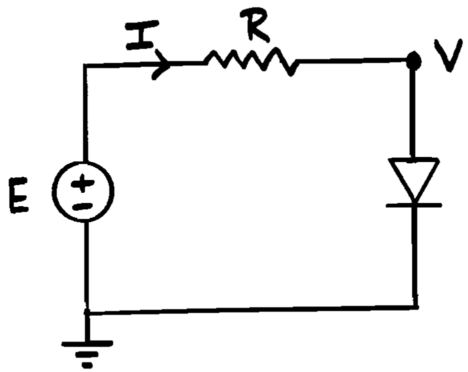
# NONLINEAR PROBLEMS




$$I = \frac{E - V}{R} \qquad (1)$$

ALSO, FOR THE DIODE:

$$I = I_s \left( e^{qV/kT} - 1 \right) \qquad (2)$$

REVERSE SATURATION CURRENT

$\frac{kT}{q} \sim 25$ mV AT 300 K



COMBINING (1) AND (2):

$$\frac{E - V}{R} = I_s \left( e^{qV/kT} - 1 \right)$$

OR,

$$V = E - R I_s \left( e^{qV/kT} - 1 \right)$$

NONLINEAR IN V

LET

$$f(V) \equiv V - E + R I_s \left( e^{qV/kT} - 1 \right)$$

THEN, THE PROBLEM TO BE SOLVED IS:

$$f(V) = 0.$$

## SUCCESSIVE SUBSTITUTION

0. GUESS $V^{(0)}$

   e.g., $V^{(0)} = 0$

   SET $k = 0$

1. SOLVE FOR $V^{(k+1)}$:

   $$V^{(k+1)} - E + RI_s\left(e^{qV^{(k)}/kT} - 1\right) = 0$$

   OR,

   $$V^{(k+1)} - V^{(k)} + f^{(k)} = 0$$

2. IF

   $$f\left(V^{(k+1)}\right)$$

   IS SMALL ENOUGH, STOP.

   ELSE:

3. SET $k = k+1$.

   GOTO 1.



SUCCESSIVE SUBSTITUTION

$E = 100$ mV, $I_s = 1\mu$A, $R = 500\ \Omega$

| k | V | f |
|---|---|---|
| 0 | 0.000000 | -0.100000 |
| 1 | 0.100000 | 0.026799 |
| 2 | 0.073201 | -0.017954 |
| 3 | 0.091155 | 0.009819 |
| 4 | 0.081336 | -0.006224 |
| 5 | 0.087560 | 0.003658 |
| 6 | 0.083902 | -0.002259 |
| 7 | 0.086162 | 0.001356 |
| 8 | 0.084805 | -0.000829 |
| 9 | 0.085634 | 0.000501 |
| 10 | 0.085133 | -0.000305 |
| 11 | 0.085438 | 0.000185 |
| 12 | 0.085253 | -0.000112 |
| 13 | 0.085365 | 0.000068 |
| 14 | 0.085297 | -0.000041 |
| 15 | 0.085339 | 0.000025 |
| 16 | 0.085314 | -0.000015 |
| 17 | 0.085329 | 0.000009 |
| 18 | 0.085320 | -0.000006 |
| 19 | 0.085325 | 0.000003 |
| 20 | 0.085322 | -0.000002 |
| 21 | 0.085324 | 0.000001 |
| 22 | 0.085323 | -0.000001 |
| 23 | 0.085323 | 0.000000 |

FIG. 1 SUCCESSIVE SUBSTITUTION

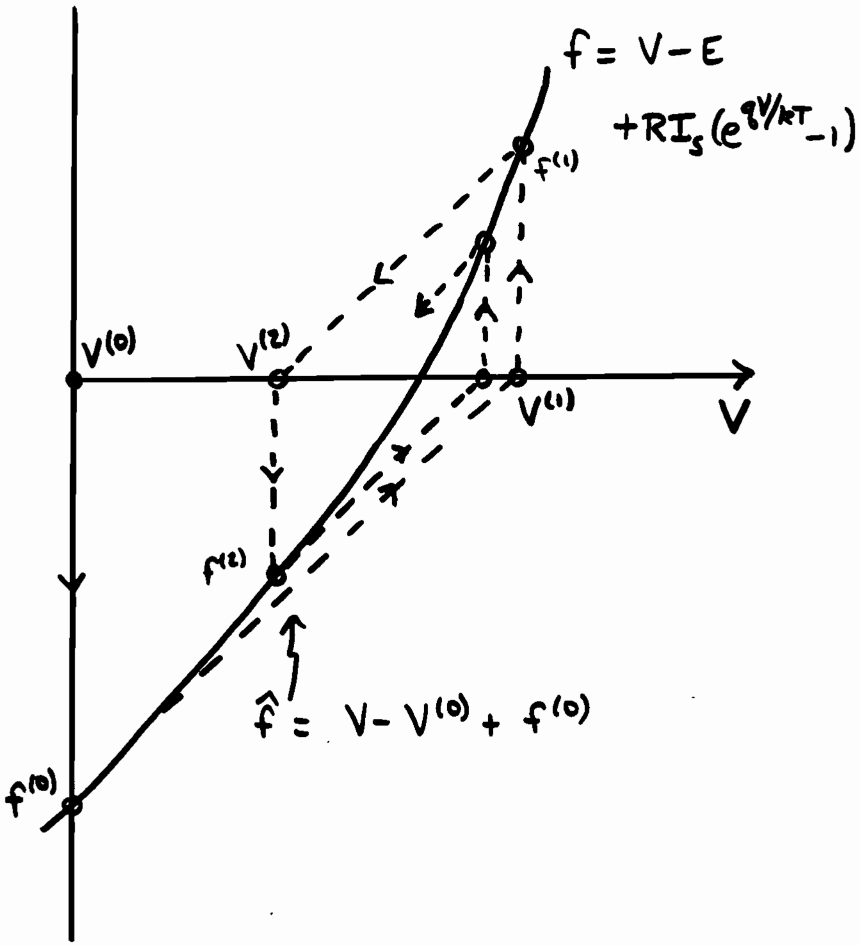
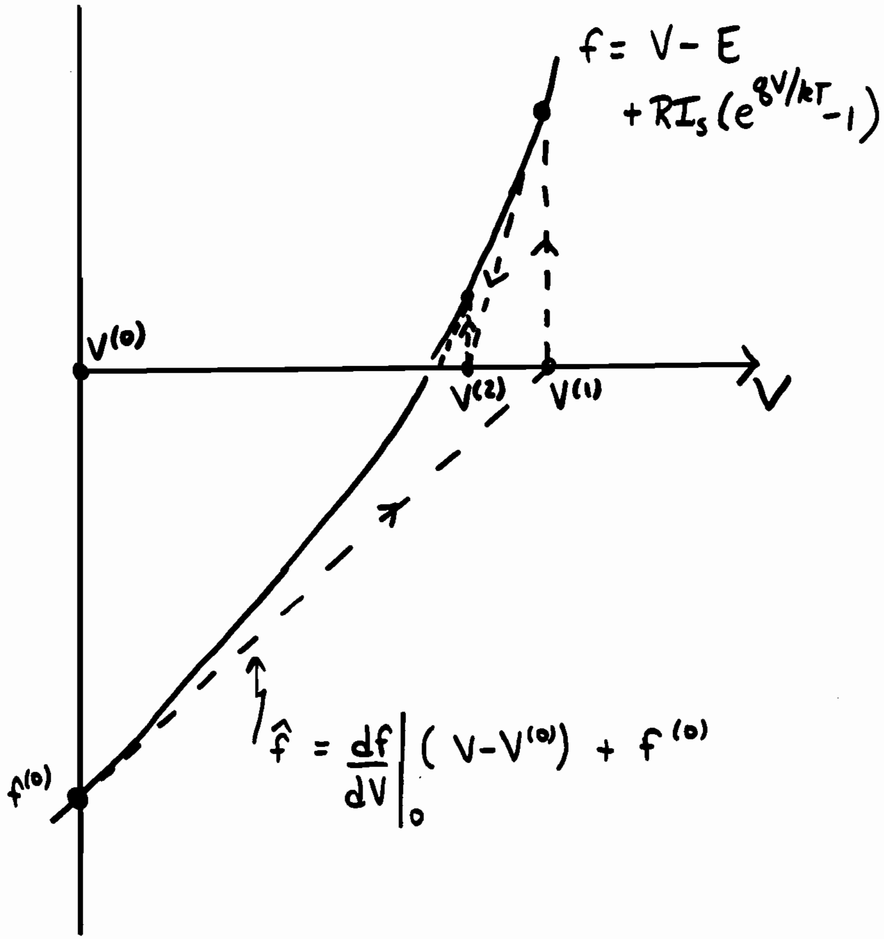



FIG. 2 NEWTON-RAPHSON



# NEWTON - RAPHSON

0. GUESS $V^{(0)}$, e.g. $V^{(0)} = 0$.

   SET $k=0$.

1. SOLVE FOR $V^{(k+1)}$:

$$f'^{(k)}\left(V^{(k+1)} - V^{(k)}\right) + f^{(k)} = 0$$

   WHERE

$$f'^{(k)} = \left.\frac{df}{dV}\right|_{V=V^{(k)}}$$

2. IF $f(V^{(k+1)})$

   IS SMALL ENOUGH, STOP.

   ELSE:

3. SET $k = k+1$.

   GOTO 1.



NEWTON-RAPHSON

$E = 100$ mV, $I_s = 1\mu A$, $R = 500\ \Omega$

| k | V | f | f' |
|---|---|---|---|
| 0 | 0.000000 | -0.100000 | 1.020000 |
| 1 | 0.098039 | 0.022779 | 2.009591 |
| 2 | 0.086704 | 0.002243 | 1.641555 |
| 3 | 0.085338 | 0.000024 | 1.607432 |
| 4 | 0.085323 | 0.000000 | 1.607077 |

# QUADRATIC CONVERGENCE:

EXPAND $f$ IN A TAYLOR SERIES ABOUT $V^{(k)}$:

$$f(V) = f^{(k)} + f'^{(k)}(V - V^{(k)}) + \frac{1}{2} f''^{(k)}(V - V^{(k)})^2 + \cdots$$

IN PARTICULAR, IF $V = V_s$, THE SOLUTION:

$$\underbrace{f(V_s)}_{=0} = f^{(k)} + f'^{(k)}(V_s - V^{(k)}) + \frac{1}{2} f''^{(k)}\underbrace{(V_s - V^{(k)})}_{\varepsilon_k = |V_s - V^{(k)}|}{}^2 + \underbrace{\cdots}_{\substack{\text{IGNORE}\\ \text{CLOSE}\\ \text{TO } V_s}}$$



THUS,

$$f^{(k)} + f'^{(k)}(V_s - V^{(k)}) + \frac{1}{2} f''^{(k)} \varepsilon_k^2 = 0$$

DIVIDE BY $f'^{(k)}$:

$$\frac{f^{(k)}}{f'^{(k)}} + (V_s - V^{(k)}) + \frac{1}{2} \frac{f''^{(k)}}{f'^{(k)}} \varepsilon_k^2 = 0$$

RE-ARRANGE:

$$V_s - \underbrace{\left( V^{(k)} - \frac{f^{(k)}}{f'^{(k)}} \right)}_{= V^{(k+1)}} = -\frac{1}{2} \frac{f''^{(k)}}{f'^{(k)}} \varepsilon_k^2$$

i.e.,

$$\underbrace{V_s - V^{(k+1)}}_{\varepsilon_{k+1} = |V_s - V^{(k+1)}|} = -\frac{1}{2} \frac{f''^{(k)}}{f'^{(k)}} \varepsilon_k^2$$

OR,

$$\varepsilon_{k+1} = \frac{1}{2}\left|\frac{f''(k)}{f'(k)}\right| \varepsilon_k^2$$

IF

$$\frac{1}{2}\left|\frac{f''(k)}{f'(k)}\right| < c$$

THEN

$$\varepsilon_{k+1} < c\,\varepsilon_k^2 .$$

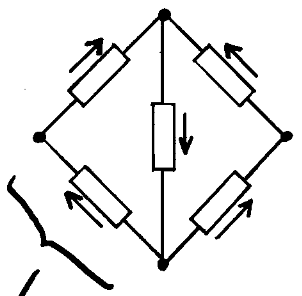
SUPPOSE

$$\varepsilon_k = 10^{-n}$$

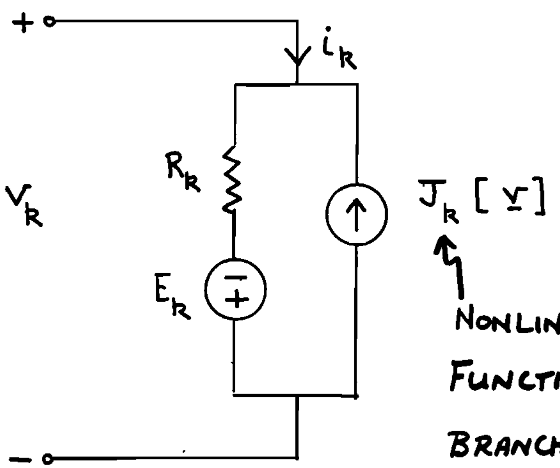
THEN

$$\varepsilon_{k+1} < c\,10^{-2n} .$$

# NONLINEAR NETWORKS



GENERAL BRANCH




NONLINEAR FUNCTION OF THE BRANCH VOLTAGES



e.g., DIODE :

$$J_k = -J_s\left(e^{qV_k/kT} - 1\right)$$

NETWORK EQUATIONS :

KCL : $\underline{\underline{A}}\,\underline{i} = 0$ ← REDUCED INCIDENCE MATRIX

KVL : $\underline{v} = \underline{\underline{A}}^T \underline{v}_n$ ← NODAL VOLTAGES

BRANCH EQUATIONS : $\underline{i} = \underline{\underline{Y}}\,\underline{v} + \underline{\underline{Y}}\,\underline{E} - \underline{J}[\underline{v}]$

COMBINING :

$$(\underline{\underline{A}}\,\underline{\underline{Y}}\,\underline{\underline{A}}^T)\underline{v}_n = \underline{\underline{A}}\left(\underline{J}[\underline{\underline{A}}^T\underline{v}_n] - \underline{\underline{Y}}\,\underline{\underline{E}}\right)$$

NONLINEAR IN $\underline{v}_n$

LET

$$\underline{f}[\underline{v}_n] \equiv (\underline{\underline{A}}\,\underline{\underline{Y}}\,\underline{\underline{A}}^T)\underline{v}_n - \underline{\underline{A}}\,\underline{J}[\underline{\underline{A}}^T\underline{v}_n] + \underline{\underline{A}}\,\underline{\underline{Y}}\,\underline{E}$$

THEN THE PROBLEM TO BE SOLVED IS:

$$\underline{f}[\underline{v}_n] = 0$$

SUCCESSIVE SUBSTITUTION:

⇒ AT EACH STEP, HOLD $\underline{J}[\underline{\underline{A}}^T\underline{v}_n]$ FIXED AND SOLVE A LINEAR PROBLEM.



## SUCCESSIVE SUBSTITUTION

0. GUESS $\underline{v}_n^{(0)}$, e.g., $\underline{v}_n^{(0)} = 0$

   SET $k = 0$.

1. SOLVE FOR $\underline{v}_n^{(k+1)}$ :

$$(\underline{\underline{A}}\,\underline{\underline{Y}}\,\underline{\underline{A}}^T)\,\underline{v}_n^{(k+1)} = \underline{\underline{A}}\,\underline{J}[\underline{\underline{A}}^T\underline{v}_n^{(k)}] - \underline{\underline{A}}\,\underline{\underline{Y}}\,\underline{E}$$

OR,

$$(\underline{\underline{A}}\,\underline{\underline{Y}}\,\underline{\underline{A}}^T)(\underline{v}_n^{(k+1)} - \underline{v}_n^{(k)}) + \underline{f}^{(k)} = 0$$

2. IF $\underline{f}^{(k+1)}$ IS SMALL ENOUGH, STOP.

   ELSE:

3. SET $k = k+1$. GOTO 1.

To get **Newton-Raphson**, replace:

$$(\underline{\underline{A}}\ \underline{\underline{Y}}\ \underline{\underline{A}}^T)(\underline{v}_n^{(k+1)} - \underline{v}_n^{(k)}) + \underline{f}^{(k)} = 0$$

By:

$$\underbrace{\frac{d\underline{f}^{(k)}}{d\underline{v}_n}}(\underline{v}_n^{(k+1)} - \underline{v}_n^{(k)}) + \underline{f}^{(k)} = 0$$

Jacobian matrix $\underline{\underline{\mathcal{F}}}^{(k)}$

$$\mathcal{F}_{ij}^{(k)} = \left.\frac{\partial f_i}{\partial v_{n_j}}\right|_{\underline{v}_n = \underline{v}_n^{(k)}}$$



Recall,

$$\underline{f} = (\underline{\underline{A}}\ \underline{\underline{Y}}\ \underline{\underline{A}}^T)\underline{v}_n - \underline{\underline{A}}\ \underline{J}[\underline{\underline{A}}^T\underline{v}_n] + \underline{\underline{A}}\ \underline{\underline{Y}}\ \underline{E}$$

Thus,

$$\frac{\partial \underline{f}}{\partial \underline{v}_n} = (\underline{\underline{A}}\ \underline{\underline{Y}}\ \underline{\underline{A}}^T) - \underline{\underline{A}}\ \underbrace{\frac{\partial \underline{J}}{\partial \underline{v}_n}}$$

$$= \frac{\partial \underline{J}}{\partial \underline{v}}\ \underbrace{\frac{\partial \underline{v}}{\partial \underline{v}_n}}$$

$$= \frac{\partial}{\partial \underline{v}_n}(\underline{\underline{A}}^T\underline{v}_n) = \underline{\underline{A}}^T$$

HENCE,

$$\underline{\underline{J}} = \frac{\partial \underline{f}}{\partial \underline{v}_n} = (\underline{\underline{A}}\,\underline{\underline{Y}}\,\underline{\underline{A}}^T) - \underline{\underline{A}}\,\frac{\partial \underline{J}}{\partial \underline{v}}\,\underline{\underline{A}}^T$$

AVAILABLE FROM DEVICE CHARACTERISTICS

SUCCESSIVE SUBSTITUTION:

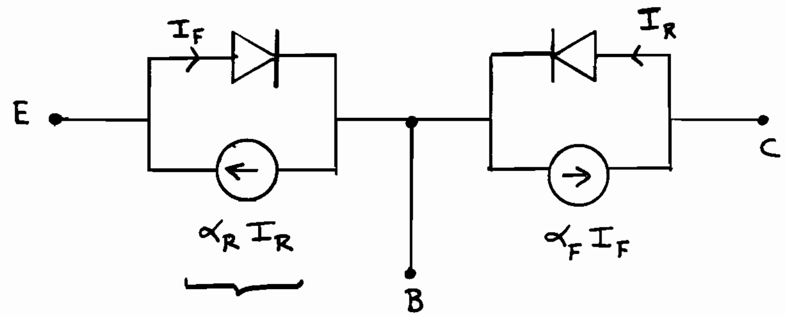
$$(\underline{\underline{A}}\,\underline{\underline{Y}}\,\underline{\underline{A}}^T)(\underline{v}_n^{(k+1)} - \underline{v}_n^{(k)}) + \underline{f}^{(k)} = 0$$

NEWTON-RAPHSON:

$$\left(\underline{\underline{A}}\,\underline{\underline{Y}}\,\underline{\underline{A}}^T - \underline{\underline{A}}\,\frac{\partial \underline{J}}{\partial \underline{v}}\,\underline{\underline{A}}^T\right)\left(\underline{v}_n^{(k+1)} - \underline{v}_n^{(k)}\right) + \underline{f}^{(k)} = 0.$$

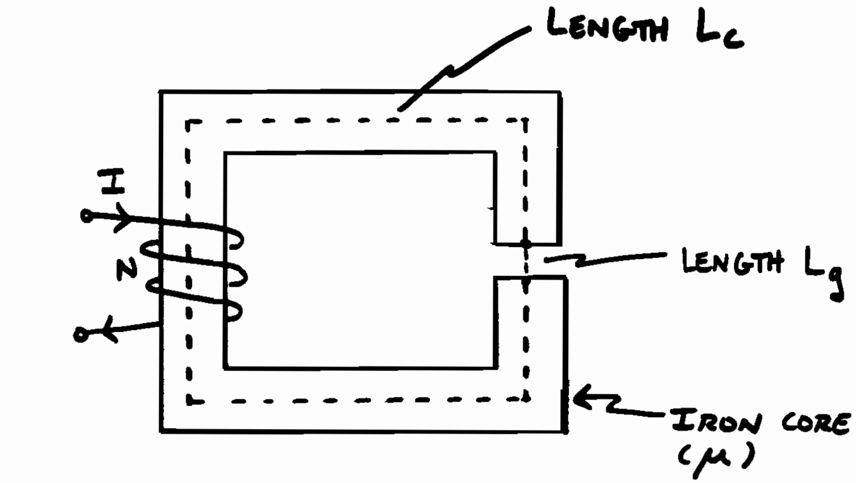


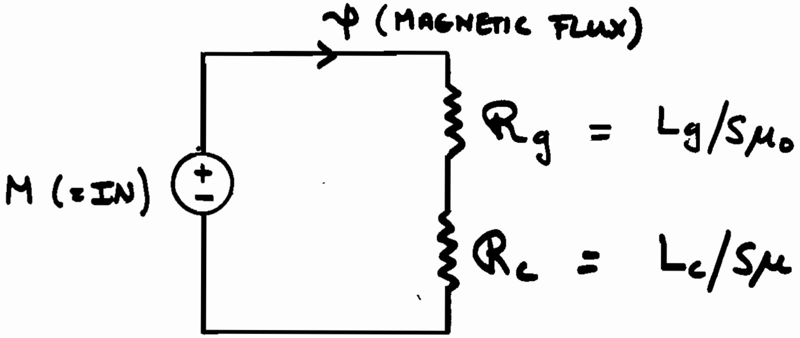
TRANSISTOR (EBERS-MOLL):




DEPENDENT CURRENT SOURCE

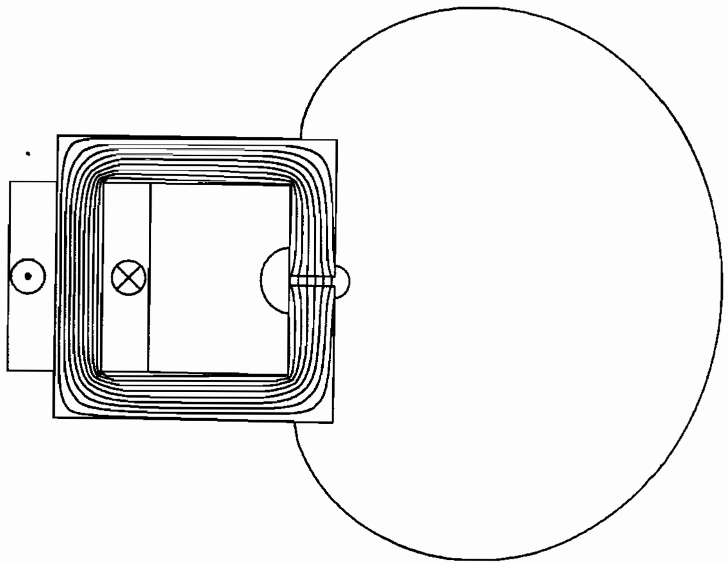
# MAGNETIC CIRCUITS




$$\Rightarrow (\mathcal{R}_g + \mathcal{R}_c)\psi = M$$

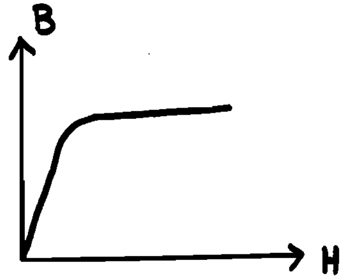




$$(\mathcal{R}_g + \mathcal{R}_c)\,\psi = M$$

$$\mathcal{R}_c = \frac{L_c}{S\mu}$$

$$\mu = \frac{B}{H}$$




$$= \mu(B)$$

$B$ = FLUX DENSITY

$$= \psi / S$$

THUS,

$$(\mathcal{R}_g + \mathcal{R}(\psi))\,\psi = M$$

NONLINEAR IN $\psi$