A LEISURELY INTRODUCTION TO SIMPLICIAL SETS

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1. Introduction

Simplicial sets, an extension of the notion of simplicial complexes, have many applications in algebraic topology, where they provide a good model for the homotopy theory of topological spaces. There are natural functors between the categories of topological spaces and simplicial sets called the total singular complex functor and geometric realization, which form an adjoint pair and give a Quillen equivalence between the usual model structures on these categories.[1]

More recently, simplicial sets have found many applications in higher category theory because simplicial sets which satisfy a certain horn lifting criterion (see Definition 5.7) provide a model for $(\infty, 1)$-*categories*, in which every cell of dimension greater than one is invertible. These categories again provide a natural setting for homotopy theory; consequently, many constructions in this setting derive their motivations from algebraic topology as well.

These higher level applications will not be discussed here. Instead, we aim to provide an elementary introduction to simplicial sets, accessible to anyone with a very basic familiarity with category theory. In Section 2, we give two equivalent definitions of simplicial sets, and in Section 3, we describe several examples. Section 4, which discusses several important adjunctions that involve the category of simplicial sets, is the heart of this paper. Rather than present the various adjunctions independently, as is commonly done, we use a categorical construction to show how all of these examples fit in the same general framework, greatly expediting the proof of the adjoint relationship. Finally, in Section 5, we define simplicial spheres and horns, which are the jumping off point for the applications to homotopy theory and higher category theory mentioned above.

One of the examples in Section 3 and some of the notation introduced in Section 5 require some familiarity with the Yoneda lemma. Section 4 assumes that the reader knows the definition of an adjunction but requires no further knowledge about them. Otherwise, comfort with the definitions of a category, functor, and natural transformation should suffice as prerequisites.

The standard references for the theory of simplicial sets and some of their many uses in algebraic topology are [3] and [9], though both sources quickly move on to more advanced material. Less well known is a brief introduction given in [8, §7], which may serve better for the reader interested in only the most rudimentary definitions.

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[1]Model structures on the category of simplicial sets will not be discussed here. See [2], [4], or the original [10] for an introduction to model categories, which includes the aforementioned examples.

2. Definitions

Let $\mathbf{\Delta}$ be the category whose objects are finite, totally ordered sets

$$[n] = \{0, 1, \ldots, n\}$$

and morphisms are order preserving functions. $\mathbf{\Delta}$ is *almost* the category of all finite ordinals but not quite; we do not include the empty ordinal and we denote the ordinal $n+1$ (above) by $[n]$ to emphasize connections with topology, which will be expounded upon below.

Definition 2.1. A *simplicial set* is a contravariant functor from $\mathbf{\Delta}$ to **Set**. More generally, for any category $\mathcal{C}$, a *simplicial object* in $\mathcal{C}$ is a functor $X : \mathbf{\Delta}^{\mathrm{op}} \to \mathcal{C}$.

Let $X : \mathbf{\Delta}^{\mathrm{op}} \to \mathbf{Set}$ be a simplicial set. It is standard to write X_n for the set $X[n]$ and call its elements *n-simplices*. We write **sSet** for the category of simplicial sets, which is simply the functor category $[\mathbf{\Delta}^{\mathrm{op}}, \mathbf{Set}]$; in particular, morphisms $f : X \to Y$ between simplicial sets are natural transformations.

The combinatorial data of a simplicial set can be simplified somewhat because the category $\mathbf{\Delta}$ has a natural generating set of morphisms. For each $n \geq 0$ there are $n+1$ injections $d^i : [n-1] \to [n]$ called the *coface* maps and $n+1$ surjections $s^i : [n+1] \to [n]$ called the *codegeneracy* maps, defined as follows:

$$d^i(k) = \begin{cases} k, & k < i \\ k+1, & k \geq i \end{cases} \quad \text{and} \quad s^i(k) = \begin{cases} k, & k \leq i \\ k-1, & k > i \end{cases}$$

for $i = 0, \ldots, n$.[2] The i-th coface map d^i misses the element i in the image, while the i-th codegeneracy map s^i sends two elements to i. These morphisms satisfy several obvious relations:

$$\begin{aligned} d^j d^i &= d^i d^{j-1}, \quad i < j \\ s^j s^i &= s^i s^{j+1}, \quad i \leq j \end{aligned}$$

$$s^j d^i = \begin{cases} 1, & i = j, j+1 \\ d^i s^{j-1}, & i < j \\ d^{i-1} s^j, & i > j+1. \end{cases} \tag{2.2}$$

It is not difficult to verify that every morphism of $\mathbf{\Delta}$ can be expressed as a composite of the coface and codegeneracy maps. If we impose further artificial requirements specifying the order in which the generating morphisms should occur, then each morphism of $\mathbf{\Delta}$ can be expressed uniquely as a composite of this particular form. Details are left to the reader (or see [8, §7.5]). The intuition that these names are meant to elicit comes from algebraic topology (see Example 3.2).

If X is a simplicial set, we write

$$\begin{aligned} d_i &= X d^i : X_n \to X_{n-1} \\ s_i &= X s^i : X_n \to X_{n+1} \end{aligned}$$

and call these the *face* and *degeneracy* maps respectively. Again, see Example 3.2 for the intuition. The d_i and s_i will then satisfy relations dual to (2.2). In fact, the data of a simplicial set is completely specified by the sets X_n and the maps d_i, s_i in the sense of the following alternative definition:

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[2] As is usual, the dependence on n is suppressed in this notation, as the codomain should be clear from the context.

Definition 2.3. A *simplicial set* X is a collection of sets X_n for each integer $n \geq 0$ together with functions $d_i : X_n \to X_{n-1}$ and $s_i : X_n \to X_{n+1}$ for all $0 \leq i \leq n$ and for each n satisfying the following relations:

$$\begin{aligned} d_i d_j = & \quad d_{j-1} d_i, \quad i < j \\ s_i s_j = & \quad s_{j+1} s_i, \quad i \leq j \end{aligned}$$

$$d_i s_j = \begin{cases} 1, & i = j, j+1 \\ s_{j-1} d_i, & i < j \\ s_j d_{i-1}, & i > j+1. \end{cases} \tag{2.4}$$

For example, this is the first definition given in [9]. In practice, one usually specifies a simplicial set X by describing the sets X_n of n-simplices and then defining the required face and degeneracy maps. Mercifully, the required relations are often obvious, and even if they are not, it is still advisable to assert that they are, after privately verifying that they do in fact hold.

3. Yoneda and Examples

The Yoneda embedding defines a functor

$$y : \mathbf{\Delta} \hookrightarrow [\mathbf{\Delta}^{\mathrm{op}}, \mathbf{Set}] = \mathbf{sSet}.$$

For each $[m] \in \mathbf{\Delta}$, we denote the image

$$y[m] = \mathbf{\Delta}(-, [m]) =: \Delta^m$$

by Δ^m. This gives us our first example of a simplicial set. From the definition $\Delta^m_n = \mathbf{\Delta}([n], [m])$, the set of maps $[n] \to [m]$ in $\mathbf{\Delta}$. The face and degeneracy maps d_i and s_i are given by precomposition in $\mathbf{\Delta}$ by d^i and s^i, respectively. Explicitly:

$$d_i : \Delta^m_n \to \Delta^m_{n-1} \text{ is the function } [n] \xrightarrow{f} [m] \mapsto [n-1] \xrightarrow{d^i} [n] \xrightarrow{f} [m] \text{ and}$$

$$s_i : \Delta^m_n \to \Delta^m_{n+1} \text{ is the function } [n] \xrightarrow{f} [m] \mapsto [n+1] \xrightarrow{s^i} [n] \xrightarrow{f} [m].$$

The Yoneda lemma tells us that y is full and faithful, which means that natural transformations $f : \Delta^n \to \Delta^m$ correspond bijectively to morphisms $f : [n] \to [m]$ in $\mathbf{\Delta}$; hence, our notation deliberately confuses the two. Explicitly the functions $f_k : \Delta^n_k \to \Delta^m_k$ are defined by postcomposition by f.

The Δ^n are the canonical representable simplicial sets. For any simplicial set X, the Yoneda lemma tells us that there is a natural bijective correspondence between elements of the set X_n and morphisms $\Delta^n \to X$ in $\mathbf{sSet}$. So an n-simplicex x can be regarded as a map $x : \Delta^n \to X$ such that $x_n(1_{[n]}) = x$; again, our notation will deliberately confuse the two. Applying this notational convention, if x is an n-simplex, the $n-1$-simplex $d_i(x)$ equals xd^i; here the left hand side uses traditional function notation for the element $x \in X_n$ while the right hand side expresses the element $d_i(x) \in X_{n-1}$ as the composite

$$\Delta^{n-1} \xrightarrow{d^i} \Delta^n \xrightarrow{x} X.$$

This notation can be a little tricky to become accustomed to but can also be very useful.

In the remainder of this section, we will give two further examples of simplicial sets, both of which will play a key role in §4.

Example 3.1. Let $\mathcal{C}$ be any small category. Define the *nerve* of $\mathcal{C}$ to be the simplicial set $\mathbb{N}\mathcal{C}$ defined as follows:

$\mathbb{N}\mathcal{C}_0 = \operatorname{ob}\mathcal{C}$
$\mathbb{N}\mathcal{C}_1 = \operatorname{mor}\mathcal{C}$
$\mathbb{N}\mathcal{C}_2 = \{$pairs of composable arrows $\to\to$ in $\mathcal{C}\}$
$\vdots$
$\mathbb{N}\mathcal{C}_n = \{$strings of n composable arrows $\to\to\cdots\to$ in $\mathcal{C}\}$.

The degeneracy map $s_i : \mathbb{N}\mathcal{C}_n \to \mathbb{N}\mathcal{C}_{n+1}$ takes a string of n composable arrows

$$c_0 \to c_1 \to \cdots \to c_i \to \cdots \to c_n$$

and obtains $n+1$ composable arrows by inserting the identity at c_i in the i-th spot. The face map $d_i : \mathbb{N}\mathcal{C}_n \to \mathbb{N}\mathcal{C}_{n-1}$ composes the i-th and $i+1$-th arrows if $0 < i < n$, and leaves out the first or last arrow for $i = 0$ or n respectively. One can verify directly that these maps satisfy the relations from §2. We decline to do so at this point, however, because this will become more obvious when we redefine the nerve of a category in §4.

Example 3.2. To begin, we define a *covariant* functor $\Delta : \boldsymbol{\Delta} \to \mathbf{Top}$ that sends $[n]$ to the standard topological n-simplex

$$\Delta_n = \{(x_0, \ldots, x_n) \in \mathbb{R}^{n+1} \mid x_0 + \cdots + x_n = 1, x_i \geq 0\}.$$

The morphisms $\Delta d^i : \Delta_n \to \Delta_{n+1}$ insert a zero in the i-th coordinate, while the morphisms $\Delta s^i : \Delta^n \to \Delta^{n-1}$ add the x_i and x_{i+1} coordinates. Geometrically, Δd^i inserts Δ_{n-1} as the i-th face of Δ_n and Δs^i projects the $n+1$ simplex Δ_{n+1} onto the n-simplex that is orthogonal to its i-th face.

Let Y be any topological space. We define a simplicial set $S(Y)$ by defining $S(Y)_n = \mathbf{Top}(\Delta_n, Y)$ to be the set of continuous maps from the standard topological n-simplex to Y. Elements of this set are called n-simplices of Y in algebraic topology, which coincides with our terminology. Precomposition by Δd^i induces a map of sets

$$d_i : \mathbf{Top}(\Delta_{n+1}, Y) \to \mathbf{Top}(\Delta_n, Y),$$

which obtains a singular n-simplex as the i-th face of a singular $n+1$-simplex. Similarly, precomposition by Δs^i induces a map of sets

$$s_i : \mathbf{Top}(\Delta_{n-1}, Y) \to \mathbf{Top}(\Delta_n, Y),$$

whose image consists of the degenerate singular n-simplices, which are continuous functions $\Delta_n \to Y$ that factor through Δ_{n-1} along Δs^i. The morphisms d_i and s_i will satisfy (2.4) because the Δd^i and Δs^i satisfy the dual relations (2.2). This makes $S(Y)$ a simplicial set. It is called the *total singular complex* of a topological space.

4. Many Important Adjunctions

In this section we will describe several important pairs of adjoint functors involving the category **sSet**. For all these examples, the same category theoretic proof will show that the desired functors are adjoints. We begin by proving this fact in its full generality and then give the examples of interest afterwards.

Let $\mathcal{E}$ be any cocomplete, locally small category and $F : \boldsymbol{\Delta} \to \mathcal{E}$ any covariant functor. We can define a functor $R : \mathcal{E} \to \mathbf{sSet}$ by setting

$$Re_n = \mathcal{E}(F[n], e),$$

the set of morphisms in $\mathcal{E}$ from $F[n]$ to e for any object e of $\mathcal{E}$. As in the previous example, $d_i : Re_n \to Re_{n-1}$ is given by precomposition by Fd^i and $s_i : Re_n \to Re_{n+1}$ is given by precomposition by Fs^i. As before, the Fd^i and Fs^i satisfy (2.2), so the d_i and s_i will satisfy (2.4). Thus, Re is a simplicial set. Levelwise postcomposition defines a natural transformation of simplicial sets and makes R a functor.

The functor R will be the right adjoint of a functor $L : \mathbf{sSet} \to \mathcal{E}$, which is most concisely described as the left Kan extension of $F : \mathbf{\Delta} \to \mathcal{E}$ along the Yoneda embedding. Explicitly, this functor is defined by using a particular type of colimit called a coend. It can also be described using the more conventional coequalizer, but this author believes the coend description makes it easier to understand the colimiting diagram.

Let X be a simplicial set. Because $\mathcal{E}$ is cocomplete, there exists *copowers*

$$X_m \cdot F[n]$$

for any $n, m \in \mathbb{Z}_{\geq 0}$, where the compower $S \cdot e$ of a set S with an object e of $\mathcal{E}$ just means the coproduct $\coprod_S e$ in $\mathcal{E}$ of copies of e indexed by S. A morphism $f : [n] \to [m]$ of $\mathbf{\Delta}$ induces a map

$$f_* : X_m \cdot F[n] \to X_m \cdot F[m],$$

which applies Ff to the copy of $F[n]$ in the component corresponding to $x \in X_m$ and includes it in the component corresponding to x in $X_m \cdot F[m]$. There is also a map

$$f^* : X_m \cdot F[n] \to X_n \cdot F[n],$$

which includes the component corresponding to $x \in X_m$ in the component corresponding to $Xf(x) \in X_n$.

Definition 4.1. Consider the diagram consisting of morphisms f_* and f^* for each $f \in \operatorname{mor} \mathbf{\Delta}$. A *wedge* under this diagram is an object e of $\mathcal{E}$ together with maps $\gamma_n : X_n \cdot F[n] \to e$ such that the squares

$$\begin{tikzcd} X_m \cdot F[n] \arrow[r, "f_*"] \arrow[d, "f^*"'] & X_m \cdot F[m] \arrow[d, "\gamma_m"] \\ X_n \cdot F[n] \arrow[r, "\gamma_n"'] & e \end{tikzcd} \tag{4.2}$$

commute for each f. The *coend* $\int^n X_n \cdot F[n]$ is defined to be a universal wedge. Equivalently, $\int^n X_n \cdot F[n]$ is a coequalizer of the diagram

$$\coprod_{f:[n]\to[m]} X_m \cdot F[n] \overset{f_*}{\underset{f^*}{\rightrightarrows}} \coprod_{[n]} X_n \cdot F[n]$$

On objects, the functor $L : \mathbf{sSet} \to \mathcal{E}$ is given by this coend:

$$LX := \int^n X_n \cdot F[n].$$

If $\alpha : X \to Y$ is a map of simplicial sets, then α induces a wedge from the diagram for X to the object LY, and the universal property of the coend gives us a map

$$L\alpha : LX \to LY;$$

this defines the functor L on morphisms. Uniqueness of the universal property will imply that L is functorial, as is always the case when we use a colimit construction to define a functor.

It remains to show that L is left adjoint to R. A morphism $\gamma : X \to Re$ of simplicial sets consists of maps $\gamma_n : X_n \to \mathcal{E}(F[n], e)$ for each n. We can use this to define morphisms $\gamma_n : X_n \cdot F[n] \to e$ in $\mathcal{E}$ by applying $\gamma_n(x)$ to the component of the copower corresponding to the element $x \in X_n$. We claim that the γ_n form a wedge under the diagram for X to the element e; we must show that 4.2 commutes for each $f \in \text{mor}\,\mathbf{\Delta}$. Unravelling the definitions, this is equivalent to commutativity of

$$\begin{tikzcd} X_m \arrow[r, "\gamma_m"] \arrow[d, "Xf"'] & \mathcal{E}(F[m], e) \arrow[d, "(Ff)^*"] \\ X_n \arrow[r, "\gamma_n"'] & \mathcal{E}(F[n], e) \end{tikzcd}$$

which is true by naturality of γ. The universal property of the coend, then gives us a map $\gamma^\flat : LX \to e$, which we use to define a map

$$\phi : \gamma \mapsto \gamma^\flat : \mathbf{sSet}(X, Re) \longrightarrow \mathcal{E}(LX, e).$$

Conversely, given an arrow $h : LX \to e$ in $\mathcal{E}$, we have maps

$$X_n \cdot F[n] \xrightarrow{\omega_n} LX \xrightarrow{h} e$$

for each n, where the ω_n are the maps of the coend wedge. We can use these to define functions

$$h_n^\sharp : X_n \to \mathcal{E}(F[n], e)$$

which taken $x \in X_n$ to the map induced by $h\omega_n$ on the corresponding component of the coproduct. We claim that $h^\sharp : X \to Re$ is a map of simplicial sets. As noted above, the left hand square below commutes if and only if the right hand one does:

$$\begin{tikzcd} X_m \arrow[r, "h_m^\sharp"] \arrow[d, "Xf"'] & \mathcal{E}(F[m], e) \arrow[d, "(Ff)^*"] \\ X_n \arrow[r, "h_n^\sharp"'] & \mathcal{E}(F[n], e) \end{tikzcd} \qquad \begin{tikzcd} X_m \cdot F[n] \arrow[r, "f_*"] \arrow[d, "f^*"'] & X_m \cdot F[m] \arrow[d, "h\omega_m"] \\ X_n \cdot F[n] \arrow[r, "h\omega_n"'] & e \end{tikzcd}$$

So $h^\sharp$ is natural, since $h\omega$ is a wedge, and this defines a map

$$\psi : h \mapsto h^\sharp : \mathcal{E}(LX, e) \longrightarrow \mathbf{sSet}(X, Re).$$

It is easy to see that both ϕ and ψ are natural in X and e and are inverses. This shows that L is left adjoint to R, as desired.

Example 4.3. Post-composition turns the total singular complex defined in Example 3.2 into a functor $S : \mathbf{Top} \to \mathbf{sSet}$. Following the above prescription, its left adjoint $|\cdot| : \mathbf{sSet} \to \mathbf{Top}$ is defined on objects by

$$|X| = \int^n X_n \times \Delta_n = \text{colim} \left(\textstyle\coprod_{f:[n]\to[m]} X_m \times \Delta_n \underset{f^*}{\overset{f_*}{\rightrightarrows}} \textstyle\coprod_{[n]} X_n \times \Delta_n \right)$$

and is called the *geometric realization* of the simplicial set X. The copower $X_n \cdot \Delta_n$ of topological spaces is equivalent to the cartesian product $X_n \times \Delta_n$ where the set X_n is given the discrete topology, so we have deviated from our previous notation to reflect this. This coend is also knows as the *tensor product* of the functors $\mathbf{\Delta}^{\mathrm{op}} \xrightarrow{X} \mathbf{Set} \xrightarrow{D} \mathbf{Top}$ and $\mathbf{\Delta} \xrightarrow{\Delta} \mathbf{Top}$, which is commonly denoted by $X \otimes_{\mathbf{\Delta}} \Delta$, ignoring the discrete topological space functor D. Such a tensor product can be defined more generally for any pair of functors $F : \mathcal{C}^{\mathrm{op}} \to \mathcal{D}$ and $G : \mathcal{C} \to \mathcal{D}$ where $\mathcal{D}$ has some monoidal product $\otimes$ to play the role of the cartesian product in **Top** and provided the desired copowers and coends exist in $\mathcal{D}$.

Example 4.4. The construction of the nerve of a category in Example 3.1 gives a functor $\mathbf{Cat} \to \mathbf{sSet}$. Let $F : \mathbf{\Delta} \to \mathbf{Cat}$ be the functor that sends $[n]$ to the category

$$\underset{\sim}{n} = \cdot \to \cdot \to \cdots \to \cdot$$

with $n+1$ objects and n generating non-identity arrows, as well as their composites and the requisite identities. Each order-preserving map in $\mathbf{\Delta}$ uniquely determines a functor between categories of this type by prescribing the object function, and the object functions of all such functors give rise to order preserving maps. Thus F is actually an embedding $\mathbf{\Delta} \hookrightarrow \mathbf{Cat}$. In this example, the right adjoint is commonly called the *categorical nerve* functor $\mathbb{N} : \mathbf{Cat} \to \mathbf{sSet}$, with

$$\mathbb{N}\mathcal{C}_n = \mathbf{Cat}(\underset{\sim}{n}, \mathcal{C}),$$

or, more colloquially, the set of strings of n composable arrows in $\mathcal{C}$. The map $s_i : \mathbb{N}\mathcal{C}_n \to \mathbb{N}\mathcal{C}_{n+1}$ inserts the appropriate identity arrow at the i-th place, and $d_i : \mathbb{N}\mathcal{C}_n \to \mathbb{N}\mathcal{C}_{n-1}$ leaves off an outside arrow if $i = 0$ or n and composes the i-th and $i+1$-th arrows otherwise.

The left adjoint is typically denoted $\tau_1 : \mathbf{sSet} \to \mathbf{Cat}$ for first truncation. Surprisingly, the image of a simplicial set X under this functor is completely determined by its 0-,1-, and 2-simplices and the maps between them. This becomes obvious when we give an alternate description of τ_1. Given a simplicial set X, define $\mathrm{ob}\,\tau_1 X$ to be X_0. Morphisms in $\tau_1 X$ are freely generated by the set X_1 subject to relations given by elements of X_2 (we'll say what this means precisely in a moment). The degeneracy map $s_0 : X_0 \to X_1$ picks out the identity morphism for each object. The face maps $d_1, d_0 : X_1 \to X_0$ assign a domain and codomain, respectively, to each arrow. To obtain $\tau_1 X$, we begin by taking the free graph on X_0 generated by the arrows X_1 and then impose the relation $h = gf$ if there exists a 2-simplex $x \in X_2$ such that $xd_2 = f, xd_0 = g$, and $xd_1 = h$:

$$\begin{tikzcd} & 1 \arrow[dr, "g"] & \\ 0 \arrow[ur, "f"] \arrow[rr, "h"'] & & 2 \end{tikzcd}$$

Composition is associative already in the free graph. The degenerate 2-simplices in the image of the maps s_0 and s_1 give witness to the fact that the identity arrows behave like identities with respect to composition. This makes $\tau_1 X$ a category.

It is not hard to verify explicitly that this definition of τ_1 gives a left adjoint to $\mathbb{N}$; then the fact that left adjoints of a given functor must necessarily by naturally isomorphic gives an economical verification that this definition of τ_1 is compatible with the definition given above.

Example 4.5. This construction can also be used to show that **sSet** is *cartesian closed*, i.e., for every simplicial set Y, the functor $- \times Y : \mathbf{sSet} \to \mathbf{sSet}$ has a right adjoint. This is true for all categories of presheaves on a small category, and the construction of the right adjoint given here is the usual one.

Fix a simplicial set Y and let $F : \mathbf{\Delta} \to \mathbf{sSet}$ be given on objects by

$$[n] \mapsto \Delta^n \times Y$$

and on morphisms by $f \mapsto f \times 1_Y$. Recall that we mentioned that the functor L is defined to be the left Kan extension of F along the Yoneda embedding. In this case, F is the composite of the Yoneda embedding with the functor $- \times Y$, so L is the functor $- \times Y$.[3]

The right adjoint R is traditionally denoted $[Y, -]$ or $(-)^Y$ and sometimes referred to as an *internal hom-set*. For a simplicial set Z, the above construction gives

$$RZ_n = [Y, Z]_n = \mathbf{sSet}(\Delta^n \times Y, Z);$$

that is, n-simplices $[Y, Z]_n$ are natural transformations $\Delta^n \times Y \to Z$. $[Y, Z]$ is a simplicial set with face and degeneracy maps

$d_i : [Y, Z]_n \to [Y, Z]_{n-1}$ given by precomposition by $d^i \times 1 : \Delta^{n-1} \times Y \to \Delta^n \times Y$

$s_i : [Y, Z]_n \to [Y, Z]_{n+1}$ given by precomposition by $s^i \times 1 : \Delta^{n+1} \times Y \to \Delta^n \times Y$

These definitions give us the desired adjunction

$$\mathbf{sSet}(X \times Y, Z) \cong \mathbf{sSet}(X, [Y, Z]).$$

In fact, this bijection is natural in all three variables, which is easily verified.

There are several other examples of adjoint pairs of functors involving **sSet** that fit into the above template that we choose not to define here but list in case the reader may encounter these elsewhere. Descriptions of the first two can be found in [3]; the third is defined in [7].

Example 4.6. Groupoids can be regarded as categories where every morphism is an isomorphism; in fact the inclusion $\mathbf{Gpd} \hookrightarrow \mathbf{Cat}$ is reflective (i.e., has a left adjoint) and, more unusually, coreflective (has a right adjoint) . Using inclusion and its left adjoint, the nerve functor for groupoids has the same form as above, yielding an adjunction

$$\Pi_1 : \mathbf{sSet} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathbf{Gpd} : \mathbb{N}$$

The left adjoint Π_1 gives the fundamental groupoid of a simplicial set.

Example 4.7. Subdivision of a simplicial set gives a functor $\mathrm{sd} : \mathbf{sSet} \to \mathbf{sSet}$ such that $|\mathrm{sd}\, \Delta^n|$ is the barycentric subdivision of $|\Delta^n|$. This functor has a right adjoint

$$\mathrm{sd} : \mathbf{sSet} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathbf{sSet} : \mathrm{Ex}$$

Example 4.8. Let **sCat** denote the category of small categories enriched in **sSet** and simplicially enriched functors. The simplicial nerve of a simplicially enriched category (which is different from the nerve functor of Example 4.4) has a left adjoint

$$\mathbb{C} : \mathbf{sSet} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathbf{sCat} : \mathbb{N}$$

which, when restricted to the representable simplicial sets, can be regarded as a *simplicial thickening* of the categories $\underset{\sim}{n}$.

[3]Alternatively, one can check directly that $X \times Y$ is a universal wedge under the diagram (4.2), with the limiting cone defined by the Yoneda lemma.

5. Spheres and Horns

We say a simplicial set Y is a subset of a simplicial set X if $Y_n \subset X_n$ for all $[n] \in \mathbf{\Delta}$ and if

$$Xf\big|_{Y_n} = Yf$$

for all $f : [m] \to [n]$ in $\mathbf{\Delta}$. In the presence of a simplicial set X, we often specify a simplicial subset by giving a set of *generators*, which will typically have the form of a subset $S \subset X_n$ for some n. The simplicial set $Y(S)$ *generated* by S is then the smallest simplicial subset of X that contains S. Its k-simplices $Y(S)_k$ will be those k-simplices of X that are in the image of S under the set of morphisms

$$\{Xf \mid f \in \mathbf{\Delta}^n_k \text{ for some } n\}.$$

We now define a few important instances of simplicial subsets of a representable simplicial set Δ^n.

Definition 5.1. The i-th face $\partial_i\Delta^n$ of Δ^n is the simplicial subset generated by $d^i \in \Delta^n_{n-1}$. Equivalently, this is the image of $d^i : \Delta^{n-1} \to \Delta^n$.

Definition 5.2. The *simplicial n-sphere* $\partial\Delta^n$ is the simplicial subset of Δ^n given by the union of the faces $\partial_0\Delta^n, \dots, \partial_n\Delta^n$. Equivalently, it is the simplicial subset generated by $\{d^0, \dots, d^n\} \subset \Delta^n_{n-1}$. Alternatively, it is the colimit of the diagram consisting of the faces $\partial_i\Delta^n$ together with the morphisms that include each of the $(n-2)$-simplices that form *their* boundary into both of the faces in which each one is contained.

The sphere $\partial\Delta^n$ has the property that $(\partial\Delta^n)_k = \Delta^n_k$ for all $k < n$; all higher simplices of $\partial\Delta^n$ are degenerate. More generally, a *n-sphere* in a simplicial set X is a map $\partial\Delta^n \to X$ of simplical sets.

Definition 5.3. The *simplicial horn* Λ^n_k is the union of all of the faces of Δ^n except for the k-th face; equivalently, it is the simplicial subset of Δ^n generated by the set $\{d^0, \dots, d^{k-1}, d^{k+1}, \dots, d^n\}$. Alternatively, it can be described as a colimit in **sSet**, analogous to that for $\partial\Delta^n$ described above, except with the face $\partial_k\Delta^n$ left out of the colimiting diagram.

The horn Λ^n_k has the property that $(\Lambda^n_k)_j = \Delta^n_j$ for $j < n-1$ and $(\Lambda^n_k)_{n-1} = \Delta^n_{n-1}\backslash\{d^k\}$, with higher simplices again being degenerate. More generally, a horn in a simplicial set X is a map $\Lambda^n_k \to X$ of simplicial sets.

Remark 5.4. For each of these simplicial sets, their geometric realization is the topological object suggested by their name; $|\partial_i\Delta^n|$ is the i-th face of the standard topological n-simplex $\Delta_n = |\Delta^n|$, $|\partial\Delta^n|$ is its boundary, and $|\Lambda^n_k|$ is the union of all faces but the k-th.

There are certain special types of simplicial sets, which are defined by various "horn filling" conditions. This is much too big a subject to really get into here; we give only the basic definitions and list a few sources for the interested reader.

Definition 5.5. A *Kan complex* is a simplicial set X such that every horn has a filler (which is not assumed to be unique). More specifically, X is a Kan complex if given a horn $\Lambda^n_k \to X$ in X there exists an extension along the inclusion $\Lambda^n_k \hookrightarrow \Delta^n$

as shown

$$\begin{tikzcd} \Lambda^n_k \arrow[r] \arrow[d, hook] & X \\ \Delta^n \arrow[ur, dashed] \end{tikzcd}$$

Lemma 5.6. *If X is a topological space, then $S(X)$ is a Kan complex.*

Proof. By the adjunction (4.3), the diagram

$$\begin{tikzcd} \Lambda^n_k \arrow[r] \arrow[d, hook] & S(X) \\ \Delta^n \arrow[ur, dashed] \end{tikzcd} \text{ in } \mathbf{sSet} \text{ corresponds to } \begin{tikzcd} {|\Lambda^n_k|} \arrow[r] \arrow[d, hook] & X \\ {|\Delta^n|} \arrow[ur, dashed] \end{tikzcd} \text{ in } \mathbf{Top}.$$

A topological (n,k)-horn is a deformation retract of the standard n-simplex $\Delta_n = |\Delta^n|$, so the lift on the right hand side exists. Passing this back along the adjunction gives us the desired lift on the left. □

Kan complexes play an important role in studying the homotopy theory of **sSet**, which is closely linked to the homotopy theory of **Top** via the adjunction (4.3). For more details see [3] or [9].

Definition 5.7. A *quasi-category* is a simplicial set X such that every *inner* horn, i.e., horn Λ^n_k with $0 < k < n$, has a filler.

Example 5.8. For any category $\mathcal{C}$, its nerve $\mathbb{N}\mathcal{C}$ is a quasi-category. In fact, it is a quasi-category with the special property that every inner horn has an *unique* filler. Conversely, any quasi-category such that every inner horn has a unique filler is isomorphic to the nerve of a category. We won't give formal proofs of these facts here (instead see [7]) but we will at least provide some intuition for why the nerve of a category has a unique filler for horns $\Lambda^2_1 \to \mathbb{N}\mathcal{C}$. This horn is often represented by the following picture:

$$\begin{tikzcd} & x_1 \arrow[dr, "g"] & \\ x_0 \arrow[ur, "f"] & & x_2 \end{tikzcd} \subset \begin{tikzcd} & x_1 \arrow[dr] & \\ x_0 \arrow[ur] \arrow[rr] & & x_2 \end{tikzcd}$$

Here $f, g \in \mathbb{N}\mathcal{C}_1$ are morphisms in $\mathcal{C}$ and $x_0, x_1, x_2 \in \mathbb{N}\mathcal{C}_0$ are objects in $\mathcal{C}$. $fd_1 = x_0$ and $fd_0 = x_1$, colloquially, x_0 is the domain of f and x_1 is its codomain, and similarly for g. The essential point that this picture communicates is that if f and g are the generating 1-simplices of a horn $\Lambda^2_1 \to \mathbb{N}\mathcal{C}$, then f and g are a composable pair of arrows in $\mathcal{C}$. The statement that this horn can be filled then simply expresses the fact that this pair necessarily has a composite gf. Composition is unique in a category, so this horn can be filled uniquely.[4]

However, it will not necessarily be true that *outer* horns $\Lambda^2_0 \to \mathbb{N}\mathcal{C}$ and $\Lambda^2_2 \to \mathbb{N}\mathcal{C}$ will have fillers. Indeed, asking the $(2,2)$-horn depicted below has a filler is

[4]However, in many higher categorical settings, composition is not required to be unique. This example gives a glimpse of why horn filling conditions and quasi-categories in particular are so useful for studying higher categories.

equivalent to asking that the arrow f have a right inverse, which is certainly not true in general.

$$
\begin{tikzcd}
 & x_1 \arrow[dr, "f"] & & & & x_1 \arrow[dr] & \\
x_2 \arrow[rr, "1"'] & & x_2 & \subset & x_2 \arrow[ur] \arrow[rr] & & x_2
\end{tikzcd}
$$

It turns out that if $\mathcal{C}$ is a *groupoid*, then these outer horns will have fillers. In fact *all* outer horns will have fillers, which says that the nerve of a groupoid is a Kan complex (see [7]).

Quasi-categories were first defined by Boardman and Vogt in [1] under the name *weak Kan complexes*. In recent years, their theory has been developed extensively by André Joyal (see [5] and [6]) and Jacob Lurie (in [7]).

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