Signals & Systems

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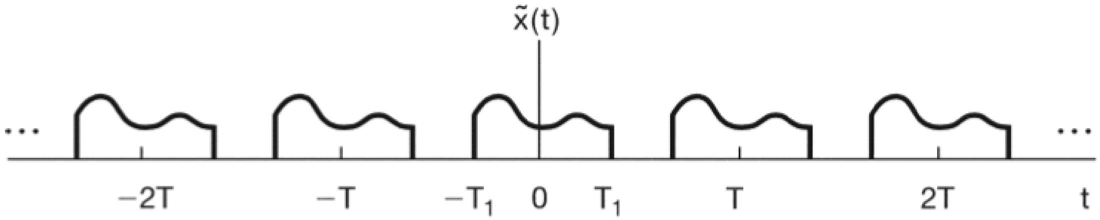
Ch. 4 Continuous-Time Fourier Transform

- Sec. 4.1 Representation of Aperiodic Signals:
  The Continuous-Time Fourier Transform
- Sec. 4.2 The Fourier Transform for Periodic Signals
- Sec. 4.3 Properties of the Continuous-Time Fourier Transform
- Sec. 4.4 The Convolution Property
- Sec. 4.5 The Multiplication Property
- Sec. 4.6 Tables of Fourier Properties and of Basic Fourier
  Transform Pairs
- Sec. 4.7 Systems Characterized by Linear Constant Coefficient
  Differential Equations

Sect. 4.1 Representation of *Aperiodic* Signals: CTFT

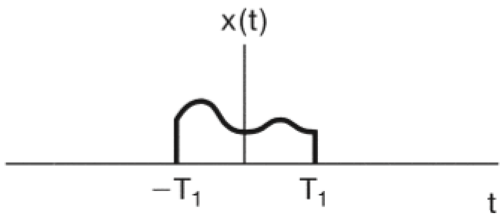
- Sect. 4.1.1 Development of the Fourier Transform Representation of an Aperiodic Signal

Fourier series can analyze a periodic signal.



(b)

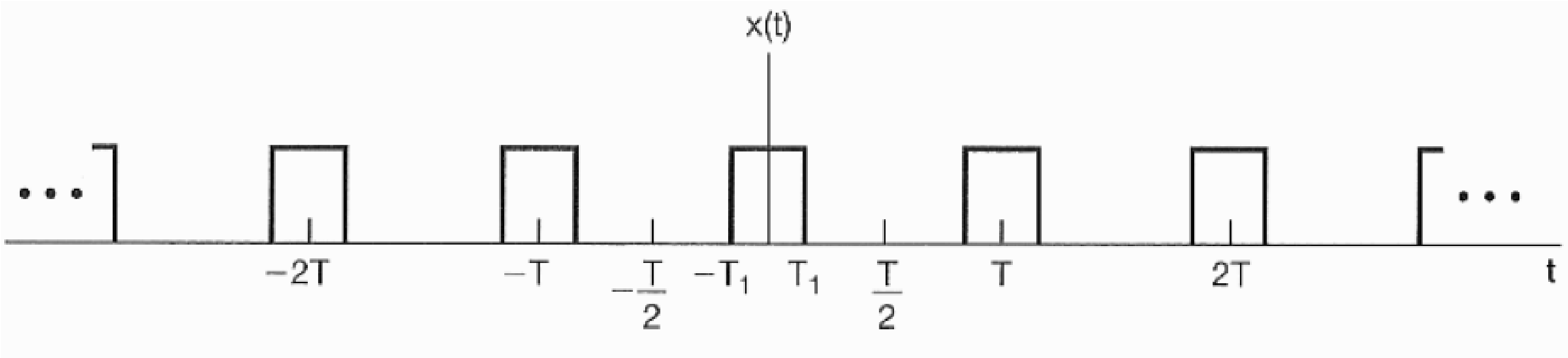
How do we analyze an aperiodic signal?



(a)

Will come back with details in few minutes…

- Sect. 4.1.1 Development of the FT Representation of an Aperiodic Signal
- Recall that…



$$x(t) = \begin{cases} 1, & |t| < T_1 \\ 0, & T_1 < |t| < T/2 \end{cases}$$

$$a_k = \frac{2\sin(k\omega_0 T_1)}{k\omega_0 T}$$

$$Ta_k = \left.\frac{2\sin(\omega T_1)}{\omega}\right|_{\omega = k\omega_0}$$

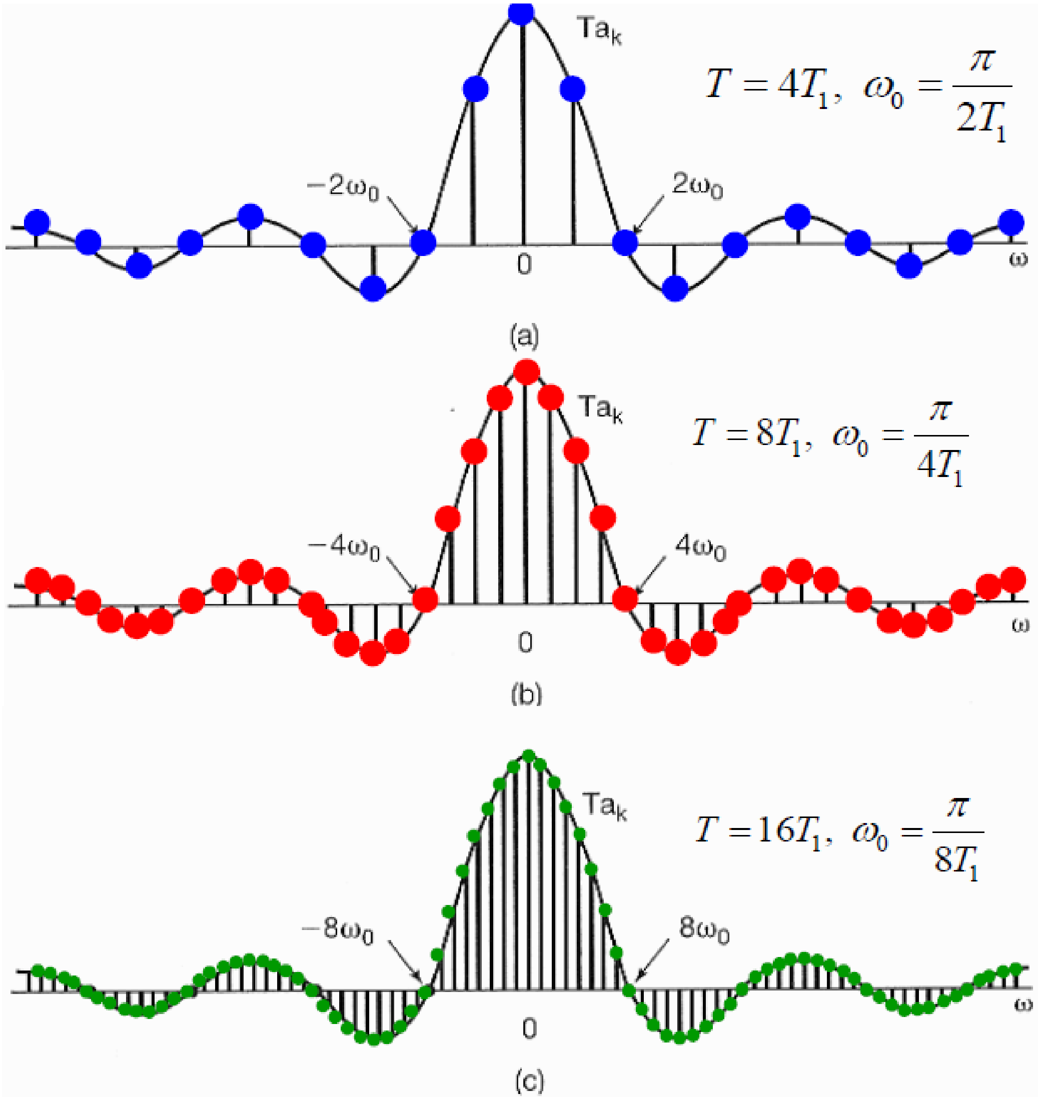
With ω thought of as a continuous variable, this equation represents the envelope of Ta_k.

- Sect. 4.1.1 Development of the FT Representation of an Aperiodic Signal
- Recall that…

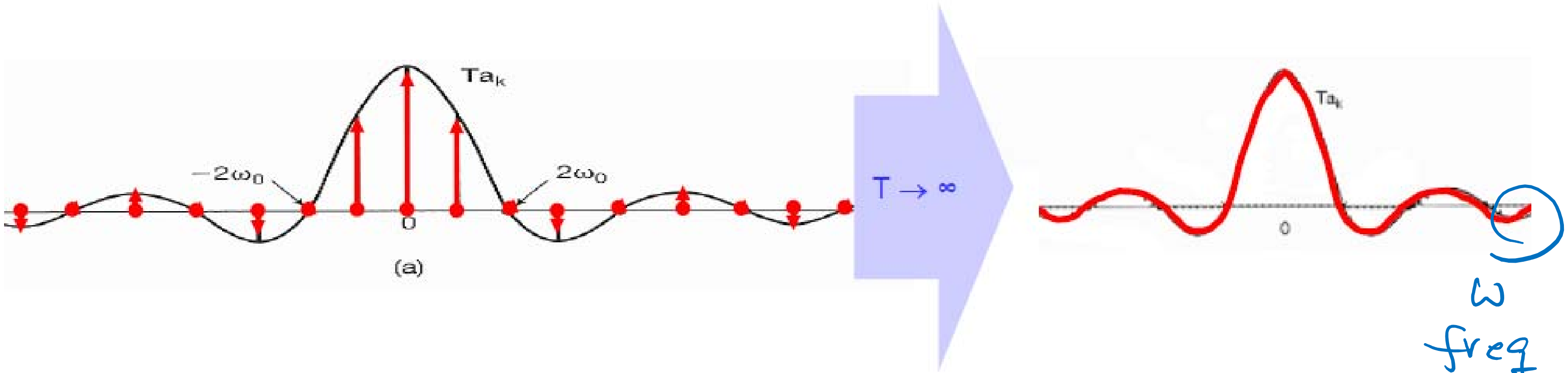
$$Ta_k = \frac{2\sin(\omega T_1)}{\omega}$$

$$\omega = k\omega_0 = k\frac{2\pi}{T}$$

$$\omega_0 = \frac{2\pi}{T}$$

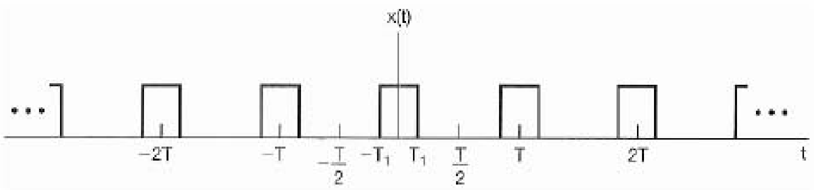


- Sect. 4.1.1 Development of the FT Representation of an Aperiodic Signal
- Pushing to the limit…

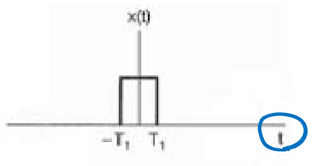


ω
freq

periodic



aperiodic



time

coeff

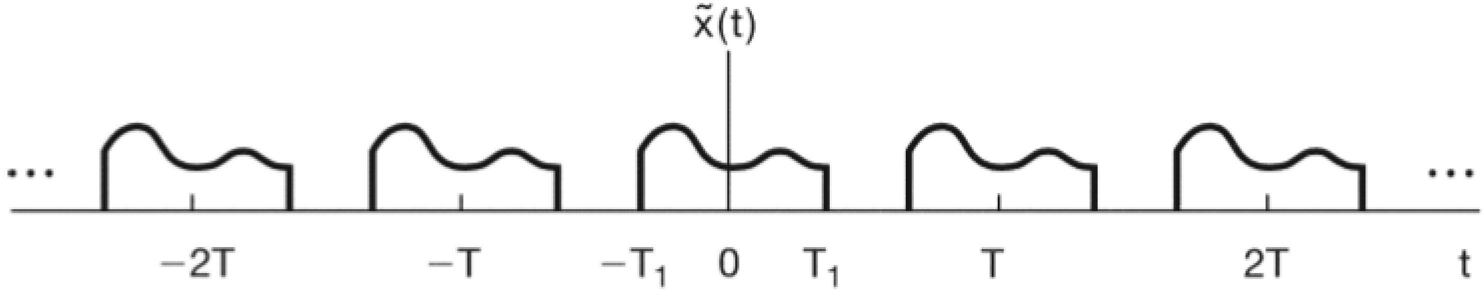
$$\text{FS}: \begin{cases} x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jkw_0 t} \\ a_k = \frac{1}{T} \int_T x(t) e^{-jkw_0 t} dt \end{cases}$$

$$\text{FT}: \begin{cases} x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\omega) e^{j\omega t} d\omega \\ X(j\omega) = \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt \end{cases}$$

Sect. 4.1 Representation of *Aperiodic* Signals: CTFT

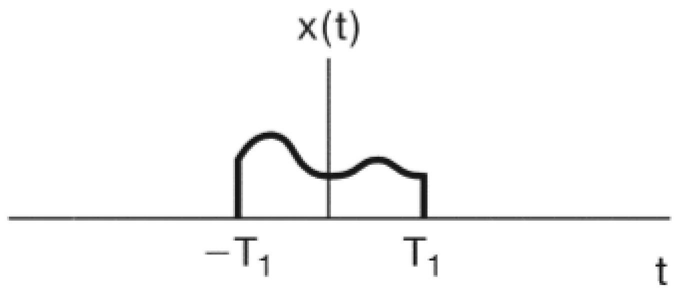
- Derivation of FT from FS

Fourier series can analyze a periodic signal.



(b)

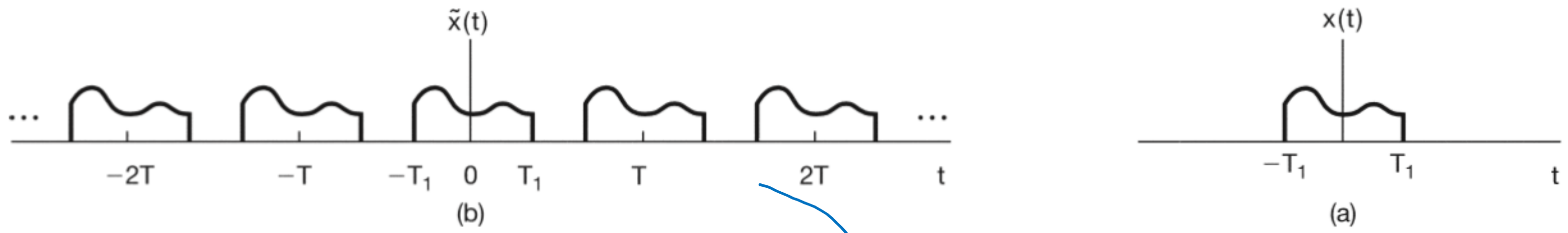
How do we analyze an aperiodic signal?



(a)

- **Sect. 4.1.1 Development of the FT Representation of an Aperiodic Signal**

- Derivation of FT from FS



$$\tilde{x}(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t} \qquad a_k = \frac{1}{T}\int_{-T/2}^{T/2} \tilde{x}(t) e^{-jk\omega_0 t}\,dt$$ where $\omega_0 = 2\pi/T$

FS

Since $\tilde{x}(t) = x(t)$ for $|t| < T/2$, and also, since $x(t) = 0$ outside this interval,

$$a_k = \frac{1}{T}\int_{-T/2}^{T/2} x(t) e^{-jk\omega_0 t}\,dt = \frac{1}{T}\int_{-\infty}^{+\infty} x(t) e^{-jk\omega_0 t}\,dt$$

Define $$\boxed{X(j\omega) = \int_{-\infty}^{+\infty} x(t) e^{-j\omega t}\,dt}$$

then $$\boxed{a_k = \frac{1}{T} X(jk\omega_0)}$$

- **Sect. 4.1.1 Development of the FT Representation of an Aperiodic Signal**
- Derivation of FT from FS

Define

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t)e^{-j\omega t}dt,$$

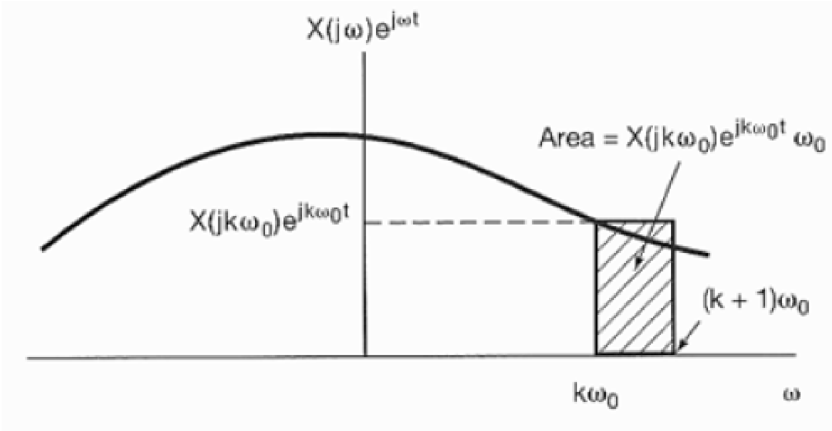
then a_k can be written as

$$a_k = \frac{1}{T}X(jk\omega_0),$$

and $\tilde{x}(t)$ can be written as

$$\tilde{x}(t) = \sum_{k=-\infty}^{+\infty} \frac{1}{T}X(jk\omega_0)e^{jk\omega_0 t}$$

$$= \frac{1}{2\pi}\sum_{k=-\infty}^{+\infty} X(jk\omega_0)e^{jk\omega_0 t}\omega_0.$$



Since $\omega_0 \to 0$ and $\tilde{x}(t) \to x(t)$ as $T \to \infty$, we have

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega$$

- **Sect. 4.1.1 Development of the FT Representation of an Aperiodic Signal**
- Fourier Transform Pairs

$$\boxed{x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)e^{j\omega t}\,d\omega}$$

Synthesis equation

$$\boxed{X(j\omega) = \int_{-\infty}^{+\infty} x(t)e^{-j\omega t}\,dt}$$

Analysis equation

Fourier transform
or Fourier integral

The equation

$$a_k = \frac{1}{T} X(jk\omega_0)$$

is equivalent to

$$\boxed{a_k = \frac{1}{T} X(j\omega)\Big|_{\omega = k\omega_0}}$$

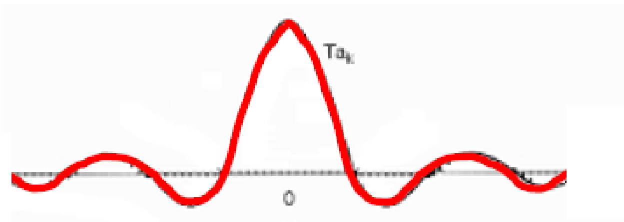
Eq. (4.10)

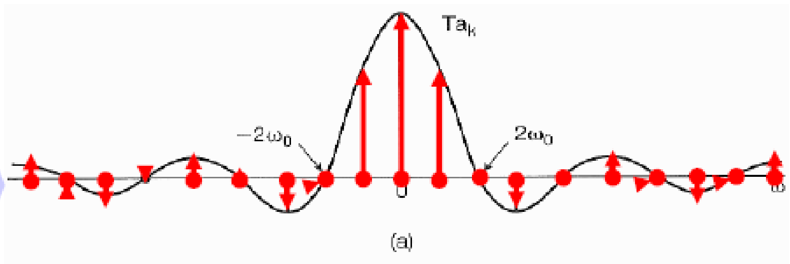
- **Sect. 4.1.1 Development of the FT Representation of an Aperiodic Signal**

- Fourier Transform vs. Fourier Series

Fourier coefficient of $\tilde{x}(t)$ can be expressed as equally spaced samples of the Fourier transform of $x(t)$.

$$\boxed{a_k = \frac{1}{T} X(j\omega)\Big|_{\omega = k\omega_0}}$$





$$\text{FT:} \begin{cases} x(t) = \dfrac{1}{2\pi} \displaystyle\int_{-\infty}^{+\infty} X(j\omega) e^{j\omega t} d\omega \\ X(j\omega) = \displaystyle\int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt \end{cases}$$

$$\text{FS}: \begin{cases} \tilde{x}(t) = \displaystyle\sum_{k=-\infty}^{+\infty} a_k e^{jkw_0 t} \\ a_k = \dfrac{1}{T} \displaystyle\int_T \tilde{x}(t) e^{-jkw_0 t} dt \end{cases}$$

- **4.1.2 Convergence of Fourier Transforms**

- Sufficient Conditions for the Convergence of FT
- Derivation of FT suggests the same convergence condition as that of FS.

Define $\hat{x}(t) = \dfrac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)e^{j\omega t}d\omega$ and $e(t) = \hat{x}(t) - x(t)$.

If $x(t)$ has finite energy (that is, it is square integrable),

$$\int_{-\infty}^{+\infty} |x(t)|^2 \, dt < \infty$$

Then we are guaranteed that $X(j\omega)$ is finite and that

$$\int_{-\infty}^{+\infty} |e(t)|^2 \, dt = 0.$$

That is, there is no energy in their difference, even if $\hat{x}(t)$ may differ significantly at individual values of t.

- **4.1.2 Convergence of Fourier Transforms**

- Sufficient Conditions for the Convergence of FT
- Dirichlet Conditions:

1. $x(t)$ be absolutely integrable; that is, $\int_{-\infty}^{+\infty} |x(t)| dt < \infty$

2. $x(t)$ has a finite number of maxima and minima within any finite interval.

3. $x(t)$ has a finite number of discontinuities within any finite interval. Furthermore, each of these discontinuities must be finite.

Sect. 4.1 Representation of *Aperiodic* Signals: CTFT

- Definitions of FT & Inverse FT:

Fourier transform:

time → freq

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t)e^{-j\omega t}dt$$

Inverse Fourier transform:

freq → time

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \cancel{x}X(j\omega)e^{j\omega t}d\omega$$

These two equations are referred to as the Fourier transform pair.

The Fourier transform can be viewed as the Fourier series in the case where the period $T \to \infty$ (i.e., the period is infinite).

- Example 4.1

$x(t) \to \boxed{h(t)} \to y(t)$

$$x(t) = e^{-at}\underline{\underline{u(t)}}, a > 0$$

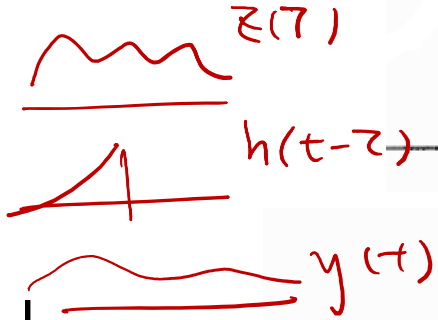
$$\Rightarrow X(j\omega) = \int_0^{\infty} e^{-at} e^{-j\omega t}\, dt$$

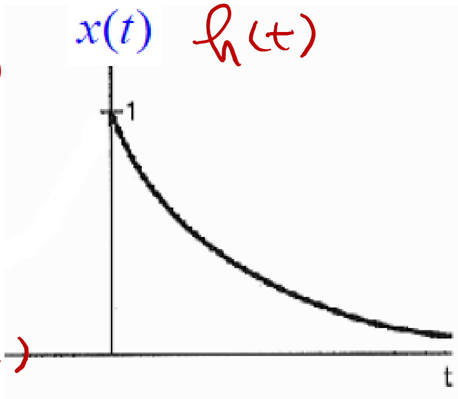
$$= -\frac{1}{a + j\omega} e^{-(a+j\omega)t} \Big|_0^{\infty}$$

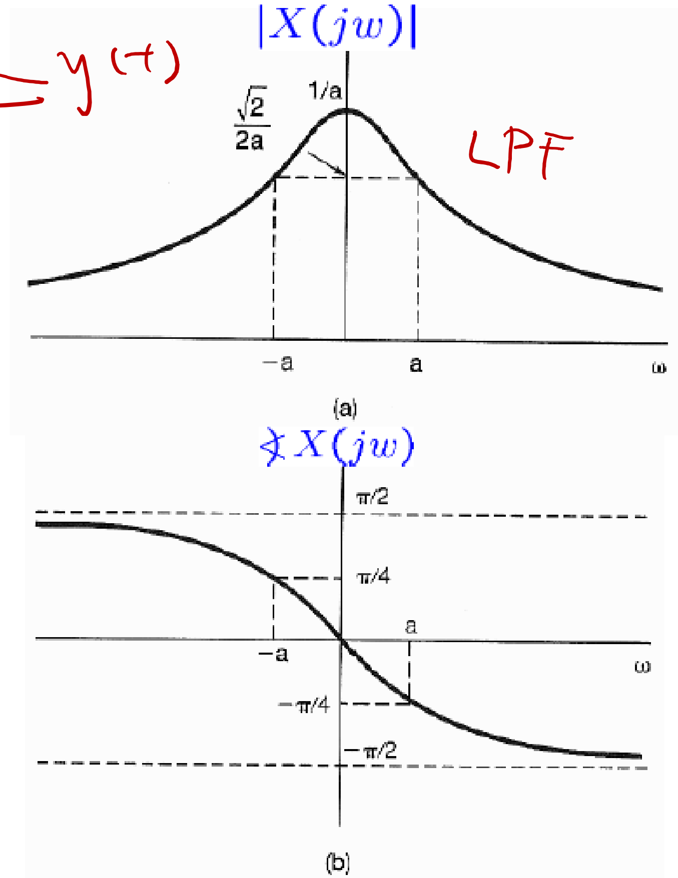
$$= \frac{1}{a + j\omega}, \quad a > 0$$

$$|X(j\omega)| = \frac{1}{\sqrt{a^2 + \omega^2}}$$

$$\angle X(j\omega) = -\tan^{-1}\left(\frac{\omega}{a}\right)$$

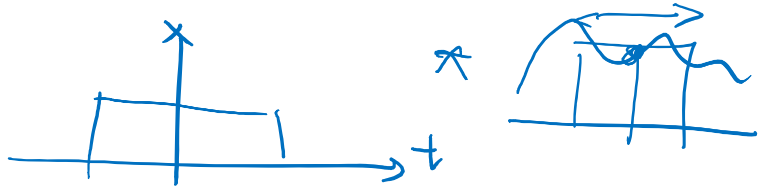






- Example 4.2

naïve LPF h(t)



$$x(t) = e^{-a|t|}, \quad a > 0$$

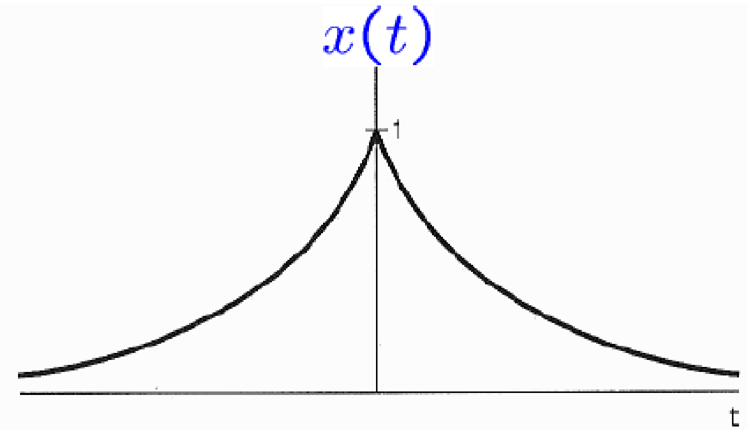
$$\Rightarrow X(j\omega) = \int_{-\infty}^{\infty} e^{-a|t|} e^{-j\omega t}\, dt$$

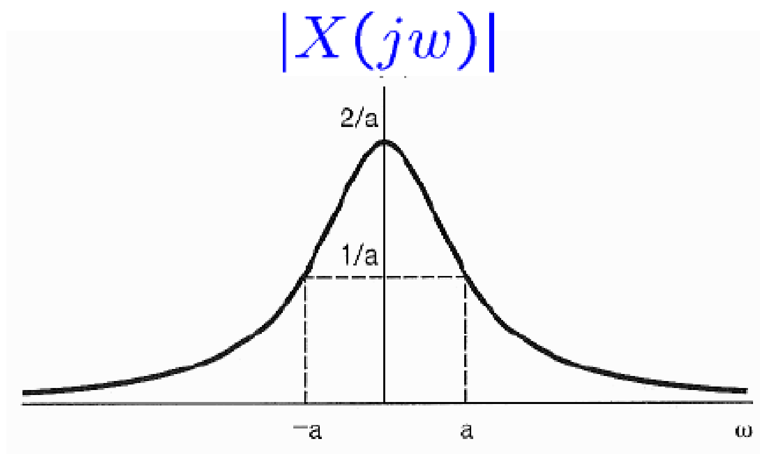
t<0

$$= \underbrace{\int_{-\infty}^{0} e^{at} e^{-j\omega t}\, dt}_{} + \underbrace{\int_{0}^{\infty} e^{-at} e^{-j\omega t}\, dt}_{}$$

$$= \frac{1}{a - j\omega} + \frac{1}{a + j\omega}$$

$$= \frac{2a}{a^2 + \omega^2}$$



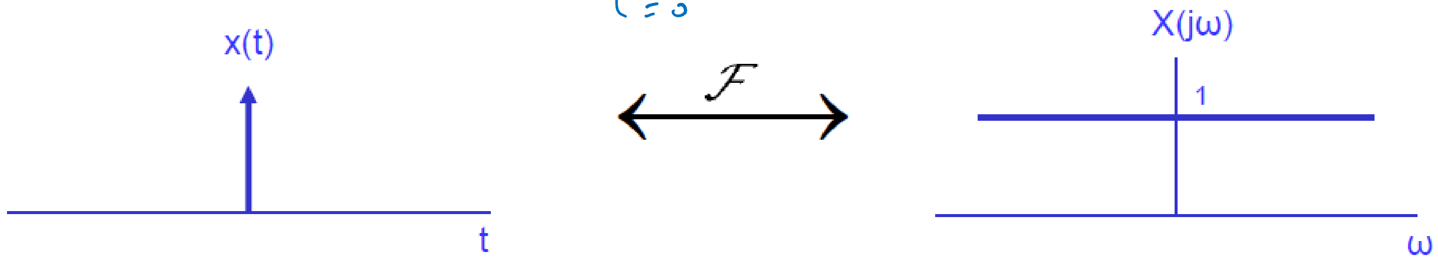


- Example 4.3a: FT of the Unit Impulse

$$x(t) = \delta(t)$$

$$\Rightarrow X(j\omega) = \int_{-\infty}^{\infty} \delta(t) e^{-j\omega t} dt = 1$$

$t = 0$



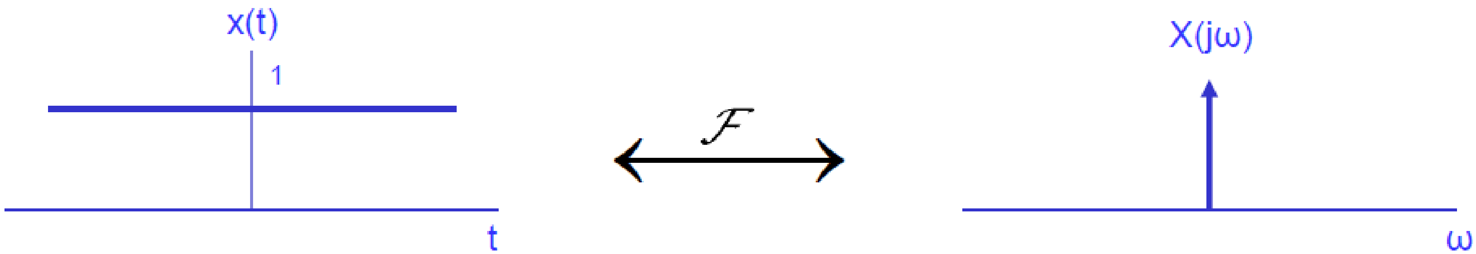
$$\boxed{\delta(t) \xleftrightarrow{\ \mathcal{F}\ } 1}$$

The impulse contains unity contributions from complex sinusoids of all frequencies, from $\omega = -\infty$ to $\omega = +\infty$.

- Example 4.3b: IFT of an Impulse Spectrum

$$X(j\omega) = 2\pi\delta(\omega)$$

$$\Rightarrow x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} 2\pi\delta(\omega)e^{j\omega t}\,d\omega = 1$$



$$\boxed{1 \xleftrightarrow{\mathcal{F}} 2\pi\delta(\omega)}$$

The frequency content of a dc signal is concentrated entirely at ω=0.

- Example 4.3b: IFT of an Impulse Spectrum
- An alternative proof

We know from Example 4.3a that

$$x(t)=\delta(t)\xleftrightarrow{\text{FT}}X(j\omega)=1.$$

Since the inverse FT of $X(j\omega)$ is $x(t)$. We have

$$x(t)=\delta(t)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}X(j\omega)e^{j\omega t}d\omega=\frac{1}{2\pi}\boxed{\int_{-\infty}^{+\infty}e^{j\omega t}d\omega.}$$

$$a(t)=\begin{cases}1\cdot 2\pi, & t=0\\ 0, & t\neq 0\end{cases}$$

Therefore,

$$\int_{-\infty}^{+\infty}e^{j\omega t}d\omega=2\pi\delta(t).\qquad\text{(Eq. 1)}$$

Now we want to find the FT of $g(t)=1$. By definition,

$$G(j\omega)=\int_{-\infty}^{+\infty}g(t)e^{-j\omega t}dt=\int_{-\infty}^{+\infty}e^{-j\omega t}dt.$$

Substituting $t=-\tau$ into the above equation and comparing it with Eq. 1, we have

$$G(j\omega)=\int_{\infty}^{-\infty}e^{j\omega\tau}(-d\tau)=\int_{-\infty}^{+\infty}e^{j\omega\tau}d\tau=2\pi\delta(\omega).$$

Thus

$$1\xleftrightarrow{\text{FT}}2\pi\delta(\omega).$$

- Example 4.4

$$x(t)=\begin{cases}1, & |t|<T_1\\ 0, & |t|>T_1\end{cases}$$

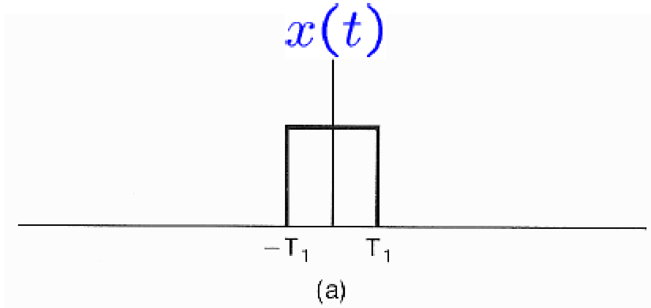
$$\Rightarrow X(j\omega)=\int_{-T_1}^{T_1} e^{-j\omega t}dt$$

$$=\frac{1}{-j\omega}e^{-j\omega t}\Big|_{-T_1}^{T_1}$$

$$=\frac{1}{-j\omega}(e^{-j\omega T_1}-e^{j\omega T_1})$$

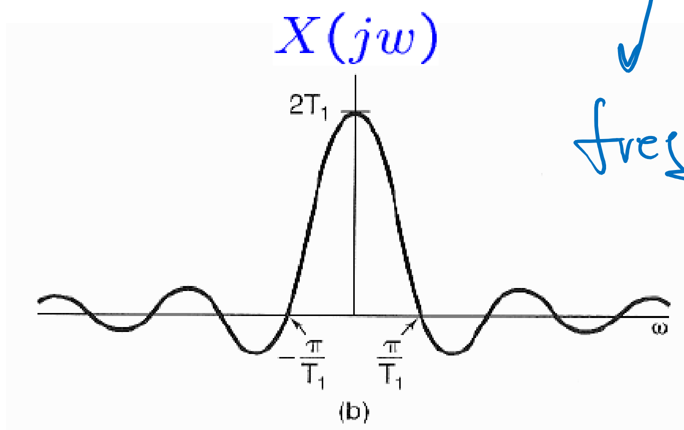
$$=\frac{1}{j\omega}(e^{j\omega T_1}-e^{-j\omega T_1})$$

$$=2\frac{\sin(\omega T_1)}{\omega}$$



time

freq



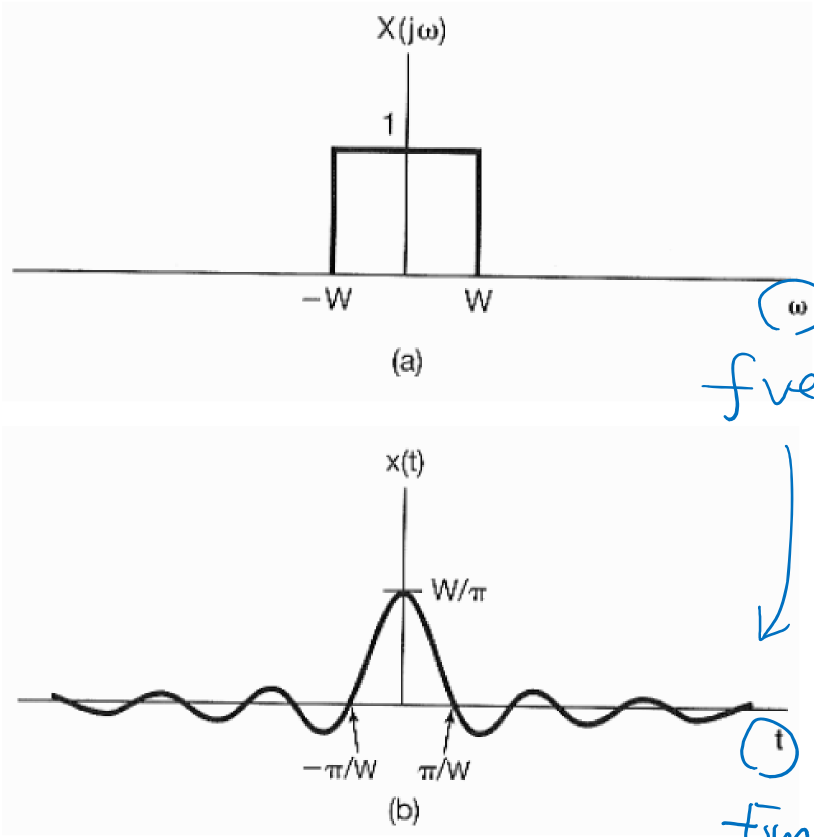
$X(j\omega)=\int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt$

- Example 4.5

$$X(j\omega)=\begin{cases}1, & |\omega|<W\\ 0, & |\omega|>W\end{cases}$$

$$\Rightarrow x(t)=\frac{1}{2\pi}\int_{-W}^{W} e^{j\omega t}\,d\omega$$

$$=\frac{\sin(Wt)}{\pi t}$$



freq

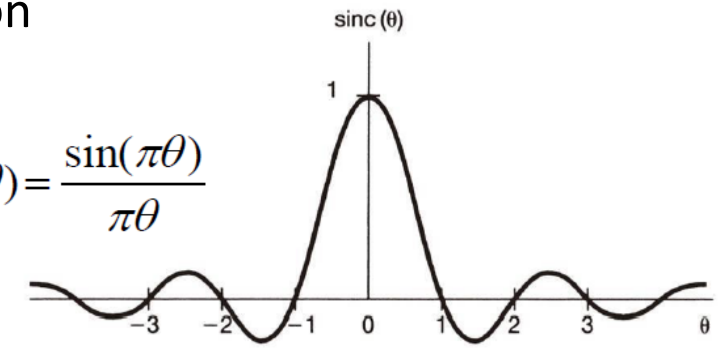
iFT

time

$$\lim_{W\to\infty}\frac{1}{2\pi}\int_{-W}^{W} e^{j\omega t}\,d\omega=\delta(t) \;\Rightarrow\; \lim_{W\to\infty}\frac{\sin(Wt)}{\pi t}=\delta(t)$$

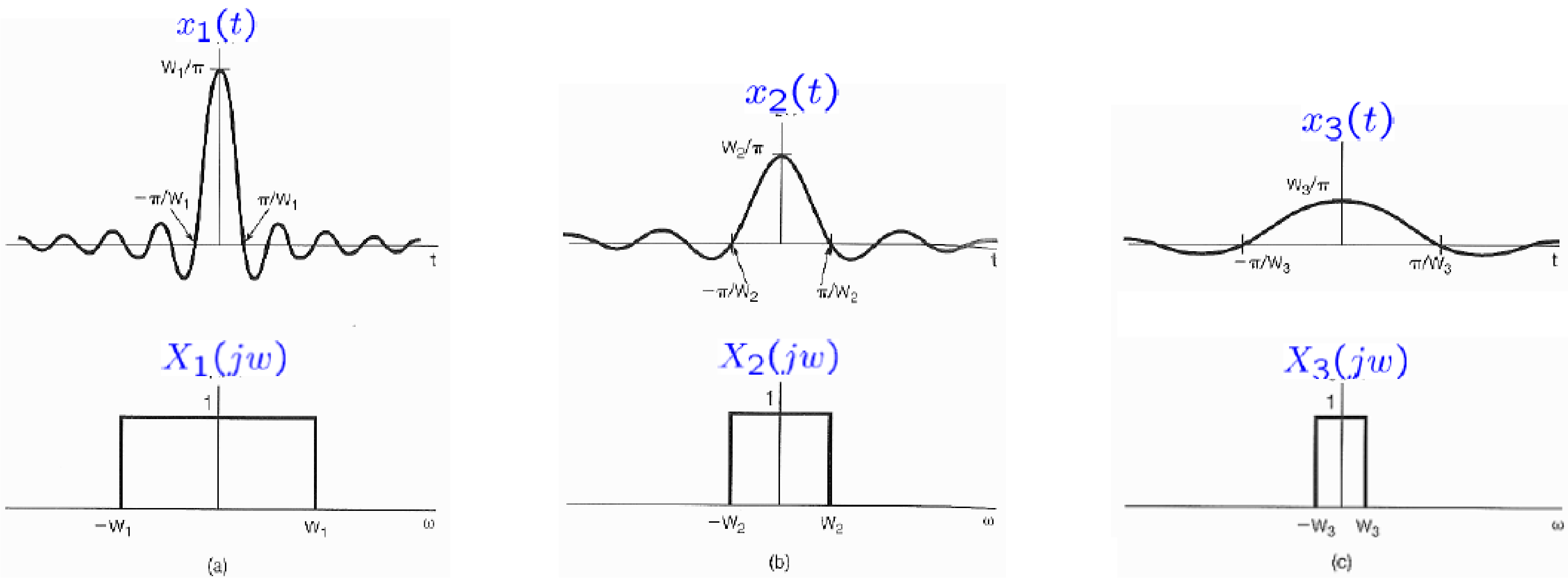
- Revisit of Sinc Function

$$\mathrm{sinc}(\theta) = \frac{\sin(\pi\theta)}{\pi\theta}$$



- From Example 4.5

$$\frac{\sin(Wt)}{\pi t} = \frac{W}{\pi}\,\mathrm{sinc}\left(\frac{Wt}{\pi}\right)$$

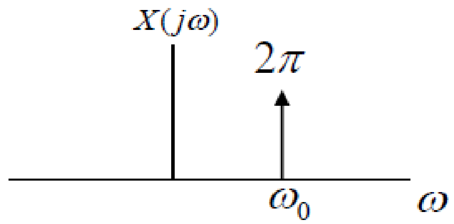


Sect. 4.2 FT for *Periodic* Signals

Consider the Fourier transform pair $x(t) \xleftrightarrow{\text{FT}} X(j\omega)$, where

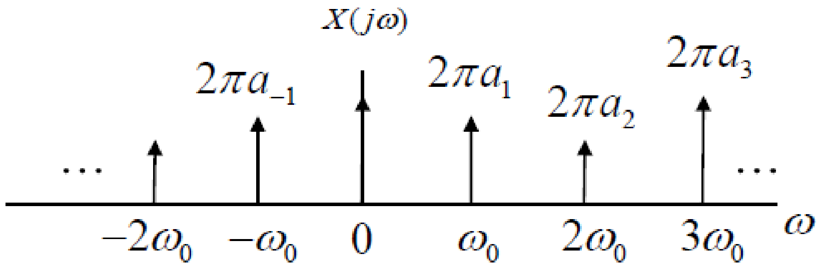
$$X(j\omega) = 2\pi\delta(\omega-\omega_0)$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} 2\pi\delta(\omega-\omega_0)e^{j\omega t}d\omega = e^{j\omega_0 t}.$$



In general, if $X(j\omega)$ is a linear combination of equally spaced impulses

$$\boxed{X(j\omega) = \sum_{k=-\infty}^{+\infty} 2\pi a_k \delta(\omega - k\omega_0),}$$



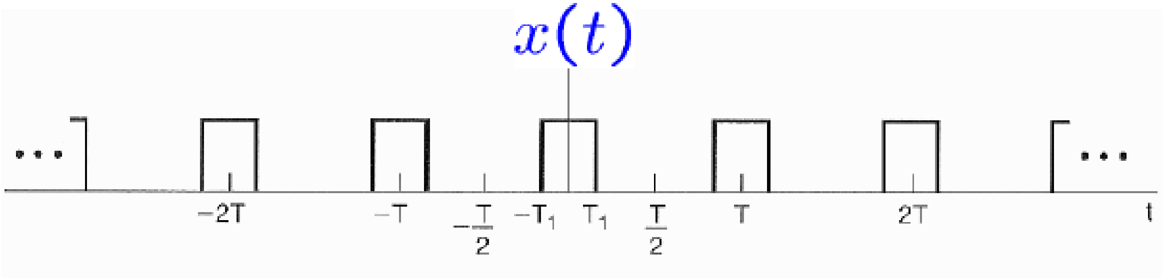
then we have

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t},$$

which is in the form of Fourier series representation.

Therefore, the FT of a periodic signal $x(t)$ is a train of equally spaced impulses occuring at $k\omega_0$, k an integer, and with areas $2\pi a_k$, where a_k are the Fourier series coefficients of $x(t)$.

- Example 4.6



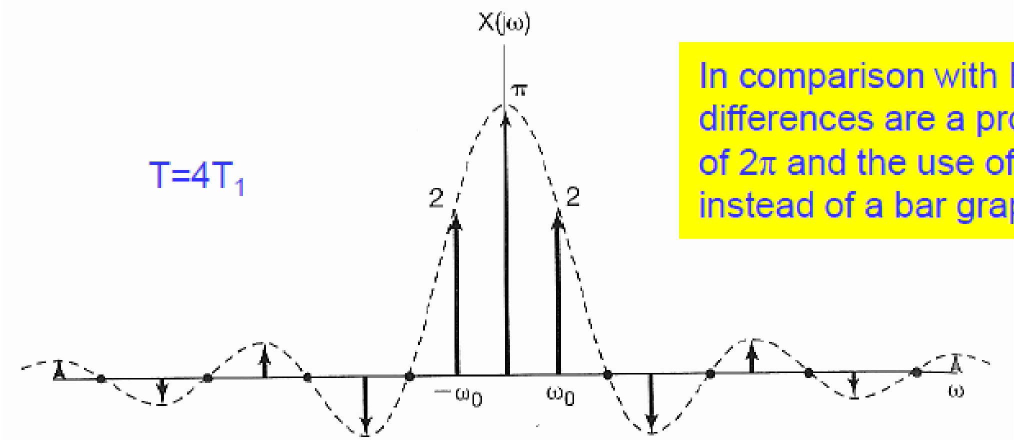
$$x(t)=\sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$$

$$a_k=\frac{\sin(k\omega_0 T_1)}{\pi k}$$

$$\Rightarrow X(j\omega)=\sum_{k=-\infty}^{+\infty} 2\pi a_k \delta(\omega-k\omega_0)$$

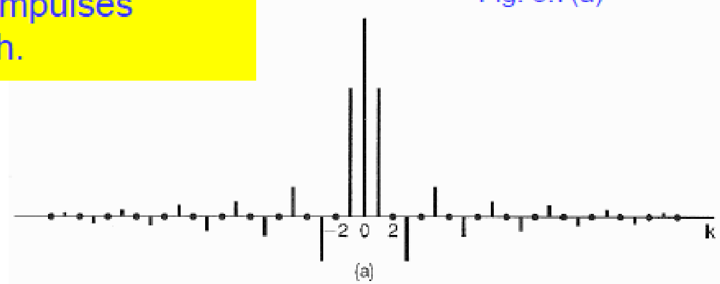
$$=\sum_{k=-\infty}^{+\infty} \frac{2\sin(k\omega_0 T_1)}{k}\delta(\omega-k\omega_0)$$

$T=4T_1$



In comparison with Fig. 3.7(a), the differences are a proportional factor of 2π and the use of impulses instead of a bar graph.

Fig. 3.7(a)

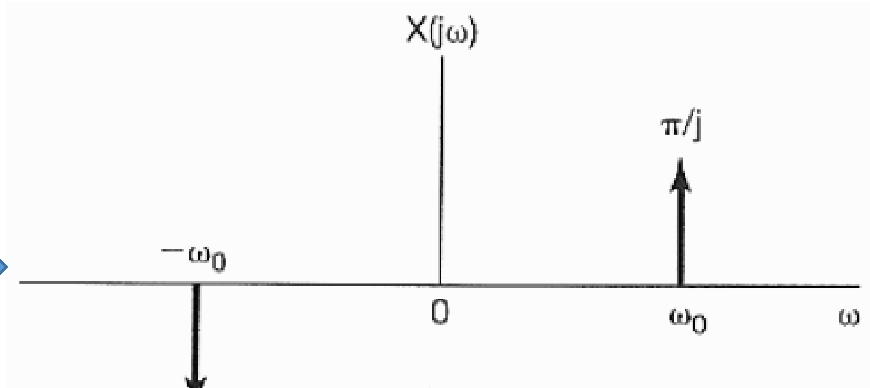


- Example 4.7

$$x(t)=\sin(\omega_0 t)=\frac{e^{j\omega_0 t}-e^{-j\omega_0 t}}{2j}$$

$$\Rightarrow a_1=\frac{1}{2j},\ a_{-1}=-\frac{1}{2j}$$

$$a_k=0,\ k\neq\pm 1$$

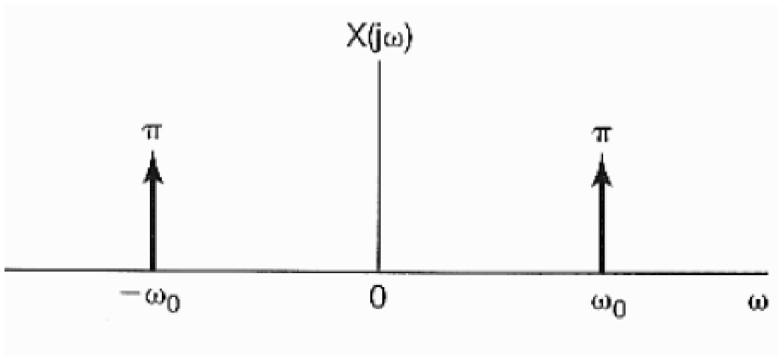


(a)

$$x(t)=\cos(\omega_0 t)=\frac{e^{j\omega_0 t}+e^{-j\omega_0 t}}{2}$$

$$\Rightarrow a_1=\frac{1}{2},\ a_{-1}=\frac{1}{2}$$

$$a_k=0,\ k\neq\pm 1$$



(b)

$$x(t)=\sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$$

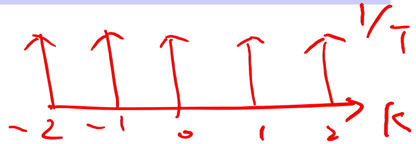
$$a_k=\frac{1}{T}\int_{-T/2}^{+T/2} x(t)e^{-jk\omega_0 t}dt$$

$$X(j\omega)=\sum_{k=-\infty}^{+\infty} 2\pi a_k \delta(\omega-k\omega_0)$$

- Example 4.8 FT of an Impulse Train

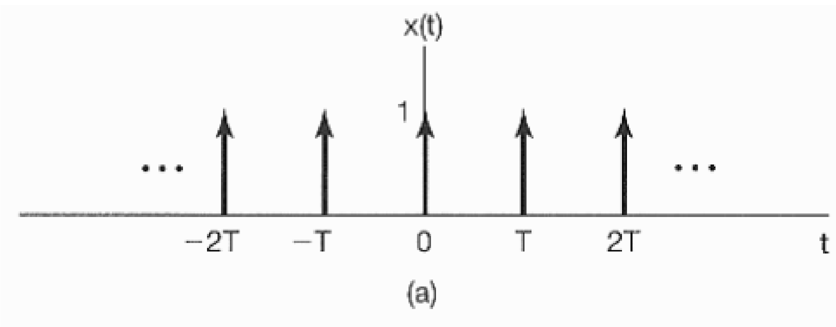
$$x(t)=\sum_{k=-\infty}^{+\infty}\delta(t-kT)$$

$$\Rightarrow a_k=\frac{1}{T}\int_{-T/2}^{+T/2}\delta(t)e^{-jk\omega_0 t}dt=\frac{1}{T} \quad \text{(see Example 3.8)}$$

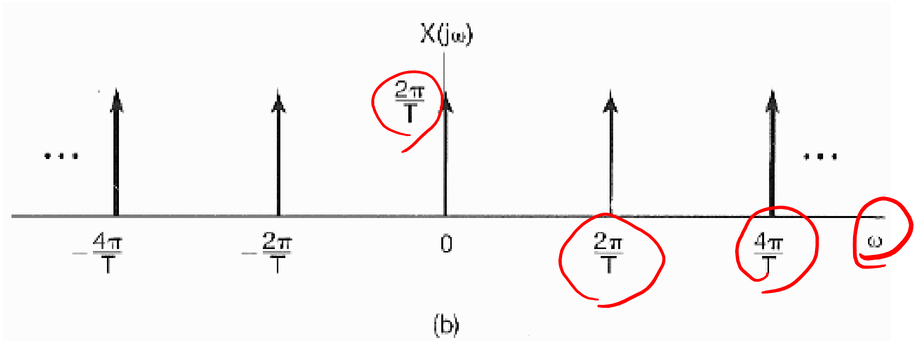


$$\Rightarrow X(j\omega)=\sum_{k=-\infty}^{+\infty}2\pi a_k\delta(\omega-k\omega_0)=\frac{2\pi}{T}\sum_{k=-\infty}^{+\infty}\delta(\omega-\frac{2\pi k}{T})$$

In addition, we have $\displaystyle\sum_{k=-\infty}^{+\infty}\delta(t-kT)=\sum_{k=-\infty}^{+\infty}\frac{1}{T}e^{jk\omega_0 t}$



(a)



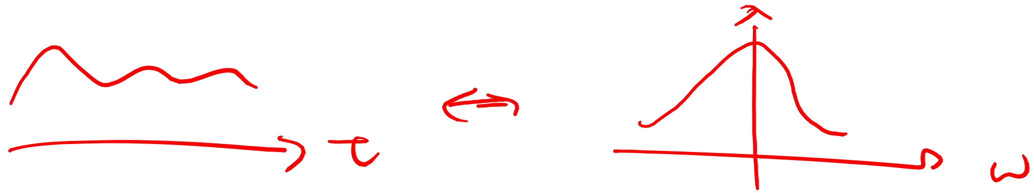
(b)

A periodic impulse train in the frequency domain with period $2\pi / T$

Sect. 4.3 Properties of CTFT

- Linearity

$$x(t) \xleftrightarrow{\mathcal{F}} X(j\omega)$$



$$y(t) \xleftrightarrow{\mathcal{F}} Y(j\omega)$$

$$\Rightarrow \ a\,x(t) + b\,y(t) \xleftrightarrow{\mathcal{F}} aX(j\omega) + bY(j\omega)$$

Sect. 4.3 Properties of CTFT

- Time Shift

$$\boxed{x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(j\omega) \;\Rightarrow\; x(t-t_0) \overset{\mathcal{F}}{\longleftrightarrow} e^{-j\omega t_0} X(j\omega)}$$

Proof:

$$x(t-t_0) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega) e^{j\omega(t-t_0)} d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} \underbrace{(e^{-j\omega t_0} X(j\omega))}_{\text{FT of } x(t-t_0)} e^{j\omega t} d\omega$$

$$F\{x(t-t_0)\} = e^{-j\omega t_0} X(j\omega) = |X(j\omega)| e^{j[\angle X(j\omega) - \omega t_0]}$$

A time shift in a signal introduces a phase shift to the FT of the signal. However, it does not change the magnitude of the FT.

Sect. 4.3 Properties of CTFT

- Frequency Shift

$$\boxed{x(t) \xleftrightarrow{\mathcal{F}} X(j\omega) \quad \Rightarrow \quad g(t)? \xleftrightarrow{\mathcal{F}} G(j\omega) = X(j(\omega-\omega_0))}$$

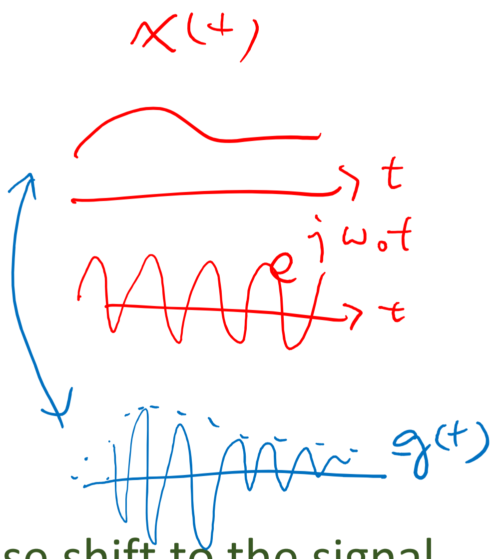
Proof:

$$g(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j(\omega-\omega_0))e^{j\omega t}\,d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\phi)e^{j(\phi+\omega_0)t}\,d\phi$$

$x(t)$

$$= e^{j\omega_0 t}\,\underbrace{\frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\phi)e^{j\phi t}\,d\phi}_{x(t)} = e^{j\omega_0 t}x(t)$$



Thus, $\boxed{e^{j\omega_0 t}x(t) \xleftrightarrow{\mathcal{F}} X(j(\omega-\omega_0))}$

A frequency shift in the FT of a signal introduces a phase shift to the signal. However, it does not change the magnitude of the signal itself.

Sect. 4.3 Properties of CTFT

- Conjugation

$$\boxed{x(t) \xleftrightarrow{\mathcal{F}} X(j\omega) \;\; \Rightarrow \;\; x^*(t) \xleftrightarrow{\mathcal{F}} X^*(-j\omega)}$$

$$X^*(j\omega) = \left[\int_{-\infty}^{+\infty} x(t)e^{-j\omega t}dt\right]^*$$

$$= \int_{-\infty}^{+\infty} x^*(t)e^{j\omega t}dt$$

Replacing ω with $-\omega$ yields

$$X^*(-j\omega) = \underbrace{\int_{-\infty}^{+\infty} \boxed{x^*(t)}e^{-j\omega t}dt}_{\text{Fourier transform of } x^*(t)}$$

Sect. 4.3 Properties of CTFT

- Conjugation
  - $x(t)$ real $\Rightarrow X(j\omega)$ conjugate symmetric

If $x(t)$ is real, then $X(j\omega)$ has conjugate symmetry

$$X(-j\omega) = X^*(j\omega)$$

Proof :

$$X^*(-j\omega) = \int_{-\infty}^{+\infty} x^*(t)e^{-j\omega t}dt = \int_{-\infty}^{+\infty} x(t)e^{-j\omega t}dt = X(j\omega)$$

$$\Rightarrow X^*(j\omega) = X(-j\omega)$$

As a result,

$$\mathcal{Re}\{X(j\omega)\} = \mathcal{Re}\{X(-j\omega)\} \quad \text{--- even}$$

$$\mathcal{Im}\{X(j\omega)\} = -\mathcal{Im}\{X(-j\omega)\} \quad \text{--- odd}$$

where $X(j\omega) = \mathcal{Re}\{X(j\omega)\} + j\mathcal{Im}\{X(j\omega)\}$.

Sect. 4.3 Properties of CTFT

- Conjugation
  - $x(t)$ even $\Rightarrow X(j\omega)$ even

If $x(t)$ is even, then $X(j\omega)$ is also even; that is,

$$X(-j\omega) = X(j\omega)$$

Proof :

$$x(t) \text{ even } \Rightarrow x(-t) = x(t)$$

$$X(-j\omega) = \int_{-\infty}^{+\infty} x(t)e^{j\omega t}dt$$

$$= \int_{-\infty}^{+\infty} x(-\tau)e^{-j\omega\tau}d\tau \qquad t = -\tau$$

$$= \int_{-\infty}^{+\infty} x(\tau)e^{-j\omega\tau}d\tau$$

$$= X(j\omega)$$

Sect. 4.3 Properties of CTFT

- Conjugation
  - $x(t)$ odd $\Rightarrow X(j\omega)$ odd

If $x(t)$ is odd, then $X(j\omega)$ is also odd; that is,

$$X(-j\omega) = -X(j\omega)$$

Proof:

$$x(t) \text{ odd } \Rightarrow x(-t) = -x(t)$$

$$X(-j\omega) = \int_{-\infty}^{+\infty} x(t)e^{j\omega t}dt$$

$$= \int_{-\infty}^{+\infty} x(-\tau)e^{-j\omega\tau}d\tau \qquad t = -\tau$$

$$= -\int_{-\infty}^{+\infty} x(\tau)e^{-j\omega\tau}d\tau$$

$$= -X(j\omega)$$

Sect. 4.3 Properties of CTFT

- Conjugation

  - $x(t)$ real and even $\Rightarrow X(j\omega)$ real and even

If $x(t)$ is both real and even, then $X(j\omega)$ is also real and even

$$X^*(j\omega) = X(-j\omega) = X(j\omega)$$

$$x(t)\text{ real } \Rightarrow x^*(t) = x(t) \Rightarrow X^*(j\omega) = X(-j\omega)$$

$$x(t)\text{ even } \Rightarrow x(-t) = x(t) \Rightarrow X(-j\omega) = X(j\omega)$$

Sect. 4.3 Properties of CTFT

- Conjugation

  - $x(t)$ real and odd $\Rightarrow$ $X(j\omega)$ pure imaginary and odd

If $x(t)$ is both real and odd, then $X(j\omega)$ is pure imaginary and odd,

$$X^*(j\omega) = X(-j\omega) = -X(j\omega)$$

$$\begin{aligned}
x(t) \text{ real } &\Rightarrow x^*(t) = x(t) \Rightarrow X^*(j\omega) = X(-j\omega) \\
x(t) \text{ odd } &\Rightarrow x(-t) = -x(t) \Rightarrow X(-j\omega) = -X(j\omega) \\
&\Rightarrow X^*(j\omega) + X(j\omega) = 0 \\
&\Rightarrow X(j\omega) \text{ pure imaginary}
\end{aligned}$$

Sect. 4.3 Properties of CTFT

- *x(t)* real

For a real function $x(t)$,

$$x(t) = x_e(t) + x_o(t)$$

$$\mathrm{F}\{x(t)\} = F\{x_e(t)\} + F\{x_o(t)\}.$$

Then, due to conjugate symmetry,

$F\{x_e(t)\}$ is a real function, and

$F\{x_o(t)\}$ is a purely imaginary function.

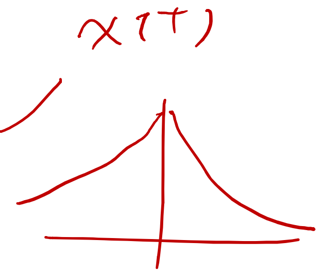
$$\Rightarrow x(t) \xleftrightarrow{\mathcal{F}} X(j\omega) = \mathcal{R}e\{X(j\omega)\} + j\mathcal{I}m\{X(j\omega)\}$$

$$x_e(t) \xleftrightarrow{\mathcal{F}} \mathcal{R}e\{X(j\omega)\}$$

$$x_o(t) \xleftrightarrow{\mathcal{F}} j\mathcal{I}m\{X(j\omega)\}$$

- Example 4.10

Given $e^{-at}u(t) \xleftrightarrow{F} \dfrac{1}{a+j\omega}$, find the FT of $e^{-a|t|}$.



$$x(t) = e^{-a|t|} = e^{-at}u(t) + e^{at}u(-t)$$

$$= 2\left[\frac{e^{-at}u(t) + e^{at}u(-t)}{2}\right]$$

$$\mathcal{E}v\left\{e^{-at}u(t)\right\} \xleftrightarrow{\mathcal{F}} \mathcal{R}e\left\{\frac{1}{a+jw}\right\}$$

$$\mathcal{O}d\left\{e^{-at}u(t)\right\} \xleftrightarrow{\mathcal{F}} j\,\mathcal{I}m\left\{\frac{1}{a+jw}\right\}$$

$$X(j\omega) = 2\,Re\left\{\frac{1}{a+j\omega}\right\} = \frac{2a}{a^2+\omega^2}$$

Sect. 4.3 Properties of CTFT

iFT $f \to t$

$$x(t)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}X(j\omega)e^{j\omega t}d\omega$$

- Differentiation

FT $t \to f$

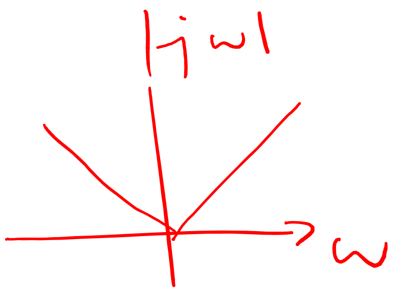
$$X(j\omega)=\int_{-\infty}^{+\infty}x(t)e^{-j\omega t}dt$$

$$\boxed{x(t) \xleftrightarrow{\mathcal{F}} X(j\omega) \quad \Rightarrow \quad \frac{d}{dt}x(t) \xleftrightarrow{\mathcal{F}} j\omega X(j\omega)}$$

Proof:

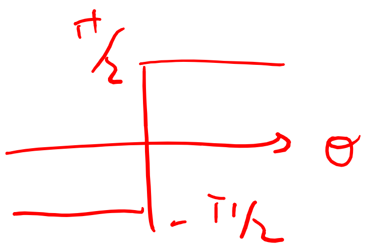
$$\frac{dx(t)}{dt}=\frac{1}{2\pi}\int_{-\infty}^{+\infty}X(j\omega)\frac{d}{dt}(e^{j\omega t})d\omega$$

$$=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\boxed{j\omega X(j\omega)}e^{j\omega t}d\omega$$



Therefore,

$$\frac{dx(t)}{dt} \xleftrightarrow{\mathcal{F}} j\omega X(j\omega)$$



Sect. 4.3 Properties of CTFT

- Integration

$$x(t) \xleftrightarrow{\mathcal{F}} X(j\omega)$$

$$\Rightarrow \int_{-\infty}^{t} x(\tau)d\tau \xleftrightarrow{\mathcal{F}} \frac{1}{j\omega}X(j\omega)+\pi X(j0)\delta(\omega)$$

$$y(t)=\int_{-\infty}^{t} x(\tau)d\tau \quad \Rightarrow \quad \boxed{\frac{dy(t)}{dt}} = x(t)$$

$j\omega \cdot Y(j\omega) = X(j\omega)$

$$Y(j\omega)=\frac{1}{j\omega}X(j\omega)$$

Indeterminate at ω=0 as a result of the differentiation that destroys the dc component of $y(t)$

The value at ω=0 is modified by including an impulse in the transform:

$$Y(j\omega)=\frac{1}{j\omega}X(j\omega)+\pi \boxed{X(j0)}\boxed{\delta(\omega)}$$

$X(j\omega)\big|_{\omega=0}$

Sect. 4.3 Properties of CTFT

- Time and Frequency Scaling

$$x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(j\omega)$$

$$x(at) \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{|a|} X\left(\frac{j\omega}{a}\right)$$

Inverse relationship between time and frequency

$$x\left(\frac{t}{b}\right) \overset{\mathcal{F}}{\longleftrightarrow} |b| X(jb\omega)$$

Example: slow playback of an audio tape

$$x(-t) \overset{\mathcal{F}}{\longleftrightarrow} X(-j\omega)$$

Reversing a signal in time also reverses its transform

Sect. 4.3 Properties of CTFT

- Duality

$$x_1(t)=\begin{cases}1, & |t|<T_1\\ 0, & |t|>T_1\end{cases} \xleftrightarrow{\mathcal{F}} X_1(j\omega)=\frac{2\sin(\omega T_1)}{\omega}$$

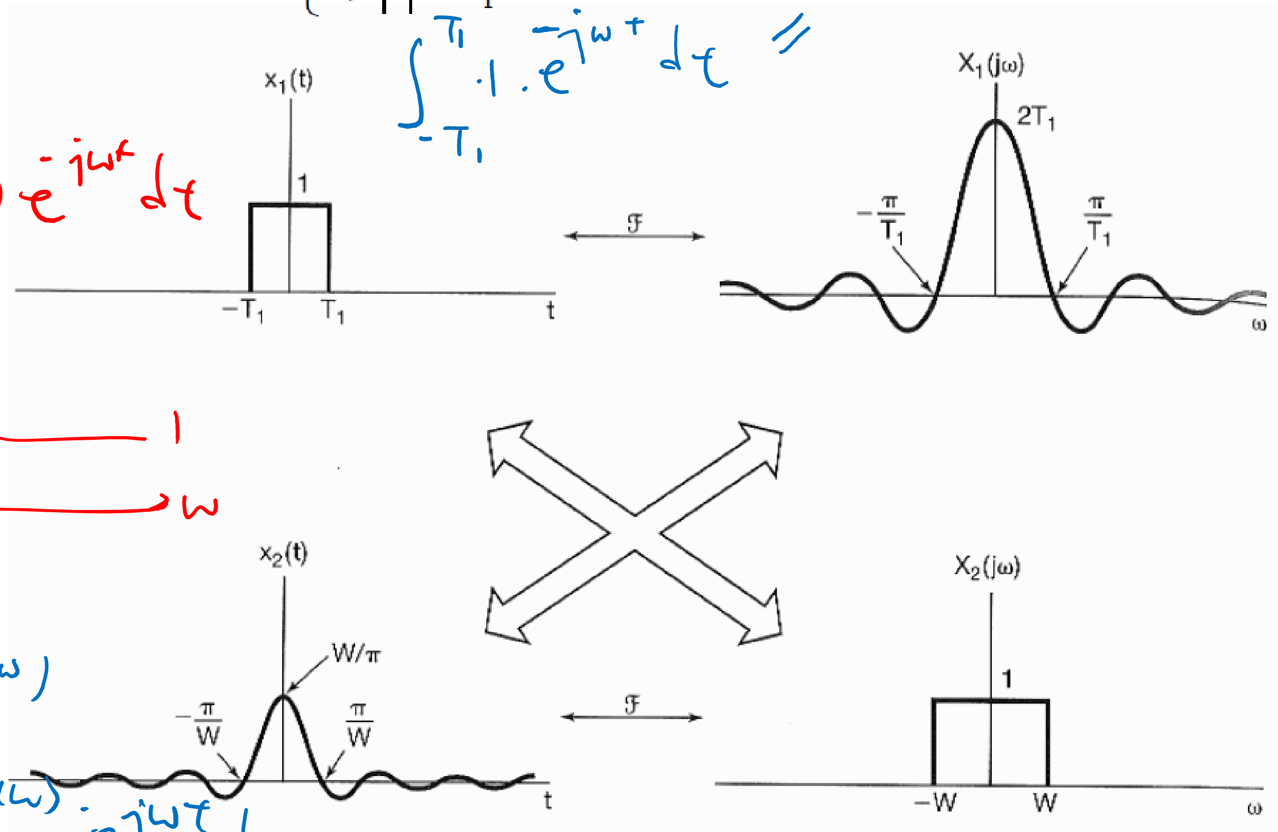
$x(t):\delta(t)$

$X(j\omega)=\int_{-\infty}^{\infty}\delta(t)e^{-j\omega t}dt$

$=1$

$\delta(t)$ t $\Leftrightarrow$ 1 ω

$\int_{-T_1}^{T_1} 1\cdot e^{-j\omega t}dt$ //



$X(j\omega)=2\pi\,\delta(\omega)$

$x(t)=\frac{1}{\cancel{2\pi}}\int_{-\infty}^{\infty}\cancel{2\pi}\,\delta(\omega)\cdot e^{j\omega t}d\omega$

$=1$

t $\Leftrightarrow$ 2π 0 ω

$$x_2(t)=\frac{\sin(Wt)}{\pi t} \xleftrightarrow{\mathcal{F}} X_2(j\omega)=\begin{cases}1, & |\omega|<W\\ 0, & |\omega|>W\end{cases}$$

Sect. 4.3 Properties of CTFT

- Parseval's relation

$$x(t) \xleftrightarrow{\mathcal{F}} X(j\omega) \quad \Rightarrow \quad \int_{-\infty}^{+\infty} |x(t)|^2 \, dt = \frac{1}{2\pi} \int_{-\infty}^{+\infty} |X(j\omega)|^2 \, d\omega$$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\omega) e^{j\omega t} \, d\omega$$

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} \, dt$$

Proof:

$$\int_{-\infty}^{+\infty} |x(t)|^2 \, dt = \int_{-\infty}^{+\infty} x(t) x^*(t) dt$$

$$= \int_{-\infty}^{+\infty} x(t) \underbrace{\left[\frac{1}{2\pi} \int_{-\infty}^{+\infty} X^*(j\omega) e^{-j\omega t} \, d\omega \right]}_{\text{IFT of } X^*(j\omega)} dt$$

$$= \frac{1}{2\pi} \int_{-\infty}^{+\infty} X^*(j\omega) \underbrace{\left[\int_{-\infty}^{+\infty} x(t) e^{-j\omega t} \, dt \right]}_{\text{FT of } x(t)} d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{+\infty} |X(j\omega)|^2 \, d\omega$$

- Example 4.11 Differentiation/Integration

$$x(t)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}X(j\omega)e^{j\omega t}d\omega$$

$$X(j\omega)=\int_{-\infty}^{+\infty}x(t)e^{-j\omega t}dt$$

$$g(t)=\delta(t)\overset{\mathcal{F}}{\longleftrightarrow}\boxed{G(j\omega)=1}$$

$$\Rightarrow x(t)=u(t)\overset{\mathcal{F}}{\longleftrightarrow}X(j\omega)=?$$

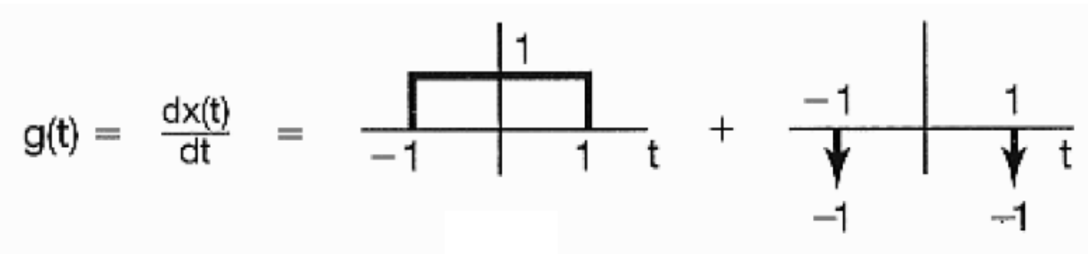
$$x(t)=u(t)=\int_{-\infty}^{t}g(\tau)d\tau$$

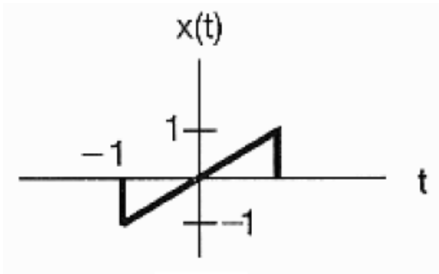
$$X(j\omega)=U(j\omega)=\frac{1}{j\omega}G(j\omega)+\pi G(0)\delta(\omega)$$

$$=\frac{1}{j\omega}+\pi\delta(\omega)$$

$$\delta(t)=\frac{d}{dt}u(t)\overset{\mathcal{F}}{\longleftrightarrow}j\omega\left[\frac{1}{j\omega}+\pi\delta(\omega)\right]=1$$

- Example 4.12 Find the FT of x(t).





$$g(t) = \frac{d}{dt} x(t)$$

$G(j0)=0$

$$G(j\omega) = \frac{2\sin(\omega)}{\omega} - e^{j\omega} - e^{-j\omega}$$

$$\Rightarrow X(j\omega) = \frac{G(j\omega)}{j\omega} + \pi G(j0)\delta(\omega)$$

$$= \frac{2\sin(\omega)}{j\omega^2} - \frac{2\cos(\omega)}{j\omega}$$

Pure imaginary and odd

- Example 4.13 Find the FT by duality.

$$g(t)=\frac{2}{1+t^2} \xleftrightarrow{\mathcal{F}} G(j\omega)=?$$

t

Recall from Example 4.2:

$$x(t)=e^{-|t|} \xleftrightarrow{\mathcal{F}} X(j\omega)=\frac{2}{1+\omega^2}$$

$$\therefore\ e^{-|t|}=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\left(\frac{2}{1+\omega^2}\right)e^{j\omega t}\,d\omega \qquad \text{replace } t \text{ by } -t$$

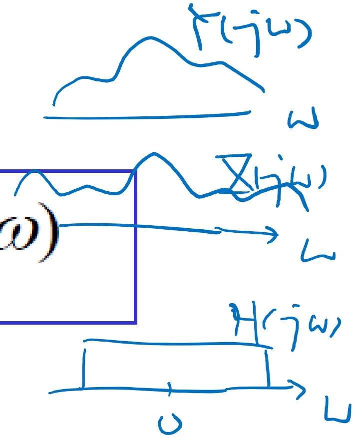
$$\Rightarrow 2\pi e^{-|t|}=\int_{-\infty}^{+\infty}\left(\frac{2}{1+\omega^2}\right)e^{-j\omega t}\,d\omega \qquad \text{interchange } t \text{ with } \omega$$

$$\Rightarrow 2\pi e^{-|\omega|}=\int_{-\infty}^{+\infty}\left(\frac{2}{1+t^2}\right)e^{-j\omega t}\,dt$$

$$\Rightarrow G(j\omega)=2\pi e^{-|\omega|}$$

$x(t) \rightarrow \boxed{h(t)} \rightarrow y(t) = x * h$

Sect. 4.3 Convolution Property



$$\boxed{y(t) = x(t) * h(t) \xleftrightarrow{\mathcal{F}} Y(j\omega) = X(j\omega)H(j\omega)}$$

$$y(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau$$

$$\Rightarrow Y(j\omega) = F\{y(t)\} = \int_{-\infty}^{+\infty}\left[\int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau\right]e^{-j\omega t}dt$$

$$= \int_{-\infty}^{+\infty} x(\tau)\left[\int_{-\infty}^{+\infty} h(t-\tau)e^{-j\omega t}dt\right]d\tau$$

$$= \int_{-\infty}^{+\infty} x(\tau)\left[e^{-j\omega\tau}H(j\omega)\right]d\tau$$

$$= H(j\omega)\underbrace{\int_{-\infty}^{+\infty} x(\tau)e^{-j\omega\tau}d\tau}_{X(j\omega)} = H(j\omega)X(j\omega)$$

- Example 4.15 Time Shift

$$x(t) \longrightarrow \boxed{h(t)=\delta(t-t_0)} \longrightarrow y(t)$$

$$h(t)=\delta(t-t_0)$$

$$\Rightarrow H(j\omega)=\int_{-\infty}^{+\infty} h(t)e^{-j\omega t}dt = e^{-j\omega t_0}$$

$$\begin{aligned} Y(j\omega) &= H(j\omega)X(j\omega) \\ &= e^{-j\omega t_0}X(j\omega) \end{aligned}$$

which is consistent with the time-shift property, and

$y(t)=x(t-t_0)$.

- Example 4.16 & 17 Differentiator/Integrator

Differenciator :

$$y(t) = \frac{d}{dt}x(t) \Rightarrow Y(j\omega) = j\omega X(j\omega)$$

$$\Rightarrow H(j\omega) = j\omega$$

Integrator :

$$\because\ y(t) = \int_{-\infty}^{t} x(\tau)d\tau = \int_{-\infty}^{\infty} x(\tau)u(t-\tau)d\tau = x(t) * u(t)$$

$$h(t) = u(t)$$

$$\Rightarrow H(j\omega) = \frac{1}{j\omega} + \pi\delta(\omega) \quad \text{(from Example 4.11)}$$

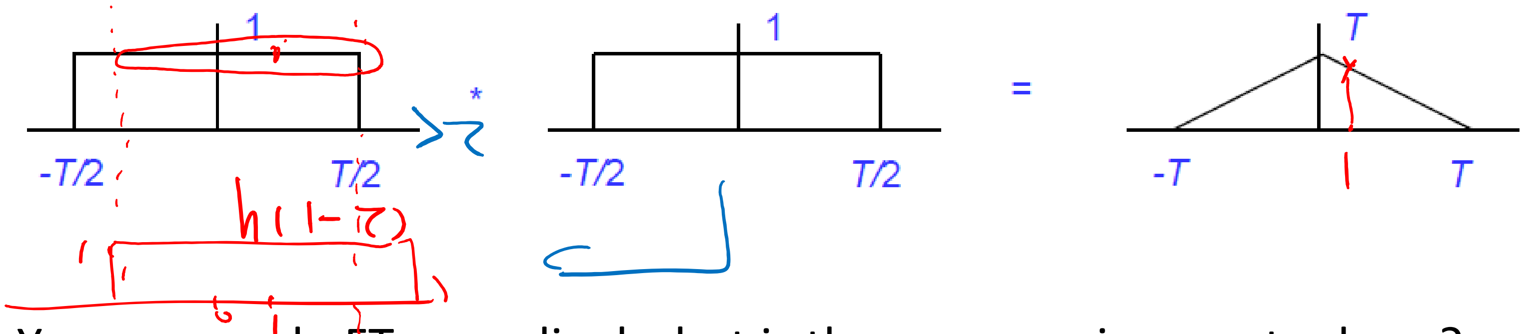
$$\Rightarrow Y(j\omega) = H(j\omega)X(j\omega) = \frac{1}{j\omega}X(j\omega) + \pi\delta(\omega)X(j\omega)$$

$$= \frac{1}{j\omega}X(j\omega) + \pi\delta(\omega)X(0)$$

← consistent with integration property

Sect. 4.3 Convolution Property (cont'd)

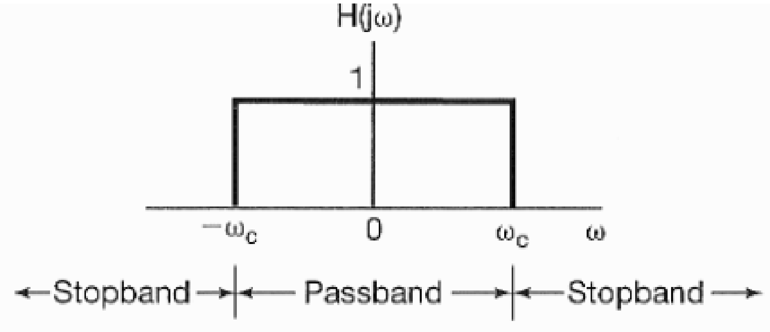
- Fourier Transform of a triangular pulse?

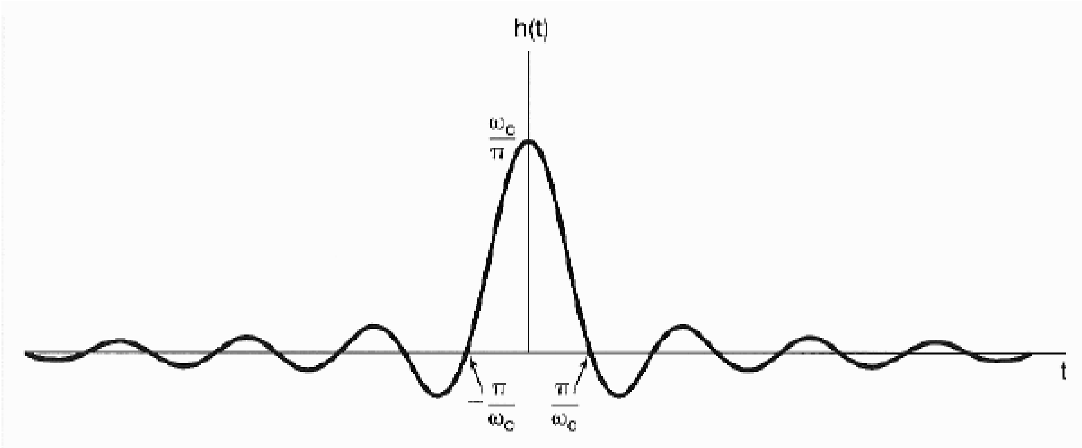


- Yes, we can do FT accordingly, but is there any easier way to do so?
- In fact, it is much easier to compute the FT of the triangular signal by the property of convolution…How?

- Example 4.18: Ideal Low-Pass Filter

$$H(j\omega)=\begin{cases}1, & |\omega|<\omega_c \\ 0, & |\omega|>\omega_c\end{cases} \qquad \Rightarrow h(t)=\frac{1}{2\pi}\int_{-\omega_c}^{+\omega_c} e^{j\omega t}\,d\omega=\frac{\sin(\omega_c t)}{\pi t}$$

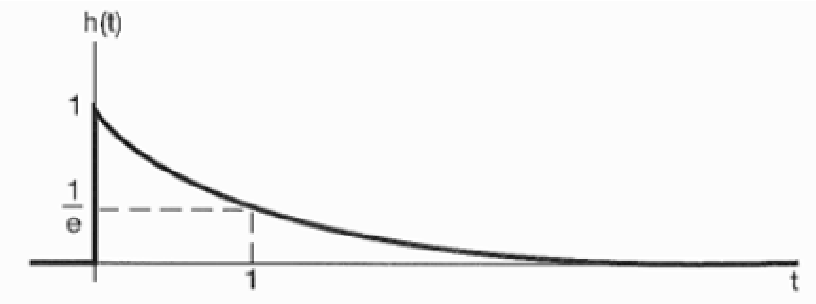




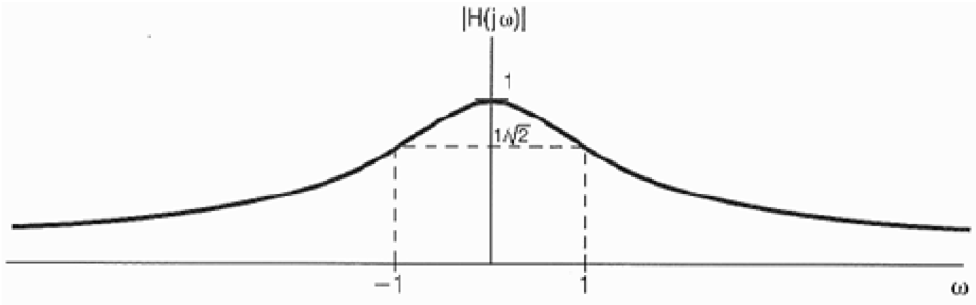
Undesirable characteristics of this ideal low-pass filter:
1) Non-causal
2) Not easy to approximate closely
3) Oscillatory response

- Practical Low-Pass Filter Design

$$h(t) = e^{-t}u(t) \xleftrightarrow{\mathcal{F}} H(j\omega) = \frac{1}{j\omega + 1}$$



(a)



(b)

- Example 4.19 Illustration of the Convolution Property

$$x(t) \longrightarrow \boxed{\begin{array}{c}\text{Filter}\\ \text{LTI System}\end{array}} \longrightarrow y(t)=?$$

$$h(t)=e^{-at}u(t), \quad a>0 \qquad \Rightarrow H(j\omega)=\frac{1}{\mathrm{a}+\mathrm{j}\omega}$$

$$x(t)=e^{-bt}u(t),\ b>0 \qquad \Rightarrow X(j\omega)=\frac{1}{b+j\omega}$$

$$\Rightarrow Y(j\omega)=H(j\omega)X(j\omega)=\left(\frac{1}{a+j\omega}\right)\left(\frac{1}{b+j\omega}\right)$$

$$\mathrm{a}\neq\mathrm{b} \ \Rightarrow Y(j\omega)=\frac{1}{b-a}\left[\frac{1}{a+j\omega}-\frac{1}{b+j\omega}\right]$$

Partial fraction expansion (see Appendix).

$$\Rightarrow y(t)=\frac{1}{b-a}\left[e^{-at}u(t)-e^{-bt}u(t)\right]$$

- Example 4.19 (cont'd)

$$a = b \Rightarrow Y(j\omega) = \frac{1}{(a + j\omega)^2}$$

since

$$e^{-at}u(t) \xleftrightarrow{\mathcal{F}} \frac{1}{a + j\omega}$$

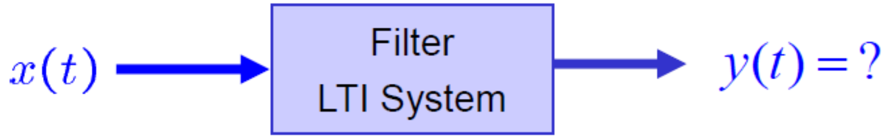
and

$$te^{-at}u(t) \xleftrightarrow{\mathcal{F}} j\frac{d}{d\omega}\left[\frac{1}{a + j\omega}\right] = \frac{1}{(a + j\omega)^2}$$

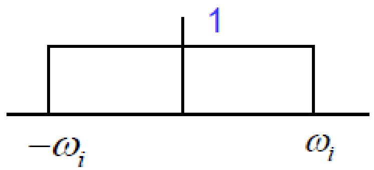
We have

$$y(t) = te^{-at}u(t).$$

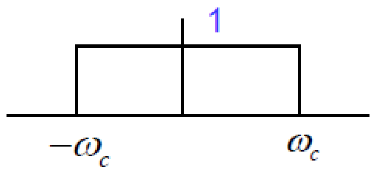
- Example 4.20



$$x(t)=\frac{\sin(\omega_i t)}{\pi t} \Rightarrow X(j\omega)=\begin{cases}1, & |\omega|\le\omega_i\\ 0, & \text{otherwise}\end{cases}$$

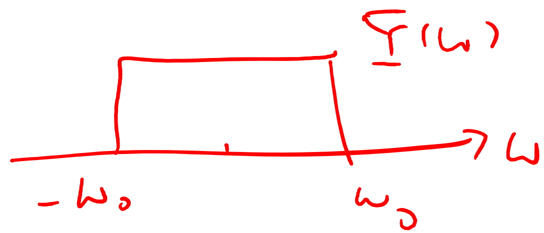


$$h(t)=\frac{\sin(\omega_c t)}{\pi t} \Rightarrow H(j\omega)=\begin{cases}1, & |\omega|\le\omega_c\\ 0, & \text{otherwise}\end{cases}$$

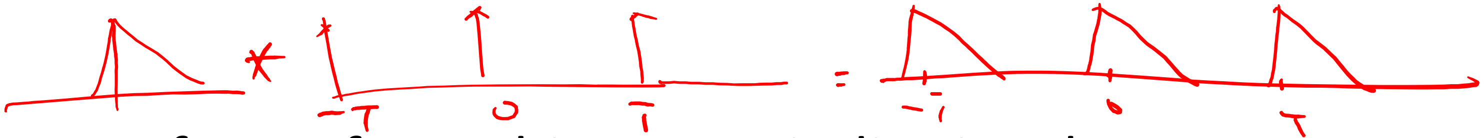


Let $\omega_0=\min(\omega_c,\omega_i)$, then

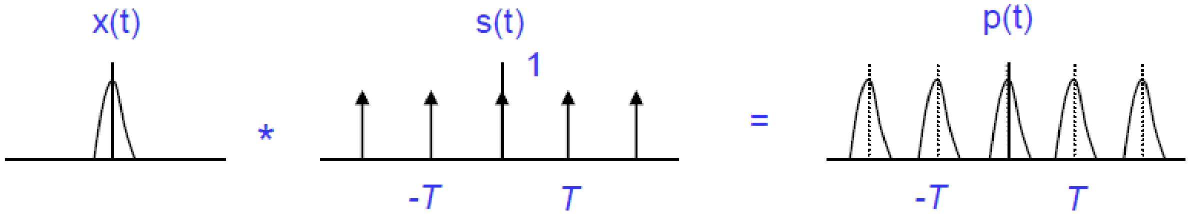
$$Y(j\omega)=H(j\omega)X(j\omega)=\begin{cases}1, & |\omega|\le\omega_0\\ 0, & \text{otherwise}\end{cases}$$



$$\Rightarrow y(t)=\begin{cases}\dfrac{\sin(\omega_c t)}{\pi t}, \omega_c\le\omega_i\\ \dfrac{\sin(\omega_i t)}{\pi t}, \omega_c\ge\omega_i\end{cases}$$



- Fourier Transform of an Arbitrary Periodic Signal



By convolution,

$$p(t) = x(t) * s(t) = x(t) * \sum_{k=-\infty}^{\infty} \delta(t - kT) = \sum_{k=-\infty}^{\infty} x(t - kT)$$

From Example 3.8 or Example 4.8, we have

$$s(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t} = \frac{1}{T} \sum_{k=-\infty}^{\infty} e^{jk\omega_0 t}$$

That is, we have two expressions for a periodic impulse train.
Express them in terms of FT, we have

$$S(j\omega) = \sum_{k=-\infty}^{\infty} e^{-jk\omega T} = \frac{1}{T} \sum_{k=-\infty}^{\infty} 2\pi \delta(\omega - k\omega_0)$$

Therefore,

$$P(j\omega) = X(j\omega) S(j\omega) = \frac{2\pi}{T} \sum_{k=-\infty}^{\infty} X(j\omega) \delta(\omega - k\omega_0)$$

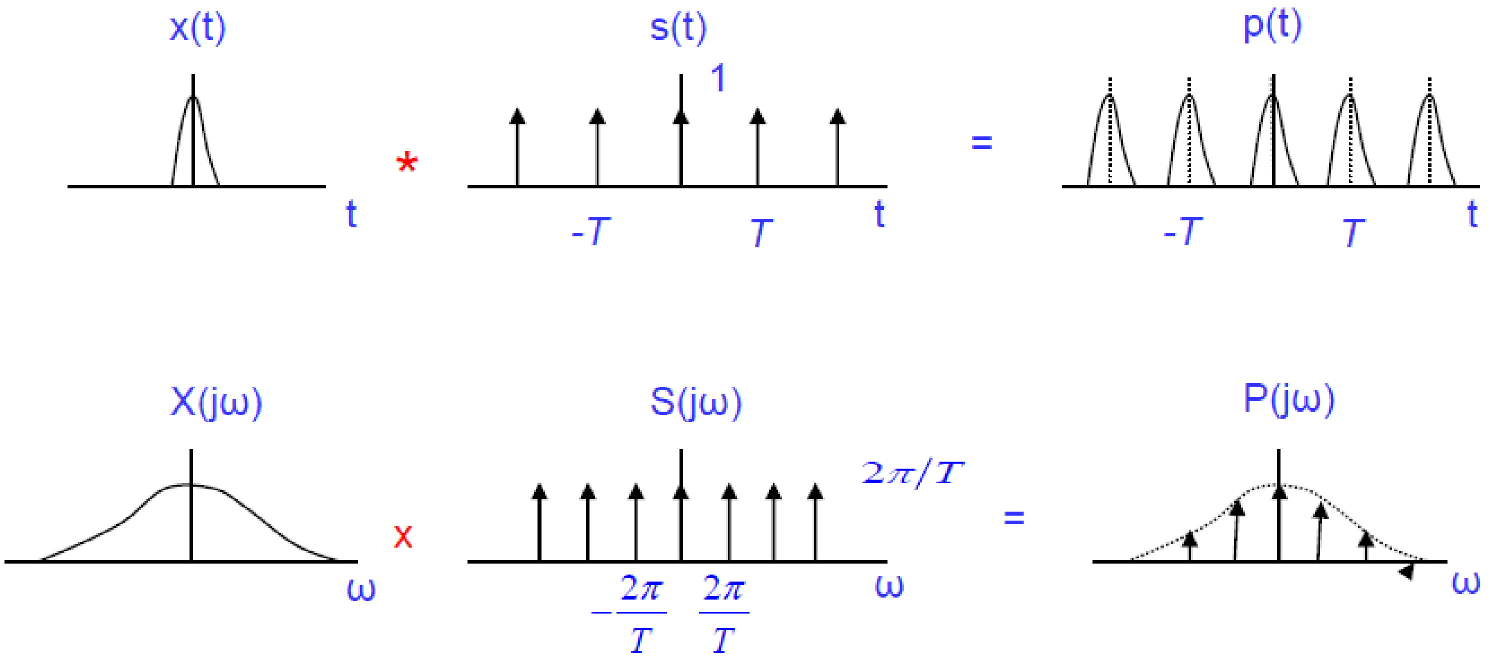
$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\omega) e^{j\omega t} d\omega$$

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt$$

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$$

$$a_k = \frac{1}{T} \int_{-T/2}^{+T/2} x(t) e^{-jk\omega_0 t} dt$$

- Fourier Transform of an Arbitrary Periodic Signal
- We see that the FT of periodic function consists of impulses in frequency at multiples of the fundamental frequency.
- Thus, CT periodic signals can be represented by a countably infinite number of complex exponentials.



- Fourier Transform of an Arbitrary Periodic Signal
- Recall that $X(j\omega)=\int_{-\infty}^{\infty}x(t)e^{-j\omega t}dt=\int_{-T/2}^{T/2}x(t)e^{-j\omega t}dt$
- Therefore, $\dfrac{X(jk\omega_0)}{T}=\dfrac{1}{T}\int_{-T/2}^{T/2}x(t)e^{-jk\omega_0 t}dt=a_k$ (=Fourier series coefficient)

  which is exactly the same equation as Eqn. (4.10). Therefore, for an arbitrary CT periodic signal, it's FT consists of impulses (located at the harmonic frequencies) whose areas are proportional to the FS coeff.

- We can also conclude that the FT of a periodic signal is related to its FS coefficients a_k by

$$P(j\omega)=\sum_{k=-\infty}^{\infty}2\pi a_k\delta(\omega-k\omega_0),$$

  which is exactly the same as Eqn. (4.22) (i.e., X(jω)).

Sect. 4.4 Multiplication Property

- Multiplication Property

$$\boxed{r(t) = x(t)y(t) \xleftrightarrow{\mathcal{F}} R(j\omega) = \frac{1}{2\pi} X(j\omega) * Y(j\omega)}$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\phi)e^{j\phi t}d\phi, \qquad y(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} Y(j\eta)e^{j\eta t}d\eta$$

$$r(t) = x(t)y(t) = \frac{1}{(2\pi)^2}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} X(j\phi)Y(j\eta)e^{j(\phi+\eta)t}d\eta d\phi \qquad \leftarrow \ \eta = \omega - \phi$$

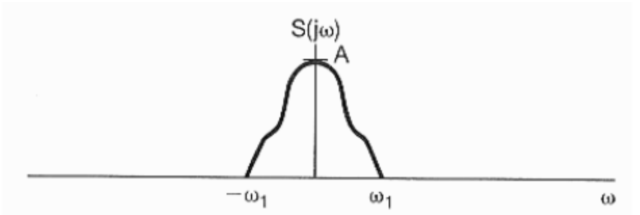
$$= (\tfrac{1}{2\pi})\underbrace{(\tfrac{1}{2\pi})\int_{-\infty}^{+\infty}\underbrace{\left[\int_{-\infty}^{+\infty} X(j\phi)Y(j\omega - j\phi)d\phi\right]}_{\text{convolution}} e^{j\omega t}d\omega}_{\text{inverse Fourier transform}}$$

Multiplication in time corresponds to convolution in frequency.

Multiply one signal by another in time is referred to as amplitude modulation.

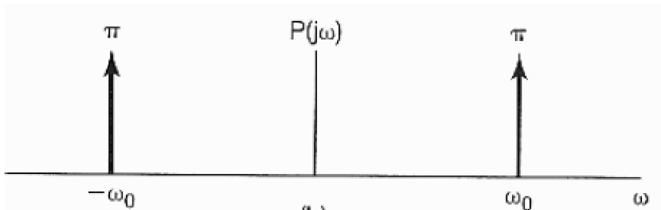
- Example 4.21 Amplitude Modulation (AM)

$$s(t) \xleftrightarrow{\mathcal{F}} S(j\omega)$$
$$p(t) \xleftrightarrow{\mathcal{F}} P(j\omega)$$
$$r(t) = s(t)p(t)$$



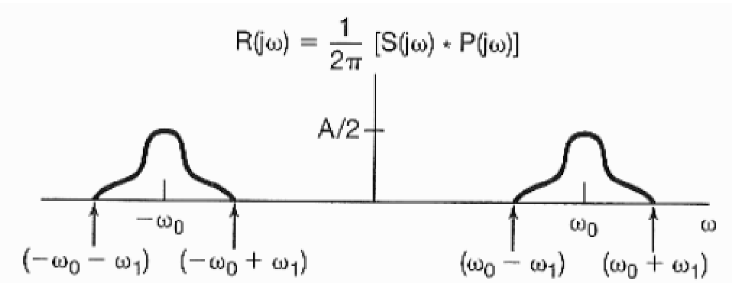
(a)

A band-limited signal



(b)

$$p(t) = \cos(\omega_0 t)$$
$$P(j\omega) = \pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0)$$



(c)

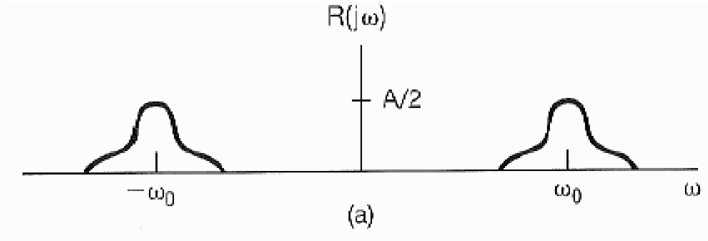
$$R(j\omega) = \frac{1}{2\pi} S(j\omega) * P(j\omega)$$
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} S(j\theta)P(j(\omega-\theta))d\theta$$
$$= \frac{1}{2}S(j(\omega-\omega_0)) + \frac{1}{2}S(j(\omega+\omega_0))$$

- Example 4.22 Demoludation

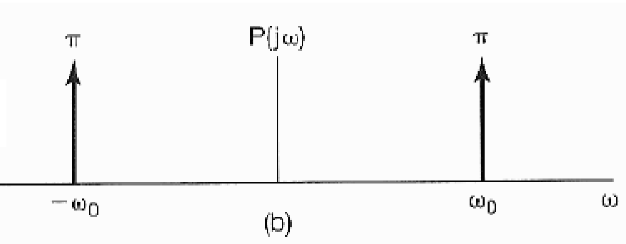
$$r(t) \overset{\mathcal{F}}{\longleftrightarrow} R(j\omega)$$

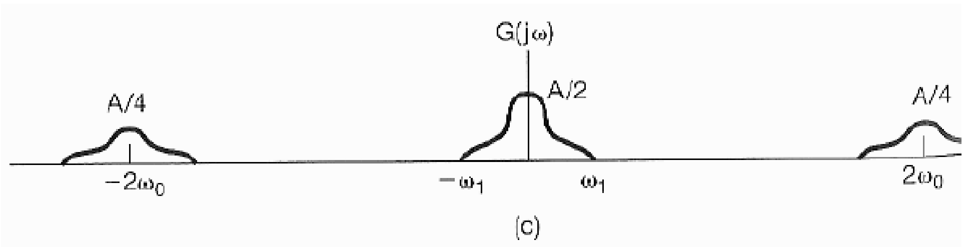
$$p(t) \overset{\mathcal{F}}{\longleftrightarrow} P(j\omega)$$

$$g(t) = r(t)p(t)$$



$$p(t) = \cos(\omega_0 t)$$





$$G(j\omega) = \frac{1}{2\pi}[R(j\omega) * P(j\omega)]$$