Signals & Systems

Spring 2019

https://sites.google.com/site/ntusands/

https://ceiba.ntu.edu.tw/1072EE2011_04

Yu-Chiang Frank Wang 王鈺強, Associate Professor

Dept. Electrical Engineering, National Taiwan University

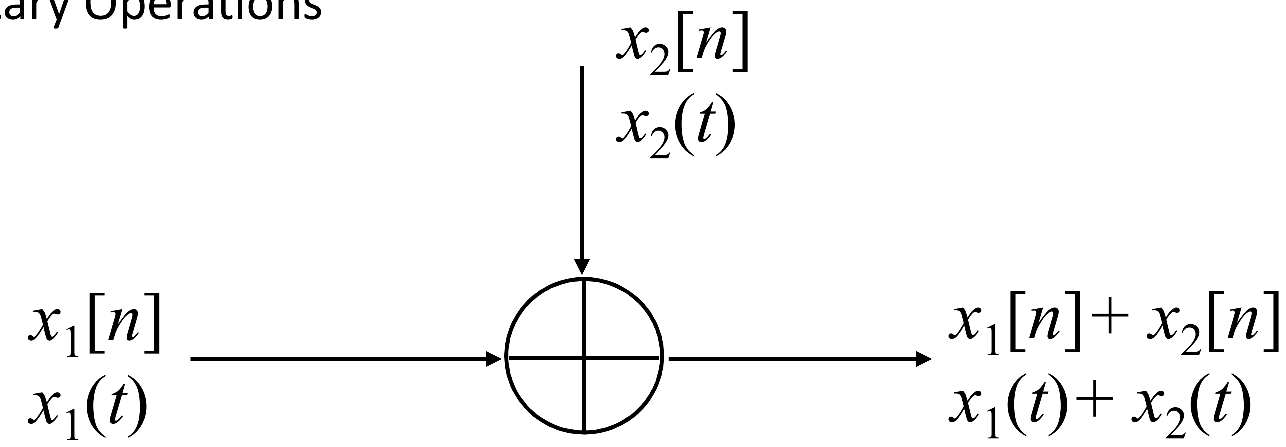
2019/03/11 & 14

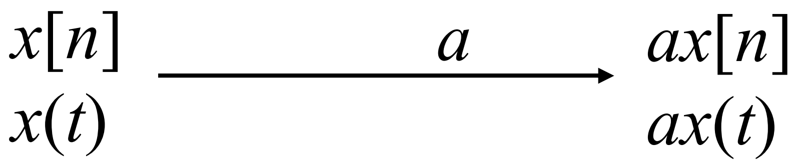
Chapter 2 Linear Time Invariant Systems

- Sec. 2.1 Discrete-time LTI Systems: The Convolution Sum
- Sec. 2.2 Continuous-time LTI Systems: The Convolution Integral
- Sec. 2.3 Properties of Linear Time-invariant Systems
- Sec. 2.4 Causal LTI Systems Described by Differential and Difference Equations
- Sec. 2.5 Singularity Functions

Block Diagram Representation

- Elementary Operations





Block Diagram Representation

- Elementary Operations (cont’d)

$$x[n] \longrightarrow \boxed{\mathbf{D}} \longrightarrow x[n\text{-}1]$$

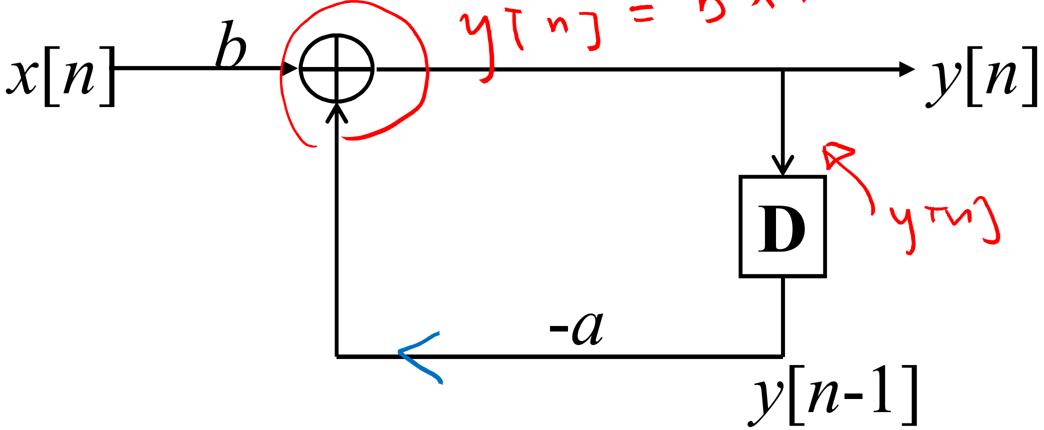
$$x(t) \longrightarrow \boxed{\frac{d}{dt}} \longrightarrow \frac{dx(t)}{dt}$$

$$x(t) \longrightarrow \boxed{\int} \longrightarrow \int_{-\infty}^{t} x(\tau)\, d\tau$$

Block Diagram Representation

- Example: $y[n] + ay[n-1] = bx[n]$

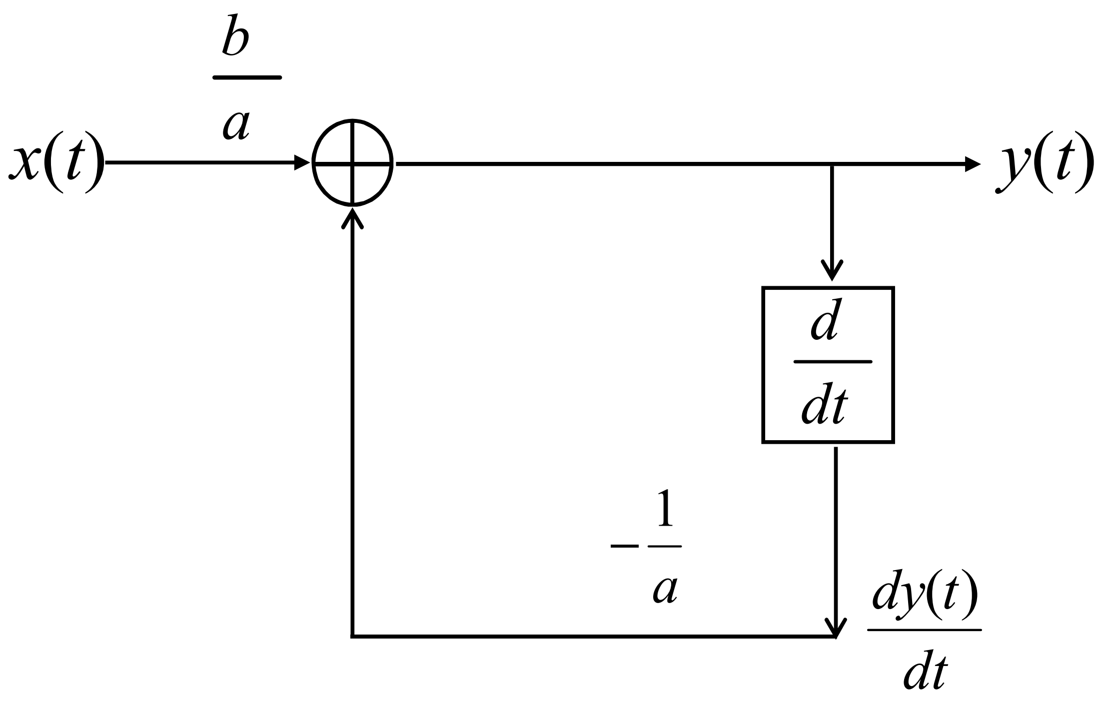
$y[n] = bx[n] + [-ay[n-1]]$



- Note that, this systems observes feedback (i.e., with memory). Initial value of the memory element = initial condition of the systems
- Initial rest condition: initial value in the memory element is zero.

Block Diagram Representation

- CT Example: $$\frac{dy(t)}{dt} + ay(t) = bx(t)$$



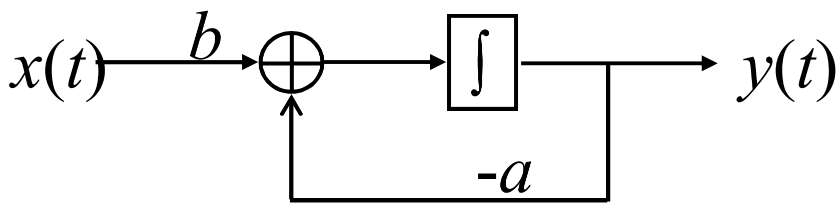
Block Diagram Representation

- CT Example: $$\frac{dy(t)}{dt} + ay(t) = bx(t)$$

- Expressed by integrator, assuming initially at rest

$$y(t) = \int_{-\infty}^{t} [bx(\tau) - ay(\tau)]d\tau$$

$$y(t) = y(t_0) + \int_{t_0}^{t} [bx(\tau) - ay(\tau)]d\tau$$



- The integrator represents the memory element.

HW #1 Prob. 2

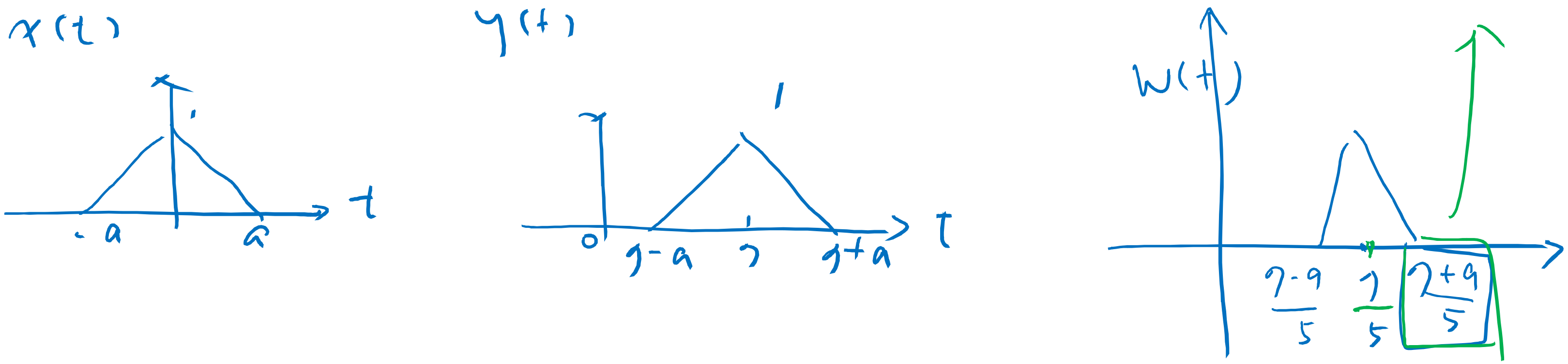
- H: $y(t) = x(t-7)$, G: $y(t) = x(5t)$
- Determine H^{-1}, G^{-1}, and F^{-1} (note that F is the cascade of H and G)

$x(t) \rightarrow \boxed{H} \xrightarrow{y(t)} \boxed{G} \rightarrow w(t)$

$= y(5t)$

$x(t) \rightarrow \boxed{F} \rightarrow w(t)$

$= x(5t-7)$

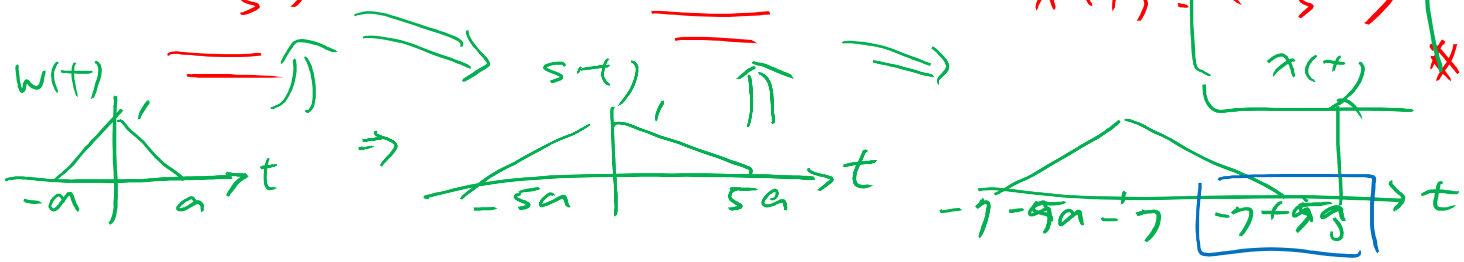


$w(t) \rightarrow \boxed{F^{-1}} \rightarrow x(t)$

$w(t) \rightarrow \boxed{G^{-1}} \xrightarrow{z(t)} \boxed{H^{-1}} \rightarrow x(t)$

S(t)

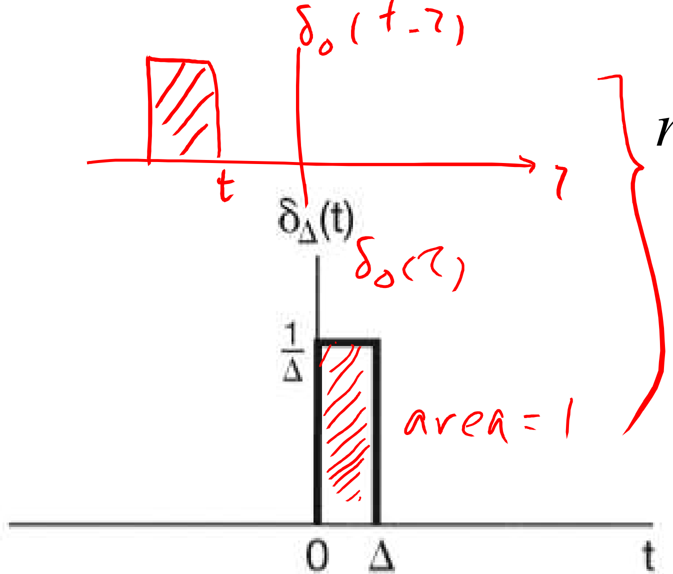
$z(t) = w\left(\frac{t}{5}\right)$ $= z(t+7)$ $\rightarrow$ $x(t) = w\left(\frac{t+7}{5}\right)$



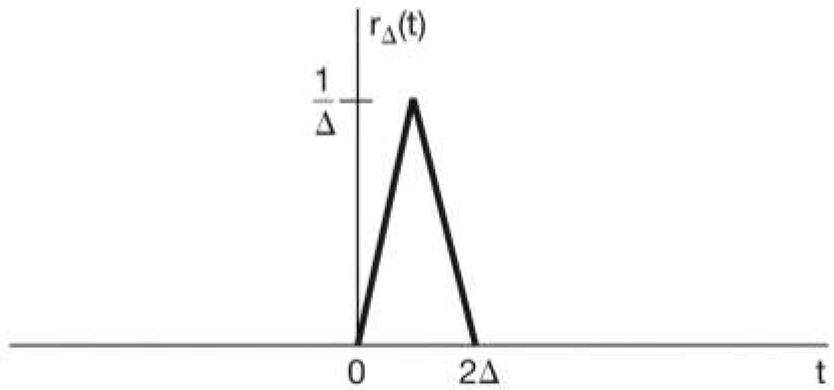
2.5 Singularity Functions: Properties of CT unit impulse

- **2.5.1 The Unit Impulse as an Idealized Short Pulse**
  - There is no explicit form of a unit impulse.
  - Instead, we can say some "functions" behave like a unit impulse.

$x(t) = x(t) * \delta(t)$ when $x(t) = \delta(t)$ $\quad \delta(t) = \delta(t) * \delta(t)$



$$r_\Delta(t) = \delta_\Delta(t) * \delta_\Delta(t)$$

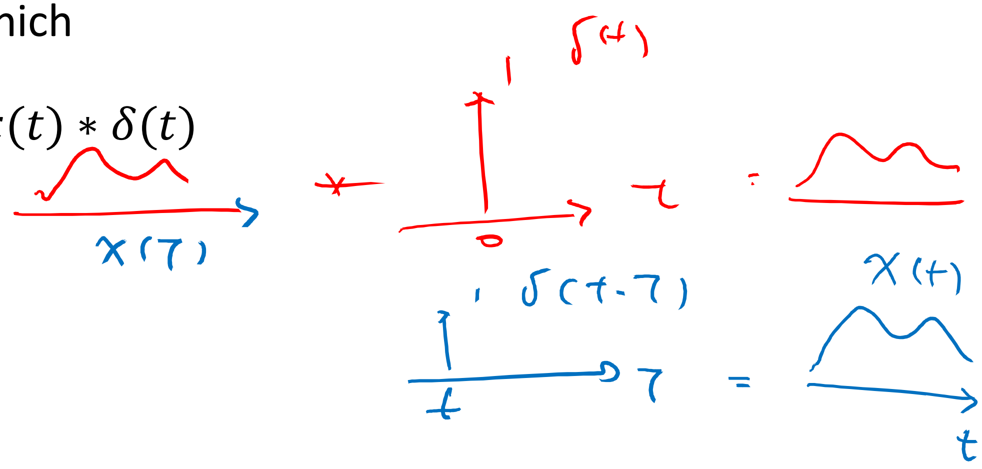


$\delta_\Delta(t)\big|_{\Delta\to 0}$ and $r_\Delta(t)\big|_{\Delta\to 0}$ can all be viewed as a unit impulse.

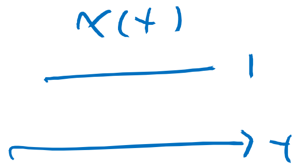
2.5 Singularity Functions: Properties of CT unit impulse

- **2.5.2 Defining the Unit Impulse through Convolution**
  - We define $\delta(t)$ as the signal for which

$$x(t) = x(t) * \delta(t)$$



is satisfied.



Let $x(t) = 1$,

$$1 = x(t) = x(t) * \delta(t) = \delta(t) * x(t)$$

$$= \int_{-\infty}^{+\infty} \delta(\tau) x(\overset{1}{\cancel{t-\tau}}) d\tau = \int_{-\infty}^{+\infty} \delta(\tau) d\tau$$

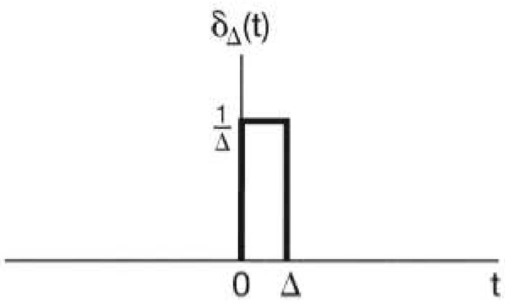
That is, the unit impulse has unit area

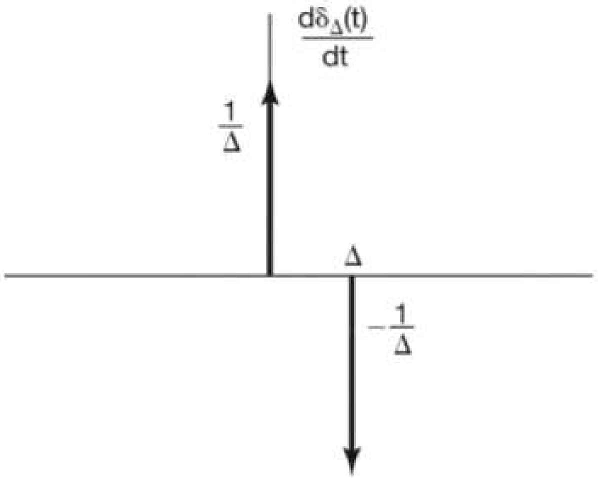
2.5 Singularity Functions: Properties of CT unit impulse

- **2.5.3 Unit Doublets and Other Singularity Functions**
    - Define

$$u_1(t) = \frac{d}{dt}\delta(t)$$

$$\frac{d}{dt}x(t) = x(t) * u_1(t)$$





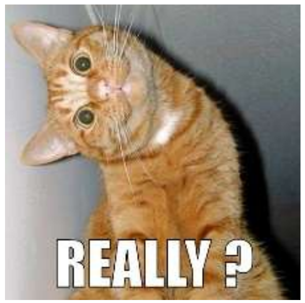
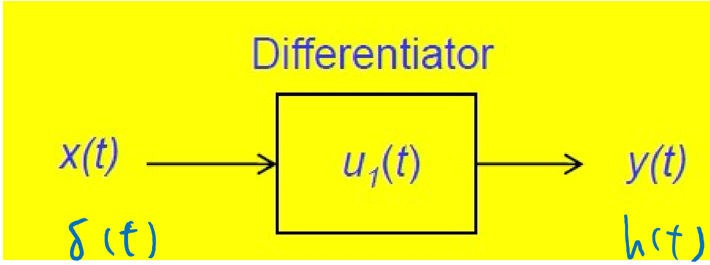


Figure 2.36 The derivative $d\delta_\Delta(t)/dt$ of the short rectangular pulse $\delta_\Delta(t)$ of Figure 1.34.

2.5 Singularity Functions: Properties of CT unit impulse



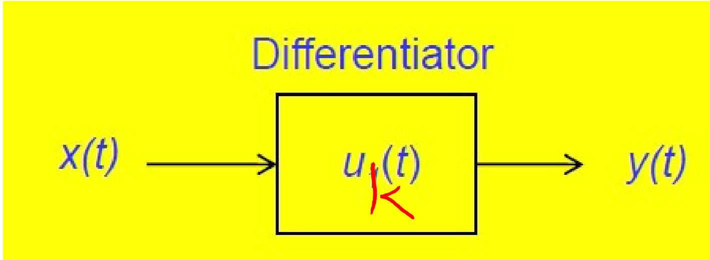
- **2.5.3 Unit Doublets and Other Singularity Functions**
  - Consider the system $\boxed{y(t) = \frac{d}{dt}x(t).}$
  - The unit impulse response of the system is the derivative of the unit impulse, which is called the unit doublet $u_1(t)$, which is defined as:

$$u_1(t) = \frac{d}{dt}\delta(t).$$

  - From the convolution representation of LTI systems, we have

$$\underbrace{\frac{d}{dt}x(t)}_{\text{output}} = \underbrace{\underbrace{x(t)}_{\text{input}} * u_1(t).}_{\text{LTI convolution}}$$

2.5 Singularity Functions: Properties of CT unit impulse



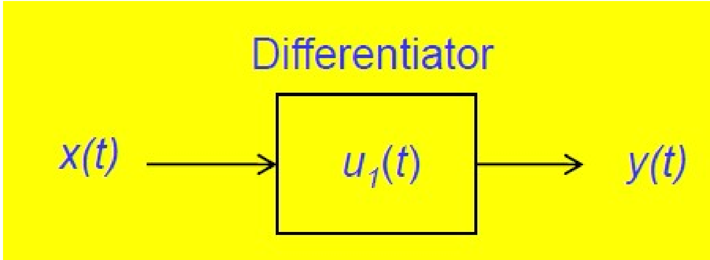
- **2.5.3 Unit Doublets and Other Singularity Functions**
  - Similarly, we may define $\frac{d^2}{dt^2}x(t) = x(t) * u_2(t)$.
  - We have $\frac{d^2}{dt^2}x(t) = \frac{d}{dt}\left(\frac{d}{dt}x(t)\right) = (x(t) * u_1(t)) * u_1(t)$.
  - Therefore, we observe

$$u_2(t) = u_1(t) * u_1(t).$$

  - In general, for the kth derivative of $\delta(t)$, we have

$$u_k(t) = \underbrace{u_1(t) * \cdots * u_1(t)}_{\text{k times}}, \mathrm{k} > 0.$$

2.5 Singularity Functions: Properties of CT unit impulse



- **2.5.3 Unit Doublets and Other Singularity Functions**
  - Consider $x(t) = 1$, we have

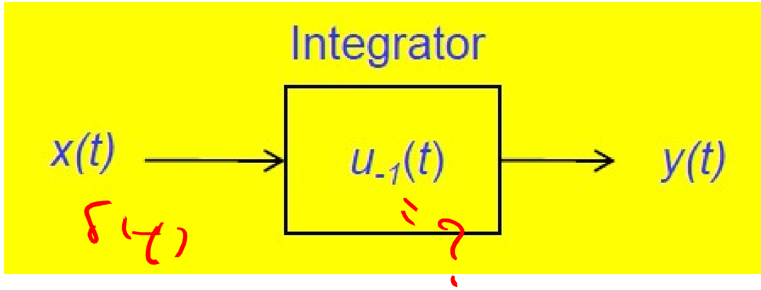
$$0 = \frac{dx(t)}{dt} = x(t) * u_1(t)$$

$$= \int_{-\infty}^{\infty} u_1(\tau) x(t-\tau) d\tau$$

$$= \int_{-\infty}^{\infty} u_1(\tau) d\tau$$

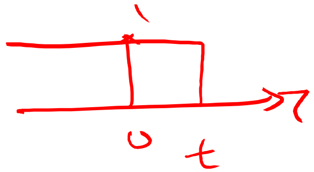
  - That is, the unit doublet has zero area.

2.5.3 Unit Doublets and Other Singularity Functions



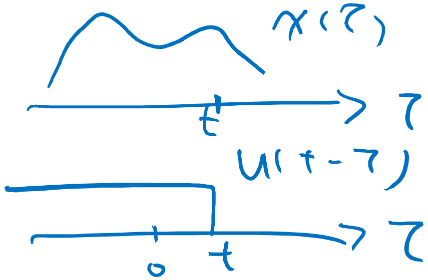
- **Integral of Unit Impulse**
  - Consider an integrator: $y(t) = \int_{-\infty}^{t} x(\tau)d\tau$
  - By definition of integral, the impulse response of an integrator is the unit step.

$$u_{-1}(t) \triangleq \int_{-\infty}^{t} \delta(\tau)d\tau = u(t)$$



  - Thus, we have the following operational definition of $u(t)$.

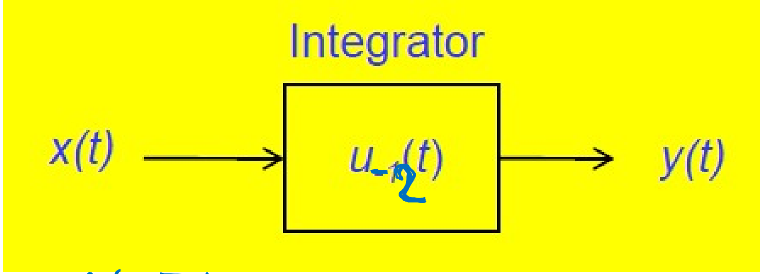
$$x(t) * u(t) = \int_{-\infty}^{t} x(\tau)d\tau$$



||

= y(t)

2.5.3 Unit Doublets and Other Singularity Functions



- **Integral of Unit Impulse**
  - Similarly, we observe

$= u_{-2}(t)$

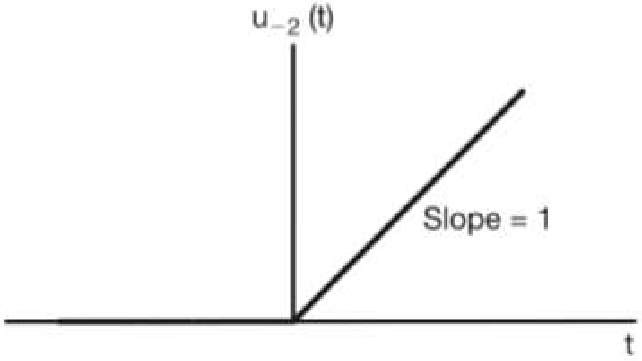
$$u_{-2}(t) = u(t) * u(t) = \int_{-\infty}^{t} u(\tau)d\tau$$

  - Since $u(t)$ equals 0 for $t < 0$ and 1 for $t \geq 0$, it follows that

$$u_{-2}(t) = \underline{tu(t)}$$

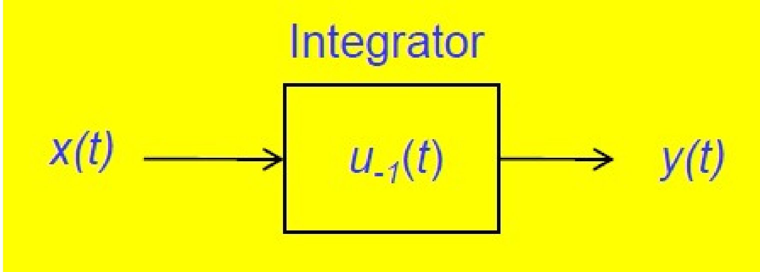
Rectified linear unit

(ReLU)



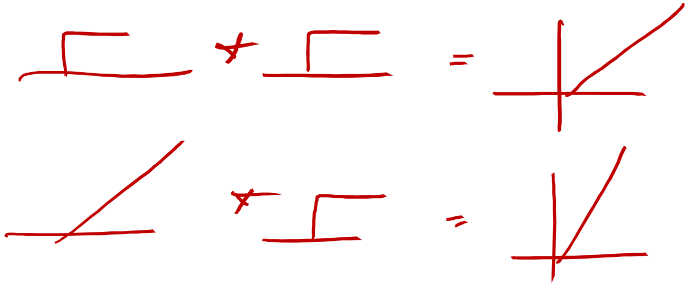
unit ramp function

2.5.3 Unit Doublets and Other Singularity Functions



- **Integral of Unit Impulse**
  - Moreover

$$x(t)*u_{-2}(t)=x(t)*u(t)*u(t)$$



$$=\left(\int_{-\infty}^{t}x(\sigma)d\sigma\right)*u(t)$$

$$=\int_{-\infty}^{t}\left(\int_{-\infty}^{\tau}x(\sigma)d\sigma\right)d\tau$$

  - In general,

$$u_{-k}=u(t)*\cdots*u(t)=\int_{-\infty}^{t}u_{-(k-1)}(\tau)d\tau$$

$$u_{-k}(t)=\frac{t^{k-1}}{(k-1)!}u(t)$$

2.5 Singularity Functions: Properties of CT unit impulse

- **Summary**

$$\delta(t) = u_0(t)$$

$$u(t) = u_{-1}(t)$$

$$u_k(t)\begin{cases} k > 0, & \text{Impulse response of a cascade of } k \text{ differentiators} \\ k < 0, & \text{Impulse response of a cascade of } |k| \text{ integrators} \end{cases}$$

$$I \rightarrow \boxed{D} \rightarrow I$$

$$u(t) * u_1(t) = \delta(t) \qquad \text{or} \quad u_{-1}(t) * u_1(t) = u_0(t)$$

$$\Rightarrow u_k(t) * u_r(t) = u_{k+r}(t)$$

2.5 Singularity Functions: Properties of CT unit impulse

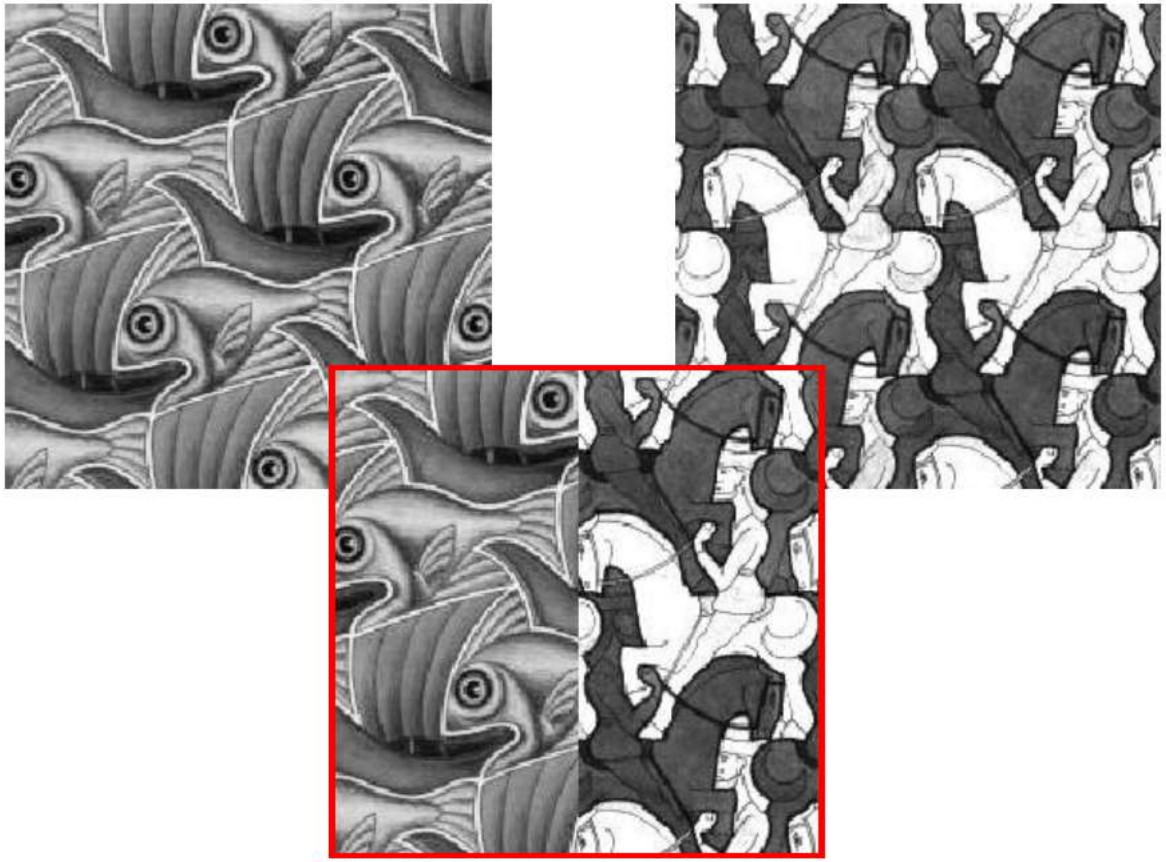
Property or Definition	Formula
(1) Integration	$\int_{-\infty}^{\infty} \delta(t)dt = 1$
(2) Relation with the unit step function	$\int_{-\infty}^{t} \delta(\tau)d\tau = u(t), \quad \frac{d}{dt}u(t) = \delta(\tau)$
(3) Convolution	$x(t) * \delta(t) = x(t)$
(4) Auto convolution	$\delta(t) * \delta(t) = \delta(t), \quad \delta(t) * \delta(t) * .... * \delta(t) = \delta(t)$
(5) Sifting (I)	$\int_{a}^{b} f(t)\delta(t-t_0)dt = f(t_0)$ if $a < t_0 < b$
(6) Sifting (II)	$f(t)\delta(t-t_0) = f(t_0)\delta(t-t_0)$
(7) Unit doublet $u_1(t)$	$u_1(t) = \frac{d}{dt}\delta(t)$ $x(t) * u_1(t) = \frac{d}{dt}x(t)$
(8) $u_k(t)$ (k is a positive integer)	$u_k(t) = \underbrace{u_1(t) * \cdots * u_1(t)}_{k \text{ times}} = \frac{d^k}{dt^k}\delta(t)$ $x(t) * u_k(t) = \frac{d^k}{dt^k}x(t)$
(9) $u_{-1}(t)$	$u_{-1}(t) = u(t),$
(10) $u_{-k}(t)$ (k is a positive integer) When $k = 2$, it is called a unit ramp function	$u_{-k}(t) = \underbrace{u(t) * \cdots * u(t)}_{k \text{ times}} = \frac{t^{k-1}}{(k-1)!}u(t),$ $x(t) * u_{-k}(t) = \int_{-\infty}^{t}\int_{-\infty}^{\tau_{k-1}} \cdots \int_{-\infty}^{\tau_2}\left(\int_{-\infty}^{\tau_1} x(\sigma)d\sigma\right)d\tau_1 d\tau_2 \ldots d\tau_{k-1}.$ (k times of integration)

Chapter 3 Fourier Series Representations of Periodic Signals

- 3.1/3.2 Preliminary & Response of LTI Systems to Complex Exponential Signals
- 3.3 Fourier Series Representation of CT Periodic Signals
- 3.4 Convergence of the Fourier Series
- 3.5 Properties of CT Fourier Series
- 3.6 Fourier Series Representation of DT Periodic Signals
- 3.7 Properties of DT Fourier Series
- 3.8 Fourier Series and LTI Systems
- 3.9 Filtering
- 3.10 Examples of CT Filters Described by Differential Equations
- 3.11 Examples of DT Filters Described by Difference Equations

Revisit:
Representation of Signals in terms of Basis Functions

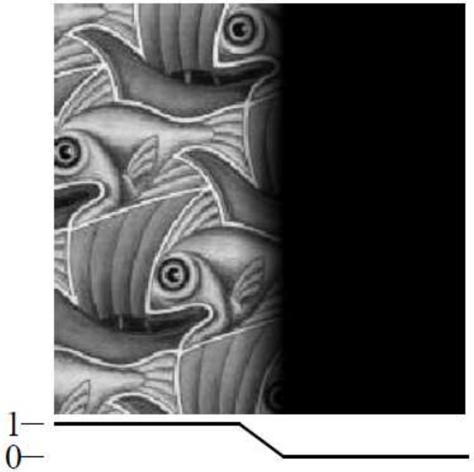
- An image-based example: image blending



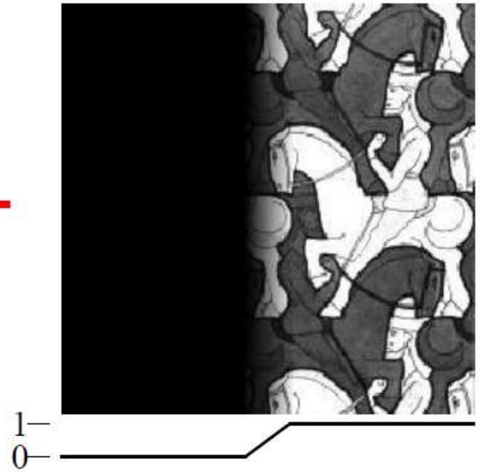
Revisit:
Representation of Signals in terms of Basis Functions

- An image-based example: alpha image blending/feathering

$\alpha =$



+



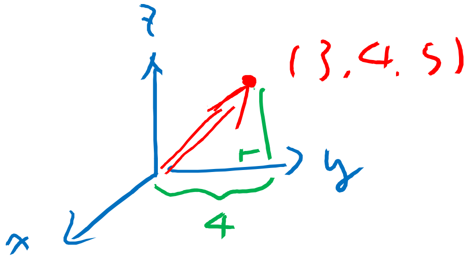
$= 1 - \alpha$

=

$$I_{blend} = \underbrace{\alpha I_{left}}_{\text{basis}} + \underbrace{(1\text{-}\alpha) I_{right}}_{\text{basis}}$$

Revisit:
Representation of Signals in terms of Basis Functions

- Detailed Remarks:



$$\begin{bmatrix}3\\4\\5\end{bmatrix} = \overset{\text{coeff}}{3}\underbrace{\begin{bmatrix}1\\0\\0\end{bmatrix}}_{= x\text{-axis}} + \overset{\text{coeff}}{4}\underbrace{\begin{bmatrix}0\\1\\0\end{bmatrix}}_{= y} + \overset{\text{coeff}}{5}\underbrace{\begin{bmatrix}0\\0\\1\end{bmatrix}}_{= z}$$

basis functions

$$\underline{a} = \sum_{i=1}^{d} \alpha_i \underline{u}_i \, , \quad \underline{a}, \underline{u}_i \in \mathbb{R}^d$$

- Will see more in Ch. 3 Fourier Series, etc.

Sect. 3.2 The Response of LTI Systems to Complex Exponential Signals

$x(t) \rightarrow \boxed{h(t)} \rightarrow y(t) = x(t) * h(t)$

- Objective
  - To represent signals as linear combinations of basic (or basis) functions
- Requirements
  - The set of basic functions can be used to construct a broad/useful range of signals.
  - The response of an LTI system to each basic signal should be *sufficiently simple* in structure, so that the response of the system to any signal can be constructed as a linear combination of such basis signals.
- Remarks
  - An input signal for which the system output is the same signal with only a change in the amplitude is referred to as an eigenfunction of the system.
  - For each eigenfunction, the associated amplitude/coefficient is referred to as the eigenvalue of the system.

"$\underset{d \times d}{\underline{A}}$" $\underset{d \times 1}{\underline{e}_i} = \underset{1 \times 1}{\lambda_i} \underset{d \times 1}{\underline{e}_i}$ $\begin{cases} \{\underline{e}_1, \ldots, \underline{e}_d\} \\ \{\lambda_1, \ldots, \lambda_d\} \end{cases}$

Sect. 3.2 The Response of LTI Systems to Complex Exponential Signals (cont'd)

basic functions

- Eigenfunctions of LTI systems: e^{st} and Z^n

Let $x(t) = e^{st}$

$$y(t) = \int_{-\infty}^{+\infty} h(\tau)x(t-\tau)d\tau$$
$$= \int_{-\infty}^{+\infty} h(\tau)e^{s(t-\tau)}d\tau$$
$$= e^{st} \underbrace{\int_{-\infty}^{+\infty} h(\tau)e^{-s\tau}d\tau}_{H(s)}$$

$$\Rightarrow y(t) = H(s)e^{st}$$

Let $x[n] = z^n$

$$y[n] = \sum_{k=-\infty}^{+\infty} h[k]x[n-k]$$
$$= \sum_{k=-\infty}^{+\infty} h[k]z^{n-k}$$
$$= z^n \underbrace{\sum_{k=-\infty}^{+\infty} h[k]z^{-k}}_{H(z)}$$

$$\Rightarrow y[n] = H(z)z^n$$

$$e^{s_k t} \rightarrow \boxed{\text{LTI}} \rightarrow H(s_k)e^{s_k t}$$

Sect. 3.2 The Response of LTI Systems to Complex Exponential Signals (cont'd)

- Response of LTI Systems to a Linear Combination of Exponential Signals

$$x(t) = a_1 e^{s_1 t} + a_2 e^{s_2 t} + a_3 e^{s_3 t}$$

$$a_1 e^{s_1 t} \longrightarrow a_1 H(s_1) e^{s_1 t}$$

$$a_2 e^{s_2 t} \longrightarrow a_2 H(s_2) e^{s_2 t}$$

$$a_3 e^{s_3 t} \longrightarrow a_3 H(s_3) e^{s_3 t}$$

By superposition property of LTI systems

$$y(t) = a_1 H(s_1) e^{s_1 t} + a_2 H(s_2) e^{s_2 t} + a_3 H(s_3) e^{s_3 t}$$

$$\Rightarrow y(t) = \sum_k a_k H(s_k) e^{s_k t}$$

Likewise, $x[n] = \sum_k a_k z_k{}^n \Rightarrow y[n] = \sum_k a_k H(z_k) z_k{}^n$

Sect. 3.2 The Response of LTI Systems to Complex Exponential Signals (cont'd)

$x(\tau)$ $\delta(-\tau+3)$

$x(t) \to \boxed{h(t)} \to y(t) = x(t-3)$

- Example 3.1 $y(t) = x(t-3)$ is an LTI system.
  - What is the impulse response of the system? $\boxed{\delta(t-3)}$ $\Rightarrow$ $\delta(\tau-3)$
  - When $x(t) = e^{j2t}$, $y(t) = e^{j2(t-3)} = e^{-j6}e^{j2t}$ with eigenvalue e^{-j6}.
  - The eigenvalue for $\boxed{e^{jst}}$ is $H(s) = \int_{-\infty}^{\infty} \delta(\tau-3)\, e^{-s\tau} d\tau = e^{-3s}$. $\to$ $e^{js(t-\tau)}$
  - When $x(t) = \cos(4t) + \cos(7t)$, we have

$$x(t) = \tfrac{1}{2}e^{j4} + \tfrac{1}{2}e^{-j4} + \tfrac{1}{2}e^{j7t} + \tfrac{1}{2}e^{-j7t}$$
$$y(t) = \tfrac{1}{2}e^{-j12}e^{j4t} + \tfrac{1}{2}e^{j12}e^{-j4t} + \tfrac{1}{2}e^{-j21}e^{j7t} + \tfrac{1}{2}e^{j21}e^{-j7t}$$
$$y(t) = \cos\big(4(t-3)\big) + \cos(7(t-3)).$$

$e^{j\theta} = \cos\theta + j\sin\theta$

Sect. 3.3 Fourier Series Representation of CT Periodic Signals

- A periodic signal $x(t)$ with period T can be represented by (aka the Fourier series representation of $x(t)$):

$$\boxed{x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}}, \quad \text{where } \omega_0 = \frac{2\pi}{T}$$

- This signal is periodic with period T because each set of harmonically related complex exponentials

$$\phi_k(t) = e^{jk\omega_0 t} = e^{jk(\frac{2\pi}{T})t}, \quad k = 0, \pm 1, \pm 2 \ldots$$

has a fundamental frequency $k\omega_0$ and is periodic with $T = 2\pi/\omega_0$.

$k = \pm 1$: the first harmonic components or, the fundamental components
$k = \pm 2$: the second harmonic components

Sect. 3.3 Fourier Series Representation of CT Periodic Signals

- Example 3.2

$$x(t) = \sum_{k=-3}^{+3} a_k e^{jk(2\pi)t}$$

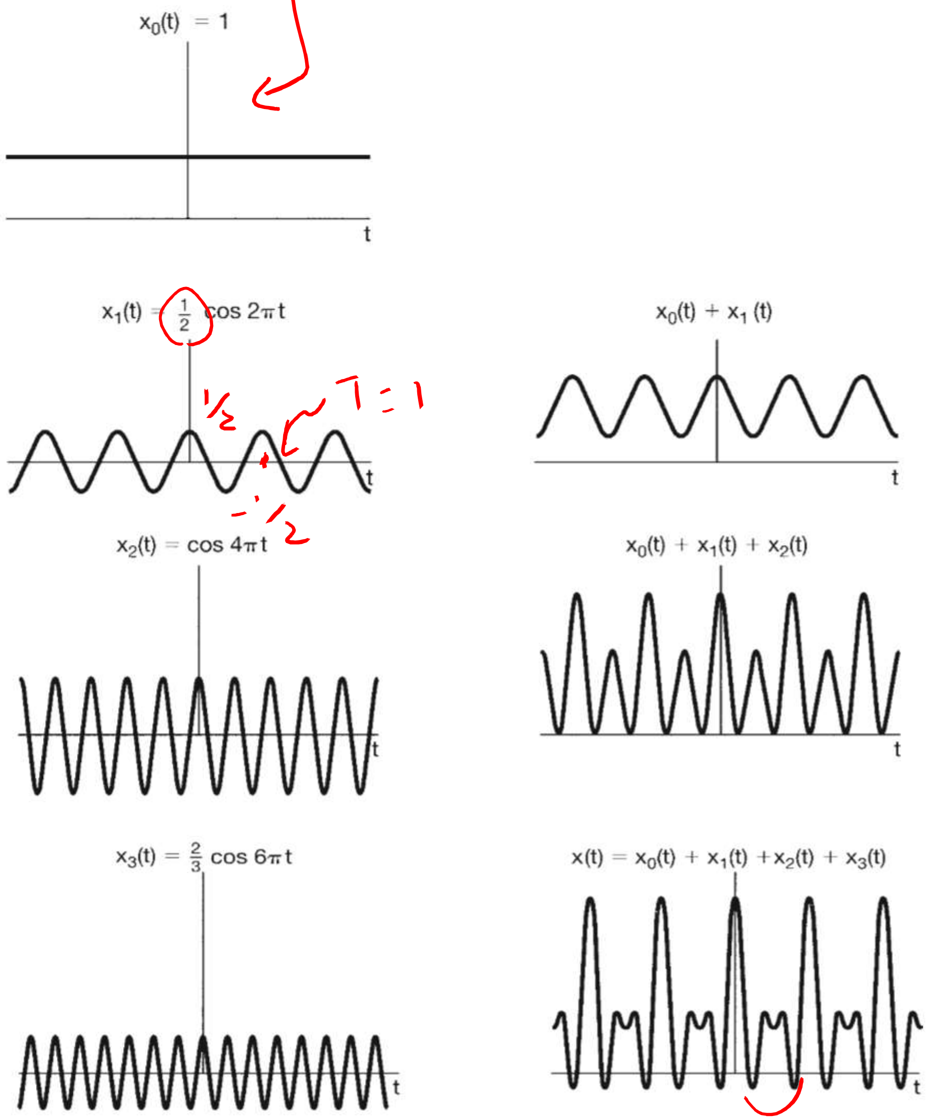
$$a_0 = 1$$
$$a_1 = a_{-1} = \tfrac{1}{4}$$
$$a_2 = a_{-2} = \tfrac{1}{2}$$
$$a_3 = a_{-3} = \tfrac{1}{3}$$

$$\Rightarrow \; x(t) = 1 + \frac{1}{4}(e^{j2\pi t} + e^{-j2\pi t}) + \frac{1}{2}(e^{j4\pi t} + e^{-j4\pi t}) + \frac{1}{3}(e^{j6\pi t} + e^{-j6\pi t})$$

$$\Rightarrow \; x(t) = 1 + \frac{1}{2}\cos 2\pi t + \cos 4\pi t + \frac{2}{3}\cos 6\pi t$$

- Example 3.2 $\Rightarrow x(t) = 1 + \frac{1}{2}\cos 2\pi t + \cos 4\pi t + \frac{2}{3}\cos 6\pi t$

k = 0



period = T = 1

$$x(t) = \sum_{k=-3}^{+3} a_k e^{jk(2\pi)t}$$

Sect. 3.3 Fourier Series Representation of CT Periodic Signals

- Determining the Fourier Series Coefficients:

$$x(t)=\sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}, \quad \text{where } \omega_0=\frac{2\pi}{T}$$

$\Rightarrow a_n = ?$

$$x(t)\boxed{e^{-jn\omega_0 t}}=\sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}\boxed{e^{-jn\omega_0 t}}$$

$$\int_0^T x(t)e^{-jn\omega_0 t}dt=\int_0^T \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t} e^{-jn\omega_0 t}dt=\sum_{k=-\infty}^{+\infty} a_k\left[\int_0^T e^{j(k-n)\omega_0 t}dt\right]$$

$$\int_0^T e^{j(k-n)\omega_0 t}dt=\int_0^T \cos((k-n)\omega_0 t)dt + j\int_0^T \sin((k-n)\omega_0 t)dt$$

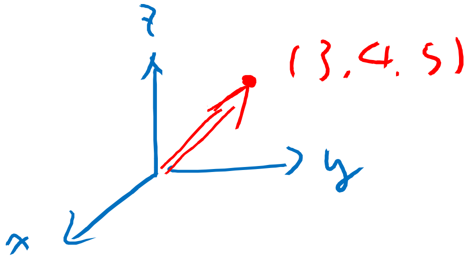
$$=\int_0^T \cos((k-n)\omega_0 t)dt + \quad j0$$

$$=\begin{cases} T, & k=n \\ 0, & k\neq n \end{cases}$$

$$\int_0^T x(t)e^{-jn\omega_0 t}dt = a_n T \quad \Rightarrow \quad a_n=\frac{1}{T}\int_T x(t)e^{-jn\omega_0 t}dt$$

Revisit:
Representation of Signals in terms of Basis Functions

- Detailed Remarks:



coeff

$$\begin{bmatrix}3\\4\\5\end{bmatrix} = 3\begin{bmatrix}1\\0\\0\end{bmatrix} + 4\begin{bmatrix}0\\1\\0\end{bmatrix} + 5\begin{bmatrix}0\\0\\1\end{bmatrix}$$

$= x\text{-axis} \qquad = y \qquad = z$

basis functions

$$\underline{a} = \sum_{i=1}^{d} \alpha_i u_i \,, \qquad a, u_i \in \mathbb{R}^d$$

- Will see more in Ch. 3 Fourier Series, etc.

Sect. 3.3 Fourier Series Representation of CT Periodic Signals

- Determining the Fourier Series Coefficients
- In summary, we have:

$$\boxed{a_k = \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}\,dt}$$ Analysis Equation

$$\boxed{x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}}$$ Synthesis Equation

$$T = \frac{2\pi}{\omega_0}$$

- CT Fourier series pair: $x(t) \overset{FS}{\leftrightarrow} a_k$
- Fourier series coefficients or spectral coefficients of $x(t)$: $\{a_k\}$
- The DC component (or constant) of $x(t)$: $a_0 = \frac{1}{T}\int_T x(t)\,dt$

Sect. 3.3 Fourier Series Representation of CT Periodic Signals

- Fourier Series of Real Periodic Signals
  - If $x(t)$ is real, then $x^*(t) = x(t)$. Thus, we have:

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k \boxed{e^{jk\omega_0 t}}$$

$$= x^*(t) = \left(\sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}\right)^*$$

$$= \sum_{k=-\infty}^{+\infty} a_k^* e^{-jk\omega_0 t} = \sum_{k=-\infty}^{+\infty} a_{-k}^* e^{jk\omega_0 t} \quad \Rightarrow \quad a_{-k}^* = a_k \quad \text{or} \quad \boxed{a_k^* = a_{-k}}$$

$(ac - bd) + j(ad + bc)$

$(ac - bd) - j(ad + bc)$

  - Therefore, we have $x(t)$ as:

$x(t)$: real

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t} = a_0 + \sum_{k=1}^{+\infty}\left[a_k e^{jk\omega_0 t} + a_{-k} e^{-jk\omega_0 t}\right]$$

$$= a_0 + \sum_{k=1}^{+\infty}\left[a_k e^{jk\omega_0 t} + \boxed{a_k^* e^{-jk\omega_0 t}}\right]$$

$$= a_0 + \sum_{k=1}^{+\infty} 2\,\mathrm{Re}\left[a_k e^{jk\omega_0 t}\right] \qquad \textbf{Why??}$$

$2[ac - bd]$

$a_k = a + bj$

$e^{\pm jk\omega_0 t} = c \pm dj$

$a_k^* = a - bj$

Sect. 3.3 Fourier Series Representation of CT Periodic Signals

- Fourier Series of Real Periodic Signals

$$\text{Let } a_k = A_k e^{j\theta_k}$$

$$e^{j\theta} = \cos(\theta) + j\sin(\theta)$$

$$\Rightarrow x(t) = a_0 + \sum_{k=1}^{+\infty} 2\,\mathrm{Re}\left\{A_k e^{j\theta_k} e^{jk\omega_0 t}\right\}$$

$$= a_0 + \sum_{k=1}^{+\infty} 2\,\mathrm{Re}\left\{A_k e^{j(k\omega_0 t+\theta_k)}\right\}$$

$$= a_0 + 2\sum_{k=1}^{+\infty} A_k \cos(k\omega_0 t + \theta_k)$$

This trigonometric form is commonly encountered.

$$\text{Let } a_k = B_k + jC_k$$

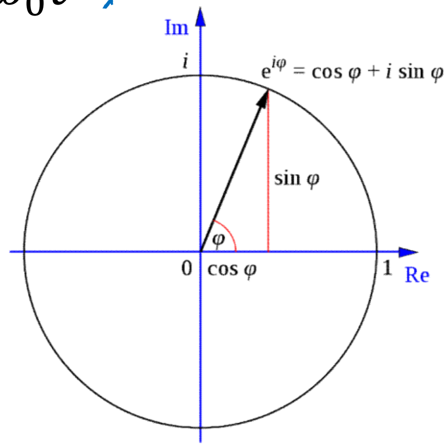
$$(a+jb)(c+jd) = (ac-bd) + j(ad+bc)$$

$$\Rightarrow x(t) = a_0 + 2\sum_{k=1}^{\infty}\left[B_k \cos(k\omega_0 t) - C_k \sin(k\omega_0 t)\right]$$

Sect. 3.3 Fourier Series Representation of CT Periodic Signals

$T = \frac{2\pi}{\omega_0}$

- Example 3.3 Compute the FS coefficients of $x(t) = \sin \omega_0 t$,
- Method #1: Use Eqn. 3.39 directly. $e^{jk\omega_0 t} \big|_{k=1}$
- Method #2: Use Euler's relation



$$\sin \omega_0 t = \frac{1}{2j}(e^{j\omega_0 t} - e^{-j\omega_0 t})$$

$$a_1 = \frac{1}{2j},\ a_{-1} = -\frac{1}{2j},\ \text{and } a_k = 0 \text{ for } k \neq \pm 1.$$

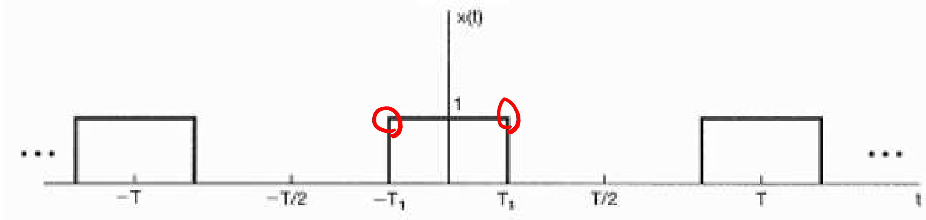
That is,

$$\sin \omega_0 t \xleftrightarrow{FS} \frac{1}{2j}(\delta(\omega - \omega_0) - \delta(\omega + \omega_0))$$

Sect. 3.3 Fourier Series Representation of CT Periodic Signals

- Example 3.5 FS of periodic square wave

$$x(t) = \begin{cases} 1, & |t| < T_1 \\ 0, & T_1 < |t| < T/2 \end{cases}$$



$$\omega_0 = \frac{2\pi}{T}$$

$$a_0 = \frac{1}{T}\int_{-T_1}^{T_1} dt = \frac{2T_1}{T}$$

$$a_k = \frac{1}{T}\int_{-T_1}^{T_1} e^{-jk\omega_0 t} dt = -\frac{1}{jk\omega_0 T} e^{-jk\omega_0 t}\Big|_{-T_1}^{T_1}$$

$$= \frac{2}{k\omega_0 T}\left[\frac{e^{jk\omega_0 T_1} - e^{-jk\omega_0 T_1}}{2j}\right]$$

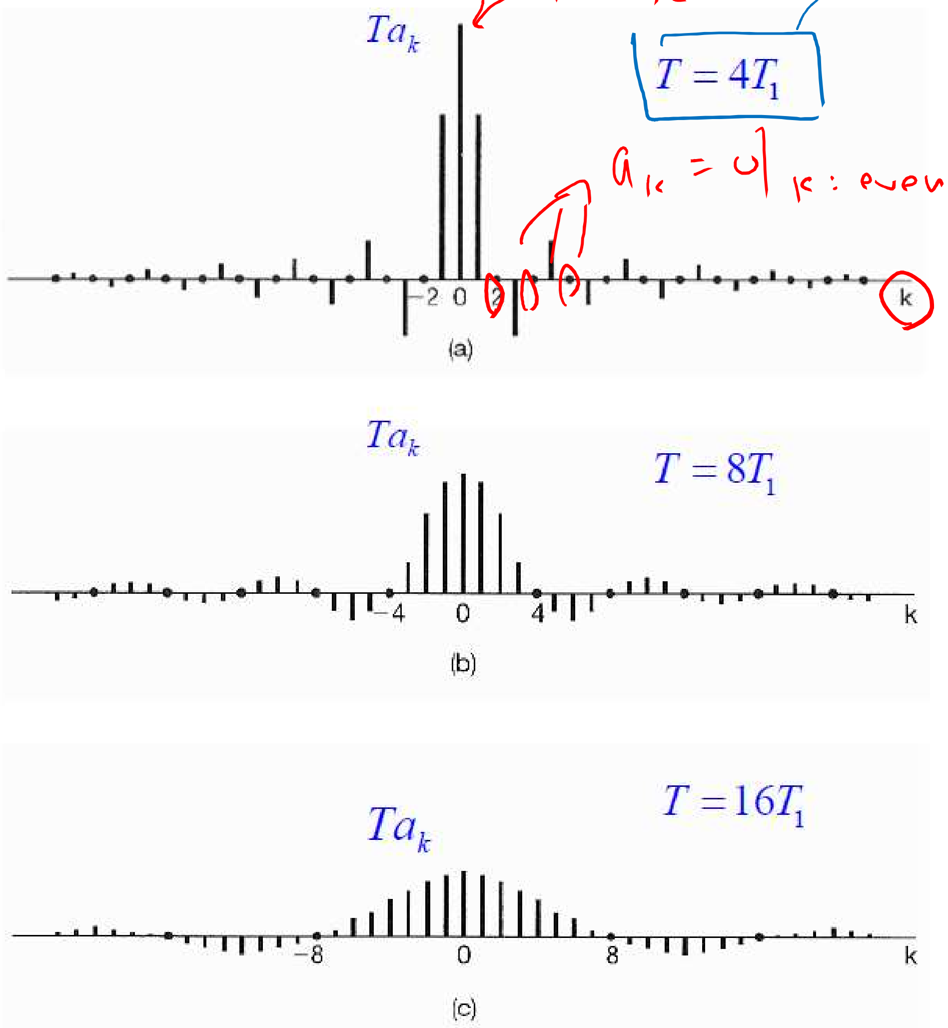
$$= \frac{2\sin(k\omega_0 T_1)}{k\omega_0 T} = \frac{\sin(k\omega_0 T_1)}{k\pi} = \frac{\sin\left(k\pi \frac{2T_1}{T}\right)}{k\pi} = ?$$

Sect. 3.3 Fourier Series Representation of CT Periodic Signals

- Example 3.5 (cont'd) What is the shape of the FS coefficients?

$a_0 = 1/2$

$a_k = 0 \quad k$: even #



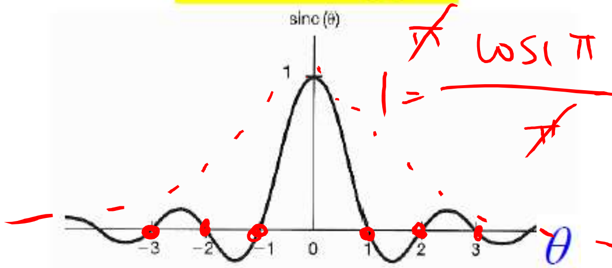
$$a_k = \frac{\sin\left(k\pi \frac{2T_1}{T}\right)}{k\pi}$$

$$= \frac{\sin(k\pi/2)}{k\pi}$$

$$\lim_{\theta \to 0} \Rightarrow \frac{d \sin(\pi\theta)}{d(\pi\theta)}$$

$$\mathrm{sinc}(\theta) = \frac{\sin(\pi\theta)}{\pi\theta}$$

$$1 = \frac{\cancel{\pi}\cos(\pi\theta)}{\cancel{\pi}}$$



Sect. 3.4 Convergence of the Fourier Series

- The FS exists if (1) $|a_k| < \infty$ and (2) the resulting infinite series $\sum_{k=-\infty}^{\infty} a_k e^{jk(2\pi/T)t}$ converges to the original signal $x(t)$.
- Fourier concluded that any periodic signal could be represented by a Fourier series.

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk(2\pi/T)t}, \quad a_k = \frac{1}{T}\int_T x(t)e^{-jk(2\pi/T)t}dt$$

- This is not quite true, but Fourier series indeed can represent an extremely large class of periodic signals.
- When does a periodic signal $x(t)$ have a Fourier series representation?

Sect. 3.4 Convergence of the Fourier Series (cont'd)

- To gain an understanding of the validity of FS representation, let's first examine the approximation of a signal by a finite series.
- Define $x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk(2\pi/T)t}$ and $e_N(t) = x(t) - \boxed{x_N(t)}$.

$\sum_{k=-N}^{N} a_k e^{jk\omega_0 t}$

- We want to minimize $E_N(t) = \int_T \ |e_N(t)|^2 dt$.
- From Problem 3.66, the error $E_N(t)$ is minimized if we choose

$$a_k = \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}dt$$

which is identical to the FS coefficient defined earlier.

- Thus, if $x(t)$ has a FS representation, the best approximation of $x(t)$ is obtained truncating its FS to the desired number of terms.
  - N increases, E_N decreases.
  - If the FS of $x(t)$ exists, then $E_N \rightarrow 0$ as $N \rightarrow \infty$.

Sect. 3.4 Convergence of the Fourier Series (cont'd)

- For its FS to exist, a periodic signal $x(t)$ must satisfy two different classes of conditions:

1. $x(t)$ has finite energy over a single period.

$$\int_T |x(t)|^2 \, dt < \infty \quad \Rightarrow \quad a_k = \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t} dt < \infty$$

2. The energy $E_N(t)$ converges to 0 as $N \to \infty$.

$$e_N(t) = x(t) - \sum_{k=-N}^{+N} a_k e^{jk\omega_0 t}, \qquad E_N(t) = \int_T |e_N(t)|^2 \, dt$$

$$\Rightarrow \quad \lim_{N\to\infty} E_N(t) = 0$$

Note: $\lim_{N\to\infty} E_N(t) = 0$ does NOT imply $x(t) = \sum_{k=-N}^{+N} a_k e^{jk\omega_0 t}$ at every value of t.

Sect. 3.4 Convergence of the Fourier Series (cont'd)

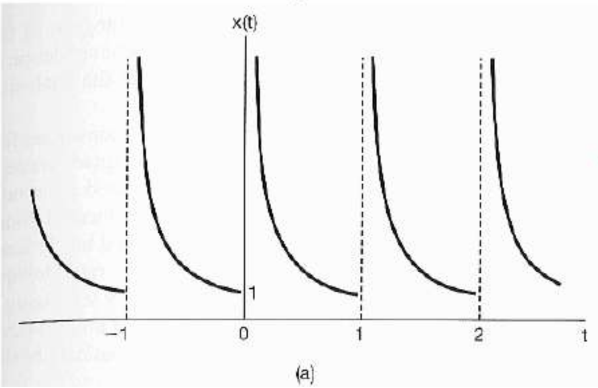
- Convergence Conditions Developed by Dirichlet
  - Condition 1

Over any period, x(t) must be absolutely integrable. That is,

$$\int_T |x(t)|\ dt < \infty \quad \Rightarrow \quad |a_k| \le \frac{1}{T}\int_T \left|x(t)e^{-jk\omega_0 t}\right| dt$$

$$= \frac{1}{T}\int_T |x(t)| dt < \infty$$

Example: $x(t) = \frac{1}{t}, \quad 0 < t \le 1$



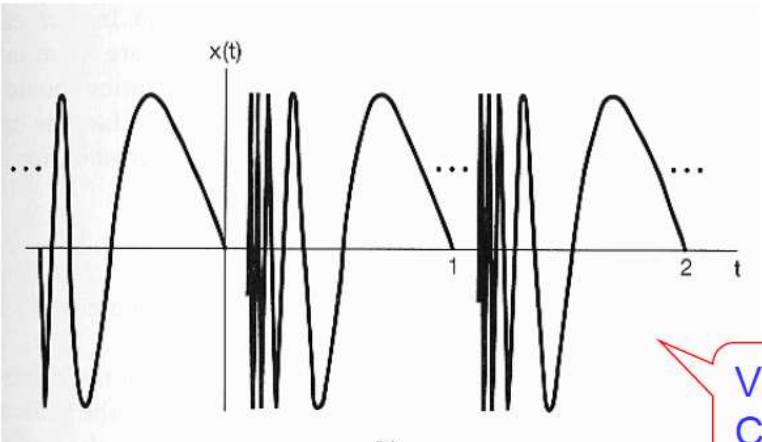
Violate Condition 1

Sect. 3.4 Convergence of the Fourier Series (cont'd)

- Convergence Conditions Developed by Dirichlet
  - Condition 2

In any finite interval, x(t) is of bounded variation. That is, there are no more than a finite number of maxima and minima during any single period of the signal.

Example: $x(t) = \sin(\frac{2\pi}{t}), \quad 0 < t \le 1$



(b)

$$\int_0^1 |x(t)|\,dt < 1$$

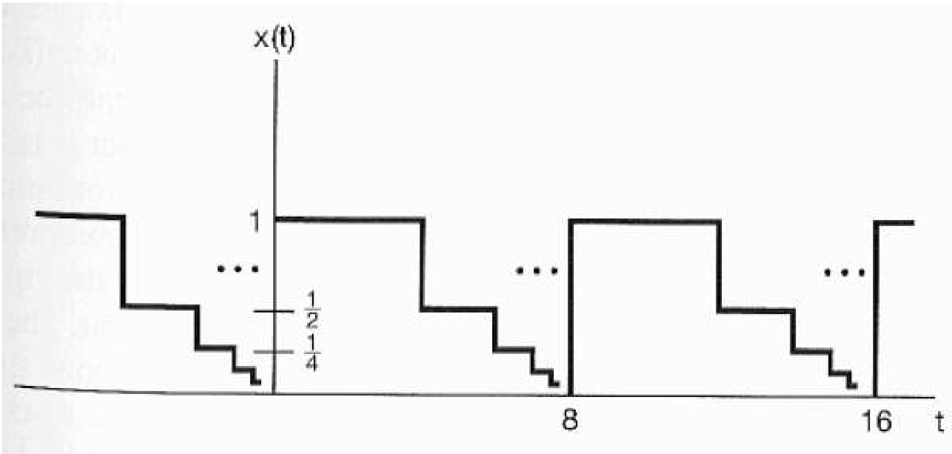
But it has an infinite number of maxima and minima in each period.

Violate Condition 2

Sect. 3.4 Convergence of the Fourier Series (cont'd)

- Convergence Conditions Developed by Dirichlet
  - Condition 3

    In any finite interval, x(t) has only finite number of discontinuities. Furthermore, each of these discontinuities is finite.



(c)

Infinite number of discontinuities in each period

Condition 3 not satisfied

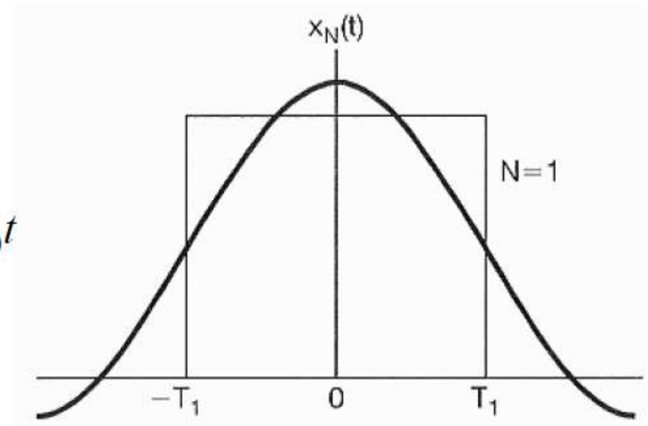
Sect. 3.4 Convergence of the Fourier Series (cont'd)

- In practice, most signals converge.
- For a periodic signal that has no discontinuity, the FS representation converges and equals the original signal at every t.
- For a periodic signal with a finite number of discontinuities in each period, the FS representation equals the original signal everywhere except at the isolated points of discontinuity, at which the FS converges to the average value of the signal on either side of the discontinuity.

Sect. 3.4 Convergence of the Fourier Series (cont'd)

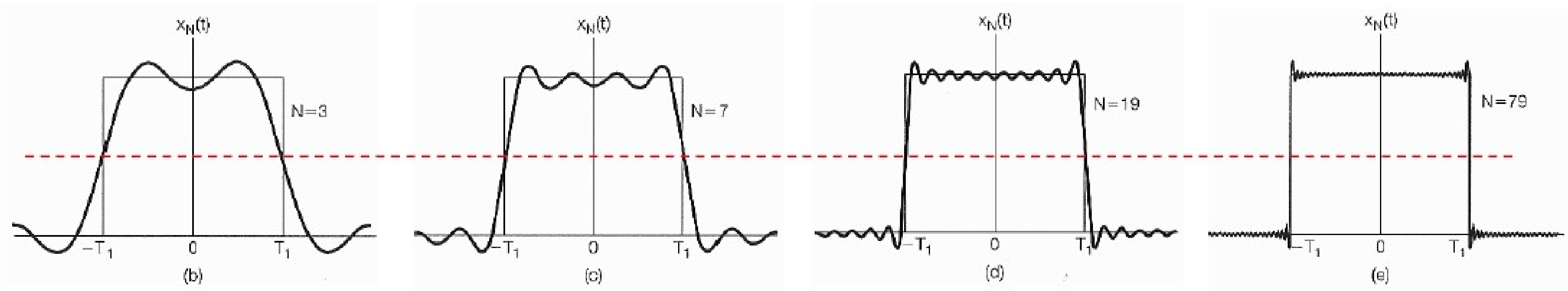
- How the FS converges for a periodic signal with discontinuities?
- In 1989, Albert Michelson (an American physicist) used his harmonic analyzer to compute the truncated Fourier series approximation for the square wave.

$$x_N(t) = \sum_{k=-N}^{+N} a_k e^{jk\omega_0 t}$$



Sect. 3.4 Convergence of the Fourier Series (cont'd)

- Michelson wrote Josiah Gibbs a letter describing his observations.
- In 1899, Gibbs showed that
  - The partial sum near discontinuity exhibits ripples, and
  - The peak amplitude remains constant with increasing N.
- Known as the Gibbs phenomenon



Sect. 3.5 Properties of CT Fourier Series

- Linearity

$x(t), y(t)$: periodic signals with period T

$$x(t) \xleftrightarrow{FS} a_k \qquad x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$$

$$y(t) \xleftrightarrow{FS} b_k \qquad y(t) = \sum_{m=-\infty}^{+\infty} b_m e^{jm\omega_0 t}$$

$$\Rightarrow z(t) = Ax(t) + By(t) \xleftrightarrow{FS} c_k = Aa_k + Bb_k \qquad z(t) = \sum_{k=-\infty}^{+\infty} c_k e^{jk\omega_0 t}$$

Sect. 3.5 Properties of CT Fourier Series (cont'd)

- Time Shifting

$x(t)$: periodic signal with period T

$$x(t) \xleftrightarrow{FS} a_k \quad \Rightarrow \quad x(t-t_0) \xleftrightarrow{FS} e^{-jk\omega_0 t_0} a_k$$

$$\begin{aligned} b_k &= \frac{1}{T}\int_T x(t-t_0)e^{-jk\omega_0 t}dt \\ &= \frac{1}{T}\int_T x(\tau)e^{-jk\omega_0(\tau+t_0)}d\tau \\ &= e^{-jk\omega_0 t_0}\frac{1}{T}\int_T x(\tau)e^{-jk\omega_0\tau}d\tau \\ &= e^{-jk\omega_0 t_0} a_k \end{aligned}$$

Sect. 3.5 Properties of CT Fourier Series (cont'd)

- Time Reversal

$$x(t) \xleftrightarrow{FS} a_k \quad \Rightarrow \quad x(-t) \xleftrightarrow{FS} a_{-k}$$

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$$

$$\Rightarrow x(-t) = \sum_{k=-\infty}^{+\infty} a_k e^{-jk\omega_0 t} = \sum_{m=-\infty}^{+\infty} a_{-m} e^{jm\omega_0 t}$$

If $x(t)$ is even, we have $x(-t) = x(t)$

$\Rightarrow \; a_{-k} = a_k$, so a_k is even

If $x(t)$ is odd, we have $x(-t) = -x(t)$

$\Rightarrow \; a_{-k} = -a_k$, so a_k is odd

Sect. 3.5 Properties of CT Fourier Series (cont'd)

- Time Scaling

$x(t)$ is periodic with period T and fundamental frequency ω_0

$\Rightarrow x(\alpha t)$ is periodic with period $\dfrac{T}{\alpha}$ and fundamental frequency $\alpha\omega_0$

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$$

$$x(\alpha t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 (\alpha t)} = \sum_{k=-\infty}^{+\infty} a_k e^{jk(\alpha\omega_0)t}$$

Sect. 3.5 Properties of CT Fourier Series (cont'd)

- Multiplication

If $x(t)$ and $y(t)$ are periodic signal with period T and

$$x(t) \xleftrightarrow{FS} a_k \quad \text{and} \quad y(t) \xleftrightarrow{FS} b_k.$$

Then $z(t)=x(t)y(t)$ is also periodic with T, and

$$z(t) \xleftrightarrow{FS} c_k = \sum_{l=-\infty}^{+\infty} a_l b_{k-l}$$

Proof:

$$x(t)y(t) = \sum_{m=-\infty}^{+\infty} \sum_{l=-\infty}^{+\infty} a_k b_l e^{jm\omega_0 t} e^{jl\omega_0 t} = \sum_{m=-\infty}^{+\infty} \sum_{l=-\infty}^{+\infty} a_m b_l e^{j(m+l)\omega_0 t}$$

$$c_k = \frac{1}{T}\int_T x(t)y(t)e^{-jk\omega_0 t}dt = \frac{1}{T}\int_T \sum_{m=-\infty}^{+\infty} \sum_{l=-\infty}^{+\infty} a_m b_l e^{j(m+l)\omega_0 t} e^{-jk\omega_0 t} dt$$

$$= \frac{1}{T}\sum_{m=-\infty}^{+\infty} \sum_{l=-\infty}^{+\infty} a_m b_l \int_T e^{j(m+l)\omega_0 t} e^{-jk\omega_0 t} dt = \sum_{m=-\infty}^{+\infty} \sum_{l=-\infty}^{+\infty} a_m b_l \delta(k-(m+l))$$

$$= \sum_{m=-\infty}^{+\infty} a_m b_{k-m}$$

Sect. 3.5 Properties of CT Fourier Series (cont'd)

- Differentiation

If $x(t)$ is a periodic signal with period T and

$$x(t) \xleftrightarrow{FS} a_k$$

then

$$\frac{d}{dt}x(t) \xleftrightarrow{FS} jk\omega_0 a_k.$$

Proof:

$$\frac{d}{dt}x(t) = \frac{d}{dt}\sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{+\infty} a_k jk\omega_0 e^{jk\omega_0 t}$$

Sect. 3.5 Properties of CT Fourier Series (cont'd)

- Integration

If $x(t)$ is a periodic signal with period T and

$$x(t) \xleftrightarrow{FS} a_k$$

then

$$\int_{-\infty}^{t} x(\tau)d\tau \xleftrightarrow{FS} \frac{1}{jk\omega_0} a_k.$$

Finite valued and periodic only if a_0=0

Proof:

Let $y(t) = \int_{-\infty}^{t} x(\tau)d\tau$. Then $\frac{d}{dt} y(t) = x(t)$.

We have $jk\omega_0 b_k = a_k$.

Therefore, $b_k = \frac{1}{jk\omega_0} a_k$.

Sect. 3.5 Properties of CT Fourier Series (cont'd)

- Conjugation and Conjugate Symmetry

If

$$x(t) \xleftrightarrow{FS} a_k,$$

then

$$x(t)^* \xleftrightarrow{FS} a_{-k}^*.$$

- $x(t)$ real $\Rightarrow x(t) = x(t)^* \Rightarrow a_{-k} = a_k^*$

  If $x(t)$ is real, then $\{a_k\}$ are conjugate symmetric.

- $x(t) = x(t)^*$ and $x(-t) = x(t) \Rightarrow a_{-k} = a_k$ and $a_{-k} = a_k^*$

  $$\Rightarrow a_k = a_k^*$$

  $x(t)$ is real and even $\Rightarrow \{a_k\}$ are real and even

- $x(t)$ is real and odd $\Rightarrow \{a_k\}$ are pure imaginary and odd

Sect. 3.5 Properties of CT Fourier Series (cont'd)

- Parseval's Relation

The total average power in a periodic signal equals the sum of the average powers in all of its harmonic components

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t} \quad \Rightarrow \quad \boxed{\frac{1}{T}\int_T |x(t)|^2\, dt = \sum_{k=-\infty}^{+\infty} |a_k|^2}$$

Proof:

$$\text{If } x(t) \xleftrightarrow{\mathcal{FS}} a_k, \text{ then } x^*(t) \xleftrightarrow{\mathcal{FS}} b_k = a^*_{-k}$$

$$\Rightarrow \; x^*(t)x(t) \xleftrightarrow{\mathcal{FS}} c_k = \sum_{m=-\infty}^{+\infty} a_m b_{k-m} = \sum_{m=-\infty}^{+\infty} a_m a^*_{m-k}$$

$$\text{and } c_k = \frac{1}{T}\int_T x^*(t)x(t)e^{-jk\omega_0 t}\, dt$$

$$\Rightarrow \frac{1}{T}\int_T x^*(t)x(t)\, dt = c_0 = \sum_{m=-\infty}^{+\infty} a_m a^*_m = \sum_{m=-\infty}^{+\infty} |a_m|^2$$