Rayleigh flow: heat addition[†]

1. Set up

Note that the flow is neither adiabatic, nor isentropic. Consider a duct with *constant cross-sectional area*. Between entrance and exit, heat is added. The basic question is, given the upstream state and the amount of heat added, what is the downstream state?

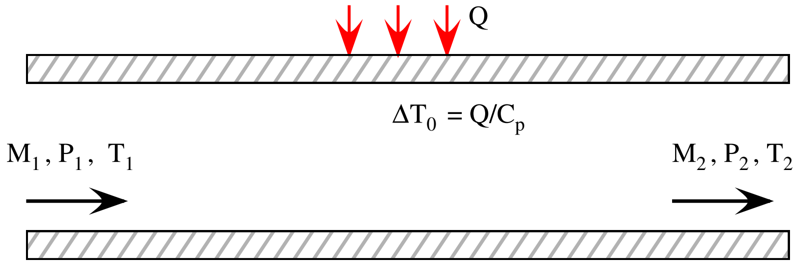

Figure 1: Configuration

2. Derivations

Momentum conservation

In general, the bulk x-momentum flux is $(P + \rho U^2) \times A$. Under the condition of constant cross sectional area, bulk momentum conservation becomes

$$P_1 + \rho_1 U_1^2 = P_2 + \rho_2 U_2^2$$

Using $a^2 = \gamma RT = \gamma P/\rho$ to replace ρ by $\gamma P/a^2$ gives

$$P_1 + \gamma P_1 M_1^2 = P_2 + \gamma P_2 M_2^2$$

Hence

$$\boxed{\frac{P_1}{P_2} = \frac{1 + \gamma M_2^2}{1 + \gamma M_1^2}} \tag{1}$$

Mass conservation

Again, for constant cross-sectional area

$$\rho_1 U_1 = \rho_2 U_2$$

With $\rho = P/RT$

$$\frac{P_1 U_1}{RT_1} = \frac{P_2 U_2}{RT_2}$$

Multiply both sides by $\sqrt{R/\gamma}$ to get

$$\frac{P_1 M_1}{\sqrt{T_1}} = \frac{P_2 M_2}{\sqrt{T_2}}$$

So temperature ratio is

$$\boxed{\frac{T_1}{T_2} = \left(\frac{P_1 M_1}{P_2 M_2}\right)^2} \tag{2}$$

[†]A.k.a. Rayleigh line https://en.wikipedia.org/wiki/Rayleigh_flow

Example:
If $M_1 = 0.4$ and $M_2 = 0.6$,

$$\frac{T_2}{T_1} = \left(\frac{M_2(5+7M_1^2)}{M_1(5+7M_2^2)}\right)^2 = 1.490$$

and

$$\frac{T_{02}}{T_{01}} = \frac{T_2}{T_1} \cdot \frac{5+M_2^2}{5+M_1^2} = 1.548$$

The total temperature must be raised by 55% to increase the Mach number from 0.4 to 0.6

Example:
If $M_1 = 1.1$ and $M_2 = 1.5$,

$$\frac{T_2}{T_1} = \left(\frac{M_2(5+7M_1^2)}{M_1(5+7M_2^2)}\right)^2 = 0.7836$$

and

$$\frac{T_{02}}{T_{01}} = \frac{T_2}{T_1} \cdot \frac{5+M_2^2}{5+M_1^2} = 0.9148$$

The total temperature must be *decreased* by 9% to increase the Mach number from 1.1 to 1.5.
Conversely if heat is added M decreases.

Can check that

$$\frac{M_2(5+7M_1^2)}{M_1(5+7M_2^2)}$$

equals 1 at $M_{1,2} = 1$ and is $\lesseqgtr 1$ for $M \gtreqless 1$.

Or look at limits: $\propto M_2/M_1$ as $M_{1,2} \to 0$ and $\propto M_1/M_2$ as $M_{1,2} \to \infty$.

3. General formulas

These formulas could be used, as is; but, to create tables, or general purpose formulas, pressure, and temperature are evaluated relative to a reference state. Let state 2 be sonic. Thus, $M_2 = 1$, $P_2 \equiv P^*$; also, drop the subscript for state 1 in Eq. 1 and Eq. 2. Then the formulas for Rayleigh flow are

$$\left.\begin{aligned} \frac{P}{P^*} &= \frac{1+\gamma}{1+\gamma M^2} &&= \frac{12}{5+7M^2} \\ \frac{T}{T^*} &= M^2\frac{(1+\gamma)^2}{(1+\gamma M^2)^2} &&= \frac{144M^2}{(5+7M^2)^2} \\ \frac{\rho^*}{\rho} = \frac{U}{U^*} &= M^2\frac{1+\gamma}{1+\gamma M^2} &&= \frac{12M^2}{5+7M^2} \end{aligned}\right\} \tag{3}$$

for air. In the last of (3), the reasoning that

$$M^2 = \frac{U^2}{\gamma P/\rho} = \frac{U(\rho U)}{\gamma P}$$

and $\rho U = \textit{constant}$ imply

$$U \propto M^2 P \rightarrow U/U^* = M^2 P/P^*$$

was used.

Heat addition will be modeled as a change in total temperature. So we need formulas for the total properties

$$\begin{aligned} \boldsymbol{\frac{T_0}{T_0^*}} &= \frac{5+M^2}{6}\frac{T}{T^*} &&= \boldsymbol{\frac{M^2(120+24M^2)}{(5+7M^2)^2}} \\ \frac{P_0}{P_0^*} &= \left(\frac{P^*}{P_0^*}\frac{P_0}{P}\right)\frac{P}{P^*} &&= \left(\frac{5+M^2}{6}\right)^{7/2}\frac{12}{5+7M^2} \end{aligned} \tag{4}$$

The term in parenthesis comes from the isentropic relations.

Comments (see figure 2, below)

Note that $T_0/T_0^* \leq 1$. Also note that $d(T_0/T_0^*)/dM^2$ is positive for $M < 1$ and negative for $M > 1$[‡]. Hence, increasing total temperature will move the flow toward sonic. Moving toward sonic means that when heat is added, M increases if the flow is subsonic, and M decreases if the flow is supersonic.

Then, note that $dP_0/dM^2 < 0$ for $M < 1$ and $dP_0/dM^2 > 0$ for $M > 1$. As M increases subsonically, or decreases supersonically, total pressure decreases; hence, heat addition always decreases total pressure (and increases entropy).

[‡] Proof:

$$1 \geq \frac{M^2(120+24M^2)}{(5+7M^2)^2} \rightarrow 25+70M^2+49M^4 \geq 120M^2+24M^4 \rightarrow 0 \geq 2M^2-M^4-1 = -(M^2-1)^2 \text{ q.e.d.}$$

$$\frac{d(T_0/T_0^*)}{dM^2} \propto (120+48M^2)(5+7M^2)^2 - 14M^2(120+24M^2)(5+7M^2) = (1-M^2)\times[600(5+7M^2)] \propto \boldsymbol{(1-M^2)}$$

Similarly

$$\frac{d(P_0/P_0^*)}{dM^2}] \propto \boldsymbol{(M^2-1)}$$

Example:
If $M = 0.2$ and $T = 300°K$, how much heat must be added to increase M to 1?

Solution:

At $M = 0.2$, $T = 300 \;\rightarrow\; T_0 = (1 + {}^1\!/_5 \times 0.2^2)300 = 302.4°K$
From (4), @ $M = 0.2$, $T_0/T_0^* = 0.17355 \rightarrow\; T_0^* = 302.4/0.17355 = 1742.4$.
$\Delta T_0 = 1,742.4 - 302.4 = \boxed{1,440.0°K}$.

To convert to heat, use $C_p = {}^7\!/_2\, R = 1004.68 J/kg°K$ to get

$$Q = 1004.68 \times 1440 = 1.447 \times 10^6 J/kg$$

E.g. if the mass flow is $0.1kg/s$, $144.7kJ/s$ must be added.

Example:
If $M = 0.2$ and $T = 300°K$, how much must T_0 increase to raise M to 0.5?

Solution:

At $M = 0.2$, $T = 300 \;\rightarrow\; T_0 = (1 + {}^1\!/_5 \times 0.2^2)300 = 302.4°K$
From (4), @ $M = 0.2$, $T_0/T_0^* = 0.17355 \rightarrow\; T_0^* = 302.4/0.17355 = 1742.4°K$.
At $M = 0.5$, $T_0/T_0^* = 0.69136 \rightarrow T_0 = 1742.4 * 0.69136 = 1204.63°K$
So $\boxed{\Delta T_0 = 902.23°K}$

Example:
If $M = 1.2$ and $T = 200°K$, how much must T_0 decrease to raise M to 1.6?

Solution:

At $M = 1.2$, $T = 200 \;\rightarrow\; T_0 = (1 + {}^1\!/_5 \times 1.2^2)200 = 257.6°K$
From (4), @ $M = 1.2$, $T_0/T_0^* = 0.97872 \rightarrow\; T_0^* = 263.2°K$.
At $M = 1.6$, $T_0/T_0^* = 0.88419 \rightarrow T_0 = 232.72°K$
So $\boxed{\Delta T_0 = -24.9°K}$

To raise the supersonic Mach number, the gas has to be cooled.

4. Last step: add heat

Energy conservation

Heat addition is modeled as an increase of the total temperature. Heating by an amount Q is represented as a change $\Delta T_0 = Q/C_p$ of total temperature

$$T_{0exit} = T_{0in} + \Delta T_0$$

Eq. 4 for T_0/T_0^* versus M is the key relation. It has to be inverted to find the Mach number after heat addition. The heat addition does not change T_0^*, so this can be determined at the inlet.

To that end, solve Eq. 4 for M as a function of T_0/T_0^*: Eq. 4 has the form of a quadratic equation $aM^4 - 2bM^2 + c = 0$ where

$$\begin{aligned} a &= \gamma^2\left(\frac{T_0}{T_0^*} - 1\right) + 1 \\ b &= 1 - \gamma\left(\frac{T_0}{T_0^*} - 1\right) \\ c &= \frac{T_0}{T_0^*} \end{aligned} \tag{5a}$$

$c > 0$ and $b > 0$. The quadratic has either 1 or 2 positive roots Fig. 2.

If $a < 0$, the only positive root is (theorem: # positive roots=# sign changes in quadratic)

$$\boxed{M_{sub}^2 = \frac{b}{a} - \frac{b}{a}\sqrt{1 - \frac{ca}{b^2}}} \tag{5b}$$

If $a > 0$ — i.e., if $T_0/T_0^* > 1 - 1/\gamma^2 = 24/49$ — a second positive root is (see figure 2, below)

$$\boxed{M_{super}^2 = \frac{b}{a} + \frac{b}{a}\sqrt{1 - \frac{ca}{b^2}}} \tag{5c}$$

As $a \to 0$ this root $\to \infty$. **N.B.:** Eq. 4 and Eqs. 5b, 5c with 5a, are the key elements of the heat addition theory. `Sample script`

> For instance, given M_{in}, T_{in} and ΔT_0 one can find M_{out}; or given any 3 find the 4^{th}.
> In choked flow, $M_{out} = 1$; given two of M_{in}, T_{in} or ΔT_0, find the other:
> $M_{out} = 1$, T_{in} and $\Delta T_0 \to M_{in}$

Instead of the using the exact formulas (5b) and (5c), one could make a plot and pick points, M, off it (or use Newton's method instead of the quadratic formula). Fig. 2 is the plot. It shows that

- Heat addition to subsonic flow increases M toward sonic, while heat addition to supersonic flow decreases M toward sonic.
- Consequently, sufficient heat addition causes the flow to choke.
- After it chokes, the exit will remain sonic, and further heat addition will reduce the inlet Mach number.

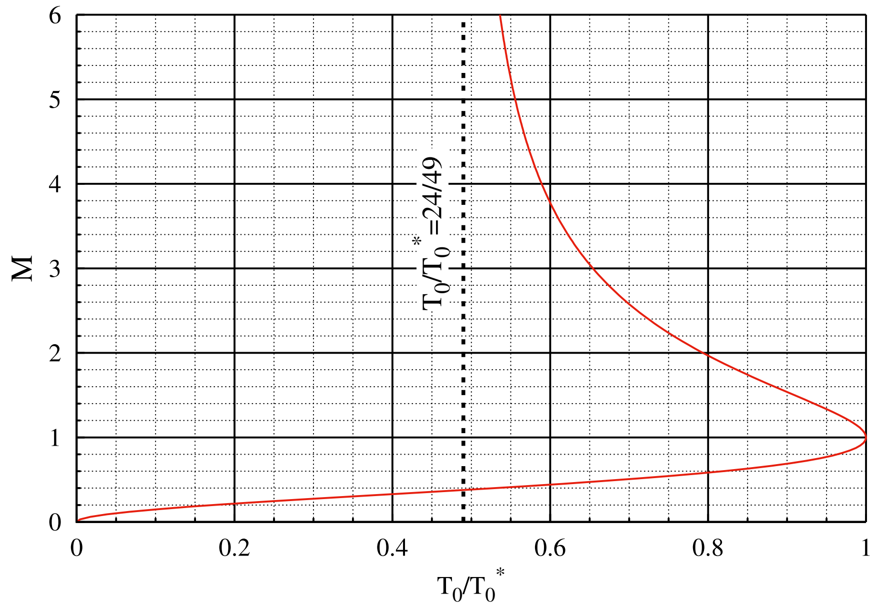

Figure 2: Heat addition drives M toward unity; leads to choking

In other words, after the flow chokes, the exit total temperature is T_0^*, by definition. Thus, starting with a given T_0 at the inlet, increases in heat addition along the duct will increase T_0^*. Hence, heat addition will cause T_0/T_0^* to decrease. Then fig. 2 shows that heat addition decreases the inlet Mach number. This is the concept of the flow being choked off by heating.

Example 1

The inlet flow to a straight channel has $U = 72m/s,\ T = 323°K$. Along the channel, $Q = 10^6 J/kg$ of heat is added. The gas is air, $R = 287.05J/kg\,°K,\ C_p = 7R/2$.

At the end of the channel what are $M_{exit},\ T_{exit},\ U_{exit},\ P_{in}/P_{exit},\ P_{0exit}/P_{0in}$?

Solution:

At inlet: $M = 72/\sqrt{7/5 * 287.05 * 323} = 0.200$, $T_0 = 323 * (1 + 1/5 * 0.2^2) = 325.58°K$.

From Eq. 4 (or fig. 2) at $M = 0.2$, $T_0/T_0^* = 0.1733$, which gives $T_0^* = 325.58/0.1733 = 1,878.7°K$.

Added heat: $\Delta T_0 = Q/C_p = 10^6/(7/2 * 287.05) = 995.35°K$.

Exit total temperature ratio: $T_0/T_0^* = (325.58 + 995.35)/1878.7 = 1320.95/1878.7 = 0.703$.

From Eq. 5b (or fig. 2), with $T_{0e}/T_0^* = 0.703$ $(a = 0.4201, b = 1.4142, c = 0.7041)$, $M_e = 0.508$.
It is the subsonic root, because the inlet is subsonic.

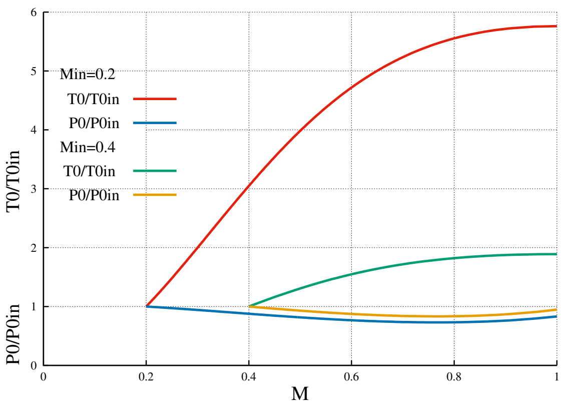

Figure 3: *Total pressure variation with* T_0/T_0^*

We have $M_e = 0.51$ and $T_{0e} = 1,321$; hence,

$$T_e = 1321/(1 + 1/5\, 0.51^2) = 1,256°K$$

$$U_e = 0.51\sqrt{7/5 * 287 * 1,256} = 361.5m/s$$

$$\frac{P_{in}}{P_{exit}} = \frac{P_{in}/P^*}{P_{exit}/P^*} = \frac{5 + 7M_{exit}^2}{5 + 7M_{in}^2} = \frac{5 + 7 * .51^2}{5 + 7 * .2^2} = 1.29$$

$$\frac{P_{0exit}}{P_{0in}} = \left(\frac{5 + M_{exit}^2}{5 + M_{in}^2}\right)^{7/2} \frac{P_{exit}}{P_{in}} = 0.90$$

Note, when heat is added, $T_0 \uparrow$ and $P_0 \downarrow$. Figure 3 shows two comment.

Example 2

a) A straight duct is choked by heat addition. The exit temperature is $600°K$. The heat addition is equivalent to $\Delta T_0 = 200°K$. What are M_{in}, U_{in}, T_{in}?

Solution:

Since $M = 1$ at the exit, the exit total temperature is $T_{0*} = 600 \times (6/5) = 720°K$.

With heat addition equivalent to $\Delta T_0 = 200°K$, at the inlet $T_0/T_{0*} = 520/720 = 13/18$.

Substituting 13/18 into eq. 5b gives $M_{in} = 0.522$. Then $T_{in} = T_{0in}/(1 + M_{in}^2/5) = 493.2°K$ and $U_{in} = M_{in} * \sqrt{1.4 * 287 * T_{in}} = 232.2m/s$.

b) If the heat addition is equivalent to $\Delta T_0 = 500°K$, what are M_{in}, U_{in}?

Solution:

At the inlet $T_0/T_{0*} = 220/720$. From eq. 5b, $M_{in} = 0.277$. Then $U_{in} = 81.8m/s$.

Comparing **a)** and **b)**: The effect of heat addition to choked flow is to lower the inlet Mach number, and velocity.

Comment: Another way to solve this is to let $M = 1$ at the inlet, and $\Delta T_0 = -500°K$.

When the flow is choked adding heat, ΔT_0, would increase T_{0exit}, but the exit is choked, so still, $T_{0exit} = T_0^*$. Thus, T_{0in}/T_0^* decreases when heat is added; it is like starting from sonic at the exit and removing heat to get the inlet Mach number.

Example 3

A straight duct is choked by heat addition. The exit temperature is measured as $T_* = 600°K$. If the inlet temperatue is measured as $300°K$, what is the inlet Mach number? How much heat was added?

Solution:

$$\begin{aligned} \frac{T^*}{T_{in}} = 2 &= \frac{(5 + 7M_{in}^2)^2}{144M_{in}^2} \\ 7M_{in}^2 - 12\sqrt{2}M_{in} + 5 &= 0 \end{aligned}$$

$M_{in} = 0.343$. $T_{0in} = (1 + {}^1/_5\, M_{in}^2)T_{in} = 307.06$; $T_{0exit} = (1 + {}^1/_5\,)T_{exit} = 720$; $\Delta T_0 = 412.9°K$

$Q = C_p \Delta T_0 = 7/2 * 287.05 * 412.9 = 4.15 \times 10^5 J/kg$

Sample script

```
#!/usr/local/bin/python3
from math import *
#————————
# Input M and T
#————————
M = float(input(" Min "))
T = float(input(" Tin(°K) "))
#-------
M2in = M*M
T0in = T*(1.+.2*M2in)
T0st = T0in*(5.+7.*M2in)**2/(120.*M2in+24.*M2in*M2in)
T0mx = T0st-T0in
print(" *** Will choke at DT0=",round(T0mx,1))
#————————
# Input Del T0
#————————
DT0 = float(input(" ENTER Del T0 "))
Tr = (T0in+DT0)/T0st
gamma=7./5.
b = 1.-gamma*(Tr-1.)  # b > 0
c = Tr
a = gamma**2*(Tr-1.)+1.  # a > 0 if Tr > 1.-1./gamma**2
M2 = b/a - b/a*sqrt(1.-c*a/b**2)
Msub = sqrt(M2)
if(a > 0.):
  M2p = b/a + b/a*sqrt(1.-c*a/b**2)
  Msup = sqrt( M2p )
else:
  Msup=0.  # just for printing
print(f"Msub=Msub:6.4f",f"Msup=Msup:7.4f")
```

```
E.g.:
   Min = 1.5
   Tin(°K)= 200
    T0in = 290.00
    T0* = 318.94
   ENTER ΔT0 (choke at 28.9)= 20
    T0e/T0* = 0.9720
    ( a= 0.9451 b= 1.0392 c= 0.9720 )
    Me = 1.2348
    T0e (°K) = 310.00
    Te (°K) = 237.56
    Ue (m/s) = 381.52
    Pe/Pin = 1.3240
    P0e/P0in = 0.9155
    (Subsonic solution Me= 0.821 )
```