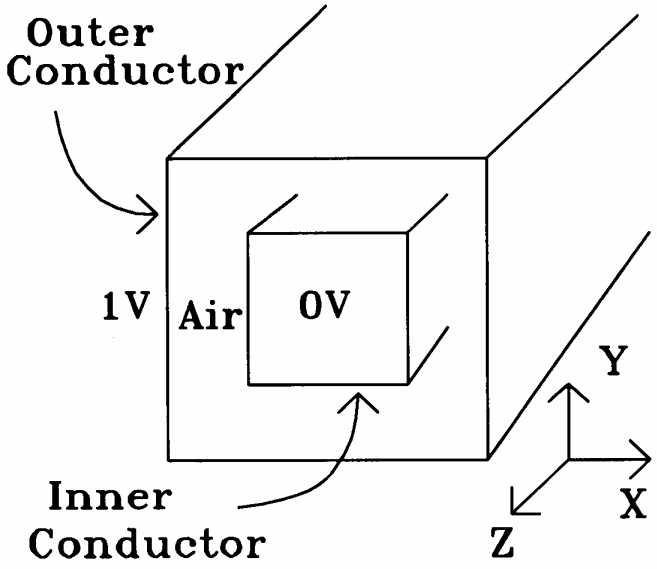
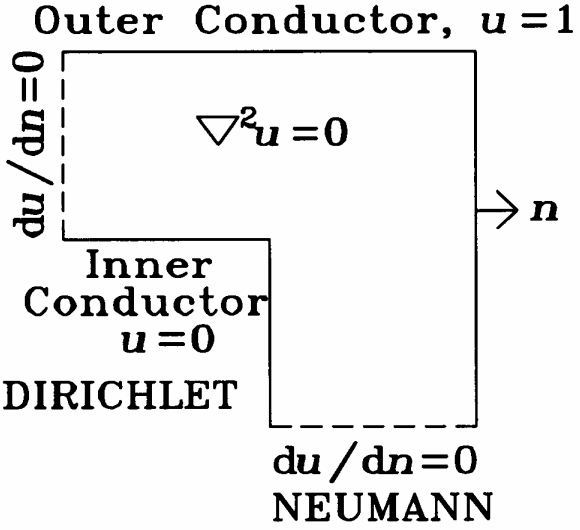
AN OUTLINE OF FIRST-ORDER TRIANGULAR FINITE ELEMENTS





$$\text{Energy} = \frac{1}{2}\int_{\Omega} \boldsymbol{E}\cdot\boldsymbol{D}\,dS = \frac{1}{2}\int_{\Omega} (-\nabla u)\cdot(-\varepsilon_0 \nabla u)\,dS$$

$$= \frac{1}{2}\int_{\Omega} \varepsilon_0 |\nabla u|^2\,dS$$

where u is the electrostatic potential.

The potential minimizes the energy, i.e. minimizes

$$W(u) = \frac{1}{2}\int_{\Omega} |\nabla u|^2\,dS$$

(Let $u(x,y)$ = true solution
Let $h(x,y)$ = function vanishing on Dirichlet surface
Let k = any scalar parameter
Then it can be shown that:

$$W(u+kh) = W(u) + k^2 W(h) \geq W(u)$$

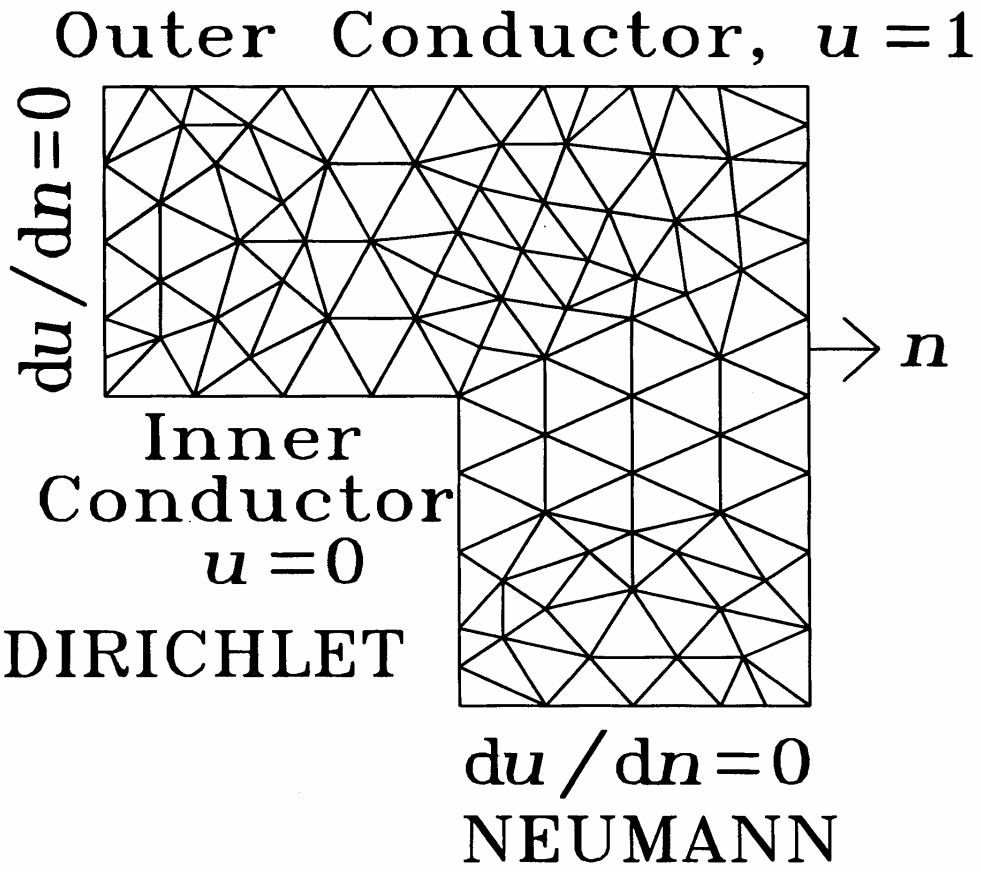
- so u minimizes W.)

GENERAL PROCEDURE

1. Break the region into triangles
2. Approximate the potential in each triangle
3. Find the energy of each triangle
4. Add triangles to get the energy of the whole system
5. Minimize the energy

GENERAL PROCEDURE

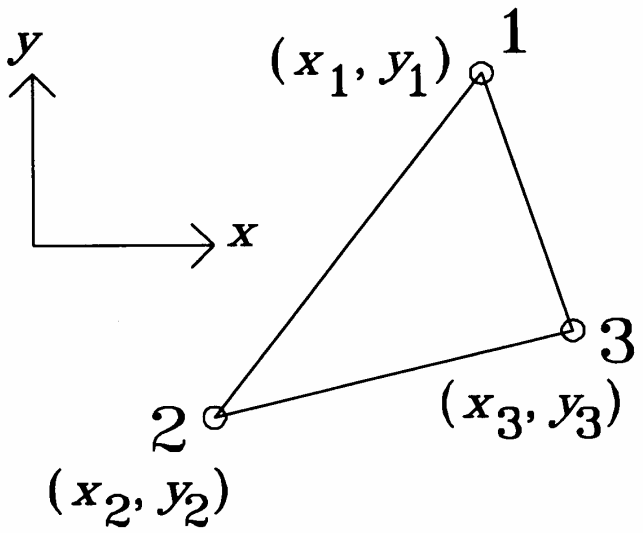
1. Break the region into triangles



GENERAL PROCEDURE

1. Break the region into triangles

2. **Approximate the potential in each triangle**



Interpolate the potential, U:

$$U = a + bx + cy$$

At vertex 1, the potential is

$$U_1 = a + bx_1 + cy_1$$

$$\begin{bmatrix} U_1 \\ U_2 \\ U_3 \end{bmatrix} = \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

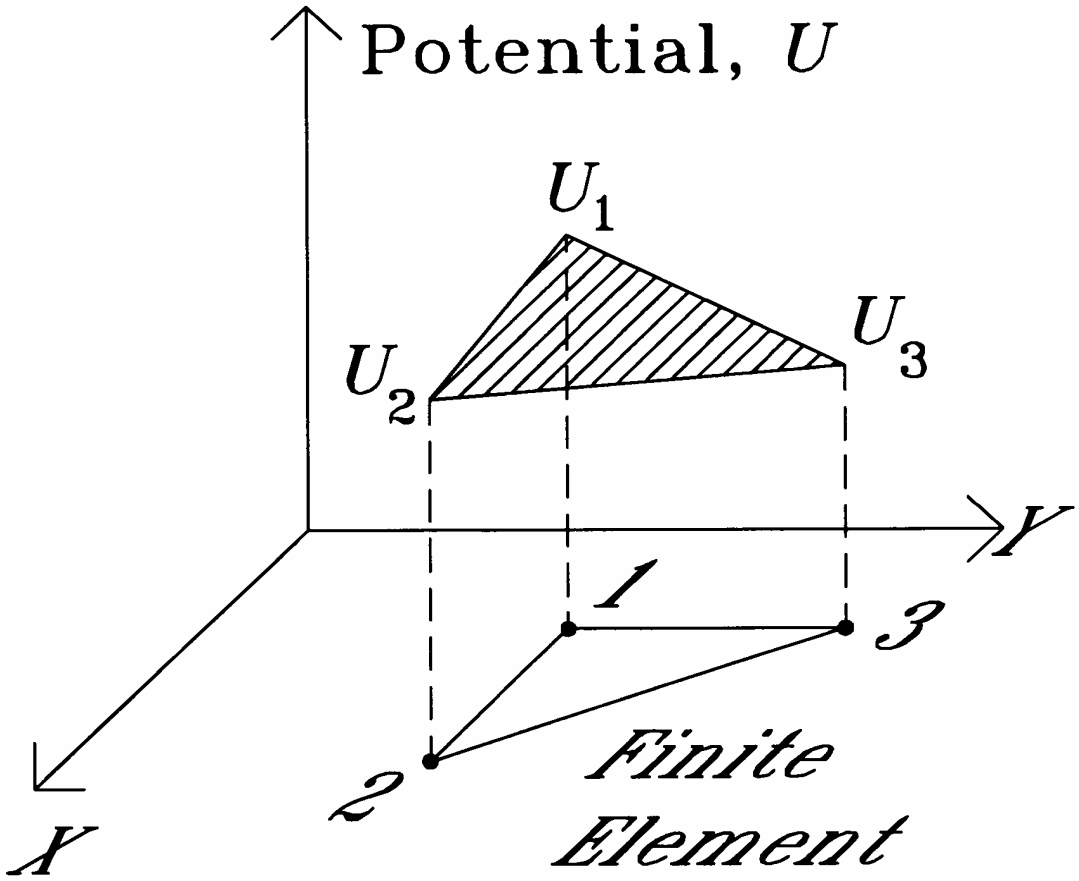
Hence, we can get a, b, c in terms of U_1, U_2, U_3

$$U = \sum_{i=1}^{3} U_i \alpha_i(x,y)$$

e.g.

$$\alpha_1(x,y) = \frac{1}{2A}\left[(x_2y_3 - x_3y_2) + (y_2 - y_3)x + (x_3 - x_2)y\right]$$

where A is the area of the triangle.



GENERAL PROCEDURE

1. Break the region into triangles
2. Approximate the potential in each triangle
3. **Find the energy of each triangle**

The contribution to the energy from triangle e is:

$$W^{(e)} = \frac{1}{2}\int_{\Delta_e} |\nabla U|^2 \mathrm{d}S$$

$$= \frac{1}{2}\sum_{i=1}^{3}\sum_{j=1}^{3} U_i U_j \int_{\Delta_e} \nabla\alpha_i \cdot \nabla\alpha_j \mathrm{d}S$$

This can be written in matrix form:

$$W^{(e)} = \frac{1}{2}\boldsymbol{U}^{\mathrm{T}}\boldsymbol{S}^{(e)}\boldsymbol{U}$$

$$\boldsymbol{U} = \begin{bmatrix} U_1 \\ U_2 \\ U_3 \end{bmatrix} \quad ; \quad \boldsymbol{S}^{(e)} = \begin{bmatrix} S_{11}^{(e)} & S_{12}^{(e)} & S_{13}^{(e)} \\ S_{21}^{(e)} & S_{22}^{(e)} & S_{23}^{(e)} \\ S_{31}^{(e)} & S_{32}^{(e)} & S_{33}^{(e)} \end{bmatrix}$$

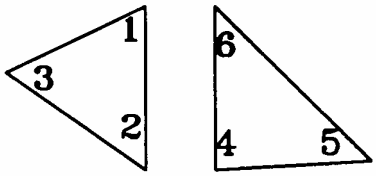
where for example

$$S_{12}^{(e)} = \frac{1}{4A}\left[(y_2 - y_3)(y_3 - y_1) + (x_3 - x_2)(x_1 - x_3)\right]$$

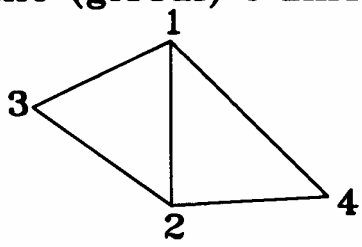
GENERAL PROCEDURE

1. Break the region into triangles
2. Approximate the potential in each triangle
3. Find the energy of each triangle
4. **Add triangles to get the energy of the whole system**

Disjoint (local) Numbering



Conjoint (global) Numbering



$$\begin{bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \\ U_5 \\ U_6 \end{bmatrix}_{\text{disjoint}} = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & 1 & & \\ & & & 1 \\ 1 & & & \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{bmatrix}_{\text{conjoint}}$$

or

$$U_{\text{dis}} = C U_{\text{con}}$$

Total energy $W = W^{(1)} + W^{(2)}$

$$= \frac{1}{2} \boldsymbol{U}_{\text{dis}}^{\text{T}} \boldsymbol{S}_{\text{dis}} \boldsymbol{U}_{\text{dis}}$$

where $\boldsymbol{S}_{\text{dis}}$ is a 6x6 matrix:

$$\boldsymbol{S}_{\text{dis}} = \begin{bmatrix} \boldsymbol{S}^{(1)} & \\ & \boldsymbol{S}^{(2)} \end{bmatrix}$$

Substituting for $\boldsymbol{U}_{\text{dis}}$ using

$$\boldsymbol{U}_{\text{dis}} = \boldsymbol{C}\boldsymbol{U}_{\text{con}}$$

gives

$$W = \frac{1}{2} \boldsymbol{U}_{\text{con}}^{\text{T}} \boldsymbol{C}^{\text{T}} \boldsymbol{S}_{\text{dis}} \boldsymbol{C} \boldsymbol{U}_{\text{con}}$$

$$= \frac{1}{2} \boldsymbol{U}_{\text{con}}^{\text{T}} \boldsymbol{S} \boldsymbol{U}_{\text{con}}$$

where $\boldsymbol{S} = \boldsymbol{C}^{\text{T}} \boldsymbol{S}_{\text{dis}} \boldsymbol{C}$

GENERAL PROCEDURE

1. Break the region into triangles
2. Approximate the potential in each triangle
3. Find the energy of each triangle
4. Add triangles to get the energy of the whole system
5. **Minimize the energy**

To find the solution, minimize the energy, i.e.

$$\frac{\partial W}{\partial U_k} = 0 \qquad \text{for each } \textit{free } k$$

Let

$$\boldsymbol{U}_{\text{con}} = \begin{bmatrix} \boldsymbol{U}_{\text{f}} & \text{free} \\ \boldsymbol{U}_{\text{p}} & \text{prescribed} \end{bmatrix}$$

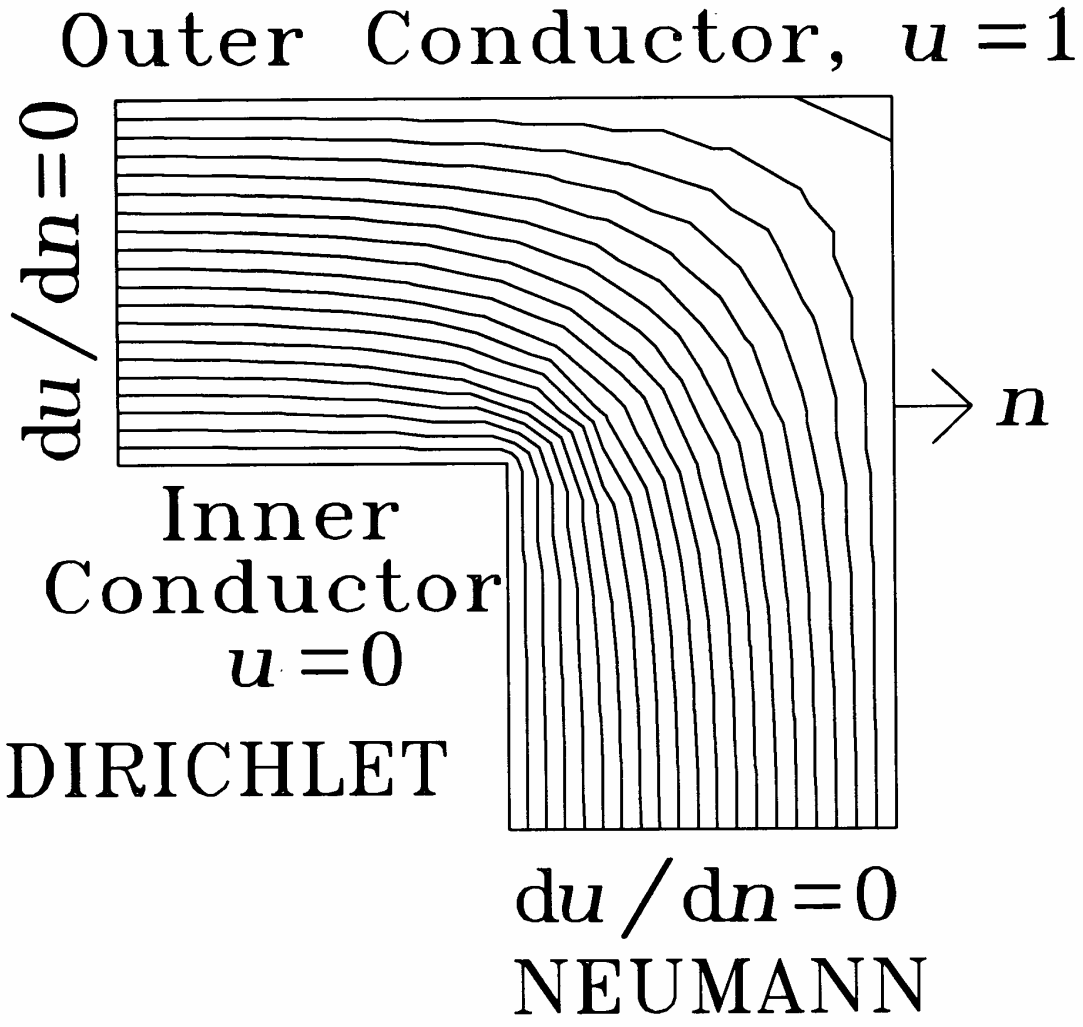
Then minimization with respect to $\boldsymbol{U}_{\text{f}}$ gives:

$$\frac{\partial}{\partial \boldsymbol{U}_{\text{f}}} \left(\begin{bmatrix} \boldsymbol{U}_{\text{f}}^{\text{T}} & \boldsymbol{U}_{\text{p}}^{\text{T}} \end{bmatrix} \begin{bmatrix} \boldsymbol{S}_{\text{ff}} & \boldsymbol{S}_{\text{fp}} \\ \boldsymbol{S}_{\text{pf}} & \boldsymbol{S}_{\text{pp}} \end{bmatrix} \begin{bmatrix} \boldsymbol{U}_{\text{f}} \\ \boldsymbol{U}_{\text{p}} \end{bmatrix} \right) = 0$$

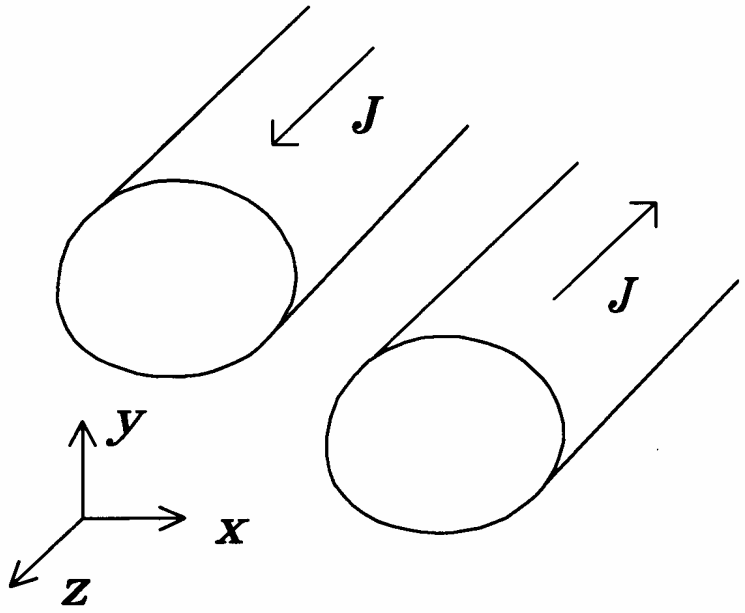
or

$$\boldsymbol{S}_{\text{ff}} \boldsymbol{U}_{\text{f}} = -\boldsymbol{S}_{\text{fp}} \boldsymbol{U}_{\text{p}}$$

This may be solved for the unknown, $\boldsymbol{U}_{\text{f}}$.


Outer Conductor, $u=1$
$du/dn=0$
n
Inner
Conductor
$u=0$
DIRICHLET
$du/dn=0$
NEUMANN

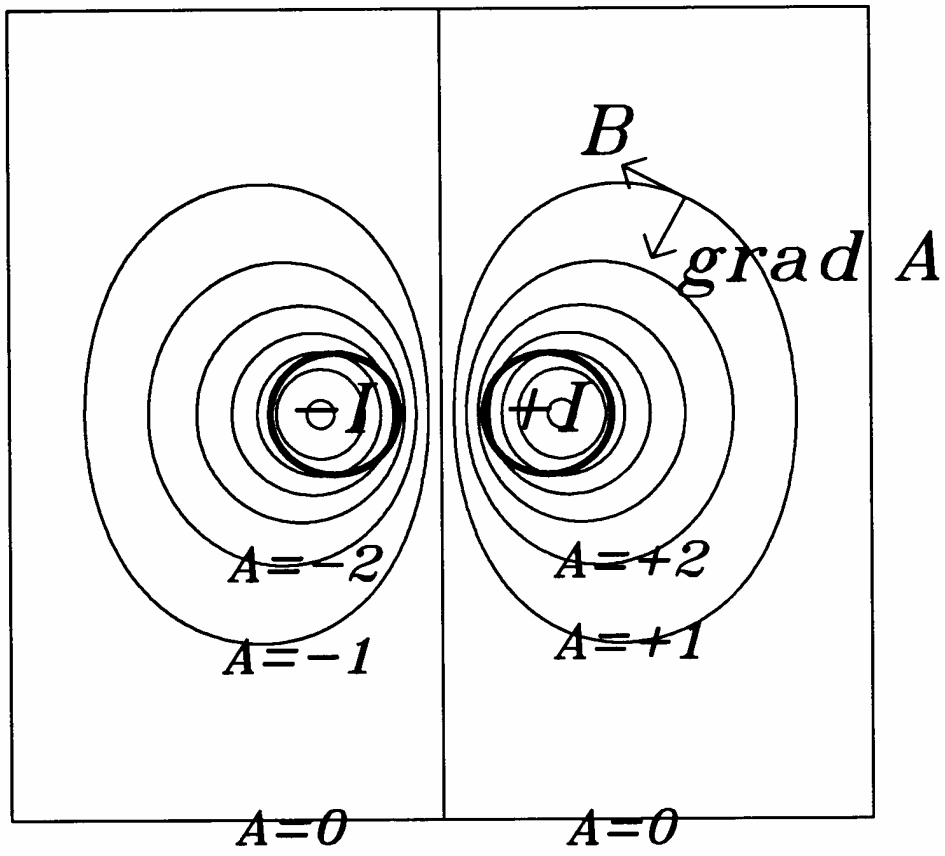
MAGNETICS IN 2D



Current density $\boldsymbol{J} = \hat{z}\,J(x,y)$

Vector potential $\boldsymbol{A} = \hat{z}\,A(x,y)$

$$\text{Flux density } \boldsymbol{B} = \nabla\times\boldsymbol{A} = \hat{x}\frac{\partial A}{\partial y} - \hat{y}\frac{\partial A}{\partial x} = \nabla A\times\hat{z}$$



For magnetostatics in free space, the differential equation is:

$$\nabla^2 A = -\mu_0 J$$

GENERAL PROCEDURE

1. Break the region into triangles
   - as before

2. Approximate the potential in each triangle
   - as before (though now the potential is A)

3. **Find the energy of each triangle**
   - **magnetic energy, not electric**
   - **different energy expression**

4. Add triangles to get the energy of the whole system
   - as before

5. Minimize the energy
   - as before

$$\text{Magnetic Energy} = \frac{1}{2}\int_{\Omega} \boldsymbol{B}\cdot\boldsymbol{H}\, dS = \frac{1}{2}\int_{\Omega} \nu_0 |\nabla A|^2 dS$$

For given current J, A minimizes this functional:

$$F(A) = \frac{1}{2}\int_{\Omega} |\nabla A|^2 dS - \mu_0 \int_{\Omega} AJ\, dS$$

In triangle e, A and J are interpolated as follows:

$$A = \sum_{i=1}^{3} A_i \alpha_i(x,y) \quad ; \quad J = J^{(e)} \quad \text{(constant)}$$

Then:

$$F^{(e)} = \frac{1}{2}\boldsymbol{A}^{\mathrm{T}}\boldsymbol{S}^{(e)}\boldsymbol{A} - \mu_0 \boldsymbol{A}^{\mathrm{T}}\boldsymbol{R}^{(e)}$$

where

$$R_i^{(e)} = J^{(e)} \int_{\Omega} \alpha_i\, dS$$

PROGRAM SIMPLE2D

MESHIN - reads and stores data

MATINIT, VECINIT - initializes global matrices

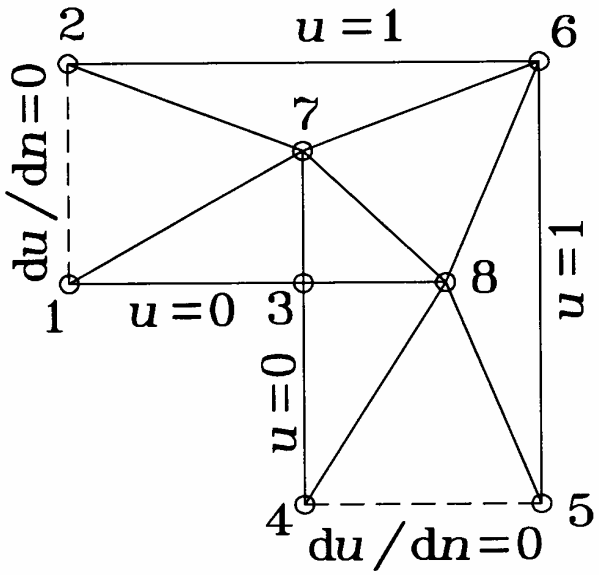
For each finite element:
ELMATR - computes local matrices $\boldsymbol{S}^{(e)}$ and $\boldsymbol{R}^{(e)}$
ELEMBD - embeds local matrices into global

EQSOLV - solves simultaneous equations

OUTPUT - prints the solution

INPUT FILE FORMAT

1	0.000	2.500
2	0.000	5.000
3	2.500	2.500
4	2.500	0.000
5	5.000	0.000
6	5.000	5.000
7	2.500	3.750
8	3.750	2.500



1	3	7	0.000
1	7	2	0.000
2	7	6	0.000
3	8	7	0.000
7	8	6	0.000
4	8	3	0.000
4	5	8	0.000
8	5	6	0.000

1	0.000
3	0.000
4	0.000
2	1.000
6	1.000
5	1.000

OUTPUT FILE FORMAT

Input Node List

N	X	Y
1	0.00000	2.50000
2	0.00000	5.00000
3	2.50000	2.50000
4	2.50000	0.00000
5	5.00000	0.00000
6	5.00000	5.00000
7	2.50000	3.75000
8	3.75000	2.50000

Input Element List

I	J	K	Source
1	3	7	0.00000
1	7	2	0.00000
2	7	6	0.00000
3	8	7	0.00000
7	8	6	0.00000
4	8	3	0.00000
4	5	8	0.00000
8	5	6	0.00000

Input Fixed Potentials

Node	Value
1	0.00000
3	0.00000
4	0.00000
2	1.00000
6	1.00000
5	1.00000

Final Solution

I	X	Y	Potential
1	0.00000	2.50000	0.00000
2	0.00000	5.00000	1.00000
3	2.50000	2.50000	0.00000
4	2.50000	0.00000	0.00000
5	5.00000	0.00000	1.00000
6	5.00000	5.00000	1.00000
7	2.50000	3.75000	0.58000
8	3.75000	2.50000	0.58000