Algorithm Design Strategies III

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Overview

- Tasks from last week – Recap + Questions ?
- The Higher-Lower Game – Analysis
- Example – Empirical Analysis
- Dynamic Programming
- Example – Fibonacci's Sequence
- Example – Linear Robot
- Example – Computing Binomial Coefficients
- Memoization
- Timing and Profiling Code

COMPUTING POWERS
BRUTE-FORCE VERSIONS

a^b – Brute-Force – Iterative algorithm

- Compute a^b, with $b \geq 0$, using

$$a^b = a \times a \times \ldots \times a$$

- Number of multiplications ?
  - Formal + Empirical analysis

a^b – Brute-Force – Iterative algorithm

```
def powerIterV1( a, b ) :

    """ Computing a**b using a loop """

    assert (type( b ) == int) and (b >= 0), "Wrong exponent!"

    assert (a != 0) or (b != 0), "Cannot compute 0**0 !"

    res = 1

    for i in range( 1, b + 1 ) :

        res *= a

    return res
```

- Number of multiplications ? b

a^b – Brute-Force – Iterative algorithm

n	2**n	#Mults	
0	1	0	
1	2	1	
2	4	2	
3	8	3	
4	16	4	
5	32	5	
6	64	6	
7	128	7	
8	256	8	
9	512	9	
10	1024	10	
11	2048	11	(linear)

a^b – Brute-Force – Recursive alg.

- Compute a^b, with $b \geq 0$, using

$$a^b = a \times a^{b-1}, \text{ with } a^0 = 1$$

- Number of multiplications ?
  - Formal + Empirical analysis
- Any gains ?

a^b – Brute-Force – Recursive alg.

```
def powerRecV1( a, b ) :
    """ Computing a**b recursively --- Direct algorithm """
    assert (type( b ) == int) and (b >= 0), "Wrong exponent!"
    assert (a != 0) or (b != 0), "Cannot compute 0**0 !"
    if b == 0 :
        return 1
    return a * powerRecV1( a, b - 1 )
```

- Number of multiplications ?

a^b – Brute-Force – Recursive alg.

n	2**n	#Mults
0	1	0
1	2	1
2	4	2
3	8	3
4	16	4
5	32	5
6	64	6
7	128	7
8	256	8
9	512	9
10	1024	10
11	2048	11

COMPUTING POWERS
DIVIDE-AND-CONQUER

a^b – Divide-And-Conquer

- Compute a^b, with $b \geq 0$, using

$$a^b = a^{b \text{ div } 2} \times a^{(b+1) \text{ div } 2}$$

- Base cases ?
- Always use two recursive calls !!

- Number of multiplications ?
  - Formal + Empirical analysis
- Is it better than the direct algorithm ?

a^b – Divide-And-Conquer

```
def powerRecV3( a, b ) :
    """ Computing a**b recursively --- Blind Div & Conq strategy"""
    # TWO base cases are needed !!
    # Otherwise, we would not stop when b == 1 !!
    assert (type( b ) == int) and (b >= 0), "Wrong exponent!"
    if b == 0 :
        return 1
    if b == 1 :
        return a
    return powerRecV3( a, b // 2 ) * powerRecV3( a, (b + 1) // 2 )
```

- Number of multiplications ?

Formal analysis

$$M(n) = M(n \text{ div } 2) + M((n+1) \text{ div } 2) + 1$$

- Easier to solve if n is a power of 2
  - $n = 2^k$, $k = \log_2 n$

$$M(n) = M(n / 2) + M(n / 2) + 1$$
$$= 2\, M(n / 2) + 1 = \ldots$$

- Closed formula ? Complexity order ?

a^b – Divide-And-Conquer

```
  n      2**n     #Mults
-------------------------
  0         1      0
  1         2      0
  2         4      1
  3         8      2
  4        16      3
  5        32      4
  6        64      5
  7       128      6
  8       256      7
  9       512      8
 10      1024      9
 11      2048     10
```

COMPUTING POWERS DECREASE-AND-CONQUER

a^b – Decrease-And-Conquer

- Compute a^b, with $b \geq 0$, using

$$a^b = a^{b \text{ div } 2} \times a^{b \text{ div } 2} \text{, if b is even}$$

$$a^b = a \times a^{(b-1) \text{ div } 2} \times a^{(b-1) \text{ div } 2} \text{, if b is odd}$$

- Base cases ?
- Use just ONE recursive call !!
- Number of multiplications ?
  - Formal + Empirical analysis

a^b – Decrease-And-Conquer

```
def powerRecV6( a, b ) :
    """ Computing a**b recursively --- Smart Dec & Conq strategy"""
    assert (type( b ) == int) and (b >= 0), "Wrong exponent!"
    if b == 0 :
        return 1
    p = powerRecV6( a, b // 2 )
    if (b % 2) == 0 :
        return p * p
    return a * p * p
```

- Number of multiplications ?

Formal analysis

M(n) = M(n div 2) + 1 , if n is even

M(n) = M((n-1) div 2) + 2 , if n is odd

- Check some examples with pencil and paper
  - Do you understand what is happening?
  - Best vs. worst cases ?
- Closed formula ? Complexity order ?
- Is it better than the previous algorithms ?

a^b – Decrease-And-Conquer

n	2**n	#Mults
0	1	0
1	2	2
2	4	3
3	8	4
4	16	4
5	32	5
6	64	5
7	128	6
8	256	5
9	512	6
10	1024	6
11	2048	7
12	4096	6
13	8192	7
14	16384	7
15	32768	8
16	65536	6
17	131072	7

ARRAY: COUNTING EVEN-VALUED ELEMENTS

Task – Counting – Recap

- Given an array with non-negative integer values

- Count the number of even-valued elements

- Implement the 3 strategies:
  - Brute-Force / Dec & C / Div & C

- Formal + Empirical analysis : Comparisons

Number of array comparisons ?

- Brute-Force
  - 1 loop / n iterations / n comparisons
- Decrease & Conquer
  - $C(0) = 0$
  - $C(n) = 1 + C(n - 1)$
- Divide & Conquer
  - $C(0) = 0$ and $C(1) = 1$
  - $C(n) = C(\ n \text{ div } 2\ ) + C(\ (n+1) \text{ div } 2\ )$

Number of array comparisons

#Elements	#COMPS	#Even-Values
1	1	0
2	2	1
4	4	2
8	8	4
16	16	8
32	32	16
64	64	32
128	128	64
256	256	128
512	512	256

ARRAY:
SEQUENTIAL SEARCH

Task – Sequential Search – Recap

- Given an array with non-negative integer values
- Use the iterative Sequential Search algorithm to look for a given value
- Formal + Empirical analysis : Comparisons
- Best / Worst / Average Cases ?

Possible cases ?

- Best case ?
  - 1 comparison → $B(n) = O(1)$
- Worst Case ?
  - n comparisons → $W(n) = O(n)$
- Average Case ?
  - Various possible scenarios
  - BUT always → $A(n) = O(n)$

Average Case

- One possible scenario
  - No repeated array values
  - Searched value belongs to the array
  - Can be any array element – Equal probability !

- Average number of comparisons ?
  - How to compute ?

- $A(n) = (n + 1) / 2 \approx n / 2$

A different scenario

```
Sequential Search on RANDOM arrays of positive integers

Searching RANDOM values
```

#Elements	#Searches	#Found	#COMPS	#Average_COMPS
1	1	0	1	1.000
2	2	0	4	2.000
4	4	0	16	4.000
8	8	0	64	8.000
16	16	0	256	16.000
32	32	2	993	31.031
64	64	3	4007	62.609
128	128	9	15726	122.859
256	256	56	57621	225.082
512	512	209	207084	404.461
1024	1024	677	658902	643.459
2048	2048	1751	1845775	901.257
4096	4096	4004	3992714	974.784

THE HIGHER-LOWER GAME

The Higher-Lower Game

- Person A chooses a random integer in [1, 100]
- Person B guesses a number in [1, 100]
- Person A says: Yes / Low guess / High guess
- Person B keeps guessing until the answer is Yes

- How long does it take ?

- Let’s play the game !!

The Higher-Lower Game

- Naïve strategy vs. smart strategy
  (Binary Search)

- How many guesses ?
  - Best case ?
  - Worst cases ?
  - Average cases ?

- Complexity order ?

Complexity Analysis of Binary Search Algorithm

- **Time Complexity:**
  - Best Case: O(1)
  - Average Case: O(log N)
  - Worst Case: O(log N)

*Best case is when the element is at the middle index of the array. It takes only one comparison to find the target element. So the best case complexity is **O(1)**.*

So, total comparisons
= 1(elements requiring 1 comparisons) + 2*(elements requiring 2 comparisons) + . . . + logN*(elements requiring logN comparisons)*
*= 1*1 + 2*2 + 3*4 + . . . + logN * (2^{logN-1})*
*= 2^{logN} * (logN – 1) + 1*
= N * (logN – 1) + 1

*Total number of cases = **N+1**.*

*Therefore, the average complexity = **(N*(logN – 1) + 1)/N+1 = N*logN / (N+1) + 1/(N+1)**. Here the dominant term is N*logN/(N+1) which is approximately **logN**. So the average case complexity is **O(logN)***

*The worst case will be when the element is present in the first position. As seen in the average case, the comparison required to reach the first element is **logN**. So the time complexity for the worst case is **O(logN)**.*

Task – Game simulation

- Implement an iterative function to simulate the game
- Repeat the game many times !!
- Count the number of guesses needed in each game
  - Histogram + Simple statiscal analysis
- Complexity order ?

One simulation scenario

```
Simulation: playing the higher-lower game 100000 times
The interval of values is [1, 100]
         1 attempts:     975 - 0.97%
         2 attempts:    2062 - 2.06%
         3 attempts:    3929 - 3.93%
         4 attempts:    7935 - 7.94%
         5 attempts:   16063 - 16.06%
         6 attempts:   31973 - 31.97%
         7 attempts:   37063 - 37.06%
MIN    - The smallest number of attempts = 1
MEDIAN - The "middle" number of attempts = 6
MEAN   - The average number of attempts  = 5.8022
MAX    - The largest number of attempts = 7
```

Best case is way far from average and worst case (which are close)

Another simulation scenario

```
Simulation: playing the higher-lower game 100000 times
The interval of values is [1, 1000]
         1 attempts:      95 - 0.10%
         2 attempts:     185 - 0.18%
         3 attempts:     379 - 0.38%
         4 attempts:     826 - 0.83%
         5 attempts:    1612 - 1.61%
         6 attempts:    3155 - 3.16%
         7 attempts:    6403 - 6.40%
         8 attempts:   12819 - 12.82%
         9 attempts:   25820 - 25.82%
        10 attempts:   48706 - 48.71%
MIN    - The smallest number of attempts = 1
MEDIAN - The "middle" number of attempts = 9
MEAN   - The average number of attempts  = 8.98709
MAX    - The largest number of attempts = 10
```

Best case is way far from average and worst case (which are close)

EXPERIMENTAL ANALYSIS - AN EXAMPLE

Example – Experimental Analysis

- Performance data for 4 sorting algorithms
- Sorting sets of n random strings

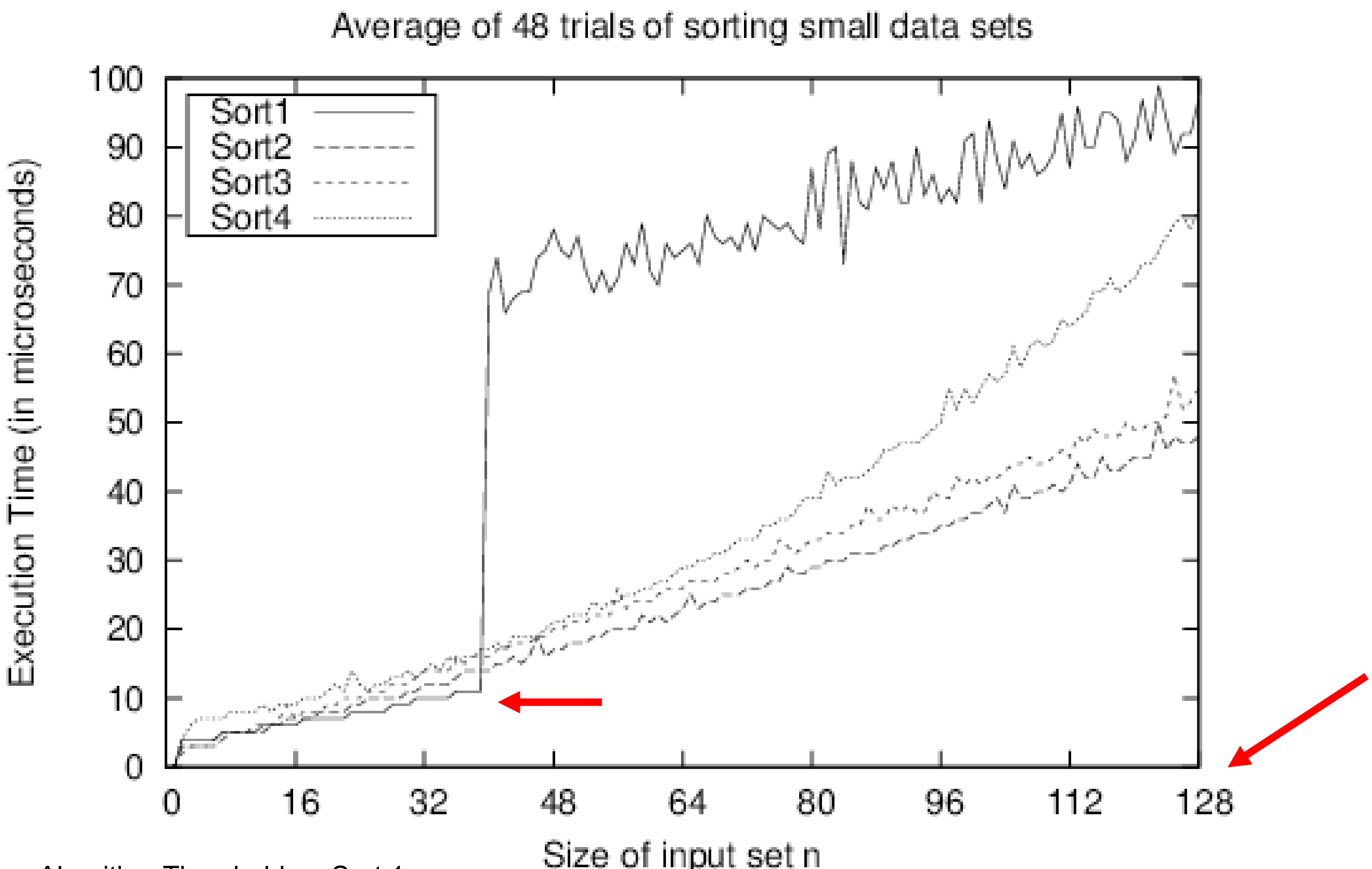
- 50 trials
- Best time and worst time were discarded

- Best algorithm ?
- Complexity order ?

- Identify the algorithms ?

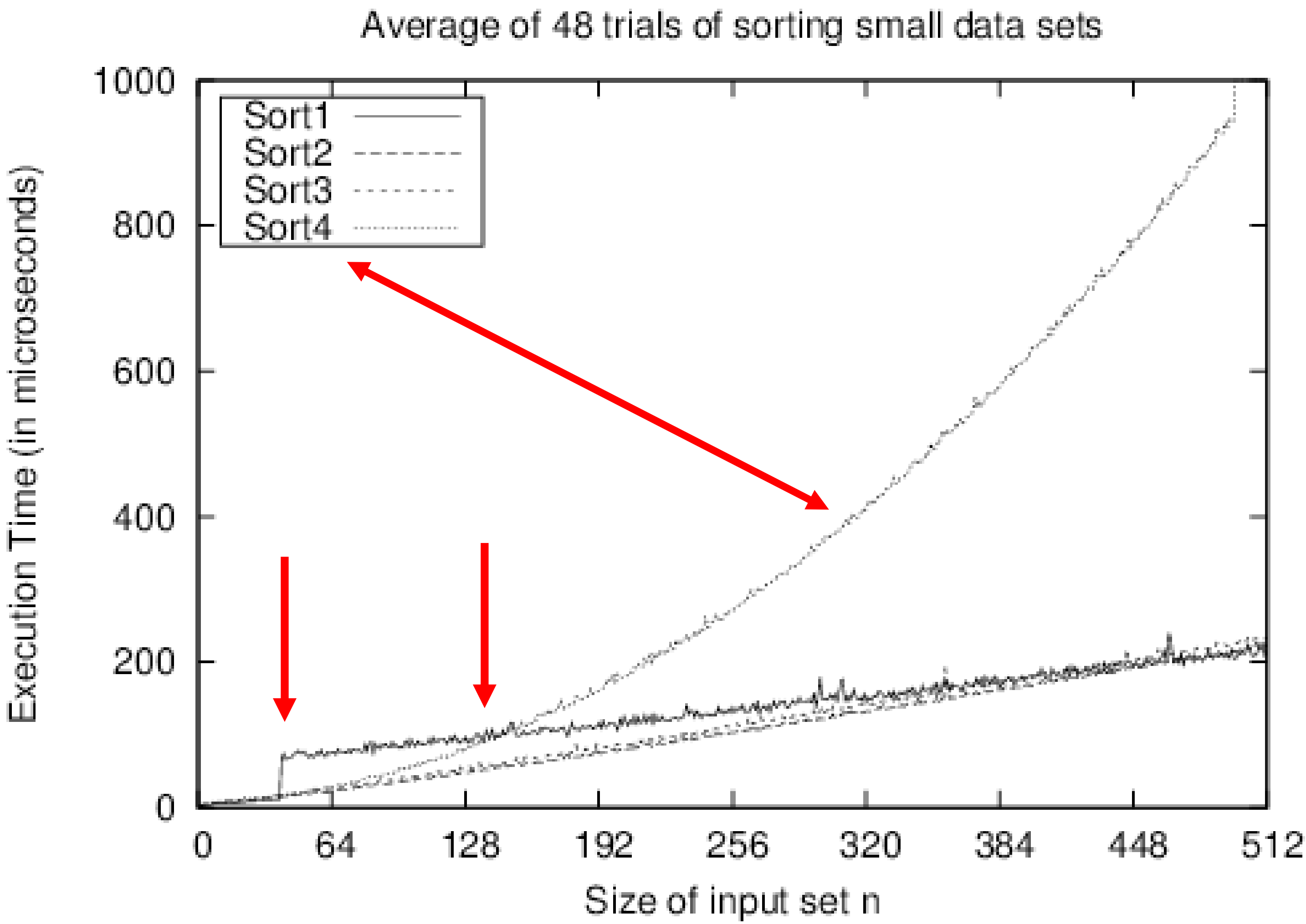
- Heineman et al. Algorithms in a Nutshell. 2nd Ed., O'Reilly, 2016

Average running time – Random data



Algorithm Threshold on Sort 1

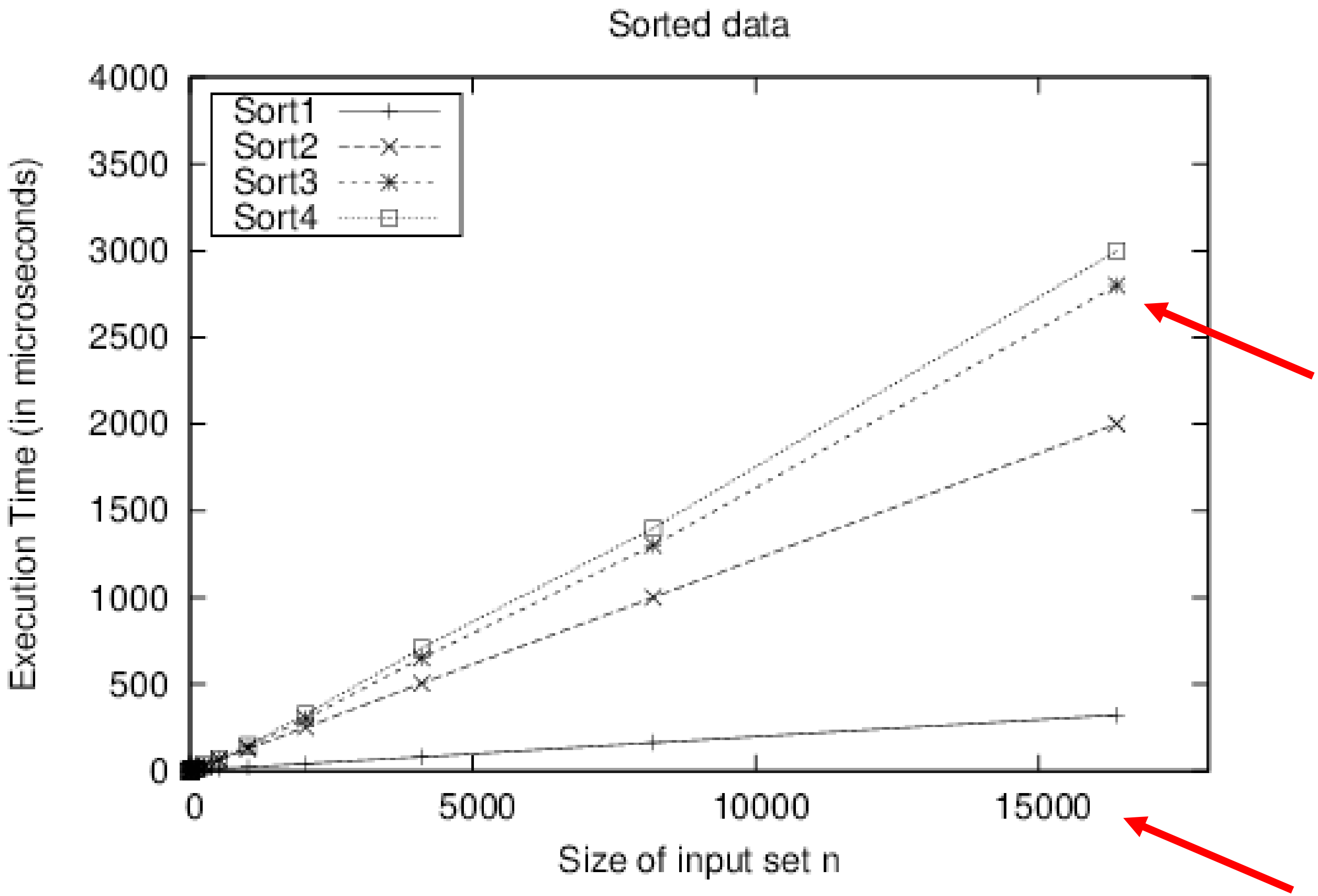
Average running time – Random data



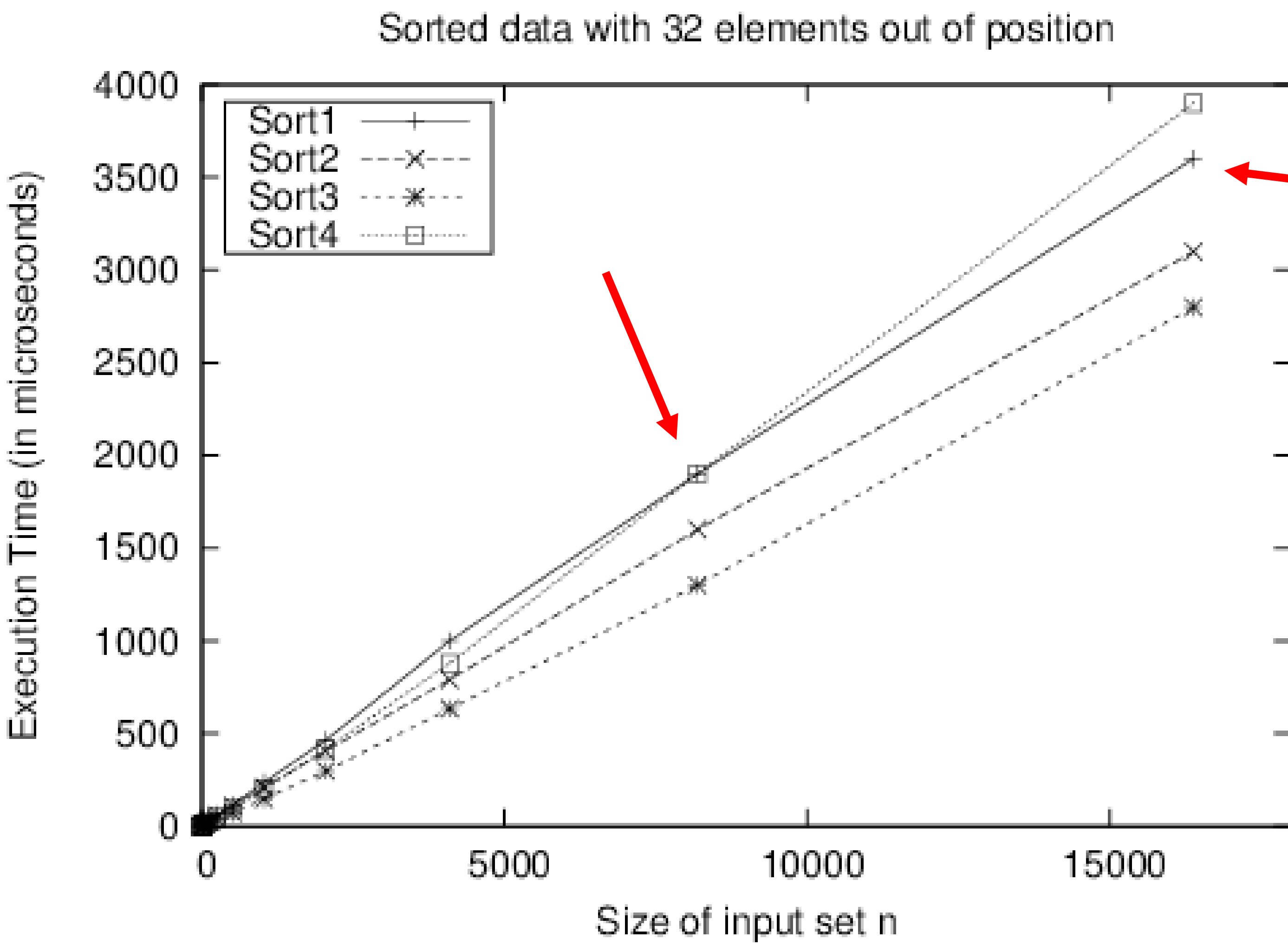
Regression analysis

- Sort-4
  - $y = 0.0053 \times n^2 - 0.3601 \times n + 39.212$
- Sort-2
  - $y = 0.05765 \times n \times log(n) + 7.9653$
- Sort-3
  - Slightly slower than Sort-2
- Sort-1
  - $n < 40$: $y = 0.0016 \times n^2 + 0.2939 \times n + 3.1838$
  - $n \geq 40$: $y = 0.0798 \times n \times log(n) + 142.7818$

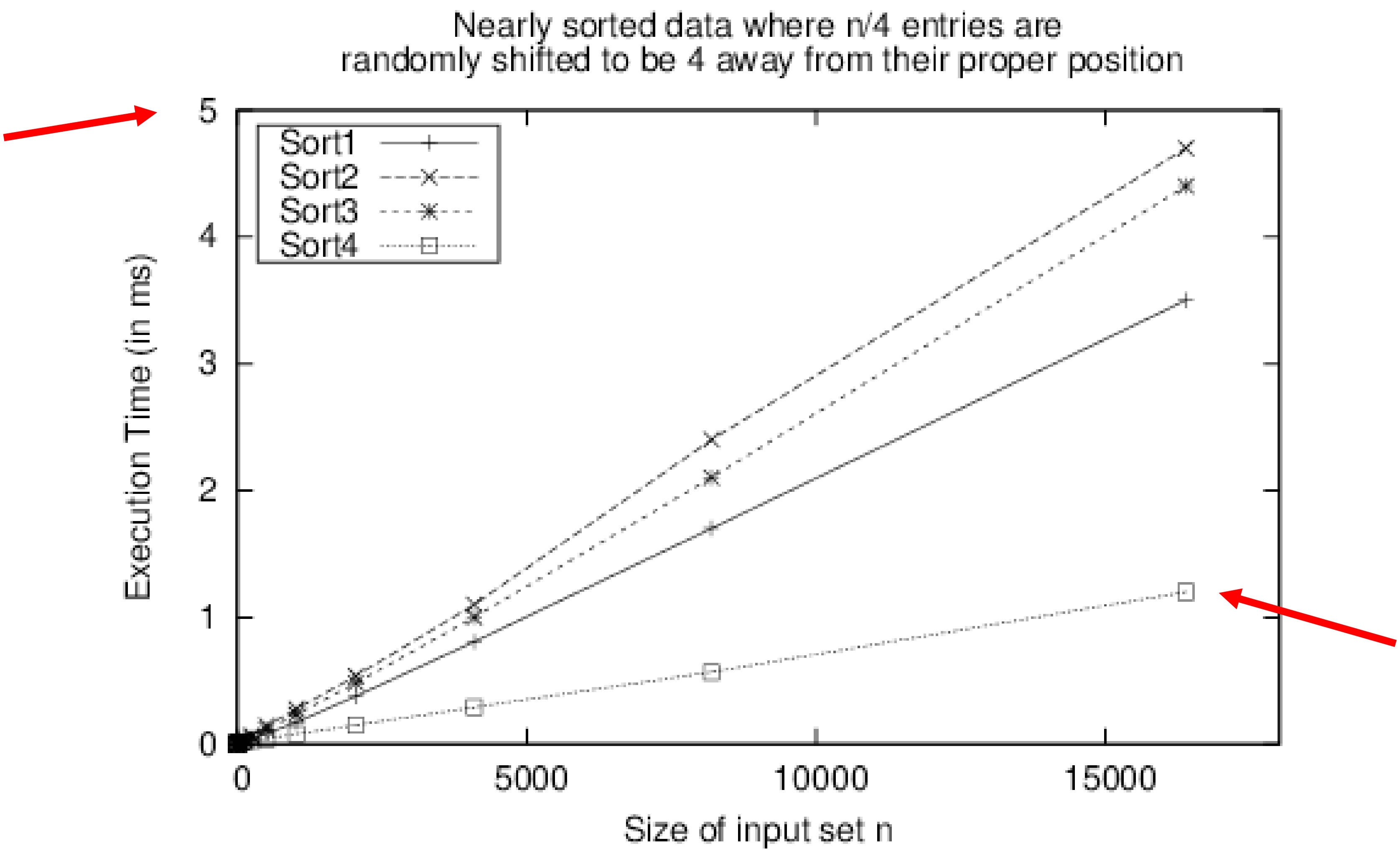
Average running time – Sorted data



Only 32 elements out of position !



Nearly-sorted



Which algorithms ?

- Sort-1 : qsort on Linux
- Sort-2 : ?
- Sort-3 : ?
- Sort-4 : ?

DYNAMIC PROGRAMMING

Dynamic Programming

- General algorithm design technique

- Apply to
  - Computing recurrences
  - Solving optimization problems

- How to store “previous” results ?
  - 2D array
  - Vector
  - A few variables

Recurrences – Top-Down

- Exploit the relationship between
  - A solution to a given problem instance
  - Solutions to smaller/simpler instances of the same problem
- Set up a recurrence !
- Decompose into smaller / simpler sub-problems
  - Parameters ?
- Identify the smallest / simplest / trivial problems
  - Base cases

Dynamic Programming – Bottom-up

- Use a recurrence: BUT go bottom-up !
- Start from the smallest / simplest / trivial problems
- Get intermediate solutions from smaller / simpler sub-problems
- Which values / results are computed in each step ?
  - How to store ?

Dynamic Programming – Advantage

- Do sub-problems overlap ?
- NOW, there is no need to repeatedly solve the same sub-problems !!
- Proceed bottom-up and store results for later use
- Compare with Divide-and-Conquer !!

COMPUTING FIBONACCI NUMBERS

Fibonacci's Sequence

- F(0) = 0 ; F(1) = 1
- F(i) = F(i – 1) + F(i – 2) ; i = 2, 3, 4,…
- F(6) = ? ➔ Number of recursive calls ?
- Do sub-problems overlap ?
- Recursion tree vs. recursion DAG !!
  (Directed Acyclic Graph)
- Complexity order ?

Tasks – V1

- Implement the recursive function of the previous slide in Python
- Count the number of additions carried out for computing a Fibonacci number
  - Use a global variable
- Table ?
- Complexity order ?

Fibonacci's Sequence

```
def fibonacci_DC( n ) :

    """ Recursive computation of Fi """

    # Global variable, for counting the number of additions

    global num_adds

    if ( n == 0 ) or ( n == 1 ) :

        return n

    num_adds += 1

    return fibonacci_DC( n - 1 ) + fibonacci_DC( n - 2 )
```

Number of additions ?

- $A(0) = 0$; $A(1) = 0$
- $A(i) = 1 + A(i - 1) + A(i - 2)$; $i = 2, 3, 4, \ldots$

- Closed formula ?

- You can get it, if you remember Discrete Mathematics…

- BUT, we can get the complexity order from the table…

Additions – Recursive version

- How fast does F(n) grow ?
  (fibonacci value)
- How fast does A(n) grow ?
  (number of additions)
- From the table we get:

  A(n) = F(n+1) – 1

- Exponential growth !!
  - Why ?

$$(1+\sqrt{5})/2 = 1{,}618034$$

n	F(n)	Ratio	A(n)	Ratio
0	0		0	
1	1		0	
2	1	1	1	
3	2	2	2	2
4	3	1,5	4	2
5	5	1,666667	7	1,75
6	8	1,6	12	1,714286
7	13	1,625	20	1,666667
8	21	1,615385	33	1,65
9	34	1,619048	54	1,636364
10	55	1,617647	88	1,62963
11	89	1,618182	143	1,625
12	144	1,617978	232	1,622378
13	233	1,618056	376	1,62069
14	377	1,618026	609	1,619681
15	610	1,618037	986	1,619048
16	987	1,618033	1596	1,618661
17	1597	1,618034	2583	1,618421
18	2584	1,618034	4180	1,618273
19	4181	1,618034	6764	1,618182
20	6765	1,618034	10945	1,618125

Fibonacci's Sequence

- $F(0) = 0$; $F(1) = 1$
- $F(i) = F(i - 1) + F(i - 2)$; $i = 2, 3, 4, \ldots$

- Use Dynamic Programming !!

- Computing F(n) using an array
  - Complexity order ?

- Can we use less memory space ?

(version with 3 variables)

```python
def fib(n):
    a = 0
    b = 1
    if (n >= 0):
        print(a, end=' ')
    if (n >= 1):
        print(b, end=' ')
    for i in range(2, n+1):
        c = a + b
        print(c, end=' ')
        a = b
        b = c
```

Tasks – V2 + V3

- Implement two iterative functions for computing F(i)
  - V2 : using an array
  - V3 : using just 3 variables

- Count the number of additions carried out

- Table ?
- Complexity order ?

Fibonacci's Sequence

i	f(i)	#ADDs-Rec	#ADDs_DP_1	#ADDs_DP_2
0	0	0	0	0
1	1	0	0	0
2	1	1	1	1
3	2	2	2	2
4	3	4	3	3
5	5	7	4	4
6	8	12	5	5
7	13	20	6	6
8	21	33	7	7
9	34	54	8	8
10	55	88	9	9
11	89	143	10	10
12	144	232	11	11
13	233	376	12	12
14	377	609	13	13
15	610	986	14	14

(Just differ in space complexity which is O(n) for the first and constant for the last)

EXAMPLE
- A LINEAR ROBOT

Another example

- Linear robot
- Can move forward by 1 meter, or 2 meters, or 3 meters

- In how many ways can it move a distance of n meters ?

- Establish the recurrence !!
  - Base cases ?

Tasks – V1 + V2 + V3

- Implement three functions for computing R(i)
  - V1 : using recursion
  - V2 : using an array
  - V3 : using a few variables – how many ?
- Count the number of additions carried out
  - Formulas ?
- Tables ?
- Complexity order ?

```python
def linear_robot_recursive(n):
  global counter
  if n == 1:
    return 1
  elif n == 2:
    return 2
  elif n == 3:
    return 4
  else:
    counter += 2
    return linear_robot_recursive(n-1) + linear_robot_recursive(n-2) + linear_robot_recursive(n-3)
```

Example – Results table

i	r(i)	#ADDs-Rec	#ADDs_DP_1	#ADDs_DP_2
1	1	0	0	0
2	2	0	0	0
3	4	0	0	0
4	7	2	2	2
5	13	4	4	4
6	24	8	6	6
7	44	16	8	8
8	81	30	10	10
9	149	56	12	12
10	274	104	14	14
11	504	192	16	16
12	927	354	18	18
13	1705	652	20	20
14	3136	1200	22	22
15	5768	2208	24	24

Base Cases

COMPUTING BINOMIAL COEFFICIENTS

Computing Binomial Coefficients

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

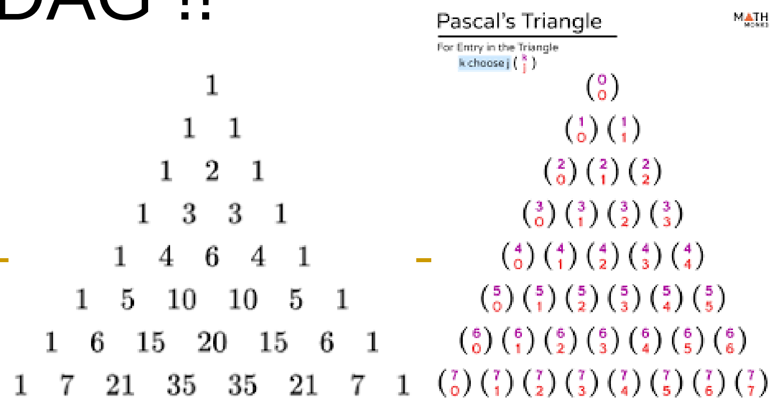
- C(n,0) = 1 ; C(n,n) = 1
- C(n,j) = C(n–1,j) + C(n–1,j–1) ; j = 1, 2,…, n – 1

- Two arguments !!
- C(4,3) = ? ➔ Number of recursive calls ?

- Do sub-problems overlap ? Yes
- Recursion tree vs. recursion DAG !!

https://visualgo.net/en/recursion

- Complexity order ?



The sum of each row of Pascal's triangle is a power of 2 starting at 2^0

Computing Binomial Coefficients

- V1 : Compute C(n,j) recursively
- V2 : Compute C(n,j) using a 2D array
  - How to proceed ?
  - Have you seen this “triangle” before ?
- Can we use less memory space ?
- And other, more efficient recurrences ?

Can also solve with 1D array with size k + 1
The final value of dp(n,k) is stored in dp[k], and returned.

Tasks – V1 + V2 + V3

- Implement three functions for computing C(n,j)
  - V1 : using recursion
  - V2 : using a 2D array
  - V3 : using a 1D array

- Count the number of additions carried out

- Tables ?
- Complexity order ?

Pascal's Triangle

```
Pascal's Triangle - Recursive Function

1
1       1
1       2       1
1       3       3       1
1       4       6       4       1
1       5       10      10      5       1
1       6       15      20      15      6       1
1       7       21      35      35      21      7       1
1       8       28      56      70      56      28      8       1
1       9       36      84      126     126     84      36      9       1
1       10      45      120     210     252     210     120     45      10      1
```

V1 – Number of additions

```
Number of Additions - Recursive Function

0
0       0
0       1       0
0       2       2       0
0       3       5       3       0
0       4       9       9       4       0
0       5       14      19      14      5       0
0       6       20      34      34      20      6       0
0       7       27      55      69      55      27      7       0
0       8       35      83      125     125     83      35      8       0
0       9       44      119     209     251     209     119     44      9       0
```

V2 – Number of additions

```
Number of Additions - Dynamic Programming - V. 1

0
0       0
1       1       1
3       3       3       3
6       6       6       6       6
10      10      10      10      10      10
15      15      15      15      15      15      15
21      21      21      21      21      21      21      21
28      28      28      28      28      28      28      28      28
36      36      36      36      36      36      36      36      36      36
45      45      45      45      45      45      45      45      45      45      45
```

V3 – Number of additions

```
Number of Additions - Dynamic Programming - V. 2

0
0        0
0        1        0
0        3        3        0
0        6        6        6        0
0        10       10       10       10       0
0        15       15       15       15       15       0
0        21       21       21       21       21       21       0
0        28       28       28       28       28       28       28       0
0        36       36       36       36       36       36       36       36       0
0        45       45       45       45       45       45       45       45       45       0
```

MEMOIZATION

Memoization

!= Dynammic Programming (Decomposition into overlapping (smaller !) sub-problems)

- Turning the results of a function into something to be remembered

- I.e., avoid repeating the calculation of results for previously processed inputs

- Use a table / array / cache to store previously computed results
  - Initialization !

Memoization is a term describing an optimization technique where you cache previously computed results, and return the cached result when the same computation is needed again.

Dynamic programming is a technique for solving problems of recursive nature, iteratively and is applicable when the computations of the subproblems overlap.

Dynamic programming is *typically* implemented using tabulation, but can also be implemented using memoization. So as you can see, neither one is a "subset" of the other.

- Time vs. space trade-off

Memoization

- Initialize all table entries to “null”
  - Not yet computed

- Whenever a result is to be computed for a given input
  - Check the corresponding table entry
  - If not “null”, retrieve the result
  - Otherwise, compute by a recursive call(s)
  - And store the result

Fibonacci's Sequence

- Initialization

```
for(i=1, i<n, i++)   f[i] = -1;
```

- Recursive function

```
int fib( int n ) {
        int r;
        if( f[n] != -1 )   return f[n];
        if( n == 1 )   r = 1;
        else if( n == 2 )   r = 1;
        else {
                r = fib( n – 2 );
                r = r + fib( n – 1 );
        }
        f[n] = r;
        return r;
}
```

The "old" Python way

```
# M. Hetland, Python Algorithms, Apress, 2010 - Chapter 8

from functools import wraps

def memo( func ) :

    cache = {}                              # Stored subproblem solutions

    @wraps(func)                            # Make wrap look like func

    def wrap( *args ) :                     # The memoized wrapper

        if args not in cache :              # Not already computed?

            cache[args] = func( *args )     # Compute & cache the solution

        return cache[args]                  # Return the cached solution

    return wrap                             # Return the wrapper
```

The "old" Python way

```python
# Testing the memoized version

fibonacci_DC = memo( fibonacci_DC )
```

i	f(i)	#ADDs_Memo
0	0	0
1	1	0
2	1	1
3	2	1
4	3	1
5	5	1
6	8	1
7	13	1
8	21	1
9	34	1
10	55	1
85	259695496911122585	1
86	420196140727489673	1
87	679891637638612258	1
88	1100087778366101931	1
89	1779979416004714189	1
90	2880067194370816120	1

The “new” Python way

- Use the functools module and the cache decorator

```
@cache
def factorial(n):
    return n * factorial(n-1) if n else 1

>>> factorial(10)      # no previously cached result, makes 11 recursive calls
3628800
>>> factorial(5)       # just looks up cached value result
120
>>> factorial(12)      # makes two new recursive calls, the other 10 are cached
479001600
```

TIMING & PROFILING CODE

timeit – Mesuring execution time

- Measure execution time of small code snippets / functions

```
$ python -m timeit "'-'.join(str(n) for n in range(100))"
10000 loops, best of 5: 30.2 usec per loop
$ python -m timeit "'-'.join([str(n) for n in range(100)])"
10000 loops, best of 5: 27.5 usec per loop
$ python -m timeit "'-'.join(map(str, range(100)))"
10000 loops, best of 5: 23.2 usec per loop
```

```
>>> import timeit
>>> timeit.timeit('"-".join(str(n) for n in range(100))', number=10000)
0.3018611848820001
>>> timeit.timeit('"-".join([str(n) for n in range(100)])', number=10000)
0.2727368790656328
>>> timeit.timeit('"-".join(map(str, range(100)))', number=10000)
0.23702679807320237
```

timeit – Mesuring execution time

- Pass a setup parameter to give access to functions you define
  - It contains an import statement

```python
def test():
    """Stupid test function"""
    L = [i for i in range(100)]

if __name__ == '__main__':
    import timeit
    print(timeit.timeit("test()", setup="from __main__ import test"))
```

The Python Profilers

- cProfile and profile provide deterministic profiling of Python programs

- A profile is a set of statistics that describes how often and for how long various parts of the program executed

- These statistics can be formatted into reports via the pstats module.

line_profiler – Line-by-line profiling

- Install line_profiler: pip install line_profiler
- In the relevant file(s), import line profiler and decorate function(s) you want to profile with @line_profiler.profile
- Set the environment variable LINE_PROFILE=1 and run your script as normal.
- When the script ends a summary of profile results, files written to disk, and instructions for inspecting details will be written to stdout

REFERENCES

References

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  - Chapter 8
- R. Johnsonbaugh and M. Schaefer, *Algorithms,* Pearson Prentice Hall, 2004
  - Chapter 8
- T. H. Cormen et al., *Introduction to Algorithms*, 3rd Ed., MIT Press, 2009
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