NUMERICAL METHODS

ECSE 543 - ASSIGNMENT 2

MIDO ASSRAN - 260505216

Question 1

The goal is to find the disjoint local **S**-matrix for each finite element triangle, and subsequently find the global conjoint **S**-matrix for the finite difference mesh composed of the triangular finite elements.

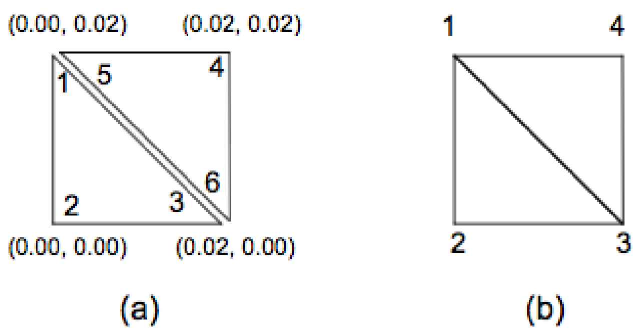

Figure 1. a) Disjoint finite elements with local numbering and vertex coordinates (x, y) in meters b) Conjoint finite element mesh with global numbering

The first step to finding the disjoint local **S**-matrix of each finite element triangle is to find the potentials in the elements. We take the potential, U, to vary linearly over the (x, y) plane - note that the assumption of a linearly varying potential within the triangular element is equivalent to assuming that the electric field is uniform within the element (this is a good assumption in parallel-plate conductor type settings). Equation (1) shows the general linear relationship for the potential - constants a, b, and c are to be determined.

$$U = a + bx + cy \tag{1}$$

Date: November 7, 2016.

Denoting the potentials at the vertices by U_v, where v is the vertex number set by the local ordering, we can solve the linear system of equations shown in equation (2) for the constants a, b, and c where the potential at local vertex v has coordinates given by (x_v, y_v).

$$\begin{bmatrix} U_1 \\ U_2 \\ U_3 \end{bmatrix} = \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} \tag{2}$$

To solve for the constants we have the closed form relationship shown in equation (3), where *adj* is used to denote the adjugate of the matrix (found by taking the transpose of its cofactor matrix), and *det* its determinant.

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \frac{adj \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix}}{det \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix}} \begin{bmatrix} U_1 \\ U_2 \\ U_3 \end{bmatrix} \tag{3}$$

The result of equation (3) gives us the constants in terms of the vertex potentials as shown in equation (4), where A_e is used to denote the area of the triangular finite element e.

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \frac{\begin{bmatrix} (x_2y_3 - x_3y_2) & (x_3y_1 - x_1y_3) & (x_1y_2 - x_2y_1) \\ (y_2 - y_3) & (y_3 - y_1) & (y_1 - y_2) \\ (x_3 - x_2) & (x_1 - x_3) & (x_2 - x_1) \end{bmatrix}}{2A_e} \begin{bmatrix} U_1 \\ U_2 \\ U_3 \end{bmatrix} \tag{4}$$

Since the potential in equation (1) can be written as

$$U = \begin{bmatrix} 1 & x & y \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

then we can directly substitute equation (4) into the above representation and rewrite the potential as:

$$U = \sum_{i=1}^{3} \alpha_i(x, y) U_i$$

where the $\alpha_i(x, y)$ (also known as the linear interpolation functions) are given by equations (5), (6), and (7),

(5) $$\alpha_1 = \frac{1}{2A_e}[(x_2y_3 - x_3y_2) + (y_2 - y_3)x + (x_3 - x_2)y]$$

(6) $$\alpha_1 = \frac{1}{2A_e}[(x_3y_1 - x_1y_3) + (y_3 - y_1)x + (x_1 - x_3)y]$$

(7) $$\alpha_1 = \frac{1}{2A_e}[(x_1y_2 - x_2y_1) + (y_1 - y_2)x + (x_2 - x_1)y]$$

and A_e is given by equation (8).

(8) $$A_e = \frac{1}{2}[(x_2y_3 - x_3y_2) + (x_3y_1 - x_1y_3) + (x_1y_2 - x_2y_1)]$$

The energy in each finite element is given by equation (9), where $W^{(e)}$ is the energy per unit length associated with finite element e, U is the potential - which in general will vary with coordinates (x, y) as was already established, and the integral is swept over A_e, which is the area occupied by element e. *Note that there the permittivity of the medium is neglected in the equation.

(9) $$W^{(e)} = \frac{1}{2}\int_{A_e} |\nabla U|^2 dS$$

Equations (10), and (11) are derived by just making a simple substitution for U in equation (9) using the derived series representation in terms of the interpolation functions and vertex potentials.

(10) $$W^{(e)} = \frac{1}{2}\sum_{i=1}^{3}\sum_{j=1}^{3} U_i \left[\int_{A_e} \nabla\alpha_i \bullet \nabla\alpha_j dS\right] U_j$$

(11) $$W^{(e)} = \frac{1}{2}U^T S^{(e)} U$$

Finally we are able to determine the local $S^{(e)}$ depicted in equation (11), whose entries are given by equation (12).

(12) $$S^{(e)}_{(i,j)} = \int_{A_e} \nabla\alpha_i \bullet \nabla\alpha_j dS$$

Therefore we have:

(13) $$S^{(e)}_{(1,1)} = \frac{1}{4A}[(y_2 - y_3)^2 + (x_3 - x_2)^2]$$

(14) $$S^{(e)}_{(1,2)} = \frac{1}{4A}[(y_2 - y_3)(y_3 - y_1) + (x_3 - x_2)(x_1 - x_3)]$$

$$S^{(e)}_{(1,3)} = \frac{1}{4A}[(y_2 - y_3)(y_1 - y_2) + (x_3 - x_2)(x_2 - x_1)] \tag{15}$$

$$S^{(e)}_{(2,2)} = \frac{1}{4A}[(y_3 - y_1)^2 + (x_1 - x_3)^2] \tag{16}$$

$$S^{(e)}_{(2,3)} = \frac{1}{4A}[(y_3 - y_1)(y_1 - y_2) + (x_1 - x_3)(x_2 - x_1)] \tag{17}$$

$$S^{(e)}_{(3,3)} = \frac{1}{4A}[(y_1 - y_2)^2 + (x_2 - x_1)^2] \tag{18}$$

$$S^{(e)}_{(1,2)} = S^{(e)}_{(2,1)}, \quad S^{(e)}_{(3,1)} = S^{(e)}_{(1,3)}, \quad S^{(e)}_{(3,2)} = S^{(e)}_{(2,3)} \tag{19}$$

Letting $S^{(L)}$ represent the disjoint matrix for the lower triangular element in Figure 1.a, and $S^{(U)}$ represent the disjoint matrix for the upper triangular element in Figure 1.b, we can apply some *plug-and-chug* to solve for the matrix entries where the local numberings relative to the derived equations are created in a counterclockwise fashion. The coordinates for the vertices in each element are:

$S^{(L)}$	
$(x1, y1)$:	$(0, 00, 0.02)$
$(x2, y2)$:	$(0.00, 0.00)$
$(x3, y3)$:	$(0.02, 0.00)$

$S^{(U)}$	
$(x1, y1)$:	$(0.02, 0.02)$
$(x2, y2)$:	$(0.00, 0.02)$
$(x3, y3)$:	$(0.02, 0.00)$

We have $A_e = \frac{1}{2}[(0.02 \cdot 0.02)] = 0.0002$, which is identical for $e = L$ and $e = U$.

$$S^{(L)}_{(1,1)} = \frac{1}{4(0.0002)}[(0.02)^2]$$

$$S^{(L)}_{(1,2)} = \frac{1}{4(0.0002)}[(0.02)(-0.02)]$$

$$S^{(L)}_{(1,3)} = \frac{1}{4(0.0002)}[0]$$

$$S^{(L)}_{(2,2)} = \frac{1}{4(0.0002)}[(-0.02)^2 + (-0.02)^2]$$

$$S^{(L)}_{(2,3)} = \frac{1}{4(0.0002)}[(-0.02)(0.02)]$$

$$S^{(L)}_{(3,3)} = \frac{1}{4(0.0002)}[(0.02)^2]$$

$$S^{(L)}_{(1,2)} = S^{(L)}_{(2,1)}, \quad S^{(L)}_{(3,1)} = S^{(L)}_{(1,3)}, \quad S^{(L)}_{(3,2)} = S^{(L)}_{(2,3)}$$

$$\boxed{S^{(L)} = \begin{bmatrix} 0.5 & -0.5 & 0 \\ -0.5 & 1 & -0.5 \\ 0 & -0.5 & 0.5 \end{bmatrix}}$$

$$S^{(U)}_{(1,1)} = \frac{1}{4(0.0002)}[(0.02)^2 + (0.02)^2]$$
$$S^{(U)}_{(1,2)} = \frac{1}{4(0.0002)}[(0.02)(-0.02)]$$
$$S^{(U)}_{(1,3)} = \frac{1}{4(0.0002)}[(0.02)(-0.02)]$$
$$S^{(U)}_{(2,2)} = \frac{1}{4(0.0002)}[(-0.02)^2]$$
$$S^{(U)}_{(2,3)} = \frac{1}{4(0.0002)}[0]$$
$$S^{(U)}_{(3,3)} = \frac{1}{4(0.0002)}[(-0.02)^2]$$
$$S^{(U)}_{(1,2)} = S^{(U)}_{(2,1)}, \quad S^{(U)}_{(3,1)} = S^{(U)}_{(1,3)}, \quad S^{(U)}_{(3,2)} = S^{U}_{(2,3)}$$
$$\boxed{S^{(U)} = \begin{bmatrix} 1 & -0.5 & -0.5 \\ -0.5 & 0.5 & 0 \\ -0.5 & 0 & 0.5 \end{bmatrix}}$$

The global conjoint **S**-matrix can be found using the disjoint finite element $S^{(e)}$ matrices. The energy of the entire finite element mesh is found by summing the energies of each individual element as is shown in equation (20).

$$W = \sum_{L,U} W^{(e)} = \frac{1}{2} U_{dis}^T S_{dis} U_{dis} \tag{20}$$

where

$$S_{dis} = \begin{bmatrix} S^{(L)} & \\ & S^{(U)} \end{bmatrix} = \begin{bmatrix} 0.5 & -0.5 & 0 & 0 & 0 & 0 \\ -0.5 & 1 & -0.5 & 0 & 0 & 0 \\ 0 & -0.5 & 0.5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -0.5 & -0.5 \\ 0 & 0 & 0 & -0.5 & 0.5 & 0 \\ 0 & 0 & 0 & -0.5 & 0 & 0.5 \end{bmatrix}$$

.

Substituting $U_{dis} = CU_{con}$ (whose relationship is shown in equation (21)) into equation (20), gives

$$W = \frac{1}{2} U_{con}^T C^T S_{dis} C U_{con}$$

where $S = C^T S_{dis} C$

$$\begin{bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \\ U_5 \\ U_6 \end{bmatrix}_{dis} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{bmatrix}_{conj} \tag{21}$$

therefore

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Carrying out the matrix multiplication we have the following for the global **S**-matrix (which was computed using MATLAB).

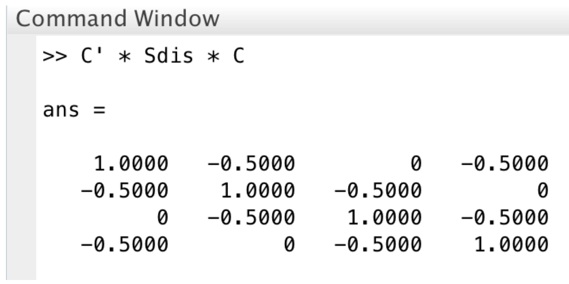

FIGURE 2. MATLAB computation of the global **S**-matrix

$$\boxed{S = \begin{bmatrix} 1 & -0.5 & 0 & -0.5 \\ -0.5 & 1 & -0.5 & 0 \\ 0 & -0.5 & 1 & -0.5 \\ -0.5 & 0 & -0.5 & 1 \end{bmatrix}}$$