Signals & Systems

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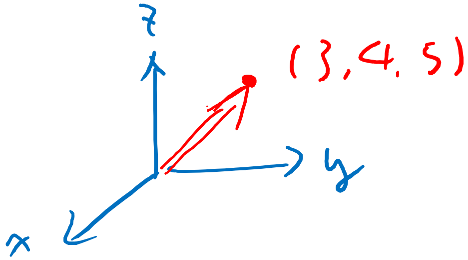
Revisit:
Representation of Signals in terms of Basis Functions

- Detailed Remarks:



coeff

$$\begin{bmatrix}3\\4\\5\end{bmatrix} = 3\underbrace{\begin{bmatrix}1\\0\\0\end{bmatrix}}_{=\ x\text{-axis}} + 4\underbrace{\begin{bmatrix}0\\1\\0\end{bmatrix}}_{=\ y} + 5\underbrace{\begin{bmatrix}0\\0\\1\end{bmatrix}}_{=\ z}$$

basis functions

$$\underline{a} = \sum_{i=1}^{d} \alpha_i \underline{u}_i \,, \quad \underline{a}, \underline{u}_i \in \mathbb{R}^d$$

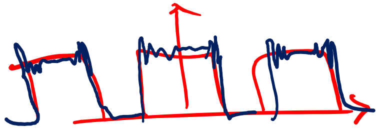
- Will see more in Ch. 3 Fourier Series, etc.

Sect. 3.3 Fourier Series Representation of CT Periodic Signals

- Determining the Fourier Series Coefficients
- In summary, we have:

$$\boxed{a_k = \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}\,dt}$$

Analysis Equation



$$\boxed{x(t) = \sum_{k=-\infty}^{\infty} a_k \boxed{e^{jk\omega_0 t}}}$$

Synthesis Equation

$$T = \frac{2\pi}{\omega_0}$$

- CT Fourier series pair: $x(t) \overset{FS}{\leftrightarrow} a_k$
- Fourier series coefficients or spectral coefficients of $x(t)$: $\{a_k\}$

Sect. 3.5 Properties of CT Fourier Series

- Linearity

$x(t), y(t)$: periodic signals with period T

$$x(t) \xleftrightarrow{FS} a_k \qquad\qquad x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$$

$$y(t) \xleftrightarrow{FS} b_k \qquad\qquad y(t) = \sum_{m=-\infty}^{+\infty} b_m e^{jm\omega_0 t}$$

$k \rightarrow \quad k \quad k$

$$\Rightarrow z(t) = Ax(t) + By(t) \xleftrightarrow{FS} c_k = Aa_k + Bb_k \qquad\qquad z(t) = \sum_{k=-\infty}^{+\infty} c_k e^{jk\omega_0 t}$$

Sect. 3.5 Properties of CT Fourier Series (cont'd)

- Time Shifting

$x(t)$: periodic signal with period T

$$x(t) \xleftrightarrow{FS} a_k \quad \Rightarrow \quad x(t-t_0) \xleftrightarrow{FS} \underbrace{e^{-jk\omega_0 t_0} a_k}_{b_k}$$

$$b_k = \frac{1}{T}\int_T x(t-t_0)e^{-jk\omega_0 t}dt$$

$$= \frac{1}{T}\int_T x(\tau)e^{-jk\omega_0(\tau+t_0)}d\tau$$

let $\tau = t - t_0$

$d\tau = dt$

$t = \tau + t_0$

$$= e^{-jk\omega_0 t_0}\left[\frac{1}{T}\int_T x(\tau)e^{-jk\omega_0 \tau}d\tau\right]$$

$$= \underbrace{e^{-jk\omega_0 t_0}}_{\text{相位差}} a_k$$

Sect. 3.5 Properties of CT Fourier Series (cont'd)

- Time Reversal

$$x(t) \xleftrightarrow{FS} a_k \quad \Rightarrow \quad x(-t) \xleftrightarrow{FS} a_{-k}$$

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$$

$$\Rightarrow x(-t) = \sum_{k=-\infty}^{+\infty} a_k e^{-jk\omega_0 t} = \sum_{m=-\infty}^{+\infty} \boxed{a_{-m}} e^{jm\omega_0 t} = \sum_k a_{-k} e^{jk\omega_0 t}$$

If $x(t)$ is even, we have $x(-t) = x(t)$

$\Rightarrow \; a_{-k} = a_k$, so a_k is even

If $x(t)$ is odd, we have $x(-t) = -x(t)$

$\Rightarrow \; a_{-k} = -a_k$, so a_k is odd

Sect. 3.5 Properties of CT Fourier Series (cont'd)

- Time Scaling

$x(t)$ is periodic with period T and fundamental frequency ω_0

$\Rightarrow x(\alpha t)$ is periodic with period $\dfrac{T}{\alpha}$ and fundamental frequency $\alpha\omega_0$

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$$

$$x(\alpha t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 (\alpha t)} = \sum_{k=-\infty}^{+\infty} a_k e^{jk(\alpha\omega_0)t}$$

$T = \dfrac{2\pi}{\omega_0}$. $T' = \dfrac{2\pi}{\alpha\omega_0} = \dfrac{T}{\alpha}$

Sect. 3.5 Properties of CT Fourier Series (cont'd)

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

$$y(t) = \sum_{k=-\infty}^{\infty} b_k e^{jk\omega_0 t}$$

$$(\alpha_0 + \alpha_1 + \alpha_2 + \dots)$$

$$(\beta_0 + \beta_1 + \beta_2 + \dots)$$

- Multiplication

  If $x(t)$ and $y(t)$ are periodic signal with period T and

  $$x(t) \xleftrightarrow{FS} a_k \quad \text{and} \quad y(t) \xleftrightarrow{FS} b_k.$$

  Then $z(t)=x(t)y(t)$ is also periodic with T, and

  $$z(t) \xleftrightarrow{FS} c_k = \sum_{l=-\infty}^{+\infty} a_l b_{k-l}$$

  Proof:

$$z(t) = x(t)y(t) = \sum_{m=-\infty}^{+\infty}\sum_{l=-\infty}^{+\infty} a_m b_l e^{jm\omega_0 t} e^{jl\omega_0 t} = \sum_{m=-\infty}^{+\infty}\sum_{l=-\infty}^{+\infty} a_m b_l e^{j(m+l)\omega_0 t}$$

$$c_k = \frac{1}{T}\int_T x(t)y(t)e^{-jk\omega_0 t}dt = \frac{1}{T}\int_T \underbrace{\boxed{\sum_{m=-\infty}^{+\infty}\sum_{l=-\infty}^{+\infty} a_m b_l e^{j(m+l)\omega_0 t}}}_{z(t)} e^{-jk\omega_0 t}dt$$

$$= \frac{1}{T}\sum_{m=-\infty}^{+\infty}\sum_{l=-\infty}^{+\infty} a_m b_l \boxed{\int_T e^{j(m+l)\omega_0 t} e^{-jk\omega_0 t}dt} = \sum_{m=-\infty}^{+\infty}\sum_{l=-\infty}^{+\infty} a_m b_l \underbrace{\delta(k-(m+l))}_{l = k-m}$$

$$= \sum_{m=-\infty}^{+\infty} a_m b_{k-m}$$

Sect. 3.5 Properties of CT Fourier Series (cont'd)

- Differentiation

If $x(t)$ is a periodic signal with period T and

$$x(t) \xleftrightarrow{FS} a_k$$

then

$$\frac{d}{dt}x(t) \xleftrightarrow{FS} jk\omega_0 a_k.$$

Proof:

$$\frac{d}{dt}x(t) = \frac{d}{dt}\sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{+\infty} \underbrace{\boxed{a_k jk\omega_0}}_{a'_k} e^{jk\omega_0 t}$$

Sect. 3.5 Properties of CT Fourier Series (cont'd)

- Integration

If $x(t)$ is a periodic signal with period T and

$$x(t) \xleftrightarrow{FS} a_k$$

then

$$\int_{-\infty}^{t} x(\tau)d\tau \xleftrightarrow{FS} \frac{1}{jk\omega_0} a_k .$$

Finite valued and periodic only if a_0=0

Proof:

Let $y(t) = \int_{-\infty}^{t} x(\tau)d\tau$. Then $\frac{d}{dt} y(t) = x(t)$.

We have $jk\omega_0 b_k = a_k$.

Therefore, $b_k = \frac{1}{jk\omega_0} a_k$.

Sect. 3.5 Properties of CT Fourier Series (cont'd)

- Conjugation and Conjugate Symmetry

$$x^*(t) = \sum_{k=-\infty}^{\infty} a_k^* \cdot e^{-jk\omega_0 t}$$

$$= \sum_k a_{-k}^* e^{jk\omega_0 t}$$

If

$$x(t) \xleftrightarrow{FS} a_k,$$

then

$$x(t)^* \xleftrightarrow{FS} a_{-k}^*.$$

- $x(t)$ real $\Rightarrow x(t) = x(t)^* \Rightarrow a_{-k} = a_k^*$

  If $x(t)$ is real, then $\{a_k\}$ are conjugate symmetric.

- $x(t) = x(t)^*$ and $x(-t) = x(t) \Rightarrow a_{-k} = a_k$ and $a_{-k} = a_k^*$

  $\Rightarrow a_k = a_k^*$

  $x(t)$ is real and even $\Rightarrow \{a_k\}$ are real and even

- $x(t)$ is ~~real and~~ odd $\Rightarrow \{a_k\}$ are pure imaginary and odd

  $a_k^* = a_{-k}$, $a_k = a_{-k}^*$

  $a_k = -a_{-k}$

Sect. 3.5 Properties of CT Fourier Series (cont'd)

- Parseval's Relation

The total average power in a periodic signal equals the sum of the average powers in all of its harmonic components

$$x(t)=\sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t} \quad \Rightarrow \quad \boxed{\frac{1}{T}\int_T |x(t)|^2\,dt=\sum_{k=-\infty}^{+\infty}|a_k|^2}$$

Proof:

$$\text{If } x(t) \xleftrightarrow{\mathcal{FS}} a_k, \text{ then } x^*(t) \xleftrightarrow{\mathcal{FS}} b_k = a^*_{-k}$$

$$\Rightarrow x^*(t)x(t) \xleftrightarrow{\mathcal{FS}} c_k=\sum_{m=-\infty}^{+\infty} a_m b_{k-m}=\sum_{m=-\infty}^{+\infty} a_m a^*_{m-k}$$

$$\text{and } c_k=\frac{1}{T}\int_T \boxed{x^*(t)x(t)e}^{-jk\omega_0 t}\,dt$$

$$\Rightarrow \frac{1}{T}\int_T x^*(t)x(t)dt=c_0=\sum_{m=-\infty}^{+\infty} a_m a^*_m=\sum_{m=-\infty}^{+\infty}|a_m|^2$$

TABLE 3.1 PROPERTIES OF CONTINUOUS-TIME FOURIER SERIES

Property	Section	Periodic Signal	Fourier Series Coefficients
		$\left.\begin{array}{l} x(t) \\ y(t) \end{array}\right\}$ Periodic with period T and fundamental frequency $\omega_0 = 2\pi/T$	a_k b_k
Linearity	3.5.1	$Ax(t) + By(t)$	$Aa_k + Bb_k$
Time Shifting	3.5.2	$x(t - t_0)$	$a_k e^{-jk\omega_0 t_0} = a_k e^{-jk(2\pi/T)t_0}$
Frequency Shifting		$e^{jM\omega_0 t}x(t) = e^{jM(2\pi/T)t}x(t)$	a_{k-M}
Conjugation	3.5.6	$x^*(t)$	a^*_{-k}
Time Reversal	3.5.3	$x(-t)$	a_{-k}
Time Scaling	3.5.4	$x(\alpha t), \alpha > 0$ (periodic with period T/α)	a_k
Periodic Convolution		$\int_T x(\tau)y(t-\tau)d\tau$	$Ta_k b_k$
Multiplication	3.5.5	$x(t)y(t)$	$\sum_{l=-\infty}^{+\infty} a_l b_{k-l}$
Differentiation		$\frac{dx(t)}{dt}$	$jk\omega_0 a_k = jk\frac{2\pi}{T}a_k$
Integration		$\int_{-\infty}^{t} x(t)\,dt$ (finite valued and periodic only if $a_0 = 0$)	$\left(\frac{1}{jk\omega_0}\right)a_k = \left(\frac{1}{jk(2\pi/T)}\right)a_k$
Conjugate Symmetry for Real Signals	3.5.6	$x(t)$ real	$\begin{cases} a_k = a^*_{-k} \\ \Re e\{a_k\} = \Re e\{a_{-k}\} \\ \Im m\{a_k\} = -\Im m\{a_{-k}\} \\ \lvert a_k\rvert = \lvert a_{-k}\rvert \\ \measuredangle a_k = -\measuredangle a_{-k} \end{cases}$
Real and Even Signals	3.5.6	$x(t)$ real and even	a_k real and even
Real and Odd Signals	3.5.6	$x(t)$ real and odd	a_k purely imaginary and odd
Even-Odd Decomposition of Real Signals		$\begin{cases} x_e(t) = \mathcal{E}v\{x(t)\} & [x(t) \text{ real}] \\ x_o(t) = \mathcal{O}d\{x(t)\} & [x(t) \text{ real}] \end{cases}$	$\Re e\{a_k\}$ $j\Im m\{a_k\}$

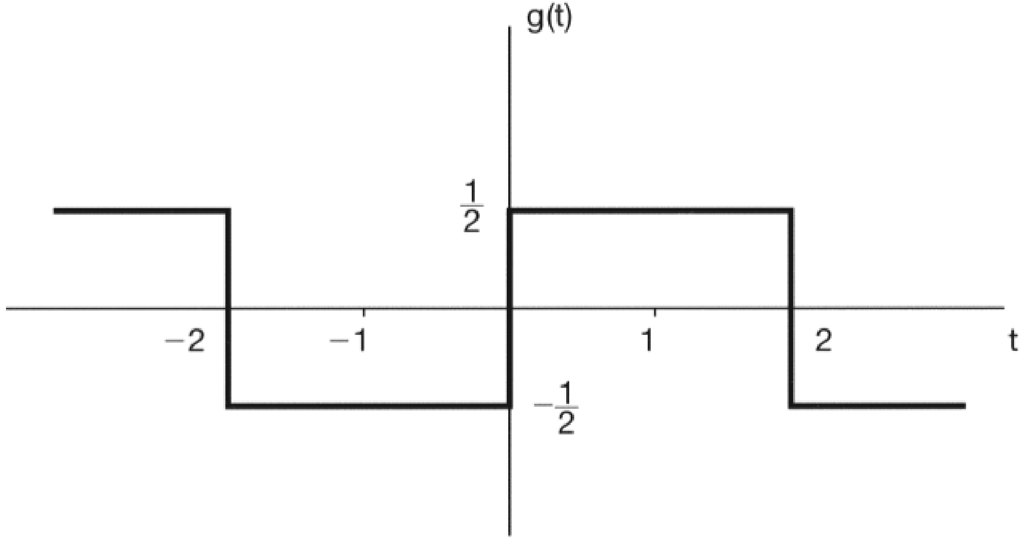
Parseval's Relation for Periodic Signals

$$\frac{1}{T}\int_T |x(t)|^2 dt = \sum_{k=-\infty}^{+\infty} |a_k|^2$$

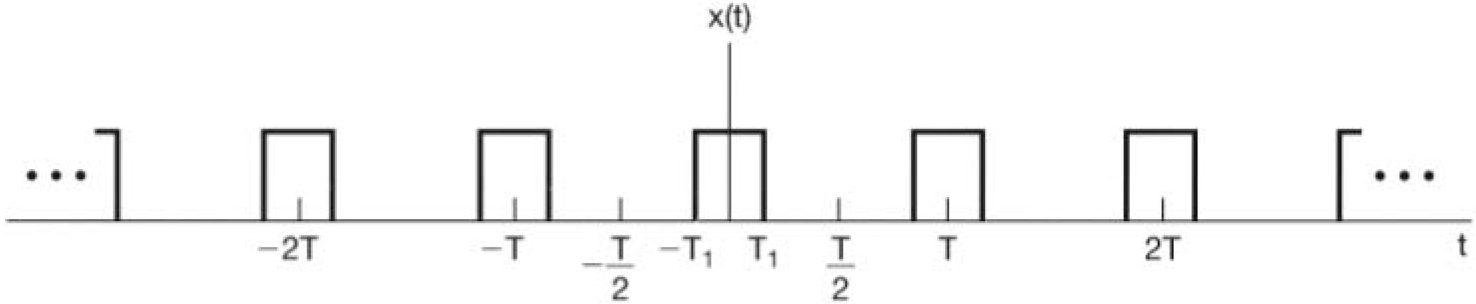
Sect. 3.5 Properties of CT Fourier Series (cont'd)

- Example 3.6

$g(t)$



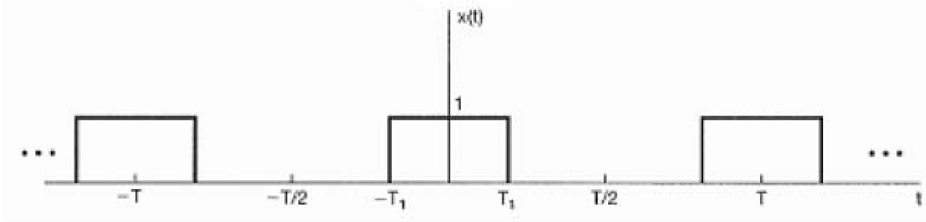
Compared to $x(t)$ in Example 3.5



$$g(t) = x(t-1) - 1/2. \qquad T = 4,\ T_1 = 1$$

- Revisit of Example 3.5: FS of periodic square wave

$$x(t) = \begin{cases} 1, & |t| < T_1 \\ 0, & T_1 < |t| < T/2 \end{cases}$$



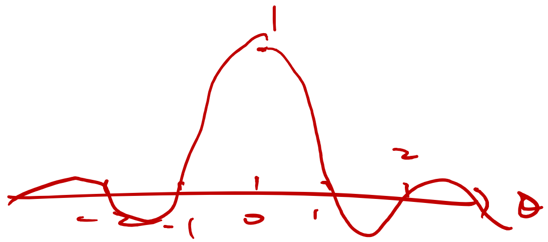
$$\omega_0 = \frac{2\pi}{T}$$

$$a_0 = \frac{1}{T}\int_{-T_1}^{T_1} dt = \frac{2T_1}{T}$$

$$a_k = \frac{1}{T}\int_{-T_1}^{T_1} e^{-jk\omega_0 t}\,dt = -\frac{1}{jk\omega_0 T} e^{-jk\omega_0 t}\Big|_{-T_1}^{T_1}$$

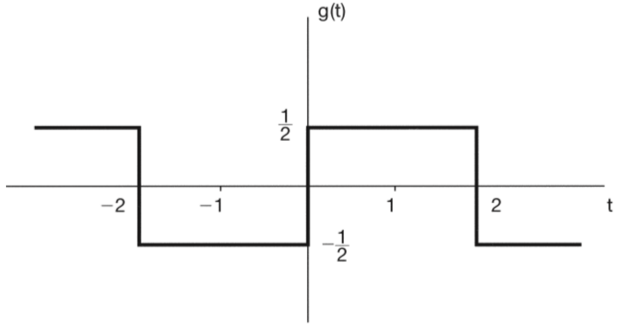
$$= \frac{2}{k\omega_0 T}\left[\frac{e^{jk\omega_0 T_1} - e^{-jk\omega_0 T_1}}{2j}\right]$$

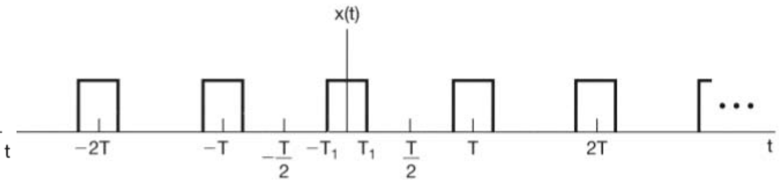
$$= \frac{2\sin(k\omega_0 T_1)}{k\omega_0 T} = \frac{\sin(k\omega_0 T_1)}{k\pi} \rightarrow \mathrm{sinc} = \frac{\sin(\pi\theta)}{\pi\theta}$$



- Example 3.6

$$g(t) = \boxed{x(t-1)} - 1/2.$$





$$T = 4,\ T_1 = 1$$

From Example 3.5

$$a_0 = \frac{1}{2} \qquad a_k = \frac{\sin(\pi k/2)}{k\pi} \qquad \omega_0 = 2\pi/T = \pi/2$$

The Fourier series of $x(t-1)$

$$b_k = a_k e^{-jk\pi/2} \rightarrow -jk\frac{2\pi}{T}\cdot T_1$$

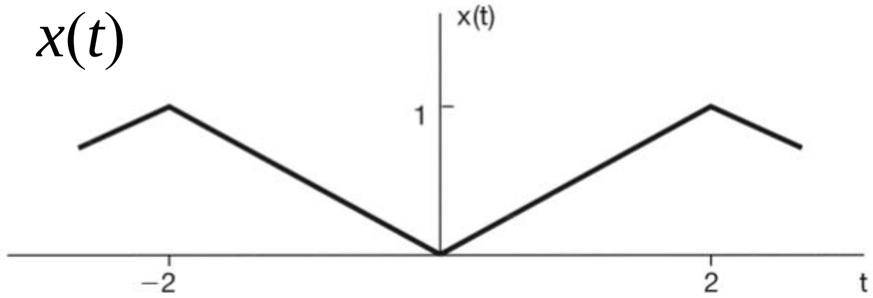
The Fourier series of 1/2

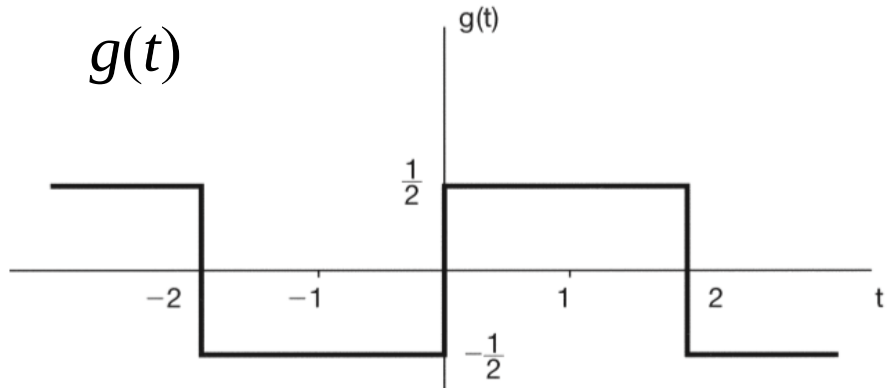
$$c_k = \begin{cases} 0, & for\ k \neq 0 \\ -\frac{1}{2}, & for\ k = 0 \end{cases}$$

The Fourier series of $x(t-1)-1/2$

$$d_k = \begin{cases} a_k e^{-jk\pi/2}, & for\ k \neq 0 \\ a_0 - \frac{1}{2}, & for\ k = 0 \end{cases} \Rightarrow d_k = \begin{cases} \frac{\sin(\pi k/2)}{k\pi} e^{-jk\pi/2}, & for\ k \neq 0 \\ 0, & for\ k = 0 \end{cases}$$

- Example 3.7





If

$$x(t) \xleftrightarrow{FS} e_k,\ g(t) \xleftrightarrow{FS} d_k,\ \omega_0 = 2\pi/T = \pi/2$$

then

$$\frac{dx(t)}{dt} = g(t) \Leftrightarrow jk\omega_0 e_k = d_k$$

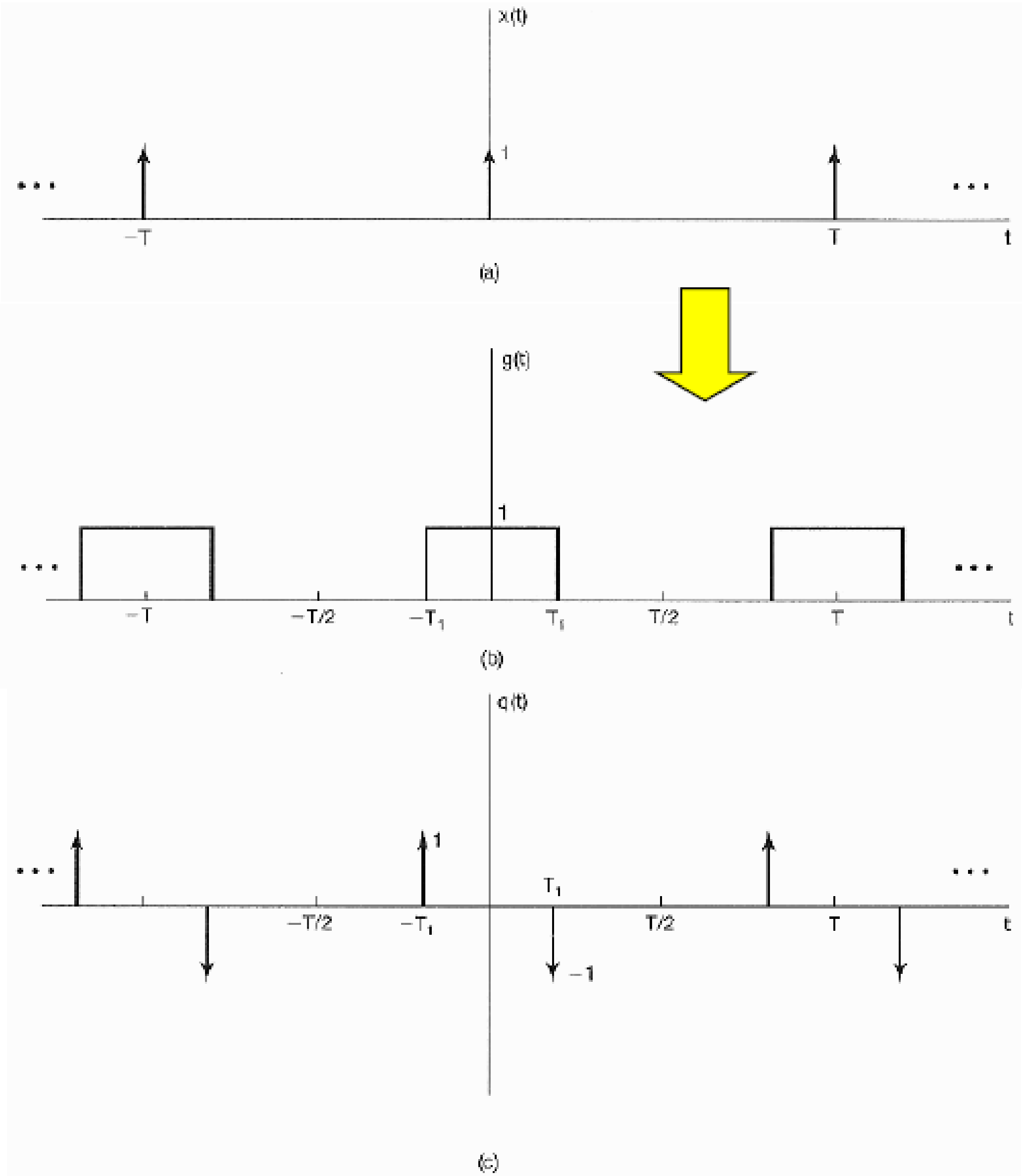
$$\boxed{e_k = \frac{2d_k}{jk\pi} = \frac{2\sin(\pi k/2)}{j(k\pi)^2} e^{-jk\pi/2}, \quad k \neq 0}$$

e_0 can be determined by finding the area under one period of $x(t)$ and dividing by the length of the period:

$$\boxed{e_0 = \frac{1}{2}}$$

This is the area under one period of x(t) divided by the length of the period.

- Example 3.8 Determine the FS impulse train



$$x(t)=\sum_{k=-\infty}^{\infty}\delta(t-kT)=\sum_{k=-\infty}^{+\infty}a_k e^{jk\omega_0 t}$$

$$x(t)\xleftrightarrow{FS}a_k$$

$$a_k=\frac{1}{T}\int_{-T/2}^{T/2}\delta(t)e^{-jk2\pi t/T}dt=\frac{1}{T}$$

$$g(t)\xleftrightarrow{FS}c_k$$

$$q(t)=\frac{d}{dt}g(t)\Rightarrow b_k=jk\omega_0 c_k$$

$$q(t)\xleftrightarrow{FS}b_k$$

$$q(t)=x(t+T_1)-x(t-T_1)$$

$$\Rightarrow b_k=e^{jk\omega_0 T_1}a_k-e^{-jk\omega_0 T_1}a_k$$

- Example 3.8 Determine the FS impulse train (cont'd)

$$b_k = e^{jk\omega_0 T_1} a_k - e^{-jk\omega_0 T_1} a_k = \frac{1}{T}\left[e^{jk\omega_0 T_1} - e^{-jk\omega_0 T_1} \right] = \frac{2j\sin(k\omega_0 T_1)}{T}$$

$$c_k = \frac{1}{jk_0\omega_0} b_k = \begin{cases} \dfrac{2j\sin(k\omega_0 T_1)}{jk\omega_0 T} = \dfrac{\sin(k\omega_0 T_1)}{k\pi}, & k \neq 0 \\ \dfrac{2T_1}{T}, & k = 0 \end{cases}$$

Obtained by computing the area under one period of the signal divided by the period.

We just showed an alternative way to obtain the FS of a periodic square wave (the first method was shown in Example 3.5).

- Example 3.9 Use the following 5 facts to determine the signal x(t).

1. x(t) is a real signal. $a_k = a_{-k}^*$
2. x(t) is periodic with period T = 4 and with FS coefficients a_k.
3. a_k=0 for |k|>1.
4. The signal with FS $b_k = e^{-j\pi k/2} a_{-k}$ odd.
5. $\frac{1}{4}\int_4 |x(t)|^2 dt = 1/2.$

$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\frac{2\pi}{4}t}$

➢ From Facts #2 and #3, we have…

$x(t) = a_0 + a_1 e^{j\frac{\pi}{2}t} + a_{-1} e^{-j\frac{\pi}{2}t}$

➢ From Fact #1, we can conclude a_0 is real and a_1 = a_{-1}^* .

➢ Thus, we have the signal as…

$x(t) = a_0 + a_1 e^{j\frac{\pi}{2}t} + (a_1 e^{j\pi/2})^*$

$= a_0 + 2\mathrm{Re}\{a_1 e^{j\frac{\pi}{2}t}\}$

- Example 3.9 Use the following 5 facts to determine the signal x(t).

1. x(t) is a real signal.
2. x(t) is periodic with period T = 4 and with FS coefficients a_k.
3. a_k=0 for |k|>1.
4. The signal with FS $b_k = \boxed{e^{-j\pi k/2}} a_{-k}$ is odd.
5. $\frac{1}{4}\int_4 |x(t)|^2 dt = 1/2$.

$e^{-jk\frac{2\pi}{4}\cdot 1}$

- a_{-k} is the FS coefficient of x(-t) and with the time shifting property of FS. From Fact #4 we conclude that b_k correspond to the signal x(-(t-1)) and are odd.
- Since x(t) is real, x(-t+1) must also be real. The FS of x(-t+1) must be pure imaginary and odd.
- Thus, $b_0 = 0$ and $b_{-1} = -b_1$.
- From Fact #5 and Parseval's Theorem, we have $|b_{-1}|^2 + |b_1|^2 = 2|b_1|^2 = 1/2$.
- Since $b_0 = 0$, Fact #4 implies $a_0 = 0$. Likewise, $a_1 = e^{-j\pi/2} b_{-1} = -jb_{-1} = jb_1$.
- If $b_1 = j/2$, then $a_1 = -1/2$ and $x(t) = -\cos(\pi t/2)$.
  If $b_1 = -j/2$, then $a_1 = 1/2$ and $x(t) = \cos(\pi t/2)$.

Sect. 3.6 Fourier Series Representation of DT Periodic Signals

- Remember that the function z^n are the eigenfunctions of discrete-time LTI systems.
- Specifically, if $z = e^{jk\omega_0}$ and $z^n = e^{jk\omega_0 n}$

  $k = N, \pm 2N, \pm 3N, \ldots$

  then $x[n]$ can be expressed as…

$$x[n] = \sum_{k} a_k e^{jk\omega_0 n} = \sum_{k} a_k e^{jk(2\pi/N)n} \qquad \omega_0 = 2\pi / N$$

- For DT periodic signals we have x[n] = x[n+N], then

$$x[n] = \sum_{k} a_k e^{jk\omega_0 n} = \sum_{k} a_k e^{jk(2\pi/N)n}$$

1

since $e^{jk(2\pi/N)n} = e^{j(k+N)(2\pi/N)n}$ $= e^{jk\frac{2\pi}{N}\cdot n} \cdot \boxed{e^{j(2\pi)\cdot n}}$

- Thus, k=<N> means k varies over a range of N successive integers. For example, $k = 0, 1, \ldots, N-1$, or $k = 3, 4, \ldots, N+2$.

Sect. 3.6 Fourier Series Representation of DT Periodic Signals

- Derivation of the FS coefficients.
- Method #1

$$x[0] = \sum_{k=<N>} a_k$$

$$x[1] = \sum_{k=<N>} a_k e^{jk\left(\frac{2\pi}{N}\right)}$$

$$x[2] = \sum_{k=<N>} a_k e^{jk2\left(\frac{2\pi}{N}\right)}$$

$$\vdots \qquad \vdots$$

$$x[N-1] = \sum_{k=<N>} a_k e^{jk(N-1)\left(\frac{2\pi}{N}\right)}$$

$x[0] \sim x[N-1]$

N linear equations and N unknowns

$a_0, \ldots, a_{N-1}$

$$\text{Also} \quad \sum_{k=<N>} e^{jm\left(\frac{2\pi}{N}\right)n} = \begin{cases} N, & m = 0, \pm \mathrm{N}, \pm 2\mathrm{N}, \ldots \\ 0, & \text{otherwise} \end{cases}$$

Sect. 3.6 Fourier Series Representation of DT Periodic Signals

- Derivation of the FS coefficients.
- Method #2

$$x[n] = \sum_{k=<N>} a_k e^{jk\left(\frac{2\pi}{N}\right)n}$$

$$\sum_{n=<N>} x[n]e^{-jr\left(\frac{2\pi}{N}\right)n} = \sum_{n=<N>} \sum_{k=<N>} a_k e^{j(k-r)\left(\frac{2\pi}{N}\right)n} = \sum_{k=<N>} a_k \sum_{n=<N>} e^{j(k-r)\left(\frac{2\pi}{N}\right)n}$$

$$\because \sum_{n=0}^{N-1} e^{j(k-r)\left(\frac{2\pi}{N}\right)n} = \begin{cases} \dfrac{1-e^{j(k-r)\left(\frac{2\pi}{N}\right)N}}{1-e^{j(k-r)\left(\frac{2\pi}{N}\right)}} = 0, & k \neq r \\ N, & \text{otherwise} \end{cases}$$

$$\Rightarrow \sum_{n=<N>} x[n]e^{-jr\left(\frac{2\pi}{N}\right)n} \boxed{= a_r N}$$

$$\Rightarrow a_r = \frac{1}{N} \sum_{n=<N>} x[n]e^{-jr\left(\frac{2\pi}{N}\right)n}$$

Sect. 3.6 Fourier Series Representation of DT Periodic Signals

- FS Representation of DT Periodic Signals

$$\boxed{x[n] = \sum_{k=<N>} a_k e^{jk\omega_0 n}}$$ Synthesis equation

$$\boxed{a_k = \frac{1}{N} \sum_{n=<N>} x[n] e^{-jk\omega_0 n}}$$ Analysis equation

$$a_k = a_{k+N}$$

$$\omega_0 = \frac{2\pi}{N}$$

$x[n] \xleftrightarrow{FS} a_k$: DT Fourier series pair

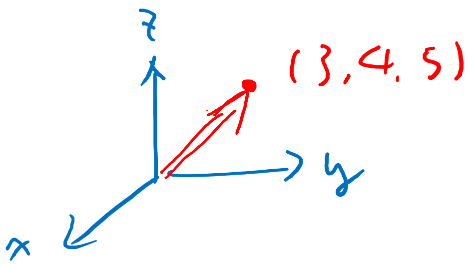
$\{a_k\}$: Fourier series coefficients

or spectral coefficients of $x[n]$

- The DT FS coefficients a_k are often referred to as the spectral coefficients of x[n] (and note that $a_k = a_{k+N}$).
- The DT FS is also called the Discrete Fourier Transform (DFT).

Revisit:
Representation of Signals in terms of Basis Functions

- Detailed Remarks:



coeff

$$\begin{bmatrix}3\\4\\5\end{bmatrix} = 3\underset{\text{x-axis}}{\underset{=}{\begin{bmatrix}1\\0\\0\end{bmatrix}}} + 4\underset{\text{y}}{\underset{=}{\begin{bmatrix}0\\1\\0\end{bmatrix}}} + 5\underset{\text{z}}{\underset{=}{\begin{bmatrix}0\\0\\1\end{bmatrix}}}$$

basis functions

$$\underline{a} = \sum_{i=1}^{d} \underset{\sim}{\alpha_i} \underline{\underline{u_i}}, \quad \underline{a}, \underline{u}_i \in \mathbb{R}^d$$

- See more in Ch. 3 Fourier Series, etc.

- Example 3.10

$$x[n]=\sum_{k=<N>} a_k e^{jk(2\pi/N)n}$$

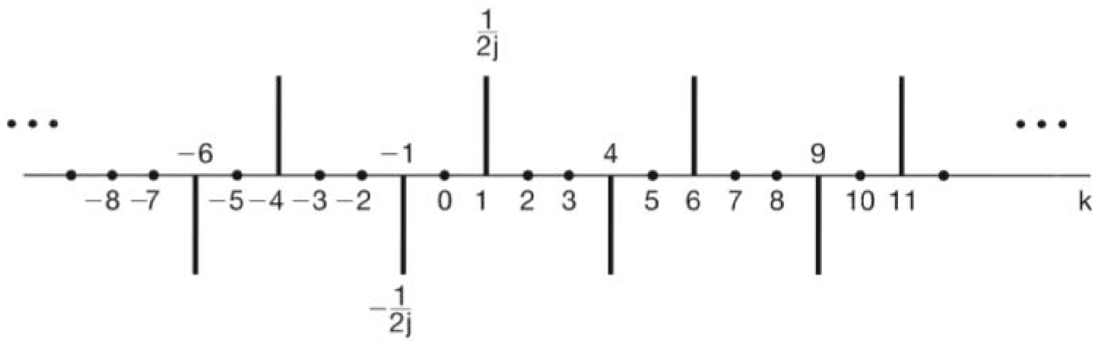
$$x[n]=\sin\omega_0 n \qquad \omega_0=2\pi/N$$

$$x[n]=\frac{1}{2j}e^{j(2\pi/N)n}-\frac{1}{2j}e^{-j(2\pi/N)n}$$

$$a_1=\frac{1}{2j},\quad a_{-1}=-\frac{1}{2j} \qquad\qquad a_k=0 \quad \text{for } k=0,\,2,\,3,\,\ldots,\,N\text{-}2$$

$$a_k=a_{k+N}$$

when $N=5$



- Example 3.11

$$x[n] = \sum_{k=<N>} a_k e^{jk(2\pi/N)n}$$

$$a_k = a_{k+N}$$

$$x[n] = 1 + \sin\left(\frac{2\pi}{N}\right)n + 3\cos\left(\frac{2\pi}{N}\right)n + \cos\left(\frac{4\pi}{N}n + \frac{\pi}{2}\right)$$

$$x[n] = 1 + \frac{1}{2j}[e^{j(2\pi/N)n} - e^{-j(2\pi/N)n}] + \frac{3}{2}[e^{j(2\pi/N)n} + e^{-j(2\pi/N)n}]$$

$\to \omega_0$

$$+\frac{1}{2}[e^{j(4\pi n/N+\pi/2)} + e^{-j(4\pi n/N+\pi/2)}].$$

$k=0$

$$x[n] = 1 + \underbrace{\left(\frac{3}{2}+\frac{1}{2j}\right)e^{j(2\pi/N)n}}_{k=1} + \underbrace{\left(\frac{3}{2}-\frac{1}{2j}\right)e^{-j(2\pi/N)n}}_{k=-1} + \underbrace{\boxed{\left(\frac{1}{2}e^{j\pi/2}\right)}}_{k=2}e^{j2(2\pi/N)n} + \underbrace{\boxed{\left(\frac{1}{2}e^{-j\pi/2}\right)}}_{k=-2}e^{-j2(2\pi/N)n}.$$

$$a_0 = 1,$$

$$a_1 = \tfrac{3}{2} + \tfrac{1}{2j} = \tfrac{3}{2} - \tfrac{1}{2}j,$$

$$a_{-1} = \tfrac{3}{2} - \tfrac{1}{2j} = \tfrac{3}{2} + \tfrac{1}{2}j,$$

$$a_2 = \tfrac{1}{2}j,$$

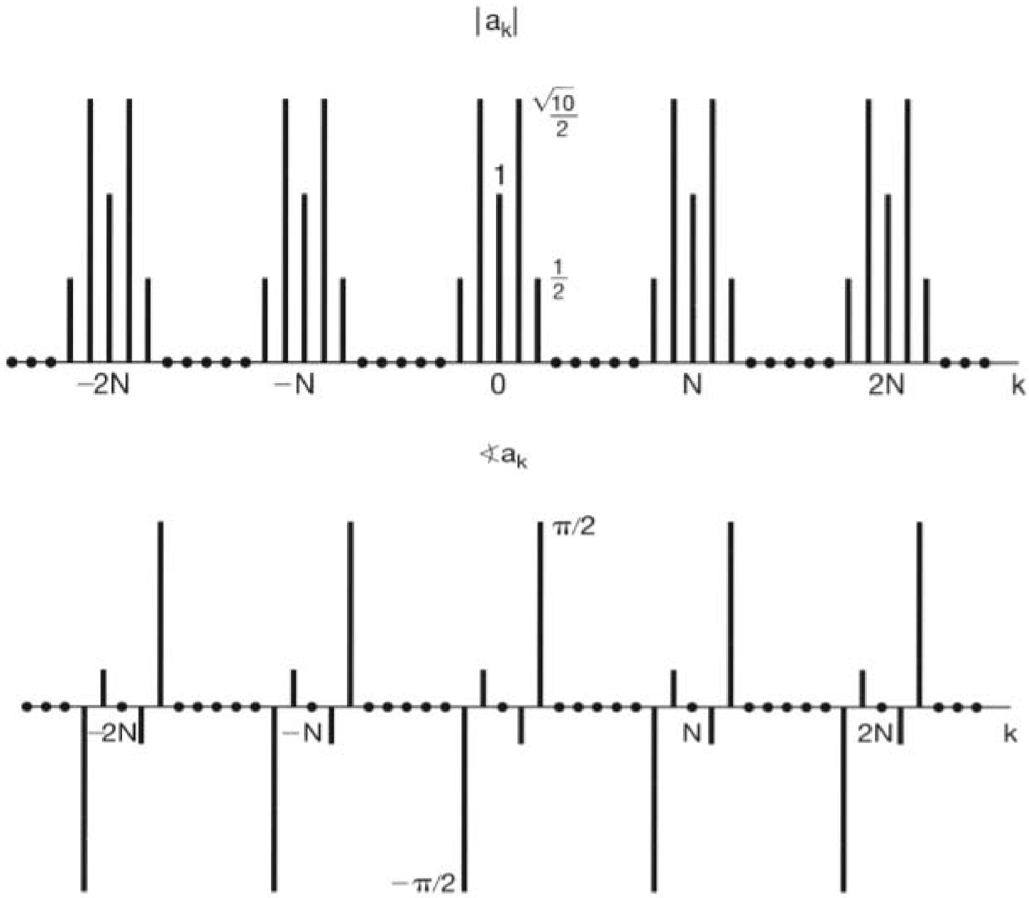
$$a_{-2} = -\tfrac{1}{2}j,$$

$$a_k = 0 \quad \text{for } k = 3, 4, \ldots, N\text{-}3$$

- Example 3.11 (cont'd)

$$x[n]=1+\left(\frac{3}{2}+\frac{1}{2j}\right)e^{j(2\pi/N)n}+\left(\frac{3}{2}-\frac{1}{2j}\right)e^{-j(2\pi/N)n}+\left(\frac{1}{2}e^{j\pi/2}\right)e^{j2(2\pi/N)n}+\left(\frac{1}{2}e^{-j\pi/2}\right)e^{-j2(2\pi/N)n}.$$

$$a_0=1,$$
$$a_1=\tfrac{3}{2}+\tfrac{1}{2j}=\tfrac{3}{2}-\tfrac{1}{2}j,$$
$$a_{-1}=\tfrac{3}{2}-\tfrac{1}{2j}=\tfrac{3}{2}+\tfrac{1}{2}j,$$
$$a_2=\tfrac{1}{2}j,$$
$$a_{-2}=-\tfrac{1}{2}j,$$
$$a_k=0 \quad \text{for } k=3,4,\ldots\ldots,N\text{-}3$$



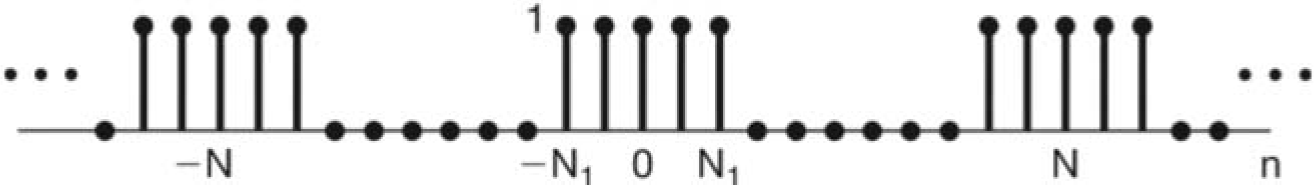
$$x[n] = \sum_{k=<N>} a_k e^{jk(2\pi/N)n}$$

$$a_k = \frac{1}{N} \sum_{n=\langle N \rangle} x[n] e^{-jk(2\pi/N)n}$$

- Example 3.12

$x[n] = 1$ for $-N_1 \le n \le N_1$,

$x[n] = 0$ for $N_1+1 \le n \le N\text{-}N_1\text{-}1$



$$a_k = \frac{1}{N}\sum_{m=0}^{2N_1} e^{-jk(2\pi/N)(m-N_1)} = \frac{1}{N} e^{-jk(2\pi/N)N_1} \sum_{m=0}^{2N_1} e^{-jk(2\pi/N)m}.$$

$$a_k = \frac{1}{N} e^{jk(2\pi/N)N_1} \left(\frac{1-e^{-jk2\pi(2N_1+1)N}}{1-e^{-jk(2\pi/N)}} \right)$$

$$= \frac{1}{N} \frac{e^{-jk(2\pi/2N)}[e^{jk2\pi(N_1+1/2)/N} - e^{-jk2\pi(N_1+1/2)/N}]}{e^{-jk(2\pi/2N)}[e^{jk(2\pi/2N)} - e^{-jk(2\pi/2N)}]}$$

$$= \frac{1}{N} \frac{\sin[2\pi k(N_1+1/2)/N]}{\sin(\pi k/N)}, \; k \ne 0, \pm N, \pm 2N \ldots$$

$$\boxed{a_k = \frac{2N_1+1}{N},} \; k = 0, \pm N, \pm 2N, \ldots$$

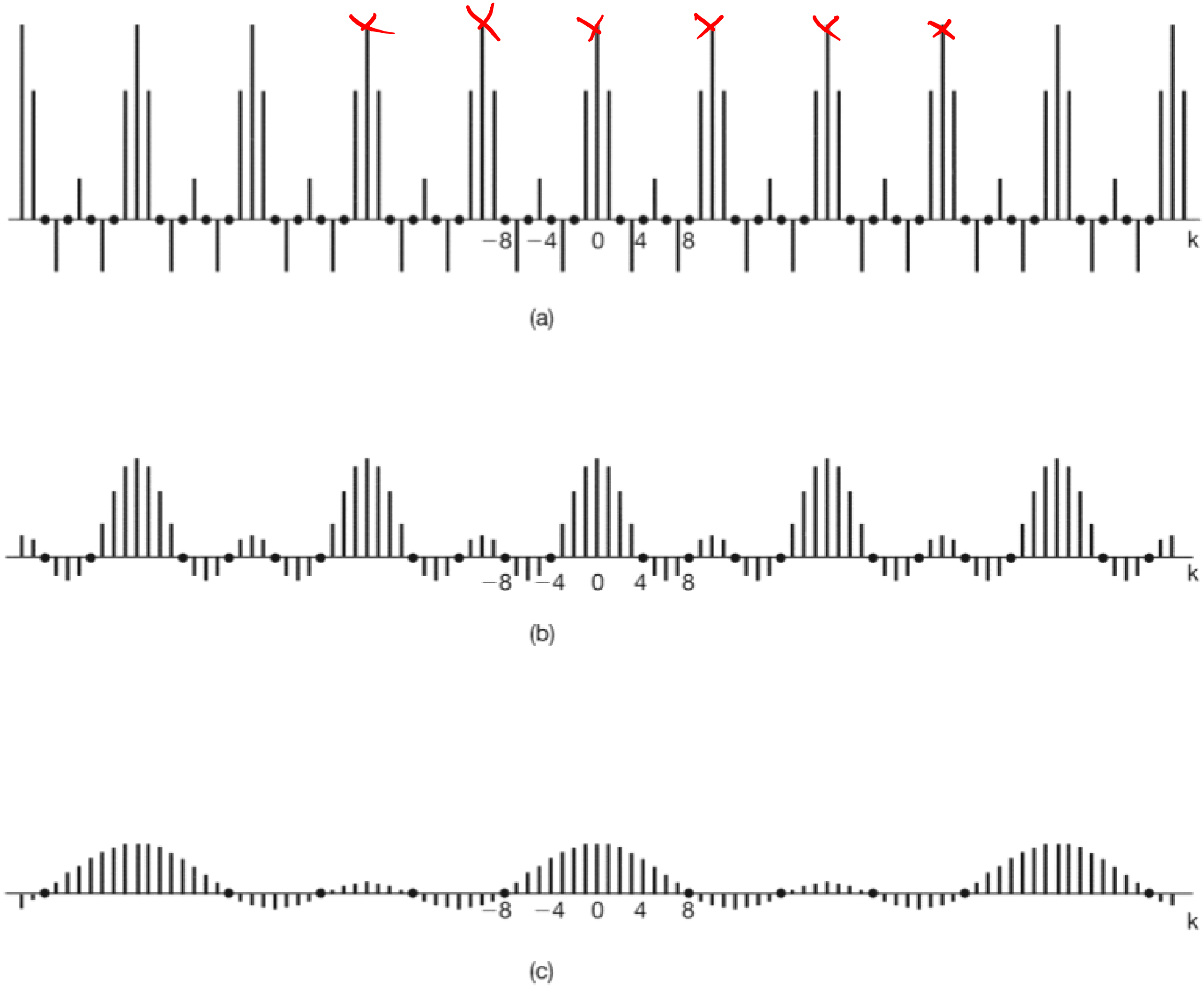
- Example 3.12 (cont'd)

$2N_1 + 1 = 5$

$N = 10$

$N = 20$

$N = 40$



Sect. 3.6 Fourier Series Representation of DT Periodic Signals

- Partial Sum

$N = 9,\ 2N_1 + 1 = 5$

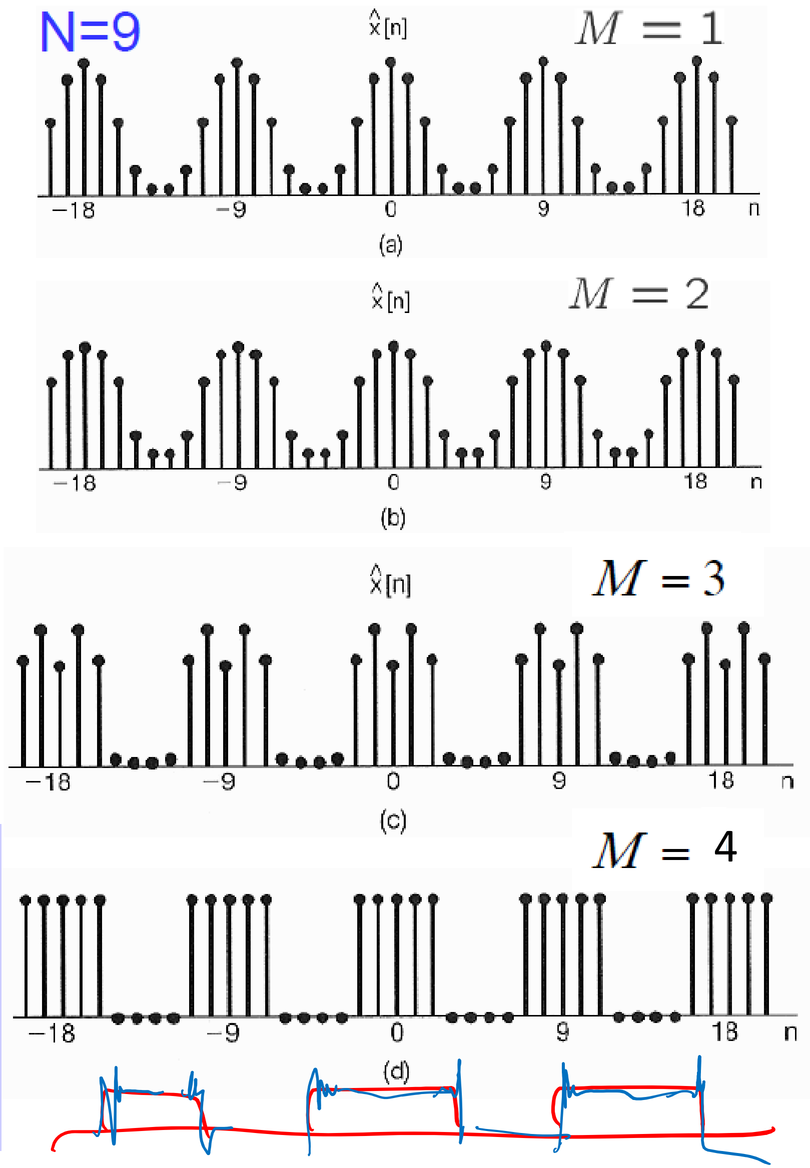
$$x[n] = \sum_{k=<N>} a_k e^{jk(\frac{2\pi}{N})n}$$

If N is odd

$$\hat{x}[n] = \sum_{k=-M}^{M} a_k e^{jk(\frac{2\pi}{N})n}$$

If N is even

$$\hat{x}[n] = \sum_{k=-M+1}^{M} a_k e^{jk(\frac{2\pi}{N})n}$$



$x(t) \leftrightarrow a_k$

Gibbs phenomenon does not exist for DT signals because DT signals are represented by a finite number of FS coefficients. For the same reason, there is no convergence issue with DTFS.

Sect. 3.7 Properties of DT Fourier Series

- Linearity

If $x[n]$ and $y[n]$ are periodic signals with period N and

$$x[n] \xleftrightarrow{FS} a_k$$

$$y[n] \xleftrightarrow{FS} b_k$$

then

$$z[n] = Ax[n] + By[n] \xleftrightarrow{FS} c_k = Aa_k + Bb_k$$

- Time Shifting

$$x[n] \xleftrightarrow{FS} a_k$$

$$\Rightarrow x[n-n_0] \xleftrightarrow{FS} e^{-jk\omega_0 n_0} a_k$$

Sect. 3.7 Properties of DT Fourier Series

- Multiplication

If $x[n]$ and $y[n]$ are periodic signals with period N, and

$$x[n] \xleftrightarrow{FS} a_k \qquad x[n] = \sum_{l=\langle N\rangle} a_l e^{jl\omega_0 n}$$

$$y[n] \xleftrightarrow{FS} b_k \qquad y[n] = \sum_{m=\langle N\rangle} b_m e^{jm\omega_0 n}$$

then x[n]y[n] are also periodic with N, and

$$\mathrm{x}[n]y[n] \xleftrightarrow{FS} d_k = \sum_{l=\langle N\rangle} a_l b_{k-l}$$

$\Rightarrow$ a periodic convolution

Sect. 3.7 Properties of DT Fourier Series

- First Difference

$$x[n] \xleftrightarrow{FS} a_k$$

$$\Rightarrow x[n-n_0] \xleftrightarrow{FS} e^{-jk\omega_0 n_0} a_k = e^{-jk(\frac{2\pi}{N})n_0} a_k$$

$$\Rightarrow x[n-1] \xleftrightarrow{FS} e^{-jk\omega_0} a_k = e^{-jk(\frac{2\pi}{N})} a_k$$

$$x[n]-x[n-1] \xleftrightarrow{FS} (1-e^{-jk(\frac{2\pi}{N})})a_k$$

Sect. 3.7 Properties of DT Fourier Series

- Parseval's Relation for DT Periodic Signals

  - As shown in Problem 3.57:

$$\frac{1}{N}\sum_{n=\langle N\rangle}|x[n]|^2=\sum_{k=\langle N\rangle}|a_k|^2$$

  - Parseval's relation
    The total average power in a periodic signal equals the sum of the average powers of its harmonic components
    (only N distinct harmonic components in DT)

Sect. 3.7 Properties of DT Fourier Series

$$x[n] \xleftrightarrow{FS} a_k$$
$$y[n] \xleftrightarrow{FS} b_k$$
$$x[n]y[n] \xleftrightarrow{FS} d_k$$

- Proof

$$d_k = \sum_{l=<N>} a_l b_{k-l}$$

$$d_k = \frac{1}{N} \sum_{n=<N>} \boxed{x[n]y[n]e}^{-j(2\pi/N)kn}$$

Let $k = 0$, we have

$x^*[n]$

$$\sum_{l=<N>} a_l b_{-l} = \frac{1}{N} \sum_{n=<N>} x[n]y[n]$$

Let $y[n] = x^*[n]$, we hava $b_l = a^*_{-l}$

Substituting it to the above equaiton yields

$$\sum_{l=<N>} a_l a_l^* = \frac{1}{N} \sum_{n=<N>} x[n]x^*[n]$$

That is, $\sum_{l=<N>} |a_l|^2 = \frac{1}{N} \sum_{n=<N>} |x[n]|^2$

TABLE 3.2 PROPERTIES OF DISCRETE-TIME FOURIER SERIES

Property	Periodic Signal	Fourier Series Coefficients
	$\left.\begin{array}{l} x[n] \\ y[n] \end{array}\right\}$ Periodic with period N and fundamental frequency $\omega_0 = 2\pi/N$	$\left.\begin{array}{l} a_k \\ b_k \end{array}\right\}$ Periodic with period N
Linearity	$Ax[n] + By[n]$	$Aa_k + Bb_k$
Time Shifting	$x[n - n_0]$	$a_k e^{-jk(2\pi/N)n_0}$
Frequency Shifting	$e^{jM(2\pi/N)n} x[n]$	a_{k-M}
Conjugation	$x^*[n]$	a^*_{-k}
Time Reversal	$x[-n]$	a_{-k}
Time Scaling	$x_{(m)}[n] = \begin{cases} x[n/m], & \text{if } n \text{ is a multiple of } m \\ 0, & \text{if } n \text{ is not a multiple of } m \end{cases}$ (periodic with period mN)	$\frac{1}{m} a_k \begin{pmatrix}\text{viewed as periodic} \\ \text{with period } mN\end{pmatrix}$
Periodic Convolution	$\sum_{r=\langle N \rangle} x[r]y[n-r]$	$N a_k b_k$
Multiplication	$x[n]y[n]$	$\sum_{l=\langle N \rangle} a_l b_{k-l}$
First Difference	$x[n] - x[n-1]$	$(1 - e^{-jk(2\pi/N)})a_k$
Running Sum	$\sum_{k=-\infty}^{n} x[k] \begin{pmatrix}\text{finite valued and periodic only} \\ \text{if } a_0 = 0\end{pmatrix}$	$\left(\frac{1}{(1 - e^{-jk(2\pi/N)})}\right) a_k$
Conjugate Symmetry for Real Signals	$x[n]$ real	$\begin{cases} a_k = a^*_{-k} \\ \Re e\{a_k\} = \Re e\{a_{-k}\} \\ \Im m\{a_k\} = -\Im m\{a_{-k}\} \\ \lvert a_k \rvert = \lvert a_{-k} \rvert \\ \sphericalangle a_k = -\sphericalangle a_{-k} \end{cases}$
Real and Even Signals	$x[n]$ real and even	a_k real and even
Real and Odd Signals	$x[n]$ real and odd	a_k purely imaginary and odd
Even-Odd Decomposition of Real Signals	$\begin{cases} x_e[n] = \mathcal{E}v\{x[n]\} & [\text{x[n] real}] \\ x_o[n] = \mathcal{O}d\{x[n]\} & [\text{x[n] real}] \end{cases}$	$\Re e\{a_k\}$ $j\Im m\{a_k\}$

Parseval's Relation for Periodic Signals

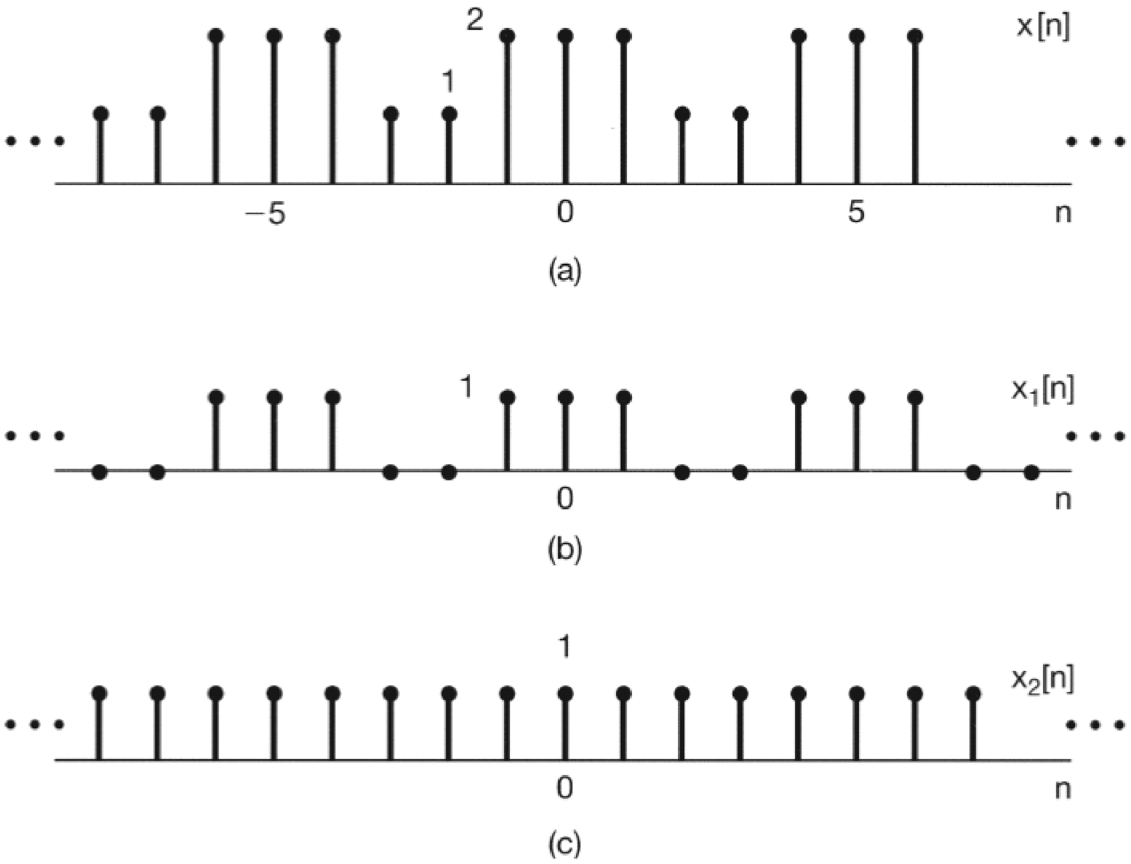
$$\frac{1}{N} \sum_{n=\langle N \rangle} \lvert x[n] \rvert^2 = \sum_{k=\langle N \rangle} \lvert a_k \rvert^2$$

$$x[n] = \sum_{k=\langle N \rangle} a_k e^{jk(2\pi/N)n}$$

$$a_k = \frac{1}{N} \sum_{n=\langle N \rangle} x[n] e^{-jk(2\pi/N)n}$$

- Example 3.13

$$x[n] = x_1[n] + x_2[n]$$



If $x[n] \overset{FS}{\longleftrightarrow} a_k$

$x_1[n] \overset{FS}{\longleftrightarrow} b_k$

$x_2[n] \overset{FS}{\longleftrightarrow} c_k$

then

$$a_k = b_k + c_k$$

- Example 3.13 (cont'd)

$$x[n] \xleftrightarrow{FS} a_k \qquad x_1[n] \xleftrightarrow{FS} b_k \qquad x_2[n] \xleftrightarrow{FS} c_k$$

$$b_k = \begin{cases} \dfrac{1}{5}\dfrac{\sin(3\pi k/5)}{\sin(\pi k/5)}, & for\ k \neq 0, \pm 5, \pm 10, \ldots \\ \dfrac{3}{5}, & for\ k = 0, \pm 5, \pm 10, \ldots \end{cases}$$

$$c_0 = \frac{1}{5}\sum_{n=0}^{4} x_2[n] = 1.$$

$$c_k = \begin{cases} 0, & for\ k \neq 0, \pm 5, \pm 10, \ldots \\ 1, & for\ k = 0, \pm 5, \pm 10, \ldots \end{cases}$$

$$\boxed{a_k = \begin{cases} b_k = \dfrac{1}{5}\dfrac{\sin(3\pi k/5)}{\sin(\pi k/5)}, & for\ k \neq 0, \pm 5, \pm 10, \ldots \\ \dfrac{8}{5}, & for\ k = 0, \pm 5, \pm 10, \ldots \end{cases}}$$

- Example 3.14

Suppose that

1. $x[n]$ is periodic with period $N = 6$.
2. $\sum_{n=0}^{5} x[n] = 2$
3. $\sum_{n=2}^{7} (-1)^n x[n] = 1$
4. $x[n]$ has the minimum power per period among the set of signals satisfying the preceding three conditions.

From $\sum_{n=0}^{5} x[n] = 2$ $\quad a_0 = \frac{1}{N} \boxed{\sum_{N} x[n] \cdot e^{0}} = \frac{1}{3}$ $\quad a_0 = 2/N = 1/3$

From $\sum_{n=2}^{7} (-1)^n x[n] = 1$ $\quad a_{k} = \frac{1}{N} \boxed{\sum_{N} x[n] \cdot e^{-jk\frac{2\pi}{6}n}}$ $\quad a_3 = 1/N = 1/6$

To minimize the power

$$P = \sum_{k=0}^{5} |a_k|^2$$

$a_1 = a_2 = a_4 = a_5 = 0$

- Example 3.14 (cont'd)

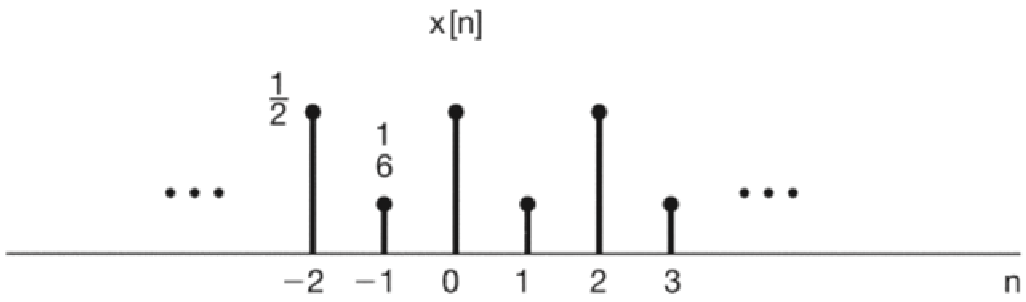
From $\sum_{n=0}^{5} x[n] = 2$ $\qquad a_0 = 2/N = 1/3$

From $\sum_{n=2}^{7} (-1)^n x[n] = 1$ $\qquad a_3 = 1/N = 1/6$

To minimize the power

$$P = \sum_{k=0}^{5} |a_k|^2 \qquad a_1 = a_2 = a_4 = a_5 = 0$$

$$x[n] = a_0 + a_3 e^{j\pi n} = (1/3) + (1/6)(-1)^n$$



Sect. 3.8 FS and LTI Systems

$y(t) = x(t) * h(t)$

$= \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau = e^{st}$

$e^{s(t-\tau)}$

$\int \boxed{}\, d\tau$

- The response of an LTI system

$$in \rightarrow \boxed{\text{LTI}} \rightarrow out$$

$$\begin{cases} \text{CT:} \quad e^{st} \rightarrow H(s)e^{st} \\ \text{DT:} \quad z^{n} \rightarrow H(z)z^{n} \end{cases}$$

$$H(s) = \int_{-\infty}^{+\infty} h(t)e^{-st}\,dt$$

$$H(z) = \sum_{k=-\infty}^{+\infty} h[k]z^{-k}$$

$\Rightarrow$ system function

- If $s = j\omega$ or $z = e^{j\omega}$:

$$H(j\omega) = \int_{-\infty}^{+\infty} h(t)e^{-j\omega t}\,dt$$

$$H(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} h[\text{n}]e^{-j\omega n}$$

$\Rightarrow$ frequency reponse

Sect. 3.8 FS and LTI Systems

- In summary

$x(t)$ $\quad$ $y(t)$

in → LTI H(s/z/ω) → *out*

$(s_i = j\omega_i \text{ or } z_i = e^{j\omega_i})$

$$\begin{cases} \text{CT:} & e^{s_i t} \rightarrow H(s_i)e^{s_i t} \\ \text{DT:} & z_i^n \rightarrow H(z_i)z_i^n \end{cases}$$

eig fun

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k \boxed{e^{jk\omega_0 t}} \quad \Rightarrow y(t) = \sum_{k=-\infty}^{+\infty} a_k H(jk\omega_0) e^{jk\omega_0 t}$$

$$x[n] = \sum_{k=<N>} a_k e^{jk(\frac{2\pi}{N})n} \quad \Rightarrow y[n] = \sum_{k=<N>} \underbrace{a_k H(e^{jk(\frac{2\pi}{N})})}\, e^{jk(\frac{2\pi}{N})n}$$

The output is also periodic with the same fundamental frequency as the input.

The Fourier coefficient of the output is simply the multiplication of the Fourier coefficient of the input with the frequency response of the system.

Sect. 3.9 Filtering

- Filtering
  - Note the relationship between the input and output:

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t} \qquad \Rightarrow y(t) = \sum_{k=-\infty}^{+\infty} \overbrace{\boxed{a_k H(jk\omega_0)}}^{b_k} e^{jk\omega_0 t}$$

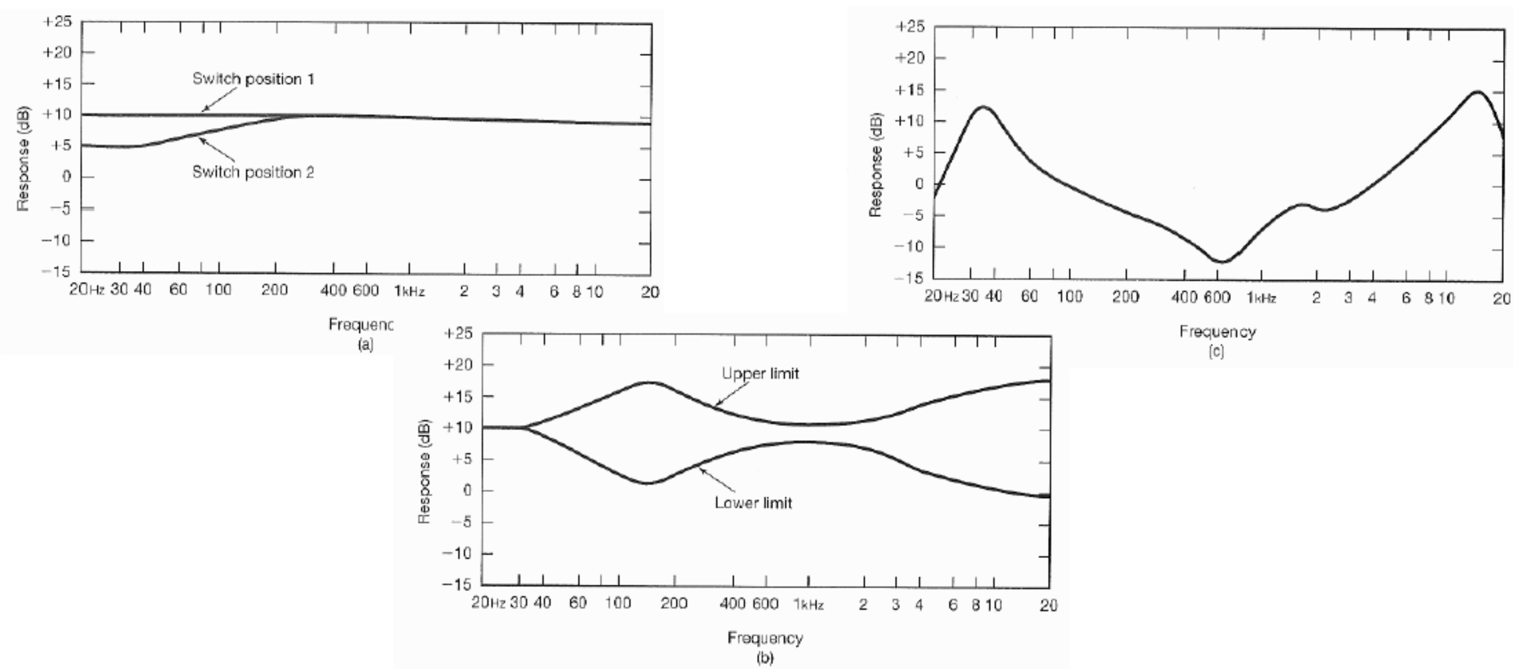
$$x[n] = \sum_{k=<N>} a_k e^{jk(\frac{2\pi}{N})n} \qquad \Rightarrow y[n] = \sum_{k=<N>} a_k H(j(\frac{2\pi}{N})k) e^{jk(\frac{2\pi}{N})n}$$

  - Change the relative amplitudes of the frequency components in a signal indicates frequency-shape filtering.

  - Or, significantly attenuate or eliminate some frequency components entirely indicates frequency-selective filtering.

Sect. 3.9 Filtering

- Filtering
  - Frequency-shaping filters (e.g., audio systems)



Sect. 3.9 Filtering

$\sum a_k e^{jk\omega t}$

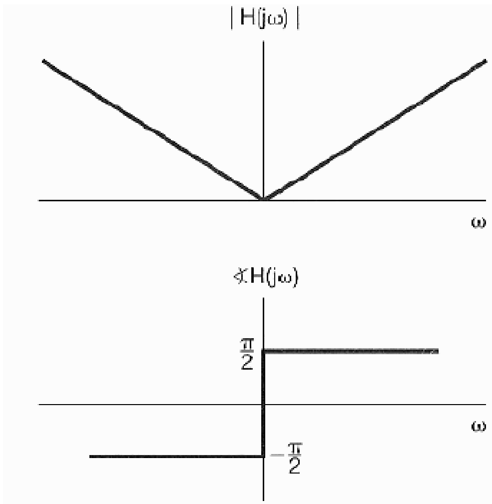
$\sum \boxed{(jk\omega)} a_k \cdot e^{jk\omega t}$

$H(jk\omega)$

- Filtering
  - Frequency-shaping filters (e.g., differentiating filters)

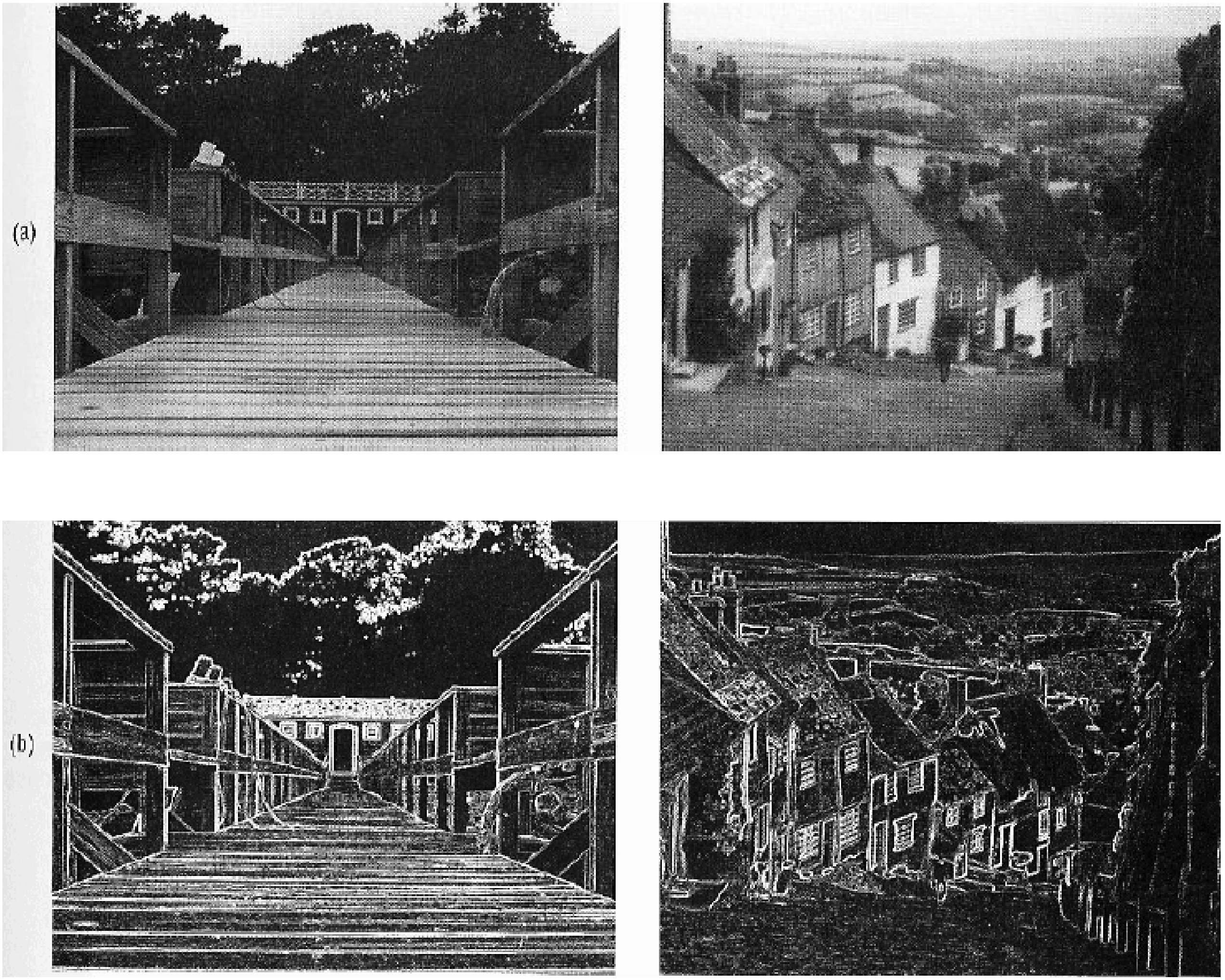
$x(t) \longrightarrow$ Differentiating Filter $\longrightarrow y(t) = \frac{d}{dt}x(t)$

$H(j\omega) = j\omega$



Sect. 3.9 Filtering

- E.g., differentiating filters enhance edges in an image.



Sect. 3.9 Filtering

- Frequency-shaping filters
- Two-point average: a naïve DT low-pass filter (LPF)

$$y[n] = \tfrac{1}{2}(x[n] + x[n-1])$$

$$\Rightarrow h[n] = \tfrac{1}{2}(\delta[n] + \delta[n-1])$$

$$\Rightarrow H(e^{j\omega}) = \tfrac{1}{2}[1 + e^{-j\omega}]$$

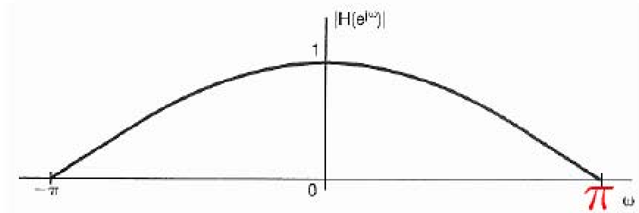
$$= \tfrac{1}{2} e^{-j(\frac{\omega}{2})}[e^{j(\frac{\omega}{2})} + e^{-j(\frac{\omega}{2})}]$$

$$= e^{-j(\frac{\omega}{2})} \cos(\frac{\omega}{2})$$

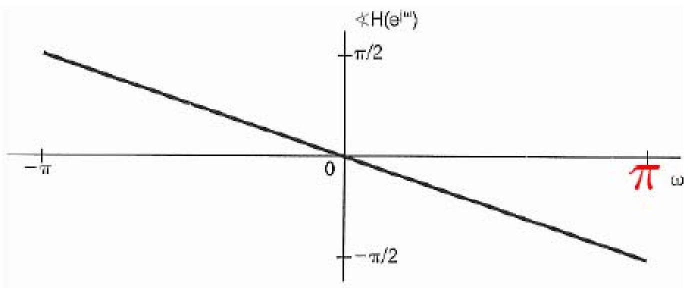
$H(e^{j\omega})$ is periodic with period 2π

$y[n] = x[n]$ at $\omega = 0$

$y[n] = 0$ at $\omega = \pi$



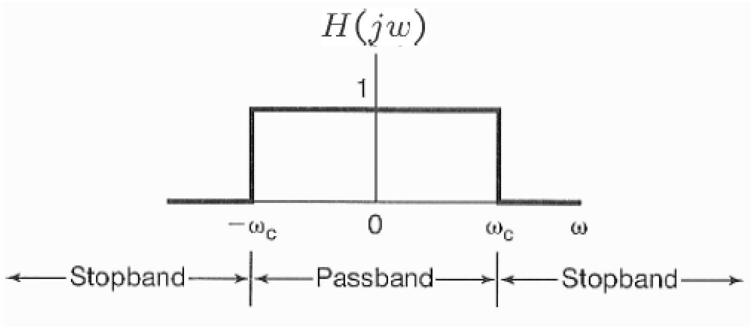
(a)



(b)

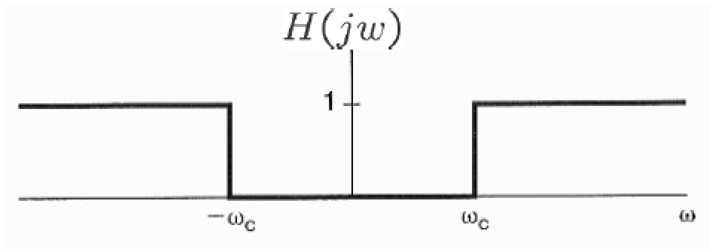
Sect. 3.9 Filtering

- Frequency-selective filters
  - Select some frequency bands and reject others



CT ideal lowpass filter

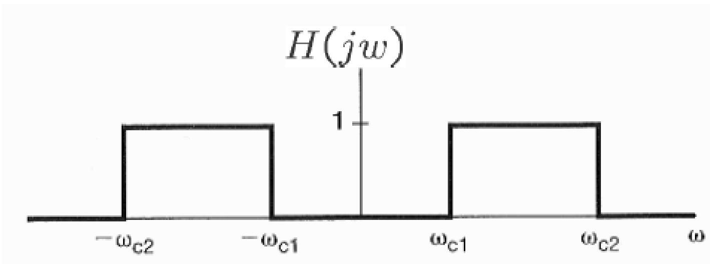
$$H(j\omega) = \begin{cases} 1, & |\omega| \le \omega_c \\ 0, & |\omega| > \omega_c \end{cases}$$



(a)

CT ideal highpass filter

$$H(j\omega) = \begin{cases} 0, & |\omega| < \omega_c \\ 1, & |\omega| \ge \omega_c \end{cases}$$



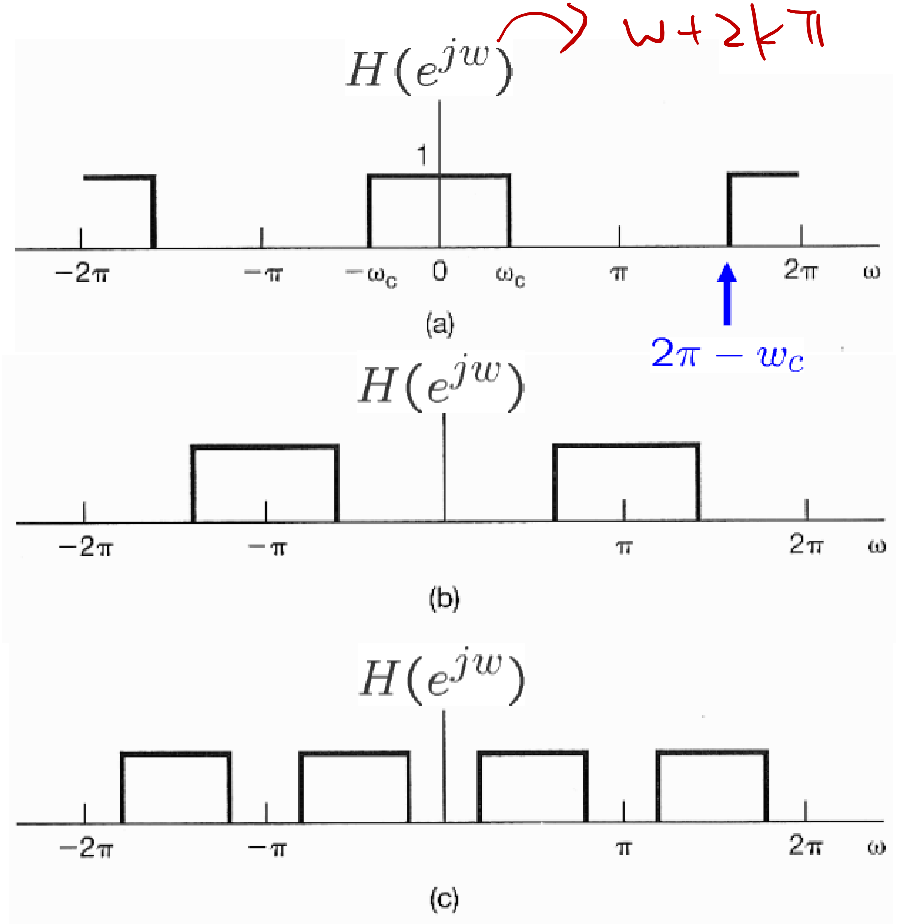
(b)

CT ideal bandpass filter

$$H(j\omega) = \begin{cases} 1, & \omega_{c1} \le |\omega| \le \omega_{c2} \\ 0, & \text{otherwise} \end{cases}$$

Sect. 3.9 Filtering

- Frequency-selective filters
  - Select some frequency bands and reject others



DT ideal lowpass filter

DT ideal highpass filter

DT ideal bandpass filter

Sect. 3.9 Filtering

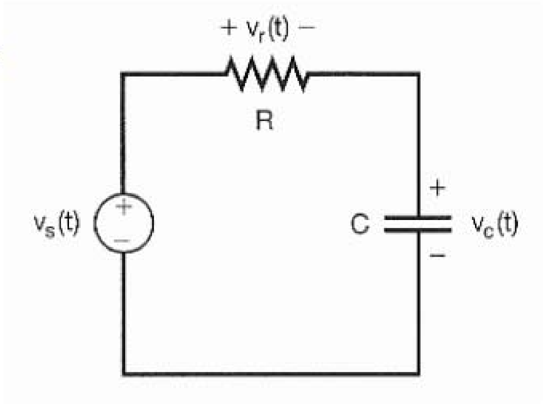
- A simple RC high pass filter (HPF) as frequency-selective filters

Input signal:

$v_s(t) = e^{j\omega t}$

$\delta(t)$

$u(t)$



Output signal

$v_r(t) = G(j\omega)e^{j\omega t}$

$h(t)$

$s(t)$

$$\Rightarrow RC\frac{d}{dt}v_r(t) + v_r(t) = RC\frac{d}{dt}v_s(t)$$

$$\Rightarrow RC\frac{d}{dt}\left[G(j\omega)e^{j\omega t}\right] + G(j\omega)e^{j\omega t} = RC\frac{d}{dt}e^{j\omega t}$$

$$\Rightarrow RCj\omega G(j\omega)e^{j\omega t} + G(j\omega)e^{j\omega t} = RCj\omega e^{j\omega t}$$

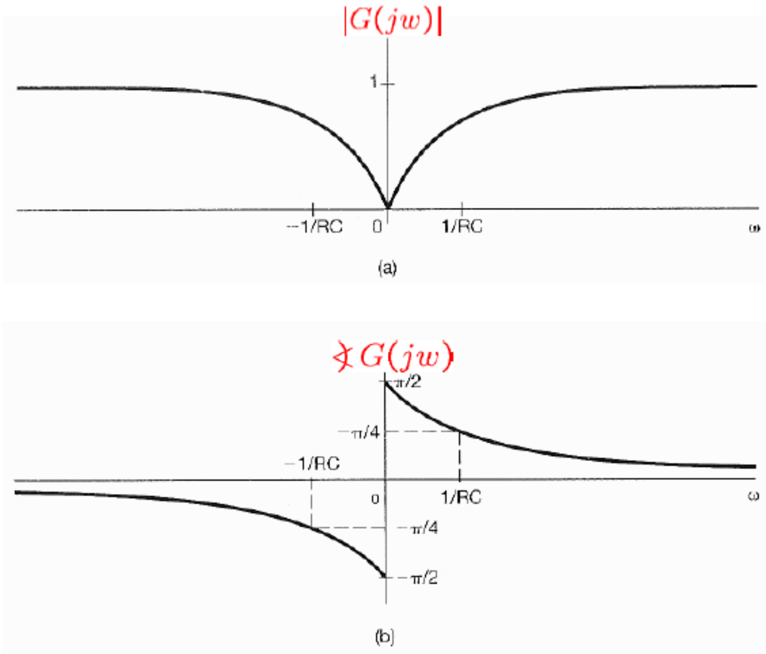
$$\Rightarrow G(j\omega)e^{j\omega t} = \frac{j\omega RC}{1 + j\omega RC}e^{j\omega t}$$

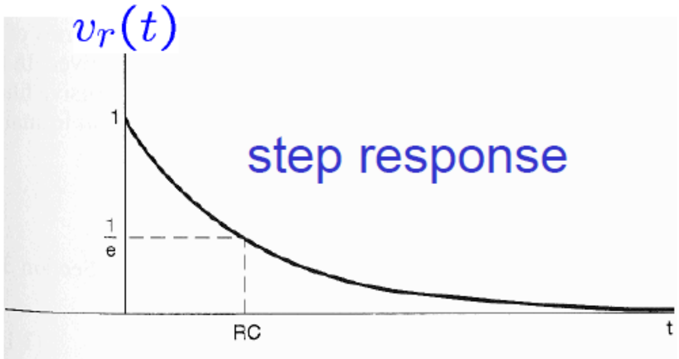
Sect. 3.9 Filtering

- A simple RC high pass filter (HPF) as frequency-selective filters

$$G(j\omega) = \frac{j\omega RC}{1 + j\omega RC}$$

$$\begin{aligned} v_r(t) &= v_s(t) - v_c(t) \\ &= u(t) - (1 - e^{-t/RC})u(t) \\ &= e^{-t/RC}u(t) \end{aligned}$$





Tradeoff between frequency shaping and response time

Sect. 3.9 Filtering

- First-order recursive DT filters

$$y[n] - ay[n-1] = x[n]$$

If $x[n] = e^{j\omega n}$, then $y[n] = H(e^{j\omega})e^{j\omega n}$

$H(e^{j\omega})$: the frequency reponse

$$\Rightarrow H(e^{j\omega})e^{j\omega n} - aH(e^{j\omega})e^{j\omega(n-1)} = e^{j\omega n}$$

$$\Rightarrow \left[1 - ae^{-j\omega}\right]H(e^{j\omega})e^{j\omega n} = e^{j\omega n}$$

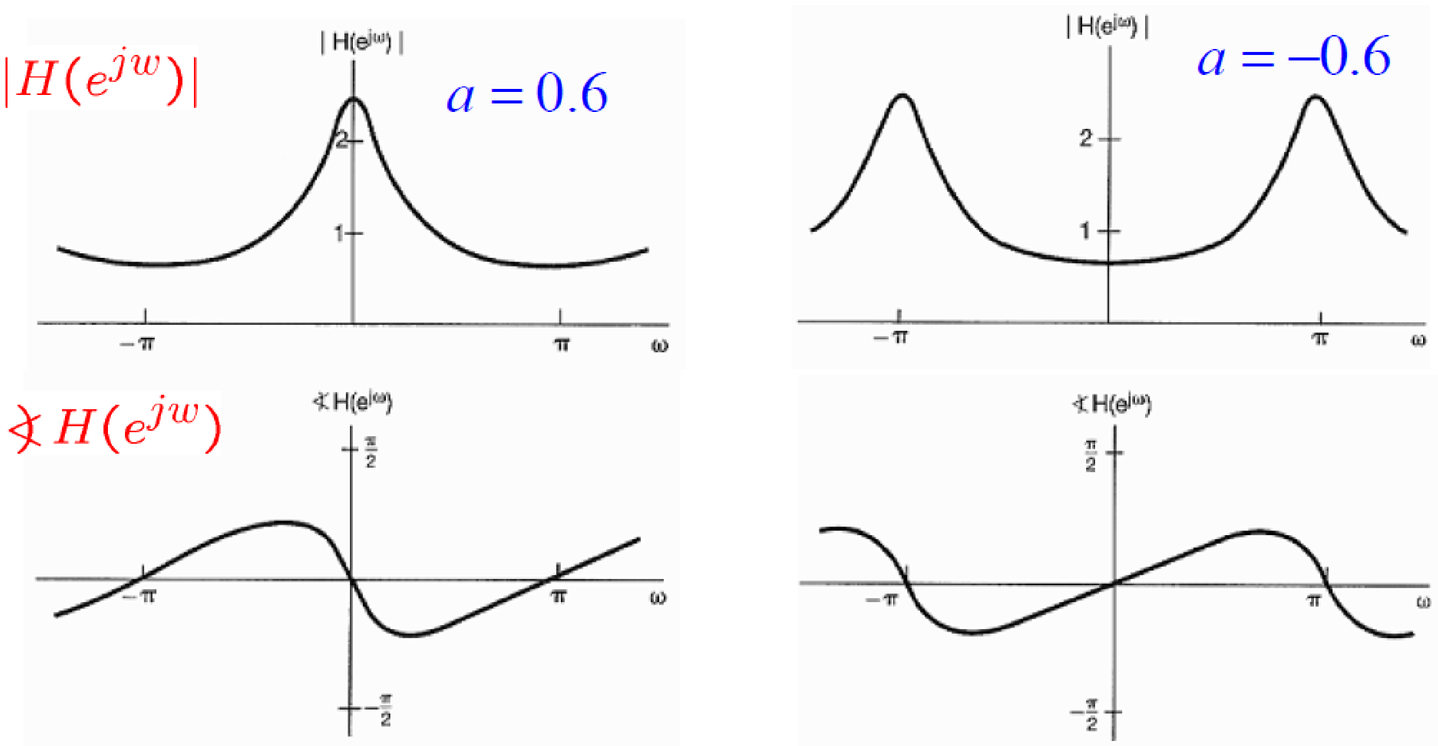
$$\Rightarrow H(e^{j\omega}) = \frac{1}{1 - ae^{-j\omega}}$$

Sect. 3.9 Filtering

- First-order recursive DT filters

$$H(e^{j\omega}) = \frac{1}{1 - ae^{-j\omega}}$$

lowpass filter: $0 < a < 1$ highpass filter: $-1 < a < 0$



Sect. 3.9 Filtering

- First-order recursive DT filters

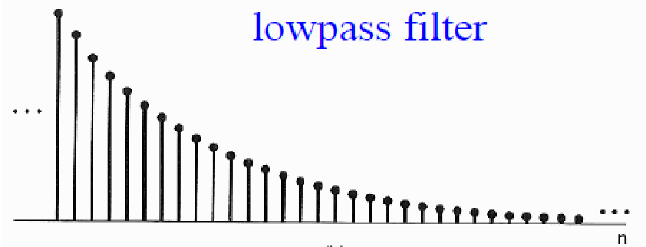
$$y[n] = ay[n-1] + x[n]$$

Impulse reponse: $h[n] = a^n u[n]$

Step reponse: $s[n] = u[n] * h[n] = \dfrac{1-a^{n+1}}{1-a} u[n]$

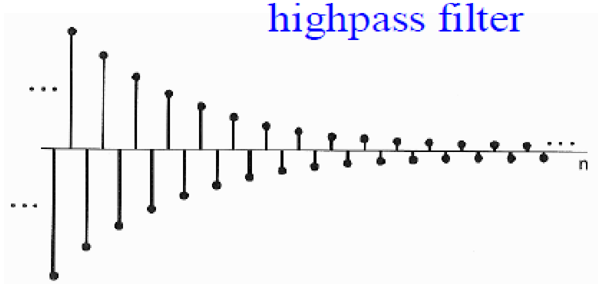
$|a|$ controls the speed of response

$0 < a < 1$
lowpass filter



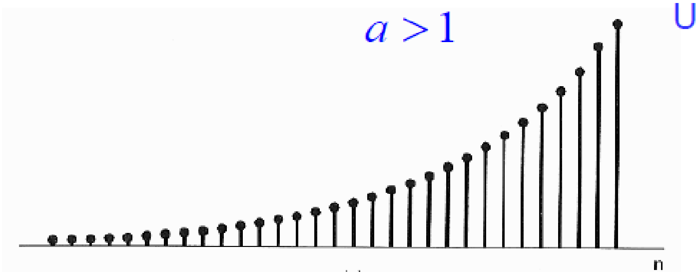
(b)

$-1 < a < 0$
highpass filter



(c)

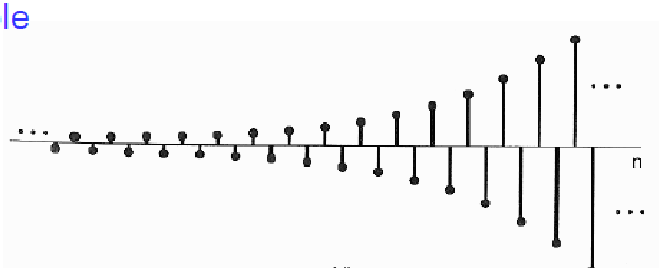
$a > 1$



(a)

Unstable

$a < -1$



(d)