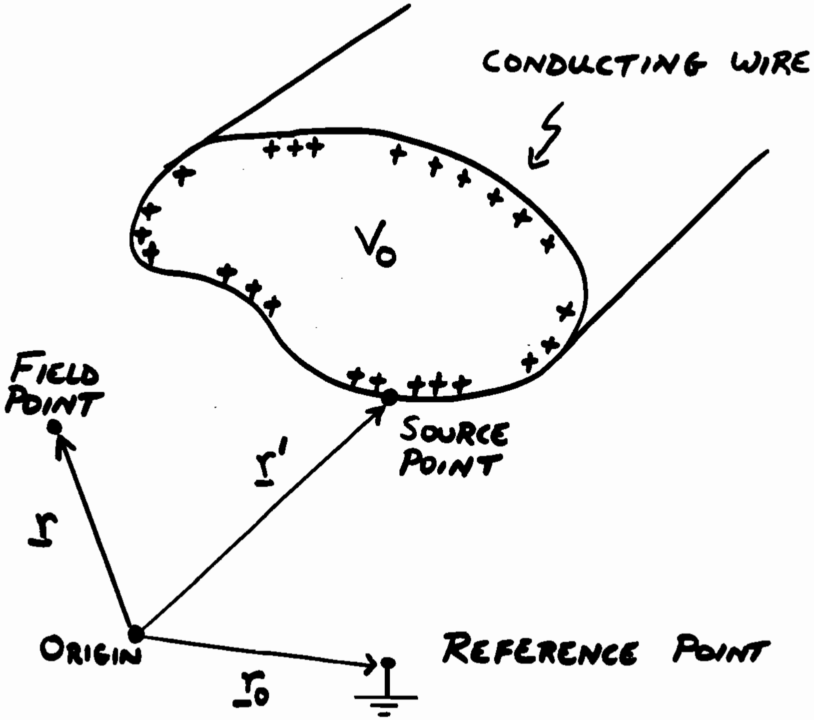
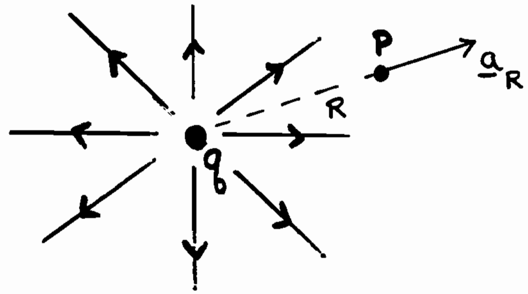
AN INTEGRAL EQUATION (BOUNDARY ELEMENT) METHOD FOR 2D ELECTROSTATICS

CONSIDER A CONDUCTING WIRE OF ARBITRARY CROSS-SECTION CHARGED TO A POTENTIAL V_0 w.r.t. REFERENCE POINT $\underline{r}_0$:



FIELD DUE TO AN INFINITELY LONG, STRAIGHT, UNIFORM LINE CHARGE q (C/m):



$$\underline{E}_P = \frac{q}{2\pi\varepsilon_0} \frac{\underline{a}_R}{R} \qquad \text{(VOLTS/m)}$$

THUS,

$$\underline{V}_P = \frac{-q}{2\pi\varepsilon_0} \ln R + \text{CONST.} \qquad \text{(VOLTS)}$$

IF WE CHOOSE A ZERO-VOLTAGE POINT AT A DISTANCE R_0 AWAY FROM q, THEN THE CONSTANT MUST BE

$$+\frac{q}{2\pi\varepsilon_0} \ln R_0$$

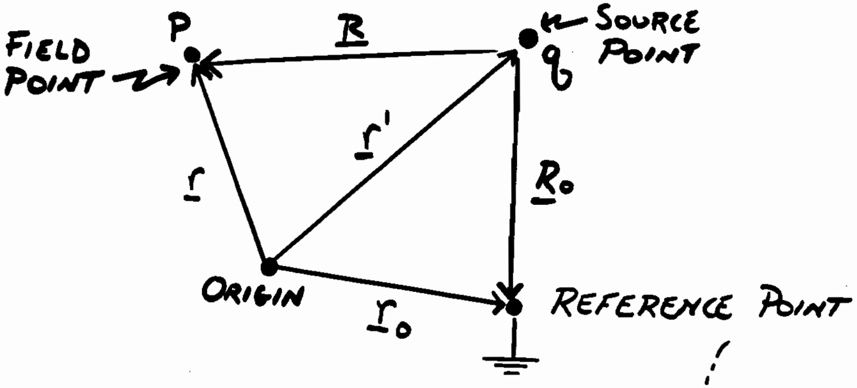
NOW, WHEN :

q IS AT $\underline{r}'$

P IS AT $\underline{r}$

AND THE REFERENCE POINT IS AT $\underline{r}_o$:

$$R = |\underline{r} - \underline{r}'|, \qquad R_o = |\underline{r}_o - \underline{r}'|$$



THEN :

$$V(\underline{r}) = -\frac{q}{2\pi\varepsilon_o}\ln|\underline{r}-\underline{r}'| \overbrace{+\frac{q}{2\pi\varepsilon_o}\ln|\underline{r}_o-\underline{r}'|}^{\text{ADDITIVE CONSTANT}}$$

$$= -\frac{q}{2\pi\varepsilon_o}\ln\frac{|\underline{r}-\underline{r}'|}{|\underline{r}_o-\underline{r}'|}$$

THE POTENTIAL AT $\underline{r}$, w.r.t. $\underline{r}_o$, DUE TO <u>ALL</u> THE DISTRIBUTION $\sigma(\underline{r}')$ (C/m²) OF <u>LINE CHARGES</u> OVER THE SURFACE OF THE WIRE IS :

$$V(\underline{r}) = -\frac{1}{2\pi\varepsilon_o}\int_C \ln\frac{|\underline{r}-\underline{r}'|}{|\underline{r}_o-\underline{r}'|}\,\sigma(\underline{r}')\,dl'$$

THE ' INDICATES INTEGRATION w.r.t. SOURCE TERM VARIABLES ($\underline{r}'$)

NOW, ON C THE POTENTIAL IS SPECIFIED: IT IS V_o.

$\Rightarrow$ DERIVE THE FOLLOWING INTEGRAL EQUATION FOR THE UNKNOWN CHARGE DENSITY σ :

FOR $\underline{r} \in C$:

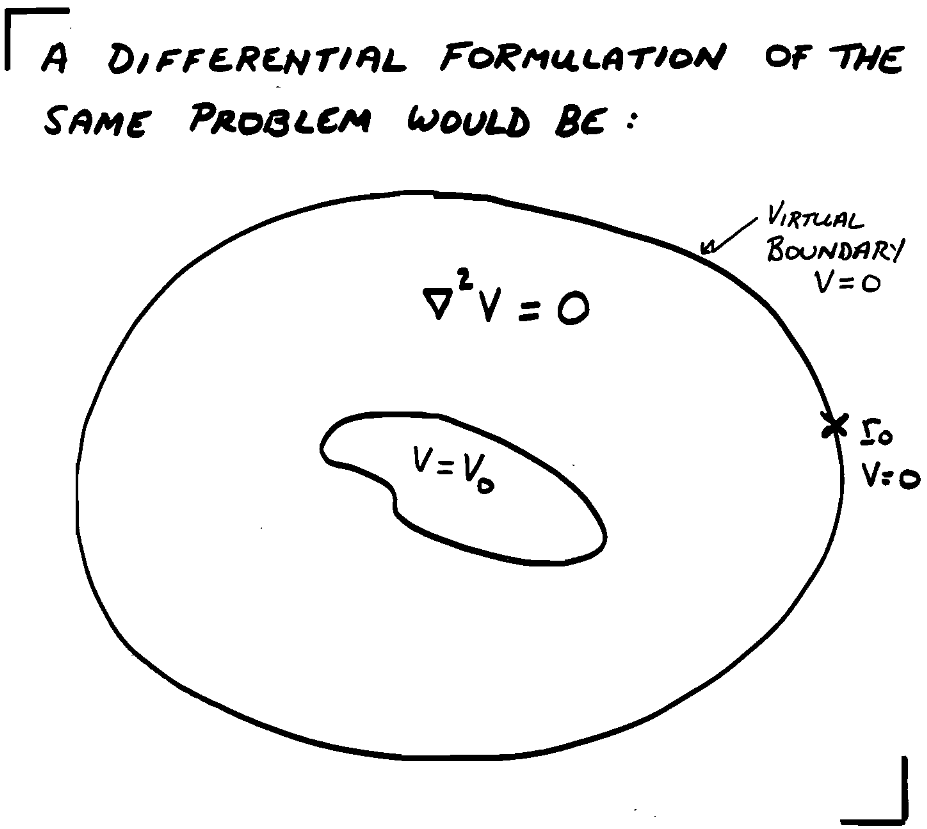
$$\boxed{V_o = -\frac{1}{2\pi\varepsilon_o}\int_C \ln\frac{|\underline{r}-\underline{r}'|}{|r_o-r'|}\,\sigma(\underline{r}')\,dl'}$$

N.B.:

$$G(\underline{r}, \underline{r}') = -\frac{1}{2\pi\varepsilon_0} \ln \frac{|\underline{r} - \underline{r}'|}{|r_0 - \underline{r}'|}$$

IS CALLED A **GREEN'S FUNCTION.**

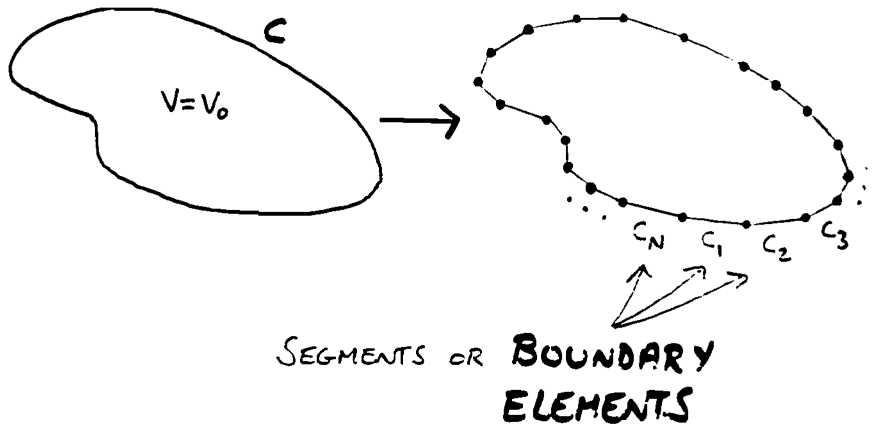
A DIFFERENTIAL FORMULATION OF THE SAME PROBLEM WOULD BE:



NUMERICAL SOLUTION

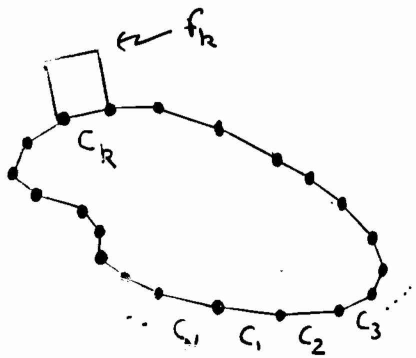
- THE FIRST STEP IS TO REPRESENT THE UNKNOWN BY A DISCRETE NUMBER **N** OF PARAMETERS: THE **COEFFICIENTS** OF SUITABLE **TRIAL FUNCTIONS.**

- IN THIS CASE, WE BUILD THE TRIAL FUNCTIONS BY FIRST APPROXIMATING THE CURVED BOUNDARY OF THE WIRE WITH **N** UNEQUAL LINE SEGMENTS:



THE N TRIAL FUNCTIONS $f_1, \ldots, f_N$ ARE DEFINED ON THIS POLYGON AS FOLLOWS :

$$f_k(\underline{r}') = \begin{cases} 1 & \text{IF } \underline{r}' \in C_k \\ 0 & \text{OTHERWISE} \end{cases}$$



i.e., EACH TRIAL FUNCTION IS NON-ZERO ON ONLY ONE SEGMENT.

THE CHARGE DENSITY $\sigma(\underline{r}')$ CAN BE APPROXIMATED AS :

$$\sigma(\underline{r}') = \sum_{k=1}^{N} \sigma_k f_k(\underline{r}') \qquad (*)$$

IT CAN BE SEEN FROM THIS THAT THE UNKNOWN COEFFICIENT σ_k IS THE VALUE OF $\sigma(\underline{r}')$ ON THE k-TH SEGMENT.

SUBSTITUTING $(*)$ INTO THE INTEGRAL EQUATION GIVES :

FOR $\underline{r} \in C$:

$$V_0 = -\frac{1}{2\pi\varepsilon_0} \sum_{k=1}^{N} \sigma_k \int_{C_k} \ln \frac{|\underline{r} - \underline{r}'|}{|\underline{r}_0 - \underline{r}'|} \, dl'$$

(NOTE : WE HAVE REPLACED INTEGRATION OVER THE SMOOTH WIRE SURFACE BY INTEGRATION OVER THE POLYGONAL APPROXIMATION TO IT.)

- IDEALLY, WE WOULD LIKE TO FIND σ_k'S THAT WOULD MAKE THIS EQUATION HOLD AT EVERY POINT $\underline{r}$ ON THE WIRE SURFACE.
- IN GENERAL, THIS IS IMPOSSIBLE.
- HOWEVER, WE CAN MAKE IT HOLD AT **N POINTS** ON C (SINCE WE HAVE **N DEGREES OF FREEDOM**).
- WE CHOOSE THESE POINTS TO BE THE **MIDPOINTS** $\underline{r}_i$ OF THE SEGMENTS, i.e.,

FOR $i = 1, \ldots, N$:

$$V_0 = -\frac{1}{2\pi\varepsilon_0} \sum_{k=1}^{N} \sigma_k \int_{C_k} \ln \frac{|\underline{r}_i - \underline{r}'|}{|\underline{r}_0 - \underline{r}'|} \, dl'$$

- THIS IS A SET OF **N EQUATIONS** IN **N UNKNOWNS**: $\sigma_1, \ldots, \sigma_N$.

$\Rightarrow$ WE CAN WRITE IT IN MATRIX FORM:

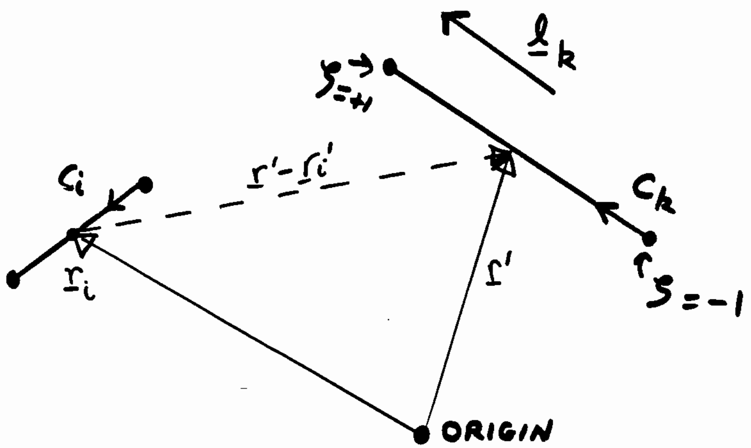
$$\underline{\underline{A}}\,\underline{\sigma} = \underline{V}$$

WHERE

$$\underline{\sigma} = \begin{pmatrix} \sigma_1 \\ \vdots \\ \sigma_N \end{pmatrix} ; \quad \underline{V} = \begin{pmatrix} V_0 \\ \vdots \\ V_0 \end{pmatrix} ;$$

$$A_{ik} = -\frac{1}{2\pi\varepsilon_0} \int_{C_k} \ln \frac{|\underline{r}_i - \underline{r}'|}{|\underline{r}_0 - \underline{r}'|} \, dl'$$

- WE KNOW HOW TO SOLVE $\underline{\underline{A}}\,\underline{\sigma} = \underline{V}$.
- THE ONLY REMAINING PROBLEM IS EVALUATING THE INTEGRALS.



$$I = \int_{C_k} \ln |\underline{r}_i - \underline{r}'|\, dl'$$

CENTER OF SEGMENT k

$$\underline{r}' = \underline{r}_k + \rho\, \underline{l}_k \qquad -1 \le \rho \le +1$$

WHERE $\underline{l}_k$ = VECTOR ALONG C_k

$|\underline{l}_k| = \frac{1}{2} \times$ LENGTH OF C_k

SO, $l' = l_k(1+\rho)$ AND $dl' = l_k\, d\rho$

THEN,

$$I = l_k \int_{\rho=-1}^{1} \ln |\underline{r}_i - \underline{r}_k - \rho\, \underline{l}_k|\, d\rho$$

$$= l_k \int_{\rho=-1}^{1} \frac{1}{2} \ln \Big[(x_i - x_k - \rho\, l_{kx})^2 + (y_i - y_k - \rho\, l_{ky})^2 \Big]\, d\rho$$

- DIFFICULT (OR IMPOSSIBLE) TO EVALUATE IN CLOSED FORM.
- INSTEAD EVALUATE NUMERICALLY.

NUMERICAL INTEGRATION (1D)

BASIC IDEA :

$$\int_{\xi=-1}^{1} f(\xi)\, d\xi \cong \sum_{i=0}^{n} w_i\, f_i$$

WEIGHTS

$f_i = f(\xi_i)$

EVALUATION POINTS (ABSCISSAS)

NEWTON-COTES INTEGRATION

- USE $n+1$ EQUALLY SPACED ABSCISSAS:

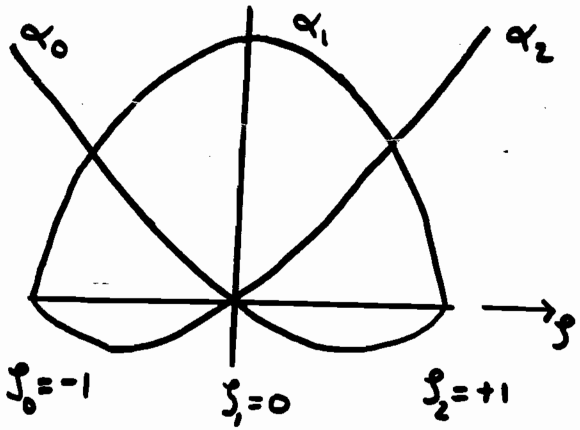
$$\xi_i = -1 + \frac{i}{n} 2$$

- APPROXIMATE f BY AN n^{th}-ORDER POLYNOMIAL THAT MATCHES f EXACTLY AT THE $n+1$ ABSCISSAS (THIS IS POSSIBLE BECAUSE AN n^{th}-ORDER POLYNOMIAL HAS $n+1$ COEFFICIENTS).

- THIS IS CONVENIENTLY DONE USING LAGRANGE INTERPOLATION POLYNOMIALS

$$L_i = \alpha_i(\xi_j) = \begin{cases} 1 & \text{WHEN } j = i \\ 0 & \text{WHEN } j \neq i \end{cases}$$

e.g., FOR $n=2$:



THE APPROXIMATION OF f IS :

$$f(\xi) \cong \sum_{i=0}^{n} f_i \, \alpha_i(\xi)$$

• FROM THIS, WE CAN GET AN APPROXIMATION TO THE INTEGRAL :

$$\int_{\xi=-1}^{1} f(\xi)\, d\xi \cong \int_{\xi=-1}^{1} \sum_{i=0}^{n} f_i \, \alpha_i(\xi)\, d\xi$$

$$= \sum_{i=0}^{n} \underbrace{\left(\int_{\xi=-1}^{1} \alpha_i(\xi)\, d\xi \right)} f_i$$

i.e.,

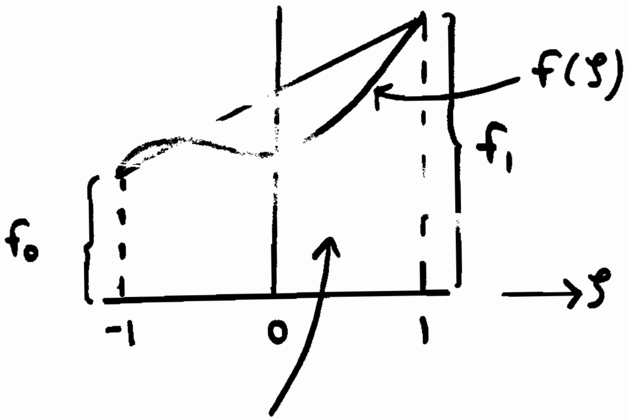
$$w_i = \int_{\xi=-1}^{1} \alpha_i(\xi)\, d\xi$$

e.g., FOR $n=1$:

$$W_0 = 1 \quad , \quad W_1 = 1$$

$$\int_{\xi=-1}^{1} f(\xi)\,d\xi \cong f_0 + f_1 .$$

(a.k.a. TRAPEZOIDAL RULE)



AREA = $\frac{1}{2}(f_0+f_1) \times 2$

• FOR $n=2$:

$$W_0 = \frac{1}{3} \quad , \quad W_1 = \frac{4}{3} \quad , \quad W_2 = \frac{1}{3}$$

$$\int_{\xi=-1}^{1} f(\xi)\,d\xi \cong \frac{1}{3} f_0 + \frac{4}{3} f_1 + \frac{1}{3} f_2$$

(a.k.a. SIMPSON'S RULE)

NOTE THAT THE n^{th}-ORDER FORMULA WILL INTEGRATE <u>EXACTLY</u> ALL POLYNOMIALS UP TO ORDER n.

Gauss-Legendre Integration

In the formula

$$\int_{\xi=-1}^{1} f(\xi)\, d\xi \cong \sum_{i=0}^{n} w_i f(\xi_i)$$

Treat both w_i and ξ_i as unknowns

$\Rightarrow$ $2n+2$ parameters

e.g., Fix $2n+2$ parameters such that the following functions are integrated exactly:

$$1,\ \xi,\ \xi^2,\ \xi^3 \ldots,\ \xi^{2n+1}$$

i.e., all polynomials up to order $\underline{2n+1}$

This gives $2n+2$ equations in $2n+2$ unknowns:

$$\int \overset{=\xi^0}{1}\, d\xi = \sum_{i=0}^{n} w_i \cdot \overset{=\xi_i^0}{1}$$

$$\int \xi^1 d\xi = \sum_{i=0}^{n} w_i \xi_i^1$$

$$\int \xi^2 d\xi = \sum_{i=0}^{n} w_i \xi_i^2$$

$$\vdots$$

$$\int \xi^{2n+1} d\xi = \sum_{i=0}^{n} w_i \xi_i^{2n+1}$$

These are <u>nonlinear simultaneous</u> equations, but are nonetheless capable of solution.

n	ξ_i	w_i
0	0.00000	2.00000
1	+0.57735 -0.57735	1.00000 1.00000
2	+0.77459 -0.77459 0.00000	0.55555 0.55555 0.88889

THE n^{th}- ORDER FORMULA WILL INTEGRATE EXACTLY ALL POLYNOMIALS UP TO ORDER $2n+1$.

RECALL,

$$I = \ell_k \int_{\xi=-1}^{1} \ln \left| \underline{r}_i - \underline{r}_k + \xi \underline{\ell}_k \right| d\xi$$

FOR $i \neq k$ WE CAN USE NUMERICAL INTEGRATION.

WHAT ABOUT THE CASE $i = k$?

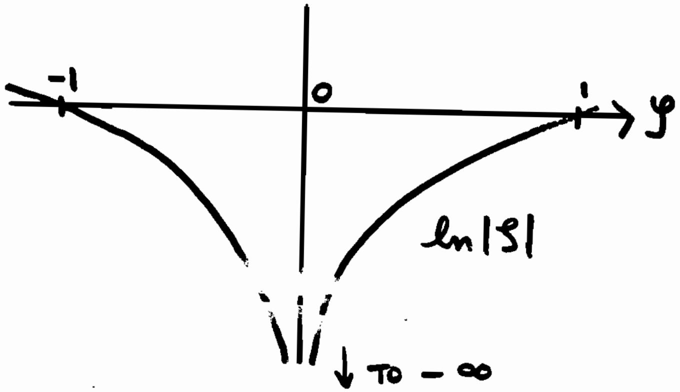
THEN :

$$I = \ell_k \int_{\xi=-1}^{1} \ln \left(|\xi| \ell_k \right) d\xi$$

$$= \ell_k \int_{\xi=-1}^{1} \left[\ln|\xi| + \ln \ell_k \right] d\xi$$

$$I = l_k \int_{\xi=-1}^{1} \ln|\xi|\, d\xi \; + \; 2 l_k \ln l_k$$

INTEGRAND $\ln|\xi|$ IS SINGULAR AT $\xi = 0$



$\Rightarrow$ WHAT WE REALLY MEAN BY

$$\int_{\xi=-1}^{1} \ln|\xi|\, d\xi$$

IS

$$\lim_{\varepsilon \to 0} \left[\int_{\xi=-1}^{-\varepsilon} \ln|\xi|\, d\xi + \int_{\xi=\varepsilon}^{+1} \ln|\xi|\, d\xi \right]$$

WHERE ε IS A SMALL, POSITIVE NUMBER.

THIS LIMIT EXISTS, EVEN THOUGH $\ln|\xi|$ IS INFINITE WHEN $\varepsilon = 0$.

$\Rightarrow$ $\ln|\xi|$ IS AN <u>INTEGRABLE SINGULARITY</u>.

- Now,

$$\int_{\xi=-1}^{-\varepsilon} \ln|\xi|\,d\xi = \int_{\xi=\varepsilon}^{1} \ln|\xi|\,d\xi$$

$$= \int_{\xi=\varepsilon}^{1} \ln\xi\,d\xi$$

$$= \left[\xi\ln\xi - \xi\right]_{\varepsilon}^{1}$$

$$= -1 - \varepsilon\ln\varepsilon + \varepsilon$$

$$\left(\text{CHECK: } \frac{d}{d\xi}(\xi\ln\xi - \xi) = \ln\xi + \frac{\xi}{\xi} - 1\right)$$

- So

$$\int_{\xi=-1}^{1} \ln|\xi|\,d\xi = 2\lim_{\varepsilon\to 0}(-1-\varepsilon\ln\varepsilon+\varepsilon)$$

- Now,

$$\lim_{\varepsilon\to 0} \varepsilon\ln\varepsilon = 0$$

- So

$$\int_{\xi=-1}^{1} \ln|\xi|\,d\xi = -2$$

$$\Rightarrow \quad I = -2\ell_k + 2\ell_k\ln\ell_k$$