How do I do this shit?

Type II:

Linear Algebra

Finding the set of solutions to linear equations

1) state the equations above each other, with matching variables above each other as so:

$$\begin{aligned} 2x + \quad y + 2z &= 4 \\ 2x + 2y \qquad &= 5 \\ 2x - \quad y + 6z &= 2 \end{aligned}$$

2) State these as an augmented matrix, and do row operations to obtain *reduced row echelon form*. (Row Operations can be adding one row to another, switching rows or multiplying a row by a scalar.)

$$A = \left[\begin{array}{ccc|c} 2 & 1 & 2 & 4 \\ 2 & 2 & 0 & 5 \\ 2 & -1 & 6 & 2 \end{array}\right] \xrightarrow{?} \left[\begin{array}{ccc|c} 1 & 0 & 2 & 1.5 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

3) Pure row-echelon form contains only leading 1's and 0's. If this is obtained, the variable corresponding to the column of the leading 1 will have the value of the value in the right-most position of the row of the leading 1 (the results-column).
4) If not obtained (as in example above) we will determine the columns containing a leading 1 (and 0's otherwise) as *pivot columns*. Other columns will be *free columns* which we will assign a parameter (t, s or other letters). Then we will do as below:

$$\begin{aligned} z &= t \\ y - 2z = 1 \rightarrow y &= 1 + 2z \rightarrow y = 1 + 2t \\ x \qquad + 2z = 1.5 \rightarrow x &= 1.5 - 2z \rightarrow x = 1.5 - 2t \end{aligned}$$

This gives us that there are <u>infinitely many solutions</u> described by:

$$\begin{aligned} x \qquad &= 1.5 - 2t \\ y \qquad &= 1 + 2t \\ z &= t \end{aligned}$$

Null-space of the solutions: (N(A))

Null-space can be computed when letting the 'result-column' be all zeros, meaning that we have actually been computing Ax=0. In other words, we can define the null-space of the coefficient matrix of the system of linear equations.

In practice we will just place the parameters outside, and multiply them onto a vector, which will make it look like:

1) For a set of solutions to linear equations above:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1.5 \\ 1 \\ 0 \end{bmatrix} + t * \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}$$

2) For null-space of the coefficient-matrix from above:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = t * \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix} \qquad B = \left\{ \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix} \right\}$$

Solving linear equations using invertible matrices:
If A is invertible in: $A*x = b$ then $x = A^{-1}*b$
In which A will have been the coefficient matrix of the system of linear equations.
(For inverse of invertible matrix, see Inverse of matrix)

Finding Rank & Dimensions
Rank(matrix)=Dim(Row Space)=Dim(Column Space)
<u>Rank</u> is the numbxer of leading 1's in the reduced row echelon form
<u>Dimensions of Row Space</u> are the number of rows containing entries that are not 0 when achieving reduced row echelon form of a matrix (f.ex. a *coefficient matrix* for a system of linear equations…)

For A= $\left[\begin{array}{ccc|c} 1 & 0 & 2 & 1.5 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$ Rank(A) = Dim(Row Space of A) = 2
Because of 2 leading 1's

Matrix Multiplication:
ROW X COLUMN
IT MATTERS WHICH ONE IS FIRST IN THE MULTIPLICATION
TO KNOW WHICH MATRIX YOU'RE GETTING FROM MULTIPLYING THEM, YOU SAY:

N X a times b X M

a = b

Output is N X M

Example:

2 X 3 times 3 X 2

3 = 3

Output is 2 X 2

Okay, let's begin :´)

$$For\ A\ with\ size\ n\ x\ m\ \&\ B\ with\ size\ m\ x\ k,$$

$$then\ AB\ wil\ be\ and\ n\ x\ k\ matrix$$

$$a_{xy}\ will\ be\ an\ entry\ of\ \ A\ at\ row\ x\ column\ y$$

$$A * B = \begin{bmatrix} a_{11} & \dots & a_{1m} \\ \dots & \dots & \dots \\ a_{n1} & \dots & a_{nm} \end{bmatrix} * \begin{bmatrix} b_{11} & \dots & b_{1k} \\ \dots & \dots & \dots \\ b_{m1} & \dots & b_{mk} \end{bmatrix}$$
$$= \begin{bmatrix} a_{11} * b_{11} + \dots + a_{1m} * b_{m1} & \dots & a_{11} * b_{1k} + \dots + a_{1m} * b_{mk} \\ \dots & \dots & \dots \\ a_{n1} * b_{11} + \dots + a_{nm} * b_{m1} & \dots & a_{n1} * b_{1k} + \dots + a_{nm} * b_{mk} \end{bmatrix}$$

Remember the entry of the number in the matrix you're calculating for:

$$\begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} 4 & 6 \\ 5 & 5 \\ 6 & 4 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

a = row 1, column 1, therefore: row 1 · column 1

b = row 1, column 2, therefore: row 1 · column 2

c = row 2, column 1, therefore: row 2 · column 1

d = row 2, column 2, therefore: row 2 · column 2

$$\begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} 4 & 6 \\ 5 & 5 \\ 6 & 4 \end{bmatrix} = \begin{bmatrix} (1 \cdot 4 + 2 \cdot 5 + 3 \cdot 6) & (1 \cdot 6 + 2 \cdot 5 + 3 \cdot 4) \\ (3 \cdot 4 + 2 \cdot 5 + 1 \cdot 6) & (3 \cdot 6 + 2 \cdot 5 + 1 \cdot 4) \end{bmatrix} = \begin{bmatrix} 32 & 28 \\ 28 & 32 \end{bmatrix}$$

THIS IS AN EXAMPLE OF WHAT A MATRIX LOOKS LIKE

$$a = \begin{bmatrix} a\_11 & a\_12 & a\_13 \\ a\_21 & a\_22 & a\_23 \\ a\_31 & a\_32 & a\_33 \end{bmatrix}$$

Basis:
A basis is a *spanning set* of the least number of vectors possible that still span the whole space. <u>Spanning means that any entity in the set/space can be computed as a linear combination of these vectors/matrices.</u>
Number of vectors in basis = number of unique entries in result matrix

1) Vectorspace □^{2} will have a basis of 2 vectors. Simplest basis would be □ = {[*1 0*], [*0 1*]}
2) Vectorspace □$_{2,2}$ will have a basis of 4 matrices. Simplest basis would be
$$B = \left\{\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}, \begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}, \begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix}, \begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}\right\}$$
3) Subspace of □$_{2,2}$ containing all symmetric matrices would have a basis of 3 matrices.
Simplest: $$B = \left\{\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}, \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}, \begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix}\right\}$$

Testing whether a set of vectors is a basis for a given vectorspace?
Prove either Non-zero determinant or linear independence! (Or look in the back of the book)

Properties of Vector Addition and Scalar Multiplication in R^n

Let **u**, **v**, and **w** be vectors in R^n, and let c and d be scalars.

1.	$\mathbf{u} + \mathbf{v}$ is a vector in R^n	Closure under addition
2.	$\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$	Commutative property of addition
3.	$(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$	Associative property of addition
4.	$\mathbf{u} + \mathbf{0} = \mathbf{u}$	Additive identity property
5.	$\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$	Additive inverse property
6.	$c\mathbf{u}$ is a vector in R^n.	Closure under scalar multiplication
7.	$c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$	Distributive property
8.	$(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$	Distributive property
9.	$c(d\mathbf{u}) = (cd)\mathbf{u}$	Associative property of multiplication
10.	$1(\mathbf{u}) = \mathbf{u}$	Multiplicative identity property

Testing whether a vector is in a subspace with basis B:
Essentially:
Is there a set of coordinates $c_1, ..., c_n$ that can be used to compute a linear combination of the basis to obtain v?
For subspace V, we try to compute:

$$V \text{ is spanned by the vectors in the basis } B = \{u, w\}$$

$$\text{if } v \text{ in } V, \text{then there will exist:}$$

$$v = c_1 * u + c_2 * w$$

1) Present as an Augmented matrix, because:
$$\begin{bmatrix}\uparrow & \uparrow\\ u & w\\ \downarrow & \downarrow\end{bmatrix} * \begin{bmatrix}c_1\\ c_2\end{bmatrix} = v \qquad \Rightarrow \qquad \left[\begin{array}{cc|c}\uparrow & \uparrow & \uparrow\\ u & w & v\\ \downarrow & \downarrow & \downarrow\end{array}\right]$$
2) Try to achieve row-echelon form

a) if you get a row that has 0's in the coefficient-matrix part and a number in the resulting-column, there will be no linear combination, and v will <u>not</u> be in the subspace because:
   i) $0*c_1 + \ldots + 0*c_n = a$ will have no viable solutions <3

Coordinates of v relative to basis B:
Will be the resulting vector $[\square_1 \ \ \square_2]$ when having done the above calculations.
Denoted $[v]_B=[c_1\ c_2]$

Nullspaces

Computing a nullspace:

1) Begin with a matrix A, and try to achieve reduced row echelon form.
2) Then:

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$x_1 = -2x_3 = -2t$$

$$x_2 = 2x_3 = 2t$$

$$x_3 = 1$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = t * \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix} \qquad B = \left\{ \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix} \right\}$$

Basis for null-space:

See above :) B={v_1, …, v_n}

Dimensions of null-space:

The dimensions of the nullspace is the number of variables minus the rank of the of the matrix.

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 \end{bmatrix}$$

For this matrix, the dimensions of the null-space will be 4-3=1 , denoted as Dim(N(A))=1

The dimension of the nullspace is the number of free columns or parameters, or otherwise the number of vectors needed in the basis to span this null-space.

Determinant of a matrix:

2x2 matrix:

$$\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = a * d - b * c$$

Summary of Equivalent Conditions for Square Matrices

If A is an $n \times n$ matrix, then the conditions below are equivalent.

1. A is invertible.
2. $A\mathbf{x} = \mathbf{b}$ has a unique solution for any $n \times 1$ matrix $\mathbf{b}$.
3. $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.
4. A is row-equivalent to I_n.
5. $|A| \neq 0$
6. $\text{Rank}(A) = n$
7. The n row vectors of A are linearly independent.
8. The n column vectors of A are linearly independent.

Other Square matrices:

1) Choose a row or a column to compute the determinant over.

$$A = \begin{bmatrix} a_{11} & \dots & a_{1j} \\ \dots & \dots & \dots \\ a_{i1} & \dots & a_{ij} \end{bmatrix}$$

    a) If possible, choose a row/column with as many 0's as possible. (See below as to why)
2) Compute the Minors of the chosen row/column
    a) You do not need to compute the Minors/cofactors of the entries which are 0 in our matrix (since they will be multiplied onto the cofactors, and thus the whole thing will become 0, no matter the value of the minor/cofactor.)
    b) $$M = \begin{bmatrix} |A\ except\ row\ 1\ column\ 1| & \dots & |A\ except\ row\ 1\ column\ j| \\ \dots & \dots & \dots \\ |A\ except\ row\ i\ column\ 1| & \dots & |A\ except\ row\ i\ column\ j| \end{bmatrix}$$
    c) Minors are the determinants of the original matrix, but removing the row and column of the entry.
    d) Thus for a 3x3 matrix you would only go one step down, whilst for a 4x4 matrix the Minors of this would be the determinants of 3x3 matrices, which would need to be computed using this new matrix' minors and entries.
3) Compute the Cofactors of the chosen row/Column
    a) i.e. change the sign +/- for every second entry starting with: (keep, change, keep, change, ….)
4) Compute the sum of the original entry from the matrix, multiplied by the Cofactor of the same entry over the whole row/column of your choice.
    a) Upper is the determinant calculated over row 1, Lower is determinant calculated over column 1:

$$\det(A) = a_{11} * |A\ except\ row\ 1\ column\ 1| + \dots + a_{1j} * |A\ except\ row\ 1\ column\ j|$$

    b) $$\det(A) = a_{11} * |A\ except\ row\ 1\ column\ 1| + \dots + a_{i1} * |A\ except\ row\ i\ column\ 1|$$

Inverse of matrix
Inverse is the matrix B given as A^{-1} that fulfills AB=BA=I .
If a matrix has an inverse it is *invertible* or *non-singular*
2 x 2 matrix:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{\underbrace{ad-bc}_{\text{determinant}}} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Other square matrices:

1) Method 1:
    a) We state the matrix [A I] (We concatenate our matrix A and the identity matrix of the same size)
    b) We Retrieve [I A^{-1}] (We do row operations until the left half of the concatenated matrix is the identity matrix, and then the right part will be the inverse of A)
2) Method 2:
    a) Solve for X in AX = I

Testing if v is an eigenvector of B
if v is an eigenvector of B, then there will exist a scalar λ such that:

$$B * v = \lambda * v$$

Since we know B & v, we can compute the matrix multiplication of B * v...
Then we see whether the computed matrix can be computed by multiplying a scalar onto v, solving for λ in:

$$B * v = \lambda * v$$

Computing eigenvalues & vectors

1) Compute the *characteristic polynomial*.
    a) $|I\lambda - A| = 0$
    b) For a 2 x 2 matrix A:

$$\begin{vmatrix} \lambda - a_1 & -a_2 \\ -a_3 & \lambda - a_4 \end{vmatrix} = (\lambda - a_1)(\lambda - a_4) - (-a_3)(-a_2) = 0$$

2) Find the roots of this polynomial. Denote these λ_1 & λ_2
3) Backsubstitute and solve:

$$\begin{bmatrix} \lambda_1 - a_1 & -a_2 \\ -a_3 & \lambda_1 - a_4 \end{bmatrix} v_1 = 0$$

$$\begin{bmatrix} \lambda_2 - a_1 & -a_2 \\ -a_3 & \lambda_2 - a_4 \end{bmatrix} v_2 = 0$$

    a) i.e. find the reduced row echelon form of $[\lambda\square - \square]$, and determine the vector v as the basis of the solution-subspace.
4) The Eigenvectors can now be described as ANY vector v, that is either $\square * \square_1$ or $\square * \square_2$ for which t cannot equal to 0.

Computing $A^n u$ for a matrix A with eigenvectors v1 & v2:

1) Determine the constants c_1 & c_2 that denote u as a linear combination of eigenvectors v_1 & v_2 giving $u = c_1 * v_1 + c_2 * v_2$
2) Now see and state, that the following is true:

$$A^n * u = A^n * (c_1 * v_1 + c_2 * v_2) = c_1 * A^n * v_1 + c_2 * A^n * v_2$$

$$= c_1 * {\lambda_1}^n * v_1 + c_2 * {\lambda_2}^n * v_2$$

$$For\ eigenvalues\ \lambda_1, \lambda_2\ and\ eigenvectors\ v_1, v_2$$

$$and\ c_1, c_2\ from\ u's\ coordinates\ relative\ to\ the\ eigenvectors.$$

if confused, also look at assignment 4 in LAO.

Test for linear independence

1) Method 1:
    a) We test whether Ax=0 implies that all entries in x are 0 (if this is the case, any vector will only have one combination of the original vectors as their linear combination, thus proving linear independence)
2) Method 2:
    a) A system of homogeneous linear equations has a unique solution if it has a non-zero determinant. (Unique solution == linear independence >>> True)

Testing for subspace

For something to be a subspace it needs to be non-empty and closed under addition and scalar multiplication. We can prove these properties for a subspace:

1) Non-empty
   We show that there is a set of values (x,y,...) (often we use something trivial like (0,0,...)) that fulfill the requirement set for the subset.
2) Closed under Addition
   a) See below in function line showing $2(x_0+x_1)=...=5(y_0+y_1)$
   b) $(0,0) \in W$ because $2 \cdot 0 = 5 \cdot 0$. If $(x_0, y_0) \in W$ and $(x_1, y_1) \in W$ then $(x_0 + x_1, y_0 + y_1) \in W$ because
   $$2(x_0 + x_1) = 2x_0 + 2x_1 = 5y_0 + 5y_1 = 5(y_0 + y_1)$$
3) Closed under Scalar Multiplication
   a) See below in function line showing $2(cx)=...=5(cx)$
   b) If $(x, y) \in W$ and $c \in \mathbb{R}$, also $c(x, y) = (cx, cy) \in W$ because
   $$2cx = c2x = c5y = 5cy$$

Properties of Vector Addition and Scalar Multiplication in R^n

Let **u**, **v**, and **w** be vectors in R^n, and let c and d be scalars.

1. $\mathbf{u} + \mathbf{v}$ is a vector in R^n	**Closure under addition**
2. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$	**Commutative property of addition**
3. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$	**Associative property of addition**
4. $\mathbf{u} + \mathbf{0} = \mathbf{u}$	**Additive identity property**
5. $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$	**Additive inverse property**
6. $c\mathbf{u}$ is a vector in R^n.	**Closure under scalar multiplication**
7. $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$	**Distributive property**
8. $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$	**Distributive property**
9. $c(d\mathbf{u}) = (cd)\mathbf{u}$	**Associative property of multiplication**
10. $1(\mathbf{u}) = \mathbf{u}$	**Multiplicative identity property**

Projection Matrix P

1) $P = A\,(A^TA)^{-1}A^T$
2) Write up A & A^T
   a) Transposing a matrix A means that the rows become the columns and the columns become the rows
      i) Transpose of a Matrix
      $$A = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}_{2 \times 3} \qquad A^T = \begin{bmatrix} a & d \\ b & e \\ c & f \end{bmatrix}_{3 \times 2}$$
3) Calculate $A^T A$
4) Determine $(A^TA)^{-1}$ (remember 1/det out in front of the matrix)
5) Keep $1/\det(A^TA)$ out in front of the whole calculation, and calculate:

a) $\square = \square\,(\square^{\square}\square)^{-1}\square^{\square}$

Page Rank

M = (1-m)(A+D)+mS

m = damping factor

A = 1 / links from j   for entries at row i column j if j links to i, 0 otherwise

D = 1/n   for all entries in columns of nodes that do not link to other nodes

S = 1/n   as all entries no matter what (what does it do, and when to use it?)

Calculus

Derivatives

Multivariable:

1) We can only do *partial derivatives*
2) Denoted $f_x(x,y)$ and $f_y(x,y)$ or otherwise

Multidimensional:

1) g(t)=(x(t),y(t))
2) Derivatives are just calculated for each direction:
3) g'(t)=(x'(t),y'(t))

Taylor Polynomials:

$$T_n = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

a = x

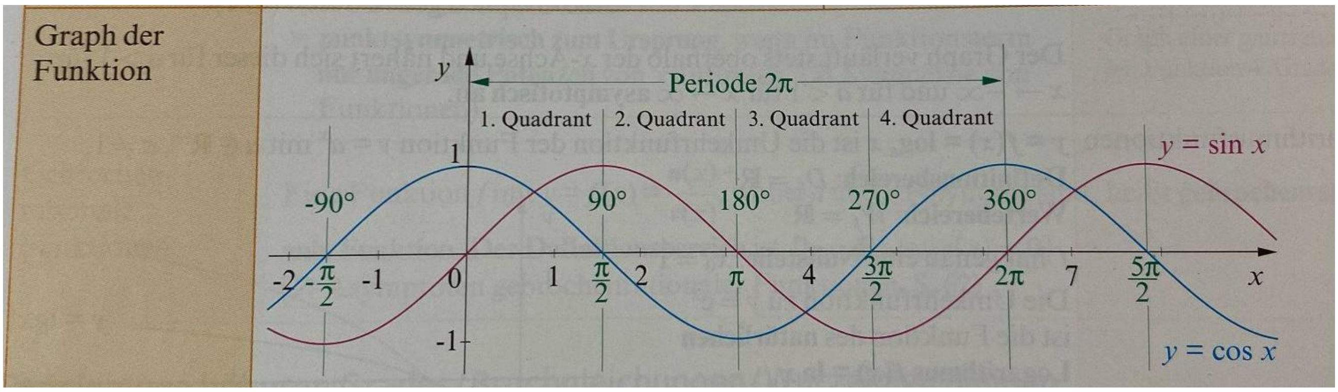
x = a

e.g. x = 0 is the same as a = 0

Order k means the upper bounds of the sum should be set to k

1) State f(x), and determine f'(x), then f''(x), then f'''(x) until you reach the upper bound.
2) Calculate f(a), f'(a), f''(a) etc. with <u>a</u> being the basepoint we are estimating the function around.
3) Insert into formula: *(here written until order 2)*

$$f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \cdots$$



Directional derivatives
Directional derivative is defined as:

Directional derivative in the u-direction:

$$D_u(x,y) = \nabla f(x,y) \cdot u = f_x(x,y) * u_1 + f_y(x,y) * u_2$$

With $f(x,y)$ being the surface and u being the vector pointing in the direction for which we want to find the derivative.

The generalized chain rule:

We are given: $f(x,y)$, $x(t)$, $y(t)$ and:

$$g(t) = f(x(t), y(t))$$

Then:

$$g'(t) = f_x\big(x(t), y(t)\big) * x'(t) + f_y\big(x(t), y(t)\big) * y'(t)$$

The normal chainrule
You use the chainrule when a function is either to the power of something, or when it is divided.
(outer function **derived** * inner function **not derived**) * the inner function **derived**

$$\big(f(g(x))\big)' = f'(g(x)) \cdot g'(x)$$

Computing gradients

$$\nabla f(x,y) = (f_x(x,y), f_y(x,y))$$

Tangent line / tangent planes

Tangent Line: for f(x,y)=k at (x_0 , y_0)

$$f(x,y) = k \qquad (x_0, y_0)$$

$$0 = \nabla f(x_0, y_0) \cdot (x - x_0, y - y_0)$$

$$0 = f_x(x_0, y_0) * (x - x_0) + f_y(x_0, y_0) * (y - y_0)$$

Tangent Plane: for f(x,y) OR f(x,y,z)=k at (x_0 , y_0)

$$f(x,y) \qquad (x_0, y_0)$$

$$z - z_0 = \nabla f(x_0, y_0) \cdot (x - x_0, y - y_0)$$

$$z - f(x_0, y_0) = f_x(x_0, y_0) * (x - x_0) + f_y(x_0, y_0) * (y - y_0)$$

OR

$$f(x,y,z) = k \qquad (x_0, y_0, z_0)$$

$$0 = \nabla f(x_0, y_0, z_0) \cdot (x - x_0, y - y_0, z - z_0)$$

$$0 = f_x(x_0, y_0, z_0) * (x - x_0) + f_y(x_0, y_0, z_0) * (y - y_0) + f_z(x_0, y_0, z_0) * (z - z_0)$$

Critical point - min, max or saddle?

1) Compute Gradient
2) Set gradient = 0
3) Solve for x & y in the equations from (2)
4) This/These are your critical points

Second derivative test:

$$a = f_{xx} \qquad b = f_{xy} = f_{yx} \qquad c = f_{yy}$$

if	then
$a > 0$ & $ac - b^2 > 0$	minimum
$a < 0$ & $ac - b^2 > 0$	maximum
$ac - b^2 < 0$	saddle point

Lagrange Multipliers

$$f(x,y)\ is\ the\ function\ we\ are\ optimizing$$

$$g(x,y) = 0\ is\ the\ constraint$$

$$\nabla L(x,y,\lambda) = \left(f_x(x,y) - \lambda * g_x(x,y)\ , f_y(x,y) - \lambda * g_y(x,y)\ , -g(x,y)\right) = (0{,}0{,}0)$$

$$f_x(x,y) - \lambda * g_x(x,y) = 0$$

$$f_y(x,y) - \lambda * g_y(x,y) = 0$$

$$-g(x,y) = 0$$

1) Solve for y in the first equation
2) Insert the y from (1) in the second equation and solve for x
3) Insert y from (1) then x from (2) in the third equationa and solve for lambda.
4) Insert this lambda back into x & y.
5) For several points explain/prove which are max & min

Arc length

$$f(t) = (x(t), y(t))$$

$$\int_{t1}^{t2} \|f'(t)\|$$

1) Compute f'(t)=(x'(t) , y'(t))
2) Find the length of this vector: $\|f'(t)\| = \sqrt{(x'(t))^2 + (y'(t))^2}$
3) Integrate with boundaries t1-t2 giving F(t2) - F(t1) with F(t) being the integrated function.

Integrals:

$f(x)$	$F(x)$
x	$\frac{1}{2}x^2$
kx	$\frac{k}{2}x^2$
k	kx
x^n	$\frac{1}{n+1}x^{n+1}$
$\frac{1}{x}$	$\ln(\|x\|)$
a^x	$\frac{a^x}{\ln(a)}$
e^x	e^x
e^{kx}	$\frac{1}{k} \cdot e^{kx}$
$\sqrt{x}$	$\frac{2}{3}x^{3/2} = \frac{2}{3}(\sqrt{x})^3$
$\ln(x)$	$x \cdot \ln(x) - x$

Example for number divided by variable:

$$\int \frac{4}{x}\, dx = \int \left(4 \cdot \frac{1}{x}\right) dx = 4 \cdot \int \frac{1}{x}\, dx = 4 \cdot \ln(x)$$

Multiple Integrals:

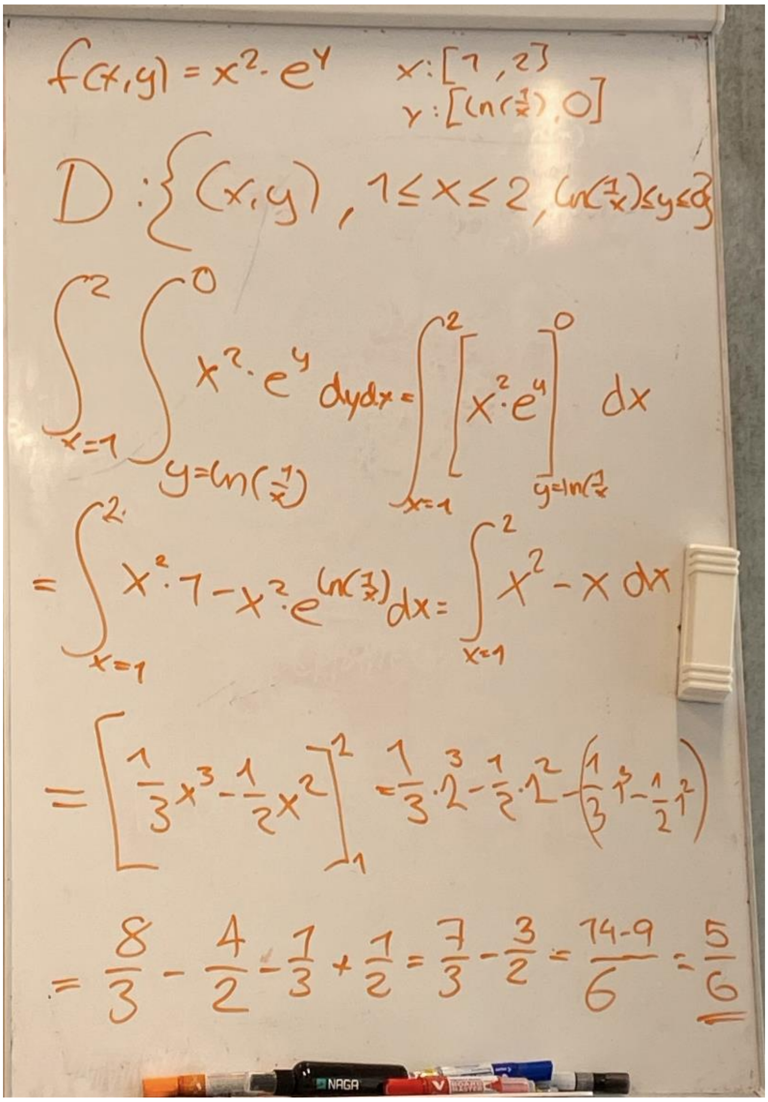
Type I:

$$D = \{(x,y)\ AND\ r_1 \le x \le r_2\ , f_1(x) \le y \le f_2(x)\}$$

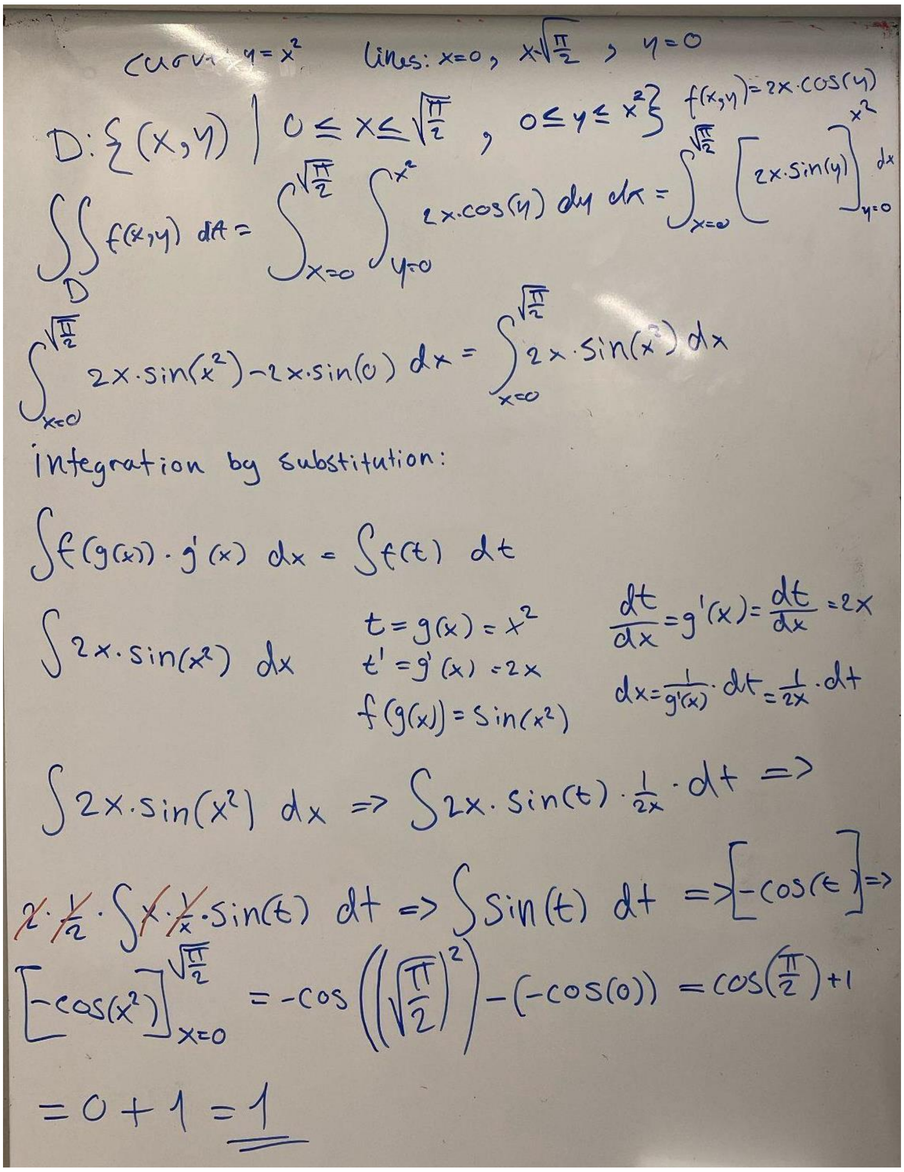
$$\iint\limits_D f(x,y)\ dA = \int_{r_1}^{r_2} \int_{f_1(x)}^{f_2(x)} f(x,y)\ dy\ dx$$

1) Find integral in accordance to y for f(x,y)
2) Insert y-boundaries on y (You will get a result that most likely contains x’s)
3) Find integral of result from (2) in accordance to x
4) Insert x-boundaries

Example 1 - :



Example 2 - March 2020 (integration by substitution):



When to use integration by substitution:
Når integranden (indmaden i integralet) indeholder et *produkt* af funktioner, og når en af dem er *sammensat*. Det er ikke i alle disse tilfælde, det vil virke, men ofte er det et forsøg værd.

Type II:

$$D = \{(x, y)\ AND\ \ f_1(y) \le x \le f_2(y), r_1 \le y \le r_2\ \}$$

$$\iint_D f(x, y)\ dA = \int_{r_1}^{r_2} \int_{f_1(y)}^{f_2(y)} f(x, y)\ dy\ dx$$

5) Find integral in accordance to x for f(x,y)
6) Insert x-boundaries on x (You will get a result that most likely contains y's)
7) Find integral of result from (2) in accordance to y
8) Insert y-boundaries