Introduction to Randomized Algorithms II

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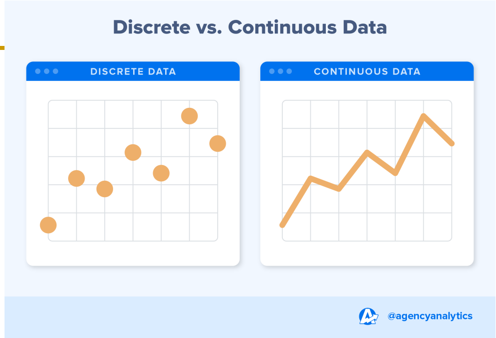
Version 0.4 – October 2024

Overview

- Discrete Probability
- Statistical Experiments and Events
- Probabilities and Random Variables
- Application Examples and Problems

DISCRETE PROBABILITY

Discrete Probability



- Chance enters into many attempts to understand the world we live in

- A theory of probability allows us to calculate the likelihood of complex events

- Probabilities are called “discrete” if we can compute the probabilities of all events by summation

Probability Space

Omega Ω

- Probability theory starts with the idea of a probability space (Ω,Pr)
  - A set Ω of all things that can happen
  - A rule assigning a probability $Pr(\omega)$ to each elementary event ω in Ω

Probability Distribution

- For a discrete probability space
  - $Pr(\omega) \geq 0$
  - $\sum Pr(\omega) = 1$
- Pr is the probability distribution
  - It distributes the total probability among the elementary events

Example – Fair dice

- Roll one fair 6-sided die

  D = { ⚀, ⚁, ⚂, ⚃, ⚄, ⚅ }

  - Each of the 6 possibilities has probability 1/6

- Roll a pair of fair dice
  - Set of elementary events : D^2 = ?
  - Probability of each event ?

Example – “Loaded” dice

- Distribution of probabilities

$$\Pr_1(\text{⚀}) = \Pr_1(\text{⚅}) = \frac{1}{4};$$
$$\Pr_1(\text{⚁}) = \Pr_1(\text{⚂}) = \Pr_1(\text{⚃}) = \Pr_1(\text{⚄}) = \frac{1}{8}$$

- Probability of each event in D^2 ?

$$\Pr_{11}(d\,d') = \Pr_1(d)\,\Pr_1(d')$$

Example – Fair die + "Loaded" die

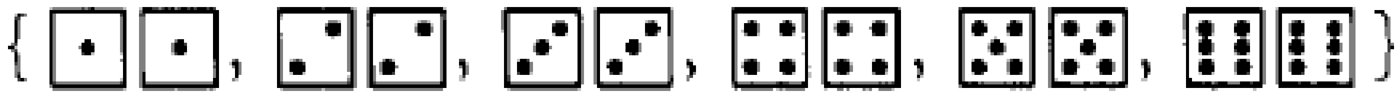
- Consider the case of one fair die and one loaded die

$$\Pr_{01}(d\,d') = \mathrm{Pro}(d)\ \Pr_1(d'), \qquad \text{where } \mathrm{Pro}(d) = \frac{1}{6}$$

- Real-world dice do not turn up equally often on each side !!
  - No perfect symmetry !!
- BUT, 1/6 is usually close to the truth…

Example – Doubles are thrown

- The event that "doubles are thrown"



- Probability of an event A

$$\Pr(\omega \in A) = \sum_{\omega \in A} \Pr(\omega)$$

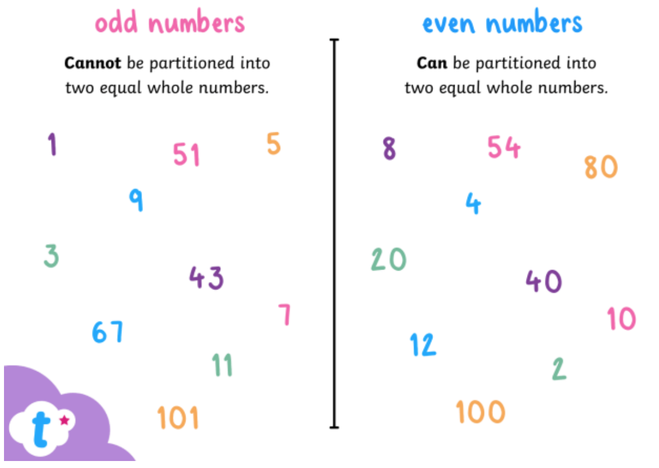
- Pr("doubles are thrown") = ?
  - When is this event more probable ?

Statistical experiments and events

- Statistical experiment
  - Repeatable experiment where the particular outcome of a trial cannot be predicted with certainty
- Sample space, *S*
  - Set with the representation of all possible outcomes of an experiment
  - I.e., set of elementary events
- Examples ?

Statistical experiments and events

- Event, *E*
  - A set of elementary events
  - Any subset of a sample space



Sample Space {1,2,3,4,5,6}
1 -> Event
2 -> Event
...

- Examples
  - {2, 4, 6} – Getting an even number when throwing a 6-sided die
    - Probability ?
  - …

Probabilities

- The probability of an event, *E*, describes the degree of uncertainty of that event

- $0 \leq P[E] \leq 1$
- $P[S] = 1$
- $P[\emptyset] = 0$

- $P[A \cup B] = P[A] + P[B]$, if *A* and *B* are disjoint

- If $(A \cap B) \neq \emptyset$, $P[A \cup B] = P[A] + P[B] - P[A \cap B]$

PROBABILITY DISTRIBUTIONS

Uniform probability distribution on S

- Each elementary event in S has the same probability

$$P[\{A\}] = P[\{B\}] = \frac{1}{|S|}$$

Simple problem

- Throwing a 6-sided fair die
- What is the probability of getting an even number ?
- What is the probability of getting a number larger than 2 ?
- What is the probability of getting an even number <u>or</u> a number larger than 2 ?

Another simple problem

- Tossing three fair coins
- What is the sample space S_3 ? If order matters?? 2^3 events
  {
  HHH,HTT, HTH, HHT
  TTT, THH, THT, TTH
  }
- How high is the probability of getting at least one head ? 1 - P(getting no heads)
- And at least two heads ? 1- P(0heads + 1head) OR
  P(2heads) + P(3heads) + P(4heads) + P(5heads) + P(6heads)
- Idea: relate to the binary representation
- Idea: triangular representation – paths

More difficult problem

Summary Formulas

1. Exact k heads:

$$P(X = k) = \binom{n}{k} \left(\frac{1}{2}\right)^n$$

2. At least k heads:

$$P(X \geq k) = \sum_{i=k}^{n} \binom{n}{i} \left(\frac{1}{2}\right)^n$$

- Tossing *n* fair coins
- How large is the probability to get "head" exactly k times ?
- How large is the probability to get "head" at least k times ?
- You can use your code to check your answers…

Binomial probability distribution

- Characterizes the probability of obtaining k "successes" in n experiments

$$S = \{0,1,2, \dots , n\}$$

$$P[X = k] = \binom{n}{k} p^k (1-p)^{n-k}, k = 0,1,2, \dots , n$$

- Check your previous answers !!

Tasks

- What is the probability of getting 6 heads in 15 tosses of a fair coin ?
  - Estimate the value with simulated experiments
  - Check that you got the correct value by computing the probability from the binomial distribution

- Now, consider that P[heads] = 2 × P[tails]

p + 2p = 1

In the formula p=P(heads)=2/3

p(tails) = 1/3

Independent events

- Two events *A* and *B* are said to be independent, if the occurrence of one does not affect the occurrence of the other

- Two events *A* and *B* are independent, if

$$P[A \text{ and } B] = P[A] \times P[B]$$

Conditional probability

- Conditional probability of event A given event B

$$P[A|B] = \frac{P[A\ \textit{and}\ B]}{P[B]}, P[B] \neq 0$$

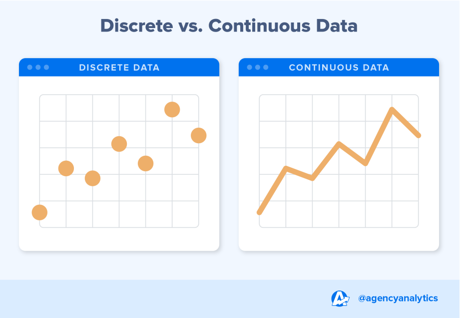
  - What happens if they are independent ? Same as P(A)

- Bayes' Theorem (Derived from P[B|A] formula)

$$P[A|B] = \frac{P[B|A]\,P[A]}{P[B]}$$

RANDOM VARIABLES

Random variable



- A random variable is a function that assigns a real number to each elementary event of the sample space
- Discrete r. v. – countable set of real values
  - Values obtained by throwing a dice numbered 1 to 6
  - Assigning 0 to tail and 1 to heads when tossing a coin
- Continuous r. v. – interval or collection of intervals
  - Examples: temperature of a room or weight of product

Example – Throwing two dice

- S(w) = sum of spots on the dice roll w
- What is the probability that the spots total 7 ?

Pr(⚀⚅) + Pr(⚁⚄) + Pr(⚂⚃)

+ Pr(⚃⚂) + Pr(⚄⚁) + Pr(⚅⚀)

- Fair dice ?
- Loaded dice ?

Example – Throwing two dice

- A random variable is characterized by the probability distribution of its values

	s Sum	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**
Fair Dice	$\mathbf{P_{r00}[S=s]}$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$
Loaded Dice	$\mathbf{P_{r11}[S=s]}$	$\frac{4}{64}$	$\frac{4}{64}$	$\frac{5}{64}$	$\frac{6}{64}$	$\frac{7}{64}$	$\frac{12}{64}$	$\frac{7}{64}$	$\frac{6}{64}$	$\frac{5}{64}$	$\frac{4}{64}$	$\frac{4}{64}$

Pr_1(⚀) = Pr_1(⚅) = $\frac{1}{4}$;

Pr_1(⚁) = Pr_1(⚂) = Pr_1(⚃) = Pr_1(⚄) = $\frac{1}{8}$

Sequence of numbers – Average value

- Mean
  - Sum of all values, divided by the number of values
- Median 6 in 1,5,6,9,10
  - Middle value, numerically
- Mode Moda
  - Value that occurs most often

statistics – Python module

- Computing mathematical statistics of numeric data
- Averages and measures of central location
  - mean(…)
  - median(…)
  - mode(…)
  - …
- Measures of spread
  - stdev(…)
  - variance(…)
  - …

Discrete random variable – Features

- Mean – Expected value

$$\mu = E[X] = \sum_{n} X_n P[X = X_n]$$

- Variance
  - Measures how far a set of numbers are spread out

$$\sigma^2 = E[(X - \mu)^2] = E[X^2] - \mu^2$$

- σ is the standard deviation

Sum of independent random vars.

- Let $Z = X + Y$ be the sum of two independent random variables, defined on the same probability space

$$E[Z] = E[X + Y] = E[X] + E[Y]$$

$$\sigma_Z^2 = E[(X + Y - \mu)^2] = \sigma_X^2 + \sigma_Y^2$$

Sum of independent random vars.

- Let S_n be the sum of n independent and identically distributed random variables

$$S_n = X_1 + X_2 + \ \dots + X_n$$

$$E[S_n] = n \ \times E[X]$$

$$\sigma_{S_n}^2 = n \ \times \sigma_X^2$$

Estimating the mean of a rand. var.

- Set of independent empirical observations
- Sample mean is

$$\hat{\mu} = \frac{1}{n}\sum X_i$$

- Keep a record of the sum as the experiment progresses
- Update the sample mean, when needed

Estimating the mean of a rand. var.

- Sample variance is

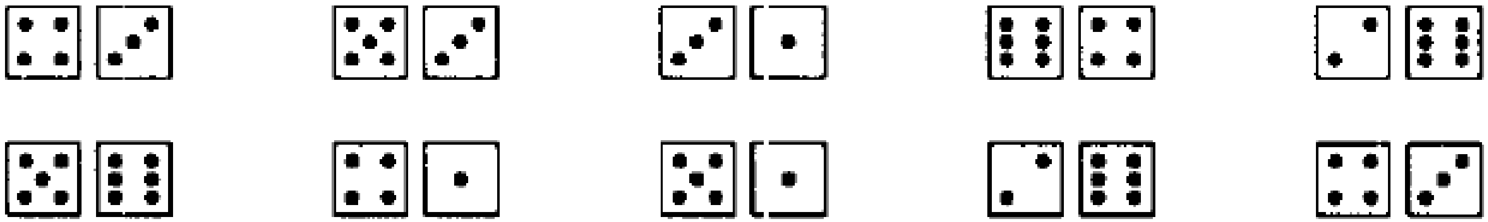
$$\hat{\sigma}^2 = \frac{1}{n-1}\sum X_i^2 - \frac{1}{n(n-1)}\left(\sum X_i\right)^2$$

- Keep a record of the sums as the experiment progresses
- Update the sample variance, when needed

- Estimate the mean as

$$\hat{\mu} \pm \hat{\sigma}/\sqrt{n}$$

Example – 10 rolls of two dice



- Sample mean of the spot sum

$$\hat{\mu} = 7.4$$ soma de todos a dividir pelo número de elementos

- Sample variance

$$\hat{\sigma}^2 \approx 2.1^2$$ $$\frac{1}{10-1}\sum X_i^2 - \frac{1}{10(10-1)}\left(\sum X_i^2\right), Xi = valor\_da\_soma \quad (6)$$

- Estimate

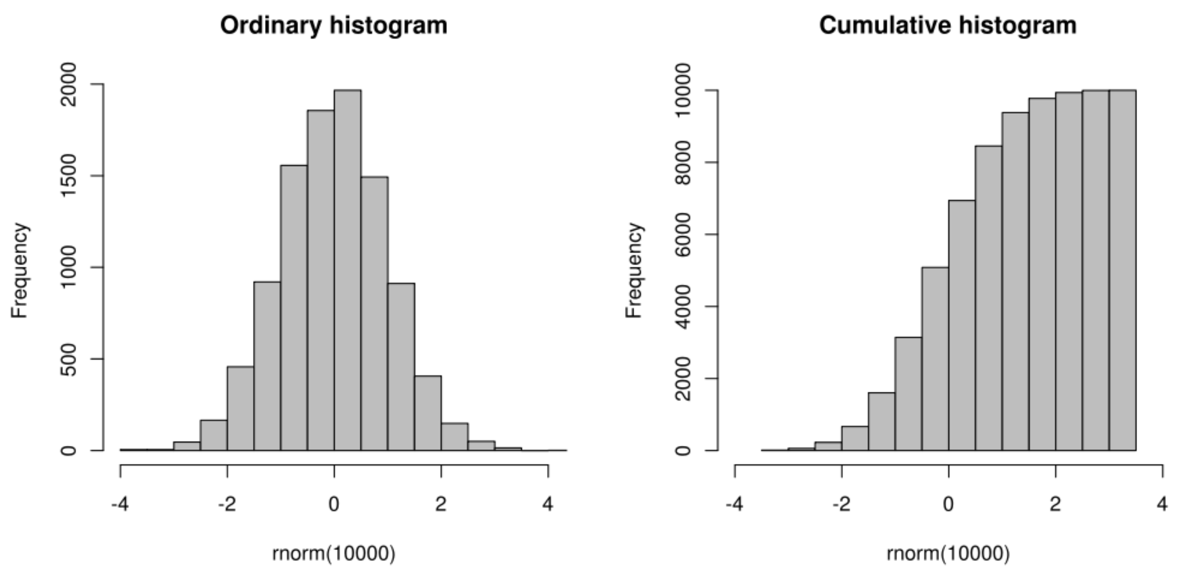
$$7.4 \pm 0.7$$

Mean Estimate

$$\hat{\mu} \pm \hat{\sigma}/\sqrt{n}$$

Histogram

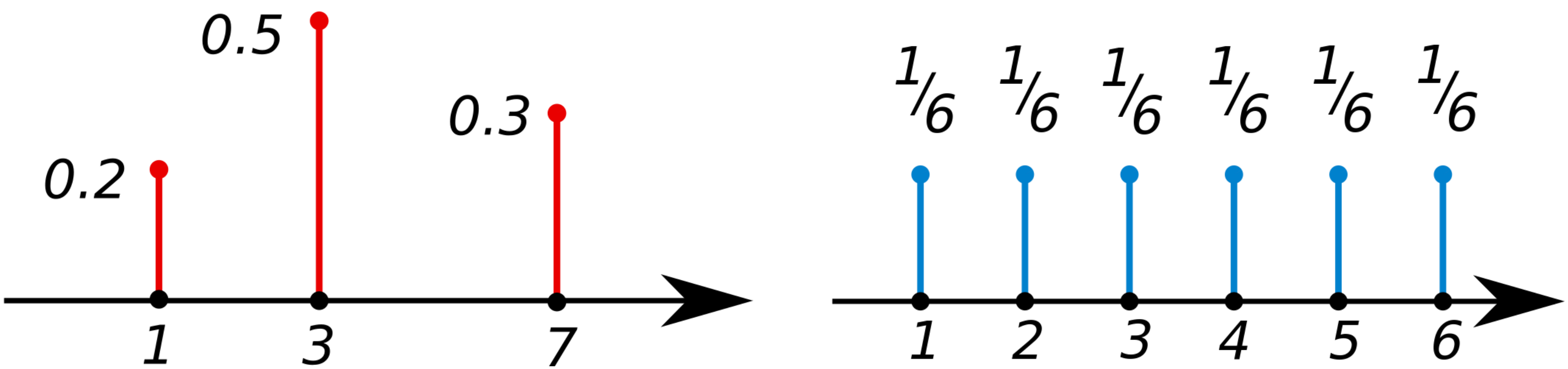
- Graphical representation of the distribution of numerical data
- May be normalized to display relative “frequencies”
- A cumulative histogram represents the cumulative number of observations



[Wikipedia]

Probability mass function

- Describes the relative likelihood of a discrete random variable to take on a given value



[Wikipedia]

APPLICATION PROBLEMS

Task – Problem 1

- Consider the statistical experiment in which a fair coin is tossed repeatedly until one of the faces appears for the second time
- An outcome of the experiment is a list of the faces that appear
  - Possible outcomes: (H, T, H) ; (T, T) ; (H, H) ; (T, H, T)
- Let *Y* denote the random variable that tells how many tosses were necessary to produce a repetition of a face
  - For the examples above: 3, 2, 2, 3
- Simulate an observation of Y

Task – Problem 2

- Consider the statistical experiment in which a fair die is thrown repeatedly until one of the faces appears for the second time
- An outcome of the experiment is a list of the faces that are thrown
  - Possible outcomes: (3, 6, 2, 6) ; (5, 5) ; (1, 4, 3, 6, 2, 4)
- Let Y denote the random variable that tells how many throws were necessary to produce a repetition of a face
  - For the examples above: 4, 2, 6
- Simulate an observation of Y

Task – Problem 3

The Birthday Paradox

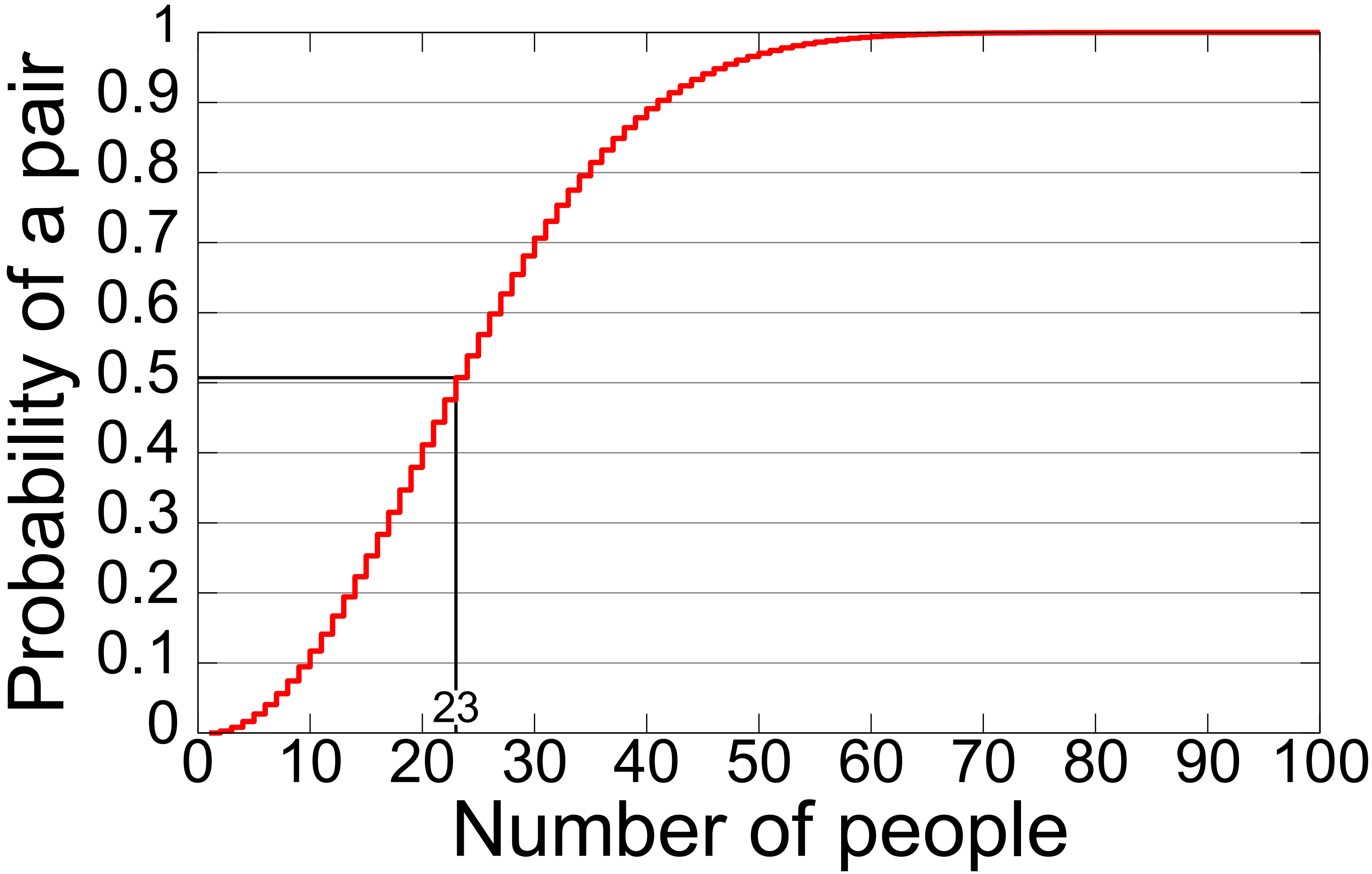
- In a party with *n* people, what is the probability of at least two of them celebrating their birthday in the same day ?

- What is the smallest *n* that guaranties that the previous probability is above 50% ? 23 people - In a room of just 23 people there's a 50-50 chance of at leaast two people having the same birthday

- Estimate the values with simulated experiments
- Consider that each birthday is equally likely

- Read about "The Birthday Paradox" !

The Birthday Paradox



[Wikipedia]

Task – Problem 4

Photo for n=2 (check your phone)

- Consider $n = 4000$
- Generate random numbers in the domain $[n]$ until two have the same value
- How many random trials did that take ?
  - Use k to represent this value
- Repeat the experiment $m = 300$ times, and record for each how many random trials that took

Task – Problem 4

- Plot that data as a *cumulative density plot*
  - *T*he x-axis records the number k of trials required, and the y-axis records the fraction of experiments that succeeded (a collision) after k trials

- Empirically estimate the expected number of k random trials in order to have a collision
  - That is, add up all values k, and divide by m

- How long did it take ?
- Carry out some tests for much larger n and m values !!

Task – Problem 5

- Consider the statistical experiment in which a fair coin is tossed repeatedly until each face has appeared at least once
- An outcome of the experiment is a list of the faces that appear
  - Possible outcomes: (H, T) ; (T, T, H) ; (H, H, H, T)
- Let *Y* denote the random variable that tells how many tosses were necessary
  - For the examples above: 2, 3, 4
- Simulate an observation of Y

Task – Problem 6

- Consider the statistical experiment in which a fair die is thrown repeatedly until each face has appeared at least once
- An outcome of the experiment is a list of the faces that appear
  - Possible outcomes: (1, 2, 4, 5, 3, 6) ; (1, 2, 1, 4, 3, 5, 3, 6)
- Let *Y* denote the random variable that tells how many throws were necessary
  - For the examples above: 6, 8
- Simulate an observation of Y

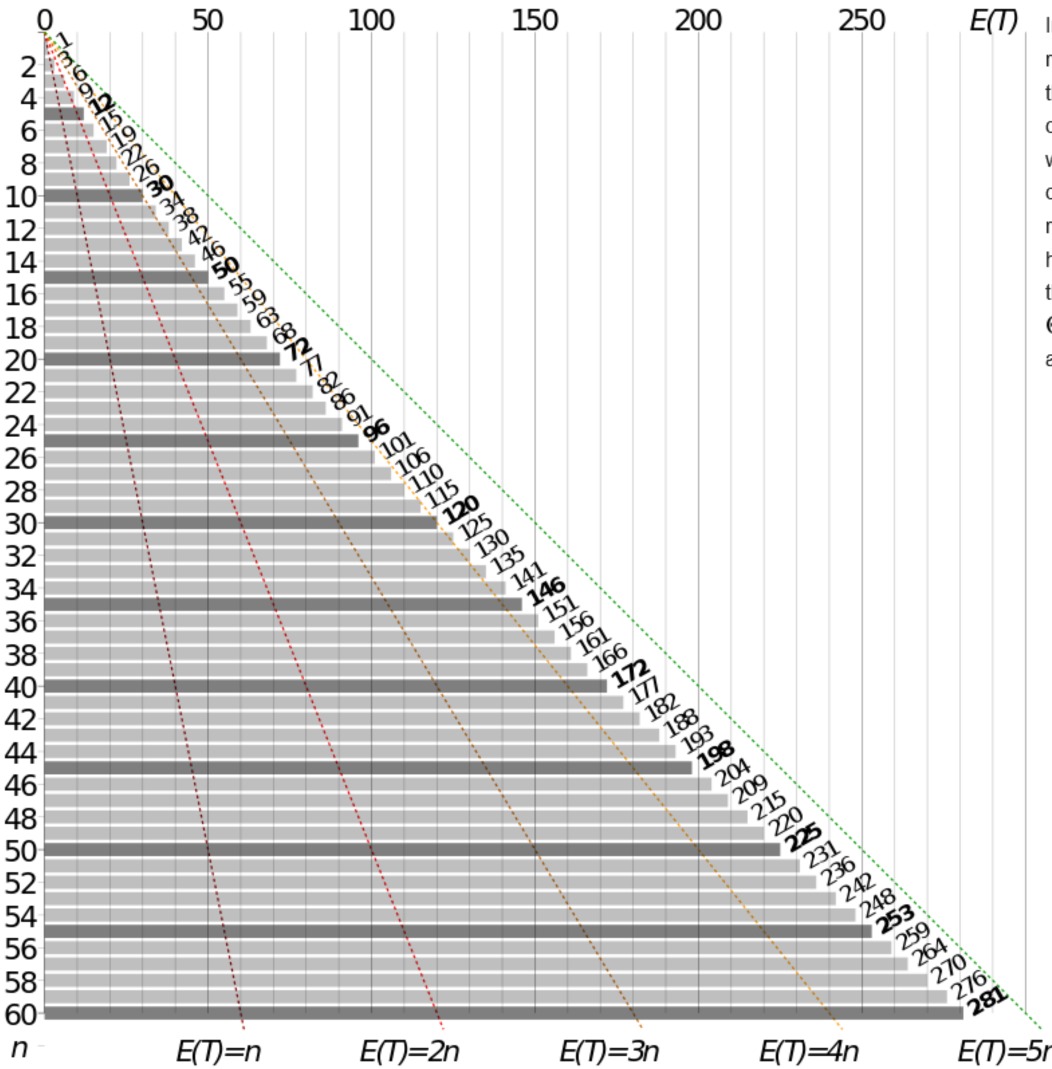
Task – Problem 7

- Consider $n = 200$
- Generate random numbers in the domain $[n]$ until every value i in $[n]$ has had one random number equal to i
- How many random trials did that take ?
  - Use k to represent this value
- Repeat the experiment $m = 300$ times, and record for each how many random trials were required to collect all values

Task – Problem 7

- Plot that data as a *cumulative density plot*
- Empirically estimate the expected number of *k* random trials in order to collect all values
- How long did it take ?
- Carry out some tests for much larger n and m values !!
- Read about "The Coupon Collectors Problem" !!

The Coupon Collectors Problem



In probability theory, the **coupon collector's problem** refers to mathematical analysis of "collect all coupons and win" contests. It asks the following question: if each box of a given product (e.g., breakfast cereals) contains a coupon, and there are *n* different types of coupons, what is the probability that more than *t* boxes need to be bought to collect all *n* coupons? An alternative statement is: given *n* coupons, how many coupons do you expect you need to draw with replacement before having drawn each coupon at least once? The mathematical analysis of the problem reveals that the expected number of trials needed grows as $\Theta(n \log(n))$.[a] For example, when n = 50 it takes about 225[b] trials on average to collect all 50 coupons.

O(n log n)

[Wikipedia]

Extra Task – Problem 8

Same as birthday paradox

- Consider a blind-folded game of darts
- n darts are thrown to m targets
- Each dart reaches one and only one target !

- What is the probability of no target being hit more than once ?

$$P = \frac{\frac{m!}{(m-n)!}}{m^n}.$$

- What is the probability of at least one target being hit at least twice ?

$$P(\text{at least one target hit at least twice}) = 1 - \frac{\frac{m!}{(m-n)!}}{m^n}.$$

REFERENCES

References

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  - Chapter 8
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Acknowledgments

- An earlier version of some of these slides was developed by Professor Carlos Bastos
- Some of the initial examples are from the "Concrete Mathematics" book
- Problems 3 and 4 are from Jeff M. Phillips' Data Mining course at the University of Utah