FINA4354 – Financial Engineering

<u>Summary report</u>

Step-Up Barrier-Protected S&P 500 ELN

2020-21 Semester 2

Faculty of Business and Economics

The University of Hong Kong

Group 2

Fu Xipeng	3035447805
Kong Wong Kai	3035478373
Shiu Chung Haang	3035483653
Tan Zhini	3035478361

Step-Up Barrier-Protected S&P 500 ELN

1. Economic motivation

After the sharp and sudden crash of equity markets in March of 2020 due to COVID-19, the stock market bounced back quickly under the large fiscal stimulus. Currently, with vaccine rollout and expectation of economic recovery, S&P 500 Index continued to hit new highs. Many investors might have a bullish view of U.S. equity markets.

However, volatility also expanded under the huge uncertainties. Firstly, S&P 500 PE ratio is high at current level, indicating a limited upside, and a correction might soon be seen in the market. Secondly, as U.S. treasury yield rises, stock market becomes less attractive, especially when U.S. equity is already expensive in absolute terms. In addition, the resurgence of COVID-19 cases globally, such as the situation in India nowadays, the hike of the capital gains tax under Biden's administration and the rise of geopolitical tensions, all contribute to the higher uncertainties and volatilities in the market.

Under such a special economic environment, we created a product that provides downside protection to investors while enabling them to gain in the bullish markets.

2. Product Design

2.1 Product Description

We introduce our product from several perspectives.

1. **Underlying asset**
   We use the S&P 500 index as the underlying because it has strong indications of bullishness and volatility. Over the tenor of 6 months, the expected return is 8.75% and the volatility is 23.01%.

2. **Price and face value**
   To standardize the product, we set both the price P and face value F to \$100.

3. **Term to maturity**
   We set the term to maturity T = 6 months since ELN products on the market usually have short tenors of 1 to 6 months. Due to the high-risk feature of ELN, investors prefer short-term tenor as they can detract from the market easily in case of losses.

4. **Payoff**
   At the maturity date t = T, a payoff X = RF is given, where the value of R satisfies:

Value of R	Condition	Explanation
l_2	$inf_{0\le t\le T}\{S_t/S_0\} < l_1$ And $S_T/S_0 < l_2$	Fixed loss if barrier is triggered
S_T/S_0	$inf_{0\le t\le T}\{S_t/S_0\} \ge l_1$ And $S_T/S_0 < l_2$	Floating loss if barrier is not triggered
	$l_2 \le S_T/S_0 < g_1$	Floating gain/loss
g_1	$g_1 \le S_T/S_0 < g_2$	Fixed gain, 1st step
$g_1 + h$	$g_2 \le S_T/S_0 < g_3$	Fixed gain, 2nd step
$g_1 + 2h$	$S_T/S_0 \ge g_3$	Fixed gain, 3rd step

Explanation of parameters:

- S_t: index price at time t, S_T/S_0 = 1 + index return
- l_1: barrier level, l_2: strike level (both are for the down-and-in European put)
- g_1, g_2, g_3: boundary points of the steps
- h: step size

5. **Commission fee**
   We take away 2% of the product's price as commission fee. i.e. commission fee = $0.02P$ = \$2 if the investor purchases one unit of this product.

2.2 Product features

1. **Step-up return**
   In normal market conditions, our product provides a high return as other ELN products do. However, if the market performs even better than the expectation, our product can provide an even higher return, a maximum of two improvements (3 steps) in total.
2. **Large protection in case of market crash**
   We provide extra protection to investors in case of market crash. When the price falls greatly below a level (e.g., 70% of starting price), a barrier will be triggered to guarantee that the loss is not lower than a floor (e.g., 85% of starting price). The floor is higher than the barrier, meaning that once the barrier is triggered, investors can have reduced losses.
3. **Dividend effect is considered**
   Since S&P 500 is a dividend-paying index, we have improved our model based on traditional B-S model and included the dividend factor, making it more realistic.

3. Product structure

Note that the replicating portfolio is based on S_0, initial index points, while the product is standardized to P = \$100. Therefore, in the following sections, we adopt a conversion rate of 1 replicating portfolio = (S_0 / P) standardized products.

3.1 Decomposition of product

To understand the payoff more clearly, we separate the scenarios into two: 1) when $S_T < S_0$, and 2) when $S_T > S_0$, where S_T = underlying price at maturity and S_0 = current stock price.

1. **If $S_T < S_0$,** the buyer faces a loss. Throughout the holding period, if the underlying price has touched the barrier, then the price barrier becomes valid and the maximum loss the buyer is limited. To create such payoff condition, a long position on European Down-and-In put option will be needed. Otherwise, a long position on stock suffices.
2. **If $S_T > S_0$,** the buyer receives a profit. To create a ladder-like payoff, a short position on European Vanilla call option, and two long positions on European Binary call options at different strike levels will be needed.

The replication process is shown below:

Cases	Payoff from stock	Options that provide extra payoff	Extra payoff	Total payoff
1. $S_T < S_0$, barrier option never triggers	S_T	Barrier option (= N/A)	0	S_T
2. $S_T < S_0$, barrier option triggers		Barrier option (= European put)	$\max(l_2 S_0 - S_T, 0)$	$S_T + \max(l_2 S_0 - S_T, 0)$
3. $S_T > S_0$, $S_T < g_1 S_0$		N/A	0	S_T
4. $S_T > S_0$, $g_1 S_0 \leq S_T < g_2$		Short call	$-\max(S_T - g_1 S_0, 0)$	$S_T - \max(S_T - g_1 S_0, 0)$
5. $S_T > F$, $g_2 S_0 \leq S_T < g_3 S_0$		Short call, Digital option	$-max(S_T - g_1 S_0, 0)$ $+hS_0$	$S_T - \max(S_T - g_1 S_0, 0)$ $+hS_0$
6. $S_T > F, S_T \geq g_3 S_0$		Short call, Digital option (2 types)	$-\max(S_T - g_1 S_0, 0)$ $+2hS_0$	$S_T - \max(S_T - g_1 S_0, 0)$ $+2hS_0$

By dividing the payoff by S_0, the results perfectly match values of R in section 2.1.4.

4. Valuation Methodology

This product can be priced based on its decomposition. Specifically, for a long position in one unit of our product, it equals to **long 1 stock, long 1 down-and-in European put, short 1 European call, long hS_0 units of 1st binary call and long hS_0 units of 2nd binary call.**

4.1 Pricing approach

We use the GBM model with dividends to price our product. In a Black-Scholes environment, the underlying is expected to follow a lognormal random walk process, which is expressed as follows:

$$dS_t = (r - q)S_t dt + \sigma S_t dB_t$$
$$where\ B_t\ is\ a\ Wiener\ process.$$

By Ito's lemma, the underlying price at maturity can be represented as follows:

$$S_T = S_0 e^{\left(r - q - \frac{\sigma^2}{2}\right)T + \sigma dB_T^{\mathbb{Q}}}$$
$$where\ B_T\ is\ a\ Wiener\ process\ under\ \mathbb{Q} - martingale.$$

1. European Vanilla Option

The European Vanilla Option Price is a closed-form function which can be represented as follows:

$$Option\ Price = \omega S_0 e^{-qT}\Phi(\omega d_1) - \omega K e^{-rT}\Phi(\omega d_2)$$
$$where\ \omega = \begin{cases} 1\ if\ it\ is\ call\ option \\ -1\ if\ it\ is\ put\ option' \end{cases}$$

2. European Vanilla Binary Option

The European Vanilla Binary Price is a closed-form function which can be represented as follows:

$$Option\ Price = N\left(\omega e^{-rT}\Phi(\omega d_2)\right)$$
$$where\ \omega = \begin{cases} 1\ if\ it\ is\ call\ option \\ -1\ if\ it\ is\ put\ option' \end{cases}$$

3. European Down-and-In Barrier Put Option

The European Down-and-In Barrier Put Price is a closed-form function which can be represented as follows:

$$Option\ price = \begin{cases} \left(\frac{H}{S_0}\right)^{\frac{2v}{\sigma^2}} \left(C\left(\frac{H^2}{S_0}, K\right) - C\left(\frac{H^2}{S_0}, H\right) - (H-K)e^{-rT}\Phi\big(d_2(H,S)\big) \right) + \Big(P(S_0,H) - (H-K)e^{-rT}\Phi\big(-d_2(S_0,H)\big)\Big) & when\ K > H \\ P(S_0,K)\ otherwise & \end{cases}$$

$$where\ C(x,y) = S_0 e^{-qT}\Phi\big(d_1(x,y)\big) - Ke^{-rT}\Phi\big(d_2(x,y)\big),$$
$$P(x,y) = -S_0 e^{-qT}\Phi\big(-d_1(x,y)\big) + Ke^{-rT}\Phi\big(-d_2(x,y)\big),$$
$$v = r - q - \frac{1}{2}\sigma^2, \qquad H\ is\ the\ barrier\ price,$$

4. **Overall Product Price**

In overall, the aggregation can be represented as follows:

$$Overall\ Price = S_0 + C_{barrier}(S_0, g_0S_0, H, r, q, \sigma, T) - C_{European}(S_0, g_1S_0, r, q, \sigma, T) + hS_0\left(C_{digital1}(S_0, g_2S_0, r, q, \sigma, T) + C_{digital2}(S_0, g_3S_0, r, q, \sigma, T)\right)$$

4.2 Parameter estimation

1. **Step range parameters g_1, g_2, g_3**

$$g_i = 1 + i \times \mu_{S\&P500,6mo}, for\ i = 1, 2, 3.$$

Where μ is the mean, or expected return of the underlying over the 6-month tenor. We applied such setting to ensure the step ranges reflect the investors' expectation on the underlying. Below the 1st step ($< g_1 = 1 + \mu$) the index fails to meet investors' expectation, while in the 1st, 2nd and 3rd step, the index surpasses investors' expectation, and extra reward is given according to the level of surprise over the expectation.

2. **Barrier and strike level l_1, l_2**

$$l_1 = 0.7, l_2 = 0.85$$

These two parameters are given as a reference and can be adjusted freely, if the issuers consider it necessary to do so.

3. **Step size h**

When assumed 2% commission fee, overall price = $0.98S_0$, so according to the formula, we have

$$h = \frac{C_{European} - C_{barrier} - 0.02S_0}{S_0\left(C_{digital1} + C_{digital2}\right)}$$

According to the calculation using R, $h = 4.90\%$, which is an acceptable number.

5. Hedging Strategy

5.1 General Approach - Delta Hedging

Since our product consists of five components, it is difficult for issuers to hedge their position with one single derivative in the market. Dolgova (2006) and Gallus (1999) have proved the effectiveness of delta hedging for exotic option in a frictionless market. Therefore, we propose delta hedging for issuers to hedge our product.

To calculate the delta of our product, we can simply apply first-order derivative to corresponding equations with respect to the underlying asset price:

$$\Delta = \frac{\partial P_t}{\partial S_t}, where\ P_t\ is\ the\ option\ price\ and\ S_t\ is\ the\ underlying\ price.$$

and add them up the find the total delta. Using R, the calculated total $\Delta = 0.7038$, which is an acceptable number.

Note that the calculated Δ represents the position of the underlying, which is an index and cannot be traded directly. Since the SPDR S&P 500 Trust ETF (SPY) is a good replication of S&P 500 index, the position of Δ underlyings is equivalent to $\Delta \times (S_t / S_{t,SPY})$, where $S_{t,SPY}$ is the value of SPY at time t.

However, there exist some problems in delta hedging:

1. Frequent re-adjustments of the delta position is required. This may incur additional transaction costs for banks.
2. Delta for barrier option will become very sensitive when time to maturity becomes shorter and the underlying pricing is near the barrier price.
3. Binary option has a sensitive delta when the price is approaching the strike price. It means that the volatility for delta hedging is very high at that time.

5.2 Alternative Approach

The alternative approach is to separate the structured product into five different components and hedge each component separately. For some components, alternate hedging methods exist and can be used to replace delta hedging in that component. Details of some suggested alternative hedging strategies are shown below:

1. Short call

European calls and puts for S&P 500 exist on the market and can be used conveniently. The short call can be done by shorting a corresponding European call with strike $g_1 S_0$ at t = 0.

2. Long DI European put

Instead of delta-hedging, another risky way is to do nothing at the beginning, and only long a European put with strike $l_2 S_0$ when there signals of a large market crash appear. The reason is that DI European put option is valuable only if the option is triggered. However, this hedging strategy brings another problem. If such signals really appear, put option can become very expensive.

3. Binary calls

For each of the two binary options, a practical way to hedge is to long multiple bullish spreads: n x (stock + put - call) at t = 0, where n is a large number. The problem is that the bullish spreads can only partly hedge the exposure. To improve the accuracy of hedging, issuers need to use a very large n, which will largely increase the transaction costs.

6. Reference

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