Non-singular word maps for linear groups

Henry Bradford, Jakob Schneider, and Andreas Thom

ABSTRACT. We study the word image of words with constants in $\mathrm{GL}(V)$ and show that it is large provided the word satisfies some natural conditions on its length and its critical constants.

There are various consequences: We prove that for every $l \geq 1$, there are only finitely many pairs (n,q) such that the length of the shortest non-singular mixed identity $\mathrm{PSL}_n(q)$ is bounded by l. We generalize the Hull–Osin dichotomy for highly transitive permutation groups to linear groups over finite fields. Finally, we show that the rank limit of $\mathrm{GL}_n(q)$ for q fixed and $n \to \infty$ is mixed identity free.

CONTENTS

Introduction		1
1.	Basic notions	4
2.	Multiply linearly transitive groups	5
3.	Diameter and dimension of the word image	9
4.	A Hull-Osin type result for linear groups	11
5.	No mixed identities for $A(q)$	14
Acknowledgments		16
References		16

Introduction

Denote by $\mathbf{F}_r = \langle x_1, \dots, x_r \rangle$ the free group on r generators. Let G be a group and let $w \in G * \mathbf{F}_r$ be a word with r variables x_1, \dots, x_r and constants in G. We say that w is a mixed identity for G if w is non-trivial and $w(h_1, \dots, h_r) = 1_G$ for all $h_1, \dots, h_r \in G$. Here we write $w(h_1, \dots, h_r)$ for the image $\varphi(w)$ of w under the unique homomorphism $\varphi \colon G * \mathbf{F}_r \to G$ which maps $x_i \mapsto h_i$ for all $i \in \{1, \dots, r\}$ and fixes G elementwise. On $G * \mathbf{F}_r$ we consider the word length, which assigns length one to the

generators $x_1^{\pm 1}, \ldots, x_r^{\pm 1}$ and length zero to elements of G. In this setting, the augmentation map $\varepsilon \colon G \ast \mathbf{F}_r \to \mathbf{F}_r$ denotes the unique homomorphism which maps all $G \ni g \mapsto 1_{\mathbf{F}_r}$ to the identity and fixes each element of \mathbf{F}_r . The elements in the kernel of ε are called *singular* and $\varepsilon(w)$ is called the content of w.

In this article we continue the study of non-singular word maps on finite groups, which we started in [18], where the second and third named authors treated the case of symmetric and alternating groups. One of the main results in [18] was that any non-singular mixed identity of S_n is of length at least $\Omega(\log(n)/\log(\log(n)))$. This is in contrast to the length of singular mixed identities, which can be of bounded length. See [3] for more information on mixed identities in this context. Non-singular word maps for compact Lie groups where studied in [11] and for linear algebraic groups in [7].

A law or an identity of a group G is a non-trivial word $w \in \mathbf{F}_r$ such that $w(h_1, \ldots, h_r) = 1_G$ for all $h_1, \ldots, h_r \in G$. We say that $w \in \mathbf{F}_r$ is a coset identity if there exists a subgroup of finite index H and cosets g_1H, \ldots, g_rH , such that $w(h_1, \ldots, h_r) = 1_G$ for all $h_i \in g_iH, i \in \{1, \ldots, r\}$. It is a major open problem to decide if every finitely generated pro-finite group that admits a coset identity also admits an identity, see for example [16, Problem 12.95] and the work of Larsen-Shalev [15, Problem 3.1]. Note that every coset identity yields a non-singular mixed identity

$$w'(x_1,\ldots,x_r) \coloneqq w(g_1x_1^d,\ldots,g_rx_r^d)$$

for d = [G:H]!. Hence, a potential strategy to answer Larsen and Shalev's question is to study the length of the shortest non-singular mixed identity of a finite group G and compare it to the length of the shortest identity. It can be shown that the answer to Larsen and Shalev's question is positive if these quantities can be bounded in terms of each other.

QUESTION 1. Does the existence of a non-singular mixed identity $w \in G * \mathbf{F}_r$ for a finite group G imply the existence of a law for G of length bounded only in terms of the length of w?

We are able to prove that the length of the shortest non-singular mixed identity for $\mathrm{PSL}_n(q)$ tends to infinity as n or q increase. Note, that the minimal length of laws for groups $\mathrm{PSL}_n(q)$ has been studied in [2]. Hence, this contributes another class of cases where this question has a positive answer, see Corollary 3.

In view of the corresponding result of Jones [10] on laws for non-abelian finite simple groups, we put forward the following conjecture:

Conjecture 1. Let $l \in \mathbb{N}$. The number of isomorphism classes of non-abelian finite simple groups satisfying a non-singular mixed identity of length $\leq l$ is finite.

Similarly to our results for symmetric groups, our analysis of word maps allows also for detailed information on the image of word maps, singular or non-singular, as long as the lengths of the critical constants (see Section 1 for a definition) are under certain control. Our main result is stated as Theorem 1. We have various applications of this result in analogy to results on permutation actions. In particular, we have estimates on the diameter and dimension of the image.

It was shown by Gordeev–Kunyavskii–Plotkin in [7] that non-singular word maps on linear algebraic groups with generic constants are dominant. As a consequence of our results, we can prove a non-trivial lower bound on the dimension of the image in case of SL_n and non-singular word maps with arbitrary constants, see Theorem 2.

In Section 4, we can prove an analogue of the Hull-Osin dichotomy [9, Theorem 1.6] for highly linearly transitive groups, which in its original form says that a highly transitive permutation group either contains a locally finite, highly transitive, normal subgroup or it is mixed identity free. We prove a similar result for highly linearly transitive groups acting on a countably infinite dimensional vector space over a finite field, see Theorem 5. There is also a quantitative version of this, generalizing a result of Le Boudec-Matte Bon [12, Proposition A.1], see Theorem 6.

It is a consequence of the original Hull–Osin dichotomy that full groups of ergodic, probability measure preserving, measurable equivalence relations on a non-atomic probability space are mixed identity free, see the discussion in Section 5. Moreover, it follows from Popa's work [17] that the unitary group of a II₁-factor does not satisfy a mixed identity. No such approach can be taken to understand the mod-p analogue of the unitary group of the hyperfinite II₁-factor as introduced by Carderi and the third author in [5]. However, we give a direct proof that this group is also mixed identity free, see Theorem 7.

Finally, this paper might be the starting point of a more systematic study of mixed identities for oligomorphic permutation groups such as $\operatorname{Sym}(\mathbb{N})$ of $\operatorname{GL}(\mathbb{F}_q^{\oplus \omega})$. Note that $\operatorname{Aut}(\mathbb{Q},<)$ satisfies a mixed identity as consequence of work of Zarzycki [19], while the automorphism group of the Rado graph and the universal Urysohn metric space does not, see Etedadialiabadi–Gao–Le Maître-Melleray [6]. See also Remark 3 for further discussion.

1. Basic notions

Let's recall some basics where we follow the notation introduced in [18]. Let G be a group. Throughout the entire article, let

$$w = c_0 x_{\iota(1)}^{\varepsilon(1)} c_1 \cdots c_{l-1} x_{\iota(l)}^{\varepsilon(l)} c_l \in G * \mathbf{F}_r = G * \langle x_1, \dots, x_r \rangle$$

be a fixed word with constants $c_j \in G$ (for $j \in \{0, ..., l\}$) with $\varepsilon(j) \in \{\pm 1\}$ and $\iota(j) \in \{1, ..., r\}$ (for $j \in \{1, ..., l\}$). Subsequently, $H \leq G$ will be the group whose elements we plug into w. Set $J_0(w) \coloneqq \{j \in \{1, ..., l-1\} \mid \iota(j) \neq \iota(j+1)\}$, $J_+(w) \coloneqq \{j \in \{1, ..., l-1\} \mid \iota(j) = \iota(j+1)\}$ and $\varepsilon(j) = \varepsilon(j+1)\}$, and $J_-(w) \coloneqq \{j \in \{1, ..., l-1\} \mid \iota(j) = \iota(j+1)\}$ and $\varepsilon(j) = -\varepsilon(j+1)\}$. We call the elements of $J_-(w)$ the critical indices and the group elements c_j for $j \in J_-(w)$ the critical constants of w. Furthermore, we assume that the above expression for w is reduced, i.e. $c_j \notin \mathbf{C}_G(H) \setminus \mathbf{1}$ for $j \in J_0(w) \cup J_+(w)$ and $c_j \notin \mathbf{C}_G(H)$ for all $j \in J_-(w)$. Here $\mathbf{C}_G(H)$ denotes the centralizer of H in G. We call the word w strong if it has no critical constants, i.e. $J_-(w) = \emptyset$. And we call it cyclically strong if w is strong and, in addition, it does not hold that $\iota(1) = \iota(l)$ and $\varepsilon(1) = -\varepsilon(l)$. Finally, we write |w| := l for the length of the word w.

Throughout, we will need some measure of distance on our groups. That is why we define the notion of a *seminorm* on a group. For a group G, this is a mapping $|\bullet|: G \to I$, where I is either the unit interval $[0,1] \subseteq \mathbb{R}$ or the set $\mathbb{N} \cup \{\infty\}$, such that $|g| \geq 0$ (non-negativity), $|gh| \leq |g| + |h|$ (subadditivity), and $|g| = |g^{-1}|$ (homogeneity) for all $g, h \in G$. Each seminorm induces a left invariant pseudometric dist: $G \times G \to I$ by dist $(g,h) := |g^{-1}h|$. A seminorm becomes a norm if |g| = 0 implies $g = 1_G$ for all $g \in G$ (positivity). In this case, it induces a metric. For each seminorm $|\bullet|$, the set $G_0 := |\bullet|^{-1}(0) \le G$ is a subgroup. The seminorm is called *invariant* if it is invariant under conjugation, i.e. $|g^h| = |h^{-1}gh| = |g|$ for all $g, h \in G$. In this case, $G_0 \subseteq G$ is a normal subgroup; then $|\bullet|$ descends to a norm on G/G_0 . Then, the corresponding pseudometric dist: $G \times G \to I$ will be bi-invariant, i.e. $\operatorname{dist}(fg, fh) = \operatorname{dist}(g, h) = \operatorname{dist}(gf, hf)$ for all $f, g, h \in G$. In this article, all seminorms will be invariant, whence all pseudometrics will be biinvariant. For every seminorm $|\bullet|: G \to I$ and a normal subgroup $N \leq G$, we define the quotient seminorm $|\bullet|: G/N \to I$ by $|\overline{g}| := \operatorname{dist}(g,N) =$ $\inf\{\operatorname{dist}(g,n) \mid n \in N\} \text{ for all } g \in G.$

For $w \in G * \mathbf{F}_r$ and a fixed seminorm on G, we define the *critical length*

$$|w|_{\text{crit}} := \min\{\operatorname{diam}(G), |c_i| \mid j \in J_-(w)\},\$$

where the diameter is measured with respect to the fixed seminorm. Besides the word length of a word, the critical length turns out to be the most important quantity controlling lower bounds on the size of the word image.

2. Multiply linearly transitive groups

Let V be a vector space over the field K and of finite or countably infinite dimension. We only consider the cases when $K = \mathbb{F}_q$ is a finite field, or $K = \mathbb{C}$ are the complex numbers. Write GL(V) for the general linear group of V. Denote by $|\bullet|$: $GL(V) \to \mathbb{N} \cup \{\infty\}$ the *projective rank seminorm*, which is defined by

$$|g| := \min_{\lambda \in K^{\times}} \operatorname{rk}(g - \lambda 1_V) \in \mathbb{N} \cup \{\infty\}$$

for $g \in GL(V)$. It descends to a *norm*, which we also denote by

$$|\bullet|: \operatorname{PGL}(V) \to \mathbb{N} \cup \{\infty\}.$$

This induces the projective rank pseudometric on GL(V) and a projective rank metric on PGL(V), both denoted by dist.

DEFINITION 1. Let V be a vector space of finite or countably infinite dimension and let $k \in \mathbb{N}$. A linear group $H \leq \operatorname{GL}(V)$ is called linearly k-transitive if for all k-element sets of linearly independent vectors $\{u_1, \ldots, u_k\}$ and $\{w_1, \ldots, w_k\}$ there is a map $h \in H$ such that $u_i.h = w_i$ for all $i \in \{1, \ldots, k\}$.

Here is our main technical result:

Theorem 1. Let V be some vector space of finite or countably infinite dimension over the field K. If K is finite, we allow $\dim(V)$ to be infinite, whereas for $K = \mathbb{C}$, we require it to be finite. Let $D, d, l, r \in \mathbb{N}$ and let the word

$$w = x_{\iota(1)}^{\varepsilon(1)} c_1 \cdots c_{l-1} x_{\iota(l)}^{\varepsilon(l)} \in \operatorname{GL}(V) * \mathbf{F}_r$$

be reduced of length $l \geq 2$. Let $H \leq \operatorname{GL}(V)$ be a linearly D-transitive group. Let $\{u_1, \ldots, u_d\}$ and $\{w_1, \ldots, w_d\}$ be d-element sets of linearly independent vectors. If $\iota(1) = \iota(l)$ and $\varepsilon(1) = -\varepsilon(l)$, then assume in addition that the linear spans of $\{u_1, \ldots, u_d\}$ and $\{w_1, \ldots, w_d\}$ intersect trivially. Then, if d satisfies

$$d \le \frac{1}{l} \cdot \min\{|w|_{\text{crit}} - 1, D\},\$$

there exist $h_1, \ldots, h_r \in H$ such that $u_i.w(h_1, \ldots, h_r) = w_i$ for $i \in \{1, \ldots, d\}$.

PROOF. We set $U = \langle u_1, \dots, u_d \rangle$ and $W := \langle w_1, \dots, w_d \rangle$. In order to prove the theorem, we will define vectors $v_{i,j}^{-\varepsilon(j)} \in V$ for $(i,j) \in \{1,\dots,d\} \times \{1,\dots,l\}$ and elements $h_1,\dots,h_r \in H$ such that

$$u_1 = v_{1,1}^{\varepsilon(1)} \overset{h_{\iota(1)}^{\varepsilon(1)}}{\mapsto} v_{1,1}^{-\varepsilon(1)} \overset{c_1}{\mapsto} v_{1,2}^{\varepsilon(2)} \overset{h_{\iota(2)}^{\varepsilon(2)}}{\mapsto} \cdots \overset{h_{\iota(l-1)}^{\varepsilon(l-1)}}{\mapsto} v_{1,l-1}^{-\varepsilon(l-1)} \overset{c_{l-1}}{\mapsto} v_{1,l}^{\varepsilon(l)} \overset{h_{\iota(l)}^{\varepsilon(l)}}{\mapsto} v_{1,l}^{-\varepsilon(l)} = w_1$$

$$\vdots$$

$$u_d = v_{d,1}^{\varepsilon(1)} \overset{h_{\iota(1)}^{\varepsilon(1)}}{\mapsto} v_{d,1}^{-\varepsilon(1)} \overset{c_1}{\mapsto} v_{d,2}^{\varepsilon(2)} \overset{h_{\iota(2)}^{\varepsilon(2)}}{\mapsto} \cdots \overset{h_{\iota(l-1)}^{\varepsilon(l-1)}}{\mapsto} v_{d,l-1}^{-\varepsilon(l-1)} \overset{c_{l-1}}{\mapsto} v_{d,l}^{\varepsilon(l)} \overset{h_{\iota(l)}^{\varepsilon(l)}}{\mapsto} v_{d,l}^{-\varepsilon(l)} = w_d.$$

This requirement will determine vectors $v_{i,j}^{\varepsilon(j)} \in V$ by the equation $v_{i,j}^{\varepsilon(j)} \coloneqq v_{i,j-1}^{-\varepsilon(j-1)}.c_{j-1}$ for $2 \le j \le l$ and $v_{i,1}^{\varepsilon(1)} \coloneqq u_i$. The choice of the vectors $v_{i,j}^{-\varepsilon(j)} \in V$ proceeds by induction on the lexicographic order \le on the set $\{1,\ldots,d\} \times \{1,\ldots,l\}$. During the procedure, we want to ensure that the sets $\{v_{i,j}^{\varepsilon} \mid i \in \{1,\ldots,d\}, j \in \{1,\ldots,l\}, \iota(j) = k\}$ are linearly independent for all $\varepsilon \in \{\pm 1\}$ and $k \in \{1,\ldots,r\}$ and we can put $v_{i,l}^{-\varepsilon(l)} \coloneqq w_i$ at the end of the procedure.

Once this is done, we can choose $h_k \in H$ such that $v_{i,j}^+.h_k = v_{i,j}^-$ whenever $\iota(j) = k$ and the picture is complete. For this, it is enough that H is linearly dl-transitive which we assumed to be the case. Thus, in order to finish the proof, it only remains to carry out the inductive process and keep track of the linear independence.

Before we start with the induction, let's first describe some auxiliary objects which are needed in the inductive procedure. Suppose that all $v_{i',j'}^{\varepsilon}$ with $(i',j') \leq (i,j)$ are already defined. We define subspaces for $i \in \{1,\ldots,d\}$, $j \in \{1,\ldots,l\}$, $k \in \{1,\ldots,r\}$, and $\varepsilon \in \{\pm\}$.

$$V_{i,j,k}^{\varepsilon} := \langle v_{i',j'}^{\varepsilon} | (i',j') \leq (i,j), \iota(j') = k \rangle$$

$$\leq \sum_{k=1}^{r} V_{i,j,k}^{\varepsilon} = \langle v_{1,1}^{\varepsilon}, \dots, v_{1,l}^{\varepsilon}, \dots, v_{i,1}^{\varepsilon}, \dots, v_{i,j}^{\varepsilon} \rangle =: V_{i,j}^{\varepsilon} \leq V,$$

so that $\dim(V_{i,j}^{\varepsilon}) \leq (i-1)l+j$. We also set $V_{i,0,k}^{\varepsilon} \coloneqq V_{i-1,l,k}^{\varepsilon}$ for $i \in \{2,\ldots,d\}$, $k \in \{1,\ldots,r\}$, $\varepsilon \in \{\pm\}$. For $\varepsilon \in \{\pm\}$ and $\iota \in \{1,\ldots,r\}$, we need to introduce the subspaces:

$$V_{\iota}^{\varepsilon} := \begin{cases} U & \text{if } (\iota, \varepsilon) = (\iota(1), \varepsilon(1)) \text{ and } (\iota, \varepsilon) \neq (\iota(l), -\varepsilon(l)) \\ W & \text{if } (\iota, \varepsilon) \neq (\iota(1), \varepsilon(1)) \text{ and } (\iota, \varepsilon) = (\iota(l), -\varepsilon(l)) \\ U + W & \text{if } (\iota, \varepsilon) = (\iota(1), \varepsilon(1)) \text{ and } (\iota, \varepsilon) = (\iota(l), -\varepsilon(l)) \\ \mathbf{0} & \text{otherwise.} \end{cases}$$

which we will need later in the proof.

We will now describe how to choose the vectors $v_{i,j}^{-\varepsilon(j)}$. In the (i,j)th step of the induction with j < l, all the vectors $v_{i',j'}^{\varepsilon}$ for (i',j') < (i,j) and $\varepsilon \in \{\pm\}$, and the vector $v_{i,j}^{\varepsilon(j)}$, are already defined. Hence the subspace $V_{i',j',k}^{\varepsilon}$ for (i',j') < (i,j) for $k \in \{1,\ldots,r\}$ and $\varepsilon \in \{\pm\}$ and the subspace $V_{i,j,\iota(j)}^{\varepsilon(j)} = V_{i,j-1,\iota(j)}^{\varepsilon(j)} + \langle v_{i,j}^{\varepsilon(j)} \rangle$ are already defined. The vector $v_{i,j}^{-\varepsilon(j)}$ is not yet defined and our aim is to construct it now.

In order to retain the linear independence condition, we need to ensure that

$$v_{i,j}^{-\varepsilon(j)} \notin V_{i,j-1,\iota(j)}^{-\varepsilon(j)} + V_{\iota(j)}^{-\varepsilon(j)} \quad \text{and} \quad v_{i,j+1}^{\varepsilon(j+1)} = v_{i,j}^{-\varepsilon(j)}.c_j \notin V_{i,j,\iota(j+1)}^{\varepsilon(j+1)} + V_{\iota(j+1)}^{\varepsilon(j+1)}.$$

For the first constraint $v_{i,j}^{-\varepsilon(j)} \notin V_{i,j-1,\iota(j)}^{-\varepsilon(j)} + V_{\iota(j)}^{-\varepsilon(j)}$, note that this subspace is contained in $V_{d,l-2}^{-\varepsilon(j)} + \langle w_d \rangle$ and so its dimension is at most dl-1. The case l=2, j=1 needs a special argument, but note that $V_{\iota(1)}^{-\varepsilon(1)} \leq W$ by its definition and hence we obtain the same bound also in this case. In order to study the second constraint, we will now distinghish two cases.

Case 1. Assume $j \in J_0(w) \cup J_+(w)$. We must ensure in addition that

$$v_{i,j}^{-\varepsilon(j)}.c_j \notin V_{i,j,\iota(j+1)}^{\varepsilon(j+1)} + V_{\iota(j+1)}^{\varepsilon(j+1)}.$$

Again, this condition excludes a subspace of dimension at most dl - 1. Indeed, we have

$$V_{i,j,\iota(j+1)}^{\varepsilon(j+1)} + V_{\iota(j+1)}^{\varepsilon(j+1)} \le \begin{cases} V_{d,l-2}^{\varepsilon(j+1)} + \langle w_d \rangle & 1 \le j \le l-2 \\ V_{d,l-1}^{\varepsilon(l)} & j = l-1, \end{cases},$$

since $V_{\iota(l)}^{\varepsilon(l)} \leq U$ by definition in the second case, which yields the required bound on the dimension in each case.

Dealing with both constraints, we must have

$$v_{i,j}^{-\varepsilon(j)} \notin (V_{i,j-1,\iota(j)}^{-\varepsilon(j)} + V_{\iota(j)}^{-\varepsilon(j)}) \cup (V_{i,j,\iota(j+1)}^{\varepsilon(j+1)} + V_{\iota(j+1)}^{\varepsilon(j+1)}).c_j^{-1}$$

and this is a union of two subspaces of dimension at most dl-1. Since $dl-1 < dl < |w'|_{\text{crit}} \le \dim(V)$, these subspaces have codimension at least 2 and we find an appropriate choice for $v_{i,j}^{-\varepsilon(j)}$ in this case.

Case 2. Assume now that $j \in J_{-}(w)$. We have to ensure that

$$\begin{split} v_{i,j}^{-\varepsilon(j)}.c_j \notin V_{i,j,\iota(j+1)}^{\varepsilon(j+1)} + V_{\iota(j+1)}^{\varepsilon(j+1)} &= V_{i,j-1,\iota(j)}^{\varepsilon(j+1)} + \langle v_{i,j}^{\varepsilon(j+1)} \rangle + V_{\iota(j+1)}^{\varepsilon(j+1)} \\ &= V_{i,j-1,\iota(j)}^{-\varepsilon(j)} + \langle v_{i,j}^{-\varepsilon(j)} \rangle + V_{\iota(j+1)}^{\varepsilon(j+1)}. \end{split}$$

In order to get this, it is enough to ensure that for each $\lambda \in K$

$$v_{i,j}^{-\varepsilon(j)}.(c_j-\lambda) \notin V_{i,j-1,\iota(j)}^{-\varepsilon(j)} + V_{\iota(j+1)}^{\varepsilon(j+1)}.$$

Note that this subspace is again of dimension bounded by dl-1, where the case l=2 requires an additional argument that we omit. We further distinguish three more cases:

Case 2.1: Let's assume that $\dim(V) = n < \infty$ and let's first deal with the case $K = \mathbb{F}_q$. Let $\lambda_k \in \mathbb{F}_q^{\times}$ for $k \in \{1, \ldots, m\}$ be the eigenvalues of the linear map c_j and d_k their respective geometric multiplicity. Then for $\lambda \notin \{\lambda_1, \ldots, \lambda_m\}, c_j - \lambda$ is regular, so the condition above excludes a union of q - m subspaces of dimension at most dl - 1. Thus, we need to exclude at most $(q - m)q^{dl-1}$ vectors.

For $\lambda = \lambda_k$ we have to exclude the inverse image of $V_{i,j-1,\iota(j)}^{-\varepsilon(j)} + V_{\iota(j+1)}^{\varepsilon(j+1)}$ under the map $c_j - \lambda$. Thus, to deal with λ_k , we need to exclude at most $q^{d_k}q^{dl-1}$ vectors. Hence, when dealing with all $\lambda \in \mathbb{F}_q$, we need to avoid at most

$$(q-m)q^{dl-1} + \sum_{k=1}^{m} q^{dl-1}q^{d_k}$$

vectors. This covers the second constraint and we need to exclude at most q^{dl-1} vectors for the first constraint. Hence, in order to show that there is a good choice for $v_{i,j}^{-\varepsilon(j)}$ it remains to show that

$$(q-m+1)q^{dl-1} + \sum_{k=1}^{m} q^{dl-1}q^{d_k} < q^n - 1.$$

By assumption $n - d_k \ge |c_j| \ge |w|_{\text{crit}} > dl$. This implies that

$$(q-m+1)q^{dl-1} + \sum_{k=1}^{m} q^{dl-1}q^{d_k} \le (q-m+1)q^{n-2} + mq^{n-2} < q^n - 1.$$

Hence, we have shown that there is a good choice for $v_{i,j}^{-\varepsilon(j)}$.

Case 2.2: The proof for $K = \mathbb{C}$ is similar. Consider

$$V_{\lambda} := \left\{ v \in V \mid v.(c_j - \lambda) \in V_{i,j-1,\iota(j)}^{-\varepsilon(j)} + V_{\iota(j+1)}^{\varepsilon(j+1)} \right\}$$

and note that the estimate $n-d_k \geq |c_j| \geq |w|_{\rm crit} > dl$ is still valid and provides a bound of n-2 on the dimension of each V_{λ_k} for λ_k an eigenvalue of c_j . In order to deal with the other choices of λ , i.e. $\lambda \coloneqq D = \mathbb{C} \setminus \{\lambda_1, \ldots, \lambda_m\}$, note that V_{λ} is a 1-dimensional algebraic family of subspaces of codimension at least 2. Thus, $\bigcup_{\lambda \in D} V_{\lambda}$ is of codimension at least 1. Thus, we have to avoid finitely many algebraic subsets of lower dimension, which is of course possible.

Case 2.3: If $K = \mathbb{F}_q$ and $\dim(V) = \infty$, there are still only finitely elements $\lambda \in \mathbb{F}_q$ to consider and the inverse image of $V_{i,j-1,\iota(j)}^{-\varepsilon(j)} + V_{\iota(j+1)}^{\varepsilon(j+1)}$ under the map $c_j - \lambda$ is still proper. In fact, together with the first constraint

there are at most q+1 of these subspaces to consider and they are all of codimension at least 2. This implies that their union has a non-trivial complement from which we can choose $v_{i,j}^{-\varepsilon(j)}$. Thus we are done for this case as well.

Thus, this procedure can be carried out until we arrive at j = l and then we put $v_{i,l}^{-\varepsilon(l)} := w_i$. This finishes the proof of the main technical result. \square

REMARK 1. For finite permutation groups $H \leq S_n$, it is a well-known consequence of the CFSG that the only examples (other than A_n and S_n) of k-transitive groups for $k \geq 4$ are the four Matthieu groups M_{11} , M_{12} , M_{23} , and M_{24} in degrees n = 11, 12, 23 and 24. There is a similar classification of multiply linearly transitive subgroups of $GL_n(q)$, following from the work of Hering [8]. Indeed the linearly 1-transitive finite groups (that is, those acting transitively on the set of non-zero vectors in $V = \mathbb{F}_q^n$) are already rather restricted: apart from a finite list of sporadic examples, and overgroups of $SL_n(q)$, they all preserve either a non-trivial symplectic form on V or a proper field extension (meaning a structure on V of an (n/m)-dimensional \mathbb{F}_{q^m} -vector space, for some proper divisor m of n), both of which are incompatible with linear 2-transitivity. The sporadic cases include one linearly 3-transitive example (an action of A_7 on \mathbb{F}_2^4); none of the others are linearly 2-transitive [13].

The rest of the paper consists of consequences that we will explain in some detail.

3. Diameter and dimension of the word image

3.1. Diameter estimates. From Theorem 1 we get the following corollary on the diameter bound of the word image.

COROLLARY 1. Let $n \in \mathbb{N}$. Let V be an n-dimensional vector space over $K = \mathbb{F}_q$ or \mathbb{C} and $w \in \mathrm{GL}(V) * \mathbf{F}_r$ be a reduced word of length $l \geq 2$. Then the word image $w(\mathrm{SL}(V)^r)$ has diameter at least

$$\operatorname{diam}(w(\operatorname{SL}(V)^r)) \ge \frac{|w|_{\operatorname{crit}}}{l} - 1$$

in the projective rank seminorm.

Equivalently,

(1)
$$|w|(\operatorname{diam}(w(\operatorname{SL}(V)^r)) + 1) \ge |w|_{\operatorname{crit}}.$$

Let's now assume that w is non-singular and try to get a lower bound on the diameter of the word image only depending on the word length and not the critical length. The argument follows closely the corresponding argument

for permutation groups in [18]. If we remove the smallest critical constant c_j in w and perform the corresponding cancellation, we get a word w' such that

$$\operatorname{dist}(w(h_1,\ldots,h_r),w'(h_1,\ldots,h_r)) \leq \operatorname{dist}(c_i,1) = |w|_{\operatorname{crit}}$$

for all $h_1, \ldots, h_r \in H$ by bi-invariance of the rank metric. Hence

$$|\operatorname{diam}(w(\operatorname{SL}(V)^r)) - \operatorname{diam}(w'(\operatorname{SL}(V)^r))| \le 2|w|_{\operatorname{crit}}.$$

so by (1) we get

$$\operatorname{diam}(w'(\operatorname{SL}(V)^r)) \le \operatorname{diam}(w(\operatorname{SL}(V)^r)) + 2|w|_{\operatorname{crit}}$$

$$\le \operatorname{diam}(w(\operatorname{SL}(V)^r)) + 2|w|(\operatorname{diam}(w(\operatorname{SL}(V)^r)) + 1).$$

Therefore, we obtain

$$\operatorname{diam}(w'(\operatorname{SL}(V)^r)) + 1 \le (1 + 2|w|)(\operatorname{diam}(w(\operatorname{SL}(V)^r)) + 1)$$

Hence, if we iterate these reductions and $w = w_0, \ldots, w_m$ is a chain of words such that w_j is an reduction of w_{j-1} $(1 \le j \le m)$ and w_m is *strong*, then we get

$$diam(w_j(SL(V)^r)) + 1 \le (1 + 2|w_{j-1}|)(diam(w_{j-1}(SL(V)^r)) + 1)$$

for all j = 1, ..., m, and by iterating we obtain

$$\begin{aligned} \operatorname{diam}(w_m(\operatorname{SL}(V)^r)) + 1 &\leq \left(\operatorname{diam}(w_0(\operatorname{SL}(V)^r)) + 1\right) \prod_{j=0}^{m-1} (1 + 2|w_j|) \\ &\leq \left(\operatorname{diam}(w_0(\operatorname{SL}(V)^r)) + 1\right) (1 + 2|w_0|)^m \\ &\leq \left(\operatorname{diam}(w(\operatorname{SL}(V)^r)) + 1\right) (1 + 2|w|)^{\lfloor |w|/2\rfloor}. \end{aligned}$$

This is since $|w_{j-1}| \ge |w_j| - 2$ so that $m \le \lfloor |w|/2 \rfloor$. Hence by Inequality (1) we get

$$n = |w_m|_{\text{crit}} \le (\operatorname{diam}(w_m(\operatorname{SL}(V)^r)) + 1)|w_m|$$

$$\le (\operatorname{diam}(w(\operatorname{SL}(V)^r)) + 1)(1 + 2|w|)^{\lfloor |w|/2\rfloor}|w|.$$

This gives

$$\frac{1}{(1+2|w|)^{\lfloor |w|/2\rfloor}|w|} \le \frac{\operatorname{diam}(w(\operatorname{SL}(V)^r))+1}{n}.$$

In particular, this implies that when |w| is bounded and $n \to \infty$, then also $\operatorname{diam}(w(\operatorname{SL}(V)^r)) \to \infty$.

COROLLARY 2. Let q be a prime power. Let w be a non-singular mixed identity for $PSL_n(q)$. Then, we have

$$|w| = \Omega\left(\frac{\log(n)}{\log\log(n)}\right).$$

COROLLARY 3. For every $l \geq 1$, there are only finitely many pairs (n, q) such that the length of the shortest non-singular mixed identity $\mathrm{PSL}_n(q)$ is bounded by l.

PROOF. It is known from [3] that any mixed identity for $\operatorname{PSL}_n(q)$ is of length $\Omega(q)$, where the implied constant is independent of n. This and the previous corollary imply the claim.

REMARK 2. By the proof of [18, Corollary 3], there exists a singular word $v \in \mathrm{PSL}_n(q) * \langle x \rangle$ of length $O(q^{3n^2})$ with $v(g) = g^{-1}$ for all $g \in \mathrm{PSL}_n(q)$. Clearly, w = vx is a non-singular word with the property w(g) = 1 for all $g \in \mathrm{PSL}_n(q)$ and content $\varepsilon(w) = x$. This shows that there is no reason to assume that non-singularity by itself or just the length of the content of a word would be enough to give a lower bound on the diameter of the word image.

3.2. Esimates of the dimension. In this section, we follow the notation from the work of Gordeev-Kunyavskii-Plotkin [7] where we stick to the special case $G = SL_n$. Their main result is [7, Theorem 1.1], which states that for a fixed non-singular equation

$$w = c_0 x_{\iota(1)}^{\varepsilon(1)} c_1 \cdots c_{l-1} x_{\iota(l)}^{\varepsilon(l)} c_l \in G * \mathbf{F}_r = G * \langle x_1, \dots, x_r \rangle$$

with $(c_0, \ldots, c_l) \in \mathrm{SL}_n(\mathbb{C})^{l+1}$, there is a Zariski dense set of choices for the constants, where image is full-dimensional. This is essentially a consequence of the fact that the maximal dimension is generically attained and the image is full-dimensional by an old result of Borel [1] for the trivial choice of constants. However, their result says nothing about the worst case scenario.

THEOREM 2. Let $w = c_0 x_{\iota(1)}^{\varepsilon(1)} c_1 \cdots c_{l-1} x_{\iota(l)}^{\varepsilon(l)} c_l \in GL_n(\mathbb{C}) * \mathbf{F}_r$ be reduced of length $l \geq 2$. Then, we have

$$\dim w(\mathrm{SL}_n(\mathbb{C})) \ge \left\lfloor \frac{|w|_{\mathrm{crit}} - 1}{l} \right\rfloor \cdot n.$$

PROOF. Note that $d = \lfloor l^{-1} | w |_{\text{crit}} - 1 \rfloor$ is a valid choice for d in Theorem 1. Hence, by linear d-transitivity of the image of w, we get that the space of $(n \times d)$ -matrices of rank d is a subquotient of the image of w. This finishes the proof since its dimension is dn.

4. A Hull-Osin type result for linear groups

The following dichotomy theorem is due to Hull and Osin [9, Theorem 1.6]. Let's recall it quickly before we explain our generalization to the linear case.

THEOREM 3 (Hull-Osin). Let Ω be a countably infinite set and let $H \leq \operatorname{Sym}(\Omega)$ be a highly transitive permutation group. Then either

$$Alt(\Omega) \leq H$$
 or H has no mixed identities.

There is also a quantitative version of this theorem due to Le Boudec–Matte Bon [12, Proposition A.1]:

THEOREM 4 (Le Boudec-Matte Bon). Let $k \in \mathbb{Z}_{\geq 1}$ and $H \leq \operatorname{Sym}(\Omega)$ be a k-transitive permutation group which satisfies a mixed identity of length l. Then either

$$Alt(\Omega) \leq H$$
 or $k < l$.

REMARK 3. Now we come back to the discussion of mixed identities for oligomorphic groups at the end of the introduction. In view of [12, Proposition A.1], an interesting case to study is the automorphism group of the random k-uniform hypergraph for $k \geq 3$, which acts (k-1)-transitively but not k-transitively in its natural action. It is an open problem if a dense subgroup could have a highly transitive action and maybe this is obstructed by the existence of a mixed identity. This provides an interesting test case for [9, Question 6.1].

For the statement of the Hull–Osin-type result in the linear case, we need the following definition: For a countably infinite dimensional vector space V over \mathbb{F}_q , we define the normal subgroup $\mathrm{GL}_{\mathrm{fin}}(V) \coloneqq \{g \in \mathrm{GL}(V) \mid |g| < \infty\}$ of $\mathrm{GL}(V)$. This group consists of all the elements $g \in \mathrm{GL}(V)$ such that there is a subspace $W \leq V$ of finite codimension with $w.g = \lambda w$ for all $w \in W$ and a fixed scalar $\lambda \in \mathbb{F}_q^{\times}$.

Before we state the linear version of Theorem 3, we need the following preparatory lemma.

LEMMA 1. Let $g \in \operatorname{GL}_{\operatorname{fin}}(V)$ act as the scalar λ on the subspace $W \leq V$ of finite codimension $n \in \mathbb{N}$ and let $U \leq V$ be a complement of W, i.e. $U \oplus W = V$. Then $U' \coloneqq U + U.g$ is a g-invariant subspace of dimension $\leq 2n$. There is a complement $W' \leq W$ of U' in V.

PROOF. Take $u' = u + w \in U'$ arbitrarily for $u \in U$, $w \in W$. Then $w = u' - u \in U' + U = U'$, hence $u'.g = u.g + w.g = u.g + \lambda w \in U.g + U' = U'$, showing that U' is g-invariant. Define W' to be a complement to $U' \cap W$ in W. Then $U' \cap W' = U' \cap W \cap W' = \mathbf{0}$ by definition and $U' + W' = U' + (U' \cap W) + W' = U' + W = V$, so $U' \oplus W' = V$. This completes the proof.

Here comes the analogue of Theorem 3 for linear groups over finite field.

Theorem 5. Let V be a countably infinite dimensional vector space over \mathbb{F}_q and let $H \leq \operatorname{GL}(V)$ be a highly linearly transitive linear group. Then either

- (i) $H \cap GL_{fin}(V)$ is a highly transitive, locally finite, normal subgroup, or
- (ii) H has no mixed identities.

We will first prove a quantitative form, generalizing the result of Le Boudec–Matte Bon to linear groups.

THEOREM 6. Let V be a countably infinite dimensional vector space over a finite field. Let $k \in \mathbb{Z}_{\geq 1}$ and let $H \leq \operatorname{GL}(V)$ be a subgroup which is linearly k-transitive and admits a mixed identity of length l. Then, at least one of the following cases hold:

- (i) $H \cap GL_{fin}(V)$ is a locally finite, linearly $(\lceil k/2 \rceil 1)$ -transitive, normal subgroup of H, or
- (ii) $k \leq 2l$.

PROOF. The inequality in Case (ii) can be proven using Theorem 1 provided all non-trivial elements of H, and hence critical constants in a potential mixed identity $w \in H * \mathbf{F}_r$, are of length at least k/2. Indeed, k > 2l implies $|w|_{\text{crit}} - 1 \ge \lceil k/2 \rceil - 1 \ge l$ and we can apply Theorem 1 with d := 1 and D := k. Hence, any such w of length at most l is not a mixed identity, contradicting an assumption in the statement of the theorem. We conclude that $k \le 2l$ must hold.

Hence, in order to show Case (i), we may suppose that there exists a critical constant $c \in H$ of non-zero length less than k/2. We may assume $k \geq 3$. Using Lemma 1, we get for some $\lambda \in \mathbb{F}_q^{\times}$ that there exists a vector space V_1 of codimension k with complement $V_0 \cong \mathbb{F}_q^k$, such that c has a matrix of the form

$$c = \begin{pmatrix} c' & 0 \\ 0 & \lambda 1_{V_1} \end{pmatrix},$$

with respect to the decomposition $V = V_0 \oplus V_1$, where $c' \in GL_k(q)$ is a non-scalar matrix. Now, since H is linearly k-transitive, it contains elements of the form

$$g = \begin{pmatrix} g' & * \\ 0 & g'' \end{pmatrix}.$$

for arbitrary $g' \in GL_k(q)$. Thus, since $SL_k(q)$ is quasi-simple and c' is non-scalar, the normal subgroup of H generated by c will contain elements of the form as above with $g' \in SL_k(q)$ arbitrary. Together with the k-linear transitivity of H, this shows that the normal subgroup generated by c is k'-linearly transitive for $k' := \lceil k/2 \rceil - 1$. Indeed, any two linearly independent

sets $\{\alpha_1, \ldots, \alpha_{k'}\}$ and $\{\beta_1, \ldots, \beta_{k'}\}$ can be moved into V_0 by some element $h \in H$. Now, since 2k' < k, there exists some element $g' \in \operatorname{SL}(V_0) \cong \operatorname{SL}_k(q)$ exchanging the two sets $\{\alpha_1.h, \ldots, \alpha_{k'}.h\}$ and $\{\beta_1.h, \ldots, \beta_{k'}.h\}$. Let $g \in \langle\langle c \rangle\rangle$, which contains g' in its upper left corner. Then, this shows that $\alpha_i.hgh^{-1} = \beta_i$ for all $i \in \{1, \ldots, k'\}$. It is clear that the normal subgroup generated by c is contained in $H \cap \operatorname{GL}_{\operatorname{fin}}(V)$ and that this group is locally finite. This finishes the proof.

PROOF OF THEOREM 5. Since H is highly linearly transitive, we can apply Theorem 6 for any $k \in \mathbb{Z}_{\geq 1}$. In particular, the existence of a mixed identity implies Case (i) in Theorem 6 for arbitrary k. Hence, $H \cap \operatorname{GL}_{\operatorname{fin}}(V)$ is a highly transitive, locally finite, normal subgroup of H.

However, the statement of Theorem 5 is a bit stronger since it says that Cases (i) and (ii) are mutually exclusive. We will now show that Case (i) arises whenever H contains an element h such that $|h| < \infty$, i.e. whenever $H \cap \operatorname{GL}_{\operatorname{fin}}(V)$ is non-trivial. Let $W \leq V$ be a subspace of codimension n such that h acts as multiplication by the fixed scalar $\lambda \in \mathbb{F}_q^{\times}$ on W. For any $g \in \operatorname{GL}(V)$, the element $h^g \in \operatorname{GL}_{\operatorname{fin}}(V)$ acts as $\lambda \operatorname{id}_{W.g}$ on W.g. Hence [h,g] acts as the identity on $W_0 \coloneqq W \cap W.g$. Then $\operatorname{codim}(W_0) \leq 2n$. By Lemma 1, we find $W' \leq W_0$ and U' a [h,g]-invariant complement of W', such that $\dim(U') = \operatorname{codim}(W') \leq 4n$. Then [h,g] can be seen as an element of $\operatorname{GL}(U') \times \{1_{W'}\} \leq \operatorname{GL}_{4n}(q) \times 1$, so $[h,x]^e \in H * \langle x \rangle = H * \mathbf{F}_1$ is a mixed identity for H, where e denotes the exponent of $\operatorname{GL}_{4n}(q)$.

5. No mixed identities for A(q)

It was proved in [4, Proof of Theorem 1.10] that the topological full group of a faithful and minimal, continuous action of a group on a Cantor set is mixed identity free. This is essentially a direct application of the Hull-Osin theorem to the action of the full group on an infinite orbit. A similar reasoning can be applied to the study of full groups of an ergodic, probability measure preserving equivalence relation on a non-atomic standard probability space. In this section, we want to prove a linear version of this, where a certain group A(q), constructed in [5], replaces the full group.

Let's briefly recall the construction of A(q), which resembles a construction of the full group of the hyperfinite equivalence relation. Let q be a prime power as usual. For $n \in \mathbb{N}$, consider $GL_{2^n}(q)$ and the diagonal embeddings

$$\iota_n \colon \operatorname{GL}_{2^n}(q) \to \operatorname{GL}_{2^{n+1}}(q); h \mapsto \begin{pmatrix} h & 0 \\ 0 & h \end{pmatrix}.$$

Write $\operatorname{GL}_{2^{\omega}}(q) = \bigcup_{n \in \mathbb{N}} \operatorname{GL}_{2^n}(q)$ for the inductive limit of the countable chain of inclusions $(\operatorname{GL}_{2^n}(q), \iota_n)_{n \in \mathbb{N}}$. Clearly, this is a countable, locally finite group. Note that the normalized rank metric is well-defined on $\operatorname{GL}_{2^{\omega}}(q)$. We denote its Cauchy completion by A(q). Note that we could have started with $\operatorname{SL}_{2^n}(q)$ and arrived at the same completion. Furthermore, A(q)/Z, where Z denotes its center $Z \cong \mathbb{F}_q^{\times}$, is the completion of the inductive limit of the groups $\operatorname{PGL}_{2^n}(q)$ with respect to the normalized projective rank metric. See [5] for more details on the group A(q). Throughout this section, we work with the normalized projective rank metric and a suitable version of Theorem 1 and its corollaries.

Now we prove the main result of this section:

Theorem 7. The group A(q)/Z does not admit a mixed identity.

PROOF. Let

$$w = c_0 x_{\iota(1)}^{\varepsilon(1)} c_1 \cdots c_{l-1} x_{\iota(l)}^{\varepsilon(l)} c_l \in A(q) * \mathbf{F}_r$$

be some reduced word with constants, where $l \in \mathbb{Z}_{\geq 2}$. We prove that w has a non-trivial word image $w(A(q)^r)$. At first, note that, by construction, we can find a large number $n = n(\varepsilon) \in \mathbb{N}$, such that we can choose constants $c'_0, \ldots, c'_l \in \mathrm{SL}_{2^n}(q)$ so that $d(c'_j, c_j) \leq \varepsilon$ for all $i \in \{0, \ldots, l\}$. Define the word

$$w' \coloneqq c_0' x_{\iota(1)}^{\varepsilon(1)} c_1' \cdots c_{l-1}' x_{\iota(l)}^{\varepsilon(l)} c_l' \in \operatorname{SL}_{2^n}(q) * \mathbf{F}_r < A(q) * \mathbf{F}_r.$$

Note that then for all $h_1, \ldots, h_r \in A(q)$, when have

$$d(w'(h_{1},\ldots,h_{r}),w(h_{1},\ldots,h_{r}))$$

$$=d(c'_{0}h_{\iota(1)}^{\varepsilon(1)}c'_{1}\cdots c'_{l-1}h_{\iota(l)}^{\varepsilon(l)}c'_{l},c_{0}h_{\iota(1)}^{\varepsilon(1)}c_{1}\cdots c_{l-1}h_{\iota(l)}^{\varepsilon(l)}c_{l})$$

$$\leq d(c'_{0}h_{\iota(1)}^{\varepsilon(1)}c'_{1}\cdots c'_{l-1}h_{\iota(l)}^{\varepsilon(l)}c'_{l},c'_{0}h_{\iota(1)}^{\varepsilon(1)}c'_{1}\cdots c'_{l-1}h_{\iota(l)}^{\varepsilon(l)}c_{l})$$

$$+\cdots+d(c'_{0}h_{\iota(1)}^{\varepsilon(1)}c_{1}\cdots c_{l-1}h_{\iota(l)}^{\varepsilon(l)}c_{l},c_{0}h_{\iota(1)}^{\varepsilon(1)}c_{1}\cdots c_{l-1}h_{\iota(l)}^{\varepsilon(l)}c_{l})$$

$$\leq \sum_{i=0}^{l}d(c'_{j},c_{j})\leq (l+1)\varepsilon,$$

Hence, we obtain

$$|\operatorname{diam}(w'(A(q)^r)) - \operatorname{diam}(w(A(q)^r))| \le 2(l+1)\varepsilon.$$

From the definition $|w'|_{\text{crit}} \geq |w|_{\text{crit}} - \varepsilon$ and for $m \geq n$, Corollary 1 yields

$$\operatorname{diam}(w'(\operatorname{SL}_{2^m}(q)^r)) \ge \frac{|w'|_{\operatorname{crit}}}{l} - \frac{1}{2^m} \ge \frac{|w|_{\operatorname{crit}} - \varepsilon}{l} - \frac{1}{2^m}$$

and hence

$$\operatorname{diam}(w(\operatorname{SL}_{2^m}(q)^r)) \ge \frac{|w|_{\operatorname{crit}} - \varepsilon}{l} - \frac{1}{2^m} - 2(l+1)\varepsilon.$$

Thus, taking $\varepsilon > 0$ small enough and $m \ge n(\varepsilon)$ large enough, we see that the word image of w is non-trivial. This finishes the proof.

Acknowledgments

We are grateful to Melissa Lee for an enlightening discussion concerning Remark 1. The third author wants to thank the Mathematisches Forschungsinstitut in Oberwolfach for its hospitality.

References

- [1] Armand Borel, On free subgroups of semisimple groups, L'Enseignement mathématique (2) 29 (1983), no. 1-2, 151–164.
- [2] Henry Bradford and Andreas Thom, Short Laws for Finite Groups of Lie Type, arXiv:1811.05401 (2018), to appear in JEMS.
- [3] Henry Bradford, Jakob Schneider, and Andreas Thom, The length of mixed identities for finite groups, arXiv:2306.14532 (2023).
- [4] ______, On the length of non-solutions to equations with constants in some linear groups, arXiv:2306.15370 (2023).
- [5] Alessandro Carderi and Andreas Thom, An exotic group as limit of finite special linear groups, Annales de l'Institut Fourier, 2018, pp. 257–273.
- [6] Mahmood Etedadialiabadi, Su Gao, François Le Maître, and Julien Melleray, *Dense locally finite subgroups of automorphism groups of ultraextensive spaces*, Advances in Mathematics **391** (2021), 107966.
- [7] Nikolai Gordeev, Boris Kunyavskiĭ, and Eugene Plotkin, Word maps, word maps with constants and representation varieties of one-relator groups, Journal of Algebra 500 (2018), 390–424.
- [8] Christoph Hering, Transitive linear groups and linear groups which contain irreducible subgroups of prime order. II, Journal of Algebra 93 (1985), no. 1, 151–164.
- [9] Michael Hull and Denis Osin, Transitivity degrees of countable groups and acylindrical hyperbolicity, Israel Journal of Mathematics 216 (2016), no. 1, 307–353.
- [10] Gareth A. Jones, *Varieties and simple groups*, Journal of the Australian Mathematical Society **17** (1974), no. 2, 163–173.
- [11] Anton Klyachko and Andreas Thom, New topological methods to solve equations over groups, Algebraic & Geometric Topology 17 (2017), no. 1, 331–353.
- [12] Adrien Le Boudec and Nicolás Matte Bon, Triple transitivity and non-free actions in dimension one, Journal of the London Mathematical Society 105 (2022), no. 2, 884–908.
- [13] Melissa Lee, Private communication.
- [14] Michael Larsen and Aner Shalev, A probabilistic Tits alternative and probabilistic identities, Algebra Number Theory 10 (2016), no. 6, 1359–1371.
- [15] _____, A probabilistic Tits alternative and probabilistic identities, Algebra & Number Theory 10 (2016), no. 6, 1359–1371.

- [16] Viktor D. Mazurov and Evgeniĭ I. Khukhro (eds.), *The Kourovka notebook*, Eighteenth edition, Russian Academy of Sciences Siberian Division, Institute of Mathematics, Novosibirsk, 2014. Unsolved problems in group theory.
- [17] Sorin Popa, Free-independent sequences in type II_1 factors and related problems, Astérisque **232** (1995), 187–202. Recent advances in operator algebras (Orléans, 1992).
- [18] Jakob Schneider and Andreas Thom, Word maps with constants on symmetric groups, Mathematische Nachrichten (2023), available at https://doi.org/10.1002/mana.202300152.
- [19] Roland Zarzycki, *Limits of Thompson's group F*, Combinatorial and Geometric Group Theory: Dortmund and Ottawa-Montreal Conferences, 2010, pp. 307–315.
 - H. Bradford, Univ. of Cambridge, Cambridge CB3 0WB, Unites Kingdom *Email address*: hb470@cam.ac.uk
 - J. Schneider, TU Dresden, 01062 Dresden, Germany *Email address*: jakob.schneider@tu-dresden.de

A. THOM, TU DRESDEN, 01062 DRESDEN, GERMANY *Email address*: andreas.thom@tu-dresden.de