Signals & Systems

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Ch. 6 Time & Frequency Characterization of Signals and Systems

- Sec. 6.1 The Magnitude-phase Representation of the Fourier Transform
- Sec. 6.2 The Magnitude-phase Rep. of Frequency Response of LTI Systems
- Sec. 6.3 Time-domain Properties of Ideal Frequency-selective Filters
- Sec. 6.4 Time-domain and Frequency-domain Aspects of Nonideal Filters
- Sec. 6.5* Time and Frequency Characterization for Some Well-known Filters (*: not in the original 2nd edition)
- Sec. 6.6 First-order and Second-order Continuous-time Systems
- Sec. 6.7 First-order and Second-order Discrete-time Systems
- Sec. 6.8 Examples of Time- and Frequency-domain Analysis of Systems

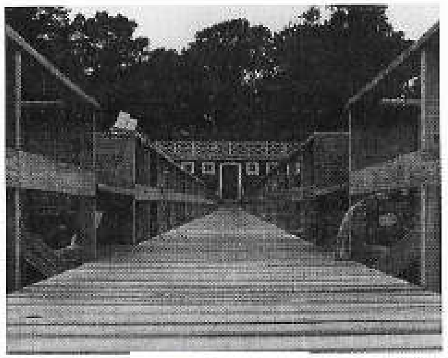
magnitude-phase representation

characters for *filters*

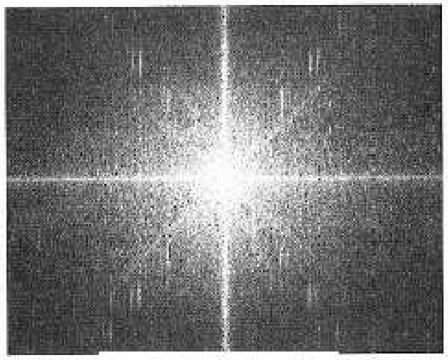
characters for *systems*

- Example 2 Impact of Phase on *Images*

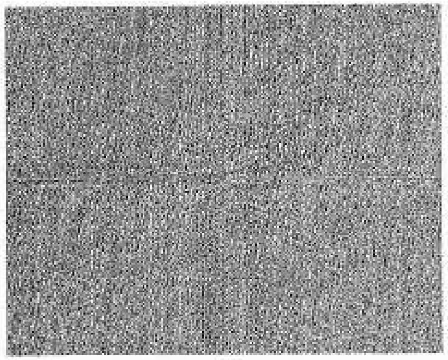
$$F(u,v) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} f[m,n] e^{-j2\pi(um+vn)}$$



image

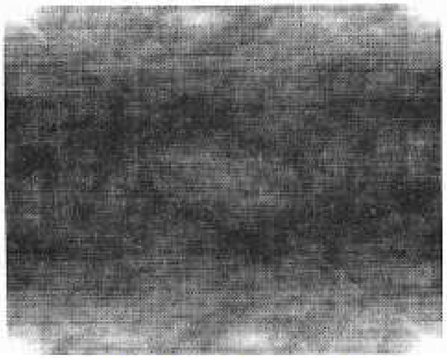


magnitude



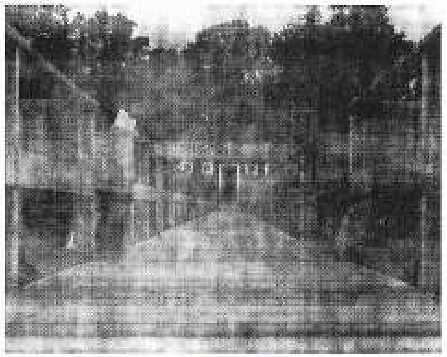
phase

new image



magnitude + zero phase

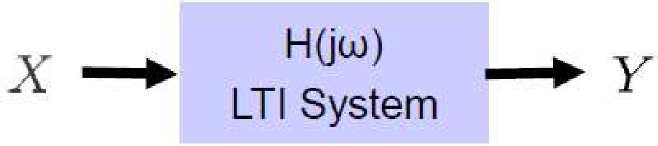
unit magnitude + phase



magnitude of a new image + phase

6.2 Magnitude-Phase Representation of Frequency Response of LTI Systems

- Effect of an LTI system on an input



$$Y(j\omega) = X(j\omega)H(j\omega)$$

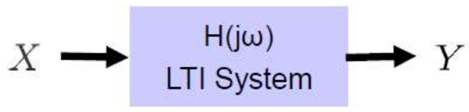
$\Rightarrow$ The system changes the complex amplitude of each frequency component of $x(t)$.

$$\begin{aligned}|Y(j\omega)|e^{j\sphericalangle Y(j\omega)} &= |X(j\omega)|e^{j\sphericalangle X(j\omega)}\,|H(j\omega)|e^{j\sphericalangle H(j\omega)} \\ &= |X(j\omega)||H(j\omega)|e^{j(\sphericalangle X(j\omega)+\sphericalangle H(j\omega))}\end{aligned}$$

$$\Rightarrow \begin{cases} |H(j\omega)|: & \text{gain of the LTI system} \\ \sphericalangle H(j\omega): & \text{phase shift of the LTI system} \end{cases}$$

If the input is changed in an unwanted manner, the effects are referred to as magnitude and phase *distortions*.

Magnitude-Phase Representation of Frequency Response of LTI Systems



- Linear vs. Non-linear Phases

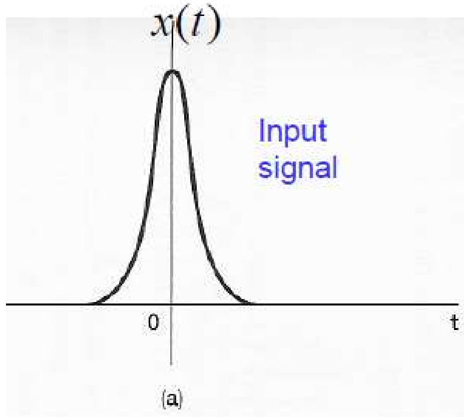
When the phase shift is a linear function of ω, we call it a linear phase shift. For example, consider

$$H_1(j\omega) = e^{-j\omega t_0}.$$

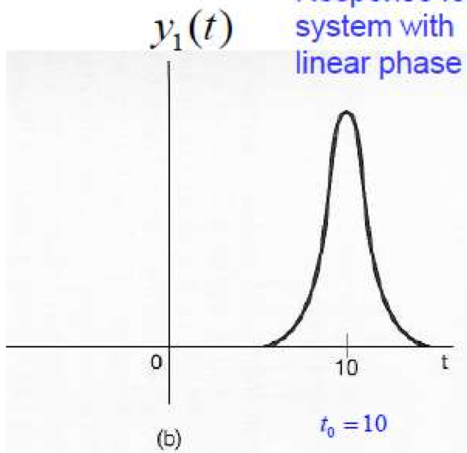
The system has unit gain and linear phase

$$|H_1(j\omega)| = 1, \ \measuredangle H_1(j\omega) = -\omega t_0.$$

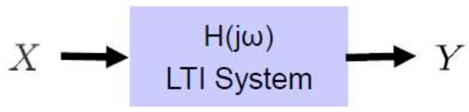
In the time domain, the system introduces a constant time shift to the signal.



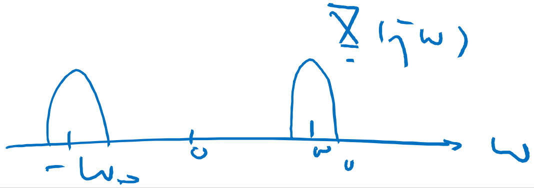
Response for a system with linear phase



Magnitude-Phase Representation of Frequency Response of LTI Systems



- Narrowband vs. Broadband Signals



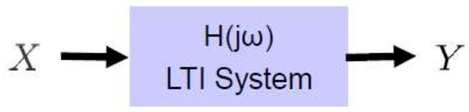
Suppose that $X(j\omega)$ is the Fourier transform of $x(t)$. If $X(j\omega)$ is zero or negligibly small outside a very small band of frequencies centered at $\omega = \pm\omega_0$, then we call $x(t)$ a *narrowband signal*. Otherwise, we call $x(t)$ a *broadband signal*.

By taking the band to be very small, we can accurately approximate the phase of this system in the band with the linear approximation.

$$\measuredangle H(j\omega) \simeq -\phi - \omega\alpha, \tag{6.20}$$

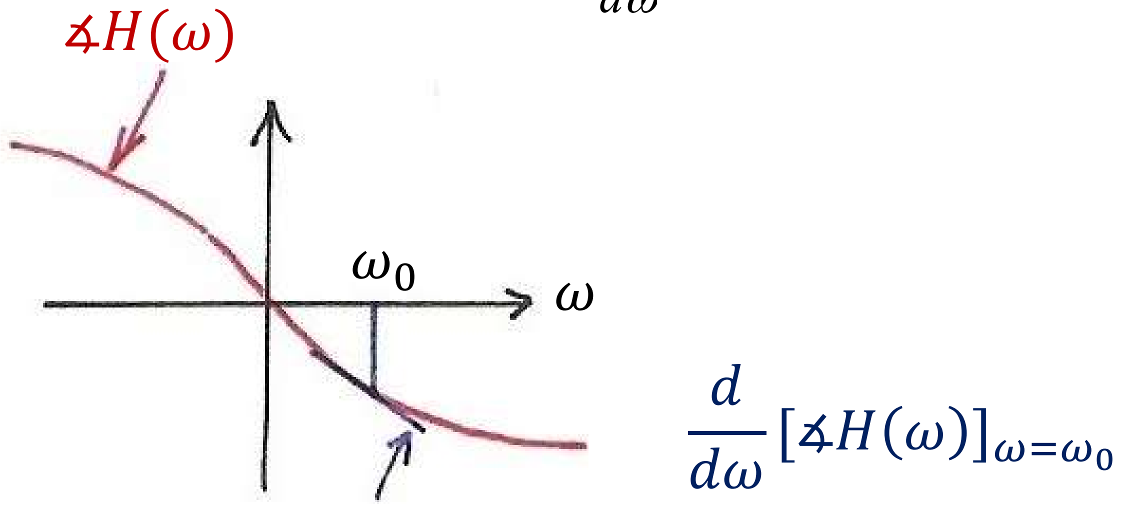
$$Y(j\omega) \simeq X(j\omega)|H(j\omega)|e^{-j\phi}e^{-j\omega\alpha}. \tag{6.21}$$

Magnitude-Phase Representation of Frequency Response of LTI Systems

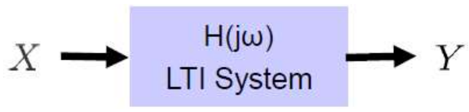


- Group Delay
  - The phase slope tells us the size of the time shift.
  - Such a time shift can be considered as a time delay.
  - E.g., $x(t-t_0) \overset{F}{\longleftrightarrow} e^{-j\omega t_0} X(j\omega)$

  - The group delay at ω is calculated as the negative slope of the phase at that frequency, i.e., $\tau(\omega) = -\dfrac{d}{d\omega}[\angle H(j\omega)]$.



Magnitude-Phase Representation of Frequency Response of LTI Systems



- Effects of LTI Systems on Narrowband Input Signals

- The concept of delay can be extended to include nonlinear phases.
- Consider a narrowband signal x(t) whose Fourier transform is zero or negligibly small outside a small band of frequencies centered at $\omega = \omega_0$.
- We can approximate the phase of the system in the band with a linear approximation centered at $\omega = \omega_0$:

$$\measuredangle H(j\omega) \simeq -\phi - \omega\alpha$$

where ϕ is a constant, so that

$$Y(j\omega) = X(j\omega)\left|H(j\omega)\right| e^{-j\phi} e^{-j\omega\alpha}$$

This time delay of α seconds is referred to as the group delay at $\omega = \omega_0$.

- Group Delay

$$\boxed{\tau(\omega) = -\frac{d}{d\omega}\{\measuredangle H(j\omega)\}}$$

- Example 6.1

$$X \longrightarrow \boxed{H1} \longrightarrow \boxed{H2} \longrightarrow \boxed{H3} \longrightarrow Y$$

An all-pass system with non-constant group delay that introduces the so-called dispersion effect where different frequencies in the input are delayed by different amounts

$$H(j\omega) = H_1(j\omega)H_2(j\omega)H_3(j\omega)$$

$$H_i(j\omega) = \frac{1 + (j\omega/\omega_i)^2 - 2j\zeta_i(\omega/\omega_i)}{1 + (j\omega/\omega_i)^2 + 2j\zeta_i(\omega/\omega_i)}$$

$$\begin{cases} |H_i(j\omega)| = 1 \\ \measuredangle H_i(j\omega) = -2\arctan\left[\dfrac{2\zeta_i(\omega/\omega_i)}{1-(\omega/\omega_i)^2}\right] \end{cases}$$

$$\Rightarrow \begin{cases} |H(j\omega)| = 1 \\ \measuredangle H(j\omega) = \measuredangle H_1(j\omega) + \measuredangle H_2(j\omega) + \measuredangle H_3(j\omega) \end{cases}$$

$$\tau(\omega) = -\frac{d}{d\omega}\{\measuredangle H(j\omega)\}$$ Group delay

- Example 6.1 (cont'd) $X \rightarrow \boxed{H1} \rightarrow \boxed{H2} \rightarrow \boxed{H3} \rightarrow Y$

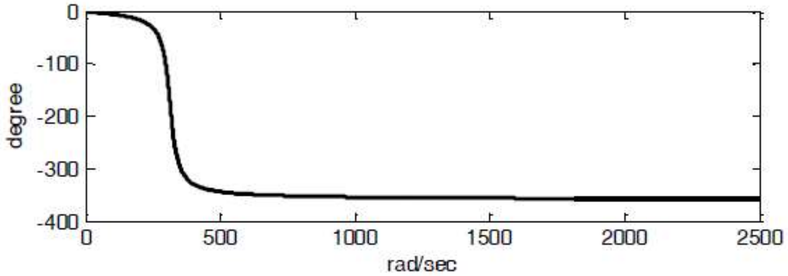
An all-pass system with non-constant group delay that introduces the so-called dispersion effect where different frequencies in the input are delayed by different amounts

$$H_1(j\omega) = \frac{1+(j\omega/\omega_1)^2 - 2j\zeta_1(\omega/\omega_1)}{1+(j\omega/\omega_1)^2 + 2j\zeta_1(\omega/\omega_1)}$$

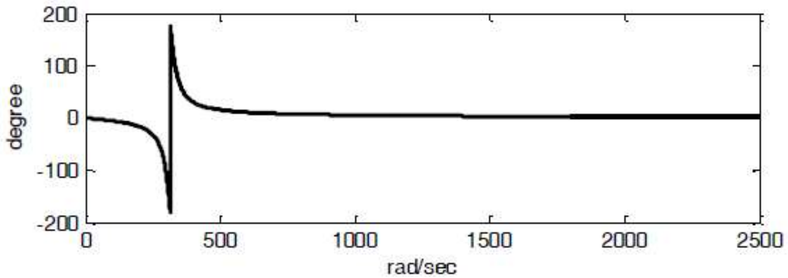
$$\Rightarrow \begin{cases} |H_1(j\omega)| = 1 \\ \sphericalangle H_1(j\omega) = -2\arctan\left[\dfrac{2\zeta_i(\omega/\omega_i)}{1-(\omega/\omega_i)^2}\right] \end{cases}$$

$$\begin{cases} \omega_1 = 315 \text{ rad/sec} \\ \zeta_1 = 0.066 \\ f_1 \approx 50 \text{ Hz} \end{cases}$$

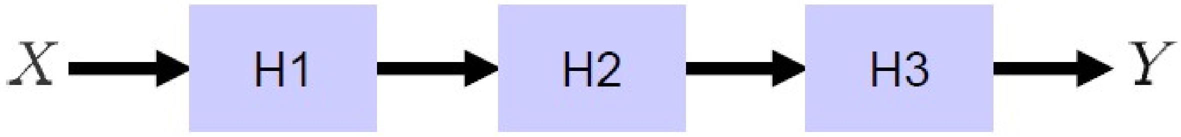
Unwrapped phase



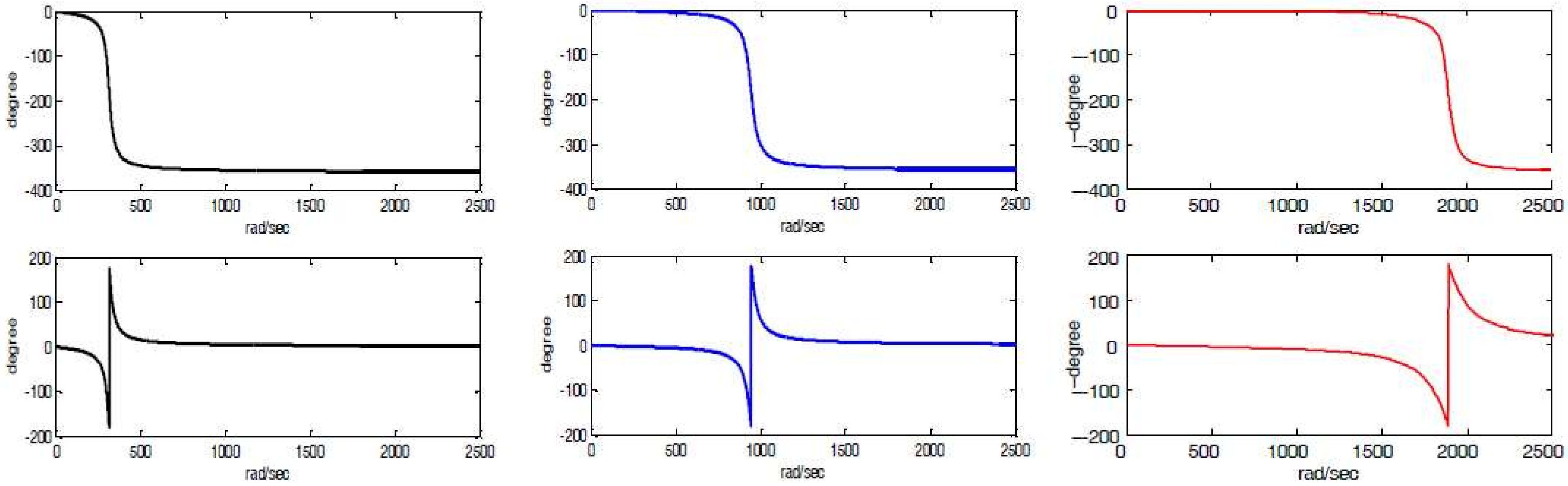
Principal phase



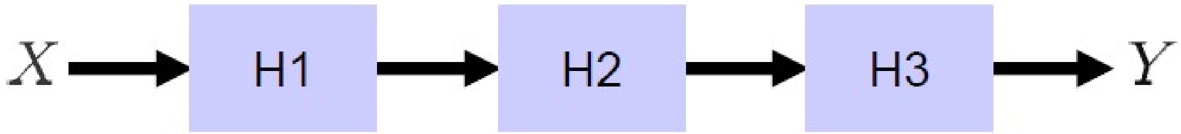
- Example 6.1 (cont'd)



$$\begin{cases} \omega_1 = 315 \quad \text{rad/sec} \\ \omega_2 = 943 \quad \text{rad/sec} \\ \omega_3 = 1888 \quad \text{rad/sec} \end{cases} \qquad \begin{cases} \zeta_1 = 0.066 \\ \zeta_2 = 0.033 \\ \zeta_3 = 0.058 \end{cases}$$

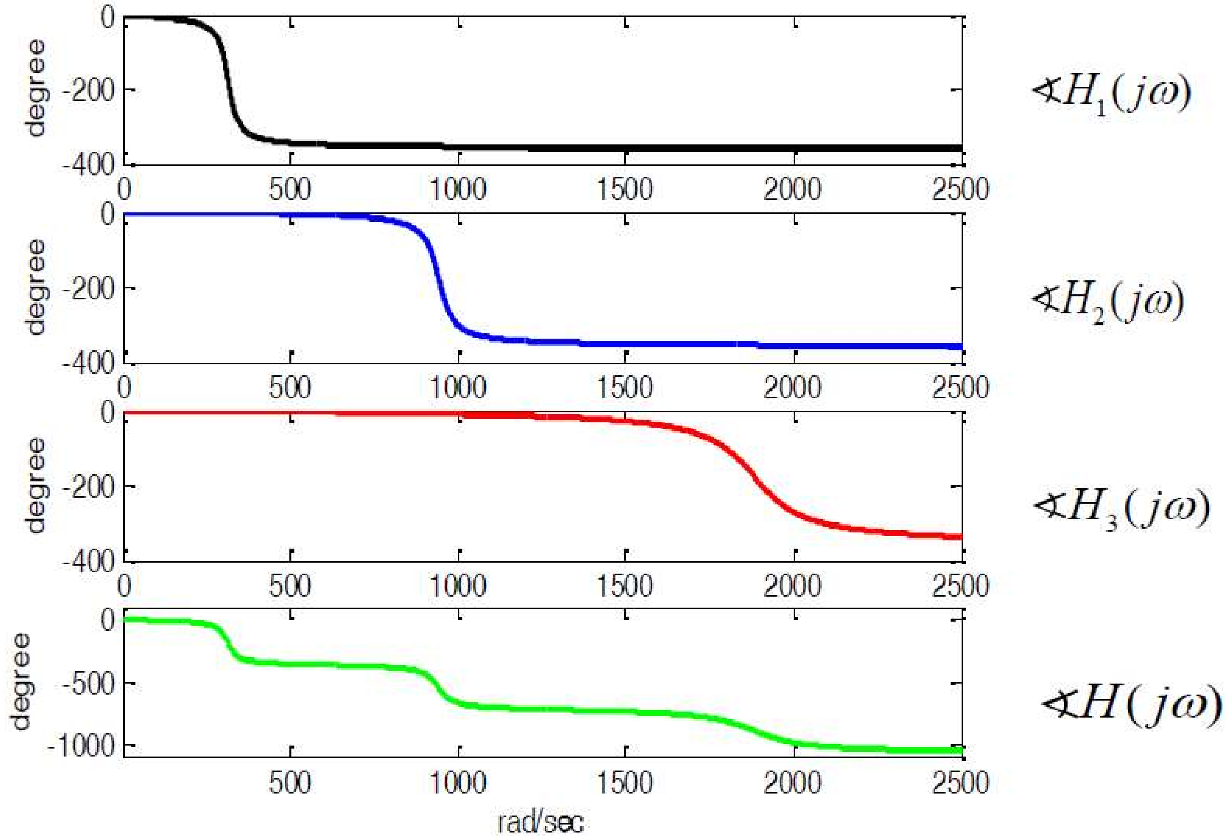


- Example 6.1 (cont'd)



$$\begin{cases} |H(j\omega)| = 1 \\ \measuredangle H(j\omega) = \measuredangle H_1(j\omega) + \measuredangle H_2(j\omega) + \measuredangle H_3(j\omega) \end{cases}$$

Unwrapped phase

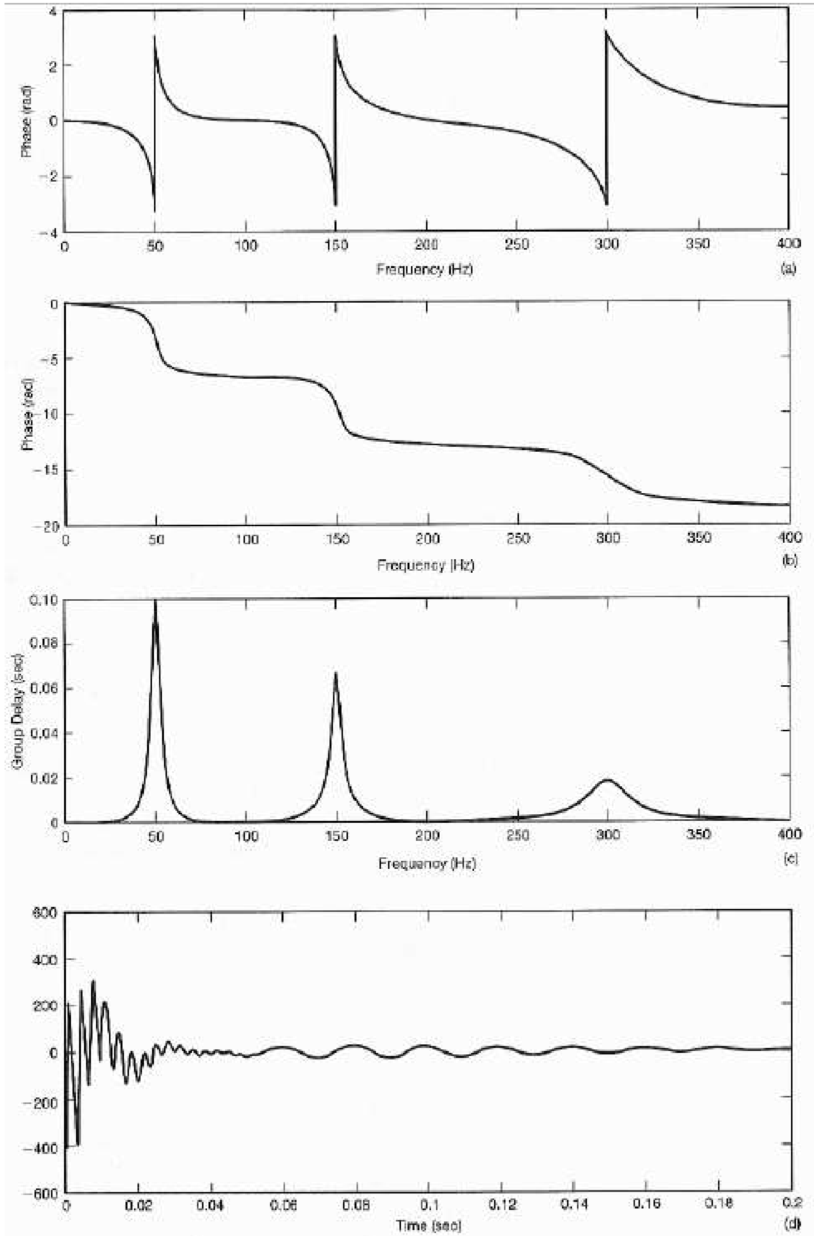


- Example 6.1 (cont'd)

Group delay:

$$\tau(\omega) = -\frac{d}{d\omega}\{\measuredangle H(j\omega)\}$$

Dispersion:
The phenomenon that different frequencies in the input are delayed by different amounts.



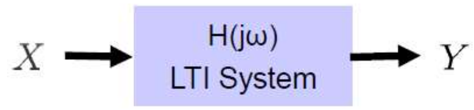
Principal phase

Unwrapped phase

Group delay

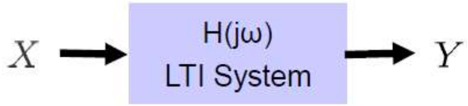
Impulse response

6.2.3 Log-Magnitude & Bode Plots

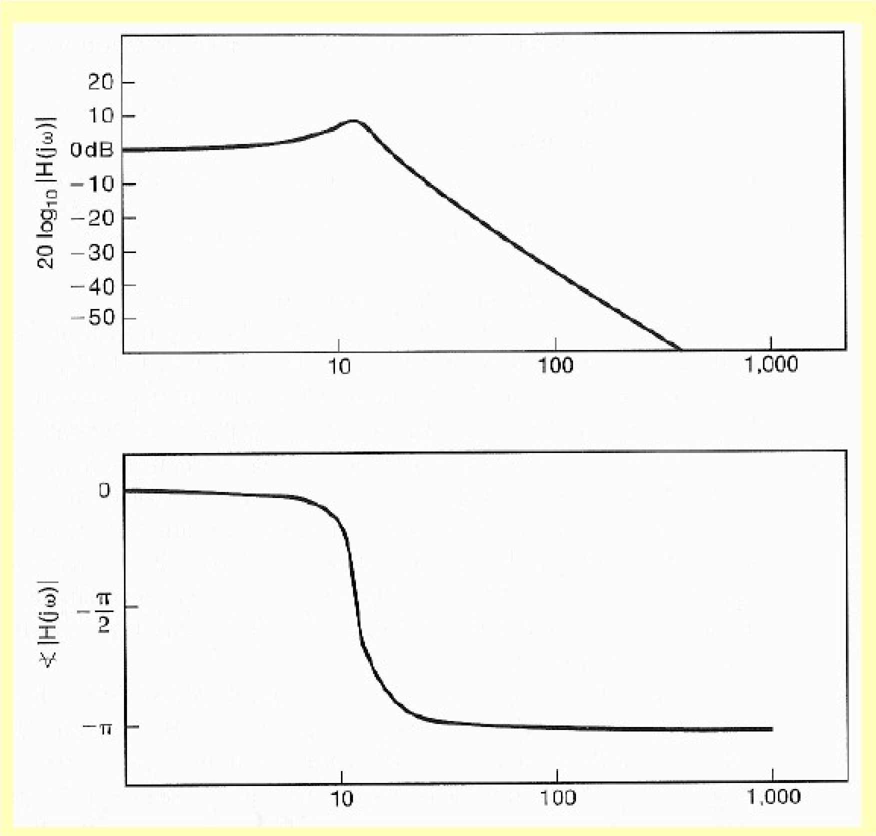


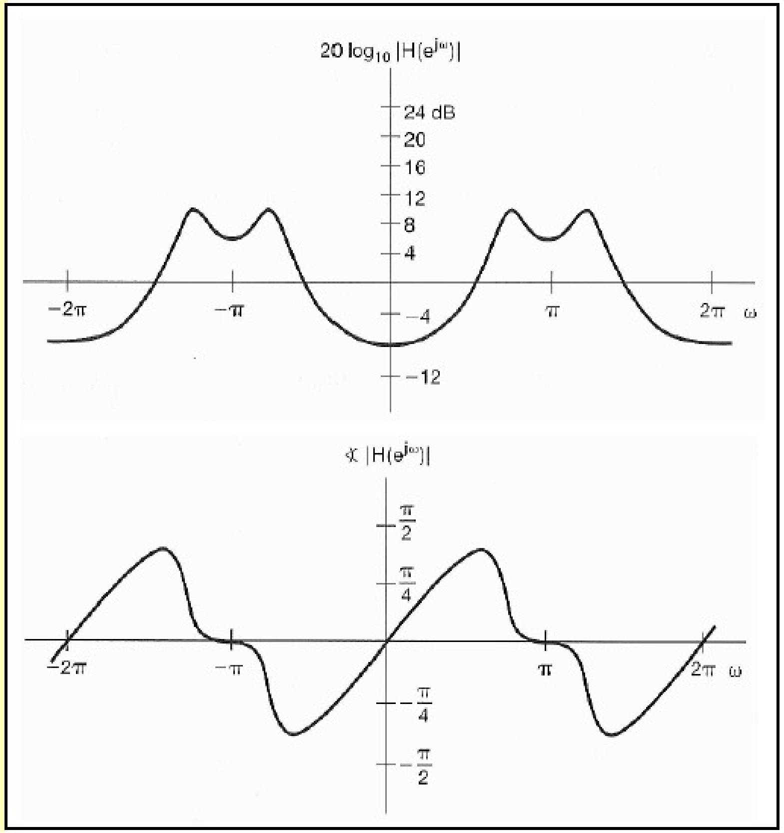
- Decibel: $20\log_{10}$
  - 0dB: a frequency response with magnitude equal to 1
  - 20dB: equivalent to a gain of 10
  - -20dB: corresponds to an attenuation of 0.1
- Bode Plots:
  - plots of $20\log_{10}|H(j\omega)|$ & $\sphericalangle H(j\omega)$ vs. $\log_{10}\omega$

6.2.3 Log-Magnitude & Bode Plots



Continuous-Time Bode plot	Discrete-Time Bode plot





$\log(w), \quad w: 0 \leftrightarrow \infty$

$(w), \quad w: -\pi \leftrightarrow \pi$

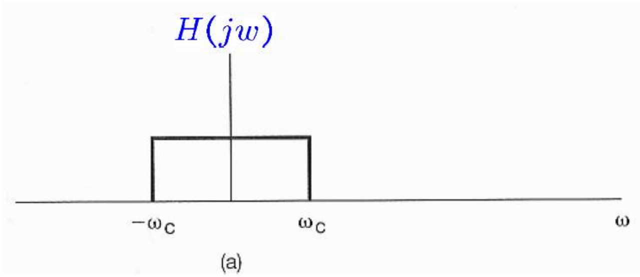
Time-Domain Properties of Ideal Frequency-Selective Filters

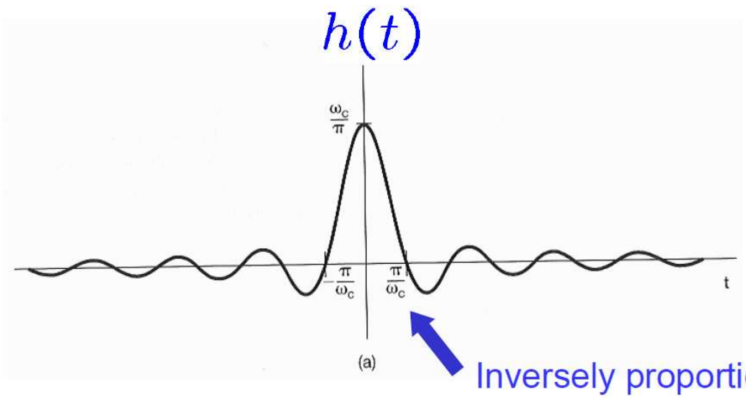
- Ideal LPF

$$H(j\omega)=\begin{cases}1, & |\omega|\le\omega_c\\ 0, & |\omega|>\omega_c\end{cases}$$

$$\Rightarrow \quad h(t)=\frac{\sin\omega_c t}{\pi t}$$

unit gain, zero phase





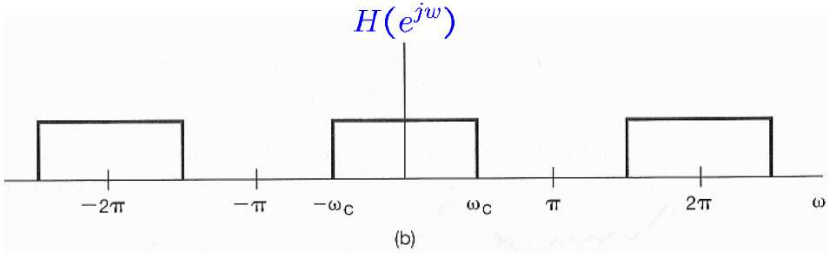
Inversely proportional to ω_c

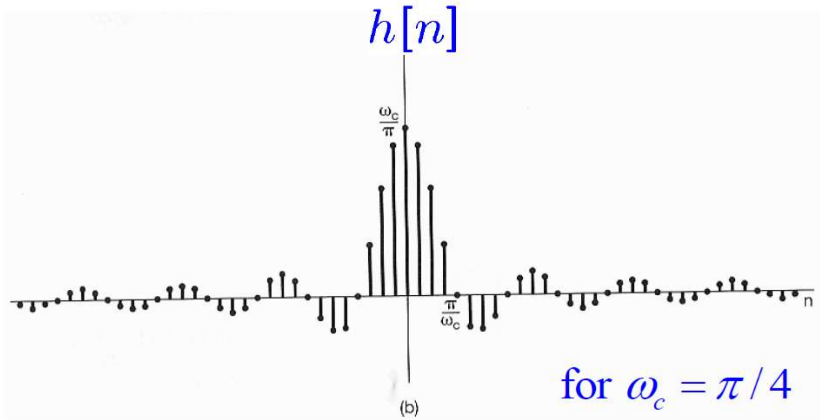
Time-Domain Properties of Ideal Frequency-Selective Filters

- Ideal LPF

$$H(e^{j\omega}) = \begin{cases} 1, & |\omega| \le \omega_c \\ 0, & \omega_c \le |\omega| \le \pi \end{cases}$$

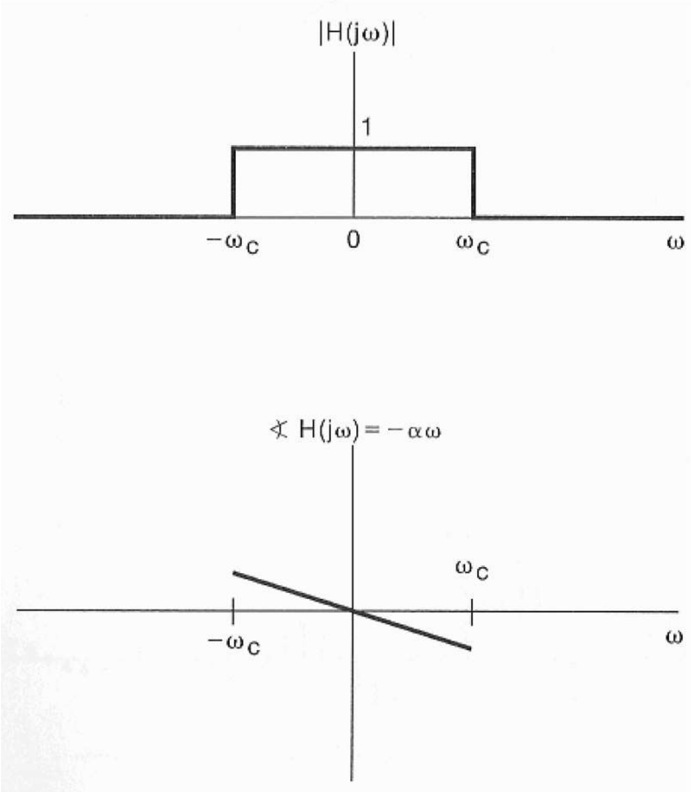
$$\Rightarrow \quad h[n] = \frac{\sin \omega_c n}{\pi n}$$

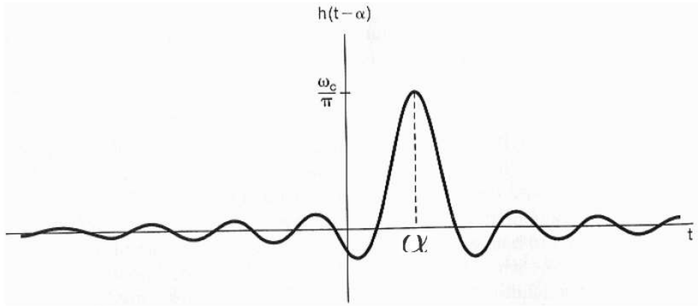




Time-Domain Properties of Ideal Frequency-Selective Filters

- Ideal LPF wit Linear Phase

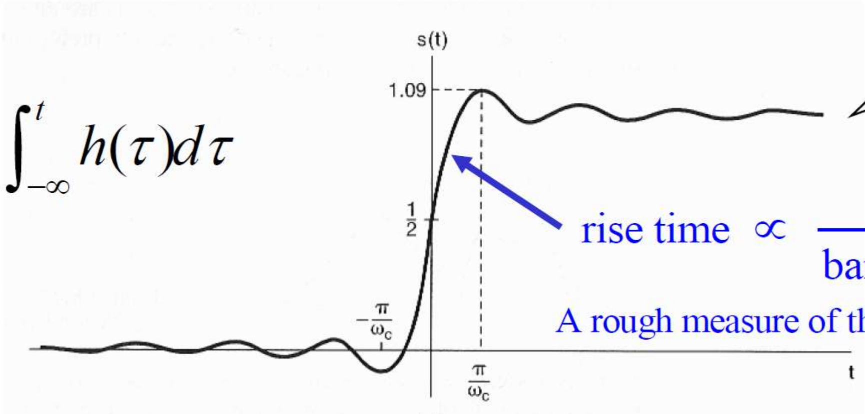




Time-Domain Properties of Ideal Frequency-Selective Filters

- Step Response of Ideal LPF

$$s(t) = \int_{-\infty}^{t} h(\tau) d\tau$$



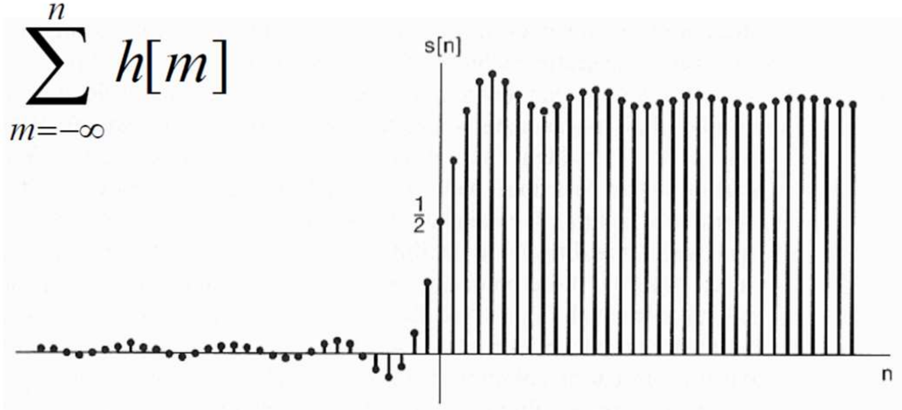
(a)

The oscillatory behavior is called ringing

$$\text{rise time} \propto \frac{1}{\text{bandwidth}}$$

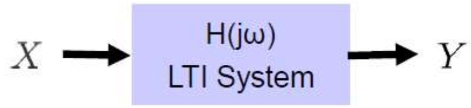
A rough measure of the response time of the filter

$$s[n] = \sum_{m=-\infty}^{n} h[m]$$



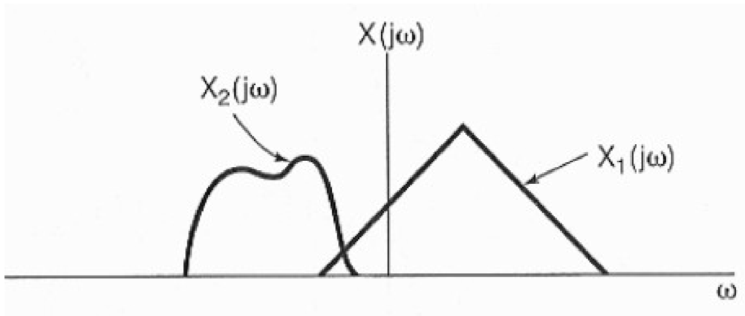
(b)

6.4 Time-Domain and Frequency-Domain Aspects of Non-Ideal Filters

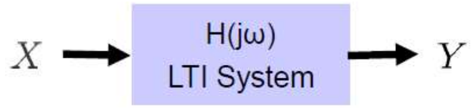


- Considerations of filter designs
  - Fidelity of the signal to be preserved/attenuated/amplified.
  - Ringing effects
  - Causality
  - Ease of implementation

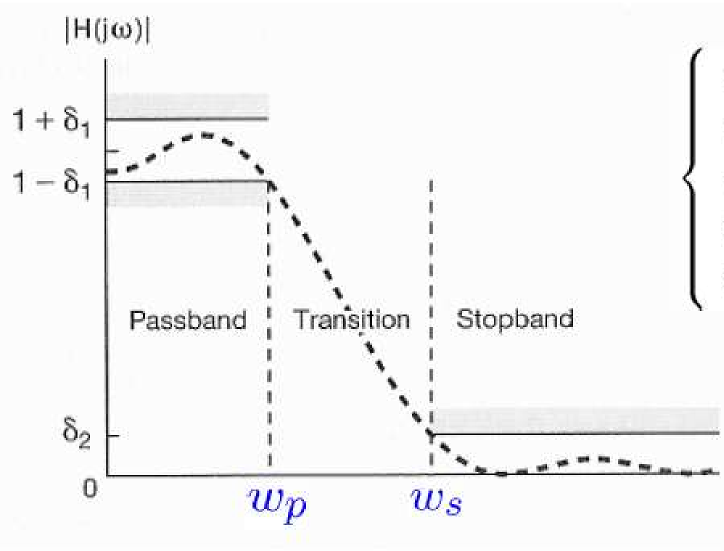
- From the above reasons, non-ideal filters are often desirable in practice.



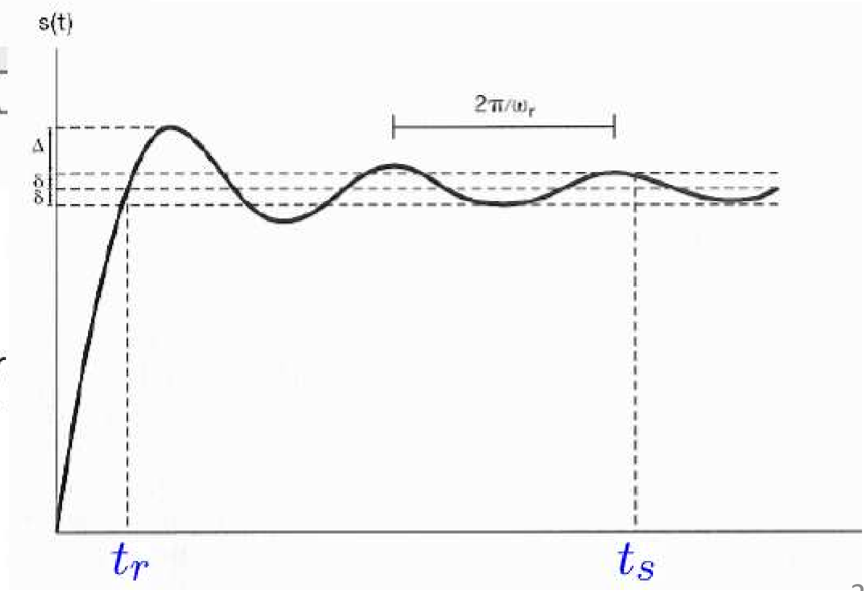
6.4 Time-Domain and Frequency-Domain Aspects of Non-Ideal Filters



- Quantification of LPF Characteristics

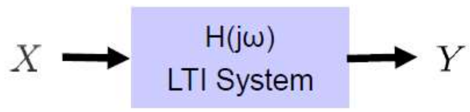


$$\begin{cases} \delta_1 : & \text{allowable passband ripple} \\ \delta_2 : & \text{allowable stopband ripple} \\ w_p : & \text{passband edge} \\ w_s : & \text{stopband edge} \\ w_s - w_p : & \text{transition band} \end{cases}$$

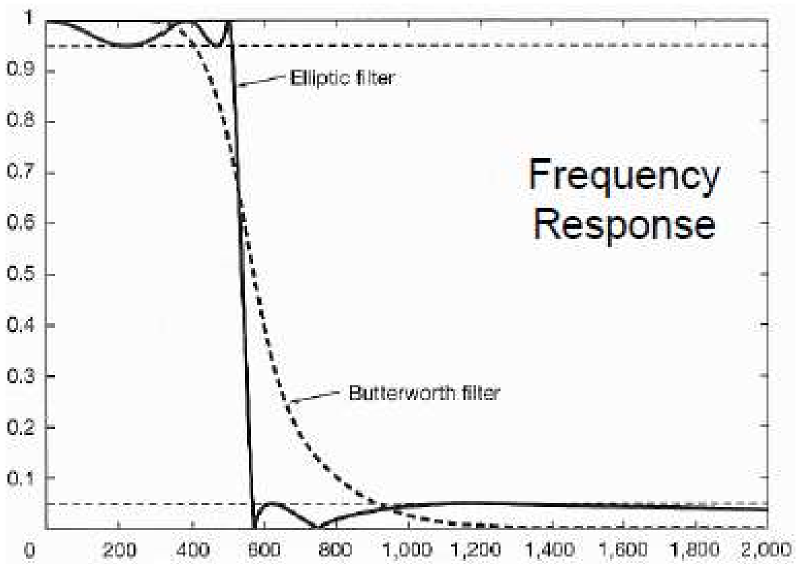


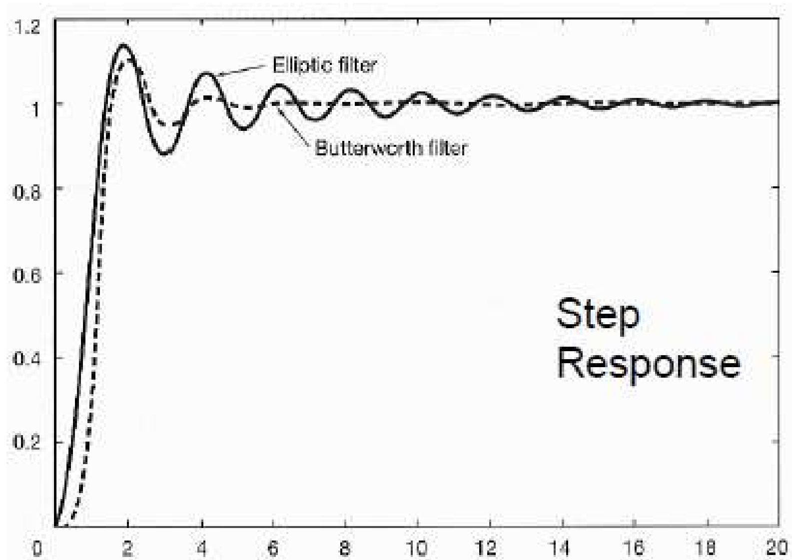
$$\begin{cases} \Delta & : & \text{overshoot} \\ \delta & : & \text{steady-state error} \\ w_r & : & \text{ringing frequency} \\ t_r & : & \text{rise time} \\ t_s & : & \text{settling time} \end{cases}$$

6.4 Time-Domain and Frequency-Domain Aspects of Non-Ideal Filters



- Example 6.3 Two Frequently Used Filters
  - Butterworth filter vs. elliptic filter
  - Fifth-order filters
  - Real-valued impulse response
  - Cutoff frequency at 500 Hz
  - Narrower transition band => more ringing





6.5* Time and Frequency Characterization for Some Well-known Filters

- Frequency selective filters
  - Ideal HPF

$$|H(j\omega)| = \begin{cases} 1 & |\omega| > \omega_c \\ 0 & |\omega| < \omega_c \end{cases}. \qquad (6.30)$$

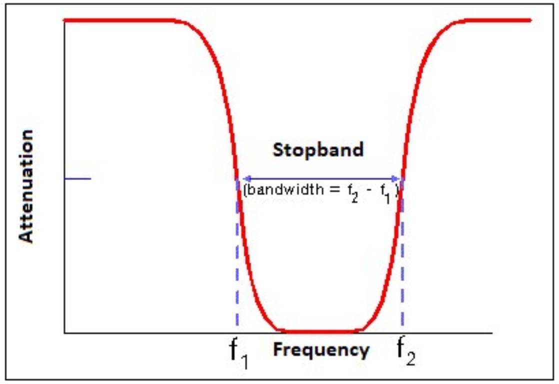
  - Ideal BPF

$$|H(j\omega)| = \begin{cases} 1 & for \quad \omega_c < |\omega| < \omega_d \\ 0 & for \quad |\omega| < \omega_c \quad or \quad |\omega| > \omega_d \end{cases}. \qquad (6.31)$$

  - Ideal Band Stop Filter

$$|H(j\omega)| = \begin{cases} 1 & for \quad |\omega| < \omega_c \quad or \quad |\omega| > \omega_d \\ 0 & for \quad \omega_c < |\omega| < \omega_d \end{cases}. \qquad (6.32)$$

6.5* Time and Frequency Characterization for Some Well-known Filters



- Frequency selective filters
  - For ideal *notch* filter, the frequency response $H(j\omega)$ is very close to that of the ideal *bandstop* filter with

$$\omega_d - \omega_c \cong 0. \tag{6.33}$$

  - The phase of the frequency response of an ideal frequency selective filter always has a linear form, as shown in Fig. 6.17:

$$\sphericalangle H(j\omega) = -\alpha\omega. \tag{6.34}$$

  - Thus, h(t) is evenly symmetric with respect to α, as shown in Fig. 6.19:

$$h(\alpha + t) = h(\alpha - t). \tag{6.35}$$

6.5* Time and Frequency Characterization for Some Well-known Filters

- For the discrete-time versions,
  - Ideal HPF

$$|H(e^{j\omega})| = \begin{cases} 1 & |\omega| \le \omega_c \\ 0 & \omega_c < |\omega| \le \pi \end{cases}. \tag{6.36}$$

  - Ideal BPF

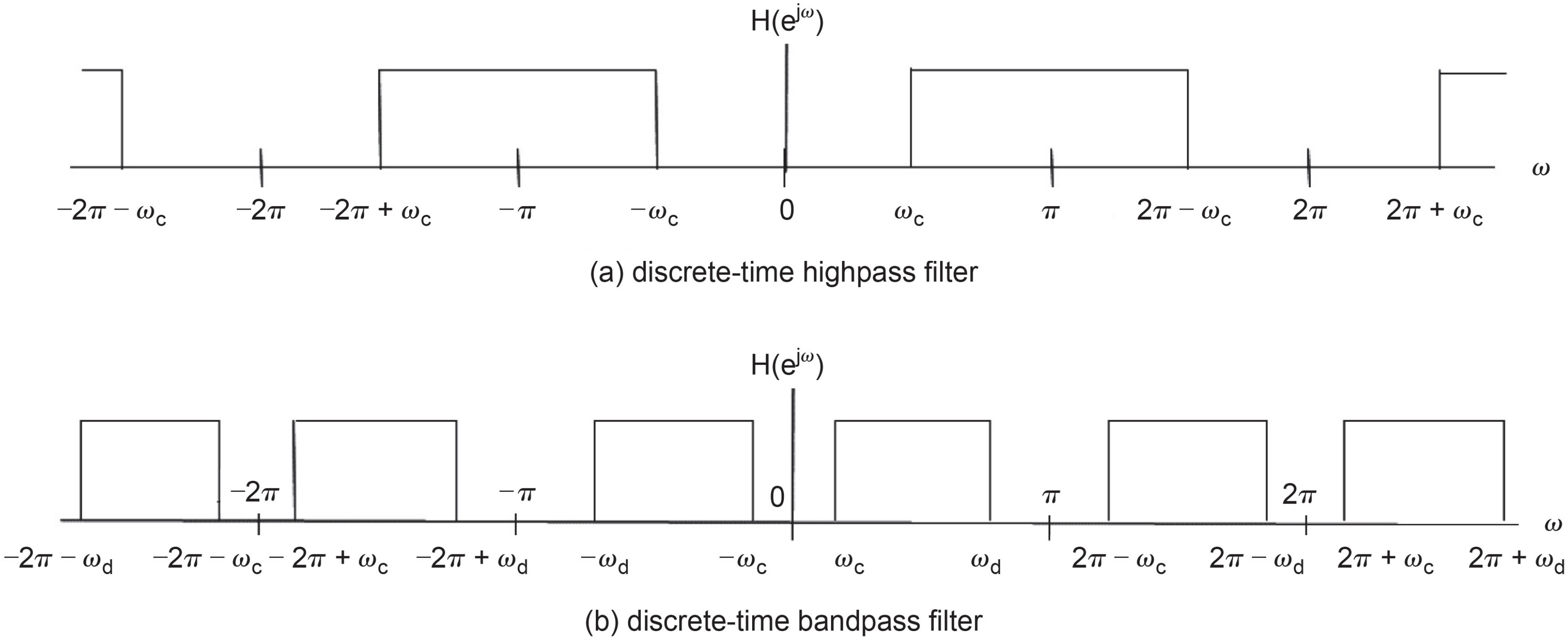
$$|H(e^{j\omega})| = \begin{cases} 1 & \omega_c \le |\omega| \le \pi \\ 0 & |\omega| < \omega_c \end{cases}. \tag{6.37}$$

  - Ideal BSF

$$|H(e^{j\omega})| = \begin{cases} 1 & \omega_c \le |\omega| \le \omega_d \\ 0 & |\omega| \le \omega_c \quad or \quad \omega_d < |\omega| \le \pi \end{cases}. \tag{6.38}$$

6.5* Time and Frequency Characterization for Some Well-known Filters

- For example,



6.5* Time and Frequency Characterization for Some Well-known Filters

- Frequency selective filters
  - For ideal discrete-time *notch* filter, the frequency response H($e^{j\omega}$) is very close to that of the ideal *bandstop* filter but

$$\omega_d - \omega_c \cong 0. \tag{6.40}$$

  - The phase of the frequency response of an ideal DT frequency selective filter always has a linear form:

$$\sphericalangle H(e^{j\omega}) = -\alpha\omega. \tag{6.41}$$

  - The frequency response of the ideal DT frequency selective filter with the linear phase in (6.41) satisfies the following symmetric relation:

$$h[\alpha + n] = h[\alpha - n]. \tag{6.42}$$

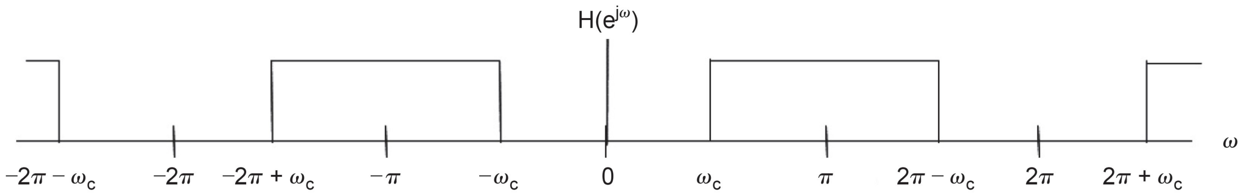
6.5* Time and Frequency Characterization for Some Well-known Filters

- Frequency selective filters
  - For the ideal DT LPF, the impulse response is

$$h[n] = \frac{\sin \omega_c n}{\pi n}, \tag{6.29}$$

  - For the ideal DT HPF, the impulse response is

$$h[n] = (-1)^n \frac{\sin(\pi - \omega_c)n}{\pi n}. \tag{6.43}$$



(a) discrete-time highpass filter

6.5* Time and Frequency Characterization for Some Well-known Filters

- 6.5.2 Frequency Response of an Edge Detection Filter
  - If h[n] is the impulse response of a difference operation, i.e., h[n] = δ[n] – δ[n – 1], then

$$H(e^{j\omega}) = 1 - e^{-j\omega} = 2e^{j(\pi/2 - \omega/2)}\sin(\omega/2). \tag{6.44}$$

  - When one applies the amplitude-phase representation, then

$$H(e^{j\omega}) = A(e^{j\omega})e^{j\angle_A H(e^{j\omega})} \tag{6.45}$$

where $\quad A(e^{j\omega}) = 2\sin(\omega/2), \qquad \angle_A H(e^{j\omega}) = \frac{\pi}{2} - \frac{\omega}{2}.$

6.6 First and Second-Order CT Systems

- First-Order CT Systems

$x(t) \longrightarrow$ LTI System $\longrightarrow y(t)$

$$\tau \frac{d}{dt} y(t) + y(t) = x(t)$$

τ: Time Constant

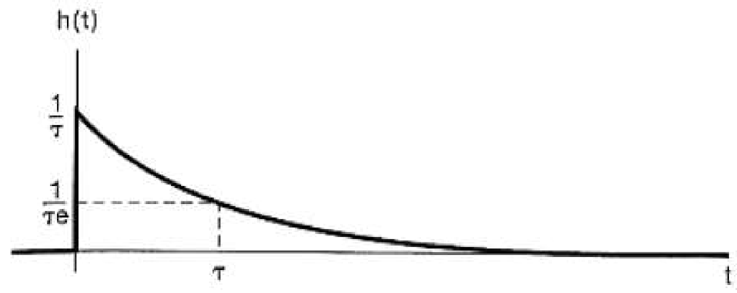
$$Y(j\omega) = X(j\omega)H(j\omega)$$

$$\Rightarrow H(j\omega) = \frac{1}{j\omega\tau + 1}$$

$$\Rightarrow h(t) = \frac{1}{\tau} e^{-t/\tau} u(t)$$

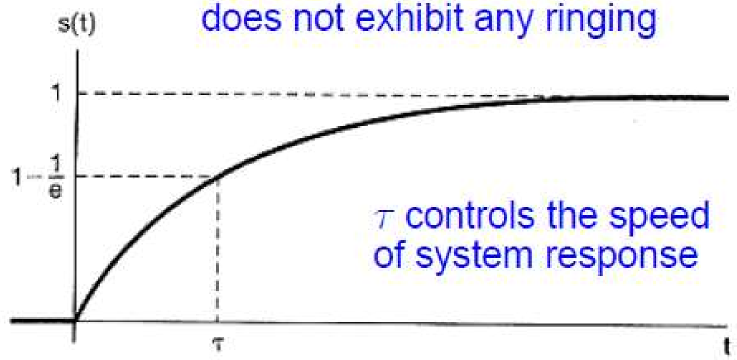
$$\Rightarrow s(t) = h(t) * u(t) \quad \text{(p. 83, Ex. 2.3)}$$

$$= \left[1 - e^{-t/\tau}\right] u(t)$$



(a)

s(t) of a first-order system does not exhibit any ringing



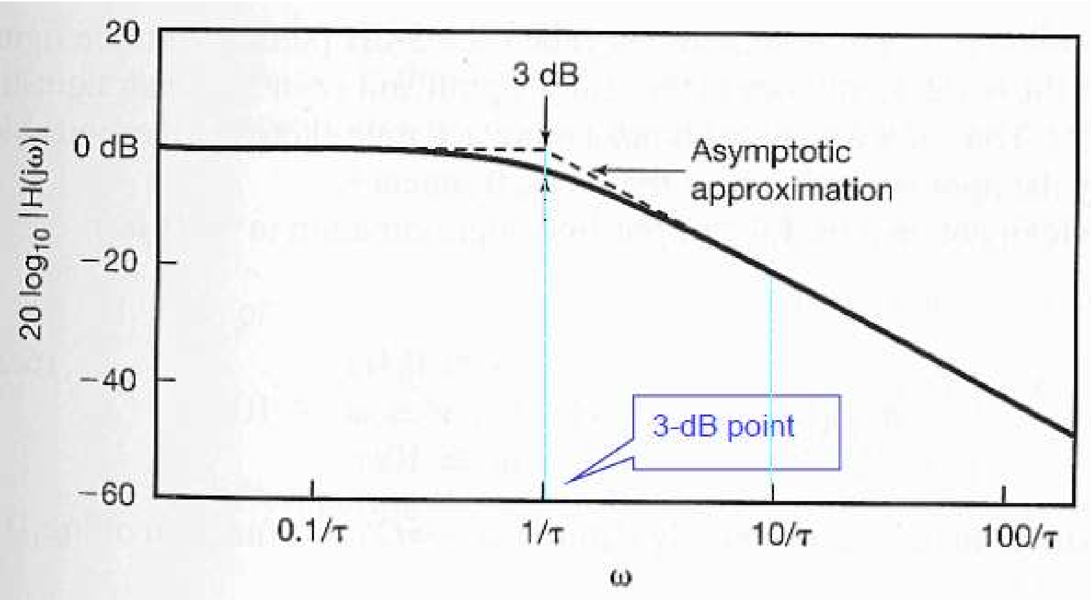
6.6 First and Second-Order CT Systems

- First-Order CT Systems

$$|H(j\omega)| = \frac{1}{\sqrt{(\omega\tau)^2 + 1}}$$

$$20\log_{10}|H(j\omega)| = -10\log_{10}\left[(\omega\tau)^2 + 1\right]$$

$$\approx \begin{cases} 0, & \omega \ll 1/\tau \\ -10\log_{10}(2) \approx -3dB, & \omega = 1/\tau \\ -20\log_{10}(\omega\tau), & \omega \gg 1/\tau \end{cases}$$

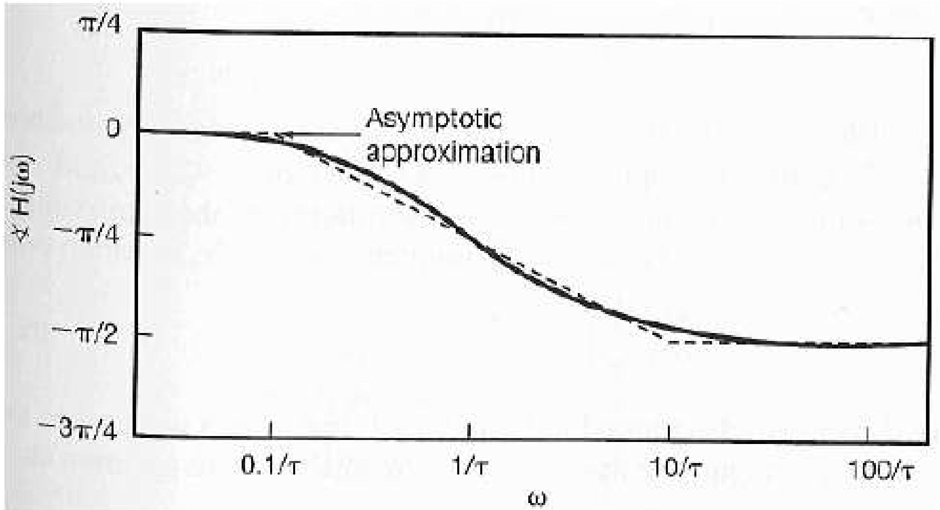


Note the inverse relationship between time and frequency

6.6 First and Second-Order CT Systems

- First-Order CT Systems

$$
\begin{aligned}
\sphericalangle H(j\omega) &= -\tan^{-1}(\omega\tau) \\
&\simeq \begin{cases} 0, & \omega \le 0.1/\tau \\ -(\pi/4)[\log_{10}(\omega\tau)+1], & 0.1/\tau \le \omega \le 10/\tau. \\ -\pi/2, & \omega \ge 10/\tau \end{cases}
\end{aligned}
\qquad (6.64)
$$



same slope

$$\frac{\theta-0}{\log\omega-\log(0.1/\tau)}=\frac{-\frac{\pi}{2}-0}{\log(10/\tau)-\log(0.1/\tau)}$$

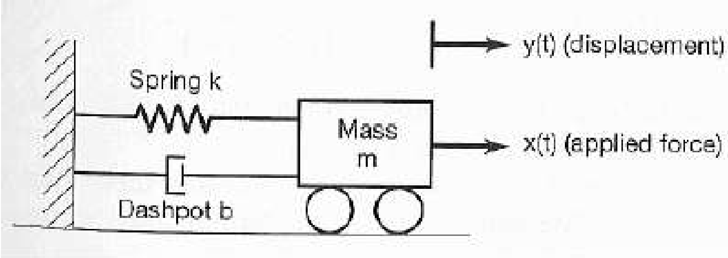
$$\theta=\frac{-\frac{\pi}{2}-0}{\log(10/\tau)-\log(0.1/\tau)}(\log\omega-\log(0.1/\tau))$$

$$=-\frac{\pi}{2}\log(10\omega\tau)\Big/\log 100$$

$$=\;-\frac{\pi}{4}(\log(\omega\tau)+1)$$

6.6 First and Second-Order CT Systems

- Second-Order CT Systems



$$\frac{d^2}{dt^2}y(t) + 2\zeta\omega_n \frac{d}{dt}y(t) + \omega_n^2 y(t) = \omega_n^2 x(t)$$

$$\omega_n = \sqrt{\frac{k}{m}}$$

$$\zeta = \frac{b}{2\sqrt{km}}$$

$$\Rightarrow H(j\omega) = \frac{\omega_n^2}{(j\omega)^2 + 2\zeta\omega_n(j\omega) + \omega_n^2}$$

$$= \frac{1}{(j\omega/\omega_n)^2 + 2\zeta(j\omega/\omega_n) + 1}$$

$$= \frac{\omega_n^2}{(j\omega - c_1)(j\omega - c_2)}$$

$$\begin{cases} c_1 = -\zeta\omega_n + \omega_n\sqrt{\zeta^2 - 1} \\ c_2 = -\zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1} \end{cases}$$

ζ : damping ratio

w_n : undamped natural frequency

$$\begin{cases} 0 < \zeta < 1: & \text{underdamped} \\ \zeta = 1: & \text{critically damped} \\ \zeta > 1: & \text{overdamped} \end{cases}$$

6.6 First and Second-Order CT Systems

- Second-Order CT Systems

- For $\zeta \neq 1, \;\Rightarrow\; c_1, c_2$: unequal:

$$\Rightarrow H(j\omega) = \frac{M}{j\omega - c_1} - \frac{M}{j\omega - c_2} \qquad M = \frac{\omega_n}{2\sqrt{\zeta^2 - 1}}$$

$$\Rightarrow h(t) = M\left[e^{c_1 t} - e^{c_2 t}\right]u(t)$$

- For $\zeta = 1, \;\Rightarrow\; c_1 = c_2 = -w_n$:

$$\Rightarrow H(j\omega) = \frac{\omega_n^2}{(j\omega + \omega_n)^2}$$

$$\Rightarrow h(t) = \omega_n^2 t e^{-\omega_n t} u(t)$$

6.6 First and Second-Order CT Systems

- Second-Order CT Systems

- For $0 < \zeta < 1$, c_1, c_2: complex: $\begin{cases} c_1 = -\zeta\omega_n + j\omega_n\sqrt{1-\zeta^2} \\ c_2 = -\zeta\omega_n - j\omega_n\sqrt{1-\zeta^2} \end{cases}$

$$H(j\omega) = \frac{\omega_n^2}{(j\omega - c_1)(j\omega - c_2)}$$

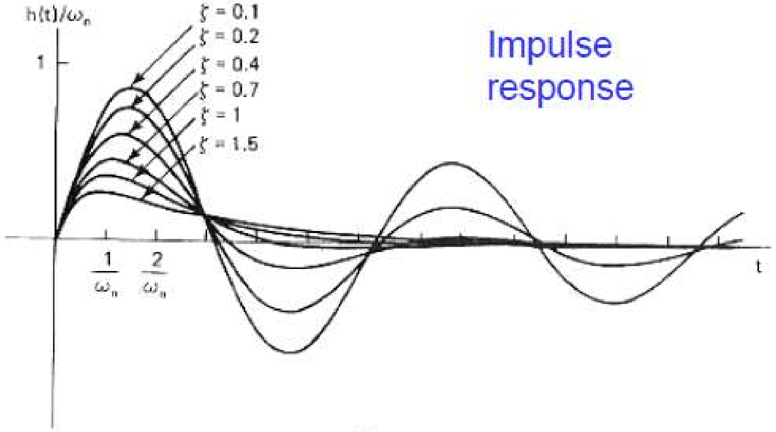
$$\Rightarrow h(t) = M\left[e^{c_1 t} - e^{c_2 t}\right]u(t)$$

$$= \frac{\omega_n e^{-\zeta\omega_n t}}{2j\sqrt{1-\zeta^2}}\left\{e^{j(\omega_n\sqrt{1-\zeta^2})t} - e^{-j(\omega_n\sqrt{1-\zeta^2})t}\right\}u(t)$$

$$= \frac{\omega_n e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\left[\sin\left(\left(\omega_n\sqrt{1-\zeta^2}\right)t\right)\right]u(t)$$

6.6 First and Second-Order CT Systems

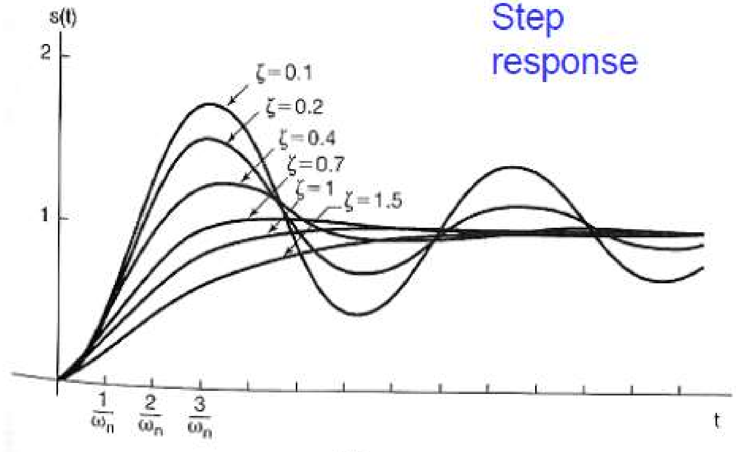
- Second-Order CT Systems



(a)

Impulse response

$$h(t)=\frac{\omega_n e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\left[\sin\left(\left(\omega_n\sqrt{1-\zeta^2}\right)t\right)\right]u(t)$$



(b)

Step response

$$s(t)=\begin{cases}\left(1+M\left[\dfrac{e^{c_1 t}}{c_1}-\dfrac{e^{c_2 t}}{c_2}\right]\right)u(t), & \zeta\neq 1\\ \left(1-e^{-\omega_n t}-\omega_n t e^{-\omega_n t}\right)u(t), & \zeta=1\end{cases}$$

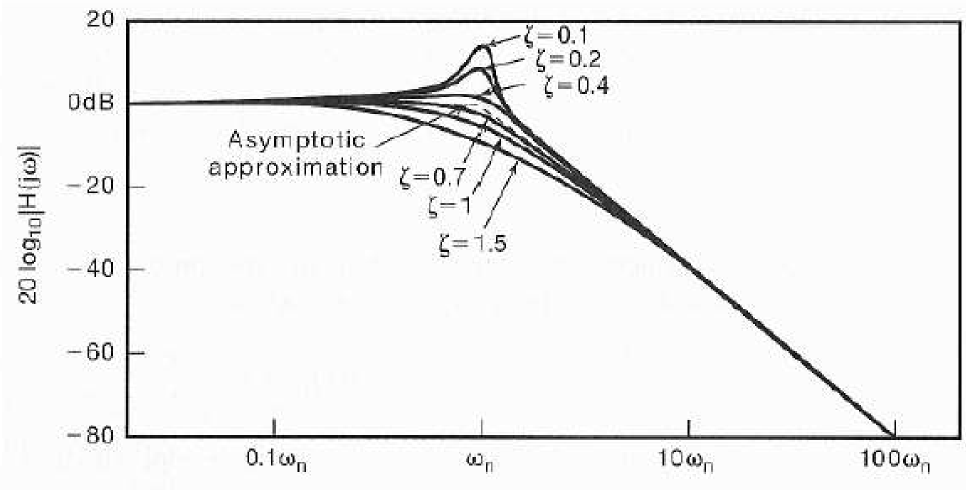
- Exhibit overshoot and ringing in the underdamped case.
- The critically damped case has the shortest settling time w/o overshoot
  → shortest settling time
- ω_n controls the time scale of the response.

6.6 First and Second-Order CT Systems

- Bode Plots of 2nd-order CT Systems

$$H(j\omega)=\frac{1}{(j\omega/\omega_n)^2+2\zeta(j\omega/\omega_n)+1} \Rightarrow \left|H(j\omega)\right|=\frac{1}{\sqrt{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2+4\zeta^2\left(\frac{\omega}{\omega_n}\right)^2}}$$

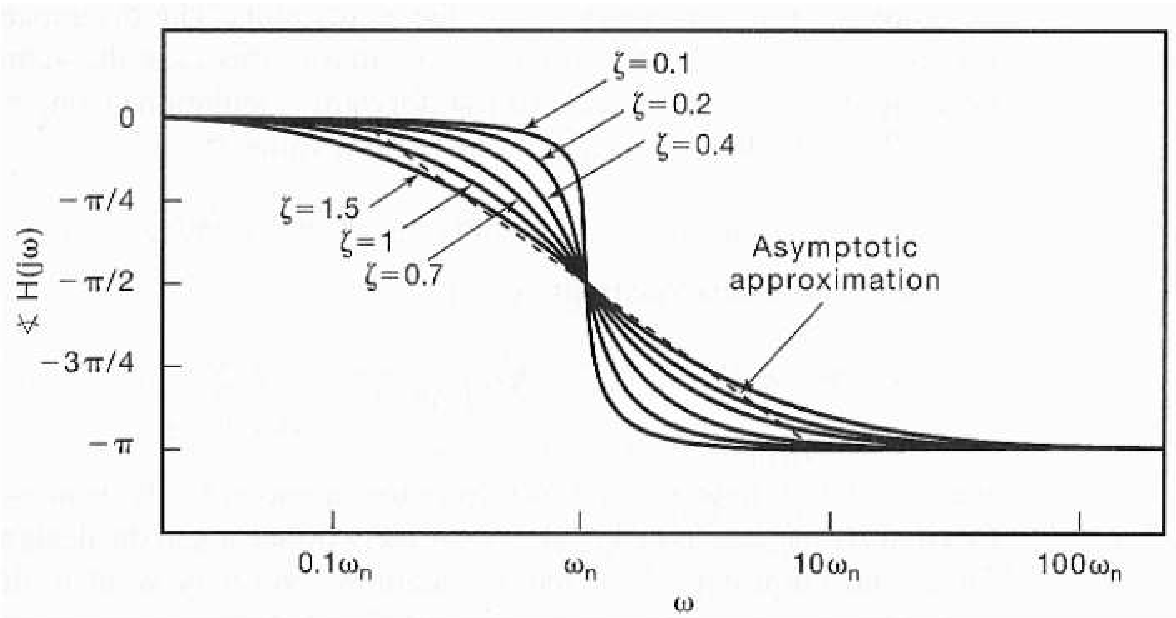
$$20\log_{10}\left|H(j\omega)\right|=-10\log_{10}\left(\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2+4\zeta^2\left(\frac{\omega}{\omega_n}\right)^2\right) \approx \begin{cases} 0, & \omega \ll \omega_n \\ -20\log_{10}(2\zeta), & \omega=\omega_n \\ -40\log_{10}(\omega)+40\log_{10}(\omega_n), & \omega \gg \omega_n \end{cases}$$



6.6 First and Second-Order CT Systems

- Bode Plots of 2nd-order CT Systems

$$\sphericalangle H(j\omega) = -\tan^{-1}\left(\frac{2\zeta(\omega/\omega_n)}{1-(\omega/\omega_n)^2}\right) \approx \begin{cases} 0, & \omega \le 0.1\omega_n \\ -(\pi/2)\left[\log_{10}(\omega/\omega_n)+1\right], & 0.1\omega_n \le \omega \le 10\omega_n \\ -\pi/2, & \omega = \omega_n \\ -\pi, & \omega \ge 10\omega_n \end{cases}$$



6.6 First and Second-Order CT Systems

- Example 6.4

$$\hat{H}(j\omega)=\frac{\omega_n^2}{j\omega^2+2j\zeta\omega\omega_n)+\omega_n^2}$$

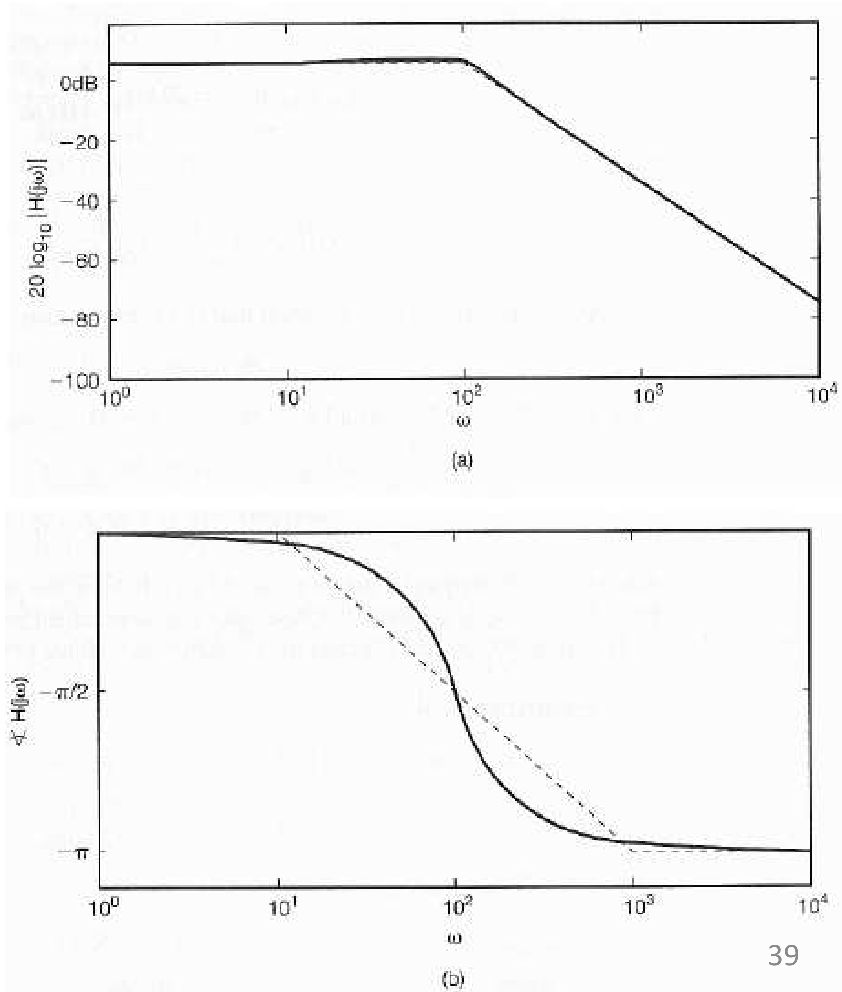
$$H(j\omega)=\frac{2\times 10^4}{(j\omega)^2+100(j\omega)+10^4}$$

$$H(j\omega)=2\times\hat{H}(j\omega)$$

$$\sphericalangle H(j\omega)=\sphericalangle\hat{H}(j\omega)$$

$$\begin{cases}\omega_n=100\\ \zeta=1/2\end{cases}$$

$$\Rightarrow 20\log_{10}\left|H(j\omega)\right|=$$
$$20\log_{10}(2)+20\log_{10}\left|\hat{H}(j\omega)\right|$$

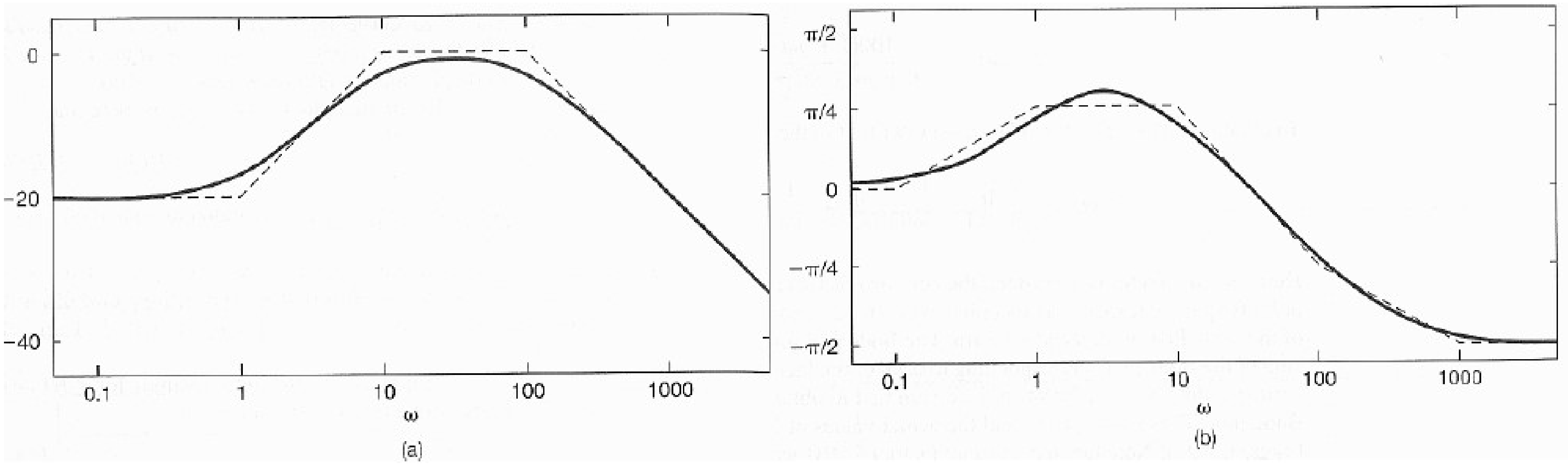


6.6 First and Second-Order CT Systems

- Example 6.5

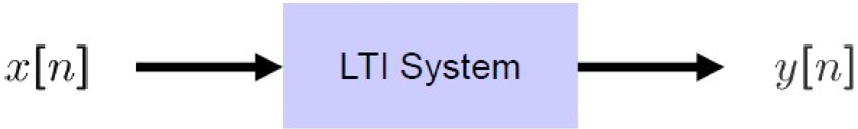
$$H(j\omega) = \frac{100(1+j\omega)}{(10+j\omega)(100+j\omega)}$$

$$= \left(\frac{1}{10}\right)(1+j\omega)\left(\frac{1}{1+j\omega/10}\right)\left(\frac{1}{1+j\omega/100}\right)$$



6.7 First and Second-Order DT Systems

- First-order DT Systems



$$Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega})$$

$$y[n] - ay[n-1] = x[n], \quad |a| < 1$$

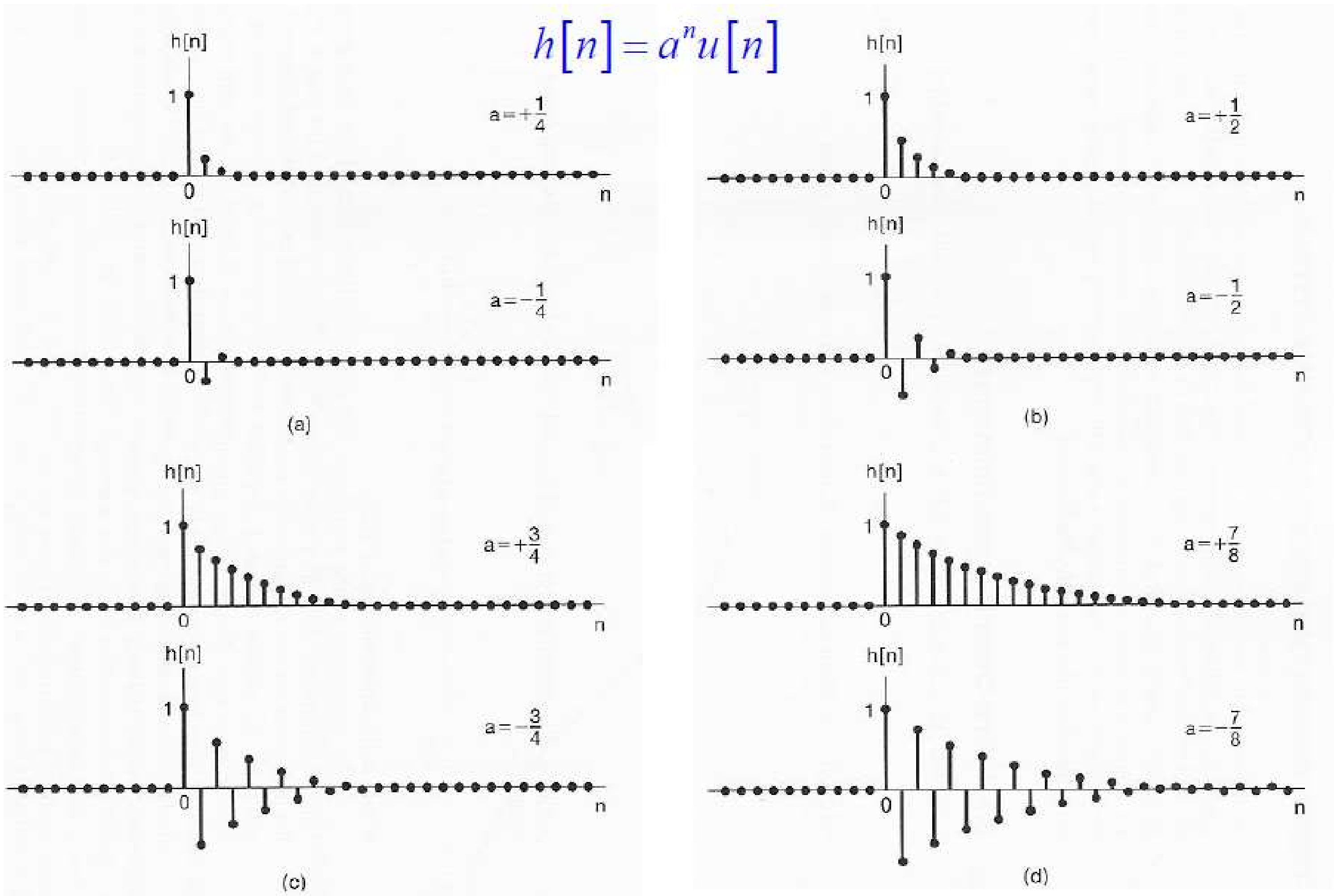
$$\Rightarrow H(e^{j\omega}) = \frac{1}{1 - ae^{-j\omega}}$$

$$\Rightarrow h[n] = a^n u[n]$$

$$\Rightarrow s[n] = h[n] * u[n] = \frac{1 - a^{n+1}}{1 - a} u[n] \quad \text{(p. 83, Example 2.3)}$$

6.7 First and Second-Order DT Systems

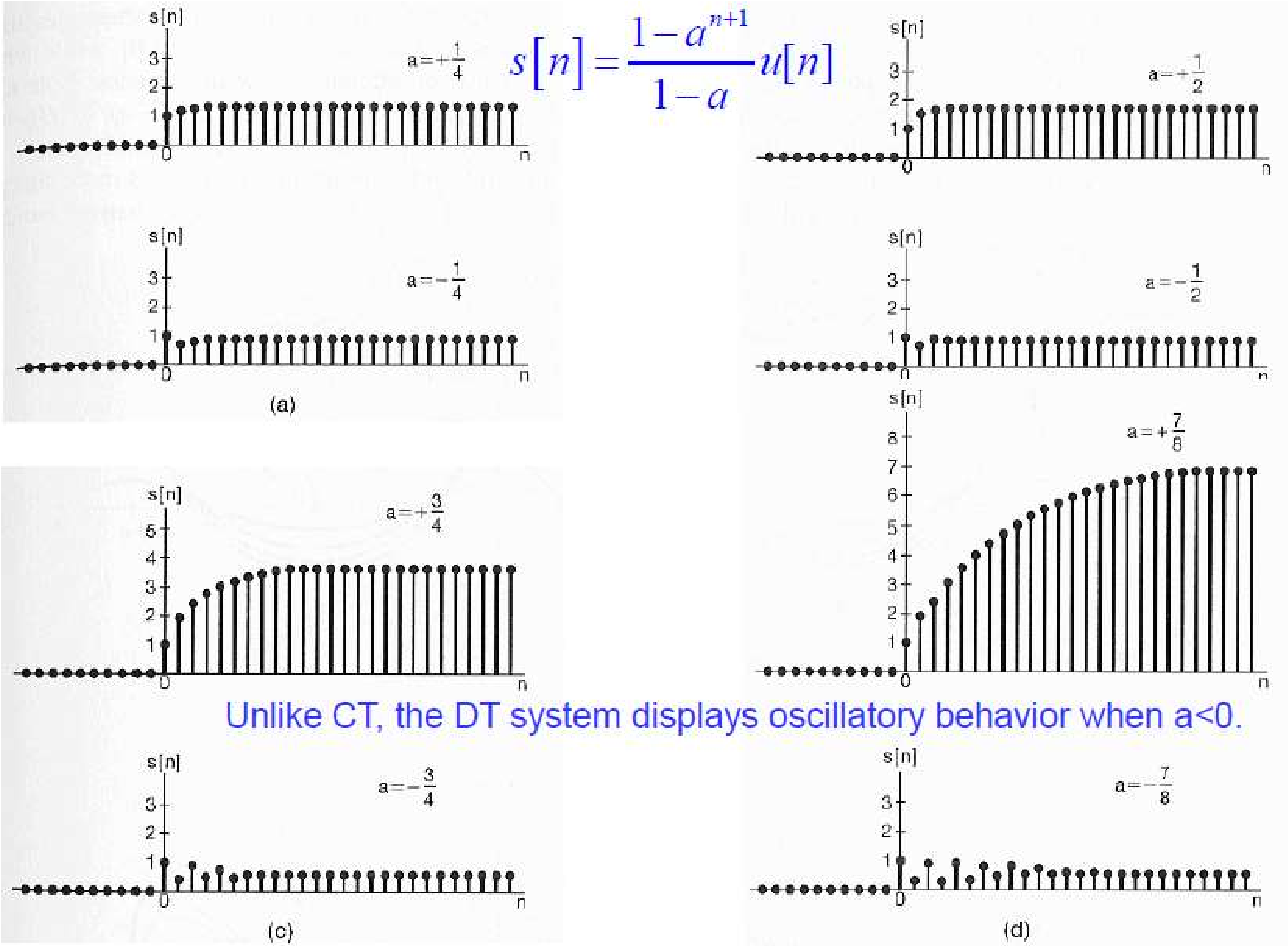
- Impulse Response of First-order DT Systems



6.7 First and Second-Order DT Systems

- Step Response of First-order DT Systems

$$s[n]=\frac{1-a^{n+1}}{1-a}u[n]$$



Unlike CT, the DT system displays oscillatory behavior when a<0.

6.7 First and Second-Order DT Systems

- Magnitude & Phase of Frequency Response of 1st-order DT Systems

$$H(e^{j\omega}) = \frac{1}{1 - ae^{-j\omega}} = \frac{1}{1 - a\cos\omega + ja\sin\omega}$$

$$\left|H(e^{j\omega})\right| = \frac{1}{\sqrt{1 + a^2 - 2a\cos\omega}}$$

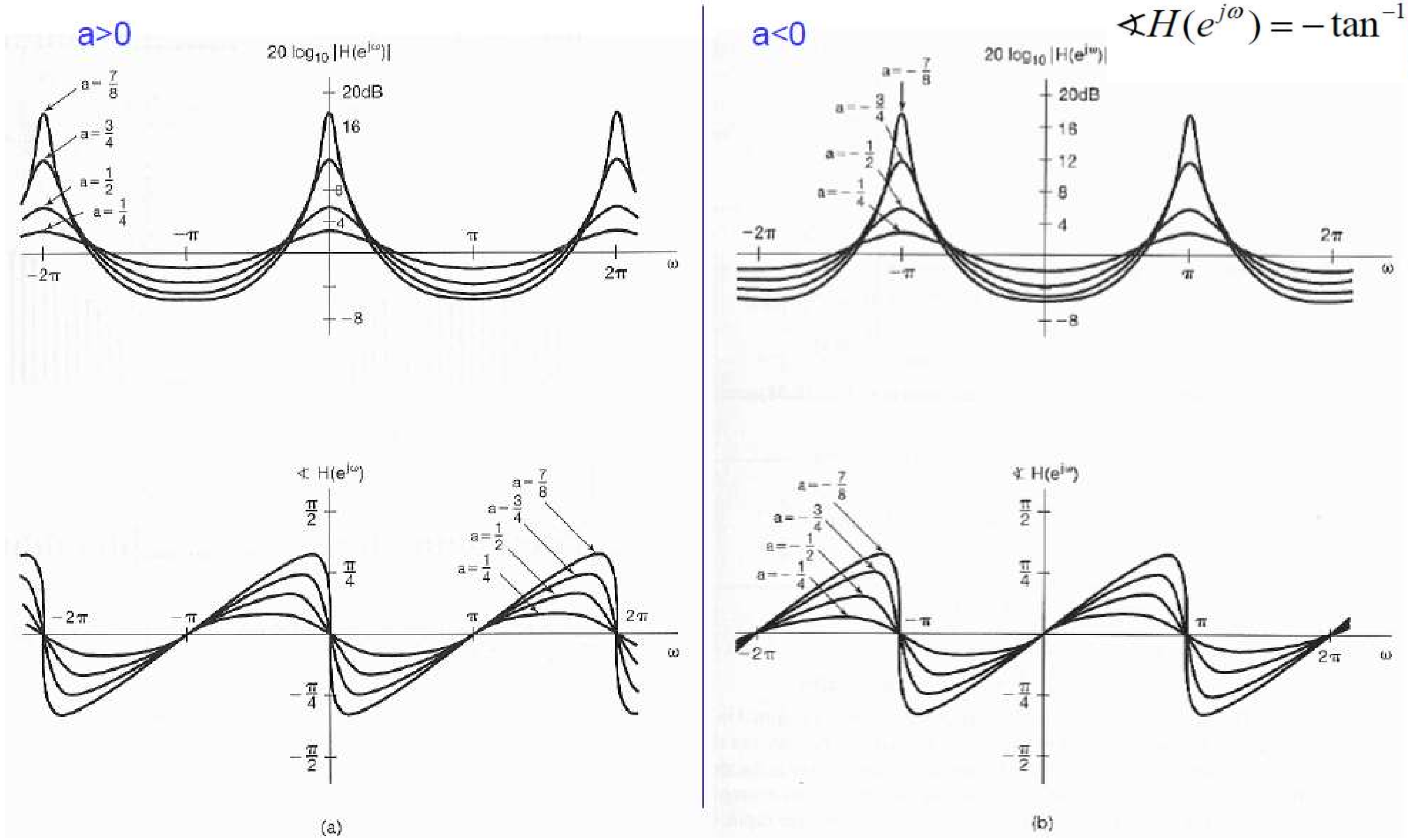
$$\sphericalangle H(e^{j\omega}) = -\tan^{-1}\left[\frac{a\sin\omega}{1 - a\cos\omega}\right]$$

6.7 First and Second-Order DT Systems

- Magnitude & Phase of Frequency Responses

$$\left|H(e^{j\omega})\right| = \frac{1}{\sqrt{1+a^2-2a\cos\omega}}$$

$$\measuredangle H(e^{j\omega}) = -\tan^{-1}\left[\frac{a\sin\omega}{1-a\cos\omega}\right]$$



6.7 First and Second-Order DT Systems

- Second-order DT Systems

$$x[n] \longrightarrow \boxed{\text{LTI System}} \longrightarrow y[n]$$

$$y[n]-2r\cos(\theta)\,y[n-1]+r^2y[n-2]=x[n],$$

$$0<r<1 \text{ and } 0\le\theta\le\pi.$$

$$\Rightarrow H(e^{j\omega})=\frac{1}{1-2r\cos(\theta)e^{-j\omega}+r^2e^{-j2\omega}}$$

$$=\frac{1}{[1-(re^{j\theta})e^{-j\omega}][1-(re^{-j\theta})e^{-j\omega}]}$$

6.7 First and Second-Order DT Systems

- Impulse Response of Second-order DT Systems

- For $\theta \neq 0$ or π:

$$\Rightarrow H(e^{j\omega}) = \frac{A}{1-(re^{j\theta})e^{-j\omega}} + \frac{B}{1-(re^{-j\theta})e^{-j\omega}}$$

$$A = \frac{e^{j\theta}}{2j\sin(\theta)}$$

$$B = -\frac{e^{-j\theta}}{2j\sin(\theta)}$$

$$\Rightarrow h[n] = [A(re^{j\theta})^n + B(re^{-j\theta})^n]\ u[n]$$

$$= r^n \frac{\sin[(n+1)\theta]}{\sin(\theta)} u[n]$$

- For $\theta = 0$:

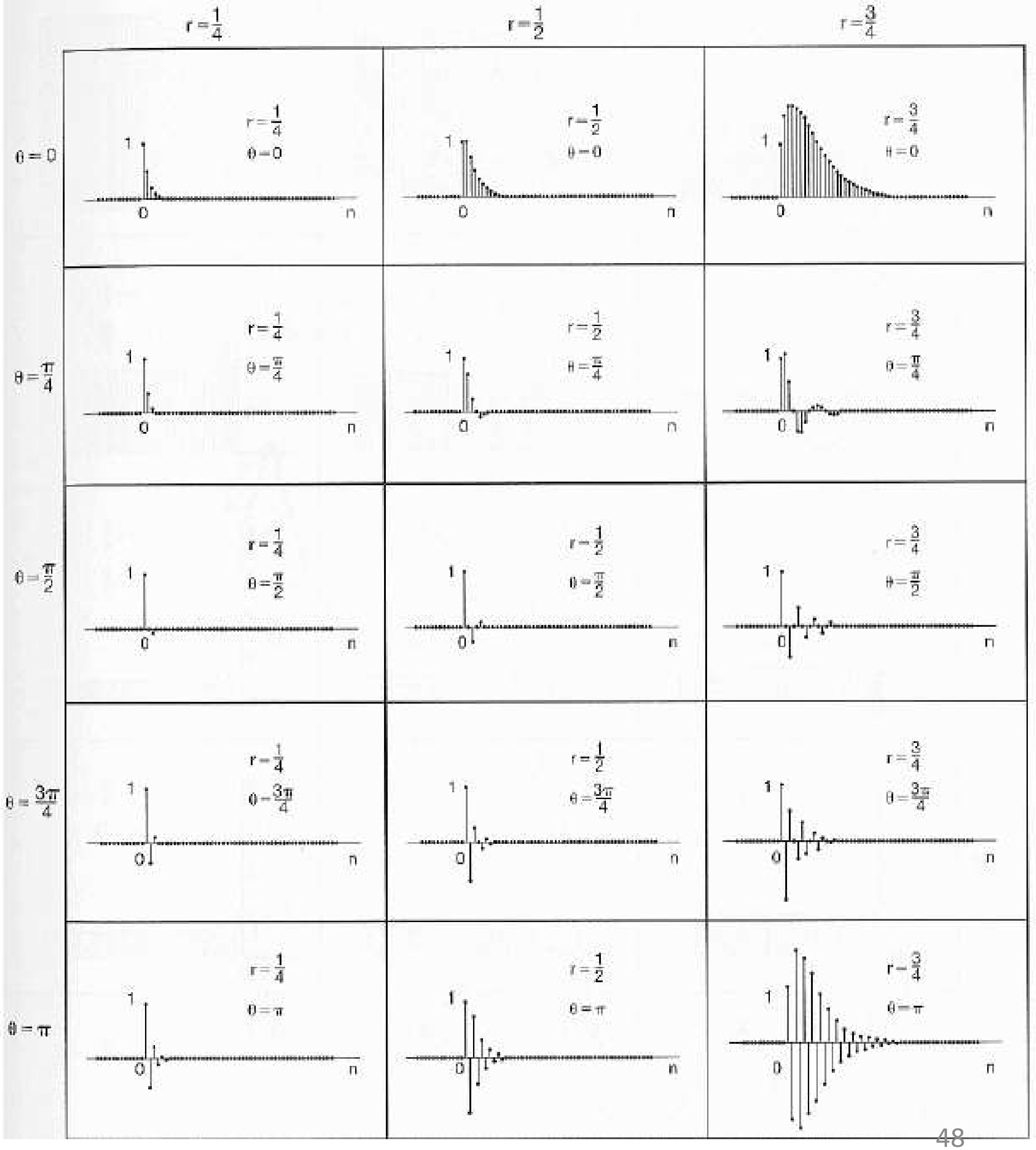
$$\Rightarrow H(e^{j\omega}) = \frac{1}{(1-re^{-j\omega})^2} \quad \Rightarrow \quad h[n] = (n+1)(r)^n u[n]$$

- For $\theta = \pi$:

$$\Rightarrow H(e^{j\omega}) = \frac{1}{(1+re^{-j\omega})^2} \quad \Rightarrow \quad h[n] = (n+1)(-r)^n u[n]$$

6.7 First and Second-Order DT Systems

Impulse Response:



- r affects the rate of decay
- θ affects the frequency of oscillation

6.7 First and Second-Order DT Systems

- Step Response of Second-order DT Systems

$$A = \frac{e^{j\theta}}{2j\sin(\theta)}$$

$$B = -\frac{e^{-j\theta}}{2j\sin(\theta)}$$

- For $\theta \neq 0$ or π:

$$s[n] = h[n] * u[n]$$

$$= \left[A\left(\frac{1-(re^{j\theta})^{n+1}}{1-re^{j\theta}} \right) + B\left(\frac{1-(re^{-j\theta})^{n+1}}{1-re^{-j\theta}} \right) \right] u[n]$$

- For $\theta = 0$:

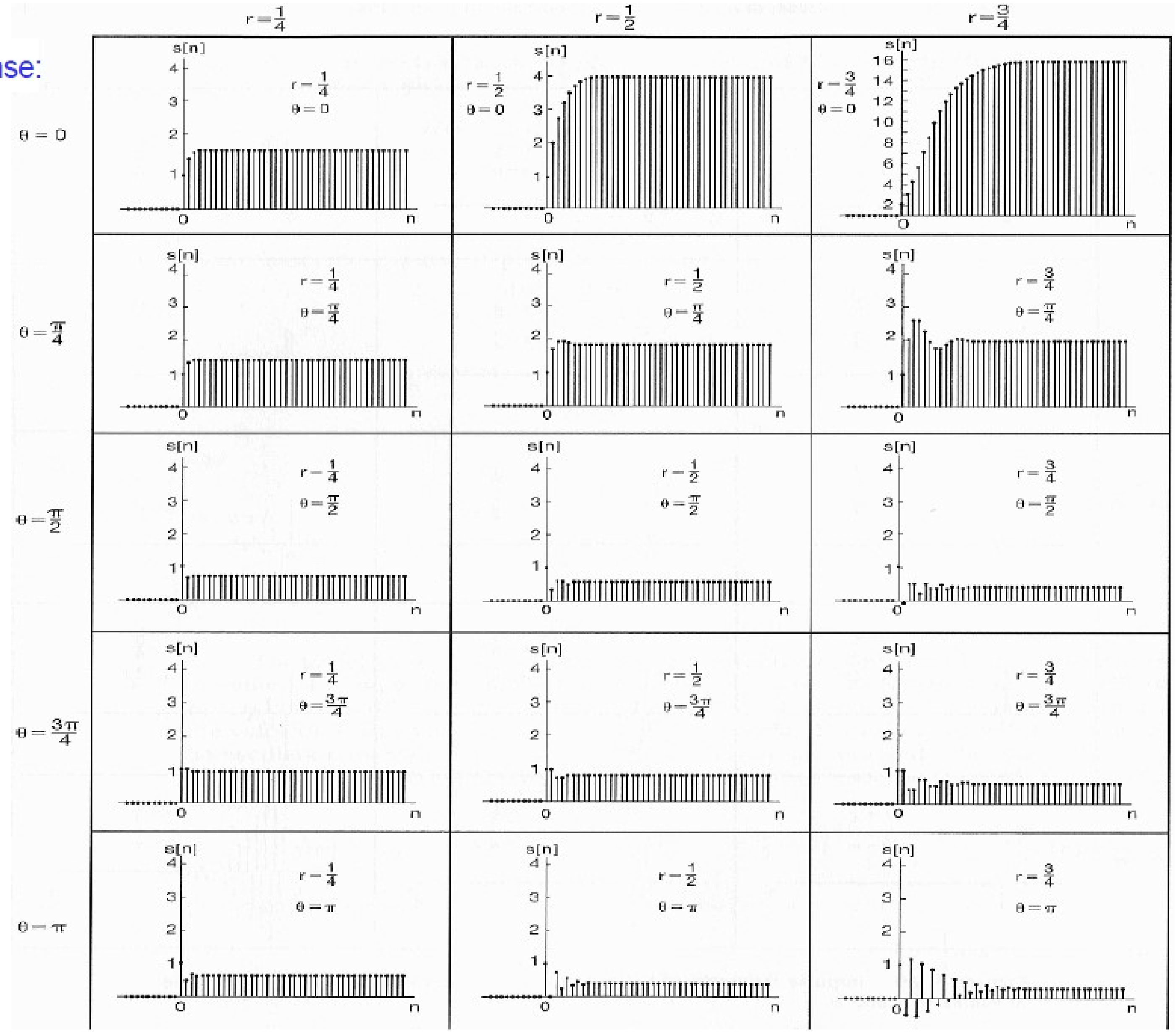
$$s[n] = \left[\frac{1}{(r-1)^2} - \frac{r}{(r-1)^2} r^n + \frac{r}{r-1}(n+1)r^n \right] u[n]$$

- For $\theta = \pi$:

$$s[n] = \left[\frac{1}{(r+1)^2} + \frac{r}{(r+1)^2}(-r)^n + \frac{r}{r+1}(n+1)(-r)^n \right] u[n]$$

6.7 First and Second-Order DT Systems

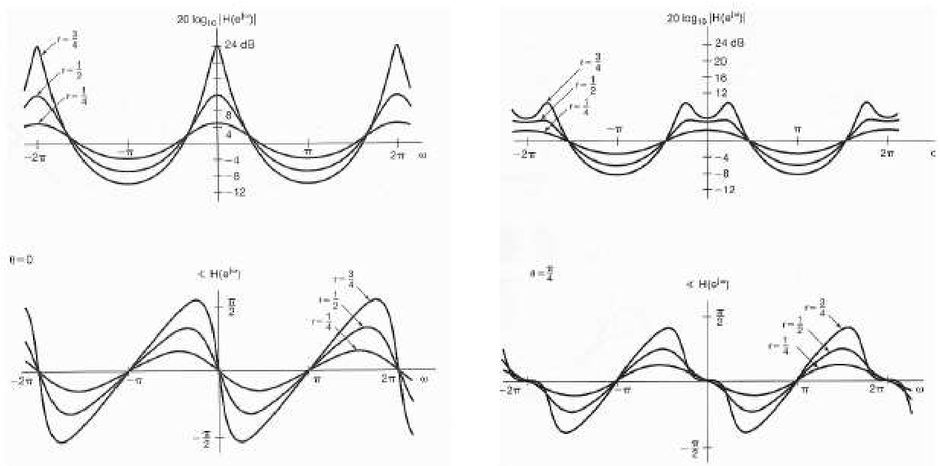
Step Response:

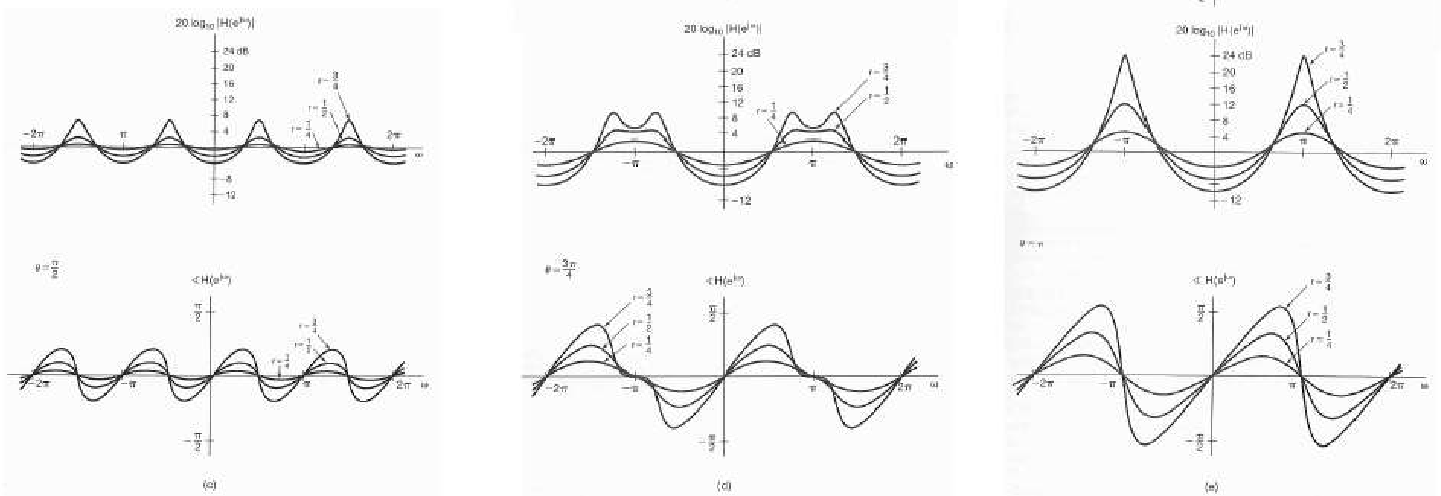


6.7 First and Second-Order DT Systems

- Magnitude & Phase of Frequency Response

$$H(e^{j\omega}) = \frac{1}{1-2r\cos(\theta)e^{-j\omega}+r^2e^{-j2\omega}}$$





6.8 Time-Domain & Frequency-Domain Analysis of Systems

- DT non-recursive filters
  - The effect of linear phase of a filter is a simple time delay
  - For IIR filters, it is not possible to design a causal, recursive filter with exactly linear phase (Problem 6.64)
  - Non-recursive or finite impulse response (FIR) filters
    > Can have exactly linear phase, at the cost of a higher order equation.

    > Example: moving average filter

$$y[n] = \frac{1}{N+M+1}\sum_{k=-N}^{M} x[n-k]$$

The corresponding impulse response is a rectangular pulse

$$\Rightarrow H(e^{j\omega}) = \frac{1}{N+M+1}\frac{\sin[\omega(M+N+1)/2]}{\sin(\omega/2)}e^{j\omega[(N-M)/2]}$$

Linear phase

Ex. 5.3, p. 365

6.8 Time-Domain & Frequency-Domain Analysis of Systems

- Problem 6.64
  For a DT filter to be causal and with exactly linear phase, its impulse response must be of finite length and its difference equation must be non-recursive.

Consider such a filter with frequency response

$$H(e^{j\omega}) = H_r(e^{j\omega})e^{-jM\omega}, -\pi < \omega < \pi$$

where M is an integer and $H_r(e^{j\omega})$ is real and even.

The signal is real and even,

$$h_r[n] = h_r[-n].$$

Using the time shift property, we have

$$h[n] = h_r[n-M]$$

6.8 Time-Domain & Frequency-Domain Analysis of Systems

- Problem 6.64 For a DT filter to be causal and with exactly linear phase, its impulse response must be of finite length and its difference equation must be non-recursive.

Also,

$$h[\mathrm{M}+n] = h_r[\mathrm{M}+n-M] = h_r[n]$$

$$h[\mathrm{M}-n] = h_r[\mathrm{M}-n-M] = h_r[-n] = h[\mathrm{M}+n]$$

Since $h[n]$ is causal, $h[n]$=0 for n<0. That is, $h[-k]$=0 for k>0.

$$h[-k] = h_r[-k-M] = h_r[k+M] = h[k+2M]$$

Thus,

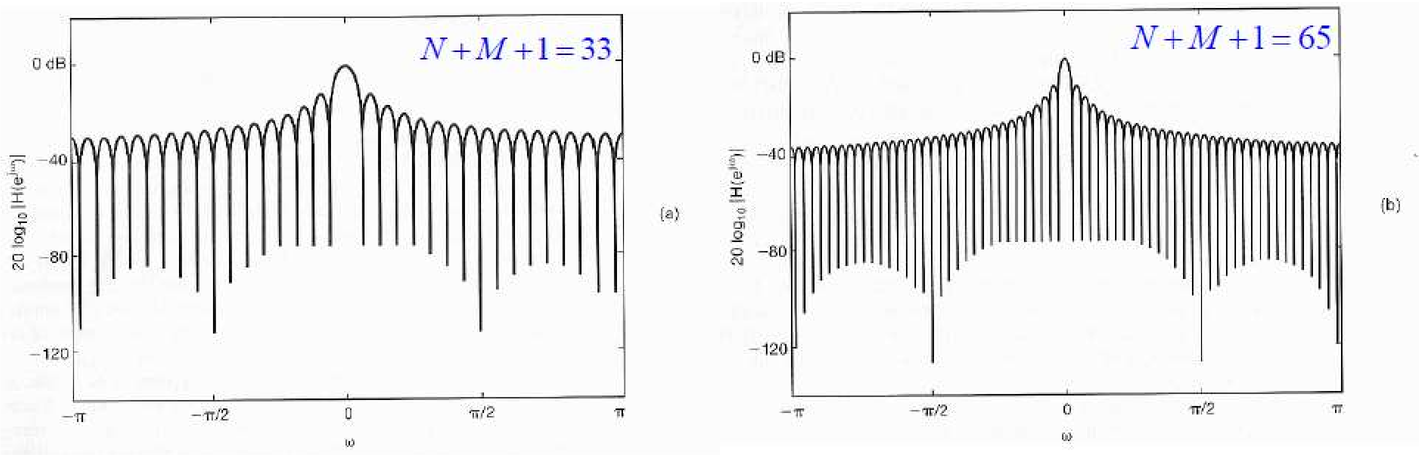
$$h[k+2M] = 0 \text{ for k>0}$$

That is

$$h[n] = 0 \text{ for } n>2M$$

6.8 Time-Domain & Frequency-Domain Analysis of Systems

- Log-Magnitude Plots

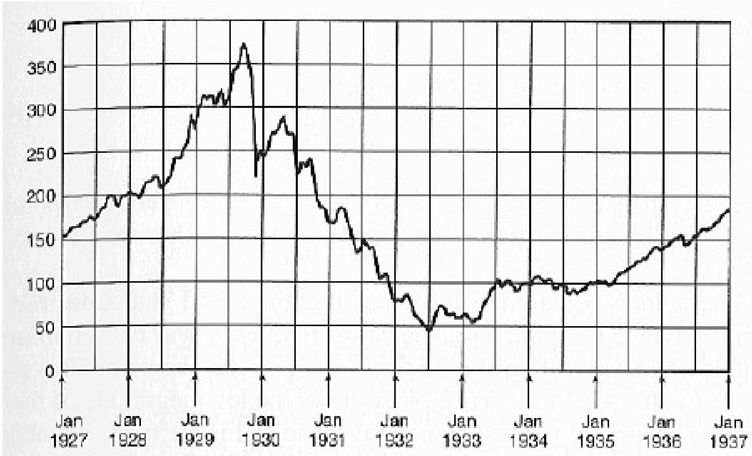


Longer impulse response, narrower passband.

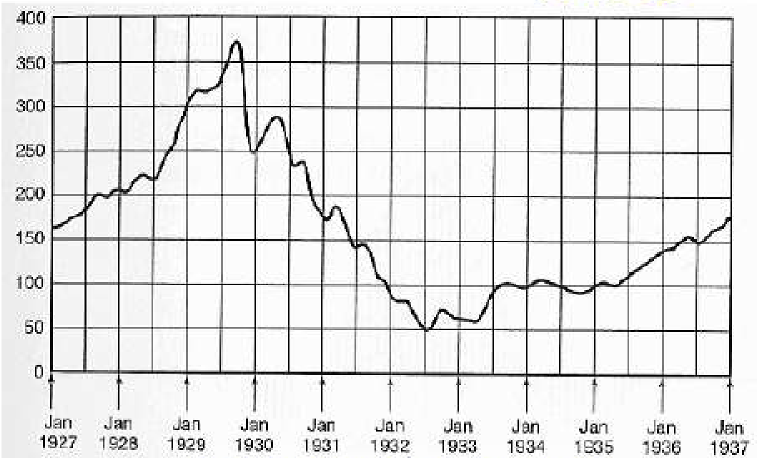
The width of the center lobe, which corresponds to the effective passband of the filter, decreases as the length of the impulse response of the filter increases—a tradeoff between frequency selectivity and the complexity of the filter.

6.8 Time-Domain & Frequency-Domain Analysis of Systems

- Moving-Average Filtering on Dow Jones Weekly Stock Market Index

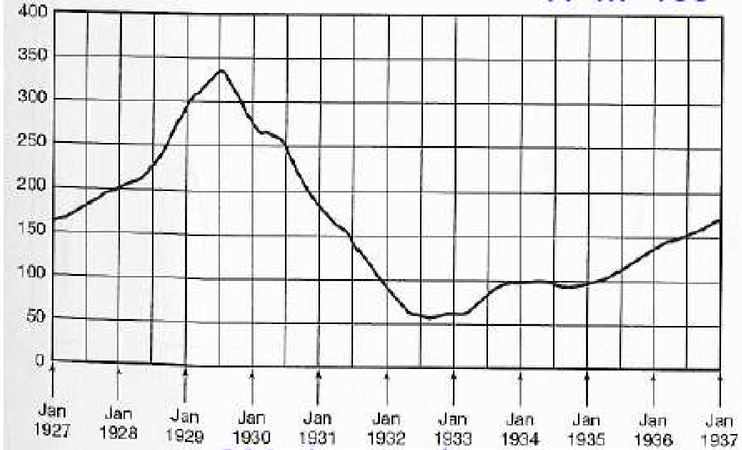


N=M=25



51-day moving average

N=M=100



201-day moving average

6.8 Time-Domain & Frequency-Domain Analysis of Systems

- General Form of DT Non-recursive Filters

$$y[n] = \sum_{k=-N}^{M} b_k x[n-k]$$

By choosing different coefficients b_k, we have a great flexibility in adjusting the frequency responses. For example, let M=N=16 and chose the filter coefficients to be:

$$b_k = \begin{cases} \dfrac{\sin(2\pi k/33)}{\pi k}, & |k| \leq 32 \\ 0, & |k| > 32 \end{cases}$$

General procedure for determining b_k can be found in the books listed on p.477.

Then the impulse response becomes:

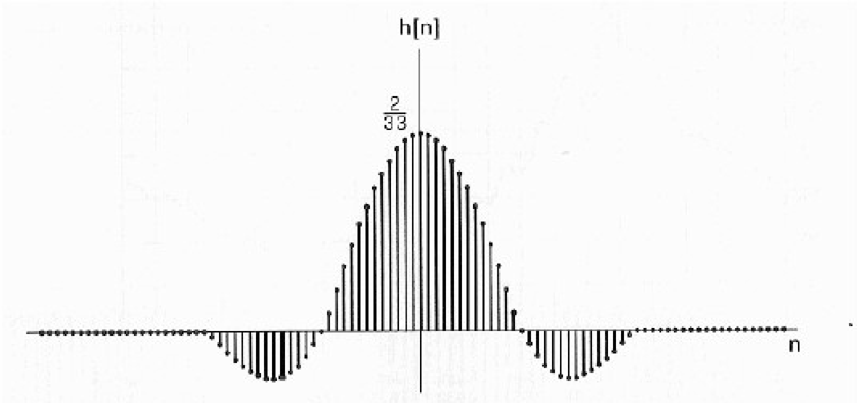
$$h[n] = \begin{cases} \dfrac{\sin(2\pi n/33)}{\pi n}, & |n| \leq 32 \\ 0, & |n| > 32 \end{cases}$$

This corresponds to truncating the ideal lowpass filter ($h_L[n] = \dfrac{\sin \omega_c n}{\pi n}$) with $\omega_c = 2\pi/33$.

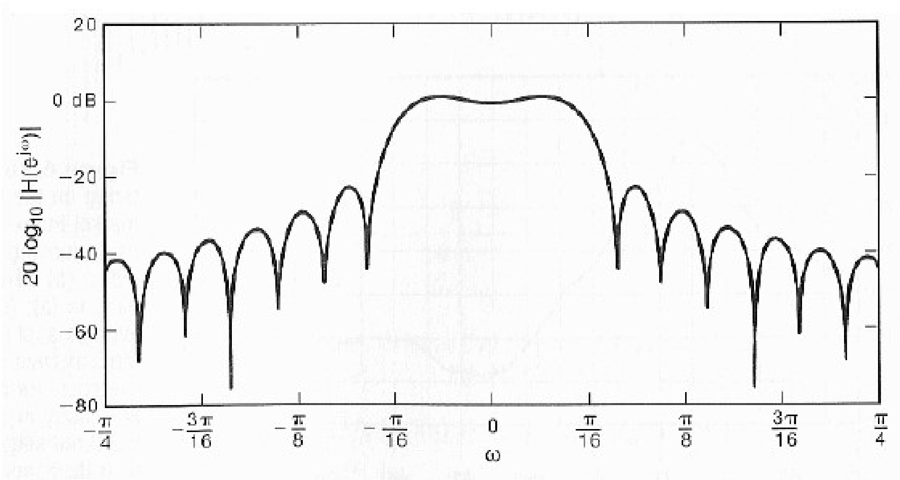
Real and even $\Rightarrow$ zero phase.

6.8 Time-Domain & Frequency-Domain Analysis of Systems

- General Form of DT Non-recursive Filters $$y[n]=\sum_{k=-N}^{M} b_k x[n-k]$$



(a)



(b)

6.8 Time-Domain & Frequency-Domain Analysis of Systems

- Comparisons