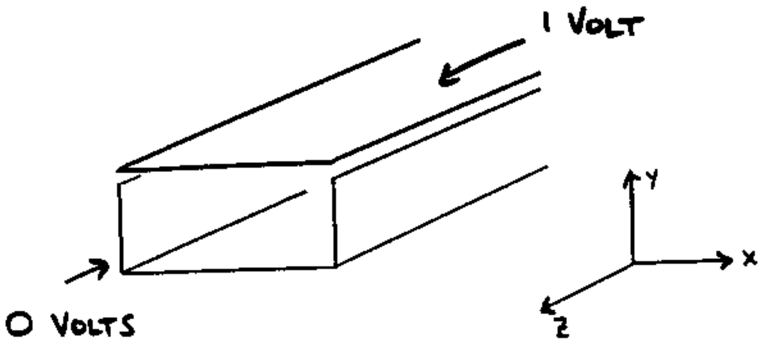
THE METHOD OF FINITE DIFFERENCES

FOR 2D, SOME OF THE EARLIEST APPLICATIONS WERE:

$$\left.\begin{array}{l}\text{RUNGE, 1908} \\ \text{RICHARDSON, 1910}\end{array}\right\} \text{POISSON EQUATION}$$

ELLIPTIC EQUATIONS

EXAMPLE: AN ELECTROSTATIC PROBLEM WITH TRANSLATIONAL SYMMETRY.



THE GOVERNING EQUATIONS OF ELECTROSTATICS ARE:

$$\nabla \times \underline{E} = 0$$

$$\nabla \cdot \underline{D} = 0$$ $\rho = 0$: CHARGE-FREE REGION

AND IN THE AIR ($\approx$ FREE-SPACE):

$$\underline{D} = \varepsilon_0 \underline{E}$$

ALSO, INSIDE CONDUCTORS

$$\underline{E} = 0$$

AND AT THE SURFACE OF CONDUCTORS

$$\underline{E}_{\text{TANGENTIAL}} = 0$$

FROM $\nabla \times \underline{E} = 0$, IT FOLLOWS:

$$\underline{E} = -\nabla \phi$$

AND SO

$$\underline{D} = -\varepsilon_0 \nabla \phi$$

AND

$$\nabla \cdot (-\varepsilon_0 \nabla \phi) = 0$$

$$\Rightarrow \quad \nabla \cdot \nabla \phi = 0$$

OR $\quad \nabla^2 \phi = 0 \quad$ IN THE <u>AIR</u>.

e.g., $\quad \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0. \quad$ (IN CARTESIAN COORDINATES.)

AT THE SURFACE OF A CONDUCTOR, $\underline{E}_{TANG} = 0$

$$\Rightarrow \quad \text{grad}\, \phi \cdot \underline{t} = 0$$

i.e., $\quad \frac{\partial \phi}{\partial t} = 0$

WHERE: $\underline{t}$ IS A UNIT VECTOR <u>TANGENT</u> TO THE SURFACE.

$\Rightarrow \quad \phi = $ CONSTANT ON THE SURFACE OF A CONDUCTOR.

FOR THE CASE OF TRANSLATIONAL SYMMETRY,

$$\frac{\partial}{\partial z} \equiv 0$$

HENCE,

$$\nabla^2 \phi \equiv \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} \quad \text{(IN CARTESIAN COORDINATES)}$$

THEN THE <u>MATHEMATICAL FORMULATION</u> IS THE FOLLOWING <u>ELLIPTIC B.V.P.</u>:

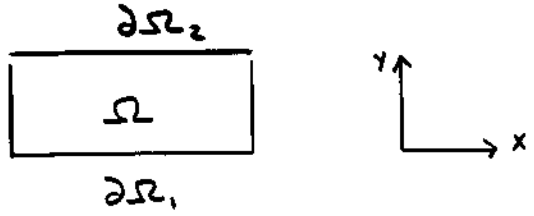
FIND $\phi(x,y)$ SUCH THAT

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \quad \text{IN } \Omega$$

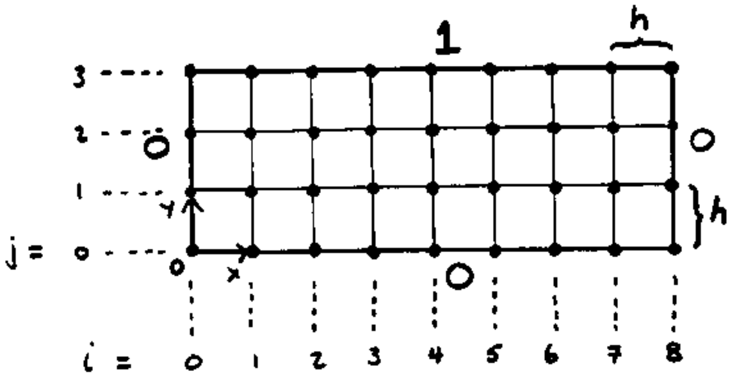
$$\phi = 0 \quad \text{ON } \partial\Omega_1$$

$$\phi = 1 \quad \text{ON } \partial\Omega_2$$

WHERE



TO SOLVE THE PROBLEM, WE SET UP A REGULAR <u>GRID</u> OR <u>MESH</u> OF <u>NODES</u> THROUGHOUT Ω :



LABEL THE NODES i,j SO THE COORDINATES OF NODE i,j ARE

$$x_i = ih$$
$$y_j = jh$$

NOW, WE WISH TO FIND THE VALUES OF ϕ AT THE NODES, i.e., TO FIND

$$\phi_{ij} = \phi(x_i, y_j)$$

WE ALREADY KNOW ϕ_{ij} ON $\partial\Omega_1$ & $\partial\Omega_2$:

$$\phi_{ij} = 0 \quad \text{FOR } j=0,\ i=0,\dots,8$$
$$\phi_{ij} = 0 \quad \text{FOR } i=0,\ j=0,\dots,2$$
$$\phi_{ij} = 0 \quad \text{FOR } i=8,\ j=0,\dots,2$$
$$\phi_{ij} = 1 \quad \text{FOR } j=3,\ i=0,\dots,8$$

TO FIND THE VALUES OF ϕ AT THE INTERIOR NODES, WE DERIVE FOR EACH SUCH NODE AN EQUATION THAT APPROXIMATES THE P.D.E.

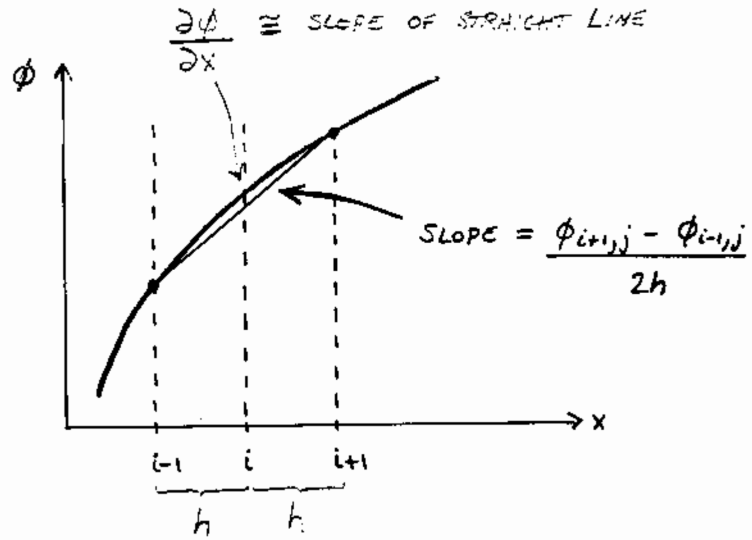
TO DO THIS WE NEED A FINITE DIFFERENCE FORMULA FOR:

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} \quad \text{AT NODE } i,j$$

e.g.,

A FINITE DIFFERENCE FORMULA FOR $\frac{\partial\phi}{\partial x}$ AT NODE i,j IS:

$$\frac{\phi_{i+1,j} - \phi_{i-1,j}}{2h}$$



WHAT IS A FINITE DIFFERENCE FORMULA FOR $\frac{\partial\phi}{\partial y}$ AT NODE i,j ?

TO DERIVE A FINITE DIFFERENCE FORMULA FOR $\dfrac{\partial^2\phi}{\partial x^2} + \dfrac{\partial^2\phi}{\partial y^2}$, WE EXPAND ϕ IN A TAYLOR SERIES ABOUT i,j :

$$\phi(x_i + h, y_j) = \phi(x_i, y_j)$$

$$+ h \left.\frac{\partial\phi}{\partial x}\right|_{i,j}$$ EVALUATED AT i,j

$$+ \frac{h^2}{2!} \left.\frac{\partial^2\phi}{\partial x^2}\right|_{i,j}$$

$$+ \frac{h^3}{3!} \left.\frac{\partial^3\phi}{\partial x^3}\right|_{i,j}$$

$$+ \mathcal{O}(h^4) \quad \text{AS } h \to 0$$

$\mathcal{O}$ NOTATION

$f(x)$ IS $\mathcal{O}(g(x))$ AS $x \to 0$

MEANS :

THERE ARE NUMBERS c, k SUCH THAT

$$|f(x)| < k\,|g(x)| \quad \forall \quad x < c.$$

e.g.,

$$f(x) = 4x^3 + 2x^2 + x$$

$$= \mathcal{O}(x) \quad \text{AS } x \to 0$$

$$f(x) = \sin \pi x = \mathcal{O}(x) \quad \text{AS } x \to 0$$

$$f(x) = 4x^2 + \sin x + \frac{1}{x} = \mathcal{O}\left(\frac{1}{x}\right) \text{ AS } x \to 0$$

ALSO

$f(x)$ IS $\mathcal{O}(g(x))$ AS $x \to \infty$ MEANS:

THERE ARE NUMBERS c, k SUCH THAT

$$|f(x)| < k\,|g(x)| \;\; \forall \;\; x > c$$

e.g.,

$$f(x) = 4x^3 + 2x^2 + x$$
$$= \mathcal{O}(x^3) \quad \text{AS} \quad x \to \infty$$

$$f(x) = \sin \pi x = \mathcal{O}(1) \quad \text{AS} \quad x \to \infty$$

$$f(x) = 4x^2 + \sin x + \frac{1}{x} = \mathcal{O}(x^2) \quad \text{AS} \quad x \to \infty$$

ALSO,

$$\phi(x_i - h, y_j) = \phi(x_i, y_j) - h \left.\frac{\partial \phi}{\partial x}\right|_{i,j} + \frac{h^2}{2!} \left.\frac{\partial^2 \phi}{\partial x^2}\right|_{i,j} - \frac{h^3}{3!} \left.\frac{\partial^3 \phi}{\partial x^3}\right|_{i,j} + \mathcal{O}(h^4)$$

NOW, ADDING $\phi(x_i + h, y_j)$ AND $\phi(x_i - h, y_j)$:

$$\phi(x_i + h, y_j) + \phi(x_i - h, y_j)$$

$$= 2\phi(x_i, y_j)$$

$$+ h^2 \left.\frac{\partial^2 \phi}{\partial x^2}\right|_{i,j}$$

$$+ \mathcal{O}(h^4)$$

OR,

$$\phi_{i+1,j} + \phi_{i-1,j} = 2\phi_{i,j} + h^2 \left.\frac{\partial^2 \phi}{\partial x^2}\right|_{i,j} + \mathcal{O}(h^4)$$

RE-ARRANGING, WE GET :

$$\left.\frac{\partial^2 \phi}{\partial x^2}\right|_{i,j} = \frac{1}{h^2}\left(\phi_{i+1,j} + \phi_{i-1,j} - 2\phi_{i,j}\right)$$

$$+ \mathcal{O}(h^2)$$

IN A SIMILAR WAY, WE GET :

$$\left.\frac{\partial^2 \phi}{\partial y^2}\right|_{i,j} = \frac{1}{h^2}\left(\phi_{i,j+1} + \phi_{i,j-1} - 2\phi_{i,j}\right)$$

$$+ \mathcal{O}(h^2)$$

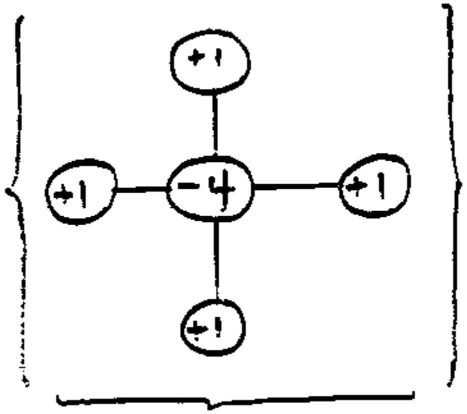
ADDING $\left.\frac{\partial^2 \phi}{\partial x^2}\right|_{i,j}$ AND $\left.\frac{\partial^2 \phi}{\partial y^2}\right|_{i,j}$, WE GET

A FINITE DIFFERENCE FORMULA FOR $\left.\nabla^2 \phi\right|_{i,j}$:

$$\nabla^2\phi\big|_{i,j} \cong \frac{1}{h^2}\Big(\phi_{i+1,j} + \phi_{i-1,j} + \phi_{i,j+1} + \phi_{i,j-1} - 4\phi_{i,j}\Big)$$

OR, PICTORIALLY :

$$\nabla^2\phi\big|_{i,j} \cong \frac{1}{h^2}$$

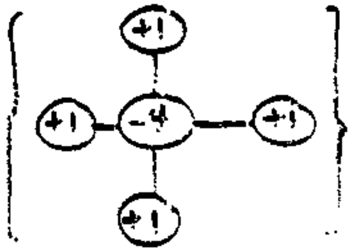


"COMPUTATIONAL MOLECULE"

THIS IS SAID TO BE A FIVE-POINT DIFFERENCE FORMULA

THIS DIFFERENCE FORMULA IS SAID TO BE A SECOND-ORDER APPROXIMATION TO $\nabla^2\phi\big|_{i,j}$, SINCE

$$\nabla^2\phi\big|_{i,j} - \frac{1}{h^2}$$



$$= \mathcal{O}(h^2) \quad \text{AS} \quad h \to 0.$$

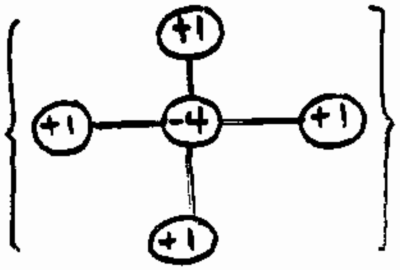
N.B. $e(x)$ IS AN n^{th}-ORDER APPROXIMATION TO $E(x)$ W.R.T. h IF n IS THE LARGEST INTEGER SUCH THAT

$$|e - E| = \mathcal{O}(h^n) \quad \text{AS} \quad h \to 0$$

THE EQUATION WE NEED FOR EACH <u>INTERIOR NODE</u> IS JUST THE FINITE DIFFERENCE APPROXIMATION TO

$$\nabla^2\phi\Big|_{i,j} = 0$$

i.e.,

$\frac{1}{h^2}$  $= 0.$

IF THERE ARE n INTERIOR NODES, THERE WILL BE n EQUATIONS, AND n UNKNOWNS.

WE HAVE REDUCED THE PROBLEM TO A SET OF ALGEBRAIC EQUATIONS.

IN MATRIX FORM:

$$\underline{\underline{A}}\,\underline{\phi}_c = \underline{b}$$

WHERE:

$$\underline{\phi}_c = \begin{bmatrix} \phi_{1,1} \\ \phi_{1,2} \\ \phi_{2,1} \\ \phi_{2,2} \\ \vdots \\ \phi_{7,1} \\ \phi_{7,2} \end{bmatrix}$$

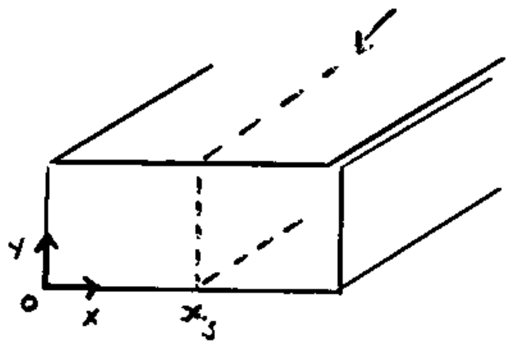
THESE n (= 14) UNKNOWNS MAY BE ORDERED IN ANY WAY.

$\underline{\underline{A}}$

	(1,1)	(1,2)	(2,1)	(2,2)	(3,1)	(3,2)	(4,1)	(4,2)	(5,1)	(5,2)	(6,1)	(6,2)	(7,1)	(7,2)
(1,1)	-4	1	1											
(1,2)	1	-4		1										
(2,1)	1		-4	1	1									
(2,2)		1	1	-4		1								
(3,1)			1		-4	1	1							
(3,2)				1	1	-4		1						
(4,1)					1		-4	1	1					
(4,2)						1	1	-4		1				
(5,1)							1		-4	1	1			
(5,2)								1	1	-4		1		
(6,1)									1		-4	1	1	
(6,2)										1	1	-4		1
(7,1)											1		-4	1
(7,2)												1	1	[illegible]

MAKING USE OF SYMMETRY

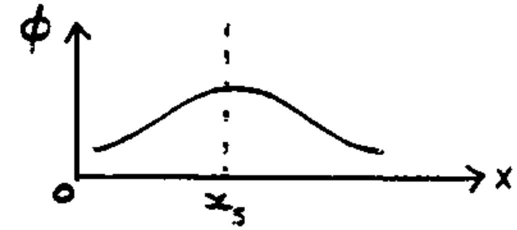
THE PROBLEM WE HAVE BEEN SOLVING HAS A PLANE OF MIRROR SYMMETRY (AS WELL AS TRANSLATIONAL SYMMETRY):



WE KNOW THAT THE POTENTIAL WILL BE EVEN ABOUT THIS PLANE OF SYMMETRY:

$$\phi(x_s + \Delta x, y) = \phi(x_s - \Delta x, y)$$

i.e.,



INTUITIVELY, WE HAVE:

$$\left.\frac{\partial \phi}{\partial x}\right|_{x_s} = 0$$

AND THIS IS INDEED WHAT HAPPENS ON A PLANE OF SYMMETRY.

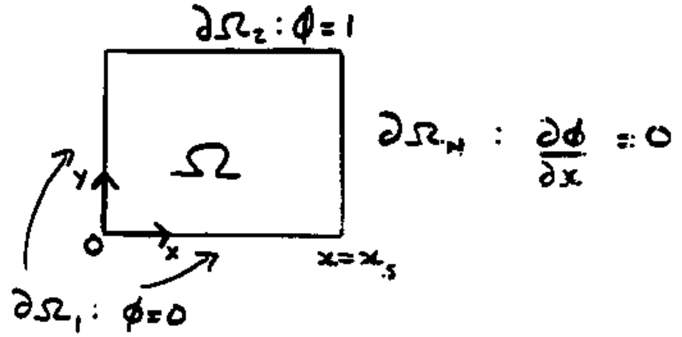
$\Rightarrow$ WHY SOLVE FOR ϕ IN BOTH HALVES?

IT IS <u>MORE EFFICIENT TO SOLVE THE FOLLOWING BVP</u>:

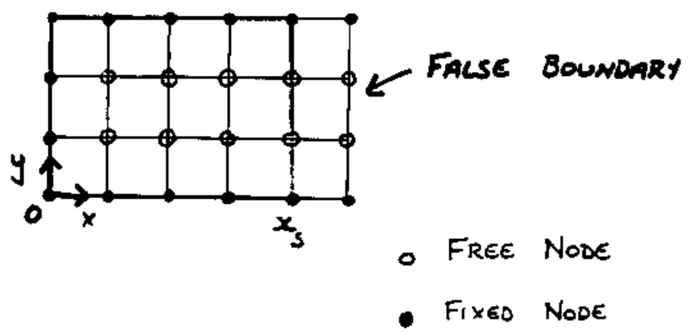
$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \quad \text{IN } \Omega$$

DIRICHLET BOUNDARY CONDITIONS
$$\begin{cases} \phi = 0 & \text{ON } \partial\Omega_1 \\ \phi = 1 & \text{ON } \partial\Omega_2 \end{cases}$$

NEUMANN BOUNDARY CONDITIONS
$$\left\{ \frac{\partial \phi}{\partial x} = 0 \quad \text{ON } \partial\Omega_N \right.$$



WHAT ABOUT THE FINITE DIFFERENCE SOLUTION?



(NOTE: AS WE SEE, BOUNDARY NODES MAY BE FREE!)

HENCE, THERE ARE <u>10 FREE NODES</u>, i.e., <u>10 UNKNOWN</u> VALUES OF POTENTIAL.

APPLYING

$$\nabla^2 \phi = 0$$

IN FINITE DIFFERENCE FORM AT EACH OF THE 8 FREE NODES IN Ω OR ON $\partial\Omega$ GIVES 8 EQNS.

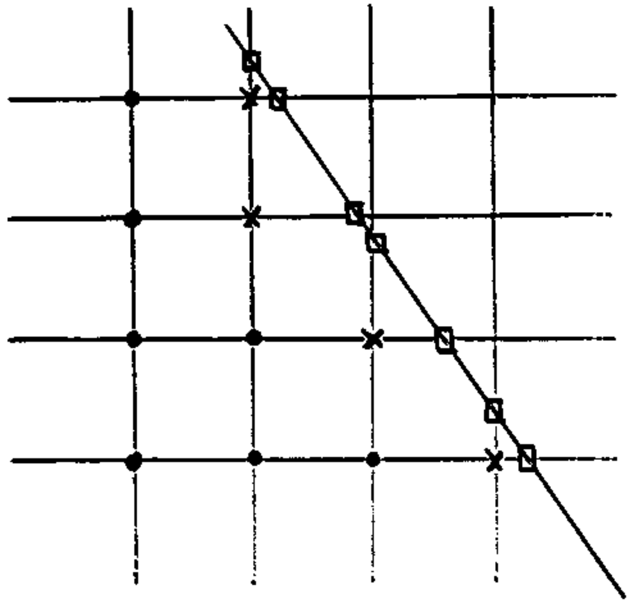
THE REMAINING 2 EQUATIONS COME FROM APPLYING THE NEUMANN BOUNDARY CONDITION:

$$\frac{\partial \phi}{\partial x} = 0$$

IN FINITE DIFFERENCE FORM AT EACH OF THE 3 FREE NODES ON $\partial\Omega_N$.

$$\frac{\partial \phi}{\partial x} = 0 \rightarrow \frac{\phi_{i+1,j} - \phi_{i-1,j}}{2h} = 0.$$

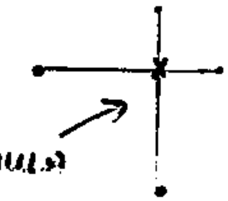
<u>ARBITRARY BOUNDARIES</u>



• REGULAR NODE

□ BOUNDARY NODE

x SPECIAL NODE



FD FORMULA FOR THIS NODE?

Some Iterative Methods of Solution of $\underline{\underline{A}}\,\underline{\phi}_c = \underline{b}$

1. Simple Relaxation

a) Jacobi Method (1844)

(a.k.a. "Simultaneous Relaxation")

Step 0: Guess $\underline{\phi}_c^{(0)}$; $k = 0$

Step 1: Construct $\underline{\phi}_c^{(k+1)}$ using

$$\phi_{i+1,j}^{(k)} + \phi_{i-1,j}^{(k)} + \phi_{i,j+1}^{(k)} + \phi_{i,j-1}^{(k)} - 4\phi_{i,j}^{(k+1)} = 0$$

OR,

$$\phi_{i,j}^{(k+1)} = \frac{1}{4}\left(\phi_{i+1,j}^{(k)} + \phi_{i-1,j}^{(k)} + \phi_{i,j+1}^{(k)} + \phi_{i,j-1}^{(k)} \right)$$

(Remember, boundary values of ϕ are fixed
Do not change with k).

Step 2:

If

$$R_{i,j}^{(k+1)} = \phi_{i+1,j}^{(k+1)} + \phi_{i-1,j}^{(k+1)} + \phi_{i,j+1}^{(k+1)} + \phi_{i,j-1}^{(k+1)} - 4\phi_{i,j}^{(k+1)}$$

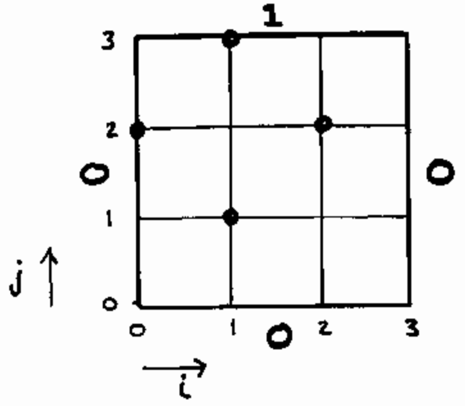
= Residual at node i,j

is small enough at each node, stop.

Otherwise, $k = k+1$ and go back to Step 1.

(Note : "Small enough" relative to $|\phi_{ij}^{(k+1)}|$)

EXAMPLE



NODE

STEP	1,1	1,2	2,1	2,2
0	0	0	0	0
1	0	0.25	0	0.25
2	0.0625	0.3125	0.0625	0.3125
3		etc...		
4				
⋮				

- SIMPLE, BUT
- SLOW TO CONVERGE

(e.g. FOR $N \times N$ GRID $\sim O(N^2)$ ITERATIONS)

b) GAUSS-SEIDEL METHOD (1874)

STEP 0: GUESS $\underline{\phi}_c^{(0)}$; $k = 0$

STEP 1: CONSTRUCT $\underline{\phi}_c^{(k+1)}$ USING

$$\phi_{i,j}^{(k+1)} = \frac{1}{4}\left(\phi_{i-1,j}^{(k+1)} + \phi_{i,j-1}^{(k+1)} + \phi_{i+1,j}^{(k)} + \phi_{i,j+1}^{(k)} \right)$$

THIS IS POSSIBLE, PROVIDED

$\phi_{i-1,j}^{(k+1)}$ AND $\phi_{i,j-1}^{(k+1)}$

ARE COMPUTED BEFORE $\phi_{i,j}^{(k+1)}$.

HENCE, THE ORDER OF COMPUTATION IS IMPORTANT.

$\Rightarrow$ a.k.a. "SUCCESSIVE RELAXATION"

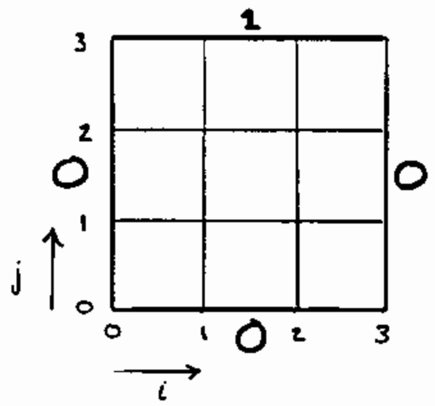
STEP 2:

IF $R_{i,j}^{(k+1)}$ IS SMALL ENOUGH AT EACH NODE, STOP.

OTHERWISE, $k = k+1$ AND GO BACK TO STEP 1.

- FASTER CONVERGENCE ~ $O(\frac{1}{2}N^2)$ ITERATIONS FOR $N \times N$ GRID.

EXAMPLE



STEP	NODE 1,1	1,2	2,1	2,2
0	0	0	0	0
1	0	0.25	0	0.3125
2	0.0625	0.34375	0.09375	0.359375
3		etc ...		
⋮				

2. OVER - RELAXATION

(OR, SUCCESSIVE OVER-RELAXATION, SOR)

IN RELAXATION METHODS, WE RELAX THE POTENTIAL AT NODE i,j TO MAKE ZERO THE QUANTITIES:

$$\phi_{i-1,j}^{(k)} + \phi_{i,j-1}^{(k)} + \phi_{i+1,j}^{(k)} + \phi_{i,j+1}^{(k)} - 4\phi_{i,j}^{(k+1)} \quad \text{(JACOBI)}$$

OR,

$$\phi_{i-1,j}^{(k+1)} + \phi_{i,j-1}^{(k+1)} + \phi_{i+1,j}^{(k)} + \phi_{i,j+1}^{(k)} - 4\phi_{i,j}^{(k+1)} \quad \text{(GAUSS-SEIDEL)}$$

IN OVER-RELAXATION, WE MAKE A LARGER RELAXATION.

LET $\bar{\phi}_{i,j}^{(k+1)}$ BE THE NEW POTENTIAL AT NODE i,j DUE TO THE GAUSS-SEIDEL SCHEME.

THEN SOR MAKES THE NEW POTENTIAL AT i,j:

$$\phi_{i,j}^{(k+1)} = (1-\omega)\,\phi_{i,j}^{(k)} + \omega\,\bar{\phi}_{i,j}^{(k+1)}$$

i.e.,

$$\phi_{i,j}^{(k+1)} = (1-\omega)\,\phi_{i,j}^{(k)} + \frac{\omega}{4}\left(\phi_{i-1,j}^{(k+1)} + \phi_{i,j-1}^{(k+1)} + \phi_{i+1,j}^{(k)} + \phi_{i,j+1}^{(k)}\right)$$

ω IS THE RELAXATION PARAMETER

($\omega < 1 \Rightarrow$ UNDER RELAXATION ; $\omega > 1 \Rightarrow$ OVER-RELAXATION)

$\omega = 1 \rightarrow$ GAUSS - SEIDEL

$\omega = 2 \rightarrow$ THE METHOD DIVERGES

SOMEWHERE BETWEEN ($1 < \omega < 2$), SOR GIVES FASTER CONVERGENCE THAN GAUSS-SEIDEL.

($\sim O(N)$ ITERATIONS FOR $\omega_{OPTIMAL}$ FOR $N \times N$ GRID).