Calculus

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Introduction

In this lecture we will be looking at calculus from a *FIRST* Tech Challenge (FTC) perspective. We will not go into rigorous mathematical proofs, but will cover the three main aspects typically taught in a Calculus I course at the undergraduate level. These topics are:

- Limits
- Derivatives
- Integrals

The following material assumes you have taken an Algebra I class at the very least. The calculus aspects will be presented with a "robotics intuition", so hopefully things aren't too complex.

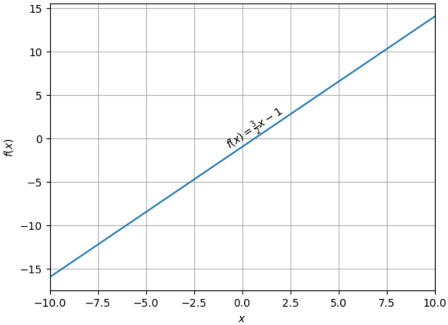
Functions

As a review, let's first discuss functions. A **function** is a rule of assignment that associates each element from one set, called the domain, with exactly one element from another set, called the codomain. It describes how input values are mapped to output values according to a specific relationship or operation.

One of the most common functions is the linear function which usually take the form

$$y = f(x) = mx + b.$$

If you've taken Algebra I, this may be familiar to you. Oftentimes the variable x is presented as an input (independent variable) and the variable y is presented as an output (dependent variable). Let's graph $f(x) = \frac{3}{2}x - 1$.

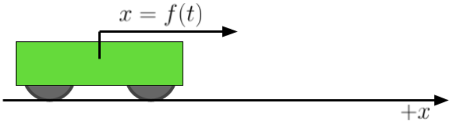


Functions

In order to later extend our math to multiple variables, let's introduce t as our independent variable from now on.

Note: in this lecture t will be associated with time. In other lectures, t will be used as a parameterization variable. It will be explicitly stated when each case arises.

Let's define the practical example where we have a robot on a track. The positive x direction of the track is defined to the right. The x position of the car is a function of time t. The x position can also be expressed $x(t)$.



We will come back to this example in a bit.

Note: let's assume t has units of seconds (s) and x has units of meters (m)

We will now look at **limits** as they are the foundation for the more interesting parts of calculus. We typically denote a limit as

$$\lim_{t \to c} f(t),$$

where t is our independent variable, $f(t)$ is our function, and c is a value. The equation above reads "the limit of $f(t)$ as t approaches c". We're essentially asking the question: "what value does $f(t)$ approach as t approaches c?"

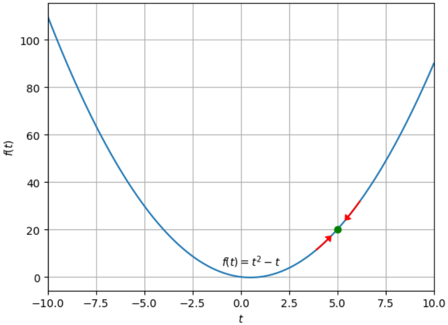
Limits

Let's look at a mathematical example. Take the function $f(t) = t^2 - t$. Let's analyze

$$\lim_{t \to 5} t^2 - t.$$

In order for the limit to be valid, we check what value $f(t)$ approaches as t approaches 5 from the left and from the right, i.e.,

$$\lim_{t \to 5^-} t^2 - t \text{ and } \lim_{t \to 5^+} t^2 - t$$



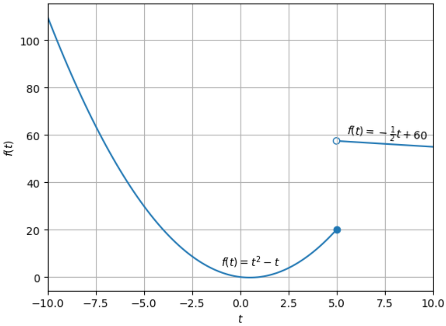
Mathematically, we can do this by plugging in values closer and closer to 5 from both directions and seeing if they approach the same value. Graphically, this is depicted with the red arrows. It should be pretty obvious from here that

$$\lim_{t \to 5} t^2 - t = 20$$

We need to be careful because some functions have a discontinuity. For example, in the function

$$f(t) = \begin{cases} t^2 - t & t \leq 5 \\ -\frac{1}{2}t + 60 & t > 5 \end{cases}$$

the limit from the previous slide no longer exists since the value of $f(t)$ approaches 57.5 as $t = 5$ is approached from the right and the the value of $f(t)$ approaches 20 as $t = 5$ is approached from the left.



We can also look at what happens to functions as the independent variable goes to infinity, i.e.,

$$\lim_{t \to \infty} f(t)$$

In this case, there is no way to check what happens to $f(t)$ "from the right". Many functions will simply blow up to infinity as t moves to infinity, however certain functions do actually have a value for this limit.

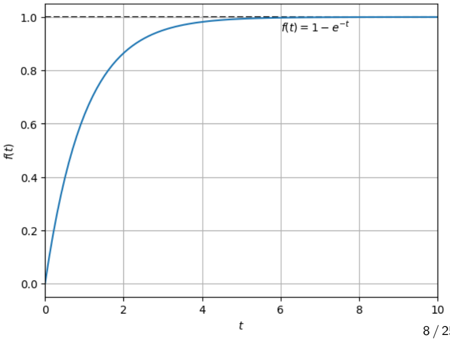
Let's look at the function

$$f(t) = 1 - e^{-t}$$

In this case, our limit as t approaches ∞ evaluates to 1, i.e.,

$$\lim_{t \to \infty} 1 - e^{-t} = 1$$

We can see this graphically as $f(t)$ asymptotically approaches 1.



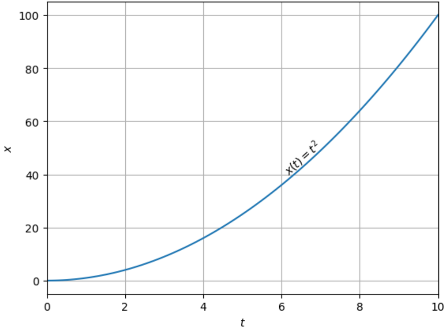
Derivatives

Let's reintroduce our cart example. Let's assume the cart moves in the positive x direction modelled by the function

$$x(t) = t^2$$

What we have here is a position versus time function, where $x(t)$ gives us our cart's position at any time t. Let's be a bit more formal and bound t to to be greater than or equal to 0, as time cannot really be negative.

$$t \geq 0$$



Derivatives

Let's say we want more information about how our cart moves. Specifically, we'd like to figure out the rate at which the cart is moving at some time T (note the capitalization).

This rate is the cart's speed, or more formally, it's velocity. In general, we can estimate the velocity of the cart by dividing it's change in position by the change in time:

$$v_x \approx \frac{(x_2 - x_1)}{(t_2 - t_1)}$$

This looks a lot like the slope formula! How do we apply this to a function that isn't linear? Well, to estimate the velocity at some time T, we can construct a line using points along the function.

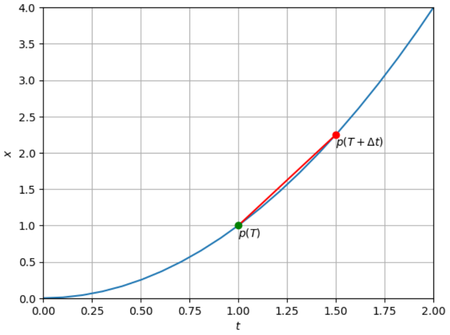
Derivatives

Let's try to estimate the velocity of the cart at $T = 1s$. We can choose a small deviation $\Delta t = 0.5s$ to generate the points:

$$p(T) = \big(T, x(T)\big) = (1.0s, 1.0m)$$
$$p(T+\Delta t) = \big((T+\Delta t, x(T+\Delta t)\big) = (1.5s, 2.25m)$$

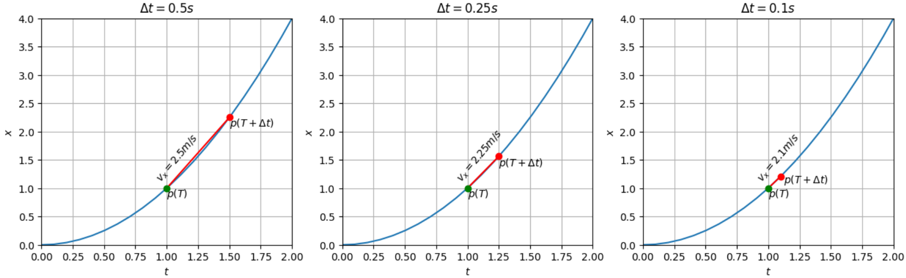
We can now estimate the velocity of the cart at 1 second by finding the slope of the line connecting $p(T+\Delta t)$ and $p(T)$:

$$\begin{aligned} v_x &\approx \frac{x(T+\Delta t) - x(T)}{(T+\Delta t) - T} \\ &= \frac{x(T+\Delta t) - x(T)}{\Delta t} \\ &= \frac{2.25m - 1.0m}{0.5s} \\ &= 2.5\frac{m}{s} \end{aligned}$$



Derivatives

The true value of the velocity at 1 second is actually $2m/s$. How can we use what we know to make our estimate better? We can decrease Δt to shrink the distance between the points! Let's take a look at what happens to our estimate as we shrink Δt.



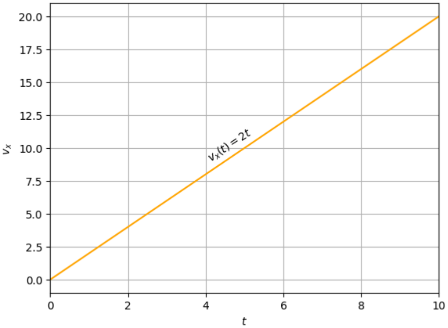
We can see that as Δt approaches 0, we get closer and closer to the true velocity value at 1 second. We just learned about a tool that pushes the independent variable towards a certain value: limits!

Derivatives

Now we can generalize our expressions such that we can find the velocity of the cart at any time t!

$$v_x(t) = \lim_{\Delta t \to 0} \frac{x(t + \Delta t) - x(t)}{\Delta t} = 2t$$

Now, solving this for some function $f(t)$ is outside the scope of this lecture, but this equation denotes the **definition of the derivative**, i.e., $v_x(t) = \dot{x}(t)$, where $\dot{x}(t)$ is the derivative of $x(t)$. The derivative is essentially just a fancy word for the slope of a function.



Derivatives

Building on the last slide, the general form of the definition of the derivative is

$$\dot{f}(t) = \lim_{\Delta t \to 0} \frac{f(t + \Delta t) - f(t)}{\Delta t},$$

where $f(t)$ is some arbitrary function and $\dot{f}(t)$ is it's derivative (with respect to t).

Derivatives

It is worth noting that the derivative of a function is expressed in a few different ways in mathematics:

$$\dot{f}(t) = \frac{df(t)}{dt} = f'(t)$$

Moreover, solving the definition of the derivative expression is often time consuming so we typically memorize tricks for different functions. The derivation for these tricks is outside the scope of this lecture, but some common derivatives to know are listed below (n is some constant):

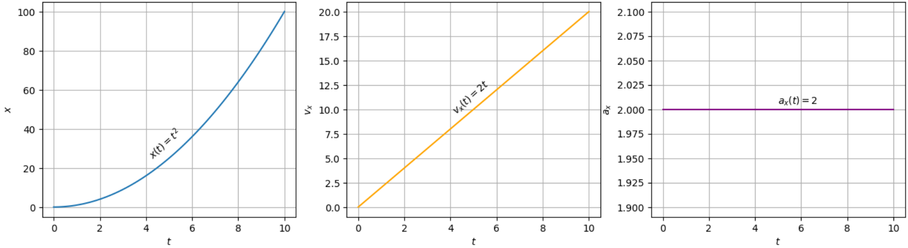
$f(t)$	$\dot{f}(t)$
n	0
t^n	nt^{n-1}
e^t	e^t
$\ln(t)$	$\frac{1}{t}$
$\cos(t)$	$-\sin(t)$
$\sin(t)$	$\cos(t)$
$\tan(t)$	$\frac{1}{\cos^2(t)}$

From this, it is trivial to find $v_x(t) = \dot{x}(t) = 2t$, the graph of which is shown on the previous slide.

What if we wanted to calculate the instantaneous acceleration at any time t? Acceleration is the rate at which the velocity of a system changes. We know that the derivative of a function gives us the rate of change of that function, so finding the acceleration of our cart is simple:

$$a_x(t) = \dot{v}_x(t) = \ddot{x}(t) = 2\frac{m}{s^2}$$

In this case the acceleration of the cart is constant. Notice that the acceleration is also the double derivative (two dots) of the position function.



Derivatives

Now, for the most important part. How do we use derivatives in code? Thankfully, we will not have to take derivatives in code. We will either take them "by hand", then apply the results in code, or we will estimate them. We will see examples of the former when we discuss path generation.

As an example of the latter, we can estimate the robot's velocity in code using only position and time information:

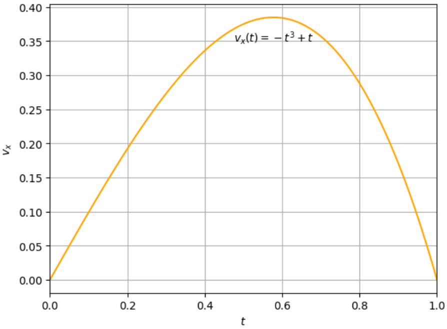
$$v_{x,k} \approx \frac{x_k - x_{k-1}}{t_k - t_{k-1}},$$

where k is the current loop iteration of the robot's main loop. Now, the accuracy of this estimation depends on the size of $t_k - t_{k-1}$. If this value is small, then we will more accurately represent the true velocity of the robot.

Integrals

Let's rework our cart example and assume we do not know the position as a function of time; we only have velocity information. Assume the car travels for 1 total second and the velocity as a function of time is given by:

$$v_x(t) = -t^3 + t$$
$$0 \leq t \leq 1$$



Can we determine the position of the car at time $t = 1$?

Let's start by attempting to estimate the position of the car. We know that

$$v_x \approx \frac{\Delta x}{\Delta t}$$

Using this equation, we can solve for Δx:

$$\Delta x = v_x \Delta t$$

We run into a problem because v_x is not constant! We need to choose a value for v_x. Since $v_x(0) = v_x(1) = 0$, let's try $v_x(0.5) = 0.375$. Our estimate of the cart's position becomes

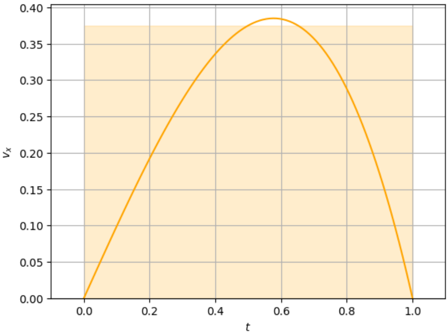
$$x(1) - x(0) = 0.375\frac{m}{s} \cdot (1s - 0s)$$

Assuming the cart starts at $0m$, i.e., $x(0) = 0$, this simplifies to

$$x(1) = 0.375\frac{m}{s} \cdot 1s = 0.375m$$

Integrals

Let's take a look at this graphically.



We see that our equation is simply one large box of width 1 and height 0.375. The true position of the car at time $t = 1$ will actually be the **total area under velocity curve**. How can we improve our estimate?

We can use multiple smaller boxes to better estimate the area under the curve.

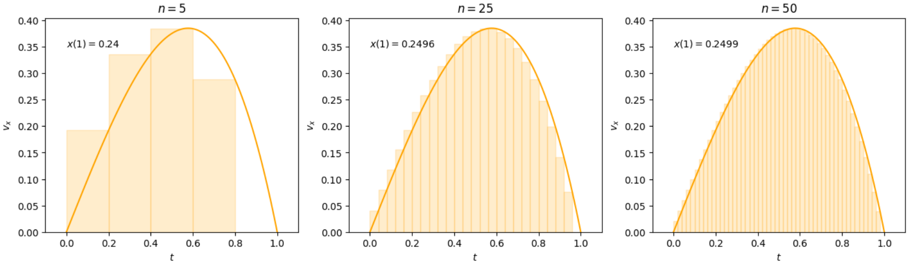
Integrals

Let's split our calculation into n boxes. We can then add these boxes to get a better estimate for the area under the curve (the position of the cart).

$$x(1) - x(0) \approx \sum_{i=1}^{n} v_x(t_0 + \Delta t \cdot i) \cdot \Delta t$$

$$\Delta t = \frac{t_f - t_0}{n}$$

In this case, our initial time is $t_0 = 0s$ and our final time is $t_f = 1s$.



Integrals

As we see from the graphs on the previous slide, increasing n gives us a more accurate representation of the area under the curve (which is actually 0.25). The representation we have constructed is known as a right **Riemann sum**.

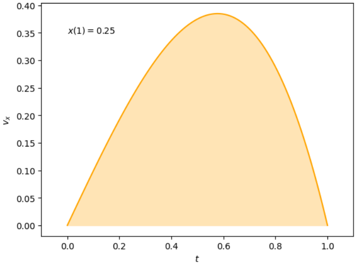
Let's apply limits to push n towards infinity!

$$x(1) - x(0) = \lim_{n \to \infty} \sum_{i=1}^{n} v_x(t_0 + \Delta t \cdot i) \cdot \Delta t = 0.25$$

This is actually equal to the **integral** of $v_x(t)$ from t_0 to t_f with respect to t:

$$\lim_{n \to \infty} \sum_{i=1}^{n} v_x\left(t_0 + \left(\frac{t_f - t_0}{n}\right) \cdot i\right) \cdot \left(\frac{t_f - t_0}{n}\right)$$

$$= \int_{t_0}^{t_f} v_x(t) dt$$



Note: dt represents an infinitesimally small Δt

Integrals

Building on the last slide, the general form of the integral is

$$F(t) = \int f(t)dt,$$

where $F(t)$ is the **antiderivative** of $f(t)$. Some common antiderivatives are

$f(t)$	$F(t)$
0	C
t^n	$\frac{t^{n+1}}{n+1} + C$
e^t	$e^t + C$
$\frac{1}{t}$	$\ln(x) + C$
$\cos(t)$	$\sin(t) + C$
$\sin(t)$	$-\cos(t) + C$

When we take an integral we add some constant C since the derivative of a constant is 0, i.e.,

$$\frac{df(t)}{dt} = \frac{d\left(f(t) + C\right)}{dt}$$

Integrals

Using this information, we can derive the function for our cart's position, $x(t)$:

$$
\begin{aligned}
x(t) &= \int v_x(t)dt \\
&= \int (-t^3 + t)dt \\
&= -\frac{1}{4}t^4 + \frac{1}{2}t^2 + C
\end{aligned}
$$

Recall the assumption that $x(0) = 0$. We can solve for C as follows:

$$
\begin{aligned}
0 &= -\frac{1}{4}(0)^4 + \frac{1}{2}(0)^2 + C \\
C &= 0
\end{aligned}
$$

We now know the position of the cart at any time t:

$$
x(t) = -\frac{1}{4}(t)^4 + \frac{1}{2}(t)^2
$$

Integrals

Similar to derivatives, we will not explicitly code integrals. We will see examples of integral estimation when we discuss proportional-integral-derivative (PID) controllers. For now, we can estimate integrals in code via addition.

As an example, let's estimate the robot's position in code using only velocity and time information:

$$x_k = x_{k-1} + v_k + (t_k - t_{k-1}),$$

where k is current loop iteration of the robot's main loop. Using this, we can continuously add to a set variable.