0^2}$	$\Re e\{s\} > 0$
12	$[\sin\omega_0 t]u(t)$	$\frac{\omega_0}{s^2+\omega_0^2}$	$\Re e\{s\} > 0$
13	$[e^{-\alpha t}\cos\omega_0 t]u(t)$	$\frac{s+\alpha}{(s+\alpha)^2+\omega_0^2}$	$\Re e\{s\} > -\alpha$
14	$[e^{-\alpha t}\sin\omega_0 t]u(t)$	$\frac{\omega_0}{(s+\alpha)^2+\omega_0^2}$	$\Re e\{s\} > -\alpha$
15	$u_n(t) = \frac{d^n\delta(t)}{dt^n}$	s^n	All s
16	$u_{-n}(t) = \underbrace{u(t) * \cdots * u(t)}_{n \text{ times}}$	$\frac{1}{s^n}$	$\Re e\{s\} > 0$

Sect. 2.5 Unit Doublets

$$u_1(t) = \frac{d}{dt}\delta(t).$$

$$\frac{d}{dt}x(t) = x(t) * u_1(t).$$

$$x(t) = \delta(t) + 2u_1(t) + e^t u(t).$$

Since $x(t) = 0$ for $t < 0$, and since singularities at the origin are included in the interval of integration, the unilateral transform for $x(t)$ is the same as the bilateral transform.

$$\Rightarrow \quad \mathcal{X}(s) = X(s) = 1 + 2s + \frac{1}{s-1} = \frac{s(2s-1)}{s-1}, \qquad \Re e\{s\} > 1.$$

9.10 The Unilateral Laplace Transform

- Example 9.38

$$\mathcal{X}(s) = \frac{s^2 - 3}{s + 2}. \Rightarrow \mathcal{X}(s) = A + Bs + \frac{C}{s + 2}.$$

$$\Updownarrow$$

$$s^2 - 3 = (A + Bs)(s + 2) + C, \Rightarrow \mathcal{X}(s) = -2 + s + \frac{1}{s + 2},$$

$$\Updownarrow$$

$$x(t) = -2\delta(t) + u_1(t) + e^{-2t}u(t) \quad \text{for} \quad t > 0^-.$$

9.10 The Unilateral Laplace Transform

- Properties of Unilateral LT

TABLE 9.3 PROPERTIES OF THE UNILATERAL LAPLACE TRANSFORM

Property	Signal	Unilateral Laplace Transform
	$x(t)$ $x_1(t)$ $x_2(t)$	$\mathcal{X}(s)$ $\mathcal{X}_1(s)$ $\mathcal{X}_2(s)$
Linearity	$ax_1(t) + bx_2(t)$	$a\mathcal{X}_1(s) + b\mathcal{X}_2(s)$
Shifting in the s-domain	$e^{s_0 t}x(t)$	$\mathcal{X}(s - s_0)$
Time scaling	$x(at), \quad a > 0$	$\frac{1}{a}\mathcal{X}\left(\frac{s}{a}\right)$
Conjugation	$x^*(t)$	$X^*(s^*)$
Convolution (assuming that $x_1(t)$ and $x_2(t)$ are identically zero for $t < 0$)	$x_1(t) * x_2(t)$	$\mathcal{X}_1(s)\mathcal{X}_2(s)$

9.10 The Unilateral Laplace Transform

- Properties of Unilateral LT

Differentiation in the time domain	$\frac{d}{dt}x(t)$	$s\mathcal{X}(s) - x(0^-)$
Differentiation in the s-domain	$-tx(t)$	$\frac{d}{ds}\mathcal{X}(s)$
Integration in the time domain	$\int_{0^-}^{t} x(\tau)\,d\tau$	$\frac{1}{s}\mathcal{X}(s)$

Initial- and Final-Value Theorems

If $x(t)$ contains no impulses or higher-order singularities at $t = 0$, then

$$x(0^+) = \lim_{s\to\infty} s\mathcal{X}(s)$$

$$\lim_{t\to\infty} x(t) = \lim_{s\to 0} s\mathcal{X}(s)$$

Recall: 9.5 Properties of Laplace Transform

- Differentiation in Time and s Domain

$$x(t) \xleftrightarrow{\mathcal{L}} X(s), \quad \text{ROC} = R$$

$$\frac{d}{dt}x(t) \xleftrightarrow{\mathcal{L}} sX(s), \quad \text{ROC contains } R$$

pole-zero cancellation may occur.

$$-tx(t) \xleftrightarrow{\mathcal{L}} \frac{dX(s)}{ds}, \quad \text{ROC} = R$$

Proof :

$$\frac{dx(t)}{dt} = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} X(s)\frac{de^{st}}{dt}ds = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} sX(s)e^{st}ds$$

$$\frac{d}{ds}\int_{-\infty}^{\infty} x(t)e^{-st}dt = \int_{-\infty}^{\infty} (-t)x(t)e^{-st}dt$$

9.10 The Unilateral Laplace Transform

Integration by parts

$$\int u\frac{\mathrm{d}v}{\mathrm{d}x}\,\mathrm{d}x = u\,v - \int v\frac{\mathrm{d}u}{\mathrm{d}x}\mathrm{d}x$$

- Differential Properties

$$\mathcal{UL}\left\{\frac{dx(t)}{dt}\right\} = \int_{0^-}^{\infty}\boxed{\frac{dx(t)}{dt}}e^{-st}dt = \int_{0^-}^{\infty}\left[\frac{d}{dt}(x(t)e^{-st}) + sx(t)e^{-st}\right]dt$$

$$y = u\,v$$

$$= x(t)e^{-st}\Big|_{0^-}^{\infty} + s\int_{0^-}^{\infty}x(t)e^{-st}dt$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}(uv)}{\mathrm{d}x} = u\frac{\mathrm{d}v}{\mathrm{d}x} + v\frac{\mathrm{d}u}{\mathrm{d}x}.$$

$$= 0 - x(0^-) + s\mathcal{X}(s)$$

$$u\frac{\mathrm{d}v}{\mathrm{d}x} = \frac{\mathrm{d}(uv)}{\mathrm{d}x} - v\frac{\mathrm{d}u}{\mathrm{d}x}.$$

$$= s\mathcal{X}(s) - x(0^-)$$

$$\int u\frac{\mathrm{d}v}{\mathrm{d}x}\mathrm{d}x = \int\frac{\mathrm{d}(uv)}{\mathrm{d}x}\mathrm{d}x - \int v\frac{\mathrm{d}u}{\mathrm{d}x}\mathrm{d}x.$$

$$\mathcal{UL}\left\{\frac{d^2x(t)}{dt^2}\right\} = \int_{0^-}^{\infty}\frac{d^2x(t)}{dt^2}e^{-st}dt = s^2\mathcal{X}(s) - sx(0^-) - x'(0^-)$$

9.10 The Unilateral Laplace Transform

- Example 9.39

$$\frac{d^2y(t)}{dt^2} + 3\frac{dy(t)}{dt} + 2y(t) = x(t)$$

$$x(t) = \alpha u(t); \quad \text{initial conditions: } y(0^-) = \beta, \quad y'(0^-) = \gamma$$

$$\Rightarrow \left[s^2\mathcal{Y}(s) - \beta s - \gamma\right] + 3s\mathcal{Y}(s) - 3\beta + 2\mathcal{Y}(s) = \frac{\alpha}{s}$$

$$\Rightarrow \mathcal{Y}(s) = \underbrace{\frac{\beta(s+3)}{(s+1)(s+2)} + \frac{\gamma}{(s+1)(s+2)}}_{\substack{\text{Zero-input response} \\ (\alpha = 0)}} + \underbrace{\frac{\alpha}{s(s+1)(s+2)}}_{\substack{\text{zero-state response} \\ (\beta = \gamma = 0)}}$$

The overall response is the superposition of the zero-input response and the zero-state response.

$$\Rightarrow \mathcal{Y}(s) = \frac{1}{s} - \frac{1}{s+1} + \frac{3}{s+2}$$

with $\alpha = 2, \beta = 3,$ and $\gamma = -5$

$$\Rightarrow y(t) = [1 - e^{-t} + 3e^{-2t}]\, u(t), \text{ for } t > 0$$

9.10 The Unilateral Laplace Transform

- Initial-Value Theorem for Unilateral LT

$$x(0^+) = \lim_{s\to\infty} \boxed{s\mathcal{X}(s)}$$

Applies only when the order of the numerator polynomial of $X(s)$ is smaller than that of the denominator polynomial.

- Final-Value Theorem

$$\lim_{t\to\infty} x(t) = \lim_{s\to 0} s\boxed{\mathcal{X}(s)}$$

Applies only if all the poles of X(s) are in the left half of the s-plane, with at most a single pole at s=0.

Ch. 10 The Z Transform

- Section 10.1 The z-Transform
- Section 10.2 The Region of Convergence for z-Transforms
- Section 10.3 The Inverse z-Transform
- Section 10.4 Geometric Evaluation of the Fourier Transform from Pole-Zero Plot
- Section 10.5 Properties of the z-Transform
- Section 10.6 Some z-Transform Pairs
- Section 10.7 Analysis & Characterization of LTI Systems Using z-Transforms
- Section 10.8 DT All-Pass, Minimum Phase System, and Spectral Factorization
- Section 10.9 System Function Algebra and Block Diagram Representations
- Section 10.10 The Unilateral z-Transform
- Section 10.11 Summary

Revisit of Laplace Transform

$x(t)$
$\parallel$

Recall that the response of a linear time-invariant system with impulse response $h(t)$ to a complex exponential input of the form e^{st} is

$y(t) = x(t) * h(t)$

$= \int_{-\infty}^{\infty} h(\tau) x(t-\tau) d\tau$

$e^{s(t-\tau)}$

$$y(t) = H(s)e^{st},$$

where

$$H(s) = \int_{-\infty}^{\infty} h(t)e^{-st}\,dt.$$ (i.e., the system function of the system)

For a general signal x(t), a transform like the one above is refereed to as the (bilateral) Laplace transform:

$$X(s) \triangleq \int_{-\infty}^{\infty} x(t)e^{-st}dt$$

where s = σ + jω is a complex variable.
Thus, Laplace transform can be viewed as an extension of CTFT.

10.1 The z-Transform

- The z-Transform

Recall that for a discrete-time LTI system with impulse response $h[n]$, the response $y[n]$ of the system to a complex exponential input of the form z^n is $= x[n]$

$$\underline{y[n] = H(z)z^n}$$

$$y[n] = \sum_{k=-\infty}^{\infty} x[n-k]\, h[k]$$

$$z^{n-k}$$

$$= \left(\sum_{k} h[k] \cdot z^{-k}\right) \cdot z^n$$

where

$$H[z] = \sum_{n=-\infty}^{+\infty} h[n]z^{-n}.$$

When $\boxed{z = e^{j\omega}}$ with $|z| = 1$, the summation corresponds to the DT Fourier transform of $h[n]$. When z is not restricted to the unit circle in the z-plane, the summation is called the z-transform of $h[n]$.

The z-transform of a general DT signal $x[n]$ is defined as:

$$X(z) \triangleq \sum_{n=-\infty}^{+\infty} x[n]z^{-n}$$

where $z = re^{j\omega}$ is a complex variable.

10.1 The z-Transform

- DTFT vs. z-Transform

DT Fourier Transform	z-Transform
$z = e^{j\omega}$	$z = re^{j\omega}$
$X(e^{j\omega}) \triangleq \sum_{n=-\infty}^{+\infty} x[n]e^{-j\omega n}$	$X(z) \triangleq \sum_{n=-\infty}^{+\infty} x[n]z^{-n}$
$x[n] \xleftrightarrow{\mathcal{F}} X(e^{j\omega})$	$x[n] \xleftrightarrow{\mathcal{Z}} X(z)$
$X(e^{j\omega}) = \mathcal{F}\{x[n]\}$	$X(z) = \mathcal{Z}\{x[n]\}$
$x[n] = \mathcal{F}^{-1}\{X(e^{j\omega})\}$	$x[n] = \mathcal{Z}^{-1}\{X(z)\}$

Evaluating the z-transform on the unit circle $z = e^{j\omega}$ yields the Fourier transform:

$$X(z)\Big|_{z=e^{j\omega}} = \mathcal{Z}\{x[n]\}\Big|_{z=e^{j\omega}} = \mathcal{F}\{x[n]\} = X(e^{j\omega})$$

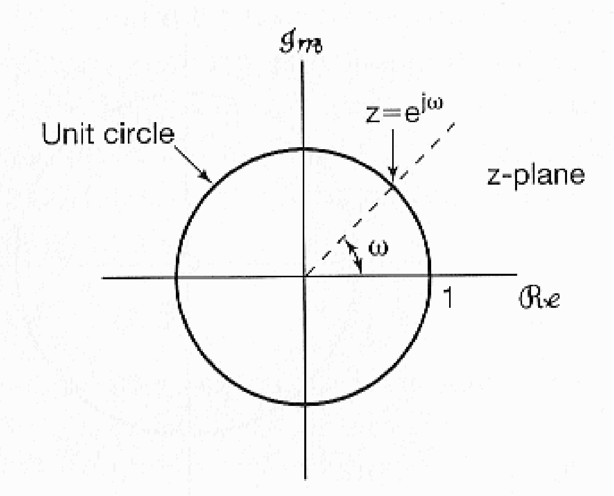
10.1 The z-Transform

- z-Transform from DTFT

$$X(re^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n](re^{j\omega})^{-n}$$

$$= \sum_{n=-\infty}^{+\infty} \left\{x[n]r^{-n}\right\} e^{-j\omega n}$$

$$= \mathcal{F}\left\{\underline{x[n]r^{-n}}\right\}$$



The z-transform of a DT signal $x[n]$ is the Fourier transform of the signal $x[n]r^{-n}$.

Region of convergence (ROC) refers to the range of values of r for which $X(z)$ converges.

10.1 The z-Transform

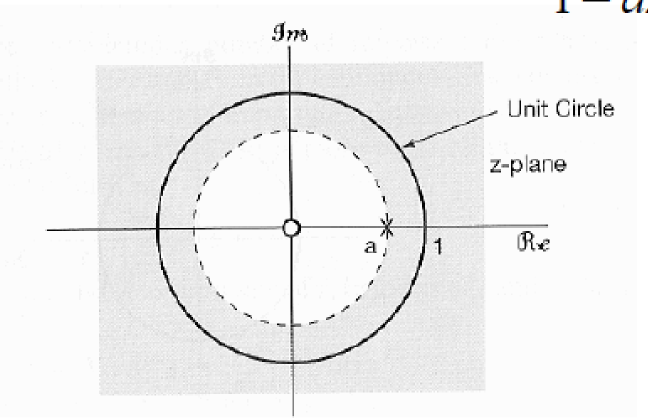
- Example 10.1

$$x[n] = a^n u[n]$$

DTFT $\Rightarrow X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} a^n u[n] e^{-j\omega n} = \sum_{n=0}^{\infty} (ae^{-j\omega})^n = \frac{1}{1-ae^{-j\omega}}$

$$\Rightarrow X(z) = \sum_{n=-\infty}^{+\infty} a^n u[n] z^{-n} = \sum_{n=0}^{\infty} (az^{-1})^n, \qquad |az^{-1}| < 1 \text{ or } |a| < |z|$$

$$= \frac{1}{1-az^{-1}} = \frac{z}{z-a}, \qquad |z| > |a|$$



For $|a| > 1$, ROC does not include the unit circle $\Rightarrow$ $\mathcal{F}\{a^n u[n]\}$ does not converge

10.1 The z-Transform

- Example 10.2

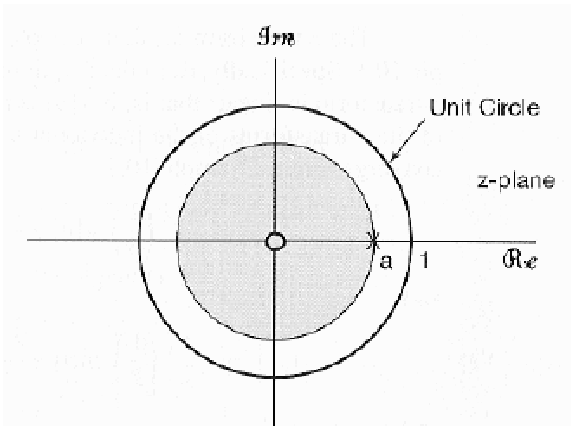
$$x[n] = -a^n u[-n-1]$$

$$X(z) = -\sum_{n=-\infty}^{+\infty} a^n u[-n-1] z^{-n}$$

$$= -\sum_{n=-\infty}^{-1} a^n z^{-n} = -\sum_{n=1}^{\infty} a^{-n} z^n = 1 - \sum_{n=0}^{\infty} \left(a^{-1}z\right)^n$$

$$= 1 - \frac{1}{1-a^{-1}z}, \qquad \left|a^{-1}z\right| < 1$$

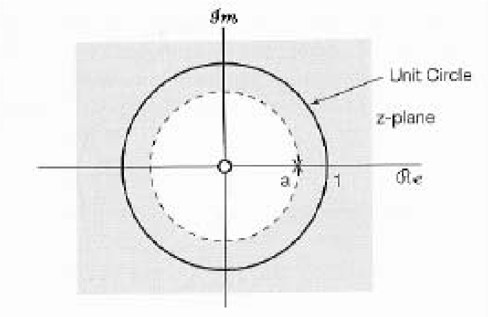
$$= \frac{1}{1-az^{-1}} = \frac{z}{z-a}, \qquad |z| < |a|$$



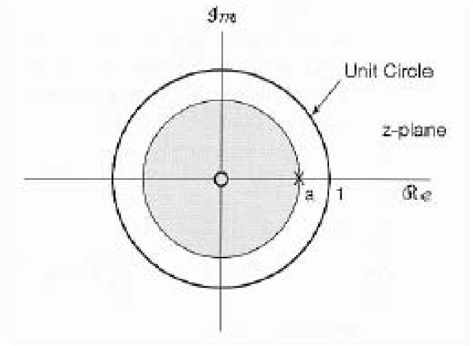
10.1 The z-Transform

- Specification of the z-Transform
  - The z-Transform is a rational function that can be characterized by its zeros and poles.
  - Specification of the z-Transform requires both the algebraic form and its regions of convergence (ROC).

$$a^n u[n] \xleftrightarrow{z} \frac{z}{z-a}, \qquad |z| > |a|$$



$$-a^n u[-n-1] \xleftrightarrow{z} \frac{z}{z-a}, \qquad |z| < |a|$$



10.1 The z-Transform

- Example 10.3

$$x[n] = 7\left(\frac{1}{3}\right)^n u[n] - 6\left(\frac{1}{2}\right)^n u[n].$$

$$X(z) = \sum_{n=-\infty}^{+\infty}\left\{7\left(\frac{1}{3}\right)^n u[n] - 6\left(\frac{1}{2}\right)^n u[n]\right\} z^{-n}$$

$$= 7\sum_{n=-\infty}^{+\infty}\left(\frac{1}{3}\right)^n u[n]z^{-n} - 6\sum_{n=-\infty}^{+\infty}\left(\frac{1}{2}\right)^n u[n]z^{-n}$$

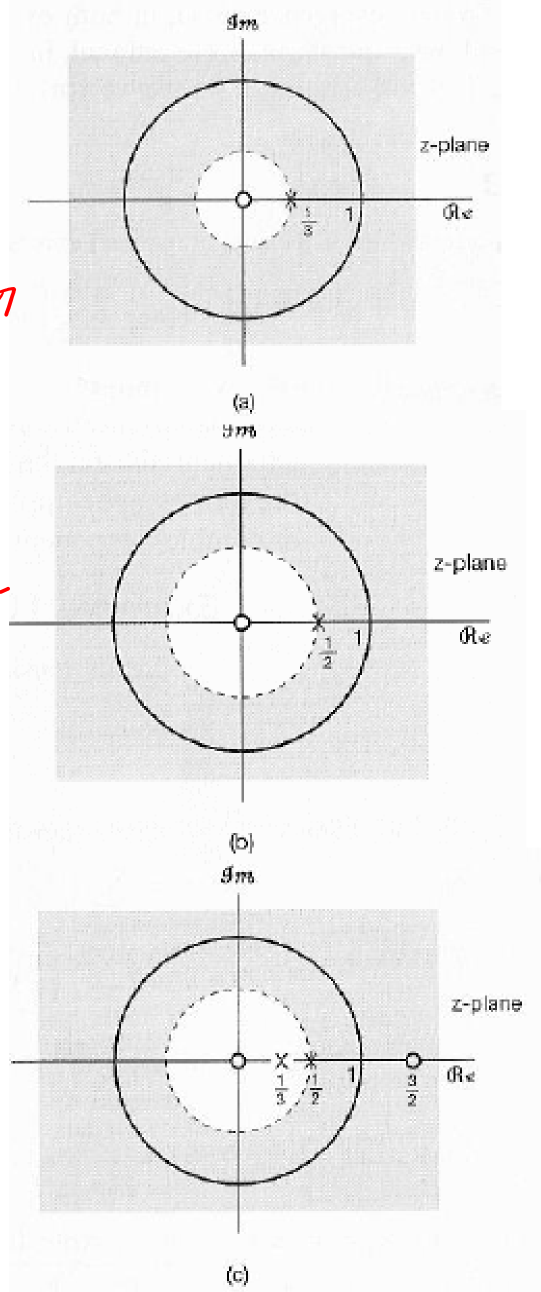
$$= 7\sum_{n=0}^{\infty}\left(\frac{1}{3}z^{-1}\right)^n - 6\sum_{n=0}^{+\infty}\left(\frac{1}{2}z^{-1}\right)^n$$

$$= \frac{7}{1-\frac{1}{3}z^{-1}} - \frac{6}{1-\frac{1}{2}z^{-1}} = \frac{1-\frac{3}{2}z^{-1}}{(1-\frac{1}{3}z^{-1})(1-\frac{1}{2}z^{-1})}$$

$$= \frac{z(z-\frac{3}{2})}{(z-\frac{1}{3})(z-\frac{1}{2})}. \qquad |z| > \frac{1}{2}$$

ROC #1

ROC #2



10.1 The z-Transform

- Example 10.4

$$x[n] = \left(\tfrac{1}{3}\right)^n \sin\left(\frac{\pi}{4}n\right)u[n]$$

$$= \tfrac{1}{2j}\left(\tfrac{1}{3}e^{j\pi/4}\right)^n u[n] - \tfrac{1}{2j}\left(\tfrac{1}{3}e^{-j\pi/4}\right)^n u[n]$$

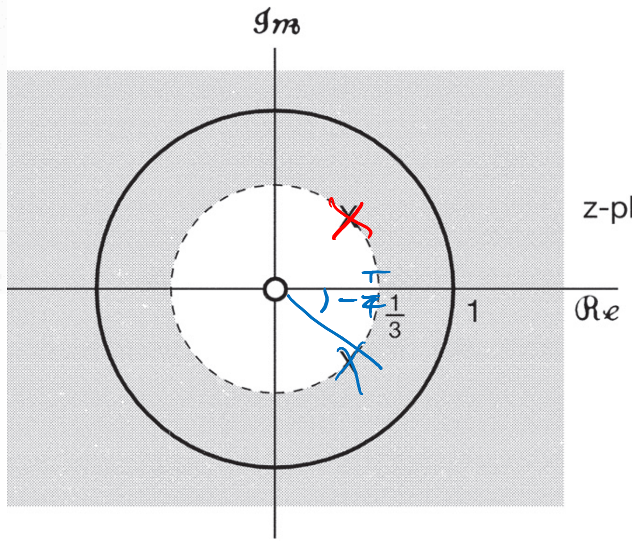
Recall

$$a^n u[n] \xleftrightarrow{z} \frac{1}{1-az^{-1}}, \qquad |z| > |a|.$$

Therefore,

$$X(z) = \frac{1}{2j}\frac{1}{1-\frac{1}{3}e^{j\pi/4}z^{-1}} - \frac{1}{2j}\frac{1}{1-\frac{1}{3}e^{-j\pi/4}z^{-1}}$$

$$= \frac{\frac{1}{3\sqrt{2}}z}{\left(z-\frac{1}{3}e^{j\pi/4}\right)\left(z-\frac{1}{3}e^{-j\pi/4}\right)}, \qquad |z| > \frac{1}{3}$$



Pole-zero plot and ROC for the z-transform in Example 10.4.

10.2 The ROC of z-Transform

- Property #1
  - The ROC of $X(z)$ consists of a ring in the z-plane centered about the origin.
  - The ROC consists of those values of $z = re^{j\omega}$ for which $x[n]r^{-n}$ has a Fourier transform that converges.
  - The ROC of the z-transform of $x[n]$ consists of the values of z for which $x[n]r^{-n}$ is absolutely summable:

$$\sum_{n=-\infty}^{+\infty} |x[n]|r^{-n} < \infty.$$

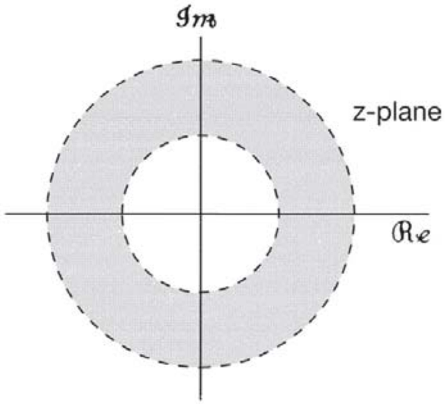


Figure 10.6 ROC as a ring in the z-plane. In some cases, the inner boundary can extend inward to the origin, in which case the ROC becomes a disc. In other cases, the outer boundary can extend outward to infinity.

- Property #2
  - The ROC of $X(z)$ does not contain any poles…

10.2 The ROC of z-Transform

- Property #3
  - If x[n] is of finite duration, then the ROC is the entire z-plane, except possibly $\boxed{z = 0 \text{ and/or } z = \infty}$.
  - Why?

A finite-duration sequence only has a finite number of nonzero values; therefore,

$$X(z) \triangleq \sum_{n=-\infty}^{+\infty} x[n]z^{-n} = \sum_{n=N_1}^{N_2} x[n]z^{-n}$$

is bounded for z not equal to 0 or ∞.

For N_1 negative and N_2 positive, the ROC does not include $z = 0$ or $z = \infty$ because

as $z \to 0$, terms involving negative power of z becomes unbounded
as $z \to \infty$, terms involving positive power of z becomes unbounded

If N_1 is zero or positive, the ROC includes $z = \infty$.

If N_2 is zero or negative, the ROC includes $z = 0$.

10.2 The ROC of z-Transform

- Example 10.5

Consider the z-transform pair

$$\delta[n] \xleftrightarrow{Z} \sum_{n=-\infty}^{\infty} \delta[n] z^{-n} = 1.$$

Its ROC consists of the entire z-plane and includes $z = 0$ and $z=\infty$. On the other hand, consider the delayed unit impulse $\delta[n-1]$, for which

$$\delta[n-1] \xleftrightarrow{Z} \sum_{n=-\infty}^{\infty} \delta[n-1] z^{-n} = z^{-1}.$$

This z-transform is well defined except at $z = 0$, where there is a pole. Thus its ROC consists of the entire z-plane (including $z = \infty$) but excludes $z = 0$. Similarly, consider

$$\delta[n+1] \xleftrightarrow{Z} \sum_{n=-\infty}^{\infty} \delta[n+1] z^{-n} = z.$$

The ROC in this case is the entire z-plane (including $z = 0$) but excludes $z = \infty$.

10.2 The ROC of z-Transform

- Properties #4 & #5
  - If x[n] is a right-sided sequence, and if the circle $|z| = r_0$ is in the ROC, then all finite values of z for which $|z| > r_0$ will also be in the ROC.
  - If x[n] is a left-sided sequence, and if the circle $|z| = r_0$ is in the ROC, then all values of z for which $0 < |z| < r_0$ will also be in the ROC.

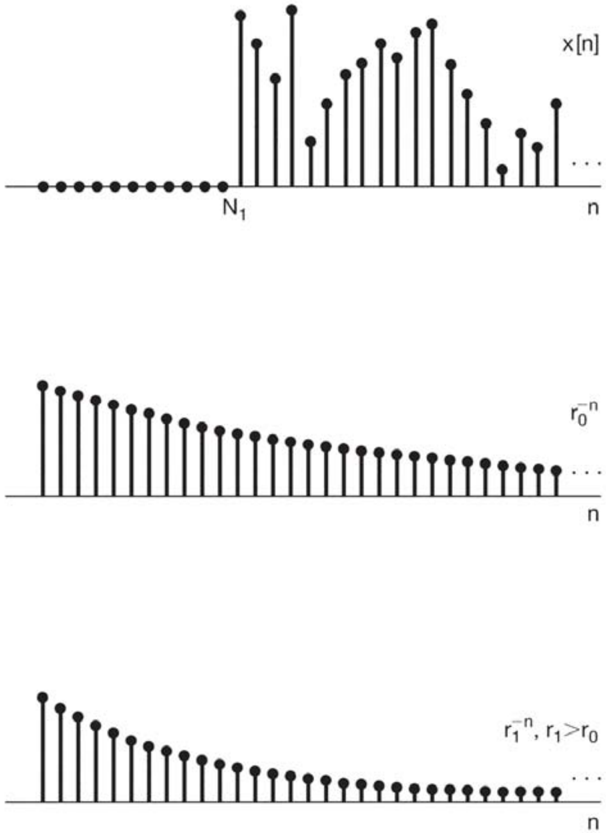


Figure 10.7 With $r_1 > r_0$, $x[n]r_1^{-n}$ decays faster with increasing n than does $x[n]r_0^{-n}$. Since $x[n] = 0$, $n < N_1$, this implies that if $x[n]r_0^{-n}$ is absolutely summable, then $x[n]r_1^{-n}$ will be also.

$$X(r_0 e^{j\omega}) = \sum_{n=N_1}^{\infty} \left\{ x[n] r_0^{-n} \right\} e^{-j\omega n} < \infty$$

$$X(r_1 e^{j\omega}) = \sum_{n=N_1}^{\infty} \left\{ x[n] r_1^{-n} \right\} e^{-j\omega n}$$

$$< \sum_{n=N_1}^{\infty} \left\{ x[n] r_0^{-n} \right\} e^{-j\omega n} < \infty$$

- For positive n, r_1^{-n} decays faster than r_0^{-n}.
- For negative n, $\sum_{n=N1}^{0} x[n] z^{-n}$ is bounded since $x[n]$ is right-sided.

10.2 The ROC of z-Transform

- Property #6
  - If x[n] is two sided, and if the circle |z| = r_0 is in the ROC, then the ROC will consist of a ring in the z-plane that includes the circle |z| = r_0.

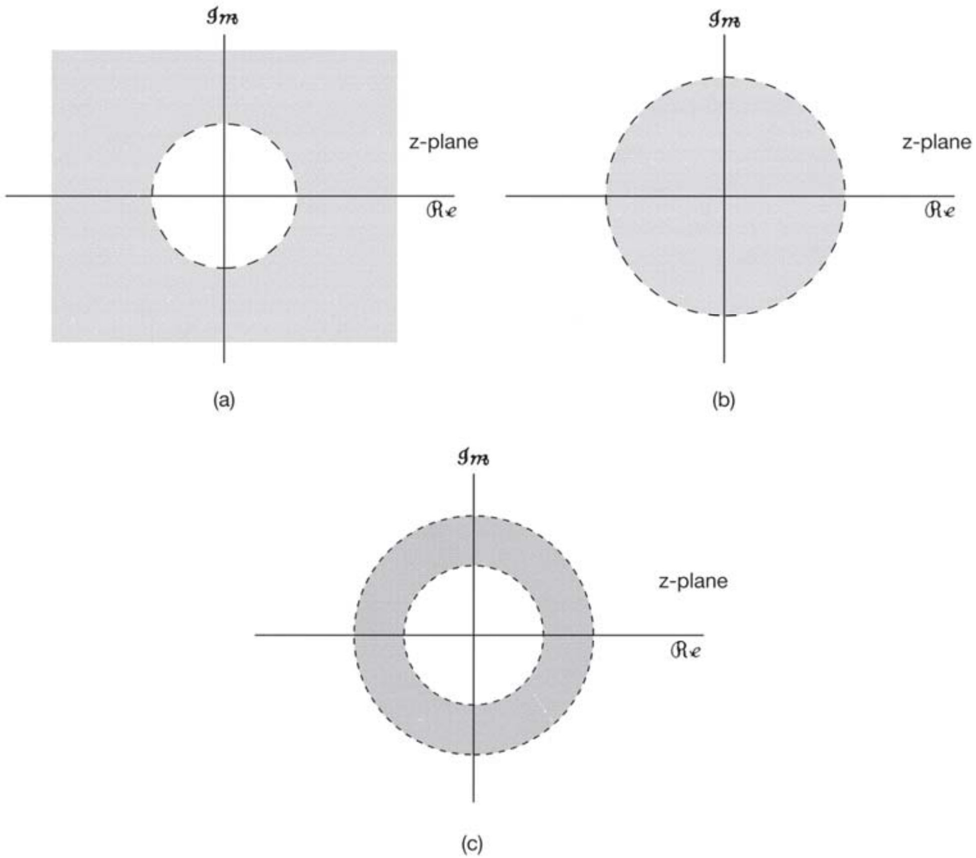


Figure 10.8 (a) ROC for right-sided sequence; (b) ROC for left-sided sequence; (c) intersection of the ROCs in (a) and (b), representing the ROC for a two-sided sequence that is the sum of the right-sided and the left-sided sequence.

10.2 The ROC of z-Transform

- Example 10.6

$$x[n] = \begin{cases} a^n, & 0 \le n \le N-1,\ a > 0 \\ 0, & \text{otherwise} \end{cases}.$$

$$\begin{aligned} X(z) &= \sum_{n=0}^{N-1} a^n z^{-n} \\ &= \sum_{n=0}^{N-1} (az^{-1})^n \\ &= \frac{1-(az^{-1})^N}{1-az^{-1}} = \frac{1}{z^{N-1}} \frac{z^N - a^N}{z-a}. \end{aligned}$$

- x[n] finite length -> entire z-plane as ROC except possibly origin/infinity
- x[n] = 0 for z<0, ROC will extent to infinity but not origin
- This is equivalent to that we have a pole of order N-1 at z=0.

10.2 The ROC of z-Transform

- Example 10.6 (cont'd)

$$X(z) = \sum_{n=0}^{N-1} a^n z^{-n}$$
$$= \sum_{n=0}^{N-1} (az^{-1})^n$$
$$= \frac{1-(az^{-1})^N}{1-az^{-1}} = \frac{1}{z^{N-1}} \frac{z^N - a^N}{z-a}.$$

- The N roots of the numerator: $z_k = ae^{j(2\pi k/N)}, \quad k = 0, 1, \ldots, N-1.$
- The root for k=0 cancels the pole at z=a. The remaining poles

$$z_k = ae^{j(2\pi k/N)}, \quad k = 1, \ldots, N-1.$$

- Example 10.6 (cont'd)

$$X(z) = \sum_{n=0}^{N-1} a^n z^{-n}$$
$$= \sum_{n=0}^{N-1} (az^{-1})^n$$
$$= \frac{1-(az^{-1})^N}{1-az^{-1}} = \frac{1}{z^{N-1}} \frac{z^N - a^N}{z-a}.$$

$$z_k = ae^{j(2\pi k/N)}, \quad k = 1, \ldots, N-1.$$

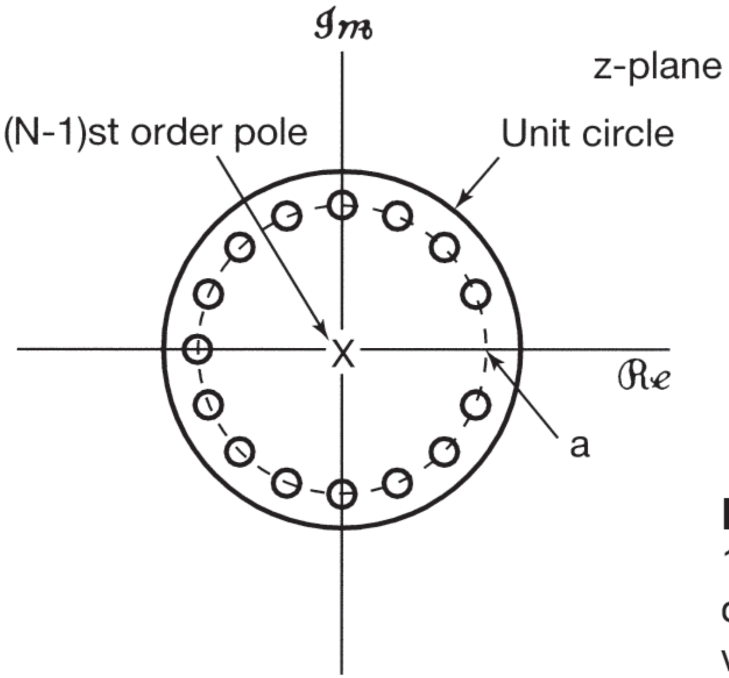


Figure 10.9 Pole-zero pattern for Example 10.6 with N = 16 and $0 < a < 1$. The region of convergence for this example consists of all values of z except z = 0.

10.2 The ROC of z-Transform

- Example 10.7

$$x[n] = b^{|n|}, \quad b > 0$$

$$= b^n u[n] + b^{-n} u[-n-1]$$

$$X(z) = \frac{1}{1 - bz^{-1}} - \frac{1}{1 - b^{-1}z^{-1}}$$

$$= \left(\frac{b^2 - 1}{b}\right) \frac{z}{(z-b)(z-b^{-1})}, \qquad b < |z| < \frac{1}{b}$$

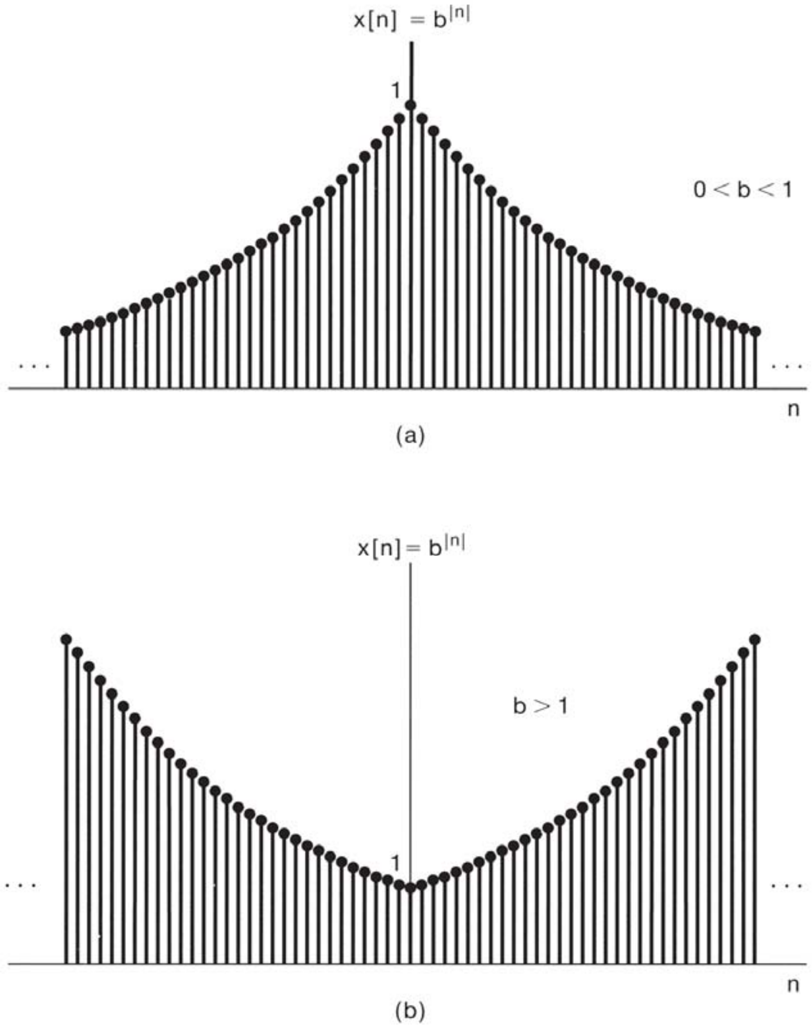
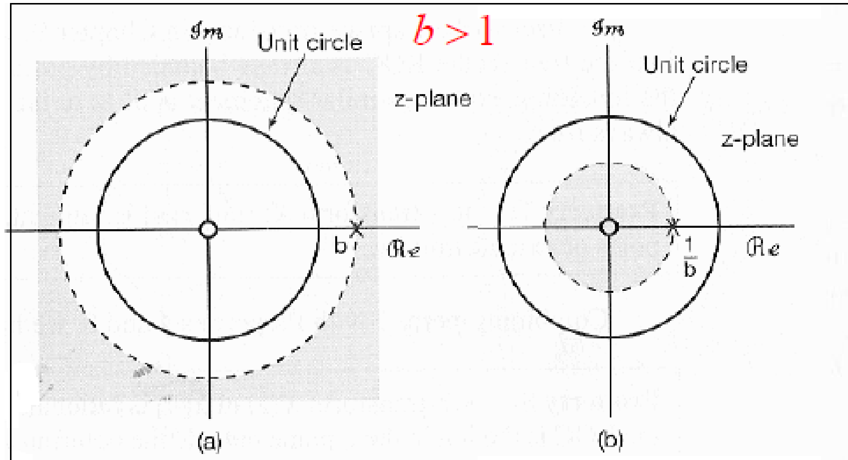


Figure 10.10 Sequence $x[n] = b^{|n|}$ for $0 < b < 1$ and for $b > 1$: (a) $b = 0.95$; (b) $b = 1.05$.

- Example 10.7 (cont'd)

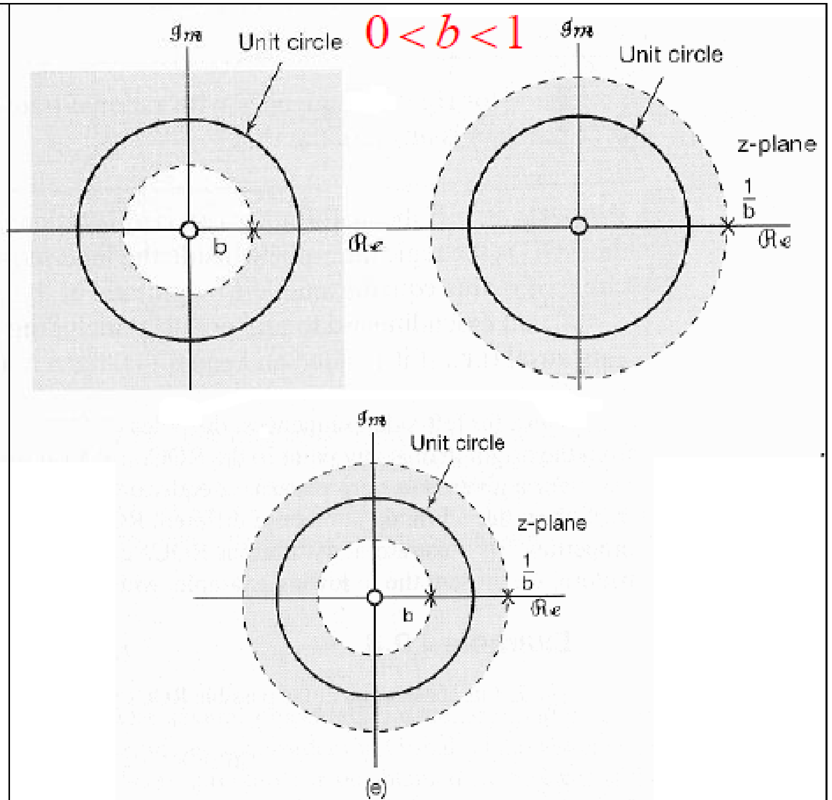
$$x[n] = b^{|n|}, \quad b > 0$$
$$= b^n u[n] + b^{-n} u[-n-1]$$
$$X(z) = \frac{1}{1-bz^{-1}} - \frac{1}{1-b^{-1}z^{-1}}$$
$$= \left(\frac{b^2-1}{b}\right)\frac{z}{(z-b)(z-b^{-1})}, \qquad b < |z| < \frac{1}{b}$$

$b > 1$

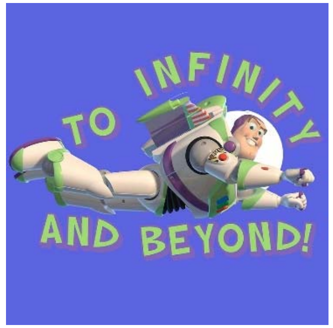


No intersection between the two ROCs

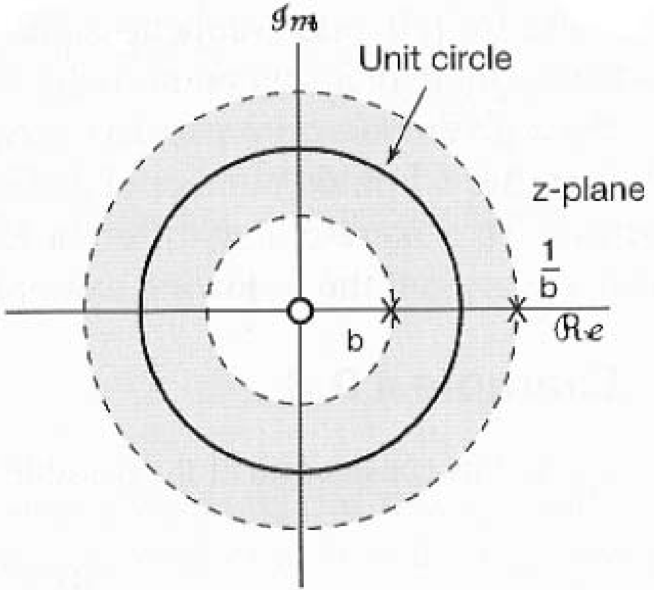
$0 < b < 1$



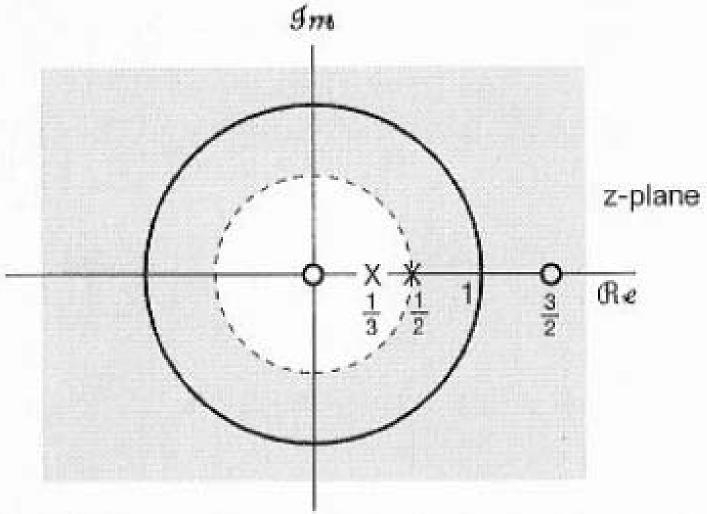
10.2 The ROC of z-Transform



- Property #7
  - If X(z) is rational, then its ROC is either bounded by poles or extends to infinity.



(e)



(c)

$$X(z)=\left(\frac{b^2-1}{b}\right)\frac{z}{(z-b)(z-b^{-1})}, \qquad b<|z|<\frac{1}{b}$$

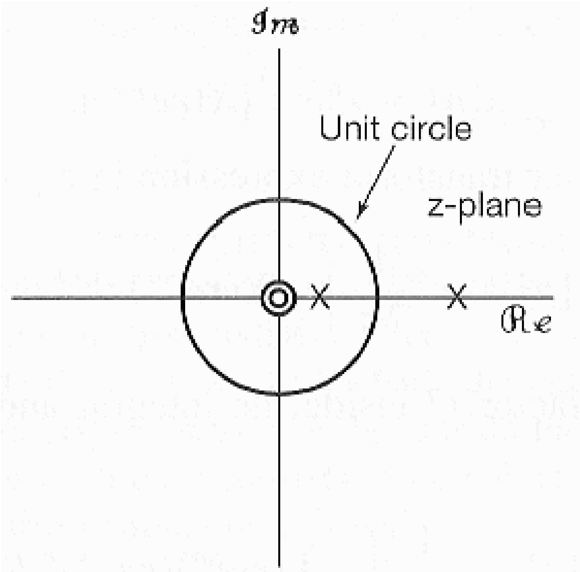
$$\frac{z\left(z-\frac{3}{2}\right)}{\left(z-\frac{1}{3}\right)\left(z-\frac{1}{2}\right)}, \qquad z>\frac{1}{2}$$

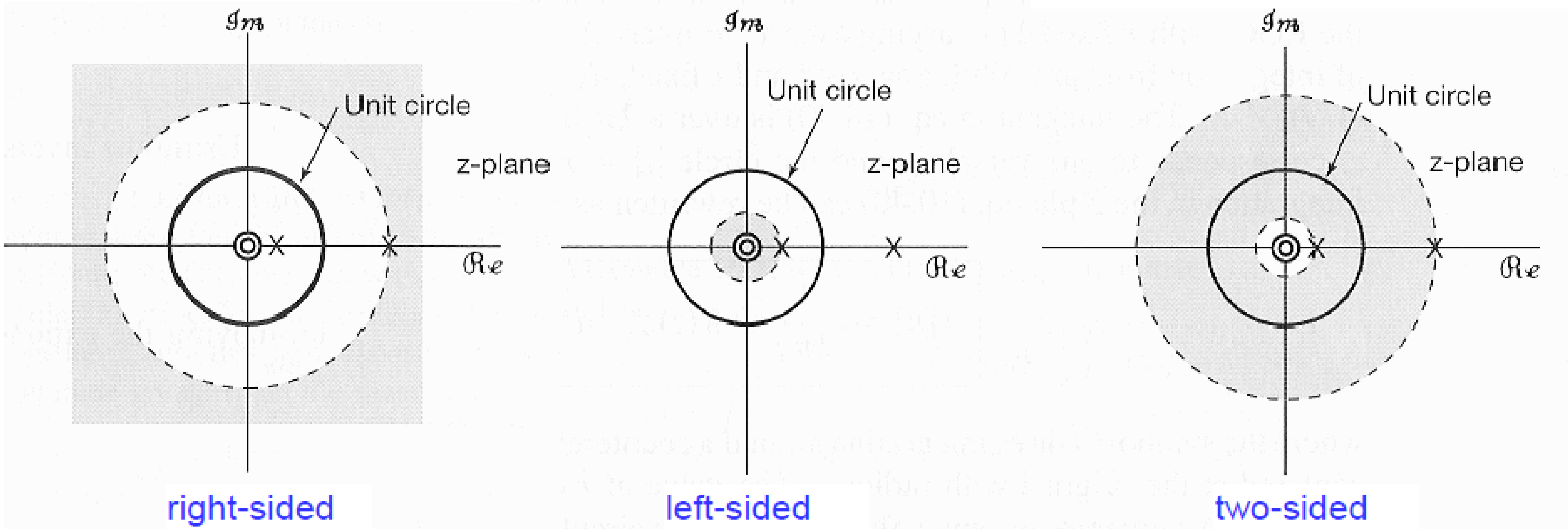
10.2 The ROC of z-Transform

- Property #8: If $X(z)$ is rational,
  - If $x[n]$ is right sided (i.e., $x[n]=0$ for $n<n_0$), then the ROC is the region in the z-plane outside the outermost pole, i.e., outside the circle of radius equal to the largest magnitude of the poles of $X(z)$.
  - Furthermore, if $x[n]$ is causal (i.e., $x[n]=0$ for $n<0$), then the ROC also includes ∞.
- Property #9: If $X(z)$ is rational,
  - If $x[n]$ is left sided, then the ROC is the region in the z-plane inside the innermost pole, i.e., inside the circle of radius equal to the smallest magnitude of the poles of $X(z)$ other than any at $z=0$, and extending inward to and possibly including $z=0$.
  - Furthermore, if $x[n]$ is anti-causal (i.e., $x[n]=0$ for $n>0$), then the ROC includes 0.

- Example 10.8

$$X(z) = \frac{1}{(1 - \frac{1}{3}z^{-1})(1 - 2z^{-1})}.$$





10.3 The Inverse z-Transform

- Method #1

Using IFT to derive IZT

$$X(re^{j\omega}) = \mathcal{F}\left\{x[n]r^{-n}\right\}$$

$$x[n]r^{-n} = \mathcal{F}^{-1}\left\{X(re^{j\omega})\right\} \qquad \forall z = re^{j\omega} \text{ inside the ROC}$$

$$x[n] = r^n \mathcal{F}^{-1}\left\{X(re^{j\omega})\right\} = r^n \frac{1}{2\pi}\int_{2\pi} X(re^{j\omega})e^{j\omega n}d\omega$$

$$= \frac{1}{2\pi}\int_{2\pi} X(re^{j\omega})\left(re^{j\omega}\right)^n d\omega \qquad z = re^{j\omega}$$

$$x[n] = \frac{1}{2\pi j}\oint X(z)z^{n-1}dz \qquad dz = jre^{j\omega}d\omega = jzd\omega$$

The integration is around a c.c.w. circular contour centered at the origin and with radius r. We may choose any value of r such that $|z| = r$ is in the ROC.

10.3 The Inverse z-Transform

- Method #2

Using partial-fraction expansion to obtain IDT

$$X(z) = \frac{A_1}{1 - a_1 z^{-1}} + \frac{A_2}{1 - a_2 z^{-1}} + \cdots + \frac{A_m}{1 - a_m z^{-1}}$$

$$\Rightarrow x[n] = A_1 a_1^n u[n] - A_2 a_2^n u[-n-1] + \cdots + x_m[n]$$

If ROC outside $z=a_1$

If ROC inside $z=a_2$

10.3 The Inverse z-Transform

- Example 10.9: IZT by partial-fraction expansion

$$X(z) = \frac{3 - \frac{5}{6}z^{-1}}{\left(1 - \frac{1}{4}z^{-1}\right)\left(1 - \frac{1}{3}z^{-1}\right)}, \qquad |z| > \frac{1}{3}$$

$$= \frac{1}{\left(1 - \frac{1}{4}z^{-1}\right)} + \frac{2}{\left(1 - \frac{1}{3}z^{-1}\right)}$$

$$\left(\frac{1}{4}\right)^n u[n] \xleftrightarrow{z} \frac{1}{\left(1 - \frac{1}{4}z^{-1}\right)}, \qquad |z| > \frac{1}{4}$$

$$2\left(\frac{1}{3}\right)^n u[n] \xleftrightarrow{z} \frac{2}{\left(1 - \frac{1}{3}z^{-1}\right)}, \qquad |z| > \frac{1}{3}$$

$$\Rightarrow x[n] = \left(\frac{1}{4}\right)^n u[n] + 2\left(\frac{1}{3}\right)^n u[n]$$

10.3 The Inverse z-Transform

- Example 10.9~10.11

	$\lvert z\rvert < \frac{1}{4}$	$\frac{1}{4} < \lvert z\rvert$
$\frac{1}{\left(1-\frac{1}{4}z^{-1}\right)}$	$-\left(\frac{1}{4}\right)^n u[-n-1]$	$\left(\frac{1}{4}\right)^n u[n]$

	$\lvert z\rvert < \frac{1}{3}$	$\frac{1}{3} < \lvert z\rvert$
$\frac{1}{\left(1-\frac{1}{3}z^{-1}\right)}$	$-\left(\frac{1}{3}\right)^n u[-n-1]$	$\left(\frac{1}{3}\right)^n u[n]$

	$\lvert z\rvert < \frac{1}{4}$	$\frac{1}{4} < \lvert z\rvert < \frac{1}{3}$	$\frac{1}{3} < \lvert z\rvert$
$\frac{1}{\left(1-\frac{1}{4}z^{-1}\right)}$	$-\left(\frac{1}{4}\right)^n u[-n-1]$	$\left(\frac{1}{4}\right)^n u[n]$	$\left(\frac{1}{4}\right)^n u[n]$
$+\frac{1}{\left(1-\frac{1}{3}z^{-1}\right)}$	$-\left(\frac{1}{3}\right)^n u[-n-1]$	$-\left(\frac{1}{3}\right)^n u[-n-1]$	$+\left(\frac{1}{3}\right)^n u[n]$

10.3 The Inverse z-Transform

- Example 10.12: IZT by power-series expansion

Given $X(z) = 4z^2 + 2 + 3z^{-1}, \quad 0 < |z| < \infty$, determine x[n].

According to the definition $X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$, we have

$$x[n] = \begin{cases} 4, & n = -2 \\ 2, & n = 0 \\ 3, & n = 1 \\ 0, & \text{otherwise} \end{cases}$$

$$= 4\delta[n+2] + 2\delta[n] + 3\delta[n-1]$$

Alternatively, by using the transform pair

$$\delta[n+n_0] \xleftrightarrow{Z} z^{n_0},$$

we can immediately obtain

$$x[n] = 4\delta[n+2] + 2\delta[n] + 3\delta[n-1].$$

In practice, we may use the Taylor's series expansion to obtain the power series.

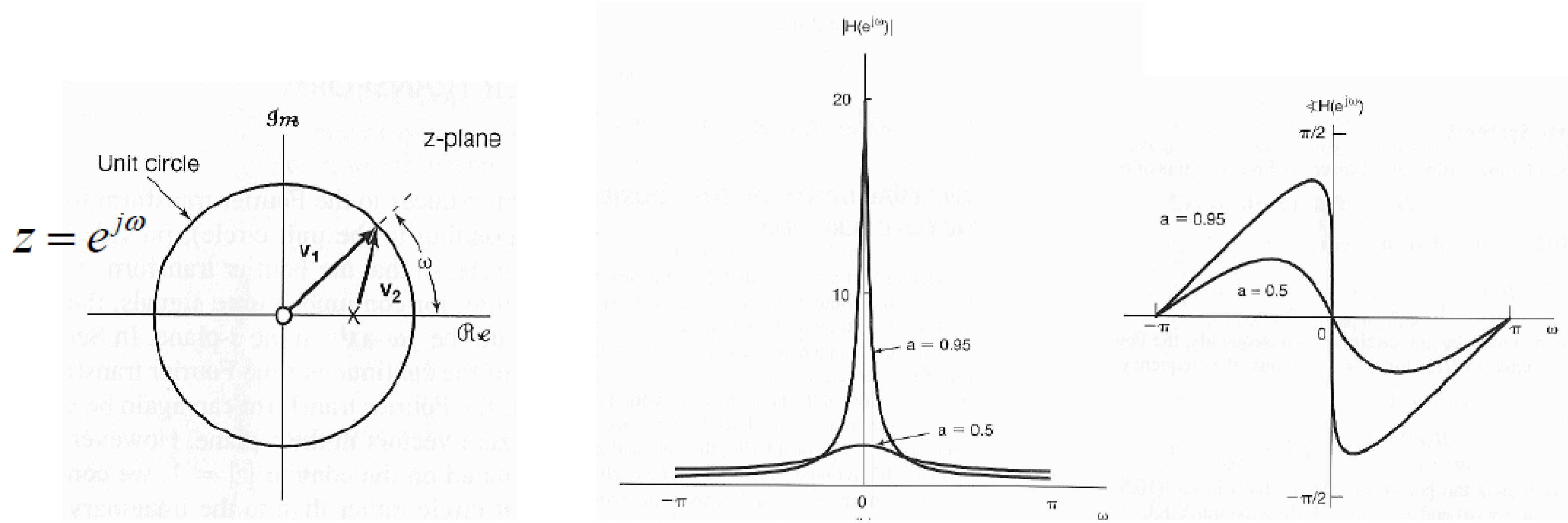
10.4 Geometric Evaluation of the Fourier Transform

- First-Order Systems

$$h[n] = a^n u[n] \xleftrightarrow{z} H(z) = \frac{1}{1 - az^{-1}} = \frac{z}{z-a}, \quad |z| > |a|.$$

$$\text{For } |a| < 1, \text{ ROC includes } |z| = 1 \Rightarrow H(e^{j\omega}) = \frac{1}{1 - ae^{-j\omega}}.$$

We can evaluate the DFT by considering the vectors from the poles and zeros of H(z) to the unit circle in the z-plane.



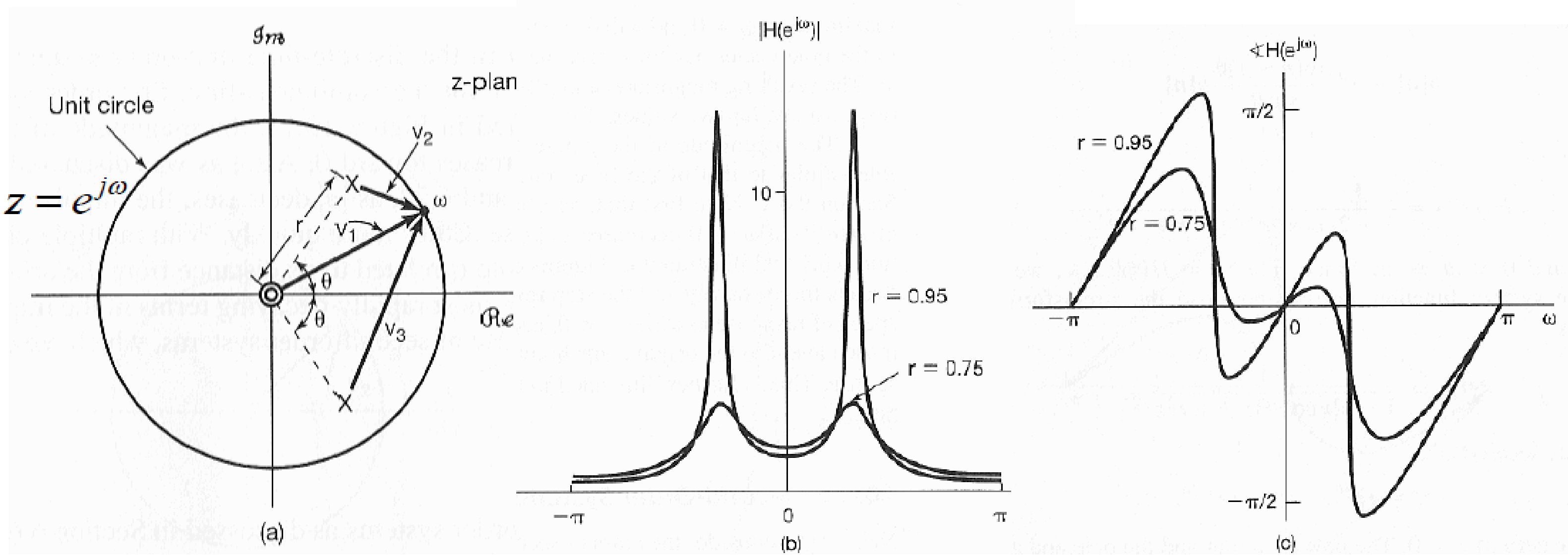
10.4 Geometric Evaluation of the Fourier Transform

- Second-Order Systems

$$r^n \frac{\sin(n+1)\theta}{\sin\theta} u[n] \xleftrightarrow{F} \frac{1}{1-2r\cos\theta e^{-j\omega}+r^2 e^{-j2\omega}}$$

$$H(z)=\frac{1}{1-2r\cos\theta z^{-1}+r^2 z^{-2}}=\frac{z^2}{(z-p_1)(z-p_2)}, \quad |z|>|a|$$

where $0<r<1$ and $0\le\theta\le\pi$. It has poles at $re^{j\theta}$ and $re^{-j\theta}$ and two zeros at $z=0$.



10.5 Properties of the z-Transform

$$X(z) = \sum_{x=-\infty}^{+\infty} x[n] z^{-n}$$

$$x[n] = \frac{1}{2\pi j} \oint X(z) z^{n-1} dz$$

- Linearity

$$x_1[n] \xleftrightarrow{\ z\ } X_1(z), \qquad \mathrm{ROC} = R_1$$

$$x_2[n] \xleftrightarrow{\ z\ } X_2(z), \qquad \mathrm{ROC} = R_2$$

$$ax_1[n] + bx_2[n] \xleftrightarrow{\ z\ } aX_1(z) + bX_2(z), \quad \text{with ROC containing } R_1 \cap R_2$$

The ROC expands when there is pole-zero cancellation.

- Time Shifting

$$x[n] \xleftrightarrow{\ z\ } X(z), \qquad \mathrm{ROC} = R$$

$$x[n-n_0] \xleftrightarrow{\ z\ } z^{-n_0} X(z), \qquad \mathrm{ROC} = R \text{ except for the possible addition or deletion of the origin or infinity}$$

For example, when $n_0>0$, poles at z=0 are introduced, and hence the origin is deleted from the ROC.

10.5 Properties of the z-Transform

$$X(z) = \sum_{n=-\infty}^{+\infty} x[n] z^{-n}$$

$$x[n] = \frac{1}{2\pi j} \oint X(z) z^{n-1} dz$$

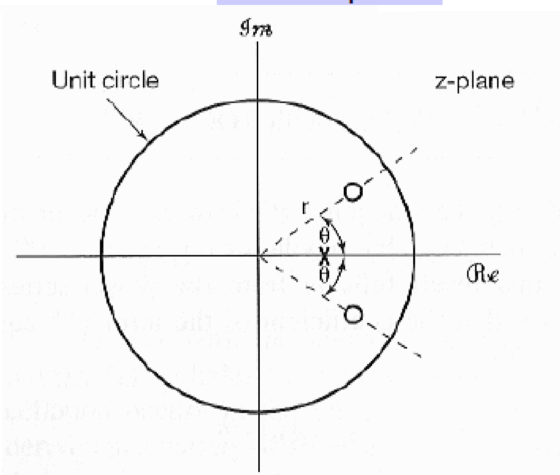
- Scaling in z

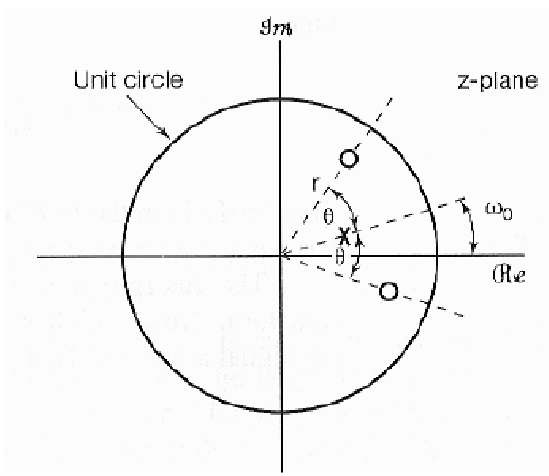
$$x[n] \xleftrightarrow{\mathcal{Z}} X(z), \qquad \text{ROC} = R$$

$$z_0^n x[n] \xleftrightarrow{\mathcal{Z}} X\left(\frac{z}{z_0}\right), \qquad \text{ROC} = |z_0| R$$

$$e^{j\omega_0 n} x[n] \xleftrightarrow{\mathcal{Z}} X(\underbrace{e^{-j\omega_0} z}_{\text{a rotation in the z-plane}}), \qquad \text{ROC} = R$$

A scaled version of R:
If z is a point in the ROC of $X(z)$, then the point $|z_0|z$ is in the ROC of $X(z/z_0)$.





10.5 Properties of the z-Transform

- Time Reversal

$$x[n] \xleftrightarrow{z} X(z), \qquad \text{ROC} = R$$

$$x[-n] \xleftrightarrow{z} X\left(\frac{1}{z}\right), \qquad \text{ROC} = \frac{1}{R}$$

If z_1 is in the ROC for x[n], then $1/z_1$ is in the ROC for x[-n].

- Time Expansion

$$x[n] \xleftrightarrow{z} X(z), \qquad \text{ROC} = R$$

$$x_{(k)}[n] = \begin{cases} x[n/k], & \text{if } n \text{ is a multiple of } k \\ 0, & \text{otherwise} \end{cases}$$

A sequence derived from x[n] by inserting k-1 zeros between successive samples of x[n].

$$x_{(k)}[n] \xleftrightarrow{z} X(z^k), \qquad \text{ROC} = R^{1/k}$$

If z is in the ROC of X(z), then $z^{1/k}$ is in the ROC of $X(z^k)$.

$$X(z^k) = \sum_{n=-\infty}^{\infty} x[n](z^k)^{-n} = \sum_{m=-\infty}^{\infty} x[\frac{m}{k}]z^{-m}, \quad m = kn$$

When m is a multiple of k, the coefficient of the term z^{-m} equals $x[m/k]$; otherwise, it is 0.

10.5 Properties of the z-Transform

- Conjugation

$$x[n] \xleftrightarrow{\mathcal{Z}} X(z), \qquad \text{ROC} = R$$

$$x^*[n] \xleftrightarrow{\mathcal{Z}} X^*\left(z^*\right), \qquad \text{ROC} = R$$

If $x(t)$ is real, $X(z) = X^*(z^*)$.
Thus if $X(z)$ has a pole (zero) at $z = z_0$,
it must have a pole (zero) at $z = z_0^*$.

- Convolution Property

$$x_1[n] \xleftrightarrow{\mathcal{Z}} X_1(z), \qquad \text{ROC} = R_1$$

$$x_2[n] \xleftrightarrow{\mathcal{Z}} X_2(z), \qquad \text{ROC} = R_2$$

$$x_1[n] * x_2[n] \xleftrightarrow{\mathcal{Z}} X_1(z)X_2\left(z\right), \qquad \text{with ROC containing } R_1 \cap R_2$$

The region $R_1 \cap R_2$ may become larger if
pole-zero cancellation occurs in the product

10.5 Properties of the z-Transform

- Differentiation in time

$$x[n] \xleftrightarrow{z} X(z), \qquad \text{ROC} = R$$

$$x[n] - x[n-1] \xleftrightarrow{z} (1 - z^{-1})X(z), \qquad \text{ROC} = R \text{ with the possible deletion of } z = 0 \text{ and/or addition of } z = 1$$

- Integration in time

$$w[n] = \sum_{k=-\infty}^{n} x[k] = u[n] * x[n]$$

Using the convolution property, we have

$$\mathrm{W}(z) = \frac{1}{1 - z^{-1}} X(z), \text{ with ROC containing at least } R \cap (|z| > 1)$$

10.5 Properties of the z-Transform

- Differentiation in the z-domain

$$x[n] \xleftrightarrow{z} X(z), \qquad \text{ROC} = R$$

$$nx[n] \xleftrightarrow{z} -z\frac{dX(z)}{dz}, \qquad \text{ROC} = R$$

$$X(z) = \sum_{n=-\infty}^{+\infty} x[n]z^{-n} \Rightarrow \frac{dX(z)}{dz} = \sum_{n=-\infty}^{+\infty} -nx[n]z^{-n-1}$$

- The Initial-Value Theorem

If $x[n] = 0$ for $n < 0$, then $x[0] = \lim_{z\to\infty} X(z)$

Proof: $X(z) = \sum_{n=-\infty}^{+\infty} x[n]z^{-n} = x[0] + x[1]z^{-1} + x[2]z^{-2}$

$$\therefore \lim_{z\to\infty} X(z) = x[0]$$

Therefore, for a causal sequence x[n], if x[0] is finite, then $\lim_{z\to\infty} X(z)$ must be finite. Consequently, the order of the numerator polynomial cannot be greater than that of the denominator polynomial.

10.7 Analysis & Characterization of LTI Systems Using z-Transforms

- Many properties of a system are tied to characteristics of the poles, zeroes, and ROC of the system.

$$H(z) = \mathcal{Z}\{h[n]\}$$

$$X(z) \longrightarrow \boxed{\text{LTI System}} \longrightarrow Y(z) = X(z)H(z)$$

$H(z)$: system function

or transfer function

10.7 Analysis & Characterization of LTI Systems Using z-Transforms

- Causality

For a causal LTI system, h[n] = 0 for n < 0. Thus h[n] is right-sided.

Since $H(z)=\sum_{n=0}^{\infty}h[n]z^{-n}$ does not include any positive power of z, the ROC of the system must include infinity.

- A DT LTI system is causal if and only if the ROC of the system function $H(z)$ is the exterior of a circle in the z-plane, including infinity
- A DT LTI system with a rational $H(z)$ is causal if and only if
  (a) ROC is exterior of a circle outside the outermost pole; and infinity must be in the ROC
  (b) Order of numerator ≤ order of denominator

Because H(z) must be finite as z approaches infinity.

10.7 Analysis & Characterization of LTI Systems Using z-Transforms

- Example 10.21 Causality Analysis

$$H(z) = \frac{1}{1-\frac{1}{2}z^{-1}} + \frac{1}{1-2z^{-1}}, \qquad |z| > 2$$

Since the ROC is the exterior of a circle outside the outermost pole, the impulse response is right-sided.

$$H(z) = \frac{2-\frac{5}{2}z^{-1}}{\left(1-\frac{1}{2}z^{-1}\right)\left(1-2z^{-1}\right)} = \frac{2z^2-\frac{5}{2}z}{z^2-\frac{5}{2}z+1}$$

Numerator degree = Denominator degree

$\Rightarrow$ The system is causal.

Check: $h[n] = \left[\left(\frac{1}{2}\right)^n + 2^n\right]u[n] \Rightarrow h[n] = 0 \text{ for } n < 0$

10.7 Analysis & Characterization of LTI Systems Using z-Transforms

- Stability
  - A DT LTI system is stable if and only if the ROC of $H(z)$ includes the unit circle of $|z| = 1$.
  - A causal LTI system with rational $H(z)$ is stable if and only if all poles of $H(z)$ lie inside the unit circle, i.e., all of the poles have magnitudes < 1.

10.7 Analysis & Characterization of LTI Systems Using z-Transforms

- Example 10.22

$$H(z) = \frac{1}{1-\frac{1}{2}z^{-1}} + \frac{1}{1-2z^{-1}}, \quad |z| > 2$$

Since the ROC does not include the unit circle $\Rightarrow$ unstable

We can check this result by noting that

$$h[n] = \left[\left(\frac{1}{2}\right)^n + 2^n\right]u[n] \to \infty, \quad \text{as } n \to \infty$$

If ROC is the region $1/2 < |z| < 2 \;\Rightarrow\; h[n] = \left(\frac{1}{2}\right)^n u[n] - 2^n u[-n-1]$

$\Rightarrow$ The system is NOT causal, but stable

If $ROC = |z| < \frac{1}{2} \;\Rightarrow\; h[n] = -\left[\left(\frac{1}{2}\right)^n + 2^n\right]u[-n-1]$

$\Rightarrow$ The system is neither causal nor stable

10.7 Analysis & Characterization of LTI Systems Using z-Transforms

- Example 10.24

$$H(z) = \frac{1}{1-(2r\cos\theta)z^{-1}+r^2z^{-2}}$$

$$\Rightarrow z_1 = re^{j\theta}, \quad z_2 = re^{-j\theta}$$

To be causal $\Rightarrow |z| > |r|$.

To be stable $\Rightarrow r < 1$.

If $r > 1$, the poles are outside the unit circle. In this case, since the ROC does not include the unit circle, the system is unstable.

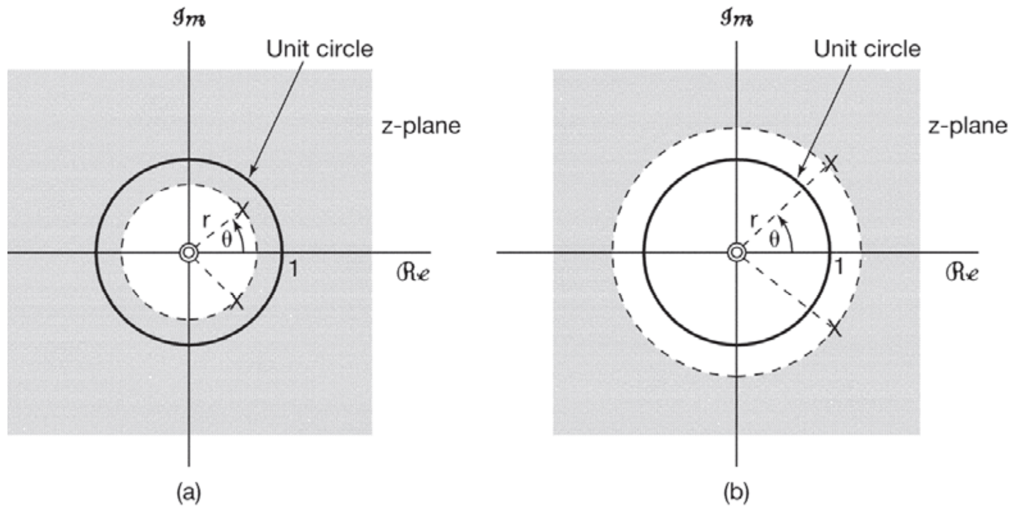


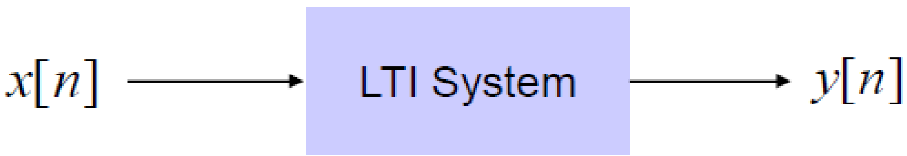
Figure 10.16 Pole-zero plot for a second-order system with complex poles: (a) $r < 1$; (b) $r > 1$.

10.7 Analysis & Characterization of LTI Systems Using z-Transforms

- LTI Systems by Linear Constant-Coeff. Difference Equations

$$a_0 y[n] + a_1 y[n-1] + \cdots + a_{N-1} y[n-N+1] + a_N y[n-N]$$
$$= b_0 x[n] + b_1 x[n-1] + \cdots + b_{M-1} x[n-M+1] + b_M x[n-M]$$

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$



$$Y(z) = X(z)H(z) \qquad\qquad H(z) = \frac{Y(z)}{X(z)}$$

10.7 Analysis & Characterization of LTI Systems Using z-Transforms

- LTI Systems by Linear Constant-Coeff. Difference Equations (cont'd)

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k] \qquad x[n-n_0] \xleftrightarrow{z} z^{-n_0} X(z)$$

$$Z\left\{\sum_{k=0}^{N} a_k y[n-k]\right\} = Z\left\{\sum_{k=0}^{M} b_k x[n-k]\right\}$$

$$\sum_{k=0}^{N} a_k Z\{y[n-k]\} = \sum_{k=0}^{M} b_k Z\{x[n-k]\}$$

$$\sum_{k=0}^{N} a_k z^{-k} Y(z) = \sum_{k=0}^{M} b_k z^{-k} X(z)$$

$$\Rightarrow H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}} \quad \begin{matrix}\text{zeros} \\ \text{poles}\end{matrix}$$

10.7 Analysis & Characterization of LTI Systems Using z-Transforms

- Example 10.25

$$y[n] - \frac{1}{2}y[n-1] = x[n] + \frac{1}{3}x[n-1]$$

$$Y(z) - \frac{1}{2}z^{-1}Y(z) = X(z) + \frac{1}{3}z^{-1}X(z) \Rightarrow H(z) = (1 + \tfrac{1}{3}z^{-1})\frac{1}{1 - \frac{1}{2}z^{-1}}$$

Two choices of ROC:

- If $|z| > 1/2 \;\Rightarrow\; h[n]$ is right-sided
- If $|z| < 1/2 \;\Rightarrow\; h[n]$ is left-sided

If ROC is the region $|z| > 1/2$, $\quad h[n] = \left(\frac{1}{2}\right)^n u[n] + \frac{1}{3}\left(\frac{1}{2}\right)^{n-1} u[n-1]$

If ROC is the region $|z| < 1/2$, $\quad h[n] = -\left(\frac{1}{2}\right)^n u[-n-1] - \frac{1}{3}\left(\frac{1}{2}\right)^{n-1} u[-n]$,

which is anticausal and unstable.