



Asymmetric Quantum Devices and Heat Transport

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“It ain’t much, but it’s honest work.”

Someone who you admire or hate

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List of publications

I) The results of this Thesis are based on the following articles

Published Articles

1. M. Pons, Y. Y. Cui, A. Ruschhaupt, **M. A. Simón** and J. G. Muga
Local rectification of heat flux
[EPL 119, 64001 \(2017\)](#).
2. A. Ruschhaupt, T. Dowdall, **M. A. Simón** and J. G. Muga
Asymmetric scattering by non-Hermitian potentials
[EPL 120, 20001 \(2017\)](#).
3. **M. A. Simón**, A. Buendía and J. G. Muga
Symmetries and Invariants for Non-Hermitian Hamiltonians
[Mathematics 6, 111 \(2018\)](#).
4. **M. A. Simón**, A. Buendía, A. Kiely, A. Mostafazadeh and J. G. Muga
S-matrix pole symmetries for non-Hermitian scattering Hamiltonians
[Phys. Rev. A 99, 052110 \(2019\)](#).
5. **M. A. Simón**, S. Martínez-Garaot, M. Pons and J. G. Muga
Asymmetric heat transport in ion crystals
[Phys. Rev. E 100, 032109 \(2019\)](#).
6. A. Alaña, S. Martínez-Garaot, **M. A. Simón** and J. G. Muga
Symmetries of ($N \times N$) non-Hermitian Hamiltonian matrices
[J. Phys. A: Math. Theor. 53, 135304 \(2020\)](#).

7. A. Ruschhaupt, A. Kiely, **M. A. Simón** and J. G. Muga
Quantum-optical implementation of non-Hermitian potentials for asymmetric scattering
Phys. Rev. A **102**, 053705 (2020).
8. **M. A. Simón**, A. Alaña, M. Pons, A. Ruiz-García and J. G. Muga
Heat rectification with a minimal model of two harmonic oscillators
Phys. Rev. E **103**, 012134 (2021).

II) Other articles produced during the Thesis period

Published Articles not included in this Thesis

1. M. Palmero, **M. A. Simón** and D. Poletti
Towards Generation of Cat States in Trapped Ions Set-Ups via FAQUAD Protocols and Dynamical Decoupling
Entropy **21**, 1207 (2019)
2. **M. A. Simón**, M. Palmero, S. Martínez-Garaot and J. G. Muga
Trapped-ion Fock-state preparation by potential deformation
Phys. Rev. Research **2**, 023372 (2020)

Introduction

Devices that are able to control the flow of energy or matter have a prominent role in technology. A key device is the rectifier, which allows currents only in one direction. A rectifier can be thought of as some kind of corridor with a trap door that you can open from left to right but is closed if you try to go back. The most notable of such kind of devices is the electric diode, which is a vital part of computers, digital devices and AC/DC current conversion systems, to name a few. With the previous examples it is easy to understand why the electric diode is so relevant: without it, much of the technology that we have today would not exist.

The electric diode is an electrical component that allows electric current to flow in one direction but acts as an insulator in the opposite one. Typically, the fundamental part in a diode is a *p-n* junction [2], which is a physical union of two different types of semiconductor materials. The first one of these semiconductors is a *p*-type semiconductor, *i.e.*, a semiconductor that has been doped with donor impurities and hence, electrical current is mainly transported by electrons. The other semiconductor is a *n*-type semiconductor *i.e.*, a semiconductor that has been doped with acceptor impurities and hence, electrical current is mainly transported by electron holes (electron vacancies with positive charge). In the absence of an external applied voltage, the junction reaches an equilibrium state in which the region near the two semiconductors interface is emptied of free charge carriers and an electric potential difference forms between the *p* and *n* semiconductors, with the *n* semiconductor being at higher potential. If a battery positive pole is connected to the *p*-side of the junction and its negative pole to the *n*-side of the junction, the potential difference between the semiconductors will decrease. If the voltage of the battery is increased further, the potential difference in the junction will eventually change its sign, allowing for a current to pass through the junction. If however, the junction is connected to the battery with its negative pole attached to the *p*-side and its positive pole attached to the *n*-side, the potential

barrier in the junction will be increased for the free charge carriers. As a result, the *p-n* junction allows current to flow through the junction only when a forward bias voltage, *i.e.* the p-side connected to the positive pole, is applied to it, and acts as an insulator otherwise.

Because of the technological impact of the diode, people have tried to find analog devices in other scenarios. For example, in optics there exists a device called optical isolator, that is used to allow one-way light propagation [1]. The functioning of this device is based on the Faraday effect, a rotation of the plane of polarization of a light wave when it traverses a material in which there is a magnetic field [3]. The angle of rotation θ_F per unit length l of the polarization plane is $\frac{d\theta_F}{dl} = VB$ where B is the component of the magnetic field along the direction of propagation of the light and V is a material-dependent constant called Verdet constant. The Faraday effect is not reciprocal, meaning that the rotation angle of the polarization plane depends on the incidence direction of the light. If the light incides in the material in the direction of the magnetic field, the polarization rotates clockwise. However, if the light incides in the direction opposite to the magnetic field, the polarization rotates anticlockwise. The optical isolator is composed by a Faraday rotator (a piece of material with an applied magnetic field) which is placed between two linear polarizers A and B at a relative angle of 45° to each other. The magnetic field B applied to the Faraday rotator is directed from A to B and has a magnitude such that the polarization rotates 45° . Light incoming from the left will be polarized by A , rotate 45° and then exit the isolator unaltered since its polarization matches that of B . However light incoming from the right gets polarized by B and rotates an additional 45° in the rotator ending up with a polarization perpendicular to A which blocks the propagation. The optic isolator is used in optics to prevent reflected light from coming back to the source, which could damage it.

At this point we can see that a common ingredient between devices which show a diode-like behaviour is some kind of internal structural asymmetry. In the case of the electric diode this asymmetry comes from the asymmetric distribution of charge carriers: electrons on the *n*-side, and holes in the *p*-side. In the case of the optical isolator the orientation of the magnetic field breaks the symmetry of the system.

This thesis is devoted to explore the possibility of designing nano devices that implement a *diodic* or rectifying mechanism to allow an asymmetric transport of

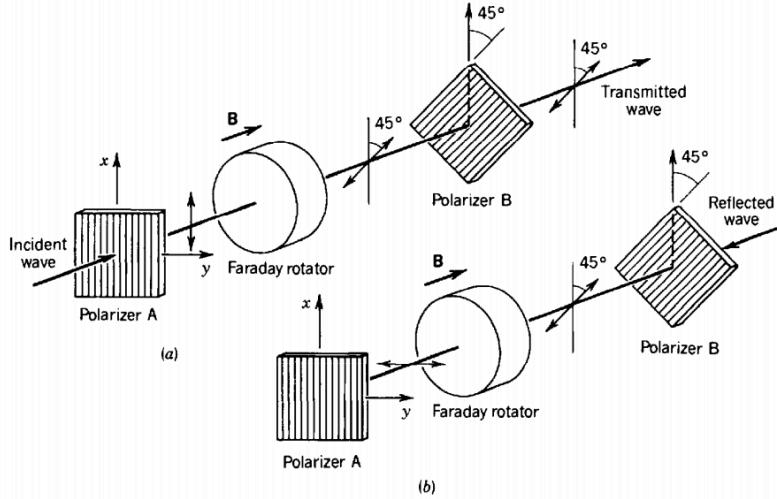


FIGURE 1: Faraday insulator, taken from [1]. Light traveling from left to right is transmitted while light propagating in the opposite direction is blocked by the polarizer A .

matter and energy. I decided to divide this thesis into two parts according to the context in which I look for asymmetric transport and particular objectives.

In the first part I work with quantum 1-dimensional scattering. The objective of this part is to design scattering potentials that lead to transmission and reflection coefficients which differ for wave packets coming from the left or the right. I will refer to this kind of potential as an atom diode, since ideally I want a potential that acts as a one-way barrier for quantum particles. In doing so I will explore also properties of non-Hermitian Hamiltonians, which are necessary for asymmetric scattering. The content of this part will be organized in the following way. In Chapter 1 I will give a brief introduction to scattering theory in 1 dimension to continue with an exposition of different asymmetric scattering devices. I will discuss how different symmetries of the scattering Hamiltonian pose restrictions on what kind of devices can be designed. In particular, non-Hermitian and non-local Hamiltonians are required to obtain asymmetric scattering. Next I will present some scattering potentials that are designed ad-hoc to obtain asymmetric devices and study their performance as a function of incident momenta. In Chapter 2 I will focus on the mathematical properties of non-local and non-Hermitian Hamiltonians. I extend the concepts of parity, time reversal and PT symmetry (parity and time reversal combined together) to an extended set of symmetry transformations with group structure. When the Hamiltonian is invariant with respect to some of these transformations, the poles of the Hamiltonian are symmetric with respect to

the imaginary axis of the complex plane. In Chapter 3 I present a possible physical realization for an asymmetric scattering Hamiltonian in a quantum optics setup.

In the second part I study heat rectification in mesoscopic systems. The objective in this part is to design an analog device to the diode that allows one-way heat propagation. In Chapter 4 I design a thermal diode composed of a chain of non-linear oscillators with a localized defect. In Chapter 5 I design a thermal diode which can be implemented on a trapped ions setup. It consists on a chain of individually trapped ions with a graded profile of the trapping frequency. This setup seizes long range interactions, due to the Coulomb force, to increase rectification power. In Chapter 6 I study the requirements for having heat rectification with a solvable model. It is generally accepted that non-linear interactions are needed in order to have rectification, since a system with non-linear interactions will have a temperature-dependent spectrum. I demonstrate, however that it is possible to have temperature dependent features in a linear system that lead to rectification.

In the last chapter of this thesis I outline the main results and state the conclusions.

Part I

Non-hermitian systems and asymmetric scattering

Introduction to Part I

Chapter 1

Asymmetric scattering by non-Hermitian potentials

The scattering of quantum particles by non-hermitian (generally nonlocal) potentials in one dimension may result in asymmetric transmission and/or reflection from left and right incidence. After extending the concept of symmetry for non-hermitian potentials, eight generalized symmetries based on the discrete Klein's four-group (formed by parity, time reversal, their product, and unity) are found. Together with generalized unitarity relations they determine selection rules for the possible and/or forbidden scattering asymmetries. Six basic device types are identified when the scattering coefficients (squared moduli of scattering amplitudes) adopt zero/one values, and transmission and/or reflection are asymmetric. They can pictorially be described as a one-way mirror, a one-way barrier (a Maxwell pressure demon), one-way (transmission or reflection) filters, a mirror with unidirectional transmission, and a transparent, one-way reflector. We design potentials for these devices and also demonstrate that the behavior of the scattering coefficients can be extended to a broad range of incident momenta.

1.1 Introduction

The current interest to develop new quantum technologies is boosting applied and fundamental research on quantum phenomena and on systems with potential applications in logic circuits, metrology, communications or sensors. Robust basic devices performing elementary operations are needed to perform complex tasks when combined in a circuit.

In this paper we investigate the properties of potentials with asymmetric transmission or reflection for a quantum, spinless particle of mass m satisfying a one-dimensional (1D) Schrödinger equation. If we restrict the analysis to transmission and reflection coefficients (squared moduli of the scattering complex amplitudes) being either zero or one, a useful simplification for quantum logic operations, there are six types of asymmetric devices, see fig. 1.1. These devices cannot be constructed with Hermitian potentials. In fact for all device types with transmission asymmetries, which are four of the six possible devices, the potentials have to be also nonlocal. Therefore, nonlocal potentials play a major role in this paper. They appear naturally when applying partitioning techniques under similar conditions to the ones leading to non-hermitian potentials, namely, as effective interactions for a subsystem or component of the full wave-function, even if the interactions for the large system are hermitian and local [4].

Symmetries can be used, similarly to their standard application in atomic physics to determine selection rules for allowed/forbidden transitions, to predict whether a certain potential may or may not lead to asymmetric scattering. The concept of symmetry, however, must be generalized when dealing with non-hermitian potentials.

The theory in this paper is worked out for particles and the Schrödinger equation but it is clearly of relevance for optical devices due to the much exploited analogies and connections between Maxwell's equations and the Schrödinger equation, which were used, e.g., to propose the realization of PT-symmetric potentials in optics [5].

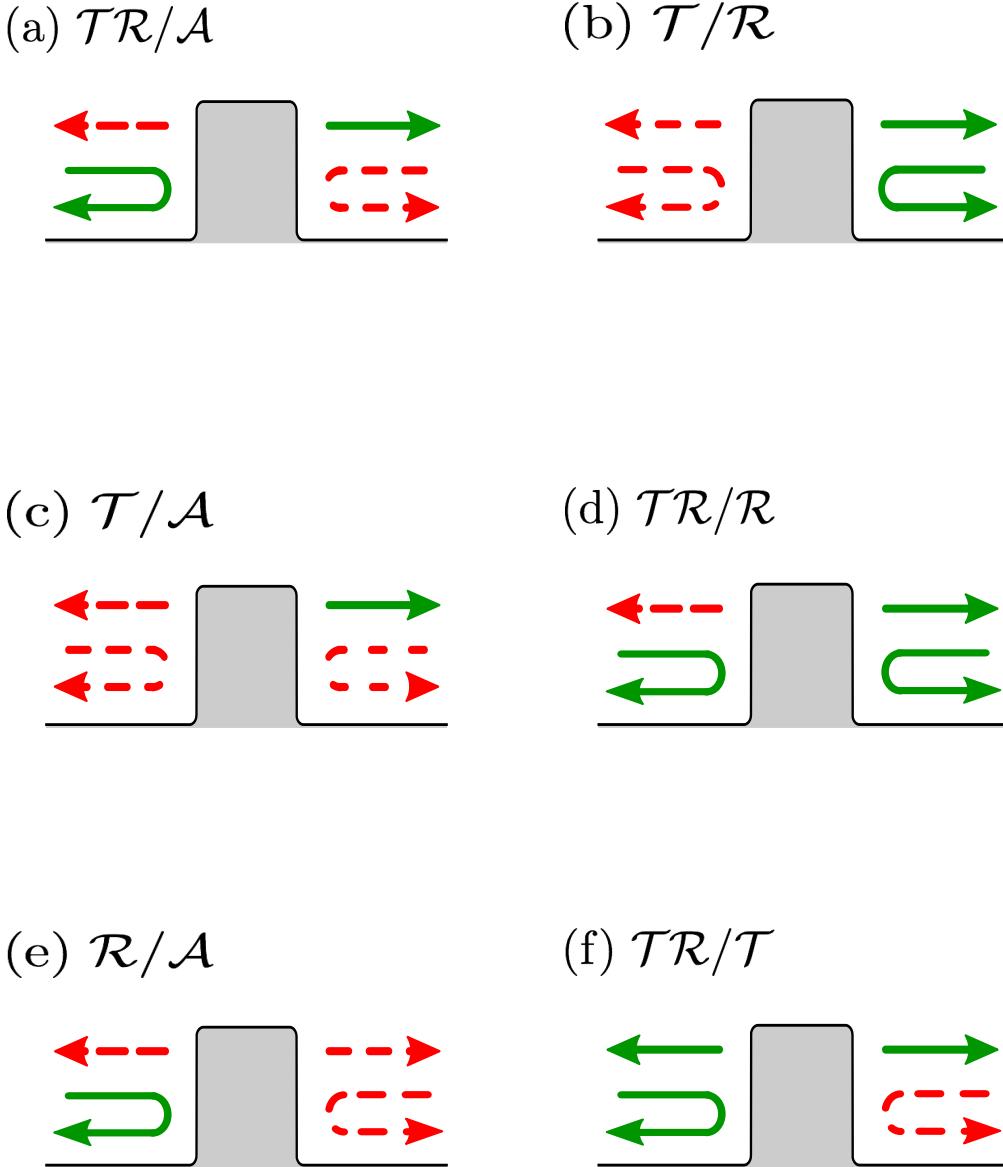


FIGURE 1.1: (Color online) Devices with asymmetric scattering (limited to scattering coefficients being 0 or 1). The dashed and continuous lines represent respectively zero or one for the moduli of the scattering amplitudes; the bended lines are for reflection amplitudes, and the straight lines for transmission: (a) One-way mirror ($\mathcal{T}\mathcal{R}/\mathcal{A}$); (b) One-way barrier (\mathcal{T}/\mathcal{R}); (c) One-Way T-filter (\mathcal{T}/\mathcal{A}); (d) Mirror & 1-way transmitter ($\mathcal{T}\mathcal{R}/\mathcal{R}$); (e) One-way R-filter (\mathcal{R}/\mathcal{A}); (f) Transparent, one-way reflector ($\mathcal{T}\mathcal{R}/\mathcal{T}$). The letter codes summarize the effect of left and right incidence, separated by a slash “/”. \mathcal{T} or \mathcal{R} on one side of the slash indicate a unit transmission or reflection coefficient for incidence from that side, whereas the absence of one or the other letter corresponds to zero coefficients. An \mathcal{A} denotes “full absorption”, i.e., both moduli of reflection and transmission amplitudes are zero for incidence from one side. For example, $\mathcal{T}\mathcal{R}/\mathcal{A}$ means unit modulus transmission and reflection from the left and total absorption from the right.

TABLE 1.1: Symmetries of the potential classified in terms of the commutativity or pseudo-hermiticity of H with the elements of Klein's 4-group $\{1, \Pi, \theta, \Pi\theta\}$ (second column). The first column sets a simplifying roman-number code for each symmetry. The relations among potential matrix elements are given in coordinate and momentum representations in the third and fourth columns. The fifth column gives the relations they imply in the matrix elements of S and/or \hat{S} matrices (S is for scattering by H and \hat{S} for scattering by H^\dagger). From them the next four columns set the relations implied on scattering amplitudes. Together with generalized unitarity relations (1.3) they also imply relations for the moduli (tenth column), and phases (not shown). The last two columns indicate the possibility to achieve perfect asymmetric transmission or reflection: “ P ” means possible (but not necessary), “No” means impossible. In some cases “ P ” is accompanied by a condition that must be satisfied.

Code	Symmetry	$\langle x V y\rangle =$	$\langle p V p'\rangle =$	$\langle p S p'\rangle =$	$T^l =$	$T^r =$	from eq. (1.3)	$ T^l =1$ $ T^r =0$	$ R^l =1$ $ R^r =0$
I	$1H = H1$	$\langle x V y\rangle$	$\langle p V p'\rangle$	$\langle p S p'\rangle$	T^l	R^l	$ T^l = T^r , R^l = R^r $	P	P
II	$1H = H^\dagger 1$	$\langle y V x\rangle^*$	$\langle p' V p\rangle^*$	$\langle p' S p\rangle$	\hat{T}^l	\hat{R}^l	$ T^l = T^r , R^l = R^r $	No	No
III	$\Pi H = H\Pi$	$\langle -x V -y\rangle$	$\langle p' V -p'\rangle$	$\langle -p S -p'\rangle$	T^l	R^l	$ T^l = T^r , R^l = R^r $	No	No
IV	$\Pi H = H^\dagger \Pi$	$\langle -y V -x\rangle^*$	$\langle -p' V -p\rangle^*$	$\langle -p \hat{S} -p\rangle$	\hat{T}^l	\hat{R}^l	$ T^l = T^r , R^l = R^r $	$P, R^r R^{l*} = 1$	$P, T^r T^{l*} = 1$
V	$\Theta H = H\Theta$	$\langle x V y\rangle^*$	$\langle -p V p\rangle^*$	$\langle -p' \hat{S} -p\rangle$	\hat{T}^l	\hat{R}^l	$ R^l = R^r $	$P, R^r R^{l*} = 1$	No
VI	$\Theta H = H^\dagger \Theta$	$\langle y V x\rangle$	$\langle -p' V -p\rangle$	$\langle -p' S -p\rangle$	T^l	R^l	$ T^l = T^r $	$P, R^{rl} =1$	No
VII	$\Theta\Pi H = H\Theta\Pi$	$\langle -x V -y\rangle^*$	$\langle p V p'\rangle$	$\langle p' \hat{S} p\rangle$	\hat{T}^l	\hat{R}^l	$ T^l = T^r $	No	$P, T^{rl}=1$
VIII	$\Theta\Pi H = H^\dagger \Theta\Pi$	$\langle -y V -x\rangle$	$\langle p' V p\rangle$	$\langle p' S p\rangle$	T^l	R^l	$ R^l = R^r $	P	No

1.2 Generalized symmetries

The detailed technical and formal background for the following can be found in a previous review on 1D scattering by complex potentials [4], a companion to this article for those readers willing to reproduce the calculations in detail. The Supplemental Material (Sec. I) provides also a minimal kit of scattering theory formulae that may be read first to set basic concepts and notation. The notation is essentially as in [4], but it proves convenient to use for the potential matrix (or kernel function) in coordinate representation two different forms, namely $\langle x|V|y\rangle = V(x, y)$. “Local” potentials are those for which $V(x, y) = V(x)\delta(x - y)$.

For hermitian Hamiltonians, symmetries are represented by the commutation of a symmetry operator with the Hamiltonian. In scattering theory, symmetry plays an important role as it implies relations among the S-matrix elements beyond those implied by its unitarity, see e.g. [6] and, for scattering in one dimension, Sec. 2.6 in [4].

Symmetries are also useful for non-hermitian Hamiltonians, but the mathematical and conceptual framework must be generalized. We consider that a unitary or antiunitary operator A represents a symmetry of H if it satisfies at least one of these relations,

$$AH = HA, \quad (1.1)$$

$$AH = H^\dagger A. \quad (1.2)$$

For a right eigenstate of H , $|\psi\rangle$, with eigenvalue E , eq. (1.1) implies that $A|\psi\rangle$ is also a right eigenstate of H , with the same eigenvalue if A is unitary, and with the complex conjugate eigenvalue E^* if A is antiunitary. Equation (1.2) implies that $A|\psi\rangle$ is a right eigenstate of H^\dagger with eigenvalue E for A unitary or E^* for A antiunitary, or a left eigenstate of H with eigenvalue E^* for A unitary, or E for A antiunitary. For real-energy scattering eigenfunctions in the continuum, the ones we are interested in here, $E^* = E$. When eq. (1.2) holds we say that H is A -pseudohermitian [7]. Parity-pseudohermiticity has played an important role as being equivalent to space-time reflection (PT) symmetry for *local* potentials [7, 8]. A large set of these equivalences will be discussed below. A relation of the form (1.2) has been also used with differential operators to get real spectra beyond PT-symmetry for local potentials [9, 10].

TABLE 1.2: Equivalences among symmetries for the potential elements. Given the symmetry of the upper row, the table provides the equivalent symmetries. For example, if II is satisfied, then III=IV holds. In words, if the potential is hermitian, parity symmetry amounts to parity pseudohermiticity. In terms of the matrix elements of the potential, if $\langle x|V|y\rangle = \langle y|V|x\rangle^*$ and also $\langle x|V|y\rangle = \langle -x|V|y\rangle$, $\forall(x,y)$, then $\langle x|V|y\rangle = \langle -y|V|-x\rangle^*$ holds as well. One may proceed similarly for all other relations. The commutation with the identity (I) is excluded as this symmetry is satisfied by all potentials.

II	III	IV	V	VI	VII	VIII
III=IV	II=IV	II=III	II=VI	II=V	II=VIII	II=VII
V=VI	V=VII	V=VIII	III=VII	III=VIII	III=V	III=VI
VII=VIII	VI=VIII	VI=VII	IV=VIII	IV=VII	IV=VI	IV=V

Here we consider A to be a member of the Klein 4-group $K_4 = \{1, \Pi, \Theta, \Pi\Theta\}$ formed by unity, the parity operator Π , the antiunitary time-reversal operator Θ , and their product $\Pi\Theta$. This is a discrete, abelian group. We also assume that the Hamiltonian is of the form $H = H_0 + V$, with H_0 , the kinetic energy operator of the particle, being hermitian and satisfying $[H_0, A] = 0$ for all members of the group, whereas the potential V may be non-local in position representation. The motivation to use Klein's group is that the eight relations implied by eqs. (1.1) and (1.2) generate all possible symmetries of a non-local potential due to the identity, complex conjugation, transposition, and sign inversion, both in coordinate or momentum representation, see table 1.1, where each symmetry has been labeled by a roman number. Interesting enough, in this classification hermiticity (symmetry II in table 1.1) may be regarded as 1-pseudohermiticity.

Examples on how to find the relations in the fifth column of table 1.1 of S - and \widehat{S} -matrix elements (for scattering by H and H^\dagger respectively) are provided in ref. [4], where the symmetry types III, VI, and VII were worked out. Similar manipulations, making use of the action of unitary or antiunitary operators of Klein's group on Möller operators, help to complete the table.

From the fifth column in table 1, equivalences among the amplitudes for left and right incidence for scattering by H , $(T^{l,r}, R^{l,r})$ or H^\dagger ($\widehat{T}^{l,r}, \widehat{R}^{l,r}$), are deduced, see the Supplemental Material and the four columns for $T^{l,r}$, and $R^{l,r}$ in table 1.1. Together with the generalized unitarity relations $\widehat{S}^\dagger S = S \widehat{S}^\dagger = 1$, which in terms

of amplitudes take the form [4]

$$\begin{aligned}\widehat{T}^l T^{l*} + \widehat{R}^l R^{l*} &= 1, \\ \widehat{T}^r T^{r*} + \widehat{R}^r R^{r*} &= 1, \\ \widehat{T}^{l*} R^r + T^r \widehat{R}^{l*} &= 0, \\ T^l \widehat{R}^{r*} + \widehat{T}^{r*} R^l &= 0,\end{aligned}\tag{1.3}$$

these equivalences between the amplitudes imply further consequences on the amplitudes' moduli (tenth column of table 1.1) and phases (not shown). The final two columns use the previous results to determine if perfect asymmetry is possible for transmission or reflection. This makes evident that hermiticity (II) and parity (III) preclude, independently, any asymmetry in the scattering coefficients; PT-symmetry (VII) or Θ -pseudohermiticity (VI) forbid transmission asymmetry (all local potentials satisfy automatically symmetry VI), whereas time-reversal symmetry (i.e., a real potential in coordinate space) (V) or PT-pseudohermiticity (VIII) forbid reflection asymmetry. A caveat is that asymmetric effects forbidden by a certain symmetry in the linear (Schrödinger) regime considered in this paper might not be forbidden in a non-linear regime [11], which goes beyond our present scope.

The occurrence of one particular symmetry in the potential (conventionally “first symmetry”) does not exclude a second symmetry to be satisfied as well. When a double symmetry holds, excluding the identity, the “first” symmetry implies the equivalence of the second symmetry with a third symmetry. We have already mentioned that Π -pseudohermiticity (IV) is equivalent to PT -symmetry (VII) for local potentials. Being local is just one particular way to satisfy symmetry VI, namely Θ -pseudohermiticity. The reader may verify with the aid of the third column for $\langle x|V|y\rangle$ in table 1.1, that indeed, if symmetry VI is satisfied (first symmetry), symmetry IV has the same effect as symmetry VII. They become equivalent. Other well known example is that for a local potential (symmetry VI is satisfied), a real potential in coordinate space is necessarily hermitian, so symmetries V and II become equivalent. These examples are just particular cases of the full set of equivalences given in table 1.2.

TABLE 1.3: Device types for transmission and/or reflection asymmetry, restricted to 1 or 0 moduli for the scattering amplitudes. The fifth column indicates the symmetries in table 1.1 that forbid the device. Figures S2, S3, S5 and S6 can be found in the supplemental material to this paper.

Device type	Left incidence	Right incidence	Code	Forbidden by	Example
One-way mirror	transmits and reflects	absorbs	$\mathcal{T}\mathcal{R}/\mathcal{A}$	II, III, IV, V, VI, VII, VIII	fig. S1
One-way barrier	transmits	reflects	\mathcal{T}/\mathcal{R}	II, III, IV, V, VI, VII, VIII	fig. S2
One-way T-filter	transmits	absorbs	\mathcal{T}/\mathcal{A}	II, III, IV, V, VI, VII	fig. 1.2, S3
Mirror&1-way transmitter	transmits and reflects	reflects	$\mathcal{T}\mathcal{R}/\mathcal{R}$	II, III, VI, VII	fig. S4
One-way R-filter	reflects	absorbs	\mathcal{R}/\mathcal{A}	II, III, IV, V, VII, VIII	[12]
Transparent 1-way reflector	transmits and reflects	transmits	$\mathcal{T}\mathcal{R}/\mathcal{T}$	II, III, V, VIII	figs. 1.3, S5

TABLE 1.4: Device types allowed for a given symmetry.

Symmetry	Allowed devices
I	All types
II	None
III	None
IV	$\mathcal{T}\mathcal{R}/\mathcal{R}, \mathcal{T}\mathcal{R}/\mathcal{T}$
V	$\mathcal{T}\mathcal{R}/\mathcal{R}$
VI	$\mathcal{R}/\mathcal{A}, \mathcal{T}\mathcal{R}/\mathcal{T}$
VII	$\mathcal{T}\mathcal{R}/\mathcal{T}$
VIII	$\mathcal{T}/\mathcal{A}, \mathcal{T}\mathcal{R}/\mathcal{R}$

Combining the information of the last two-columns in table 1.1 with the additional condition that all scattering coefficients be 0 or 1 we elaborate table 1.3, which provides the symmetries that do not allow the implementation of the devices in fig. 1.1. The complementary table 1.4 gives instead the symmetries that allow, but do not necessarily imply, a given device type. The device denominations in fig. 1.1 or table 1.3 are intended as short and meaningful, and do not necessarily coincide with some extended terminology, in part because the range of possibilities is broader here than those customarily considered, and because we use a 1 or 0 condition for the moduli. For example, a device with reflection asymmetry and with $T^r = T^l = 1$ would in our case be a particular “transparent, one-way reflector”, as full transmission occurs from both sides. This effect has however become popularized as “unidirectional invisibility” [13, 14]. A debate on terminology is not our main concern here, and the use of a code system as the one proposed will be instrumental in avoiding misunderstandings.

1.3 Designing potentials for asymmetric devices

We will show how to design non-local potentials leading to the asymmetric devices. For simplicity we look for non-local potentials $V(x, y)$ with local support that vanish for $|x| > d$ and $|y| > d$.

Inverse scattering proceeds similarly to [15], by imposing an ansatz for the wavefunctions and the potential in the stationary Schrödinger equation

$$\frac{\hbar^2 k^2}{2m} \psi(x) = -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} \psi(x) + \int_{-d}^d dy V(x, y) \psi(y). \quad (1.4)$$

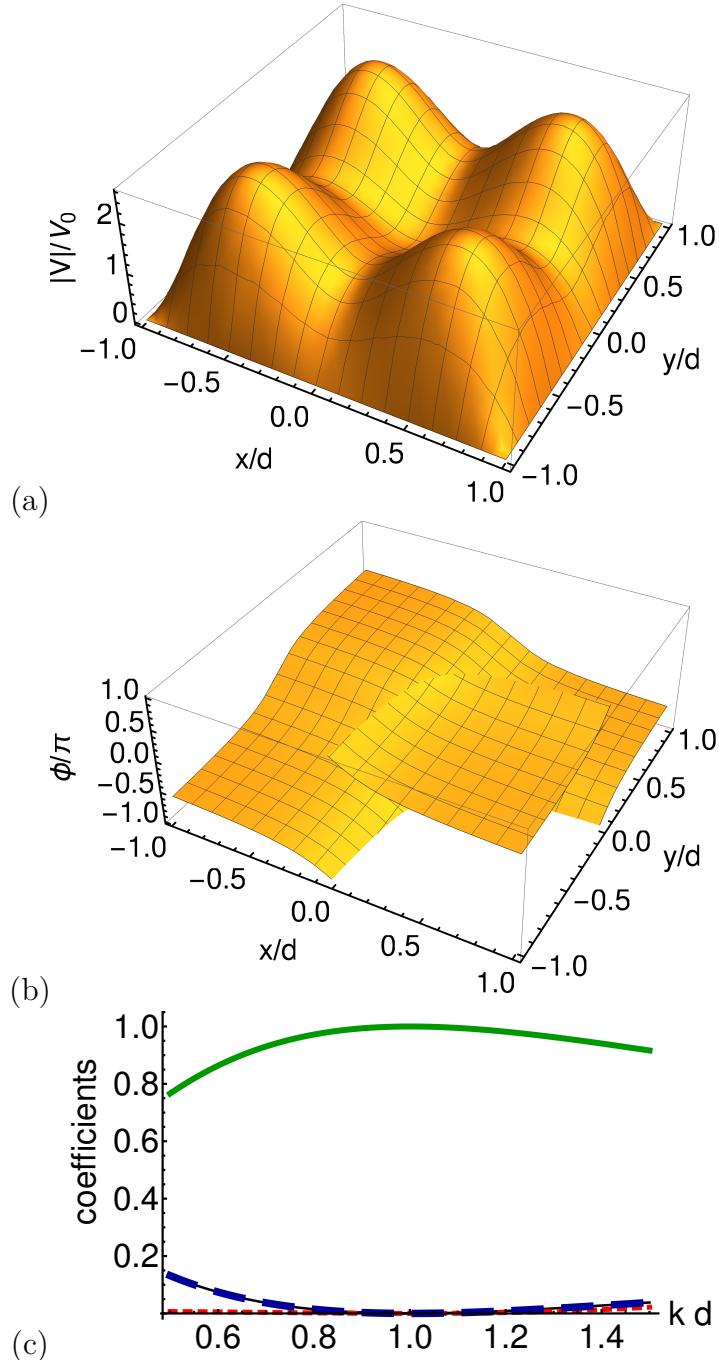


FIGURE 1.2: (Color online) One-way T-filter (\mathcal{T}/\mathcal{A} , $|T^l| = 1, T^r = R^l = R^r = 0$) with potential $V(x, y) = |V(x, y)|e^{i\phi(x, y)}$ set for $k_0 = 1/d$. (a) Absolute value $|V(x, y)|$; (b) Argument $\phi(x, y)$; (c) Transmission and reflection coefficients: $|R^l|^2$ (black, solid line), $|T^l|^2$ (green, solid line), $|R^r|^2$ (blue, tick, dashed line), $|T^r|^2$ (red, dotted line). $V_0 = \hbar^2/(2md^3)$.

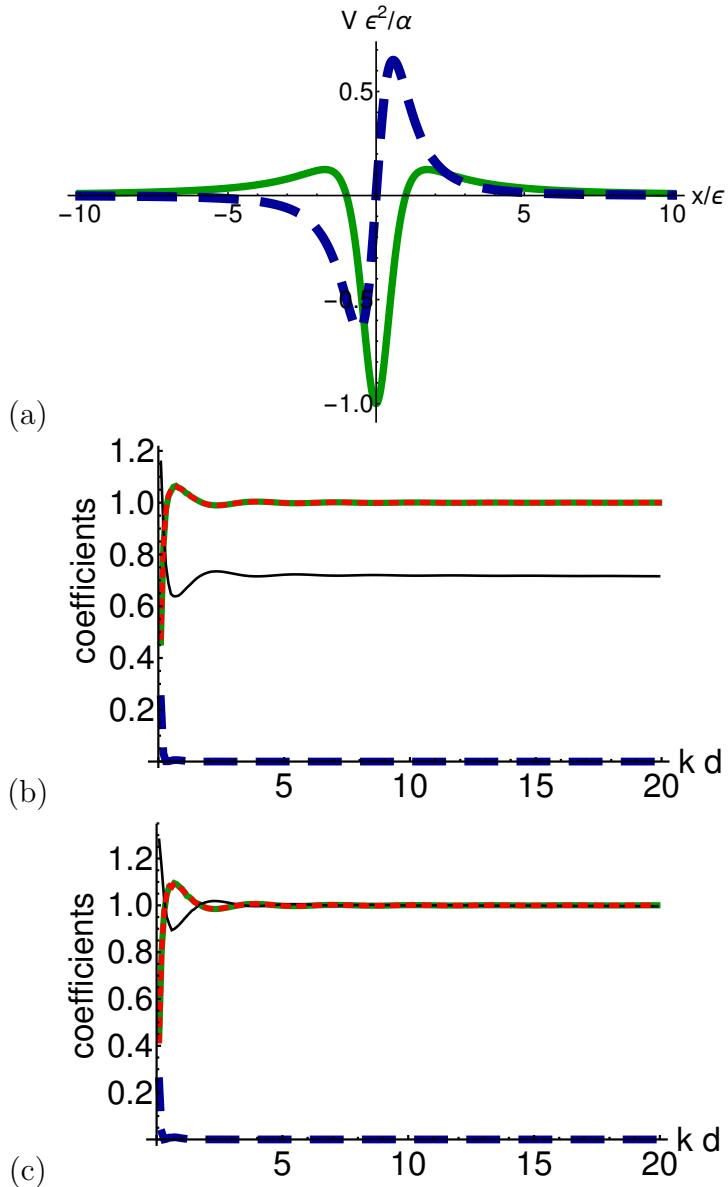


FIGURE 1.3: (Color online) Transparent 1-way reflector with a local PT potential: (a) Approximation of the potential (1.8), real part (green solid line), imaginary part (blue dashed line). (b,c) Transmission and reflection coefficients versus momentum kd ; left incidence: $|R^l|^2$ (black, solid line), $|T^l|^2$ (green, solid line); right incidence: $|R^r|^2$ (blue, tick, dashed line), $|T^r|^2$ (red, dotted line, coincides with green, solid line). $\epsilon/d = 10^{-4}$. (b) $\alpha = 1.0 \hbar^2/(4\pi m)$ (c) $\alpha = 1.225 \hbar^2/(4\pi m)$ (the black, solid line coincides here mostly with the red, dotted and green, solid lines).

The free parameters are fixed making use of the boundary conditions. The form of the wavefunction incident from the left is $\psi_l(x) = e^{ikx} + R^l e^{-ikx}$ for $x < -d$ and $\psi_l(x) = T^l e^{ikx}$ for $x > d$, where $k = p/\hbar$. The wavefunction incident from the right is instead $\psi_r(x) = e^{-ikx} T^r$ for $x < -d$ and $\psi_r(x) = e^{-ikx} + R^r e^{ikx}$ for $x > d$.

Our strategy is to assume polynomial forms for the two wavefunctions in the interval $|x| < d$, $\psi_l(x) = \sum_{j=0}^5 c_{l,j} x^j$ and $\psi_r(x) = \sum_{j=0}^5 c_{r,j} x^j$, and also a polynomial ansatz of finite degree for the potential $V(x, y) = \sum_i \sum_j v_{ij} x^i y^j$. Inserting these ansatzes in eq. (1.4) and from the conditions that $\psi_{l,r}$ and their derivatives must be continuous, all coefficients $c_{l,j}$, $c_{r,j}$ and v_{ij} can be determined. Symmetry properties of the potential can also be imposed via additional conditions on the potential coefficients v_{ij} . For example we may use this method to obtain a one-way T-filter (\mathcal{T}/\mathcal{A}) device (third device in table 1.3) with a nonlocal PT-pseudohermitian potential (symmetry VIII of table 1.1) for a chosen wavevector $k = k_0$. The absolute value and argument of the resulting potential $V(x, y)$ are shown in figs. 1.2(a) and 1.2(b) together with its scattering coefficients as function of the incident wave vector, fig. 1.2(c). As can be seen in fig. 1.2(c) the imposed scattering coefficients are fulfilled exactly for the chosen wavevector. They are also satisfied approximately in a neighborhood of k_0 . In the Supplemental Material, Sec. II, we give further details about the construction of this potential and we work out other asymmetric devices of fig. 1.1.

1.4 Extending the scattering asymmetry to a broad incident-momentum domain

The inversion technique just described may be generalized to extend the range of incident momenta for which the potential works by imposing additional conditions and increasing correspondingly the number of parameters in the wavefunction ansatz, for example we may impose that the derivatives of the amplitudes, in one or more orders, vanish at k_0 , or 0/1 values for the coefficients not only at k_0 but at a series of grid points k_1, k_2, \dots, k_N , as in [4, 15–17].

Here we put forward instead a method that provides a very broad working-window domain. While we make formally use of the Born approximation, the exact numerical computations demonstrate the robustness and accuracy of the approach to achieve that objective by making use of an adjustable parameter in the

potential. The very special role of the Born approximation in inverse problems has been discussed and demonstrated in [18–20]. Specifically we study a transparent one-way reflector \mathcal{TR}/\mathcal{T} . Our aim is now to find a local PT-symmetric potential such that asymmetric reflection results, $T^l = T^r = 1, R^r = 0, |R^l| = 1$ for a broad range of incident momenta. A similar goal was pursued in [21] making use of a supersymmetric transformation, without imposing $|R^l| = 1$.

In the Born approximation and for a local potential $V(x)$, the reflection amplitudes take the simple form

$$R^l = -\frac{2\pi im}{p} \langle -p|V|p\rangle, \quad R^r = -\frac{2\pi im}{p} \langle p|V|-p\rangle. \quad (1.5)$$

Defining the Fourier transform

$$\tilde{V}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx V(x) e^{-ikx} \quad (1.6)$$

we get for $k = p/\hbar > 0$:

$$R^l = -\frac{\sqrt{2\pi}im}{k\hbar^2} \tilde{V}(-2k), \quad R^r = -\frac{\sqrt{2\pi}im}{k\hbar^2} \tilde{V}(2k). \quad (1.7)$$

Assuming that the potential is local and PT-symmetrical, we calculate the transition coefficient from them using generalized unitarity as $|T|^2 = 1 - R^{r*}R^l$.

To build a \mathcal{TR}/\mathcal{T} device we demand: $\tilde{V}(k) = \sqrt{2\pi}\alpha k$ ($k < 0$) and $\tilde{V}(k) = 0$ ($k \geq 0$). By inverse Fourier transformation, this implies

$$\begin{aligned} V(x) &= -\alpha \frac{\partial}{\partial x} \lim_{\epsilon \rightarrow 0} \frac{1}{x - i\epsilon} = \alpha \lim_{\epsilon \rightarrow 0} \frac{1}{(x - i\epsilon)^2} \\ &= \alpha \lim_{\epsilon \rightarrow 0} \left[\frac{x^2 - \epsilon^2}{(x^2 + \epsilon^2)^2} + i \frac{2x\epsilon}{(x^2 + \epsilon^2)^2} \right], \end{aligned} \quad (1.8)$$

which is indeed a local, PT -symmetric potential for α real. α is directly related to the reflection coefficient, within the Born approximation, $R^l = 4\pi im\alpha/\hbar^2$. As the Born approximation may differ from exact results we shall keep α as an adjustable parameter in the following.

In a possible physical implementation, the potential in eq. (1.8) will be approximated by keeping a small finite $\epsilon > 0$, see fig. 1.3(a). Then, its Fourier transform is $\tilde{V}(k) = \sqrt{2\pi}\alpha k e^{\epsilon k}$ ($k < 0$) and $\tilde{V}(k) = 0$ ($k \geq 0$). In figs. 1.3(b) and (c), the resulting coefficients for $\epsilon/d = 10^{-4}$ and two different values of α are

shown. These figures have been calculated by numerically solving the Schrödinger equation exactly. Remarkably, the Born approximation contains all the information required to build the required potential shape up to a global factor. Such a prominent role of the Born approximation in inverse problems has been noted in different applications [18–20]. For a range of α , the potential gives $|R^r| \approx 0$, a nearly constant $|R^l|^2$, and $|T^r| = |T^l| \approx 1$ in a broad k -domain, see fig. 1.3(b). Adjusting the value of α , fig. 1.3(c), sets $|R^l| \approx 1$ as desired.

1.5 Discussion

Scattering asymmetries are necessary to develop technologically relevant devices such as one-way mirrors, filters and barriers, invisibility cloaks, diodes, or Maxwell demons. So far much effort has been devoted to build and apply local PT-symmetric potentials but the possible scattering asymmetries with them are quite limited. We find that six device types with asymmetric scattering are possible when imposing 0 or 1 scattering coefficients. PT-symmetry can only realize one of them, but this symmetry is just one among eight possible symmetries of complex non-local potentials. The eight symmetries arise from the discovery that Klein’s four-group $\{1, \Pi, \Theta, \Theta\Pi\}$, combined with two possible relations among the Hamiltonian, its adjoint, and the symmetry operators of the group, eqs. (1) and (2), produce all possible equalities among potential matrix elements after complex conjugation, coordinate inversion, the identity, and transposition. In other words, to have all possible such equalities, the conventional definition of a symmetry A in terms of its commutation with the Hamiltonian H is not enough, and A -pseudohermiticity must be considered as well on the same footing. Extending the concept of what a symmetry is for complex, non-local potentials is a fundamental, far-reaching step of this work. This group theoretical analysis and classification is not only esthetically pleasing, but also of practical importance, as it reveals the underlying structure and span of the possibilities available in principle to manipulate the asymmetrical response of a potential for a structureless particle.

We provide potentials for the different asymmetric devices including an example that works in a broad domain of incident momenta. Although the present theory is for the scattering of quantum particles, the analogies between quantum physics and optics suggest to extend the concepts and results for optical asymmetric devices.

Interesting questions left for future work are the inclusion of other mechanisms for transmission and reflection asymmetries (for example nonlinearities [11, 22], and time dependent potentials [23, 24]), or a full discussion of the phases of the scattering amplitudes in addition to the moduli emphasized here. In this paper the properties of the scattering amplitudes have been worked out assuming that the operator A in the symmetry relations in eqs. (1.1) and (1.2) is a unitary/antiunitary operator in Klein's group. We may generalize the study to include more general operators, possibly including differential operators, as was done in [25] for phase transitions of optical potentials, or the operator that swaps internal states or waveguides [26, 27].

We shall also examine in a complementary paper the physical realization of complex nonlocal effective potentials. In a quantum optics scenario, simple examples were provided in [28] based on applying the partitioning technique [29, 30] to the scattering of a particle with internal structure. The experimental realization of all new symmetries and devices may be challenging, e.g. to engineer the nonlocality in optics, but there is much to gain. We may expect progress similar to the successful evolution from theory to actual devices in the sequence from the first mathematical models of PT-symmetric potentials [31], to the proposal of an optical realization [5], and to the actual experiments [32], even if considerable time lapses were needed between the three steps.

Chapter 2

Symmetries of non-hermitian potentials

The following chapters will be used for this one:

- ChapterSymAndInvariants.tex
- ChapterRealEigenValues.tex
- ChapterDiscreteNNSymmetries.tex

2.1 Introduction

Now, you can write the content of the chapter and organize it into sections.

Chapter 3

Physical Implementation of non-hermitian and non-local hamiltonians

Non-Hermitian, one-dimensional potentials which are also non-local, allow for scattering asymmetries, namely, asymmetric transmission or reflection responses to the incidence of a particle from left or right. The symmetries of the potential imply selection rules for transmission and reflection. In particular, parity-time (PT) symmetry or the symmetry of any local potential do not allow for asymmetric transmission. We put forward a feasible quantum-optical implementation of non-Hermitian, non-local, non-PT potentials to implement different scattering asymmetries, including transmission asymmetries.

3.1 Introduction

The asymmetric response of diodes, valves, or rectifiers to input direction is of paramount importance in many different fields and technologies, from hydrodynamics to microelectronics, as well as in biological systems. We expect a wealth of applications of such response asymmetries also in the microscopic quantum realm, in particular in circuits or operations carrying or processing quantum information with moving atoms. So far devices such as Maxwell demons, which let atoms pass one way, have been instrumental, first as ideal devices to understand the second law [33, 34], and also as practical sorting devices [35–43].

Asymmetric transmission and reflection probabilities for one-dimensional (1D) particle scattering off a potential center are not possible if the Hamiltonian is Hermitian [4, 44]. Non-Hermitian (NH) Hamiltonians representing effective interactions have a long history in nuclear, atomic, and molecular physics, and have become common in optics, where wave equations in waveguides could simulate Schrödinger equations [5, 22, 45]. Non-Hermitian Hamiltonians constructed by analytically continuing Hermitian ones are useful and efficient tools to find resonances [46]. They can also be set phenomenologically, e.g. to describe gain and loss [5], or be found as effective Hamiltonians for a subspace from a Hermitian Hamiltonian of a larger system by projection [4, 29, 35].

Much of the recent interest in Non-Hermitian Hamiltonians focuses on parity-time (PT) symmetric Hamiltonians [8, 31] because of their spectral properties and useful applications, mostly in optics [21, 22, 45], but alternative symmetries are also being studied [9, 10, 47–53]. Symmetry operations on NH Hamiltonians can be systematized into group structures [48, 50, 51]. In particular for 1D particle scattering off a potential center, the different Hamiltonian symmetries imply selection rules for asymmetric transmission and reflection [48, 50].¹ Whereas hermiticity does not allow for any asymmetry in transmission and reflection probabilities, PT symmetry or the symmetry of local potentials, technically “pseudohermiticity with respect to time reversal” [48], do not allow for asymmetric transmission [4, 44], see symmetries II (Hermiticity), VII (PT symmetry), and VI (time-reversal pseudohermiticity) in table 3.1. (Here a “local potential” is defined as one whose only non-zero elements in coordinate representation are diagonal, whereas a non-local one

¹Throughout the paper we assume a linear theory for systems whose wave equation is linear in the wavefunction.

has non-zero nondiagonal elements.) Thus non-local, non-PT, and non-Hermitian potentials are needed to implement a rich set of scattering asymmetries, and in particular asymmetric transmission.

In this paper we put forward a physical realization of effective NH, non-local Hamiltonians which do not posses PT symmetry. Non-local potentials for asymmetric scattering had been constructed as mathematical models [48], but a physical implementation had been so far elusive. Using Feshbach's projection technique it is found that the effective potentials for a ground-state atom crossing a laser beam in a region of space are generically non-local and non-Hermitian. Shaping the spatial-dependence of the, generally complex, Rabi frequency, and selecting a specific laser detuning allows us to produce different potential symmetries and asymmetric scattering effects, including asymmetric transmission.

After a lightning review of Hamiltonian symmetries and the corresponding scattering selection rules in Sec. 3.2, we shall explain in Sec. 3.3 how to generate different NH symmetries in a quantum optical setting of an atom impinging on a laser illuminated region. Finally we provide specific example devices (constructed using numerical optimisation) with different asymmetric scattering responses in Sec. 3.4. Realistic experimental parameters are also examined. The asymmetric behavior can be intuitively understood based on a classical approximation of the motion and the non-commutativity of rotations on the Bloch sphere, which gives good estimates for the potential parameters, see Sec. 3.5.

3.2 Symmetries of Scattering Hamiltonians

We consider one-dimensional scattering Hamiltonians $H = H_0 + V$, where H_0 is the kinetic energy for a particle of mass m , and V is the potential, which is assumed to decay fast enough on both sides so that H has a continuous spectrum and scattering eigenfunctions. These eigenfunctions may be chosen so that asymptotically, i.e., far from the potential center, they are superpositions of an incident plane wave and a reflected plane wave on one side, and a transmitted plane wave on the other side. Reflected and transmitted waves include corresponding amplitudes, whose squared-modulii (scattering coefficients hereafter) sum to one for Hermitian potentials. Instead, NH potentials may produce absorption or gain.

There are eight different symmetries that H could fulfill, see table 3.1, with the forms

$$AH = HA, \quad (3.1)$$

$$AH = H^\dagger A, \quad (3.2)$$

where A is a unitary or antiunitary operator in the Klein four-group $K_4 = \{1, \Pi, \theta, \Pi\theta\}$ [48]. Relation (3.2) is called here A -pseudohermiticity of H [7, 48]. The operators 1 , Π , θ and $\Pi\theta$ are the identity, parity, time reversal, and the consecutive (commuting) application of both operators. Acting on position eigenvectors $|x\rangle$, $\Pi c|x\rangle = c| -x\rangle$, and $\theta c|x\rangle = c^*|x\rangle$, for any complex number c . Note that symmetry I is a trivial symmetry and is satisfied for all Hamiltonians.

The eight symmetries may be regarded as the invariance of the Hamiltonian with respect to eight symmetry operations that form the Abelian group E8 [50]. They are all operations that can be done by inversion, transposition, complex conjugation, and their combinations. Making use of generalized unitarity relations and the relations implied by the symmetries on S -matrix elements, the transmission and reflection amplitudes for right and left incidence, T^r , R^r and T^l , R^l , can be related to each other, as well as their modulii [48]. “Right and left incidence” are here shorthands for “incidence *from* the right”, and “incidence *from* the left”, respectively.

The possible asymmetric responses are allowed or forbidden, according to selection rules, by the symmetries of the Hamiltonian. If we impose that the transmission and reflection coefficients have only 0 or 1 values, a convenient reference scenario for devices intended to manage quantum-information applications, six possible scattering asymmetries may be identified [48], see table 3.2. It is useful to label them according to the response to incidence from the left/right. The possible responses are encoded in the letters \mathcal{A} , for “absorption”, and \mathcal{T} and \mathcal{R} for “transmission” and “reflection”, separated by “/”. The letters on the left of / are for left incidence, and the ones on the right are for right incidence. For example \mathcal{T}/\mathcal{A} means transmission for left incidence and absorption for right incidence. From the selection rules [48], it is possible to determine which symmetries allow for a given device, see table 3.2.

The relations between the symmetries and “reciprocity” are surely worth spelling out, in view of many works and discussions in optics [13, 54–56]. “Reciprocity” is a

somewhat vague term with different meanings for different authors and communities, the reviews [57] and [58] give some useful background. A primary formulation regards reciprocity as the property of detecting the same effects when interchanging source and detector without changing the scatterer. This concept has lead to different formalizations that fix in more detail what is exactly meant by “same effects” and “interchanging source and detector”. In 1D scattering problems we may first distinguish a reciprocity for scattering amplitudes or for scattering coefficients (their modulus squared). We shall hereafter focus on coefficients as in the rest of the paper. Another distinction can be made between reflection and transmission reciprocities, namely, a system with $|R^l|^2 = |R^r|^2$ would be “reflection reciprocal” and if $|T^l|^2 = |T^r|^2$ the system would be transmission reciprocal.² A formal definition of reciprocity is that, for some antiunitary operator K [58],

$$HK = KH^\dagger. \quad (3.3)$$

It follows that the scattering transition matrix obeys in momentum representation [58]

$$\langle p|\mathsf{T}|p'\rangle = \langle Kp'|\mathsf{T}|Kp\rangle. \quad (3.4)$$

In our symmetry classification, symmetries VI and VIII obey by definition reciprocity conditions of the form (3.3). Inserting the results in the exact forms of transmission and reflection amplitudes, which depend on diagonal and non-diagonal elements of the transition matrix, respectively, see e.g. [4], different physical consequences follow: In symmetry VI, $K = \Theta$, $|Kp\rangle = |-p\rangle$, and the reciprocity condition implies transmission reciprocity. In symmetry VIII, $K = \Pi\Theta$ and $|Kp\rangle = |p\rangle$, so the reciprocity condition implies reflection reciprocity. A first relevant observation is this: an arbitrary reciprocity condition of the form (3.3), does not necessarily imply symmetrical transmission. A second point is that “scattering selection rules”, i.e., the set of forbidden phenomena, or compulsory relations among right and left coefficients, see table 1 in [48], depend as well on generalized unitary relations. Putting together the effect of symmetries on transition or S -matrix elements and generalized unitarity relations, it turns out that symmetries II (Hermiticity) and III (parity) are not capable of any, reflection or transmission, asymmetry; symmetries VI (time reversal pseudohermiticity) and

²Incidentally, for reflection reciprocity, “interchanging source and detector” has to be understood in momentum space rather than spatially.

TABLE 3.1: Conditions leading to specific symmetries in the potential (3.8). A given symmetry also implies others, see the last column.

Symmetry	Conditions	Implies
(I) $1H = H1$	none	-
(II) $1H = H^\dagger 1$	$q = -q^*$ (i.e. $\text{Re } q = 0$)	I
(III) $\Pi H = H\Pi$	$\Omega(x) = e^{i\phi}\Omega(-x)$	I
(IV) $\Pi H = H^\dagger \Pi$	$q = -q^* \& \Omega(x) = e^{i\phi}\Omega(-x)$	III, II, I
(V) $\Theta H = H\Theta$	$q = -q^* \& \Omega(x) = e^{i\phi}\Omega(x)^*$	VI, II, I
(VI) $\Theta H = H^\dagger \Theta$	$\Omega(x) = e^{i\phi}\Omega(x)^*$	I
(VII) $\Theta\Pi H = H\Theta\Pi$	$q = -q^* \& \Omega(x) = e^{i\phi}\Omega(-x)^*$	VIII, II, I
(VIII) $\Theta\Pi H = H^\dagger \Theta\Pi$	$\Omega(x) = e^{i\phi}\Omega(-x)^*$	I

TABLE 3.2: Device types for transmission and/or reflection asymmetry in the first row (see main text for nomenclature, binary values (0 or 1) for the transmission and reflection coefficients are considered here as an ideal case). The second row gives the corresponding symmetries that allow each device.

$$\begin{array}{ccccccc} \mathcal{T}\mathcal{R}/\mathcal{A} & \mathcal{T}/\mathcal{R} & \mathcal{T}/\mathcal{A} & \mathcal{T}\mathcal{R}/\mathcal{R} & \mathcal{R}/\mathcal{A} & \mathcal{T}\mathcal{R}/\mathcal{T} \\ \text{I} & \text{I} & \text{I}, \text{VIII} & \text{I}, \text{VIII} & \text{I}, \text{VI} & \text{I}, \text{IV}, \text{VI}, \text{VII} \end{array}$$

VII (PT symmetry) allow for reflection asymmetry but not for transmission asymmetry; symmetries V (time-reversal symmetry) and VIII (PT pseudohermiticity) allow for transmission asymmetry but not for reflection asymmetry, whereas I (trivial symmetry) and IV (parity pseudohermiticity) allow for both scattering asymmetries. Note also the importance on non-locality for asymmetric transmission: All local potentials do satisfy automatically symmetry VI, and are therefore necessarily transmission reciprocal. Let us insist once more than all these results are for linear (Schrödinger) dynamics. Nonlinearity allows to break down these selection rules [11, 13, 56].

3.3 Effective non-local potential for the ground state of a two-level atom

The key task is now to physically realize some of the potential and device types described in the previous section. We start with a two-level atom with ground level $|1\rangle$ and excited state $|2\rangle$ impinging onto a laser illuminated region. For a full account of the model and further references see [59]. The motion is assumed one dimensional, either because the atom is confined in a waveguide or because the direction x is uncoupled to the others. We only account explicitly for atoms

before the first spontaneous emission in the wavefunction [60–62]. If the excited atom emits a spontaneous photon it disappears from the coherent wavefunction ensemble. We assume that no resetting into the ground state occurs. The physical mechanism may be an irreversible decay into a third level [63], or atom ejection from the waveguide or the privileged 1D direction due to random recoil [64]. The state $\Phi_k = \begin{pmatrix} \phi_k^{(1)} \\ \phi_k^{(2)} \end{pmatrix}$ for the atom before the first spontaneous emission impinging with wavenumber k in a laser adapted interaction picture, obeys, after applying the rotating wave approximation, an effective stationary Schrödinger equation with a time-independent Hamiltonian [28, 59] $\mathcal{H}\Phi_k(x) = E\Phi_k(x)$, where

$$\mathcal{H} = H_0\mathbf{1} + \mathcal{V} = \frac{1}{2m} \begin{pmatrix} p^2 & 0 \\ 0 & p^2 \end{pmatrix} + \mathcal{V}(x), \quad (3.5)$$

$$\mathcal{V}(x) = \frac{\hbar}{2} \begin{pmatrix} 0 & \Omega(x) \\ \Omega(x)^* & -(2\Delta + i\gamma) \end{pmatrix}. \quad (3.6)$$

We assume perpendicular incidence of the atom on the laser sheet for simplicity, oblique incidence is treated e.g. in [41, 59]. Here $E = \hbar^2 k^2 / 2m$ is the energy, and $\Omega(x)$ is the position-dependent, on-resonance Rabi frequency, where real and imaginary parts may be controlled independently using two laser field quadratures [65]; γ is the inverse of the life time of the excited state; $\Delta = \omega_L - \omega_{12}$ is the detuning (laser angular frequency minus the atomic transition angular frequency ω_{12}); $p = -i\hbar\partial/\partial x$ is the momentum operator; and $\mathbf{1} = |1\rangle\langle 1| + |2\rangle\langle 2|$ is the unit operator for the internal-state space. Complementary projectors $P = |1\rangle\langle 1|$ and $Q = |2\rangle\langle 2|$ are defined to select ground and excited state components. Using the partitioning technique [29, 30, 66], we find for the ground state amplitude $\phi_k^{(1)}$ the equation

$$E\phi_k^{(1)}(x) = H_0\phi_k^{(1)}(x) + \int dy \langle x, 1 | \mathcal{W}(E) | y, 1 \rangle \phi_k^{(1)}(y), \quad (3.7)$$

where $\mathcal{W}(E) = P\mathcal{V}P + P\mathcal{V}Q(E + i0 - Q\mathcal{H}Q)^{-1}Q\mathcal{V}P$, is generically non local and energy dependent. Specifically, we have now achieved a physical realization of an effective (in general) non-local, non-Hermitian potential whose kernel has the form

$$V(x, y) = \langle x, 1 | \mathcal{W}(E) | y, 1 \rangle = \frac{m}{4} \frac{e^{i|x-y|q}}{iq} \Omega(x)\Omega(y)^*, \quad (3.8)$$

where $q = \frac{\sqrt{2mE}}{\hbar}(1 + \mu)^{1/2}$, $\text{Im } q \geq 0$, and $\mu = \frac{2\Delta + i\gamma}{2E/\hbar}$. Eq. (3.8) is worked out in momentum representation to do the integral using the residue theorem. This is a generalized, non-local version of the effective potentials known for the ground state [63, 67], which are found from Eq. (3.8) in the large μ limit [28]. The reflection

and transmission amplitudes $R^{r,l}$ and $T^{r,l}$ may be calculated directly using the potential (3.8) or as corresponding amplitudes for transitions from ground state to ground state in the full two-level theory see Appendix D.

3.3.1 Possible symmetries of the non-local potential

The necessary conditions for the different symmetries of the potential (3.8) are outlined in the second column of table 3.1. For example, symmetry III (parity) requires that $V(x,y) = V(-x,-y)$ [48]. Inserting the functional form of the potential from Eq. (3.8) into this condition, it results in the requirement $\Omega(x)\Omega(y)^* = \Omega(-x)\Omega(-y)^*$. This is fulfilled if $\Omega(x) = \Omega(-x)e^{i\phi}$ with some arbitrary phase freedom ϕ .

Since $\Omega(x)$ does not depend on q , symmetries IV, V and VII imply that symmetry II is obeyed as well (Hermiticity). Moreover symmetry III (parity) should be discarded for our purpose since it does not allow for asymmetric transmission or reflection [48]. This leaves us with three interesting symmetries to explore: VI, which allows for asymmetric reflection; VIII which allows for asymmetric transmission, and I, which in principle allows for arbitrary asymmetric responses, except for physical limitations imposed by the two-level model see Appendix D.

As seen from table 3.1, $\text{Re}(q) = 0$ makes the potential Hermitian so we shall avoid this condition. If $\gamma = 0$, $\mu \in \mathbb{R}$. Hence $\mu + 1 < 0$ gives $\text{Re}(q) = 0$ and $\mu + 1 > 0$ gives $\text{Im}(q) = 0$. $\mu + 1 > 0$ amounts to a condition on the detuning compared to the incident energy, namely $\Delta > -E/\hbar$. In the following examples we implement potentials with symmetries VIII, VI, and I, with detunings and energies satisfying the condition $\mu + 1 > 0$.

3.4 Design of asymmetric devices

We will now apply this method to physically realize non-local potentials of the form (3.8). We shall work out explicitly a \mathcal{T}/\mathcal{A} device with symmetry VIII, a \mathcal{R}/\mathcal{A} device with symmetry VI, and a “partial”- $\mathcal{T}\mathcal{R}/\mathcal{A}$ device, having 1/2 transmission and reflection coefficients from the left, with symmetry I. The \mathcal{T}/\mathcal{A} and the “partial”- $\mathcal{T}\mathcal{R}/\mathcal{A}$ device have transmission asymmetry so they cannot be built

with local or PT -symmetric potentials. Let us motivate the effort with some possible applications, relations and analogies of these devices. \mathcal{T}/\mathcal{A} and \mathcal{R}/\mathcal{A} are, respectively, transmission and reflection filters. They are analogous to half-wave electrical rectifiers that either let the signal from one side “pass” (transmitted) or change its sign (reflected) while suppressing the other half signal. They may play the role of half-rectifiers in atomtronic circuits. A \mathcal{T}/\mathcal{A} device allows us, for example, to empty a region of selected particles, letting them go away while not letting particles in. The “atom diode” devices worked out e.g. in [35, 38, 39, 41] were of type \mathcal{R}/\mathcal{A} . As the mechanism behind them was adiabatic, a broad range of momenta with the desired asymmetry could be achieved. In comparison the current approach is not necessarily adiabatic so it can be adapted to faster processes.

As for the “partial”- \mathcal{RT}/\mathcal{A} device, it reflects and transmits from one side while absorbing from the other side. In an optical analogy an observer from the left perceives it as a darkish mirror. An observer from the right “sees” the other side because of the allowed transmission but cannot be seen from the left since nothing is transmitted from right to left. Our device is necessarily “partial” one as there cannot be net probability gain because of the underlying two-level system, and a “full” version with both reflection and transmission coefficients equal to one would need net gain.

The three devices are worked out for $\gamma = 0$, a valid approximation for hyperfine transitions. We assume for the Rabi frequencies the forms

$$\begin{aligned}\Omega_{\text{VIII}}(x) &= a[g(x + x_0) + ig(x - x_0)], \\ \Omega_{\text{VI}}(x) &= bg(x + x_0) + cg(x - x_0), \\ \Omega_{\text{I}}(x) &= -ibg(x + x_0) + cg(x - x_0),\end{aligned}\tag{3.9}$$

in terms of smooth, realizable Gaussians $g(x) = \exp[-x^2/w^2]$. We fix $2d$ as an effective finite width of the potential area beyond which the potential is negligible and assumed to vanish. We will express in the following the different length parameters as a multiple of d to keep results general. In addition, we will use as a scaling factor for the velocity $v_d = \hbar/(md)$, and for time $\tau = md^2/\hbar$.

In the following calculations we fix the width of the Gaussians to be $w = \sqrt{2d}/10$. We always first set a target velocity v_0 to achieve the desired asymmetric scattering response. The real parameters a, b, c, x_0 in Eq. (3.9), and Δ are

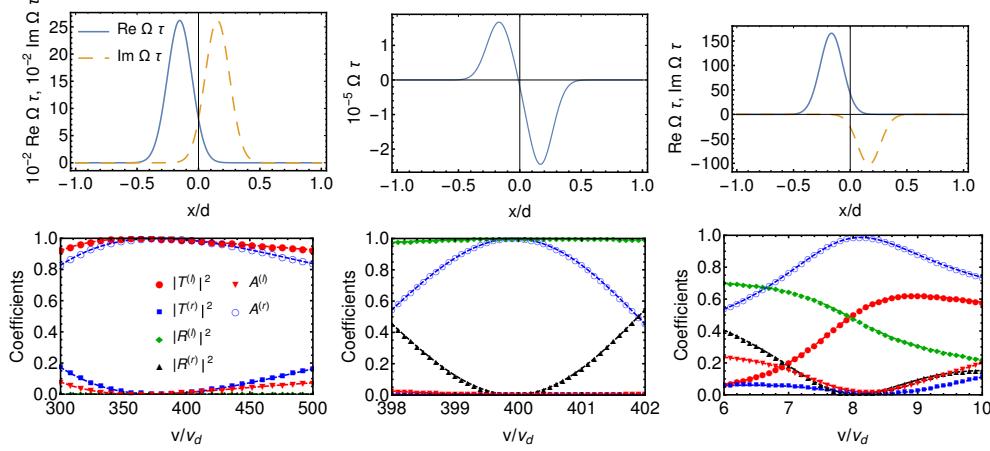


FIGURE 3.1: Left column: \mathcal{T}/\mathcal{A} device with symmetry VIII. Top: $\Omega_{\text{VIII}}(x)$; Bottom: transmission and reflection coefficients. $v_0/v_d = 400$, $a\tau = 2618.19$, $x_0/d = 0.1532$, $\tau\Delta = 1413.01$. Middle column: \mathcal{R}/\mathcal{A} device with symmetry VI. Top: $\Omega_{\text{VI}}(x)$ (it is real); Bottom: transmission and reflection coefficients. $v_0/v_d = 400$, $b\tau = -244516.1$, $c\tau = 167853.9$, $x_0/d = 0.1679$, $\tau\Delta = 193.508$. Right column: “Partial”- \mathcal{TR}/\mathcal{A} device with symmetry I. Top: $\Omega_{\text{I}}(x)$, real (blue, solid line) and imaginary parts (orange, dashed line); Bottom: transmission and reflection coefficients. $v_0/v_d = 8$, $b\tau = 102.6520$, $c\tau = 165.8355$, $x_0/d = 0.1648$, $\tau\Delta = 90.5337$. In all cases $\tau = md^2/\hbar$ and $v_d = \hbar/(md)$.

then numerically optimized with the GRAPE (Gradient Ascent Pulse Engineering) algorithm [68, 69].

The Rabi frequencies will fulfill the indicated symmetries VIII, VI, and I. $\Omega_{\text{VI}}(x)$ should not be even (i.e. $b \neq c$) to avoid symmetry II. In addition, $\Omega_{\text{I}}(x)$ should not fulfill any other symmetry than I. The corresponding Rabi frequencies $\Omega(x)$ are depicted in Figs. 3.1, top row. The scattering coefficients are shown in the bottom row. Fig. 3.1 demonstrates that the three potentials satisfy the asymmetric response conditions imposed at the selected velocity and also in a region nearby.

The “partial”- \mathcal{TR}/\mathcal{A} device fulfills $|T^l|^2 = |R^l|^2 = 1/2$ and full absorption from the right. The potential we use for that device has symmetry I only, i.e., “no symmetry” other than the trivial commutation with the identity. No other potential symmetry would allow this type of device.

The effective non-local potential $V(x, y)$, see Eq. (3.8), corresponding to the v/v_d ratios used for the three devices is shown in Fig. 3.2. Note that the non-local potential has dimensions energy/length, so we divide the absolute value by a factor $V_0 = \hbar^2/(md^3)$ to plot a dimensionless quantity.

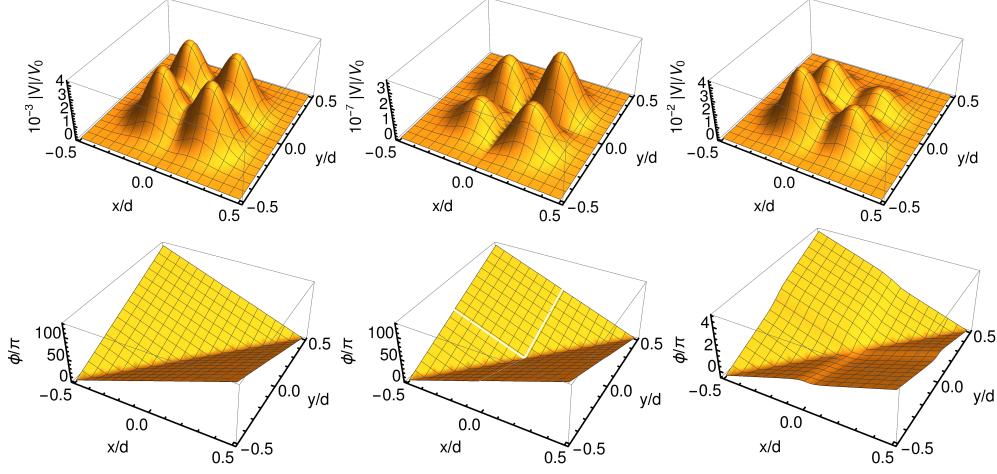


FIGURE 3.2: Nonlocal potentials $V(x, y)$: absolute value (top), argument (bottom). Left column: Potential for \mathcal{T}/\mathcal{A} device with symmetry VIII. Middle column: Potential for \mathcal{R}/\mathcal{A} device with symmetry VI. Right column: “Partial”- $\mathcal{T}\mathcal{R}/\mathcal{A}$ device with symmetry I. $V_0 = \hbar^2/(md^3)$.

In the parameter optimization we see that increasing the velocities further does not pose a problem for the \mathcal{T}/\mathcal{A} device, it is more challenging for a \mathcal{R}/\mathcal{A} device, and it is quite difficult for the partial- $\mathcal{R}\mathcal{T}/\mathcal{A}$ device. The device \mathcal{T}/\mathcal{A} is feasible for an experimental implementation as the ratio v_0/v_d can be easily increased to desired values, for reasonable values of the Rabi frequency and laser waist [70].

Moreover the velocity width with the desired behavior is much broader for \mathcal{T}/\mathcal{A} . Therefore a \mathcal{T}/\mathcal{A} device is the best candidate for an experimental implementation. As a check of feasibility, let us assume a Beryllium ion. Its hyperfine structure provides a good two-level system for which we can neglect decay (i.e. $\gamma \approx 0$ is indeed realistic). We have $m = 1.49 \times 10^{-26}$ kg and set a length $d = 10 \mu\text{m}$ compatible with the small laser waists (in this case $1.4 \mu\text{m}$) achieved for individual ion addressing [70]. The scaling factors take the values

$$\begin{aligned} v_d &= 0.67 \text{ mm/s,} \\ \tau &= 1.49 \times 10^{-2} \text{ s,} \end{aligned}$$

which gives $v \approx 27 \text{ cm/s}$ for $v/v_d = 400$, (again, we see no major obstacle to get devices for higher velocities, in particular the classical approximations in Sec. 3.5 can be used to estimate the values of the parameters) and Rabi frequencies, see Fig. 3.1, in the hundreds of kHz range. The relative ion-laser beam velocity could be as well implemented by moving the beam in the laboratory frame.

3.5 Classical approximation for \mathcal{T}/\mathcal{A} device

In a \mathcal{T}/\mathcal{A} device such as the one presented an incident plane wave from the left ends up as a pure transmitted wave with no reflection or absorption. However, a wave incident from the right is fully absorbed. How can that be? Should not the velocity-reversed motion of the transmitted wave lead to the reversed incident wave? For a more intuitive understanding we may seek help in the underlying two-level model. In the larger space the potential is again local and Hermitian. A simple semiclassical approximation is to assume that the particle moves with constant speeds $\pm v$ for left ($v > 0$) or right ($-v < 0$) incidence, so that at a given time it is subjected to the 2×2 time-dependent potentials $\mathcal{V}(\pm vt)$. The incidence from the left and right give different time dependences for the potential. The scattering problem then reduces to solving the time-dependent Schrödinger equation for the amplitudes of a two-level atom with time-dependent potential, i.e. to solving the following time-dependent Schrödinger equation ($\gamma = 0$)

$$i\hbar \frac{\partial}{\partial t} \chi_{\pm}(t) = \mathcal{V}(\pm vt) \chi_{\pm}(t), \quad (3.10)$$

with the appropriate boundary conditions $\chi_+(-\infty) = \chi_-(-\infty) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. The solutions for $v/v_d = 400$ are shown in Fig. 3.3. In Fig. 3.3(a), $\chi_+(t)$ (left incidence) is depicted: the particle ends with high probability in the ground state at final time. In Fig. 3.3(b), $\chi_-(t)$ (right incidence) demonstrates the ground state population is transferred to the excited state. Projected onto the ground-state level alone, this corresponds to full absorption of the ground state population at final time.

For an even rougher but also illustrative picture, again in a semiclassical time-dependent framework, we may substitute the smooth Gaussians for $\text{Re}(\Omega)$ and $\text{Im}(\Omega)$ in Fig. 3.1 by two simple, contiguous square functions of height $\Omega > 0$ and width $\tilde{w} > 0$. Then, the 2×2 potential at a given time is, in terms of Pauli matrices,

$$\mathcal{V}(x) = \frac{\hbar}{2} \Delta(\sigma_Z - \mathbf{1}) + \frac{\hbar}{2} \begin{cases} \Omega \sigma_X & -\tilde{w} < x < 0 \\ -\Omega \sigma_Y & 0 < x < \tilde{w} \\ 0 & \text{otherwise} \end{cases} \quad (3.11)$$

where $x = \pm vt$ and let $T = 2\tilde{w}/v$.

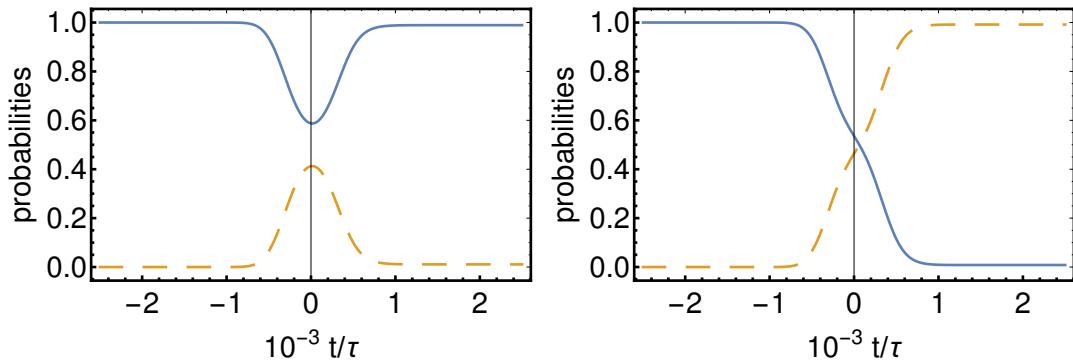


FIGURE 3.3: Simplified model of the asymmetric \mathcal{T}/\mathcal{A} device with symmetry VIII: (a) $\chi_+(t)$, (b) $\chi_-(t)$; ground-state population $|\chi_{\pm(t),1}|^2$ (blue, solid line), excited- $|\chi_{\pm(t),2}|^2$ (orange, dashed line). $v/v_d = 400$, $a\tau = 2618.19$, $x_0/d = 0.1532$, $\tau\Delta = 1413.01$.

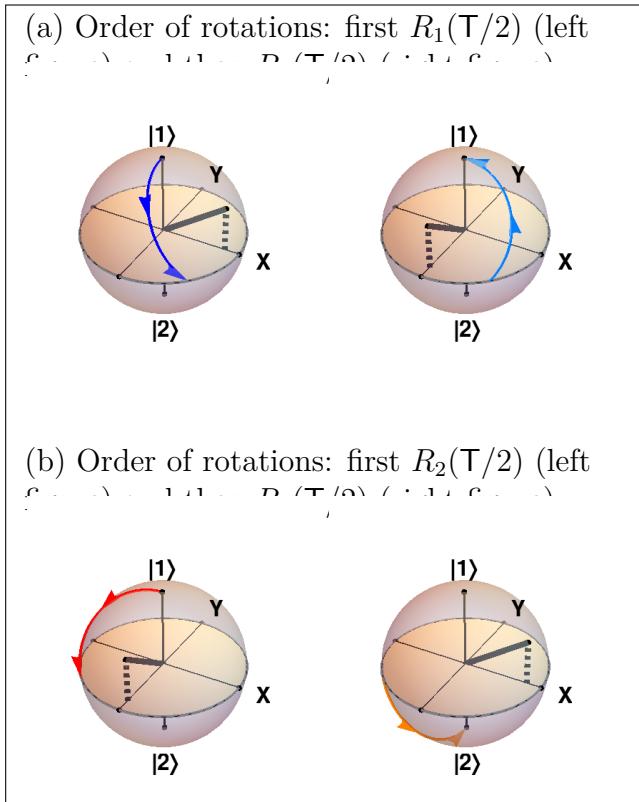


FIGURE 3.4: Simplified time-dependent model of the asymmetric \mathcal{T}/\mathcal{A} device with symmetry VIII: Bloch sphere explaining non time-reversal invariance, see text for details. The state trajectories are depicted in two-steps on the sphere. The rotation axes are also depicted. (a) The process simulates incidence from the left. The state starts and ends in $|1\rangle$. (b) The process simulates incidence from the right. The state starts at $|1\rangle$ and ends at $|2\rangle$.

The time-evolution of this process, $\chi_{\pm}(t)$, up to a phase factor may be regarded as two consecutive rotations $R_j = e^{-i\beta \mathbf{n}_j \cdot \boldsymbol{\sigma}/2}$ ($j = 1, 2$), with $\beta = \frac{T}{2}\sqrt{\Omega^2 + \Delta^2}$, of the two-level state on the Bloch sphere about the axes

$$\mathbf{n}_1 = \frac{1}{\sqrt{\Omega^2 + \Delta^2}}(\Omega, 0, \Delta), \quad (3.12)$$

$$\mathbf{n}_2 = \frac{1}{\sqrt{\Omega^2 + \Delta^2}}(0, -\Omega, \Delta). \quad (3.13)$$

The initial state at time $t = -T/2$ is again $\chi_+(-T/2) = \chi_-(-T/2) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. The unitary time-evolution operator to reach the final time $T/2$ takes the form $e^{i\Delta T/2}R_2R_1$ for incidence from the left (χ_+) and $e^{i\Delta T/2}R_1R_2$ for incidence from the right (χ_-). The time T and the parameters Ω, Δ will be fixed to reproduce the results of the full calculation with the exact model, namely, so that the system starts in the ground state to end either in the ground state ($|\chi_+(T/2)|^2 = 1$) or in the excited state by performing the rotations in one order or the reverse order ($|\chi_-(T/2)|^2 = 0$). This gives $\Omega/\Delta = \sqrt{2}$ and $T = 4\pi/(3\sqrt{3}\Delta)$. It follows that $\mathbf{n}_1 = \frac{1}{\sqrt{3}}(\sqrt{2}, 0, 1)$ and $\mathbf{n}_2 = \frac{1}{\sqrt{3}}(0, -\sqrt{2}, 1)$.

The different outcomes can thus be understood as the result of the non-commutativity of rotations on the Bloch sphere, see Fig. 3.4: In Fig. 3.4(a), first the rotation $R_1(T/2)$ and then the rotation $R_2(T/2)$ are performed. Starting in the ground state $|1\rangle$, the system ends up in the excited state $|2\rangle$. In Fig. 3.4(b), first the rotation $R_2(T/2)$ and then the rotation $R_1(T/2)$ are performed: now the system starts and ends in the ground state $|1\rangle$.

These results can be even used to approximate the parameters of the potential in the quantum setting. As an approximation of the height a we assume that the area $a \int_{-\infty}^{\infty} dx g(x) = a\sqrt{\pi}w$ is equal to $\tilde{w}\Omega = T v_0 \Omega / 2 = v_0 \pi (2/3)^{3/2}$. This results in an approximation $a \approx \frac{v_0}{w} \sqrt{\pi} (2/3)^{3/2}$. As an additional approximation, we assume that $(a/\sqrt{2})/\Delta \approx \Omega/\Delta = \sqrt{2}$, so we get $\Delta \approx a/2 \approx \frac{v_0}{2w} \sqrt{\pi} (2/3)^{3/2}$. A comparison between these approximations and the numerically achieved parameters, see Fig. 3.5, shows a good agreement over a large velocity range. This allows one to find good initial values for further numerical optimization.

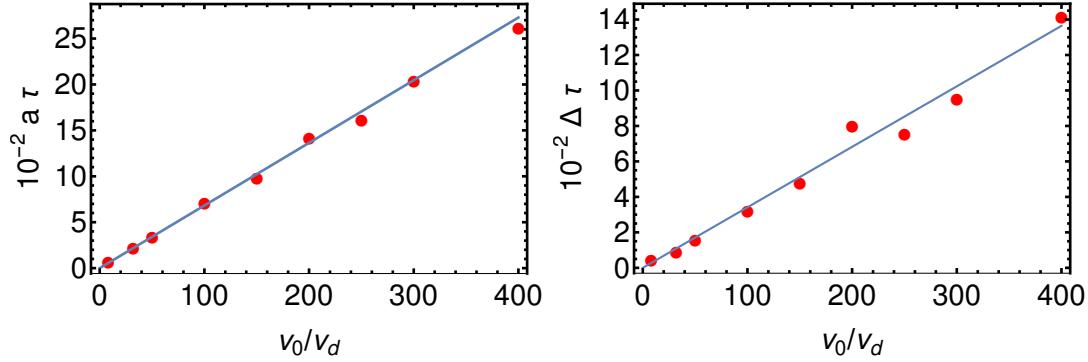


FIGURE 3.5: Asymmetric \mathcal{T}/\mathcal{A} device with symmetry VIII: comparison between numerically achieved parameters (red dots) and approximated parameters (blue, solid lines) versus velocity v_0 . (a) Height of Rabi frequency a , (b) detuning Δ .

3.6 Discussion

Non-Hermitian Hamiltonians display many interesting phenomena which are impossible for a Hermitian Hamiltonian acting on the same Hilbert space. In particular, in the Hilbert space of a single, structureless particle on a line formed by square integrable normalizable functions, Hermitian Hamiltonians do not allow, within a linear theory, for asymmetric scattering transmission and reflection coefficients. However, non-Hermitian Hamiltonians do. Since devices of technological interest, such as one-way filters for transmission or reflection, one-way barriers, one-way mirrors, and others, may be built based on such scattering response asymmetries, there is both fundamental interest and applications in sight to implement Non-Hermitian scattering Hamiltonians. This paper is a step forward in that direction, specifically we propose a quantum-optical implementation of potentials with asymmetric scattering response. They are non-local and non-PT symmetrical, which allows for asymmetric transmission.

In general the chosen Hilbert space may be regarded as a subspace of a larger space. For example, the space of a “structureless particle” in 1D is the ground-state subspace for a particle with internal structure, consisting of two-levels in the simplest scenario. It is then possible to regard the Non-Hermitian physics in the reduced space as a projection of the larger space, which may itself be driven by a Hermitian or a Non-Hermitian Hamiltonian. We have seen the Hermitian option in our examples, where we assumed a zero decay constant, $\gamma = 0$, for the excited state. A non-zero γ implies a Non-Hermitian Hamiltonian in the larger two-level space. The description may still be enlarged, including quantized field modes to account for the atom-field interaction with a Hermitian Hamiltonian. As

an outlook, depending on the application, there might be the need for a more fundamental and detailed descriptive level. Presently we discuss the desired physics (i.e., the scattering asymmetries) at the level of the smallest 1D space of the ground state, while taking refuge in the two-level space to find a feasible physical implementation.

Part II

Heat rectification in mesoscopic systems

Introduction to Part II

Radiation, heat and electricity are prominent mechanisms of energy transport. In particular, the two last mechanisms have a dominant role in technology. Modern information processing rests on electronic devices like the diode and the transistor. However, there is not an existing analogous technology to the diodes and transistors based on control of heat currents driven by phonons, *i.e.* the quasiparticles describing vibrational modes. An explanation could be that phonons are more difficult to control than electrons since, contrary to them, they do not have mass or electrical charge.

Energy transport due to phonons implies a rich set of physical mechanisms worth exploring to design phononic devices. Together with advancements in nanotechnology, this could boost the growth of phononics. The thermal rectifier, or thermal diode would be a primary building block to develop phononic devices and that is why I decided to study it.

Thermal rectification is the physical phenomenon, analogous to electrical current rectification in diodes, in which heat current through a device or medium (the thermal diode or rectifier) is not symmetric with respect to the exchange of the bath temperatures at the boundaries. It was first observed in 1936 by Starr in a junction between copper and cuprous oxide [71]. The theoretical work started much later using as rectifiers simple anharmonic chain models with different segments [72, 73]. These papers sparked much research that continues to this day. Research on thermal rectification has gained a lot of attention in recent years as a key ingredient to build prospective devices to control heat flows similarly to electrical currents [74, 75]. There are proposals to engineer thermal logic circuits [76] in which information, stored in thermal memories [77], would be processed in thermal gates [78]. Such thermal gates, as their electronic counterparts, will

require thermal diodes and thermal transistors to operate [79, 80]. Heat rectifying devices would also be quite useful in nano electronic circuits, letting delicate components dissipate heat while being protected from external heat sources [74].

Most work on thermal diodes has been theoretical with only a few experiments like Refs. [81–84]. A relevant attempt to build a thermal rectifier was based on a graded structure made of carbon and boron nitride nanotubes that transports heat between a pair of heating/sensing circuits [81]. One of the ends of the nanotube is loaded with a deposition of another material, which makes the heat flow better from the loaded end to the unloaded end. However, rectifications were small, with rectification factors (relative heat-flow differentials) around 7%.

Much of the theoretical effort in thermal rectification research has been aimed at improving the rectification factors and the features of the rectifiers. The first approach to designing thermal diodes consisted in using chains of oscillators segmented into two or more regions with different properties [72, 73, 85, 86], which is reminiscent of the idea of the *p-n* junction in electric diodes. It was soon realized, however, that the performance of segmented rectifiers was very sensitive to the size of the device, *i.e.*, rectification decreases with increasing the length of the rectifier [86]. To overcome this limitation two ideas were proposed. The first one consists in using graded rather than segmented chains, *i.e.*, chains where some physical property varies continuously along the site position such as the mass of particles in the chain [87–95]. The second one uses chains with long-range interactions (LRI), such that all the sites in the lattice interact with each other [88, 96, 97]. The rationale behind was that in a graded system new asymmetric, rectifying channels are created, while the long-range interactions create also new transport channels, avoiding the usual decay of heat flow with size [88]. Besides a stronger rectification power, LRI graded chains are expected to have better heat conductivity than segmented ones. This is an important point for technological applications, because devices with high rectification factors are not useful if the currents that flow through them are very small.

Another main focus of the theoretical research in thermal rectification is the search for the fundamental factors that contribute to the emergence of rectification. Historically the crucial elements for having rectification were identified as the presence of some structural asymmetry in the system and of nonlinear (anharmonic) forces [75, 85, 86, 98–102], which lead to a temperature dependence of the phonon bands or power spectral densities. A match or mismatch of the phonon bands of

neighboring parts of the chain implies corresponding good or bad conduction so the sign of the temperature bias may affect the conduction and lead to rectification when the spectra of the parts are affected differently by the bias reversal. However more recent research in the topic pointed out that anharmonicity is not a necessary condition for an asymmetric match/mismatch and therefore for rectification [103]. Rectification also occurs in simple (minimalistic) harmonic models that incorporate some structural asymmetry and temperature-dependence of the model parameters [103]. This dependence may indeed result from an underlying, more intricate anharmonic system by linearization of the stochastic dynamics [103, 104], or it may have a different origin [105].

In the chapters of this part I will present the contributions I made together with my group and other collaborators to the field of thermal rectification in chains of oscillators. In Chapter 4 I present a model of a thermal rectifier that relies on a localized impurity in the middle of the chain. With this approach, the proposal that I present in this chapter diverges from the prevalent approach of using segmented chains. In Chapter 5 I make a proposal for a thermal rectifier in a chain of trapped ions with a graded frequency distribution. This prototype seizes the combined power of long range interactions, which are naturally present in trapped ion chains due to the Coulomb force, and graded structures. This proposal brings the added value of being possible to be experimentally realized, since trapped ions constitute a well-developed and tested architecture for fundamental research, quantum information processing and quantum technologies such as detectors or metrology. This architecture is in principle scalable in driven ion circuits, see e.g. [106]. Controllable heat rectification in this context would be a useful asset for energy management. Finally, in Chapter 6 I study heat transport in a solvable model of two connected oscillators to explore the origin of thermal rectification. It is believed that non-linear interactions are needed in order to have rectification, since a system with non-linear interactions will have a temperature-dependent spectra. I show, however that it is possible to have temperature dependent features in a linear system that lead to rectification, in agreement with other works like [103].

Chapter 4

Heat Rectification with local impurities

In this chapter I present an atom-chain thermal rectifier. The chain is homogeneous except for boundary effects and a local modification of the interactions at one site, the “impurity”. The rectification mechanism is due here to the localized impurity, the only asymmetrical element of the structure, apart from the externally imposed temperature bias, and does not rely on putting in contact different materials or other known mechanisms such as grading or long-range interactions. The effect survives if all interaction forces are linear except the ones for the impurity.

4.1 Introduction

In spite of much work on thermal rectification after the first model proposal in 2002 by Terraneo *et al.* [72], the manipulation of phononic heat fluxes is still far from being completely controlled as no efficient and feasible thermal diodes have been found [88, 103]. A wealth of research is underway to meet the challenge posed by a “near standstill” of the field [88, 103], combined with the prospects of widespread and impactful practical applications. Together with experimental progress, at this stage work exploring new models is important to test possibilities that may become feasible as control capabilities improve [74]. In this chapter I propose, motivated by previous work on “atom diodes” [35, 36], a rectifying scheme based on the effect of a local defect, or impurity, in an otherwise homogeneous system.

To date, there have been several proposals of systems that could be used to rectify heat flows at the nanoscale. A common scheme is based on coupling two or more different homogeneous segments, modelled with chains of atoms with non-linear interactions (which in this context means non-linear forces, i.e., anharmonic potentials) [72, 73, 86, 100, 107] or with a temperature and position dependent conductivity assuming expressions for the heat current [107, 108]. Other systems proposed are graded materials, such as a chain with an uneven distribution of mass [81, 88, 98], and long-range interactions have been shown to be able to amplify the rectification and avoid or mitigate the decay of the effect with system size [88, 97]. Also, recent models and experiments use asymmetrical homogeneous or inhomogeneous nanostructures and, in particular, graphene [109, 110]. Finally, I mention for completeness theoretical works that consider the use of a quantum system [74], such as a three-level system, with each level coupled to a thermal bath [80], or a double well with different frequencies to implement the asymmetry [99].

The model proposed in this chapter is a one-dimensional chain of atoms where all, except one of them, are trapped in on-site harmonic potentials and interact with their nearest neighbours by Morse potentials (or also by harmonic potentials in a simplified version). Unlike most chain models, the structural asymmetry is, except for the boundaries, only in the impurity, which is subjected to a different on-site potential and interaction with one of its neighbors. The chain is connected to thermal baths at different temperatures at the boundaries. Let us clarify the terminology: The structural asymmetry I have in mind amounts, to

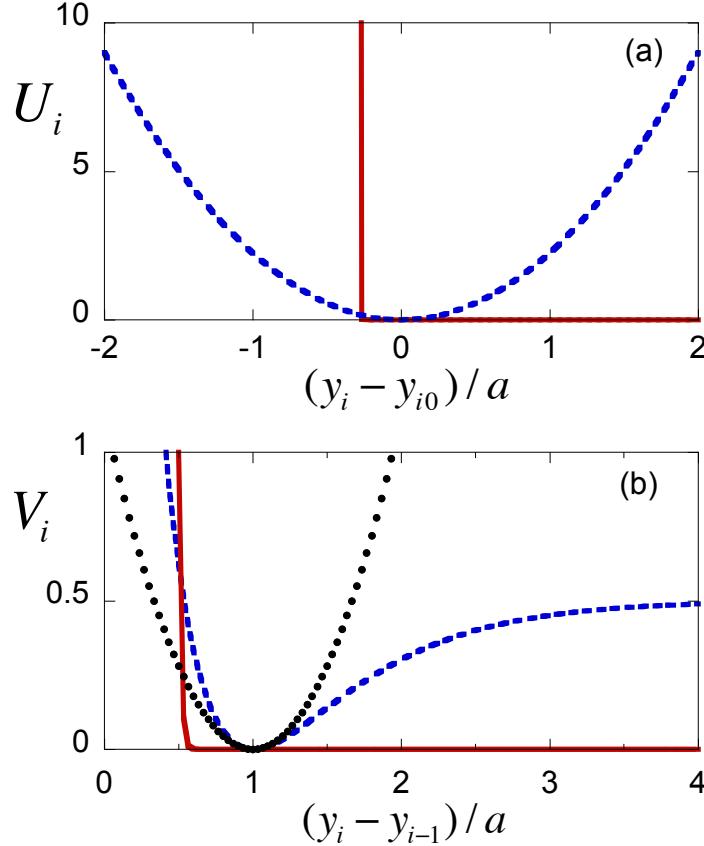


FIGURE 4.1: (a) On-site potentials: harmonic potential centered at the equilibrium position of each atom (dashed blue line) as a function of the displacement from this position $x_i = y_i - y_{i0}$ in a -units, and the on-site potential for the impurity, $i = N/2 + 1$ (N even, red solid line). (b) Interaction potentials as a function of the distance between nearest neighbors: Morse potential (blue dashed line) valid for all atoms except for $i = N/2 + 1$, N even, where the modified potential (red solid line) is used. The harmonic approximation of the Morse potential is also depicted (eq. (4.5), black dots, only used for fig. 4.5, below). Parameters: $D = 0.5$, $g = 1$, $\gamma = 45$, $d = 100$ and $b = 105$, used throughout the chapter.

different interaction potentials of one or several atoms in the chain (other than the boundary atoms) with neighbors on the right and on the left, and/or asymmetric onsite potentials. In this regard chains with Morse interatomic potentials per se are not considered structurally asymmetric here as long as their parameters are the same on both sides of a given atom, even if a single Morse potential is not parity-symmetric [111].

The rest of the chapter is organized as follows. In section 4.2 I shall describe the homogeneous 1D chain, without the impurity. For this system, I numerically solve the dynamical equations, to show that the usual heat conduction applies. In

section 4.3 I modify the potentials for one of the atoms and demonstrate the rectification effect. I also observe rectification when all the interaction Morse potentials are substituted by harmonic oscillators. Finally, in section 4.4 I summarize the results of this chapter and present the conclusions.

4.2 Homogeneous one-dimensional chain

I start with a homogeneous 1D chain with N atoms coupled at both extremes to heat baths, at different temperatures T_h and T_c for “hot” and “cold” respectively. The baths are modeled with a Nosé-Hoover method as described in [112]. Atoms 1 and N represent the first and the N -th atom in the chain, from left to right, that will be in contact with the baths. All the atoms are subjected to on-site potentials and to nearest-neighbor interactions, and their equilibrium positions y_{i0} are assumed to be equally spaced by a distance a . $x_i = y_i - y_{i0}$, $i = 1, \dots, N$, represent the displacements from the equilibrium positions of the corresponding atoms with positions y_i .

The classical Hamiltonian of the atom chain can be written in a general form as

$$H = \sum_{i=1}^N H_i, \quad (4.1)$$

with

$$\begin{aligned} H_1 &= \frac{p_1^2}{2m} + U_1(x_1) + V_L, \\ H_i &= \frac{p_i^2}{2m} + U_i(x_i) + V_i(x_{i-1}, x_i) \quad i = 2, \dots, N-1, \\ H_N &= \frac{p_N^2}{2m} + U_N(x_N) + V_N(x_{N-1}, x_N) + V_R, \end{aligned} \quad (4.2)$$

where the p_i are the momenta, $U_i(x_i)$ is the on-site potential for the i th atom, and $V_i(x_{i-1}, x_i)$ represents the atom-atom interaction potential. V_R and V_L are the interactions coupling the boundary atoms to the Nosé-Hoover thermostats, see [112].

There are a large number of 1D models that obey this general Hamiltonian. Different choices of the trapping and interaction potentials would give different conductivity behaviors. I choose a simple form of the Hamiltonian in which each

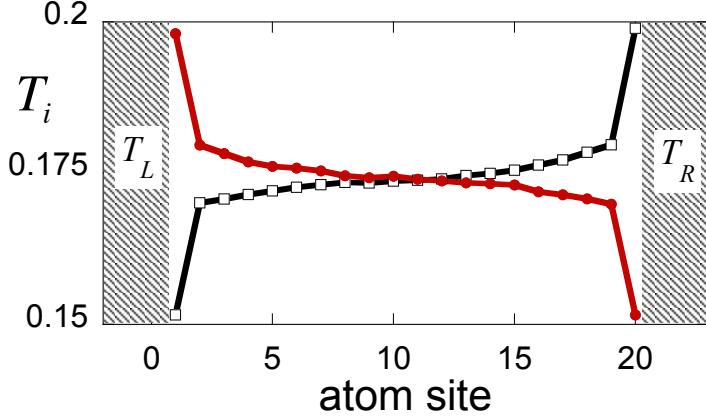


FIGURE 4.2: Symmetric temperature profiles for a homogeneous chain, without impurity. For $T_h = T_L$, $T_c = T_R$ (red solid dots) the (absolute value of) the heat flux is $J_{L \rightarrow R}$, equal to $J_{R \rightarrow L}$ for the reverse configuration of the bath temperatures, $T_h = T_R$, $T_c = T_L$ (black empty squares). Parameters as in fig. 4.1.

atom is subjected to a harmonic on-site potential and a Morse interaction potential between nearest neighbors (see fig. 4.1, dashed lines),

$$U_i(x_i) = \frac{1}{2}m\omega^2x_i^2, \quad (4.3)$$

$$V_i(x_{i-1}, x_i) = D \{e^{-\alpha[x_i - x_{i-1}]} - 1\}^2, \quad (4.4)$$

where ω is the trapping angular frequency, and D and α are time independent parameters of the Morse potential. A “minimalist version” of the model where V becomes the harmonic limit of eq. (4.4), dotted line in fig. 1, will also be considered in the final discussion,

$$V_i(x_{i-1}, x_i) = k(x_i - x_{i-1})^2/2, \quad k = 2D\alpha^2. \quad (4.5)$$

For convenience, dimensionless units are used and the mass of all particles is set to unity.

I start by studying the homogeneous configuration with no impurity and potentials (4.3) and (4.4), solving numerically the dynamical equations for the Hamiltonian (4.1) with a Runge-Kutta-Fehlberg algorithm. I have chosen a low number of atoms, $N = 20$, with thermal baths at $T_h = 0.20$ and $T_c = 0.15$ at both ends of the chain with 16 thermostats each. The real temperature is related to the dimensionless one through $T_{real} = Tma^2\omega^2/k_B$ so, for typical values $m \approx 10^{-26}$

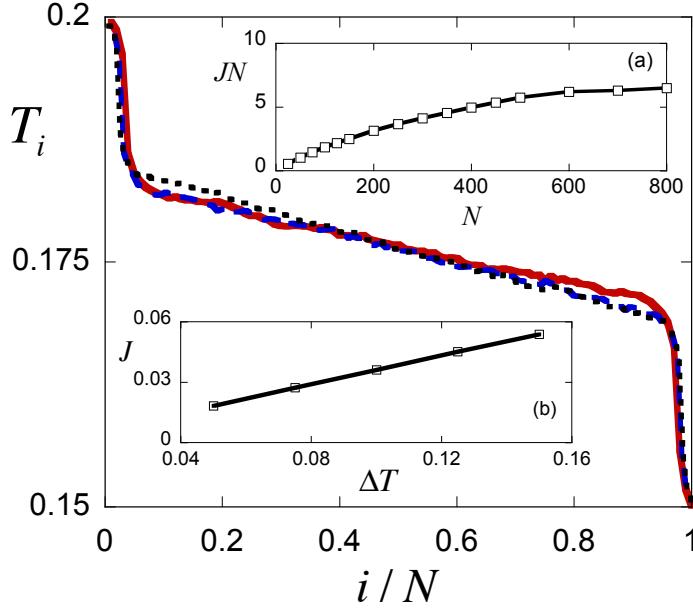


FIGURE 4.3: Temperature profile along the homogeneous chain for different number of atoms: 100 (dotted black line), 125 (dashed blue line) and 150 (solid red line). The atom sites have been rescaled with the total number of atoms. The time averages have been carried over a time interval of $\approx 2 \times 10^6$ after a transient of $\approx 1 \times 10^5$. In the inset (a), the product JN vs. N demonstrates that for long chains JN is independent of N . In (b) the linear dependence of J with ΔT for a fixed number of atoms, $N = 100$, is shown. Parameters as in fig. 4.1.

kg , $\omega \approx 10^{13} \text{ s}^{-1}$, $a \approx 10^{-10} \text{ m}$, and using $k_B = 1.38 \times 10^{-23} \text{ JK}^{-1}$, the dimensionless temperatures 0.15, 0.20, translate into 100, 150 K. It is advisable to use temperatures around these values in order to ensure that the displacements of the particles are realistic [113].

First I demonstrate the conductivity behavior of the model. To this end, I calculate the local heat flux J_i and temperature T_i , performing the numerical integration for long enough times to reach the stationary state. The local temperature is found as the time average $T_i = \langle p_i^2/m \rangle$, whereas J_i , from the continuity equation [114], is given by

$$J_i = -\dot{x}_i \frac{\partial V(x_{i-1}, x_i)}{\partial x_i}. \quad (4.6)$$

From now on I only consider the time average $\langle J_i(t) \rangle$, which converges to a constant value for all sites once the system is in the stationary nonequilibrium state. I depict the temperature profiles, for $N = 20$, first with $T_L = T_h$ and $T_R = T_c$ (L and R stand for left and right) and after switching the positions of the thermal baths in fig. 4.2. The profiles are symmetric, as expected, and the heat flux does not have

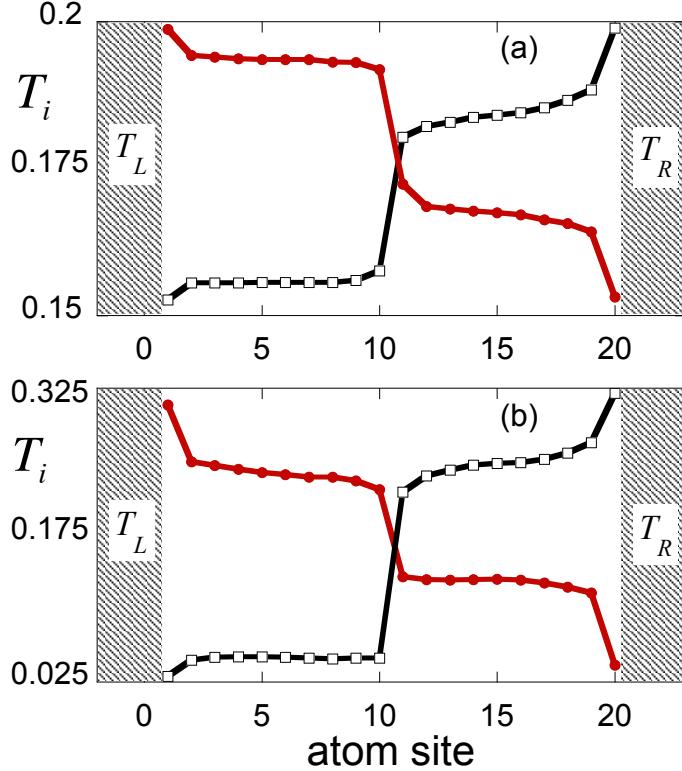


FIGURE 4.4: Temperature profile for the chain of $N = 20$ atoms, with an impurity in the $N/2 + 1$ position, with $T_L = T_h$ and $T_R = T_c$ (circles) and with the thermostat baths switched (squares). Parameters as in fig. 4.1. (a) $T_c = 0.15$, $T_h = 0.2$. $J_{L \rightarrow R} = 0.00769$ vs $J_{R \rightarrow L} = 0.00581$, with gives a rectification $R = 31\%$; (b) $T_c = 0.025$, $T_h = 0.325$. $J_{L \rightarrow R} = 0.0499$ vs $J_{R \rightarrow L} = 0.0140$, with $R = 256\%$.

a preferred direction [72, 114]. Denoting the absolute values of the fluxes from the left (when $T_L = T_h$) as $J_{L \rightarrow R}$, and from the right (when $T_R = T_h$) as $J_{R \rightarrow L}$, I find that $J_{L \rightarrow R} = J_{R \rightarrow L} = J = 1.6 \times 10^{-2}$, in the dimensionless units, consistent with the values found in other models [72, 114].

The profile of the temperature is linear with boundary nonlinearities at the edges, close to the thermal baths, due to the boundary conditions [115]. In fig. 4.3 I depict T_i vs i/N for $N = 100, 125$ and 150 with the same boundary conditions. For these larger atom numbers I have connected the first 3 and the last 3 atoms to the Nosé-Hoover baths. In the inset (a) of fig. 4.3 the product JN is plotted vs. N showing that in a low N limit there is a well defined conductivity per unit length whereas for longer chains, JN tends to be constant which indicates a normal thermal conductivity independent of the length. Fixing the number of atoms to 100, as in the inset (b) of fig. 4.3, I observe a linear dependence between the flux and ΔT .

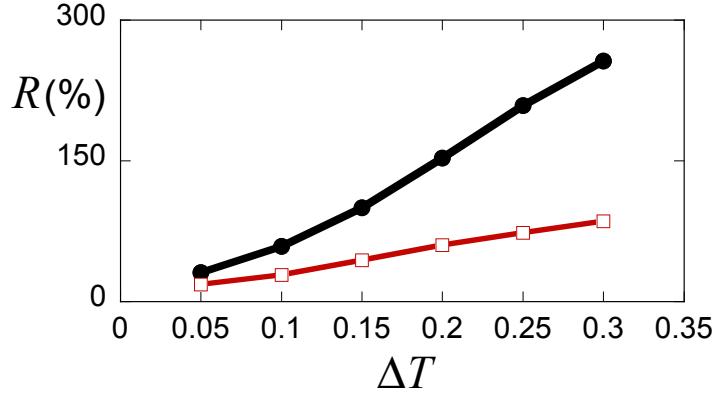


FIGURE 4.5: Rectification factor R as a function of the temperature difference between ends of the chain of atoms, ΔT . I have changed both T_h and T_c according to $T_c = 0.15 - (\Delta T - 0.05)/2$ and $T_h = 0.2 + (\Delta T - 0.05)/2$, with $N = 20$, keeping the rest of parameters as in fig. 4.1. Interatomic potentials: Morse potential, eq. (4.4) (black line with circles, see the temperature profiles of extreme points in fig. 4.4); harmonic potential, eq. (4.5) (red line with squares).

4.3 Impurity-based thermal rectifier

To rectify the heat flux I modify the potentials for site $j = N/2 + 1$ with N even, as

$$U_j(x_j, t) = de^{-b[x_j(t)+a/3]}, \quad (4.7)$$

$$V_j(x_{j-1}, x_j, t) = ge^{-\gamma[x_j(t)-x_{j-1}(t)+a/2]}. \quad (4.8)$$

All the parameters involved, d, b , and g, γ are time independent. In fig. 4.1 the modifications introduced with respect to the ordinary sites are shown (solid lines). The different on-site and interaction terms introduce soft-wall potentials (instead of hard-walls to aid integrating the dynamical equations) that make it difficult for the impurity to transmit its excitation to the left whereas left-to-right transmission is still possible. This effect is facilitated by the relative size of the coefficients, $a/3 < a/2$, that determine the position of the walls. These positions imply that an impurity excited by a hot right bath cannot affect its left cold neighbour near its equilibrium position at the $j - 1$ site. However, if the left $j - 1$ atom is excited from a hot bath on the left, it can get close enough to the impurity to kick it and transfer kinetic energy.

After extensive numerical simulations, I have chosen the values of these parameters as in fig. 4.1, such that the conductivity in the forward direction, $J_{L \rightarrow R}$, and

the rectification factor, defined as $R = (J_{L \rightarrow R} - J_{R \rightarrow L})/J_{R \rightarrow L} \times 100$, are both large for $T_h = 0.2$, $T_c = 0.15$. A large R without a large $J_{L \rightarrow R}$ could in fact be useless [74]. Note that the parameters are not necessarily the optimal combination, which in any case would depend on the exact definition of “optimal” (technically on how $J_{L \rightarrow R}/J$ and R are weighted and combined in a cost function and on the limits imposed on the parameter values). This definition is an interesting question but it goes beyond the focus of this chapter, which is to demonstrate and discuss the effect of the localized impurity.

I have used again $N = 20$ atoms connected to baths of 16 thermostats each, with the same temperatures as for the homogeneous chain, and numerically solved the dynamical equations to calculate the local temperature and the heat flux for both configurations of the baths. The interatomic potential for the regular atoms is the Morse potential (4.4). In fig. 4.4(a), the temperature profiles show a clear asymmetry between $L \rightarrow R$ and $R \rightarrow L$. Specifically, I find $J_{L \rightarrow R} = 7.6 \times 10^{-3}$ and $J_{R \rightarrow L} = 5.8 \times 10^{-3}$ which gives $R = 31\%$. The effect decays with longer chains, with, for example, $R = 19\%$ for $N = 100$, and $R = 17.8\%$ for $N = 150$.

These temperature profiles depend on the difference between the bath temperatures, see e.g. fig. 4.4(b). Increasing the temperature gap, but keeping T_h low enough so that the displacement of the atoms from their equilibrium positions is realistic, I find higher values of R . Figure 4.5 shows the strong dependence of R with ΔT (black circles). I have changed both T_h and T_c so that the mean temperature $(T_c + T_h)/2$ remains constant.

4.4 Discussion

I have presented a scheme for thermal rectification using a one-dimensional chain of atoms which is homogeneous except for the special interactions of one of them, the impurity, and the couplings with the baths at the boundaries. These proof-of-principle results for an impurity-based rectification mechanism may encourage further exploration of the impurity-based rectification, in particular of the effect of different forms for the impurity on-site potential and its interactions with neighboring atoms. In contrast to the majority of chain models, the structural asymmetry in the present model is only in the impurity. The idea of a localized effect was already implicit in early works on a two-segment Frenkel-Kontorova

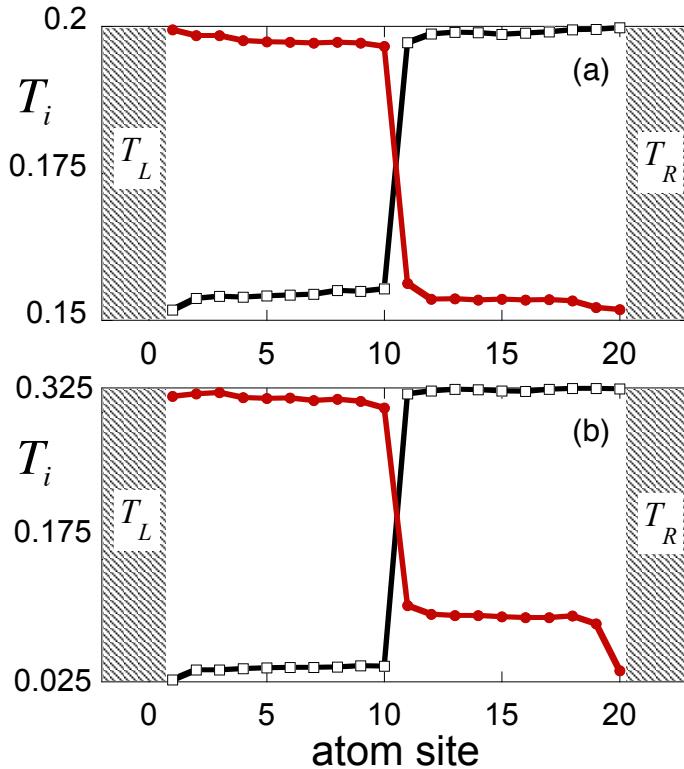


FIGURE 4.6: Temperature profile for a harmonic interacting chain of $N = 20$ atoms, with an impurity in the $N/2 + 1$ position, with $T_L = T_h$ and $T_R = T_c$ (circles) and with the thermostat baths switched (squares), for (a) $\Delta T = 0.05$ and (b) $\Delta T = 0.3$. The corresponding rectification factors are (a) $R = 18\%$ and (b) $R = 85\%$. Parameters regarding the impurity are the same as in fig. 4.1.

model [73, 86], where rectification depended crucially on the interaction constant coupling between the two segments. However, the coupling interaction was symmetrical and the asymmetry was provided by the different nature (parameters) of the segments put in contact. Also different from common chain models are the potentials chosen here. Instead of using the Morse potential as an on-site model, see e.g. [72], I have considered a natural setting where this potential characterizes the interatomic interactions, and the on-site potential is symmetrical with respect to the equilibrium position, and actually harmonic. The numerical results indicate that this model is consistent with normal conduction, and also helps to isolate and identify the local-impurity mechanism for rectification. In this regard it is useful to consider a further simplification, in the spirit of the minimalists models proposed by Pereira [103], so as to distill further the essence of the local rectification mechanism. If the Morse interatomic interaction is substituted by the corresponding harmonic interaction, see the black dotted line in fig. 4.1(b), the rectification effect remains, albeit slightly reduced, see fig. 4.5. The chain is then

perfectly linear with the only nonlinear exception localized at the impurity. The temperature dependent feature mentioned in [103] as the second necessary condition for rectification besides asymmetry, is here localized in the impurity too, and consists of a different capability to transfer kinetic energy depending on the temperatures on both sides of the impurity. Figure 4.6 shows temperature profiles for the purely harmonic chain to be compared with the Morse-interaction chain in fig. 4.4. Flatter profiles are found on both sides of the impurity, as corresponds to the abnormal transport expected for harmonic chains [116]. It would be interesting to combine the impurity effect with other rectification mechanisms (such as grading, long-range interactions, or use of different segments), or with more impurities in series to enhance further the rectification effect.

Even though the motivation was to mimic the effect of a localized atom diode that lets atoms pass only one way, unlike the atom diode [35], all interactions in the present model are elastic. The model may be extended by adding an irreversible, dissipative element so as to induce not only rectification but a truly Maxwell demon for heat transfer [39, 117]. On the experimental side, one dimensional chains of neutral atoms in optical lattices can be implemented with cold atoms [118]. An impurity with different internal structure could be subjected to a different on-site potential imprinted by a holographic mask [119], and asymmetrical interatomic interactions could be implemented by trapping a controllable polar molecule or mediated by atoms in parallel lattices [120].

Chapter 5

Heat Rectification in graded chains of trapped ions

I numerically demonstrate heat rectification for linear chains of ions in trap lattices with graded trapping frequencies, in contact with thermal baths implemented by optical molasses. The local temperatures and heat currents are calculated finding the stationary state by solving a system of algebraic equations. This approach is much faster than the usual method that integrates the dynamical equations of the system and averages over noise realizations.

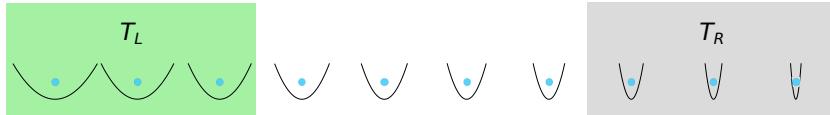


FIGURE 5.1: Schematic representation of the frequency-graded chain of trapped ions proposed as a thermal rectifier. The left and right ends of the chain are in contact with optical molasses at temperatures T_L and T_R (green and grey boxes respectively). Each ion is in an individual trap. The (angular) frequencies of the traps increase homogeneously from left to right, starting from ω_1 and ending at $\omega_1 + \Delta\omega$. The ions interact through the Coulomb force, which is long range, and therefore all the ions interact among them, even distant neighbors. By default I use 15 ions.

5.1 Introduction

Much effort has been aimed at improving the rectification factors and the features of the rectifiers. Some works relied, as in [72], on a chain segmented into two or more regions with different properties, but using other lattice models such as the Frenkel-Kontorova (FK) model [85, 86]. However, the rectification power of segmented rectifiers decays fast as their size increase [86]. As discussed in II, two workarounds have been proposed: chains with a graded distribution for some structural property [87–95] and long range interactions [88, 96, 97]. Additionally, long range interactions fix the problem of decaying maximum currents as the size of the rectifier increases.

In this chapter I propose to bridge the gap between mathematical models and actual physical systems exploring the implementation of a heat rectifier in a realistic, graded system with long-range interactions: a chain of ultracold ions in a segmented Paul trap with graded microtraps for each ion. Long-range interactions are due to the Coulomb forces, and the baths at the ends of the chain may be implemented with optical molasses, see Fig. 5.1. The trapping frequencies of the microtraps are controlled individually in order to create a graded and asymmetric trap-frequency profile along the chain. This asymmetry will lead to a heat flow that depends on the sign of the temperature difference of the baths. Heat transport in trapped ion chains has been studied in several works [121–125] and interesting phenomena like phase transitions have been investigated [121–124]. The idea of using locally-controlled traps is already mentioned in [121] to implement disorder and study its effects. The device I present here may be challenging to implement, but at reach with the current technology, in particular that of microfabricated traps [126–128]. Thus the setting is thought for a small, realistic

number of controllable ions.

The chapter is organized as follows. In Section 5.2 I describe the physical system of trapped ions with graded trap frequencies. I also set the stochastic dynamics due to the action of lasers at the chain edges. In Section 5.3 I implement an efficient method to find the steady state using Novikov's theorem and solving an algebraic system of equations. In Section 5.4 I present simulations of this system exhibiting thermal rectification and discuss the dependence with ion number, different options for the ion-laser coupling, and the advantages/disadvantages of using a graded frequency profile instead of a segmented one. Finally, in Section 5.5 I summarize the conclusions, and discuss connections with other works.

5.2 Physical System

Consider a linear lattice of N individual harmonic traps of (angular) trapping frequencies ω_n evenly distributed along the x axis at a distance a from each other. Each trap contains a single ion that interacts with the rest via Coulomb potentials. All the ions are of the same species, with mass m and charge q . The Hamiltonian that describes the dynamics of the system is (I consider only one-dimensional motion along the chain axis)

$$H(\mathbf{x}, \mathbf{p}) = \sum_{n=1}^N \left[\frac{p_n^2}{2m} + \frac{m\omega_n^2}{2}(x_n - x_n^{(0)})^2 \right] + V_{int}(\mathbf{x}), \quad (5.1)$$

where $\{x_n, p_n\}$, position and momentum of each ion, are the components of the vectors \mathbf{x}, \mathbf{p} , $x_n^{(0)} = na$ are the centers of the harmonic traps, and V_{int} is the sum of the Coulomb interaction potential between all pairs of ions,

$$V_{int}(\mathbf{x}) = \frac{1}{2} \sum_n \sum_{l \neq n} V_C(|x_n - x_l|), \quad (5.2)$$

with $V_C(|x_n - x_l|) = \frac{q^2}{4\pi\varepsilon_0} \frac{1}{|x_n - x_l|}$. The ends of the chain are in contact with two thermal reservoirs at temperatures T_L for the left bath and T_R for the right bath respectively. The action of the reservoirs on the dynamics of the chain is modeled via Langevin baths at temperatures T_L and T_R [116, 129]. The equations of motion

of the chain, taking into account the baths and the Hamiltonian, are

$$\begin{aligned}\dot{x}_n &= \frac{1}{m} p_n, \\ \dot{p}_n &= -m\omega_n^2(x_n - x_n^{(0)}) - \frac{\partial V_{int}}{\partial x_n} - \frac{\gamma_n}{m} p_n + \xi_n(t),\end{aligned}\tag{5.3}$$

where γ_n and $\xi_n(t)$ are only non-zero for the ions in the end regions, in contact with the left and right baths in the sets $\mathcal{L} = \{1, 2, \dots, N_L\}$ and $\mathcal{R} = \{N - (N_R - 1), \dots, N - 1, N\}$, see Fig. 5.1. The γ_n are friction coefficients and $\xi_n(t)$ are uncorrelated Gaussian noise forces satisfying $\langle \xi_n(t) \rangle = 0$ and $\langle \xi_n(t) \xi_m(t') \rangle = 2D_n \delta_{nm} \delta(t - t')$, D_n being the diffusion coefficients. These Gaussian forces are formally the time derivatives of independent Wiener processes (Brownian motions) $\xi_n(t) = \sqrt{2D_n} \frac{dW_n}{dt}$ [122, 130] and Eq. (5.3) is a stochastic differential equation (SDE) in the Stratonovich sense [130].

The baths are physically implemented by optical molasses consisting of a pair of counterpropagating Doppler-cooling lasers [122]. The friction and diffusion coefficients for the ions in contact with the baths are given by [122, 131, 132]

$$\begin{aligned}\gamma_n &= -4\hbar k_{L,R}^2 \left(\frac{I_{L,R}}{I_0} \right) \frac{2\delta_{L,R}/\Gamma}{[1 + (2\delta_{L,R}/\Gamma)^2]^2}, \\ D_n &= \hbar^2 k_{L,R}^2 \left(\frac{I_{L,R}}{I_0} \right) \frac{\Gamma}{1 + (2\delta_{L,R}/\Gamma)^2}, \\ n &\in \mathcal{L}, \mathcal{R},\end{aligned}\tag{5.4}$$

where k_L (k_R) and I_L (I_R) are the wave vector and intensity of the left (right) laser. δ_L (δ_R) is the detuning of the left (right) laser with respect to the angular frequency ω_0 of the atomic transition the laser is exciting, and Γ is the corresponding natural line width of the excited state. The expressions in Eq. (5.4) are valid only if the intensities of the lasers are small compared to the saturation intensity I_0 , $I_{L,R}/I_0 \ll 1$. In this bath model, the friction term in Eq. (5.3) comes from the cooling action of the laser and the white noise force $\xi_n(t)$ corresponds to the random recoil of the ions due to spontaneous emission of photons [131, 132]. Using the diffusion-dissipation relation $D = \gamma k_B T$ [133], the temperatures of the optical molasses baths are given by

$$T_{L,R} = -\frac{\hbar\Gamma}{4k_B} \frac{1 + (2\delta_{L,R}/\Gamma)^2}{(2\delta_{L,R}/\Gamma)},\tag{5.5}$$

with k_B being the Boltzmann constant. If the laser intensities are low enough,

the temperatures of the baths are controlled by modifying the detunings. When $\delta = \delta_D = -\Gamma/2$ the optical molasses reach their minimum temperature possible, the Doppler limit $T_D = \hbar\Gamma/(2k_B)$. Note that away from the Doppler limit the same temperature may be achieved for two different values of detuning. These two possibilities imply different couplings (two different pairs of γ and D values) and thus different physical effects that will be studied in Sec. 5.4.3.

5.3 Calculation of the stationary heat currents

The local energy of each site is defined by

$$H_n = \frac{1}{2m}p_n^2 + \frac{1}{2}m\omega_n^2(x_n - x_n^{(0)})^2 + \frac{1}{2}\sum_{l \neq n}V_C(|x_n - x_l|). \quad (5.6)$$

Differentiating H_n with respect to time I find the continuity equation

$$\dot{H}_n = \frac{p_n}{m}\left[\xi_n(t) - \gamma_n\frac{p_n}{m}\right] - \frac{1}{2m}\sum_{l \neq n}\frac{\partial V_C(|x_n - x_l|)}{\partial x_n}(p_n + p_l). \quad (5.7)$$

Two different contributions can be distinguished: $j_n^B \equiv \frac{p_n}{m}[\xi_n(t) - \gamma_n\frac{p_n}{m}]$, which is the energy flow from the laser reservoir to the ions at the edges of the chain (only for $n \in \mathcal{L}, \mathcal{R}$), and $\dot{H}_n^{int} \equiv -\frac{1}{2m}\sum_{l \neq n}\frac{\partial V_C(|x_n - x_l|)}{\partial x_n}(p_n + p_l)$, which gives the “internal” energy flow due to the interactions with the rest of the ions. In the steady state $\langle \dot{H}_n \rangle = 0$, and therefore

$$\langle j_n^B \rangle + \langle \dot{H}_n^{int} \rangle = 0, \quad (5.8)$$

where $\langle \dots \rangle$ stands for the expectation value with respect to the ensemble of noise processes $\boldsymbol{\xi}(t)$ ($\boldsymbol{\xi}$ represents a vector with components ξ_n). Equation (5.8) implies that, in the steady state, the internal rates \dot{H}_n^{int} vanish for the inner ions of the chain because $j_n^B = 0$ for $n \notin \mathcal{L}, \mathcal{R}$. In chains with nearest-neighbor (NN) interactions, $\langle \dot{H}_n^{int} \rangle$ simplifies to two compensating and equal-in-magnitude contributions that define the homogeneous heat flux across the chain. For long-range interactions this is not so and defining the flux is not so straightforward. A formal possibility is to impose nearest-neighbor interatomic interactions for some atoms in the chain [88], but this approach is not realistic in the current system so I define

instead the heat currents for the left and right baths as

$$\begin{aligned} J_L(t) &= \sum_{n \in \mathcal{L}} \langle j_n^B \rangle, \\ J_R(t) &= \sum_{n \in \mathcal{R}} \langle j_n^B \rangle, \end{aligned} \quad (5.9)$$

respectively. These expressions are in general time-dependent. In the steady state $J_{L,\text{steady}}$ and $J_{R,\text{steady}}$ should cancel each other since the local energies stabilize and internal energy flows cancel. I will use either $J_{L,\text{steady}}$ or $J_{R,\text{steady}}$ to calculate the total energy flow in the chain, always taking the absolute value, i.e., $J \equiv |J_{L,\text{steady}}| = |J_{R,\text{steady}}|$. J is defined as J_\rightarrow when the hot bath is on the left and J_\leftarrow when it is on the right.

To compute the average heat fluxes of the baths $\langle j_n^B \rangle$ in Eq. (5.9) I need the averages $\langle p_n(t)\xi_n(t) \rangle$. Instead of explicitly averaging $p_n(t)\xi_n(t)$ over different realizations of the white noise I use Novikov's theorem [130, 134, 135]. Novikov's theorem states that the ensemble average (over the realizations of the noise) of the product of some functional $\phi(t)$, which depends on a set of n_{noise} Gaussian noises $\xi_i(t)$ with zero mean value, $\langle \xi_i(t) \rangle = 0$, and the noise itself, is given by

$$\langle \xi_i(t)\phi(t) \rangle = \sum_{j=1}^{n_{\text{noise}}} \int_0^t dt' \langle \xi_i(t)\xi_j(t') \rangle \left\langle \frac{\delta\phi(t)}{\delta\xi_j(t')} \right\rangle, \quad (5.10)$$

where $\delta\phi(t)/\delta\xi_j(t')$ is the functional derivative of $\phi(t)$ with respect to the j -th component of the noise. Since the noises are independent from each other and δ -correlated I will have

$$\begin{aligned} \langle \xi_i(t)\phi(t) \rangle &= 2D_i \underbrace{\int_0^t dt' \delta(t-t') \left\langle \frac{\delta\phi(t)}{\delta\xi_i(t')} \right\rangle}_{= \frac{1}{2} \lim_{t' \rightarrow t^-} \left\langle \frac{\delta\phi(t)}{\delta\xi_i(t')} \right\rangle} \\ &= D_i \lim_{t' \rightarrow t^-} \left\langle \frac{\delta\phi(t)}{\delta\xi_i(t')} \right\rangle. \end{aligned} \quad (5.11)$$

The $1/2$ factor from the integral in Eq. (5.11) comes from the assumption that the dirac delta is the limit of even correlation functions when the correlation time goes to 0. The notation $\lim_{t' \rightarrow t^-}$ stands for the limit when t' goes to t from below ($t' < t$). To evaluate the functional derivatives of the position and of the position $x_n(t)$ and momentum $p_n(t)$ coordinates with respect to the white noises I integrate Eq. (5.3) to have its formal solution as a functional depending on the

white Gaussian noises $\xi_n(t)$,

$$\begin{aligned} x_n(t) &= x_n(0) + \frac{1}{m} \int_0^t ds p_n(s), \\ p_n(t) &= p_n(0) + \int_0^t ds \left[-\frac{\partial H}{\partial x_n}(s) - \frac{\gamma_n}{m} p_n(s) + \xi_n(s) \right]. \end{aligned} \quad (5.12)$$

Taking the functional derivatives of Eq. (5.12) I get for $t' < t$

$$\begin{aligned} \frac{\delta x_n(t)}{\delta \xi_m(t')} &= \frac{1}{m} \int_{t'}^t ds \frac{\delta p_n(s)}{\delta \xi_m(t')}, \\ \frac{\delta p_n(t)}{\delta \xi_m(t')} &= \delta_{nm} - \int_{t'}^t ds \frac{\delta}{\delta \xi_m(t')} \left[\frac{\partial H}{\partial x_n}(s) + \frac{\gamma_n}{m} p_n(s) \right]. \end{aligned} \quad (5.13)$$

The limits in the integrals in Eq. (5.13) go from t' to t since the functional derivatives are 0 when $s < t'$, because the values of the noise in the future cannot affect a signal in the present, which would break causality. I have also used that $\frac{\delta \xi_n(t)}{\delta \xi_m(t')} = \delta_{nm} \delta(t - t')$. Taking the limit $t' \rightarrow t^-$ in Eq. (5.12) I obtain the values of the functional derivatives of x_n and p_n , $\lim_{t' \rightarrow t^-} \delta x_n(t)/\delta \xi_m(t') = 0$ and $\lim_{t' \rightarrow t^-} \delta p_n(t)/\delta \xi_m(t') = \delta_{nm}$ (δ_{nm} is the usual Kronecker delta symbol). Thus I have $\langle x_n(t) \xi_m(t) \rangle = 0$ and $\langle p_n(t) \xi_m(t) \rangle = \delta_{nm} D_m$, which gives for the heat flow from the baths

$$\langle j_n^B \rangle = \frac{1}{m} \left[D_n - \gamma_n \frac{\langle p_n^2 \rangle}{m} \right]. \quad (5.14)$$

In all simulations I check that $|J_{L,\text{steady}}| = |J_{R,\text{steady}}|$ within the numerical tolerance of the computer. To measure the asymmetry of the heat currents I use the rectification factor R defined as

$$R = \frac{J_\rightarrow - J_\leftarrow}{\max(J_\rightarrow, J_\leftarrow)}. \quad (5.15)$$

R values may go from -1 to 1 (In the figures I depict it in % between -100% and 100%). If there is no rectification $J_\rightarrow = J_\leftarrow$ and $R = 0$. For perfect rectification in the right (left) direction, $J_\rightarrow \gg J_\leftarrow$ ($J_\rightarrow \ll J_\leftarrow$), and $R = 1$ ($R = -1$). Take note that other definitions of rectification factors exist in many works on asymmetric heat transfer so comparisons should be done with care.

This model does not show the antithermodynamical behavior observed in other models [136, 137], and heat is found to flow in all cases from the hot to the cold bath.

5.3.1 Algebraic, small-oscillations approach to calculate the steady state

To find the temperature profiles and heat currents in the steady state the usual approach is to solve the SDE system in Eq. (5.3) up to long times and for many realizations of the white noises $\xi(t)$. In that way the ensemble averages $\langle p_n(t \rightarrow \infty)^2 \rangle$, necessary for both the temperature profiles and heat currents, are computed. This standard route implies a heavy computational effort, in particular when I want to study the heat transport for several bath configurations, frequency increments and chain parameters. It is possible to circumvent this difficulty and find ensemble averages like $\langle x_n x_m \rangle$, $\langle x_n p_m \rangle$, $\langle p_n p_m \rangle$ (second order moments) without integrating any SDE [138–140]. The idea is to impose the condition $d \langle \dots \rangle / dt = 0$ for all the second order moments and linearize the dynamical equations of the system around equilibrium. A system of linear algebraic equations for the moments results, that can be easily solved without solving the SDE many times.

To linearize the SDE in Eq. (5.3) I approximate the potential energy of the Hamiltonian in Eq. (5.1), $V(\mathbf{x}) = V_{int}(\mathbf{x}) + m \sum_n \omega_n^2 (x_n - x_n^{(0)})^2 / 2$, by its harmonic approximation around the equilibrium positions \mathbf{x}^{eq} , defined by $\frac{\partial V(\mathbf{x})}{\partial \mathbf{x}} \Big|_{\mathbf{x}=\mathbf{x}^{eq}} = 0$. The approximate potential (ignoring the zero-point energy) is

$$V(\mathbf{x}) \approx \frac{1}{2} \sum_{n,m} K_{nm} (x_n - x_n^{eq})(x_m - x_m^{eq}), \quad (5.16)$$

with $K_{nm} = \frac{\partial^2 V(\mathbf{x})}{\partial x_n \partial x_m} \Big|_{\mathbf{x}=\mathbf{x}^{eq}}$ being the Hessian matrix entries of $V(\mathbf{x})$ around the equilibrium configuration [141]

$$K_{nm} = \begin{cases} m\omega_n^2 + 2 \left(\frac{q^2}{4\pi\varepsilon_0} \right) \sum_{l \neq n} \frac{1}{|x_n^{eq} - x_l^{eq}|^3} & \text{if } n = m \\ -2 \left(\frac{q^2}{4\pi\varepsilon_0} \right) \frac{1}{|x_n^{eq} - x_m^{eq}|^3} & \text{if } n \neq m \end{cases}. \quad (5.17)$$

Note that this approximation does not modify the two main features of the system, namely asymmetry and long-range interactions, which are manifest in the asymmetric distribution of ω_n and the non-zero off-diagonal elements of the K matrix, respectively. In the following I will use $y_n = x_n - x_n^{eq}$ to simplify the notation.

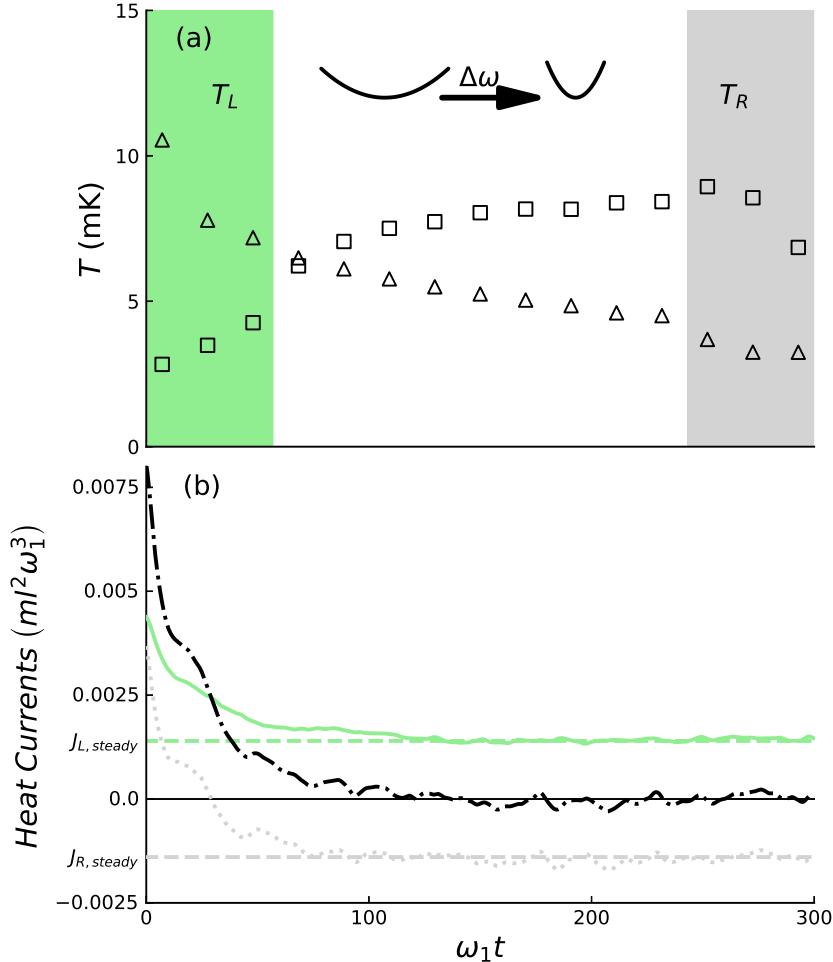


FIGURE 5.2: (a) Temperatures of the ions in the stationary state for a graded chain with the parameters described in section 5.4.1. The temperature profiles found with the algebraic method (Eq. (5.19)) are indistinguishable from the ones found solving the Langevin equation (Eq. (5.3)). Empty triangles (squares) correspond to $T_L = T_H$ ($T_L = T_C$) and $T_R = T_C$ ($T_R = T_H$). (b) Heat currents as a function of time for $T_L = T_H$ and $T_R = T_C$, see Eq. (5.9): $J_L(t)$ (solid green line) from the left reservoir into the chain; $J_R(t)$ (dotted grey line) from the right reservoir into the chain (negative except at very short times); $J_L(t) + J_R(t)$ (dotted-dashed black line), which must go to zero in the steady state. The three lines tend to stationary values marked by horizontal lines. Parameters: $\omega_1 = 2\pi \times 50$ kHz, $a = 50$ μm , $\delta_H = -0.02\Gamma$, and $\delta_C = -0.1\Gamma$, which gives temperatures $T_H \approx 12$ mK and $T_C \approx 3$ mK. $\Delta\omega = 0.5\omega_1$. In all figures $\Gamma = 2\pi \times 41.3$ MHz.

The linearized dynamics around the equilibrium positions are given by

$$\begin{aligned}\dot{y}_n &= \frac{1}{m} p_n, \\ \dot{p}_n &= - \sum_l K_{nl} y_l - \frac{\gamma_n}{m} p_n + \xi_n(t).\end{aligned}\tag{5.18}$$

Now, I set $d\langle\cdots\rangle/dt = 0$ for all the moments. Using Eq. (5.18) and applying Novikov's theorem I find

$$\begin{aligned}\langle p_n p_l \rangle - \gamma_l \langle y_n p_l \rangle - \sum_m K_{lm} \langle y_n y_m \rangle &= 0, \\ \sum_m [K_{nm} \langle y_m p_l \rangle + K_{lm} \langle y_m p_n \rangle] + \frac{1}{m} (\gamma_l + \gamma_n) \langle p_n p_l \rangle &= 2\delta_{nl} D_n.\end{aligned}\tag{5.19}$$

The system (5.19) is linear in the second order moments so it can be solved numerically to find the steady-state values of the moments. Besides Eq. (5.19) I have that $\langle y_n p_l \rangle = -\langle y_l p_n \rangle$, which follows from Eq. (5.18) and $d\langle y_n y_m \rangle / dt = 0$. Since there are $\frac{1}{2}N(N-1)$ independent $\langle y_n p_l \rangle$ moments, I choose the ones with $n < l$. Similarly, the moments $\langle y_n y_l \rangle$ and $\langle p_n p_l \rangle$ contribute with $\frac{1}{2}N(N+1)$ independent variables each and I choose the ones with $n \leq m$. Thus there are in total $\frac{1}{2}N(3N+1)$ independent moments that I arrange in the vector

$$\boldsymbol{\eta} = \left[\langle y_1 y_1 \rangle, \langle y_1 y_2 \rangle, \dots, \langle y_N y_N \rangle, \right. \\ \left. \langle p_1 p_1 \rangle, \langle p_1 p_2 \rangle, \dots, \langle p_N p_N \rangle, \right. \\ \left. \langle y_1 p_2 \rangle, \langle y_1 p_3 \rangle, \dots, \langle y_{N-1} p_N \rangle \right]^T.\tag{5.20}$$

There are the same number of independent equations as independent moments: N^2 equations correspond to the first line in Eq. (5.19), and $\frac{1}{2}N(N+1)$ equations to the second line because of the symmetry with respect to n, l . The system of equations (5.19) may be compactly written as $\mathbf{A}\boldsymbol{\eta} = \mathbf{B}$, where \mathbf{A} and \mathbf{B} are a $\frac{1}{2}N(3N+1)$ square matrix and vector.

5.4 Numerical Results

I now display the results of the simulations. To find the temperature profiles and the currents in the steady state I use the algebraic method described in section 5.3.1. I also check that the results coincide with those by solving Eq. (5.3) for many different realizations of the noisy forces $\xi(t)$ and averaging. The code for all the numerical simulations has been written in the language *Julia* [142, 143]. In particular, to solve the Langevin equation, I used *Julia*'s package *DifferentialEquations.jl* [144].

To model the baths and the chain I use atomic data taken from ion trap experiments [145, 146]. I consider 15 $^{24}\text{Mg}^+$ ions in all figures except in Fig. 5.6. The three leftmost and three rightmost ions are illuminated by Doppler cooling lasers. The Doppler cooling lasers excite the transition $3s^2 S_{1/2} \rightarrow 3p^2 P_{1/2}$, with angular frequency $\omega_0 = 2\pi \times 1069$ THz and excited state line width $\Gamma = 2\pi \times 41.3$ MHz [122]. For this ionic species and atomic transition the Doppler limit is $T_D = 1$ mK. The intensities of the laser beams are small compared to the saturation intensity I_0 so that Eq. (5.4) holds. I take $I_n/I_0 = 0.08$ for the ions in the laser beams, whereas $I_n = 0$ for the rest.

The temperatures T_L, T_R of the left and right laser baths are controlled with their detunings δ_L, δ_R with respect to the atomic transition. I fix two values for the detunings, δ_H and δ_C , such that $T_H > T_C$ (hot and cold baths, also source and drain) and I define $J_\rightarrow (J_\leftarrow)$ as the stationary heat current in the chain when $T_L = T_H$ and $T_R = T_C$ ($T_L = T_C$ and $T_R = T_H$).

Except in Sec. 5.4.5 I consider a graded frequency profile. If the frequency of the leftmost trap is ω_1 , the frequency of the n th trap will be $\omega_n = \omega_1 + \Delta\omega \frac{n-1}{N-1}$ up to $\omega_1 + \Delta\omega$ for the rightmost trap. In Sec. 5.4.5 I compare the graded chain to a segmented chain, where the left half of the chain has trapping frequencies ω_1 while the other half has $\omega_1 + \Delta\omega$.

5.4.1 Evolution to steady state

To compare the results by solving Eq. (5.3) and averaging and those from the algebraic method I simulated a frequency graded chain with a trapping frequency $\omega_1 = 2\pi \times 50$ kHz for the leftmost ion, see Fig. 5.2. The number of ions interacting

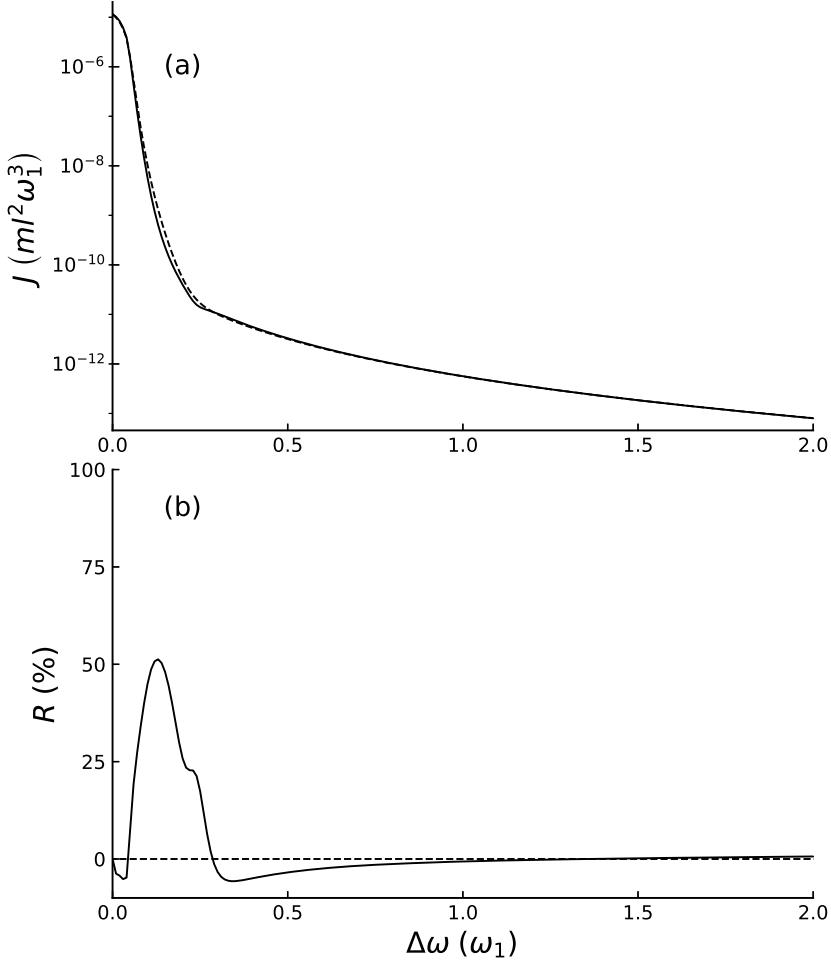


FIGURE 5.3: Graded chain of $N = 15$ $^{24}\text{Mg}^+$ ions. (a) Stationary fluxes for different frequency increments: J_{\rightarrow} (for $T_L = T_H$ and $T_R = T_C$, dashed line); J_{\leftarrow} (for $T_L = T_C$ and $T_R = T_H$, solid line) (b) Rectification factor. Parameters: $\omega_1 = 2\pi \times 1$ MHz, $l = 5.25$ μm , $a = 4.76 l$ (25 μm), $\delta_H = -0.02 \Gamma$, and $\delta_C = -0.1 \Gamma$.

with the laser beams (three on each bath) is consistent with the lattice constant and typical waists of Gaussian laser beams [145, 146]. To set the trap distance I fix first the characteristic length $l = \left(\frac{q^2}{4\pi\varepsilon_0 m} \frac{1}{\omega_1^2}\right)^{1/3}$ as the distance for which the Coulomb repulsion of two ions equals the trap potential energy for an ion at a distance l away from the center of its trap. If $a < l$, the Coulomb repulsion of the ions is stronger than the trap confinement which makes the ions jump from their traps. With the parameters used in this section I have $l = 38.7 \mu\text{m}$ and set $a = 1.29 l = 50 \mu\text{m}$. The detunings of the *hot* and *cold* lasers are $\delta_H = -0.02 \Gamma$, and $\delta_C = -0.1 \Gamma$ which gives temperatures $T_H \approx 12$ mK and $T_C \approx 3$ mK. I fix the value $\Delta\omega = 0.5 \omega_1$ for the frequency increment.

The results of the two methods are in very good agreement. In the scale of Fig.

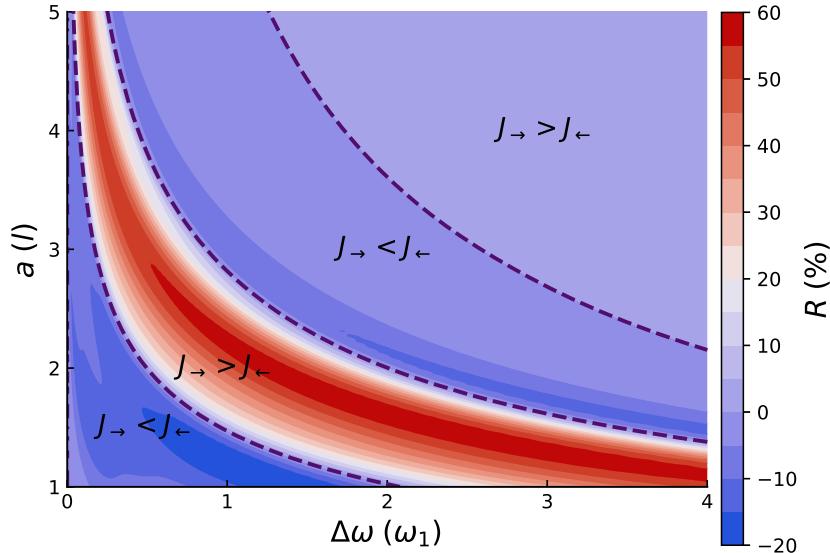


FIGURE 5.4: Rectification factor in a graded chain of $N = 15$ $^{24}\text{Mg}^+$ ions for different trap distances and frequency increment. The dashed lines are for $R = 0$ and delimit the regions $J_{\rightarrow} > J_{\leftarrow}$ and $J_{\rightarrow} < J_{\leftarrow}$. The parameters are $\omega_1 = 2\pi \times 1$ MHz, $l = 5.25 \mu\text{m}$, $\delta_H = -0.02 \Gamma$, and $\delta_C = -0.1 \Gamma$.

5.2 (a) the calculated local temperatures are undistinguishable. In the calculation based on solving the dynamics I had to integrate Eq. (5.3) for $N_{\text{trials}} = 1000$ realizations of white noise $\xi(t)$. The method based on the system of moments shortened the calculation time with respect to the dynamical trajectories by a factor of 1/700. In fact, the time gain is even more important because the dynamical method requires further processing, performing a time averaging to compute the stationary flux in addition to noise averaging, see Fig. 5.2 (b).

Additionally, the relaxation to the steady state slows down when the frequencies of the traps increase since the deterministic part of the Langevin equation dominates the dynamics over the stochastic part, entering an under-damped regime. In contrast, this increase does not affect the algebraic method.

5.4.2 Rectification in frequency graded chains

In this subsection I demonstrate rectification for the frequency graded chain. I used the method described in section 5.3.1 for $^{24}\text{Mg}^+$ ions with the same parameters for the baths used before. I fix the trapping frequency of the leftmost trap to $\omega_1 = 2\pi \times 1$ MHz, and a trap spacing $a = 4.76 l$ ($25 \mu\text{m}$) (the characteristic length is $l = 5.25 \mu\text{m}$). Figure 5.3 depicts the results with these parameters in a graded chain. Figure 5.3 (a) shows that both J_{\rightarrow} and J_{\leftarrow} decrease rapidly as

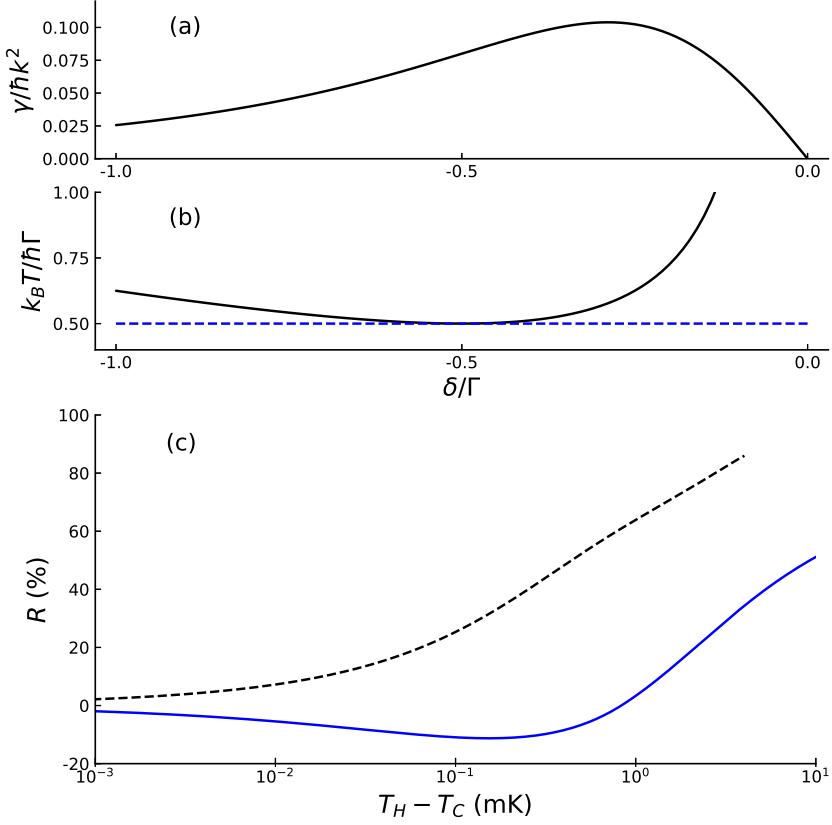


FIGURE 5.5: (a) Friction coefficient defined in Eq. (5.4). (b) Bath temperature defined in Eq. (5.5). (c) Rectification as a function of the temperature difference between the hot and cold baths $T_H - T_C$ for δ_H below (dashed black line) and above (solid blue line) the Doppler limit, and $\delta_C = \delta_D$ (Doppler limit).

Parameters: $\omega_1 = 2\pi \times 1$ MHz, $\Delta\omega = 0.15\omega_1$, $l = 5.25\mu\text{m}$, $a = 4.76l$.

the frequency increment is increased. The rectification reaches its maximum value for a frequency difference of $\Delta\omega \approx 0.1\omega_1$. The fluxes cross so there are some points where the rectification is exactly zero, besides the trivial one at $\Delta\omega = 0$, at $\Delta\omega = 0.05\omega_1$, $0.3\omega_1$, $1.3\omega_1$. At these points the direction of rectification reverses, presumably as a consequence of the changes in the match/mismatch of the temperature dependent local power spectra. The change of rectification direction occurs for all the choices of parameters, as displayed in Fig. 5.4. Figure 5.4 gives the rectification factor for different trap distances and frequency increments. 0-rectification curves separate regions with different rectification direction. The second region in Fig. 5.4 (starting from the left) would be the most interesting one to build a thermal diode, since rectification reaches its largest values there.

For small values of $\Delta\omega$ there is little asymmetry in the chain and therefore modest rectification is expected whereas a very large $\Delta\omega$ implies very high trapping

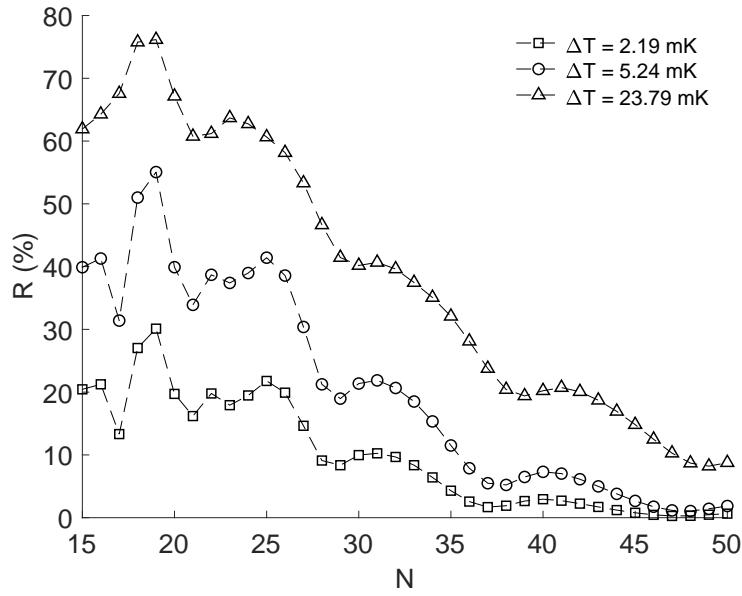


FIGURE 5.6: Rectification factor for different bath temperature differences ΔT as the number of ions is increased. The detuning of the cold bath laser is set to the Doppler limit $\delta_C = -\Gamma/2$. $\omega_1 = 2\pi \times 1$ MHz, $\Delta\omega = 0.15\omega_1$, $l = 5.25\mu\text{m}$, $a = 4.76l$.

frequencies on the right implying too strong a confinement and vanishing interactions. This bottleneck decreases the fluxes in both directions and the rectification. However, since $\Delta\omega$ is controllable, and the range of values of $\Delta\omega$ for which rectification is larger can be also controlled with the intertrap distance a , see Fig. 5.4, the existence of a rectification window does not imply a major limitation.

5.4.3 Same bath temperatures, different bath couplings

As already mentioned below Eq. (5.5), above and below the detuning $\delta_D = -\Gamma/2$ corresponding to the Doppler limit temperature, the optical molasses allow for two different couplings (two pairs of friction and diffusion coefficients in Eq. (5.4)) between the ions and the laser corresponding to the same bath temperature. This duality may be seen explicitly in Fig. 5.5. Specifically Fig. 5.5 (a) depicts the variation of the friction coefficient for values of δ around δ_R , and Fig. 5.5 (b) the corresponding temperatures. Interestingly, the different couplings imply different rectification factors. If I set $\delta_C = \delta_D = -\Gamma/2$, i.e., the cold bath is cooled to the Doppler limit, δ_H can be chosen to be below or above δ_D for the same temperature T_H . The corresponding rectification factors for the two choices are shown in Fig. 5.5 (c), which demonstrates that significant rectification can be

achieved by choosing $\delta_H < \delta_D$ for temperature increments that are smaller than or of the order of $T_C = T_D$, for example $R \approx 20\%$ for $\Delta T = 0.1T_C$, or $R \approx 60\%$ for $\Delta T = T_C$. Finding good rectification at low (relative) temperature differences is considered to be one on the challenges in asymmetric heat transport research [147].

5.4.4 Dependence with ion number

Keeping in mind that scaling the frequency-graded ion chain to a large numbers of ions is not a realistic option in this setting, it is nevertheless important to study the dependence with ion number from small to moderate numbers. In Fig. 5.6 I observe an overall trend in which the rectification decreases with the number of ions in the chain (while it increases with temperature bias ΔT in the studied range). This effect is easy to understand, as increasing N while keeping the total variation of the trapping frequency $\Delta\omega$ constant, the frequency gradient decreases. This lowers the asymmetry in the chain and the rectification factor. Oscillations with N superimposed to the global trend are more visible at the smaller N values giving an optimal N value at $N = 19$.

5.4.5 Graded versus segmented

I have also compared the performance of the graded thermal diode and a segmented version in which the left half of the chain is trapped with frequency ω_1 and the right half (including the middle ion) with $\omega_1 + \Delta\omega$. Even though the optimal rectification in Fig. 5.7 (a) for the segmented chain is larger than for the graded chain, the fact that the fluxes are generally much larger for the graded chain, see Fig. 5.7 (b), makes the graded chain more interesting for applications.

5.5 Discussion

In this chapter I have numerically demonstrated heat rectification in a chain of ions trapped in individual microtraps with graded frequencies, connected at both ends to thermal baths created by optical molasses. An alternative to implement a

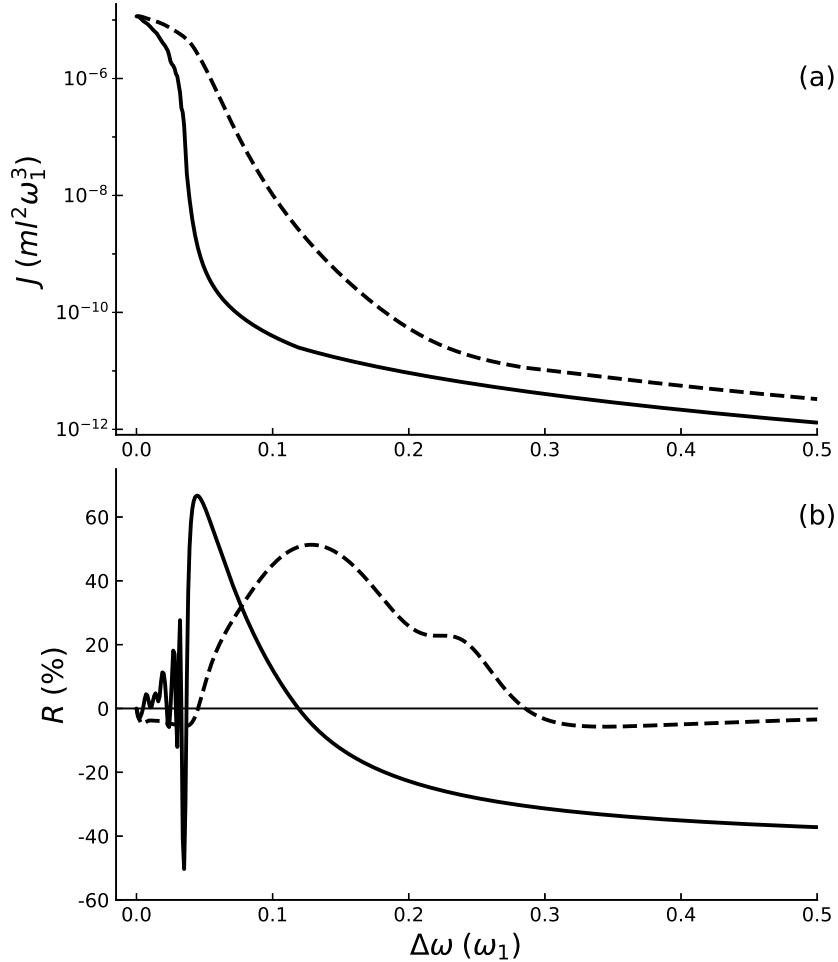


FIGURE 5.7: Comparison of graded and segmented chains with $N = 15$ $^{24}\text{Mg}^+$ ions. (a) Maximum of J_{\rightarrow} and J_{\leftarrow} for the graded and segmented chain for different frequency increments. (b) Rectification factor: graded chain (dashed lines); segmented chain (solid lines). Parameters: $\omega_1 = 2\pi \times 1$ MHz, $l = 5.25$ μm , $a = 4.76 l$, $\delta_H = -0.02 \Gamma$, and $\delta_C = -0.1 \Gamma$.

graded frequency profile in the lab could be combining a collective Paul trap for all the ions with on-site dipolar laser forces [121, 148–150].

A goal of this chapter is to connect two communities, ion trappers and researchers on heat-rectification models. The results found are encouraging and demonstrate the potential of a trapped-ion platform to experimentally investigate heat rectification schemes. Trapped ions are quite interesting to this end because they are highly controllable, and may easily adopt several features to enhance rectification, such as the ones explored here (long-range interactions and an asymmetrical gradation), or others such as time dependent forces [75, 151], or different nonlinearities in onsite forces. The limitations and application domain should also be clear, the proposed platform is circumscribed to cold temperatures of the order

of hundreds of μK to mK achieved by Doppler cooling. In this sense it is not aimed at competing with (it is rather complementary to) proposals for which experiments [81–84] or simulations [147, 152, 153] demonstrate thermal rectification at room temperature or for hundreds of K. Also, the number of ions should realistically be kept small so the proposed ion chain is not aimed at achieving a macroscopic diode length, but at playing a role in thermal diode research and in the context of ion-trapped based quantum technologies.

Methodologically, the calculation of the steady state has been performed with an algebraic approach much faster than the time-consuming integration and averaging over noise and time of the dynamical equations. The algebraic approach linearizes the forces around equilibrium positions which, in this system and for the realistic parameters considered is well justified and tested numerically. The results found provide additional evidence that simple linear models may rectify heat flow [103]. I underline that this linear model is, arguably, even simpler than some linear “minimalist, toy models” in [103] that showed rectification (the on-site forces are already linear from the start and the temperature dependence of explicit model parameters is only in the coefficients of the Langevin baths), with the important bonus of being also realistic.

Chapter 6

Rectification in a minimal model

In this chapter I present a minimalist model system which shows rectification. It is composed of two unequal atoms subjected to linear forces and in contact with effective Langevin baths induced by optical molasses. Analytic expressions of the heat currents in the steady state and the rectification coefficient are spelled out. Asymmetric heat transport is found in this linear system if both the bath temperatures and the temperature dependent bath-system couplings are exchanged. The model can be realized with two ions in either common or individual traps. This physical setting allows for a natural temperature dependence of the coupling to the baths. I also explore the parameter space of the model to optimize asymmetric heat current and find conditions for maximal rectification. High rectification corresponds to a good match of the power spectra of the ions for forward temperature bias and mismatch for reverse bias, which may be understood by the behavior of dissipative normal modes.

6.1 Introduction

As previously discussed in the introduction to this part of the thesis, the presence of thermal rectification in the first models was explained by the temperature dependences of the phonon bands (power spectra) of the different segments of the chain [72, 73]. This temperature dependence of the phonon bands brings two different extreme situations. The phonon spectra of neighboring parts of the chain match, meaning they overlap significantly, which implies a good transfer of energy. Conversely, if the spectra of the neighboring parts mismatch, meaning they do not overlap significantly, there will be a bad thermal conduction. If the spectra of the parts of the chain are affected differently by the temperature, then the magnitude of the net heat current will change with the sign of the applied temperature bias. Temperature dependence of the phonon bands occurs naturally in systems where there are non-linear (anharmonic) interactions, therefore non-linearities have been regarded recurrently as an essential element for rectification [75, 85, 86, 98–102]. However, it was later pointed out by Pereira [103] that anharmonicity is not a necessary condition for rectification. Rectification can also take place in harmonic systems where there is some structural asymmetry and temperature-dependence of the system parameters.

In this chapter I put forward a minimalist model for rectification composed of two neighboring atoms of different mass interacting harmonically and in contact with thermal baths with temperature dependent coupling. The model admits a natural realization in terms of two trapped ions subjected to Doppler cooling lasers, which provide the necessary temperature dependence of the coupling parameters. Apart from the possibility of a physical realization, another interesting feature is the analytical treatment, which facilitates greatly the exploration in parameter space to identify regimes of maximal rectification. The explicit solution of the stationary regime also provides tools for a better understanding of the physics and enhanced control. For example the match or mismatch of the spectra of the two masses for forward and reverse bias configurations, which will be made evident for the parameters with maximal rectification, may be analyzed in terms of dissipative normal modes characterized by complex eigenvalues.

The model may be compared and related to other simple models. The localization of the structural asymmetry in a small spatial region, by a “defect”, “impurity”, or asymmetrical molecule has been proposed e.g. in several anharmonic

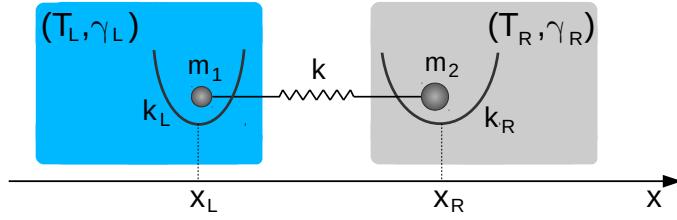


FIGURE 6.1: Diagram of the model described in Section 6.2. Two masses are coupled to each other through a spring constant k . Each mass is harmonically trapped and connected to a bath characterized by its temperature T_i and its friction coefficient γ_i .

models [102, 154, 155]. Segal and Nitzan proposed models with some similarities to the one presented in this chapter [101, 102], specifically an anharmonic chain with different couplings to both baths. They also worked out quantum models [101, 102] in terms of an N -level system asymmetrically coupled to the baths. Both types of models have “harmonic limits”, which in the chain is reached by making the potentials harmonic, and in the quantum one by taking N to infinity assuming equispaced levels. The asymmetrical couplings however, were not interchanged when reversing the temperature bias (in these models that interchange would have suppressed the asymmetry because the forward bias configuration becomes a mirror image of the reverse bias one), so that the harmonic limit did not give any rectification.

The chapter is organized as follows: In Section 6.2 I describe the physical model and its dynamical equations. In Section 6.3 I introduce the covariance matrix and derive the equation that it satisfies in the steady state. In Section 6.4 I solve the covariance matrix equation and find analytical expressions for the steady-state temperatures of the masses and heat currents. In Section 6.5 I relate the parameters of the model to those of Doppler cooled trapped ions. In Section 6.6 I look for configurations with high rectification. I also study the power spectra of the oscillators, which confirms the match or mismatch pattern for rectification. Finally, in Section 6.7 I summarize the results and present the conclusions.

6.2 Physical Model

The physical model consists of two masses m_1 and m_2 coupled to each other by a harmonic interaction with spring constant k and natural length x_e . The masses m_1 and m_2 are confined by harmonic potentials centered at x_L, x_R with spring constants k_L, k_R respectively (see Fig. 6.1). The Hamiltonian describing this model is

$$H = \frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2} + V(x_1, x_2), \quad (6.1)$$

with $V(x_1, x_2) = \frac{k}{2}(x_1 - x_2 - x_e)^2 + \frac{k_L}{2}(x_1 - x_L)^2 + \frac{k_R}{2}(x_2 - x_R)^2$, where $\{x_i, p_i\}_{i=1,2}$ are the position and momentum of each mass. Switching from the original coordinates x_i to displacements with respect to the equilibrium positions of the system $q_i = x_i - x_i^{eq}$, where x_i^{eq} are the solutions to $\partial_{x_i}V(x_1, x_2) = 0$, the Hamiltonian can be written as

$$\begin{aligned} H = & \frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2} + \frac{k + k_L}{2}q_1^2 \\ & + \frac{k + k_R}{2}q_2^2 - kq_1q_2 + V(x_1^{eq}, x_2^{eq}). \end{aligned} \quad (6.2)$$

Dropping the constant term, this has the form of the Hamiltonian of a system around a stable equilibrium point,

$$H = \frac{1}{2}\vec{p}^\top \mathbb{M}^{-1} \vec{p} + \frac{1}{2}\vec{q}^\top \mathbb{K} \vec{q}, \quad (6.3)$$

where $\vec{q} = (q_1, q_2)^\top$, $\vec{p} = (p_1, p_2)^\top$, $\mathbb{M} = \text{diag}(m_1, m_2)$ is the mass matrix of the system and \mathbb{K} is the Hessian matrix of the potential at the equilibrium point, i.e., $\mathbb{K}_{ij} = \partial_{x_i, x_j}^2 V(\vec{x}) \Big|_{\vec{x}=\vec{x}^{eq}}$. In this model $\mathbb{K}_{11} = k + k_L$, $\mathbb{K}_{22} = k + k_R$ and $\mathbb{K}_{12} = \mathbb{K}_{21} = -k$. The generic form (6.3) can be adapted to different physical settings, in particular to two ions in individual traps, or to two ions in a common trap.

The masses are in contact with Langevin baths, which will be denoted as L (for left) and R (for right), at temperatures T_L and T_R for the mass m_1 and m_2 respectively (see Fig. 6.1). The equations of motion of the system, taking into

account the Hamiltonian and the Langevin baths are

$$\begin{aligned}\dot{q}_1 &= \frac{p_1}{m_1}, \quad \dot{q}_2 = \frac{p_2}{m_2}, \\ \dot{p}_1 &= -(k + k_L)q_1 + kq_2 - \frac{\gamma_L}{m_1}p_1 + \xi_L(t), \\ \dot{p}_2 &= -(k + k_R)q_2 + kq_1 - \frac{\gamma_R}{m_2}p_2 + \xi_R(t),\end{aligned}\tag{6.4}$$

where γ_L, γ_R are the friction coefficients of the baths and $\xi_L(t), \xi_R(t)$ are Gaussian white-noise-like forces. The Gaussian forces have zero mean over noise realizations ($\langle \xi_L(t) \rangle = \langle \xi_R(t) \rangle = 0$) and satisfy the correlations $\langle \xi_L(t)\xi_L(t') \rangle = 0$, $\langle \xi_L(t)\xi_R(t') \rangle = 2D_L\delta(t-t')$, $\langle \xi_R(t)\xi_R(t') \rangle = 2D_R\delta(t-t')$. D_L and D_R are the diffusion coefficients, which satisfy the fluctuation-dissipation theorem, $D_L = \gamma_L k_B T_L$, $D_R = \gamma_R k_B T_R$, where k_B is the Boltzmann constant.

It is useful to define the phase-space vector $\vec{r}(t) = (\vec{q}, \mathbb{M}^{-1}\vec{p})^\top$ (note that $\vec{v} = \mathbb{M}^{-1}\vec{p}$ is just the velocity vector). The equations of motion are

$$\dot{\vec{r}}(t) = \mathbb{A} \vec{r}(t) + \mathbb{L} \vec{\xi}(t),\tag{6.5}$$

with

$$\begin{aligned}\mathbb{A} &= \begin{pmatrix} \mathbb{0}_{2 \times 2} & \mathbb{1}_{2 \times 2} \\ -\mathbb{M}^{-1}\mathbb{K} & -\mathbb{M}^{-1}\mathbb{T} \end{pmatrix}, \\ \mathbb{L} &= \begin{pmatrix} \mathbb{0}_{2 \times 2} \\ \mathbb{M}^{-1} \end{pmatrix},\end{aligned}\tag{6.6}$$

and $\vec{\xi}(t) = (\xi_L(t), \xi_R(t))^\top$, $\mathbb{T} = \text{diag}(\gamma_L, \gamma_R)$. $\mathbb{0}_{n \times n}$ and $\mathbb{1}_{n \times n}$ are the n -dimensional squared 0 matrix and identity matrix respectively. With the vector notation the correlation of the white-noise forces can be written as

$$\left\langle \vec{\xi}(t) \vec{\xi}(t')^\top \right\rangle = 2\mathbb{D}\delta(t-t'),\tag{6.7}$$

where $\mathbb{D} = \text{diag}(D_L, D_R)$.

6.3 Covariance matrix in the steady state

In order to look for the steady state currents in the system I will make use of the covariance matrix $\mathbb{C}(t)$ which is defined as

$$\mathbb{C}(t) = \langle \vec{r}(t) \vec{r}(t)^T \rangle. \quad (6.8)$$

The relation between the kinetic temperatures of the masses, $T_1(t)$ and $T_2(t)$, and the covariance matrix is given by

$$\begin{aligned} T_1(t) &\equiv \frac{\langle p_1^2(t) \rangle}{m_1 k_B} = \frac{m_1 C_{3,3}(t)}{k_B}, \\ T_2(t) &\equiv \frac{\langle p_2^2(t) \rangle}{m_2 k_B} = \frac{m_2 C_{4,4}(t)}{k_B}. \end{aligned} \quad (6.9)$$

As discussed in section 5.3.1, the covariances of the positions and velocities of the masses could be calculated integrating eq. (6.5) over an ensemble of noise realizations $\vec{\xi}(t)$ and taking the corresponding expectation values in the steady state. However, in section 5.3.1 I also pointed out the tremendous cost of doing this, so I introduced a way to linearize the system that let us find the steady-state covariances as the solutions for a set of algebraic equations [138–140]. In this case the system dynamics are already linear, so I can directly use the steady-state method described in 5.3.1. Besides, the solutions of the steady-state equations will be analytical for this two oscillators model.

To begin, I differentiate $\mathbb{C}(t)$ with respect to time and use Eq. (6.5) to get

$$\begin{aligned} \frac{d}{dt} \mathbb{C}(t) &= \mathbb{A} \mathbb{C}(t) + \mathbb{C}(t) \mathbb{A}^T \\ &+ \mathbb{L} \left\langle \vec{\xi}(t) \vec{r}(t)^T \right\rangle + \left\langle \vec{r}(t) \vec{\xi}(t)^T \right\rangle \mathbb{L}^T. \end{aligned} \quad (6.10)$$

The solution of Eq. (6.10) can be used to find the kinetic temperatures of the masses at all times but I am only interested in the steady-state values, *i.e.* the asymptotic values for $t \rightarrow \infty$. In the steady state, the covariance matrix is constant ($\frac{d}{dt} \mathbb{C}(t) = 0$), therefore it satisfies

$$\mathbb{A} \mathbb{C}^{s.s.} + \mathbb{C}^{s.s.} \mathbb{A}^T = -\mathbb{L} \left\langle \vec{\xi} \vec{r}^T \right\rangle^{s.s.} - \left\langle \vec{r} \vec{\xi}^T \right\rangle^{s.s.} \mathbb{L}^T, \quad (6.11)$$

with $\{\cdot\}^{s.s.} \equiv \lim_{t \rightarrow \infty} \{\cdot\}(t)$. The two terms $\langle \vec{\xi} \vec{r}^\top \rangle^{s.s.}$ and $\langle \vec{r} \vec{\xi}^\top \rangle^{s.s.}$ in (6.11) can be calculated using Novikov's theorem [134] as explained in section 5.3. I get

$$\langle \vec{\xi}(t) \vec{r}(t)^\top \rangle = \mathbb{D} \mathbb{L}^\top. \quad (6.12)$$

I have therefore, the following algebraic equation for the steady-state covariance matrix

$$\mathbb{A} \mathbb{C}^{s.s.} + \mathbb{C}^{s.s.} \mathbb{A}^\top = -\mathbb{B}, \quad (6.13)$$

with $\mathbb{B} = 2\mathbb{L}\mathbb{D}\mathbb{L}^\top$. By definition, the covariance matrix is symmetric, but there are also additional restrictions imposed by the equations of motion and the steady-state condition, which reduce the dimensionality of the problem of solving Eq. (6.13) [105]. Since $d\langle q_i q_j \rangle / dt = 0$ in the steady state, I have

$$\begin{aligned} \langle p_1 q_1 \rangle^{s.s.} &= \langle p_2 q_2 \rangle^{s.s.} = 0, \\ \frac{\langle p_1 q_2 \rangle^{s.s.}}{m_1} &= -\frac{\langle q_1 p_2 \rangle^{s.s.}}{m_2}. \end{aligned} \quad (6.14)$$

Taking (6.14) into account, the steady-state covariance matrix takes the form

$$\mathbb{C}^{s.s.} = \begin{pmatrix} \langle q_1^2 \rangle^{s.s.} & \langle q_1 q_2 \rangle^{s.s.} & 0 & \frac{\langle p_2 q_1 \rangle^{s.s.}}{m_2} \\ \langle q_1 q_2 \rangle^{s.s.} & \langle q_2^2 \rangle^{s.s.} & -\frac{\langle p_2 q_1 \rangle^{s.s.}}{m_2} & 0 \\ 0 & -\frac{\langle p_2 q_1 \rangle^{s.s.}}{m_2} & \frac{\langle p_1^2 \rangle^{s.s.}}{m_1^2} & \frac{\langle p_1 p_2 \rangle^{s.s.}}{m_1 m_2} \\ \frac{\langle p_2 q_1 \rangle^{s.s.}}{m_2} & 0 & \frac{\langle p_1 p_2 \rangle^{s.s.}}{m_1 m_2} & \frac{\langle p_2^2 \rangle^{s.s.}}{m_2^2} \end{pmatrix}. \quad (6.15)$$

The explicit set of equations for the components of $\mathbb{C}^{s.s.}$ can be found in Appendix E.

6.4 Solutions

In this section I use the solution to Eq. (6.13) to write down the temperatures and currents in the steady state. I use the software *Mathematica* to find analytic expressions for the temperatures,

$$\begin{aligned} T_1 &= \frac{T_L \mathcal{P}_{1,L}(k) + T_R \mathcal{P}_{1,R}(k)}{\mathcal{D}(k)}, \\ T_2 &= \frac{T_L \mathcal{P}_{2,L}(k) + T_R \mathcal{P}_{2,R}(k)}{\mathcal{D}(k)}, \end{aligned} \quad (6.16)$$

where $\mathcal{D}(k) = \sum_{n=0}^2 \mathcal{D}_n k^n$ and $\mathcal{P}_{i,(L/R)}(k) = \sum_{n=0}^2 a_{i,n,(L/R)} k^n$ are polynomials in the coupling constant k with coefficients

$$\begin{aligned}
\mathcal{D}_0 &= a_{1,0,L} = a_{2,0,R} \\
&= \gamma_L \gamma_R [h^{(1)}(\gamma_L k_R + \gamma_R k_L) + (m_1 k_R - m_2 k_L)^2], \\
\mathcal{D}_1 &= a_{1,1,L} = a_{2,1,R} \\
&= \gamma_L \gamma_R [h^{(0)} h^{(1)} + 2(m_1 - m_2)(m_1 k_R - m_2 k_L)], \\
\mathcal{D}_2 &= h^{(0)} h^{(2)}, \\
a_{1,2,L} &= \gamma_L (m_2 h^{(1)} + \gamma_R (m_1 - m_2)^2), \\
a_{1,2,R} &= h^{(1)} m_1 \gamma_R, \\
a_{2,2,L} &= h^{(1)} m_2 \gamma_L, \\
a_{2,2,R} &= \gamma_R (m_1 h^{(1)} + \gamma_L (m_1 - m_2)^2), \\
a_{1,0,R} &= a_{1,1,R} = a_{2,0,L} = a_{2,1,L} = 0,
\end{aligned} \tag{6.17}$$

where

$$h^{(n)} \equiv \gamma_R m_1^n + \gamma_L m_2^n. \tag{6.18}$$

The currents from the baths to the masses are given by eq. (5.14)

$$J_L = k_B \frac{\gamma_L}{m_1} (T_L - T_1), \quad J_R = k_B \frac{\gamma_R}{m_2} (T_R - T_2), \tag{6.19}$$

with T_i given by Eq. (6.16). Since, in the steady state, $J_L = -J_R$ I will use the shorthand notation $J \equiv J_L$. Substituting Eq. (6.16) into Eq. (6.19) I get for the heat current

$$J = \kappa (T_L - T_R), \tag{6.20}$$

where $\kappa = k_B k^2 \gamma_L \gamma_R h^{(1)} / \mathcal{D}(k)$ acts as an effective thermal conductance.

6.5 Relation of the Model to a trapped ion setup

In this section I discuss the realization of the model with a pair of trapped ions. I consider two different setups: two ions in a collective trap, and two ions in individual traps. Later in Section 6.6 I shall focus on two ions in individual traps to illustrate the analysis of rectification.

In both setups I assume strong confinement in the radial direction, making the effective dynamics one-dimensional. I will also assume that the confinement in the axial direction is purely electrostatic, which makes the effective spring constant independent of the mass of the ions [156]. Additionally, I will relate the temperatures and friction coefficients of the Langevin baths to those corresponding to Doppler cooling.

6.5.1 Collective trap

Consider two ions of unit charge with masses m_1 and m_2 trapped in a collective trap. Assuming strong radial confinement and purely electrostatic axial confinement, both ions feel the same harmonic oscillator potential with trapping constant k_{trap} [156]. The potential describing the system is

$$V_{\text{collective}} = \frac{1}{2}k_{trap}(x_1^2 + x_2^2) + \frac{\mathcal{C}}{x_2 - x_1}, \quad (6.21)$$

with $\mathcal{C} = Q^2/(4\pi\varepsilon_0)$. The equilibrium positions for this potential are

$$x_2^{eq} = -x_1^{eq} = \left(\frac{1}{2}\right)^{2/3} \left(\frac{Q^2}{4\pi\varepsilon_0 k_{trap}}\right)^{1/3}. \quad (6.22)$$

Assuming small oscillations of the ions around the equilibrium positions, the Hessian matrix of the system is

$$\begin{aligned} \mathbb{K}_{1,2} &= -\frac{Q^2}{2\pi\varepsilon_0} \frac{1}{(x_2^{eq} - x_1^{eq})^3} = -k_{trap}, \\ \mathbb{K}_{1,1} &= k_{trap} + \frac{Q^2}{2\pi\varepsilon_0} \frac{1}{(x_2^{eq} - x_1^{eq})^3} = 2k_{trap}, \\ \mathbb{K}_{2,2} &= k_{trap} + \frac{Q^2}{2\pi\varepsilon_0} \frac{1}{(x_2^{eq} - x_1^{eq})^3} = 2k_{trap}. \end{aligned} \quad (6.23)$$

Using Eq. (6.23) I can relate the parameters of this physical setup to those of the model described in Section 6.2,

$$k_L = k_R = k = k_{trap}. \quad (6.24)$$

6.5.2 Individual on-site traps

I can make the same assumptions for the axial confinement as in the previous subsection but now each of the ions is in an individual trap with spring constants $k_{trap,L}$ and $k_{trap,R}$ respectively. The potential of the system is

$$\begin{aligned} V_{individual} = & \frac{1}{2}k_{trap,L}(x_1 - x_L)^2 + \frac{1}{2}k_{trap,R}(x_2 - x_R)^2 \\ & + \frac{\mathcal{C}}{x_2 - x_1}, \end{aligned} \quad (6.25)$$

where x_L and x_R are the center positions of the on-site traps. The elements of the Hessian matrix in the equilibrium position are

$$\begin{aligned} \mathbb{K}_{1,2} &= -\frac{Q^2}{2\pi\varepsilon_0} \frac{1}{(x_2^{eq} - x_1^{eq})^3}, \\ \mathbb{K}_{1,1} &= k_{trap,L} + \frac{Q^2}{2\pi\varepsilon_0} \frac{1}{(x_2^{eq} - x_1^{eq})^3}, \\ \mathbb{K}_{2,2} &= k_{trap,R} + \frac{Q^2}{2\pi\varepsilon_0} \frac{1}{(x_2^{eq} - x_1^{eq})^3}. \end{aligned} \quad (6.26)$$

Comparing the parameters in Eq. (6.26) with those in the model described in Section 6.2 I identify

$$\begin{aligned} k_L &= k_{trap,L}, \\ k_R &= k_{trap,R}, \\ k &= \frac{Q^2}{2\pi\varepsilon_0} \frac{1}{(x_2^{eq} - x_1^{eq})^3}. \end{aligned} \quad (6.27)$$

In this case, the analytic expressions for the equilibrium positions are more complicated. I get for the distance between the equilibrium positions of the ions

$$\begin{aligned} (x_2 - x_1)^{(eq)} &= \frac{1}{3} \Delta x_{LR} \\ &- \frac{1}{6} \left[\frac{2^{2/3} \zeta}{k_{trap,L} k_{trap,R} (k_{trap,L} + k_{trap,R})} \right. \\ &\left. + \frac{2^{4/3} k_{trap,L} k_{trap,R} (k_{trap,L} + k_{trap,R}) (x_R - x_L)^2}{\zeta} \right], \end{aligned} \quad (6.28)$$

where $\Delta x_{LR} = (x_R - x_L)$ and $\zeta = (Y - \eta)^{(1/3)}$, with

$$\begin{aligned} Y &= 3\sqrt{3} \left\{ \mathcal{C} k_{trap,L}^4 k_{trap,R}^4 (k_{trap,L} + k_{trap,R})^7 \right. \\ &\times \left. [4k_{trap,L} k_{trap,R} \Delta x_{LR}^3 + 27\mathcal{C} (k_{trap,L} + k_{trap,R})] \right\}^{1/2}, \\ \eta &= k_{trap,L}^2 k_{trap,R}^2 (k_{trap,L} + k_{trap,R})^3 \\ &\times [2k_{trap,L} k_{trap,R} \Delta x_{LR}^3 + 27\mathcal{C} (k_{trap,L} + k_{trap,R})]. \end{aligned} \quad (6.29)$$

In this setup, the coupling between the ions k can be controlled by changing the distance between the on-site traps.

6.5.3 Optical molasses and Langevin baths

As discussed in section 5.2, the Langevin dynamics can be implemented in a trapped-ion setup with the technique of optical molasses [131, 132, 157, 158]. The damping-like force in the Langevin equation comes from the absorption of the photons by the ions, while the diffusive force ($\vec{\xi}(t)$) is originated by the random recoil of the ions when they emit photons. To recall what was explained in 5.2, the effect of the optical molass amounts to having an effective temperature T_{molass} and effective friction coefficient γ_{molass} which are controlled with the laser intensity I and frequency detuning δ with respect to the selected internal atomic transition. The following expressions describe the molass effect [122, 131, 132]

$$\begin{aligned} \gamma_{molass}(I, \delta) &= -4\hbar \left(\frac{\delta + \omega_0}{c} \right)^2 \left(\frac{I}{I_0} \right) \frac{2\delta/\Gamma}{[1 + (2\delta/\Gamma)^2]^2}, \\ T_{molass}(\delta) &= -\frac{\hbar\Gamma}{4k_B} \frac{1 + (2\delta/\Gamma)^2}{(2\delta/\Gamma)}, \end{aligned} \quad (6.30)$$

where ω_0 is the frequency of the selected internal atomic transition of the ion, Γ is the natural width (decay rate) of the excited state, and I_0 is the saturation intensity. For fixed Γ and I , γ_{molass} depends on δ , and thus, indirectly, on the temperature T_{molass} .

6.6 Looking for rectification

There is rectification if the flux J for the forward temperature bias is different from the flux \tilde{J} for reverse bias with the baths exchanged. To measure rectification, I will use the rectification coefficient $0 \leq R \leq 1$ defined as

$$R \equiv \frac{|J| - |\tilde{J}|}{\max(|J|, |\tilde{J}|)} = 1 - \min\left(\frac{\kappa}{\tilde{\kappa}}, \frac{\tilde{\kappa}}{\kappa}\right), \quad (6.31)$$

where $\kappa, \tilde{\kappa}$ are the heat conductance in the forward and reversed (after the baths are exchanged) configurations. The important point here is to define what is meant by *exchanging the baths*. In the following I will consider that a bath is characterized, not only by its temperature T but also by its coupling to the system by means of the friction coefficient γ , so, exchanging the baths is achieved by exchanging both the temperatures and the friction coefficients, as summarized in Table 6.1. For generic models this choice is a matter of definition, but for trapped ions it is a natural way to proceed.

When implementing temperatures and friction coefficients by lasers according to Eq. (6.30), the exchange operation is straightforward when the two ions are either of the same species or isotopes of each other, since the only required action to exchange temperatures is to exchange the detunings without modifying the intensities. The detuning exchange in turn automatically exchanges the friction coefficients. However, for two different species, which involve two different atomic transitions, the laser wavelengths and the decay rates Γ depend on the species. Then, exchanging the temperatures by modifying the detunings, keeping the laser intensities constant, does not necessarily imply an exchange of the friction coefficients. Nevertheless it is possible to adjust the laser intensities so that the friction coefficients get exchanged and that is the assumption hereafter. In terms of the analysis of rectification in Ref. [103], I am adding a temperature-dependent feature

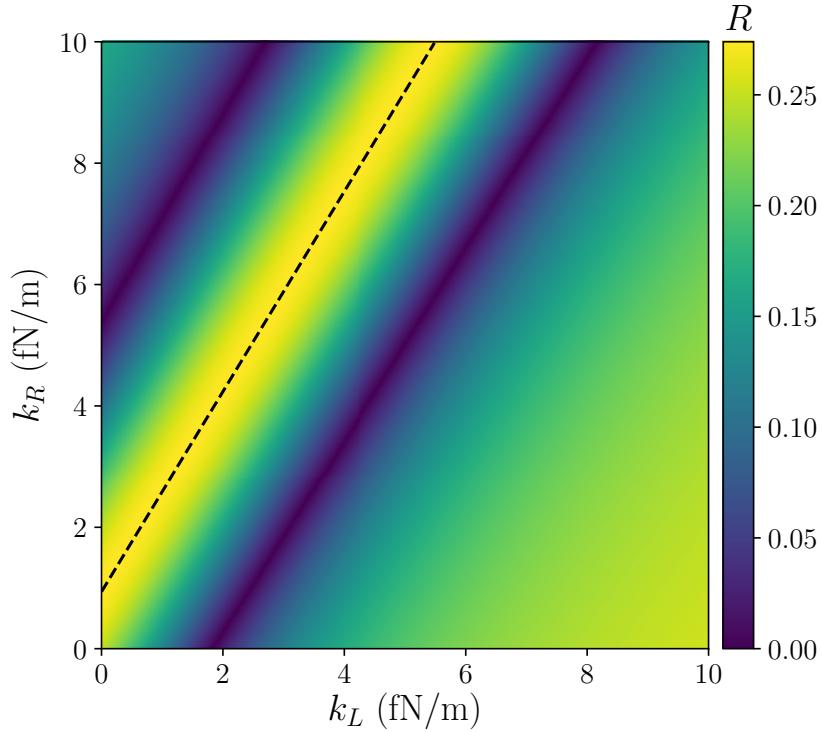


FIGURE 6.2: Rectification, R , in the $k_L k_R$ plane for $k = 1.17$ fN/m, $\gamma_L = 6.75 \times 10^{-22}$ kg/s, and $\gamma_R = 4.64\gamma_L$, $m_1 = 24.305$ a.u., $m_2 = 40.078$ a.u. The dashed line represents Eq. (6.32).

TABLE 6.1: Definition of forward and reversed (exchanged) bath configurations. The variables with a tilde are the ones for the “reversed” configuration.

	forward	reversed
Bath Friction	γ_L, γ_R	$\tilde{\gamma}_L = \gamma_R, \tilde{\gamma}_R = \gamma_L$
Bath Temperature	T_L, T_R	$\tilde{T}_L = T_R, \tilde{T}_R = T_L$

to the system, namely, the friction coefficients depend on the bath temperature and are exchanged as the baths are reversed.

6.6.1 Parametric exploration

I have explored thoroughly the space formed by the parameters of the model $m_1, m_2, k, k_L, k_R, \gamma_L, \gamma_R$, to find and maximize asymmetric heat transport. I have fixed the values of some of the parameters to realistic ones while varying the rest. Unless stated otherwise the masses are $m_1 = 24.305$ a.u. and $m_2 = 40.078$ a.u., which correspond to Mg and Ca, whose ions are broadly used in trapped-ion physics. According to eqs. (6.20) and (6.31) does not formally depend on the bath temperatures in this model for given friction coefficients. Of course the friction

coefficients depend on the temperature indirectly, but also on laser intensities, see Eq. (6.30), so in the parametric space $m_1, m_2, k, k_L, k_R, \gamma_L, \gamma_R$ there is no need to specify the bath temperatures to analyze the rectification in the following. The bath temperatures will be needed though to calculate the power spectra, and play an implicit role in the central assumption that their exchange implies an exchange of friction coefficients.

Figure 6.2 depicts the values of the rectification after sweeping the $k_L k_R$ plane for fixed values of k, γ_L , and γ_R . There is a ridge in the k_L, k_R plane for which the rectification is maximal. $\partial_{k_L} R = 0$ may be solved explicitly but the solution is too long to be displayed here. In a weak dissipation regime ($\gamma_L/m_1 \ll \sqrt{k/m_1}$, $\gamma_R/m_2 \ll \sqrt{k/m_2}$), a Taylor series around $(\gamma_L, \gamma_R) = (0, 0)$ gives in zeroth order a straight line for the ridge,

$$\frac{k + k_R}{m_2} = \frac{k + k_L}{m_1}. \quad (6.32)$$

Eq. (6.32) implies the resonance condition $\omega_L = \omega_R$ for the effective oscillation frequencies $\omega_L = \sqrt{(k + k_L)/m_1}$ and $\omega_R = \sqrt{(k + k_R)/m_2}$, see Eq. (6.2). The lowest order correction to Eq. (6.32) implies a small shift of the line, keeping the same slope,

$$\frac{k + k_R}{m_2} = \frac{k + k_L}{m_1} + \frac{(m_2\gamma_L + m_1\gamma_R)(m_1\gamma_L + m_2\gamma_R)}{2m_1m_2(m_2^2 - m_1^2)}. \quad (6.33)$$

In a trapped-ion context the condition (6.32) may be imposed by adjusting the distance of the traps for fixed k_L and k_R . Besides the line of maximum rectification, Fig. 6.2 also shows two lines where rectification is zero. At these lines forward and backward fluxes cross. Solving $R = 0$ with a Taylor series around $(\gamma_L, \gamma_R) = (0, 0)$ gives, up to second order in friction coefficients, the two approximate solutions

$$\begin{aligned} k_R &= k_L \left[\frac{m_2}{m_1} \pm \frac{1}{2k} \sqrt{\frac{m_2\gamma_L\gamma_R^3}{m_1^3}} \right] \\ &\quad + k \left[\frac{m_2}{m_1} \left(1 \pm \frac{2m_1m_2\gamma_R + (m_1^2 + m_2^2)\gamma_L}{2\sqrt{m_1m_2^3\gamma_L\gamma_R}} \right) - 1 \right] \\ &\quad \pm \frac{1}{2} \sqrt{\frac{m_2\gamma_L\gamma_R^3}{m_1^3}} + \gamma_R \frac{(m_1^2 + m_2^2)\gamma_L + m_1m_2\gamma_R}{2m_1^2(m_2 - m_1)}. \end{aligned} \quad (6.34)$$

The term $\pm \frac{1}{2k} \sqrt{m_2 \gamma_L \gamma_R^3 / m_1^3}$ in Eq. (6.34) makes the slopes of the two zero-rectification lines different from each other and also from the maximum-rectification line. This difference is however hardly noticeable for weak dissipation as in Fig. 6.2.

Interestingly, along the maximum line (6.32) the rectification no longer depends on the spring constants of the model, see Eqs. (6.20) and (6.31),

$$R = \begin{cases} 1 - \frac{a+g}{1+ag} & \text{if } a > 1, g > 1 \text{ or } a < 1, g < 1 \\ 1 - \frac{1+ag}{a+g} & \text{if } a > 1, g < 1 \text{ or } a < 1, g > 1, \end{cases} \quad (6.35)$$

it only depends on the mass and friction coefficient ratios a and g

$$a = m_2/m_1, \quad g = \gamma_R/\gamma_L. \quad (6.36)$$

Besides a high value of R , it is desirable to have a significant current J_{max} [105]. Using again Eq. (6.32) in the expression for the currents (6.20), the maximum current $J_{max} = \max(|J|, |\tilde{J}|)$ is

$$J_{max} = \begin{cases} \frac{k_B g \gamma_L k^2 |T_L - T_R|}{(a+g)(g\gamma_L^2(k_L+k)+k^2 m_1)} & \text{if } \begin{cases} a > 1, g > 1 \\ \text{or } a < 1, g < 1 \end{cases} \\ \frac{k_B g \gamma_L k^2 |T_L - T_R|}{(1+ag)(g\gamma_L^2(k_L+k)+k^2 m_1)} & \text{if } \begin{cases} a > 1, g < 1 \\ \text{or } a < 1, g > 1 \end{cases} \end{cases} \quad (6.37)$$

Now let us analyze how the parameters a and g affect the maximum current J_{max} in (6.37). To do this, I divide the ag plane in four quadrants by the axes $a = 1$ and $g = 1$ (in those axes $R = 0$). In Eq. (6.37) the parameter a appears only in the denominator, thus for a higher a , a smaller current is found. The quadrants with $a < 1$ will be better for achieving large currents. g appears both in the numerator and denominator so there is no obvious advantageous quadrant for this parameter.

Equation (6.35) is symmetric upon the transformations $a \leftrightarrow 1/a$ and $g \leftrightarrow 1/g$. Using a logarithmic scale for a and g , the resulting R map is symmetric with respect to the $a = 1$ and $g = 1$ axes. I can thus limit to analyze the quadrant $a > 1, g > 1$.

Figure 6.3(a) shows the rectification given by Eq. (6.35) in terms of a and g . If one follows the direction of the gradient vector of R , $\vec{\nabla} R \equiv (\partial_a R, \partial_g R)$, which

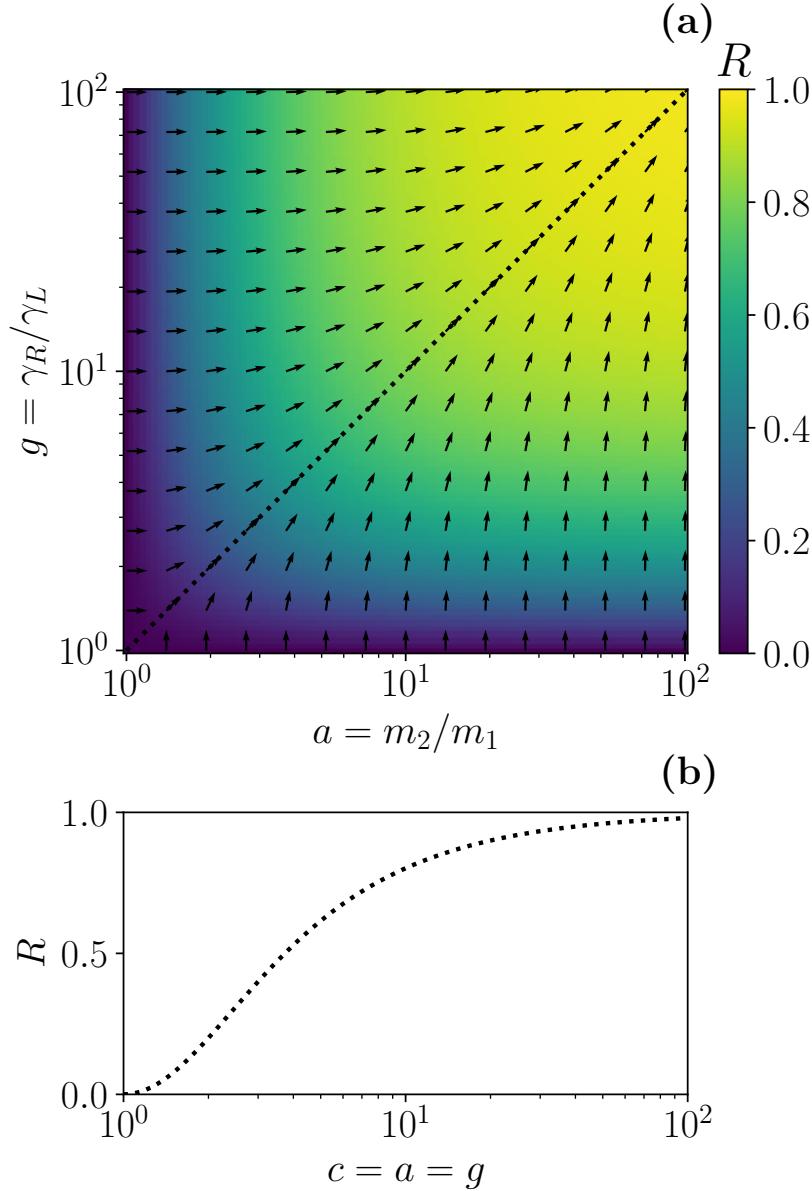


FIGURE 6.3: (a) Rectification factor, R , given by Eq. (6.35) as a function of the mass and friction ratios a and g . (b) Rectification factor along the diagonal dotted line $a = g$ in (a).

is indicated in Fig. 6.3 by the black arrows, it becomes clear that the fastest way of increasing R is following the diagonal dotted line $a = g$. Therefore I decided to vary a and g following a common value $c = a = g$. The effect of varying the common value c may be seen in Fig. 6.3(b), which shows the rectification. As can be seen in Fig. 6.3(b) the rectification increases when the mass and friction ratios increase (R tends to one for large c).

6.6.2 Spectral match/mismatch approach to rectification

If there is a good match between the phonon spectra of the ions (i.e., their peaks overlap in a broad range of frequencies) for a certain baths configuration, and mismatch when the baths exchange, the system will present heat rectification [72, 73]. I have studied the phonon spectra of the model for several sets of parameters exhibiting no rectification or strong rectification. The spectra are calculated through the spectral density matrix. For a real-valued stochastic process $\vec{x}(t)$, its spectral density matrix is defined as [138]

$$\mathbb{S}_{\vec{x}}(\omega) \equiv \left\langle \vec{X}(\omega) \vec{X}^T(-\omega) \right\rangle, \quad (6.38)$$

with $\vec{X}(\omega)$ being the Fourier transform of $\vec{x}(t)$ (I use the convention of factors of 1 and $1/(2\pi)$ for the transform and the inverse transform). A justification of the use of the spectral density matrix to understand heat transport arises from the Wiener-Khinchin theorem [138], which says that the correlation matrix of a stationary stochastic process in the steady state is the inverse Fourier transform of its spectral density matrix $\langle \vec{r}(t) \vec{r}^T(t + \tau) \rangle = \mathcal{F}^{-1}[\mathbb{S}_{\vec{r}}(\omega)](\tau)$. Thus the covariance matrix in the steady state is

$$\mathbb{C}^{s.s.} = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega \mathbb{S}_{\vec{r}}(\omega). \quad (6.39)$$

Eq. (6.39) directly connects the spectral density matrix to the steady-state temperature since $T_1^{s.s.} = m_1 C_{3,3}^{s.s.}/k_B$ and $T_2^{s.s.} = m_2 C_{4,4}^{s.s.}/k_B$, and, therefore, to the heat currents, see Eqs.(6.9) and (6.19).

The Fourier transform of the vector process $\vec{r}(t)$ describing the evolution of the system, see Eq. (6.5), is $\vec{R}(\omega) = (i\omega - \mathbb{A})^{-1} \mathbb{L} \vec{\Xi}(\omega)$ with $\vec{\Xi}(\omega)$ being the Fourier transform of the white noise $\vec{\xi}(t)$. Note that $\vec{\Xi}(\omega)$ is not square-integrable, however its spectral density is $\mathbb{S}_{\vec{\xi}}(\omega) = 2\mathbb{D}$ [138], which is flat as expected for a white noise. Therefore, the spectral density matrix of the system is

$$\mathbb{S}_{\vec{r}}(\omega) = 2(\mathbb{A} - i\omega)^{-1} \mathbb{L} \mathbb{D} \mathbb{L}^T (\mathbb{A} + i\omega)^{-T}. \quad (6.40)$$

The imaginary part of the eigenvalues of the dynamical matrix \mathbb{A} correspond to the peaks in the spectrum whereas the real part dictates their width. Eq. (6.40) gives after direct computation

$$\mathbb{S}_{\vec{r}}(\omega) = 2k_B \frac{\gamma_L T_L \mathbb{S}_L(i\omega) + \gamma_R T_R \mathbb{S}_R(i\omega)}{(m_1 m_2)^2 P_{\mathbb{A}}(i\omega) P_{\mathbb{A}}(-i\omega)}, \quad (6.41)$$

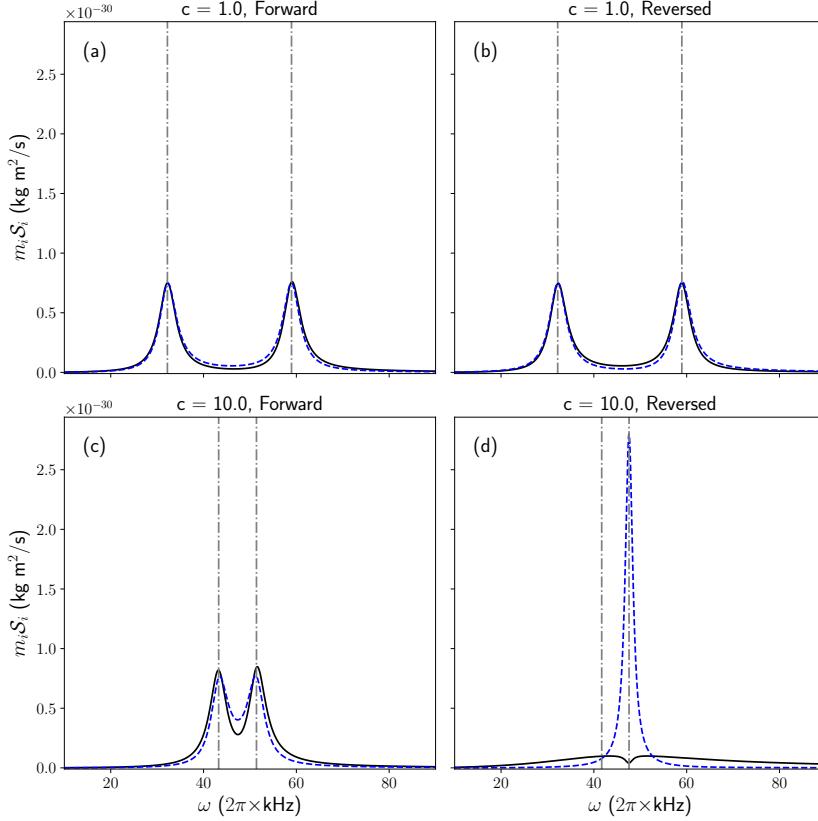


FIGURE 6.4: Spectral densities of the velocities of the ions (r_3 and r_4) corresponding to $T_L = \tilde{T}_R = 2$ mK, $T_R = \tilde{T}_L = 1$ mK, and two values of c in Fig. 6.3(b): (a), (b) for $c = 1$ and (c), (d) for $c = 10$. Solid, black lines are for the left ion spectral density $\mathcal{S}_1(\omega)$ and dashed, blue lines for the right ion spectral density $\mathcal{S}_2(\omega)$. Dot-dashed, vertical lines mark the frequencies of the normal modes of the system. The spectra are multiplied by their corresponding masses so that the areas are proportional to the steady-state temperatures, see Eq. (6.39). (a) and (b) correspond to $R = 0$: the overlap between the phonon bands is the same in forward and reversed configurations. (c) and (d) correspond to $R \approx 0.8$: in the forward configuration (c) the phonons match better than in the reversed configuration (d).

where $P_{\mathbb{A}}(\lambda)$ is the characteristic polynomial of the dynamical matrix \mathbb{A} and $\mathbb{S}_L(\omega)$, $\mathbb{S}_R(\omega)$ are the matrix polynomials in the angular frequency ω whose coefficients are defined in Appendix F. I show the spectral densities for the velocities, $\mathcal{S}_1 \equiv \mathbb{S}_{3,3}(\omega) = \langle R_3(\omega)R_3(-\omega) \rangle$ for the left ion, and $\mathcal{S}_2 \equiv \mathbb{S}_{4,4}(\omega) = \langle R_4(\omega)R_4(-\omega) \rangle$ for the right ion, since they are the elements related to the calculation of the heat

current using Eq. (6.39),

$$\begin{aligned}\mathcal{S}_1(\omega) &= 2k_B \frac{\gamma_R k^2 T_R \omega^2 + \gamma_L T_L [\omega^4 (\gamma_R^2 - 2km_2 - 2k_R m_2) + \omega^2 (k + k_R)^2 + m_2^2 \omega^6]}{(m_1 m_2)^2 P_{\mathbb{A}}(i\omega) P_{\mathbb{A}}(-i\omega)}, \\ \mathcal{S}_2(\omega) &= 2k_B \frac{\gamma_L k^2 T_L \omega^2 + \gamma_R T_R [\omega^4 (\gamma_L^2 - 2km_1 - 2k_L m_1) + \omega^2 (k + k_L)^2 + m_1^2 \omega^6]}{(m_1 m_2)^2 P_{\mathbb{A}}(i\omega) P_{\mathbb{A}}(-i\omega)}.\end{aligned}\quad (6.42)$$

Figure 6.4 depicts a series of plots of the spectra given by Eq. (6.42), corresponding to two points in Fig. 6.3(b). (The calculation for the reverse bias is done with the substitutions in Table 6.1.) For $c = 1$ (Fig. 6.4(a) and (b)) there is no rectification, since the spectra match in the forward (a) and reversed (b) configurations. However, for $c = 10$ (Fig. 6.4(c) and (d), $R \approx 0.8$) the picture is very different: there is a good match between the spectra in the forward configuration but not for the reversed configuration. It is interesting to analyze how the system changes from $c = 1$ to $c = 10$ (see fig. 6.5) using the dissipative normal modes of the system, which may be found by diagonalizing the dynamical matrix \mathbb{A} , Eq. (6.6). The frequencies of the peaks in Fig. 6.4 are given by the imaginary part of the eigenvalues $\lambda_{\mathbb{A}} = \lambda_r + i\lambda_i$ of \mathbb{A} . Likewise, the width depends on the real part of the eigenvalues. For the forward configuration, the normal frequencies (position of the peaks) come closer to each other as c is increased, while the widths remain practically constant. To understand why the real part remains practically constant, I recall that I have chosen to work with spring constants that satisfy Eq. (6.32) and making the mass and friction coefficient ratios equal to c , *i.e.*, $c \equiv m_2/m_1 = \gamma_R/\gamma_L$. The (dissipative) terms in \mathbb{A} responsible for the real parts in the eigenvalues are, for the forward configuration, γ_L/m_1 and $\gamma_R/m_2 = (c\gamma_L)/(cm_1) = \gamma_L/m_1$, which are constant for every value of c . On the contrary, in the reverse bias configuration the dissipative terms in the dynamical matrix are $\tilde{\gamma}_L/m_1 = \gamma_R/m_1 = c\gamma_L/m_1$ and $\tilde{\gamma}_R/m_2 = \gamma_L/(cm_1)$, with opposite behavior with respect to c . The real parts of the eigenvalues also behave quite differently for reverse bias, one of them gets closer to the imaginary axis for $c = 10$, see Fig. 6.4 (d), where this mode concerns mostly the right ion, the only one excited at the peak frequency, while the other eigenvalue moves far from the imaginary axis so a peak is not noticeable at the imaginary value (left dotted-dashed line) any more.

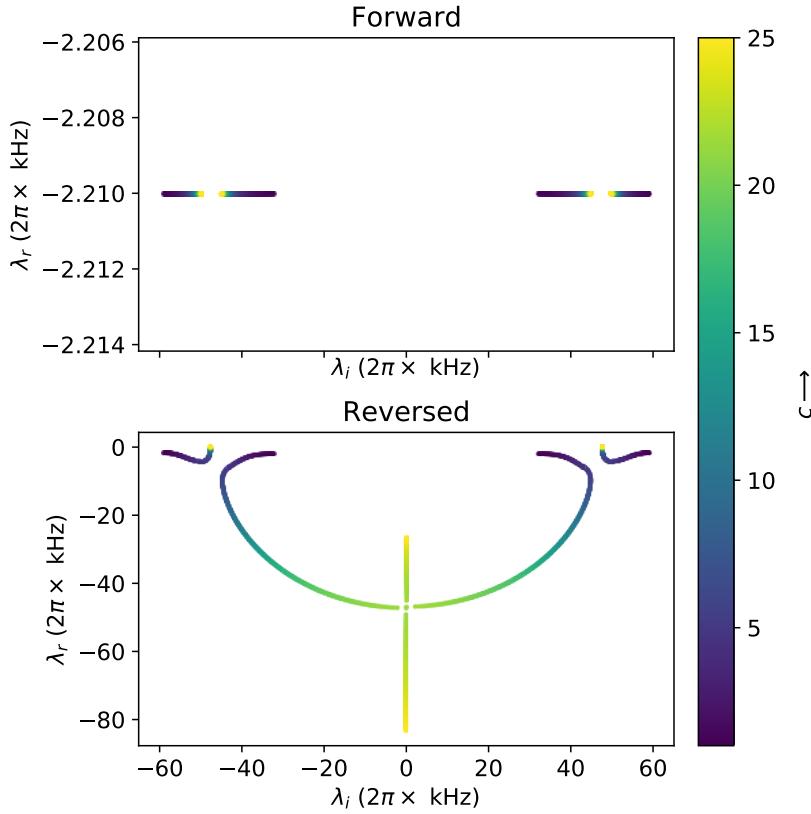


FIGURE 6.5: The evolution of the dissipative normal modes of the system as the asymmetry parameter c increases is shown in this plot. λ_r , λ_i stand for the real and imaginary parts of the eigenvectors of the dynamical matrix \mathbb{A} .

6.7 Discussion

I have studied heat rectification in a model composed of two coupled harmonic oscillators connected to Langevin baths, which could be realized with trapped ions and optical molasses. This simple model allows analytical treatment but still has enough complexity to examine different ingredients that can produce rectification. The results reported in this chapter demonstrate in a simple but realistic model that harmonic systems can rectify heat current if they have features which depend on the temperature [103]. I implement this notion of temperature-dependent features by defining the baths exchange operation as an exchange of both temperatures and coupling parameters of the baths to the system. The temperature dependence of the bath-system coupling occurs naturally in laser-cooled trapped ion setups.

I have also studied the phonon spectra of the system, aided by a normal mode

analysis, comparing the match/mismatch of the phonon bands, to reach the conclusion that the band match/mismatch description for heat rectification is also valid for systems which are purely harmonic, as long as there are temperature-dependent features. I hope this chapter sheds more light into the topic of heat rectification and that encourages more research regarding its physical implementation on chains of trapped ions.

Conclusions

My pleasure.

Appendix

Appendix A

Review of scattering theory formalism

A detailed overview of scattering theory can be found in [6] and its extension to NH systems in [4]. Scattering theory describes the interaction of an incoming wave packet with a localized potential. In general, the spectrum of scattering Hamiltonians (as defined at the beginning of section ??) has both a discrete part and a continuum with real, positive energies. The eigenstates of the continuous spectrum are constructed by the action on plane waves of the Möller operators $|p^\pm\rangle = \Omega_\pm |p\rangle$ and $|\widehat{p}^\pm\rangle = \widehat{\Omega}_\pm |p\rangle$, where

$$\begin{aligned}\Omega_+ &= \lim_{t \rightarrow -\infty} e^{iHt/\hbar} e^{-iH_0 t/\hbar}, \\ \Omega_- &= \lim_{t \rightarrow \infty} e^{iH^\dagger t/\hbar} e^{-iH_0 t/\hbar}, \\ \widehat{\Omega}_+ &= \lim_{t \rightarrow -\infty} e^{iH^\dagger t/\hbar} e^{-iH_0 t/\hbar}, \\ \widehat{\Omega}_- &= \lim_{t \rightarrow \infty} e^{iHt/\hbar} e^{-iH_0 t/\hbar},\end{aligned}\tag{A.1}$$

and a regularization of the limit is implied, see e.g. [4]. The Möller operators satisfy the isometry relation $\widehat{\Omega}_\pm^\dagger \Omega_\pm = 1$ and the intertwining relations $H\Omega_+ = \Omega_+ H_0$ and $H^\dagger \Omega_- = \Omega_- H_0$. By using the intertwining relations, it is easy to see that $|p^+\rangle$ and $|\widehat{p}^-\rangle$ are right eigenvectors of H while $|\widehat{p}^+\rangle$ and $|p^-\rangle$ are left eigenvectors of H , all with positive energy $E_p = p^2/2m$. In the following we will assume that the Hamiltonian admits a basis of biorthonormal right/left eigenstates $\{|\psi_n\rangle, |\phi_a\rangle\}$ with energies E_n satisfying $\langle\phi_n|\psi_m\rangle = \delta_{n,m}$ for the discrete part. The stationary scattering states are also biorthonormal, i.e. $\langle\widehat{p}^+|q^+\rangle = \langle\widehat{p}^-|q^-\rangle = \delta(p - q)$ and

together with the eigenstates of the discrete spectrum they give the resolution of the identity

$$\begin{aligned} 1 &= \sum_n |\psi_n\rangle\langle\phi_n| + \int_{-\infty}^{\infty} dp |p^+\rangle\langle\hat{p}^+| \\ &= \sum_n |\psi_n\rangle\langle\phi_n| + \int_{-\infty}^{\infty} dp |\hat{p}^-\rangle\langle p^-|. \end{aligned} \quad (\text{A.2})$$

There is no degeneracy in the discrete spectrum of one-dimensional systems, whereas the continuum is doubly degenerate, e.g. with continuum eigenfunctions incident from the right or the left. We shall explicitly make use of this property in what follows. Using the resolution of the identity in terms of discrete eigenstates and the stationary scattering states, the Hamiltonian can be expanded as

$$H = \sum_n E_n |\psi_n\rangle\langle\phi_n| + \frac{1}{2m} \int_{-\infty}^{\infty} dp p^2 |p^+\rangle\langle\hat{p}^+|. \quad (\text{A.3})$$

We call the first and the second terms of (A.3) the discrete, H_d , and continuous, H_c , parts of the Hamiltonian respectively. A central object is the scattering operator (or matrix), $S \equiv \Omega_-^\dagger \Omega_+$ for scattering processes by H and $\hat{S} \equiv \hat{\Omega}_-^\dagger \hat{\Omega}_+$ for H^\dagger . Unhatted quantities refer to scattering by H , while hatted ($\hat{}$) quantities refer to scattering by its Hermitian conjugate H^\dagger . The scattering operator gives the probability of an incident state $|\psi_{in}\rangle$ to be scattered (by H or H^\dagger) into a state $|\psi_{out}\rangle$ as $|\langle\psi_{out}|S|\psi_{in}\rangle|^2$ or $|\langle\psi_{out}|\hat{S}|\psi_{in}\rangle|^2$. Although the scattering operator is not unitary for NH Hamiltonians, S and \hat{S} obey the generalized unitarity relations $\hat{S}^\dagger S = S \hat{S}^\dagger = 1$ which collapses to the usual unitarity condition ($S = \hat{S}$) if $H = H^\dagger$. If the Hamiltonian is symmetric or pseudohermitian with respect to a linear/antilinear operator A , the Möller and scattering operators transform according to the intertwining relations in Table A.1. The intertwining relations of the Möller operators give the transformation rules for scattering states under A and provide interesting relations between the different transmission/reflection coefficients.

Also relevant to scattering theory is the transition operator, which is defined as

$$T_{op}(E) = V + VG(E)V \quad (\text{A.4})$$

where $G(E) = (E - H)^{-1}$ is Green's operator. The transition operator satisfies

	<i>A</i> linear	<i>A</i> antilinear
$AH = HA$	$\begin{aligned} A\Omega_{\pm} &= \Omega_{\pm}A \\ AS &= SA \end{aligned}$	$\begin{aligned} A\Omega_{\pm} &= \widehat{\Omega}_{\mp}A \\ AS &= \widehat{S}^{\dagger}A \end{aligned}$
$AH = H^{\dagger}A$	$\begin{aligned} A\Omega_{\pm} &= \widehat{\Omega}_{\pm}A \\ AS &= \widehat{S}A \end{aligned}$	$\begin{aligned} A\Omega_{\pm} &= \Omega_{\mp}A \\ AS &= S^{\dagger}A \end{aligned}$

TABLE A.1: Transformation rules of the Möller and scattering operators under symmetries with linear or antilinear operators.

$T_{op}^{\dagger}(z) = \widehat{T}_{op}(z^*)$ and its matrix elements in momentum representation are related to the scattering operator by

$$\langle p | S | p' \rangle = \delta(p - p') - 2i\pi\delta(E_p - E_{p'}) \langle p | T_{op}(+) | p' \rangle, \quad (\text{A.5})$$

where $T_{op}(\pm)|p'\rangle = \lim_{\epsilon \rightarrow 0^+} T_{op}(E_p \pm i\epsilon)|p'\rangle$. This operator can then be used to define the scattering amplitudes for real p as

$$\begin{aligned} R^l(p) &= -\frac{2\pi im}{p} \langle -p | T_{op}(\text{sign}(p)) | p \rangle, \\ T^l(p) &= 1 - \frac{2\pi im}{p} \langle p | T_{op}(\text{sign}(p)) | p \rangle, \\ R^r(p) &= -\frac{2\pi im}{p} \langle p | T_{op}(\text{sign}(p)) | -p \rangle, \\ T^r(p) &= 1 - \frac{2\pi im}{p} \langle -p | T_{op}(\text{sign}(p)) | -p \rangle, \end{aligned} \quad (\text{A.6})$$

where $R^{l,r}$ is the left/right reflection amplitude and $T^{l,r}$ is the left/right transmission amplitude. We assume that the amplitudes admit analytic continuations. The generalized unitarity relation of the scattering operators give the following set of equations for the amplitudes

$$\begin{aligned} \widehat{T}^l(p)T^{l*}(p) + \widehat{R}^l(p)R^{l*}(p) &= 1, \\ \widehat{T}^r(p)T^{r*}(p) + \widehat{R}^r(p)R^{r*}(p) &= 1, \\ \widehat{T}^{l*}(p)R^r(p) + T^r(p)\widehat{R}^{l*}(p) &= 0, \\ T^l(p)\widehat{R}^{r*}(p) + \widehat{T}^{r*}(p)R^l(p) &= 0, \end{aligned} \quad (\text{A.7})$$

where p is taken to be real and nonnegative. The Dirac deltas in Eq. (A.5)

make clear that the S matrix only connects momentum eigenstates having the same kinetic energy. Factoring out the Dirac delta of energy using $\delta(p - p') = \frac{|p|}{m} \delta(E_p - E_{p'}) \delta_{pp'}$ ($\delta_{pp'}$ is to be understood as a Kronecker delta of the signs of the momenta) we can write $\langle p | S | p' \rangle = \frac{|p|}{m} \delta(E_p - E_{p'}) \langle \mathbf{p} | \mathsf{S} | \mathbf{p}' \rangle$ in terms of the two-dimensional vectors $|\mathbf{p}\rangle \equiv (1, 0)^T$ and $|-\mathbf{p}\rangle \equiv (0, 1)^T$ that correspond to the states $|p\rangle$ and $| -p \rangle$ for $p > 0$ [4]. The previous relation defines the on-the-energy-shell S matrix as

$$\begin{aligned} \langle \mathbf{p} | \mathsf{S} | \mathbf{p}' \rangle &= \delta_{pp'} - \frac{2i\pi m}{|p|} \langle p | T_{op}(+) | p' \rangle \\ &\Downarrow \\ \mathsf{S} &= \begin{pmatrix} T^l(p) & R^r(p) \\ R^l(p) & T^r(p) \end{pmatrix}. \end{aligned} \quad (\text{A.8})$$

$\widehat{\mathsf{S}}$ can be defined similarly. The on-the-energy-shell scattering matrix S inherits the generalized unitarity relation of the scattering operator S , i.e., $\widehat{\mathsf{S}}^\dagger \mathsf{S} = 1$. Equation (A.7) is just this generalized unitarity relation written for all matrix elements.

Appendix B

Alternative formulation of A -pseudohermitian symmetries as ordinary (commuting) symmetries

Symmetry relations like (??) (for A either linear or antilinear) may also be expressed as ordinary (commuting) symmetries, generalizing for scattering Hamiltonians the work in [159–161]. In other words, for a Hamiltonian H and a linear hermitian (antilinear hermitian) operator A satisfying (??) we can find an antilinear (linear) operator B that commutes with H . In this appendix we explicitly construct the operators B from the Hamiltonian both for A linear and antilinear in the first and second sections respectively.

Let us assume for now that besides A (linear or antilinear) there exists an invertible and hermitian antilinear operator τ that also satisfies (??). With A and τ let us define the operator $B = A^{-1}\tau$ that will be antilinear (linear) for A linear (antilinear). As defined, B commutes with the Hamiltonian, because

$$\begin{aligned} BH &= A^{-1}\tau H \\ &= A^{-1}H^\dagger\tau \\ &= HA^{-1}\tau \\ &= HB. \end{aligned} \tag{B.1}$$

B is not generally Hermitian unless τ commutes with A^{-1} .

The main task to define B is to find the antilinear operator τ that satisfies (??). This can be achieved if the eigenvectors of the Hamiltonian and its adjoint form bases of the Hilbert space that are biorthonormal. In [161] the expression of τ for a discrete spectrum (with no degeneracy) is found as

$$\tau_d |\zeta\rangle = \sum_n \langle \zeta | \phi_n \rangle |\phi_n\rangle, \quad (\text{B.2})$$

where the d subscript indicates that the Hamiltonian has a discrete spectrum. The action of the operator in Eq. (B.2) on a vector in an eigenspace amounts to complex conjugation of its coordinate representation. τ_d is clearly antilinear, Hermitian (for antilinear operators hermicity is defined as $\langle \chi | \tau_d \zeta \rangle = \langle \zeta | \tau_d \chi \rangle$), and invertible. It can be checked that the relation $\tau_d H = H^\dagger \tau_d$ is satisfied.

To generalize this to Hamiltonians whose spectrum includes a continuous part, we have to build an antilinear operator τ that acts in both the subspaces, \mathcal{H}_d and \mathcal{H}_c , that are respectively spanned by the eigenfunctions associated with the discrete (point) and continuous spectra of the Hamiltonian. We propose to take $\tau = \tau_d + \tau_c$. The operators τ_d and τ_c act on complementary subspaces of the Hilbert space: while τ_d maps \mathcal{H}_d to \mathcal{H}_d and annihilates states in \mathcal{H}_c , τ_c maps \mathcal{H}_c to \mathcal{H}_c and annihilates states in \mathcal{H}_d . Specifically, we take τ_d to be given by Eq. (B.2) with n denoting the eigenvectors of the Hamiltonian associated with the discrete part of the spectrum. To construct τ_c , note that to satisfy $\tau H = H^\dagger \tau$ (Eq. (??)) it has to transform right scattering eigenvectors into some linear combination of left scattering eigenvectors in the same energy shell. This is so because

$$H^\dagger \tau |p^+\rangle = \tau H |p^+\rangle = \tau E_p |p^+\rangle = E_p \tau |p^+\rangle. \quad (\text{B.3})$$

To fulfill the last requirement we set

$$\tau_c |\zeta\rangle = \int_{-\infty}^{\infty} dp [C_+(p) \langle \zeta | \hat{p}^+ \rangle |\hat{p}^+\rangle + C_-(p) \langle \zeta | \hat{p}^+ \rangle |-\hat{p}^+\rangle], \quad (\text{B.4})$$

where $C_+(p)$ and $C_-(p)$ are complex coefficients. It is straightforward to check that $\tau_c |p^+\rangle = C_+(p) |\hat{p}^+\rangle + C_-(p) |-\hat{p}^+\rangle$. The operator in (B.4) is clearly antilinear because of the antilinearity of the inner product with respect to its first argument. Hermicity of τ requires $C_-(p) = C_-(-p)$. The condition that τ must be invertible restricts the coefficients in Eq. (B.4) further. Consider the *on shell* representation

of τ_c , $\langle p^+ | \tau q^+ \rangle = \frac{|p|}{m} \delta(E_p - E_q) C_{p,q}$, with $C_{p,q} \equiv \delta_{p,q} C_+(q) + \delta_{p,-q} C_-(q)$, or in matrix form

$$C(p) = \begin{pmatrix} C_+(p) & C_-(p) \\ C_-(p) & C_+(-p) \end{pmatrix}. \quad (\text{B.5})$$

Since τ has to be invertible, $C(p)$ must be invertible as well. This implies $C_+(p)C_+(-p) - C_-(p)C_-(p) \neq 0$.

In the following sections we construct expressions for B , in section 1 for A linear, and in section 2 for A antilinear.

B.1 Pseudohermicity with linear operators

In [159–161] it is shown that pseudohermitian Hamiltonians, i.e. those satisfying (??) for $A = \eta$ with η a hermitian and invertible linear operator, possess an energy spectrum whose complex eigenvalues come in complex-conjugate pairs. Moreover the eigenspaces associated with the eigenvalues E and E^* have the same degeneracy and η maps one to the other. Conversely, if the complex part of the spectrum of H contains only complex-conjugate pairs, it can be shown that there exists an η for which the Hamiltonian satisfies (??). These results hold for a general class of diagonalizable Hamiltonians with a discrete spectrum. For these Hamiltonians we can identify η with

$$\begin{aligned} \eta_d = & \sum_{n_0} |\phi_{n_0}\rangle\langle\phi_{n_0}| \\ & + \sum_n \left[|\phi_{n_-}\rangle\langle\phi_{n_+}| + |\phi_{n_+}\rangle\langle\phi_{n_-}| \right], \end{aligned} \quad (\text{B.6})$$

where the states $|\psi_{n_0}\rangle$ ($|\phi_{n_0}\rangle$) correspond to the right (left) eigenstates of H with real energy E_{n_0} . $|\psi_{n+/n-}\rangle$ (respectively $|\phi_{n+/n-}\rangle$) correspond to the right (respectively left) eigenvectors whose eigenvalue $E_{n+/n-}$ has a positive/negative imaginary part. This gives $\eta_d |\psi_{n_0}\rangle = |\phi_{n_0}\rangle$, $\eta_d |\psi_{n_+}\rangle = |\phi_{n_-}\rangle$ and $\eta_d |\psi_{n_-}\rangle = |\phi_{n_+}\rangle$. Clearly η_d is compatible with pseudohermicity since it maps right eigenvectors associated with the eigenvalue E into left eigenvectors with eigenvalue E^* and the pseudohermicity relation (??) is satisfied. To generalize (B.6) for a scattering Hamiltonian we must add an additional term η_c which acts on the subspace of scattering states and is compatible with the hermiticity and invertibility of $\eta = \eta_d + \eta_c$. Since η should transform the right scattering states into left ones

in the same energy shell, $\eta_c |p^+\rangle$ should be a linear combination of both $|\widehat{p}^+\rangle$ and $|-\widehat{p}^+\rangle$. Accordingly, we propose

$$\eta_c = \int_{-\infty}^{\infty} dp \left[\Lambda_+(p) |\widehat{p}^+\rangle\langle\widehat{p}^+| + \Lambda_-(p) |-\widehat{p}^+\rangle\langle\widehat{p}^+| \right], \quad (\text{B.7})$$

where $\Lambda_+(p), \Lambda_-(p)$ are complex coefficients depending on the momentum p . Hermiticity of η requires $\Lambda_+(p) \in \mathbb{R}$ and $\Lambda_-^*(p) = \Lambda_-(-p)$.

Since η_c connects scattering states with the same energy it admits the following *on-shell* representation, $\langle q^+ | \eta_c | p^+ \rangle = \frac{|p|}{m} \delta(E_q - E_p) \Lambda_{q,p}(p)$, with $\Lambda_{q,p}(p) \equiv \delta_{q,p} \Lambda_+(p) + \delta_{q,-p} \Lambda_-(p)$, or in matrix form

$$\Lambda(p) = \begin{pmatrix} \Lambda_+(p) & \Lambda_-^*(p) \\ \Lambda_-(p) & \Lambda_+(-p) \end{pmatrix}. \quad (\text{B.8})$$

Since η has to be invertible, this implies that the determinant of $\Lambda(p)$ should not vanish i.e. $\Lambda_+(p)\Lambda_+(-p) - \Lambda_-(p)\Lambda_-^*(p) \neq 0$. The inverse of η is then $\eta^{-1} = \eta_d^{-1} + \eta_c^{-1}$ with

$$\begin{aligned} \eta_d^{-1} &= \sum_{n_0} |\psi_{n_0}\rangle\langle\psi_{n_0}| \\ &\quad + \sum_n [|\psi_{n_+}\rangle\langle\psi_{n_-}| + |\psi_{n_-}\rangle\langle\psi_{n_+}|] \end{aligned} \quad (\text{B.9})$$

$$\eta_c^{-1} = \int_{-\infty}^{\infty} dp \left[\Lambda_+^{(-1)}(p) |p^+\rangle\langle p^+| + \Lambda_-^{(-1)}(p) |-\widehat{p}^+\rangle\langle\widehat{p}^+| \right], \quad (\text{B.10})$$

where the complex coefficients $\Lambda_{\pm}^{(-1)}(p)$ are taken from the inverse of $\Lambda(p)$

$$\Lambda^{-1}(p) = \begin{pmatrix} \Lambda_+^{(-1)}(p) & \Lambda_-^{(-1)*}(p) \\ \Lambda_-^{(-1)}(p) & \Lambda_+^{(-1)}(-p) \end{pmatrix}. \quad (\text{B.11})$$

Using the orthogonality between the subspace of discrete (bound) and scattering states we find the final expression for B

$$\begin{aligned} B |\zeta\rangle &= \eta_d^{-1} \tau_d |\zeta\rangle + \eta_c^{-1} \tau_c |\zeta\rangle \\ &= \sum_{n_0} \langle \zeta | \phi_{n_0} \rangle |\psi_{n_0}\rangle + \sum_n \langle \zeta | \phi_{n_+} \rangle |\psi_{n_-}\rangle + \sum_n \langle \zeta | \phi_{n_-} \rangle |\psi_{n_+}\rangle \\ &\quad + \int_{-\infty}^{\infty} dp \langle \zeta | \widehat{p}^+ \rangle \left[\tilde{C}_+(p) |p^+\rangle + \tilde{C}_-(p) |-\widehat{p}^+\rangle \right], \end{aligned} \quad (\text{B.12})$$

with $\tilde{C}_\pm(p) = C_+(p)\Lambda_\pm^{(-1)}(p) + C_-(p)\Lambda_\mp^{(-1)}(-p)$. Note that the resulting operator B is antilinear.

B.2 Pseudohermicity with antilinear operators

In this section we will consider the case where the operator A appearing in Eq. (??) is antilinear. In ref. [161] this is called antipseudohermicity, but we will not use this terminology in order to avoid confusion with antihermicity ($H = -H^\dagger$). The effect of A on a right eigenvector of H is to transform it into its corresponding biorthonormal partner, i.e. the left eigenvector corresponding to the same energy

$$\begin{aligned} H^\dagger A |\psi_n\rangle = AH |\psi_n\rangle &= AE_n |\psi_n\rangle = E_n^* A |\psi_n\rangle \\ &\Downarrow \\ A |\psi_n\rangle &\propto |\phi_n\rangle. \end{aligned} \quad (\text{B.13})$$

A also admits a decomposition similar to Eq. (B.2). The Hamiltonian satisfies Eq. (??) with respect to τ . One can check that $A^{-1}\tau$ is a linear symmetry of the Hamiltonian, $HA^{-1}\tau - A^{-1}\tau H = 0$. The expansion of A on the discrete and scattering basis is

$$A |\xi\rangle = \sum_n g_n \langle \xi | \phi_n \rangle |\phi_n\rangle + \int dp \langle \xi | \hat{p}^+ \rangle [G_+(p) |\hat{p}^+\rangle + G_-(p) |-\hat{p}^+\rangle], \quad (\text{B.14})$$

with $A |p^+\rangle = G_+(p) |\hat{p}^+\rangle + G_-(p) |-\hat{p}^+\rangle$ and $g_n = \langle \psi_n | A \psi_n \rangle$. As examples we have found the expressions of $B = A^{-1}\tau$ for $A_T = \Theta$ (time reversal) and $A_{PT} = \Pi\Theta$ (PT). In both cases we have $A_T^{-1} = A_T$ and $A_{PT}^{-1} = A_{PT}$.

B.2.1 PT Symmetry

The action of $B_{PT} = A_{PT}\tau$ on an arbitrary state is

$$B_{PT} |\zeta\rangle = A_{PT}\tau |\zeta\rangle = A_{PT} \left\{ \sum_n \langle \zeta | \phi_n \rangle |\phi_n\rangle + \int dp [C_+(p) \langle \zeta | \hat{p}^+ \rangle |\hat{p}^+\rangle + C_-(p) \langle \zeta | \hat{p}^+ \rangle |-\hat{p}^+\rangle] \right\} \quad (\text{B.15})$$

Using $A_{PT}H = H^\dagger A_{PT}$ and table A.1 we have $A_{PT}|\widehat{p}^\pm\rangle = |\widehat{p}^\mp\rangle$. Note that The “-” right scattering states can be expressed in terms of the “+” right scattering states as

$$\begin{aligned} |\widehat{p}^-\rangle &= \int dq |q^+\rangle \langle \widehat{q}^+| \widehat{p}^-\rangle \\ &= \int dq |q^+\rangle \langle q| \widehat{\Omega}_+^\dagger \widehat{\Omega}_- |p\rangle \\ &= \int dq |q^+\rangle \langle q| \widehat{S}^\dagger |p\rangle \\ &= |p^+\rangle \langle p| \widehat{S}^\dagger |p\rangle + |-p^+\rangle \langle -p| \widehat{S}^\dagger |p\rangle. \end{aligned} \quad (\text{B.16})$$

With all this, the final form of B_{PT} is

$$\begin{aligned} B_{PT} &= \sum_n (g_n^*)^{-1} |\psi_n\rangle \langle \phi_n| \\ &+ \int dp \left[\tilde{C}_+^*(p) |p^+\rangle \langle \widehat{p}^+| + \tilde{C}_-^*(p) |-p^+\rangle \langle \widehat{p}^+| \right], \end{aligned} \quad (\text{B.17})$$

with $C_\pm^*(p) = C_\pm(p) \langle \pm p| \widehat{S}^\dagger | \pm p\rangle + C_\mp(p) \langle \pm p| \widehat{S}^\dagger | \mp p\rangle$.

B.2.2 Time-Reversal Symmetry

For time-reversal symmetry,

$$B_T |\zeta\rangle = A_T \tau |\zeta\rangle = A_T \left\{ \sum_n \langle \zeta| \phi_n \rangle |\phi_n\rangle + \int dp [C_+(p) \langle \zeta| \widehat{p}^+ \rangle |\widehat{p}^+\rangle + C_-(p) \langle \zeta| \widehat{p}^+ \rangle |-p^+\rangle] \right\}. \quad (\text{B.18})$$

Since the time-reversal operator satisfies the relation (??) with the Hamiltonian, Table A.1 implies $A_T|\widehat{p}^\pm\rangle = |-p^\mp\rangle$. The linear symmetry operator can be expressed as in Eq. (B.17) but in this case $\tilde{C}_\pm^*(p) = C_\pm^*(p) \langle \pm p| \widehat{S}^\dagger | \mp p\rangle + C_\mp^*(p) \langle \pm p| \widehat{S}^\dagger | \pm p\rangle$.

Appendix C

Properties of separable potentials

C.1 Transition operator

For a separable potential $V = V_0 |\phi\rangle\langle\chi|$, the transition operator becomes

$$T_{op} = \alpha |\phi\rangle\langle\chi| \quad (\text{C.1})$$

where $\alpha = V_0 + V_0^2 \langle\chi| G(E) |\phi\rangle$. Then using the Lippmann-Schwinger equation we get that

$$\begin{aligned} T_{op}(E) &= V + VG_0(E)T_{op}(E) \\ &= [V_0 + \alpha V_0 \langle\chi| G_0(E) |\phi\rangle] |\phi\rangle\langle\chi| \end{aligned} \quad (\text{C.2})$$

where $G_0(E) = (E - H_0)^{-1}$ is the Green's operator for free motion. Solving for α now gives

$$\alpha = \frac{V_0}{1 - V_0 \langle\chi| G_0(E) |\phi\rangle} = \frac{V_0}{1 - V_0 Q_0(E)}. \quad (\text{C.3})$$

C.2 S-matrix eigenvalues

The eigenvalues for the S-matrix are given by Eq. (??) in terms of the reflection and transmission amplitudes. For a separable potential, using Eq. (A.6), we can

simplify the transmission and reflection coefficients as

$$\begin{aligned}
 T^l &= 1 - \frac{2\pi im}{p} \alpha \phi(p) \chi^*(p), \\
 T^r &= 1 - \frac{2\pi im}{p} \alpha \phi(-p) \chi^*(-p), \\
 R^l &= -\frac{2\pi im}{p} \alpha \phi(-p) \chi^*(p), \\
 R^r &= -\frac{2\pi im}{p} \alpha \phi(p) \chi^*(-p).
 \end{aligned} \tag{C.4}$$

If we now define

$$\Gamma = \frac{2\pi im}{p} \alpha [\phi(p) \chi^*(p) + \phi(-p) \chi^*(-p)], \tag{C.5}$$

we can write the eigenvalues as simply

$$S_j = 1 - \frac{\Gamma - (-1)^j \Gamma}{2}. \tag{C.6}$$

Note that $S_2 = 1$ for all p . Clearly the following relation must also always hold for the reflection and transmission amplitudes,

$$T^l + T^r - T^l T^r + R^l R^r = 1. \tag{C.7}$$

C.3 Uniqueness of bound state

A separable potential can only have at most one bound state $|\psi_E\rangle$. In momentum representation,

$$\begin{aligned}
 \langle p | \psi_E \rangle &= \langle p | \frac{V_0}{E - H_0} |\phi\rangle \langle \chi | \psi_E \rangle \\
 &= \frac{M}{p^2 - q_B^2} \langle p | \phi \rangle,
 \end{aligned} \tag{C.8}$$

where $M = -2mV_0 \langle \chi | \psi_E \rangle$ and $q_B^2 = 2mE < 0$. Suppose there is a second bound state $|\psi_{E'}\rangle$, with corresponding quantities M' and $q_{B'}^2$. Then,

$$\langle \psi_{E'} | \psi_E \rangle = MM' \int_{-\infty}^{\infty} dp |\langle p | \phi \rangle| \frac{1}{p^2 - q_B^2} \frac{1}{p^2 - q_{B'}^2}. \quad (\text{C.9})$$

Since $MM' \neq 0$ and the integral is positive the overlap cannot be zero so there cannot be two bound states.

Appendix D

Numerical calculation of transmission and reflection coefficients

Here we will discuss how to numerically solve the stationary Schrödinger equation for the two-level system by the invariant imbedding method [162, 163].

Let the potential $\mathcal{V}(x)$ be non-zero in the region $-d < x < d$. We introduce the following dimensionless variables: $\bar{k} = (2mE)^{1/2}2d/\hbar$, $\bar{x} = x/(2d) + 1/2$, $\bar{\Omega}(\bar{x}) = (4md^2/\hbar)\Omega(x)$ and $\bar{\Gamma} = (4md^2/\hbar)(\gamma - 2i\Delta)$. The non-Hermitian dimensionless Hamiltonian for the system takes the form

$$\bar{\mathcal{H}} = \bar{\mathcal{H}}_0 + \bar{\mathcal{V}}(\bar{x}), \quad (\text{D.1})$$

$$\bar{\mathcal{H}}_0 = -\frac{\partial^2}{\partial \bar{x}^2} + \begin{pmatrix} 0 & 0 \\ 0 & -i\bar{\Gamma} \end{pmatrix}, \quad (\text{D.2})$$

$$\bar{\mathcal{V}}(\bar{x}) = \begin{pmatrix} 0 & \bar{\Omega}(\bar{x}) \\ \bar{\Omega}(\bar{x})^* & 0 \end{pmatrix}. \quad (\text{D.3})$$

To set the matrices we use as in the main text the convention for internal states $|1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $|2\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. To simplify the notation, we will from now on drop the bars above variables and operators for the remaining part of this section A. The

corresponding stationary Schrödinger equation is now

$$\begin{aligned} k^2 \psi^{(1)}(x) &= -\frac{\partial^2}{\partial x^2} \psi^{(1)}(x) + \Omega(x) \psi^{(2)}(x), \\ k^2 \psi^{(2)}(x) &= -\frac{\partial^2}{\partial x^2} \psi^{(2)}(x) + \Omega(x)^* \psi_1(x) - i\Gamma \psi^{(2)}(x). \end{aligned}$$

Let us denote as $|\Psi_\alpha(x)\rangle$ the wave vector for the atom impinging in internal level α , $\alpha = 1, 2$. This vector has ground and excited state components, generically $\langle \beta | \psi_\alpha(x) \rangle$, $\beta = 1, 2$, which are still functions of x . We can define the matrices $F(x)$ and $\tilde{F}(x)$ as

$$F_{\beta,\alpha}(x) = \langle \beta | \psi_\alpha(x) \rangle, \quad \tilde{F}_{\beta,\alpha}(x) = \langle \beta | \tilde{\psi}_\alpha(x) \rangle, \quad (\text{D.4})$$

so the stationary Schrödinger equation can be rewritten as

$$\begin{aligned} [k^2 - \mathcal{H}_0 - \mathcal{V}(x)] F(x) &= 0, \\ [k^2 - \mathcal{H}_0 - \mathcal{V}(x)] \tilde{F}(x) &= 0. \end{aligned} \quad (\text{D.5})$$

D.1 Free motion, $\mathcal{V} = 0$

When $\mathcal{V}(x) = 0$ we get

$$\begin{aligned} [k^2 - \mathcal{H}_0] |\psi_\alpha(x)\rangle &= 0, \\ [k^2 - \mathcal{H}_0] |\tilde{\psi}_\alpha(x)\rangle &= 0, \end{aligned} \quad (\text{D.6})$$

for $\alpha = 1, 2$. We can write down the solutions for particles “coming” from the left $|\psi_\alpha(x)\rangle$ in internal state $|\alpha\rangle$ as

$$|\psi_1(x)\rangle = \begin{pmatrix} \frac{1}{\sqrt{k}} e^{ikx} \\ 0 \end{pmatrix}, \quad |\psi_2(x)\rangle = \begin{pmatrix} 0 \\ \frac{1}{\sqrt[4]{k^2+i\Gamma}} e^{i\sqrt{k^2+i\Gamma}x} \end{pmatrix},$$

where we assume the branch $\text{Im} \sqrt{k^2 + i\Gamma} \geq 0$. $|\psi_2(x)\rangle$ is a regular traveling wave only for real $\sqrt{k^2 + i\Gamma}$. If the square root has an imaginary part, $|\psi_2(x)\rangle$ decays from left to right. The solutions for incidence from the right $|\tilde{\psi}_\alpha(x)\rangle$ in internal

state $|\alpha\rangle$ are similarly

$$|\tilde{\psi}_1(x)\rangle = \begin{pmatrix} \frac{1}{\sqrt{k}}e^{-ikx} \\ 0 \end{pmatrix}, |\tilde{\psi}_2(x)\rangle = \begin{pmatrix} 0 \\ \frac{1}{\sqrt[4]{k^2+i\Gamma}}e^{-i\sqrt{k^2+i\Gamma}x} \end{pmatrix}.$$

The normalization is chosen in such a way that the dimensionless probability current is constant (and equal) for all solutions with real $\sqrt{k^2 + i\Gamma}$.

The solutions are given by $F(x) = h_+(x)$ and $\tilde{F}(x) = h_-(x)$, where

$$h_{\pm}(x) = \begin{pmatrix} \frac{1}{\sqrt{k}}e^{\pm ikx} & 0 \\ 0 & \frac{1}{\sqrt[4]{k^2+i\Gamma}}e^{\pm i\sqrt{k^2+i\Gamma}x} \end{pmatrix}. \quad (\text{D.7})$$

The Wronskian is $W(h_+, h_-)(x) = 2i$ so that these are linearly independent solutions.

D.2 General case

To solve the general case, we construct the Green's function defined by

$$(k^2 - \mathcal{H}_0)G_0(x, x') = \delta(x - x')\mathbf{1}. \quad (\text{D.8})$$

It is given by

$$\begin{aligned} G_0(x, x') &= W^{-1} \begin{cases} h_+(x)h_-(x') & x > x', \\ h_+(x')h_-(x) & x' > x, \end{cases} \\ &= -\frac{i}{2} \begin{pmatrix} \frac{1}{k}e^{ik|x-x'|} & 0 \\ 0 & \frac{e^{i\sqrt{k^2+i\Gamma}|x-x'|}}{\sqrt{k^2+i\Gamma}} \end{pmatrix}. \end{aligned} \quad (\text{D.9})$$

The Green's function allows us to solve for $F(x)$ and $\tilde{F}(x)$ in integral form,

$$\begin{aligned} F(x) &= h_+(x) + \int_{-\infty}^{\infty} dx' G_0(x, x') \mathcal{V}(x') F(x'), \\ \tilde{F}(x) &= h_-(x) + \int_{-\infty}^{\infty} dx' G_0(x, x') \mathcal{V}(x') \tilde{F}(x'). \end{aligned} \quad (\text{D.10})$$

D.3 Asymptotic form of the solutions

From Eq. (D.10) we find the following asymptotic forms of $F(x)$ and $\tilde{F}(x)$:

$$\begin{aligned} F_\eta(x) &= \begin{cases} h_+(x) + h_-(x)R & x < 0 \\ h_+(x)T & x > 1 \end{cases}, \\ \tilde{F}_\eta(x) &= \begin{cases} h_-(x)\tilde{T} & x < 0 \\ h_-(x) + h_+(x)\tilde{R} & x > 1 \end{cases}, \end{aligned} \quad (\text{D.11})$$

where the R and T matrices for incidence from the left are given by

$$\begin{aligned} R &= W^{-1} \int_0^1 dx' h_+(x') \mathcal{V}(x') F(x'), \\ T &= \mathbf{1} + W^{-1} \int_0^1 dx' h_-(x') \mathcal{V}(x') F(x'), \end{aligned} \quad (\text{D.12})$$

whereas, for right incidence,

$$\begin{aligned} \tilde{R} &= W^{-1} \int_0^\eta dx' h_-(x') \mathcal{V}(x') \tilde{F}_\eta(x'), \\ \tilde{T} &= \mathbf{1} + W^{-1} \int_0^\eta dx' h_+(x') \mathcal{V}(x') \tilde{F}_\eta(x'). \end{aligned} \quad (\text{D.13})$$

In particular, for left incidence in the ground-state, we get if $x < 0$,

$$|\psi_1(x)\rangle = \begin{pmatrix} \frac{1}{\sqrt{k}} e^{ikx} \\ 0 \end{pmatrix} + \begin{pmatrix} R_{1,1} \frac{1}{\sqrt{k}} e^{-ikx} \\ R_{2,1} \frac{1}{\sqrt[4]{k^2+i\Gamma}} e^{-i\sqrt{k^2+i\Gamma}x} \end{pmatrix}, \quad (\text{D.14})$$

and, if $x > 1$,

$$|\psi_1(x)\rangle = \begin{pmatrix} T_{1,1} \frac{1}{\sqrt{k}} e^{ikx} \\ T_{2,1} \frac{1}{\sqrt[4]{k^2+i\Gamma}} e^{i\sqrt{k^2+i\Gamma}x} \end{pmatrix}. \quad (\text{D.15})$$

When $\sqrt{k^2 + i\Gamma}$ is real, the elements of T and R in Eqs. (D.14) and (D.15) are transmission and reflection amplitudes for waves traveling away from the interaction region. However when $\text{Im}\sqrt{k^2 + i\Gamma} > 0$ the waves for the excited state 2 are evanescent. In scattering theory parlance the channel is “closed”, so the $T_{2,1}$ and $R_{2,1}$ are just proportionality factors rather than proper transmission and reflection amplitudes for travelling waves. By continuity however, it is customary to keep the same notation and even terminology for closed or open channels.

In a similar way, for right incidence in the ground state and $x > 1$,

$$|\tilde{\psi}_1(x)\rangle = \begin{pmatrix} \frac{1}{\sqrt{k}}e^{-ikx} \\ 0 \end{pmatrix} + \begin{pmatrix} \tilde{R}_{1,1}\frac{1}{\sqrt{k}}e^{ikx} \\ \tilde{R}_{2,1}\frac{1}{\sqrt[4]{k^2+i\Gamma}}e^{i\sqrt{k^2+i\Gamma}x} \end{pmatrix}, \quad (\text{D.16})$$

whereas, for $x < 0$,

$$|\tilde{\psi}_1(x)\rangle = \begin{pmatrix} \tilde{T}_{1,1}\frac{1}{\sqrt{k}}e^{-ikx} \\ \tilde{T}_{2,1}\frac{1}{\sqrt[4]{k^2+i\Gamma}}e^{-i\sqrt{k^2+i\Gamma}x} \end{pmatrix}. \quad (\text{D.17})$$

Note that alternative definitions of the amplitudes may be found in many works, without momentum prefactors.

The amplitudes relevant for the main text are $T^l = T_{1,1}$, $T^r = \tilde{T}_{1,1}$, $R^l = R_{1,1}$, and $R^r = \tilde{R}_{1,1}$. The following subsection explains how to compute them.

D.4 Differential equations for R and T matrices

To solve for R and T we will use cut-off versions of the potential,

$$\mathcal{V}_\eta(x) = \begin{cases} \mathcal{V}(x) & 0 \leq x \leq \eta, \\ 0 & \text{Otherwise} \end{cases}, \quad (\text{D.18})$$

where $0 \leq \eta \leq 1$, and corresponding matrices

$$\begin{aligned} R_\eta &= W^{-1} \int_0^\eta dx' h_+(x') \mathcal{V}(x') F_\eta(x'), \\ T_\eta &= \mathbf{1} + W^{-1} \int_0^\eta dx' h_-(x') \mathcal{V}(x') F_\eta(x'), \\ \tilde{R}_\eta &= W^{-1} \int_0^\eta dx' h_-(x') \mathcal{V}(x') \tilde{F}_\eta(x'), \\ \tilde{T}_\eta &= \mathbf{1} + W^{-1} \int_0^\eta dx' h_+(x') \mathcal{V}(x') \tilde{F}_\eta(x'). \end{aligned} \quad (\text{D.19})$$

Taking the derivative of these matrices with respect to η , we find a set of four coupled differential equations,

$$\frac{dR_\eta}{d\eta} = W^{-1}\tilde{T}_\eta h_+(\eta)\mathcal{V}(\eta)h_+(\eta)T_\eta, \quad (\text{D.20})$$

$$\frac{dT_\eta}{d\eta} = W^{-1} \left[h_-(\eta) + \tilde{R}_\eta h_+(\eta) \right] \mathcal{V}(\eta)h_+(\eta)T_\eta, \quad (\text{D.21})$$

$$\frac{d\tilde{R}_\eta}{d\eta} = W^{-1} \left[h_-(\eta) + \tilde{R}_\eta h_+(\eta) \right] \mathcal{V}(\eta) \left[h_-(\eta) + h_+(\eta)\tilde{R}_\eta \right], \quad (\text{D.22})$$

$$\frac{d\tilde{T}_\eta}{d\eta} = W^{-1}\tilde{T}_\eta h_+(\eta)\mathcal{V}(\eta) \left[h_-(\eta) + h_+(\eta)\tilde{R}_\eta \right]. \quad (\text{D.23})$$

The initial conditions are $R_0 = \tilde{R}_0 = 0$ and $T_0 = \tilde{T}_0 = \mathbf{1}$.

D.5 Improving numerical efficiency

The equations (D.22) and (D.23) involve only matrices for incidence from the right, they do not couple to any left-incidence matrix, whereas the equations for left incidence amplitudes involve couplings with amplitudes for right incidence. This asymmetry is due to the way we do the potential slicing. The asymmetry is not “fundamental” but we can use it for our advantage to simplify calculations. We can solve equations (D.22) and (D.23) to get amplitudes for right incidence. To get amplitudes for left incidence we use a mirror image of the potential and solve also these two equations. Thus it is enough to find an efficient numerical method to solve equations (D.22) and (D.23). In principle, one can now solve these differential equations from $\eta = 0$ to 1 to get all reflection and transmission amplitudes using the boundary conditions $\tilde{R}_0 = 0$ and $\tilde{T}_0 = \mathbf{1}$. However due to the exponential nature of the free-space solutions $h_{\pm}(x)$ especially if $\text{Im}\sqrt{k^2 + i\Gamma} > 0$, this is not very efficient numerically.

To avoid this problem we make new definitions,

$$\begin{aligned} \hat{S}_\eta &= \mathbf{1} + h_+(\eta)\tilde{R}_\eta h_-^{-1}(\eta), \\ \hat{T}_\eta &= h_+(0)\tilde{T}_\eta h_-^{-1}(\eta), \\ \hat{\mathcal{V}}(\eta) &= W^{-1}h_+^2(0)\mathcal{V}(\eta), \\ \hat{Q} &= ih_+^{-2}(0). \end{aligned} \quad (\text{D.24})$$

Rewriting the equations (D.22) and (D.23) in terms of these new variables we get

$$\begin{aligned}\frac{d\hat{S}_\eta}{d\eta} &= -2\hat{Q} + \hat{Q}\hat{S}_\eta + \hat{S}_\eta [\hat{Q} + \hat{\mathcal{V}}(\eta)\hat{S}_\eta], \\ \frac{d\hat{T}_\eta}{d\eta} &= \hat{T}_\eta [\hat{Q} + \hat{\mathcal{V}}(\eta)\hat{S}_\eta],\end{aligned}\quad (\text{D.25})$$

with initial conditions $\hat{T}_0 = \hat{S}_0 = \mathbf{1}$.

Let us consider solely incidence in the ground state. For right incidence in the ground state, the reflection coefficients and transmission coefficient are

$$\begin{aligned}\tilde{R}_{1,1} &= e^{-2ik} [(\hat{S}_{\eta=1})_{1,1} - 1], \\ \tilde{R}_{2,1} &= \frac{\sqrt[4]{k^2 + i\Gamma}}{\sqrt{k}} e^{-ik - i\sqrt{k^2 + i\Gamma}} (\hat{S}_{\eta=1})_{2,1}, \\ \tilde{T}_{1,1} &= e^{-ik} (\hat{T}_{\eta=1})_{1,1}, \\ \tilde{T}_{2,1} &= \frac{\sqrt[4]{k^2 + i\Gamma}}{\sqrt{k}} e^{-ik} (\hat{T}_{\eta=1})_{2,1}.\end{aligned}\quad (\text{D.26})$$

D.6 Bounds from unitarity

The S -matrix

$$S = \begin{pmatrix} T_{11} & T_{12} & \tilde{R}_{11} & \tilde{R}_{12} \\ T_{21} & T_{22} & \tilde{R}_{21} & \tilde{R}_{22} \\ R_{11} & R_{12} & \tilde{T}_{11} & \tilde{T}_{12} \\ R_{21} & R_{22} & \tilde{T}_{21} & \tilde{T}_{22} \end{pmatrix}\quad (\text{D.27})$$

is unitary for Hermitian Hamiltonians, in particular when $\gamma = 0$. Unitarity implies relations among the matrix elements and in particular

$$1 \geq |R_{11}|^2 + |T_{11}|^2,\quad (\text{D.28})$$

$$1 \geq |\tilde{R}_{11}|^2 + |\tilde{T}_{11}|^2,\quad (\text{D.29})$$

$$1 \geq |\tilde{R}_{11}|^2 + |T_{11}|^2,\quad (\text{D.30})$$

$$1 \geq |R_{11}|^2 + |\tilde{T}_{11}|^2.\quad (\text{D.31})$$

While the first two equations (D.28) and (D.29) are rather obvious because of probability conservation, the last two equations (D.30) and (D.31) are less so, and set physical limits to the possible asymmetric devices that can be constructed in the ground state subspace.

Appendix E

Full set of steady-state equations for the components of $\mathbb{C}^{S.S}$

Here is the full set of equations for the covariance matrix elements in the steady state. We use the shorthand notation $\langle \dots \rangle = \langle \dots \rangle^{s.s}$, i.e., all averages are in the steady state.

$$\begin{aligned}
& \frac{2k \langle p_2 q_1 \rangle}{m_1 m_2} + \frac{2\gamma_L \langle p_1^2 \rangle}{m_1^3} = \frac{2D_L}{m_1^2}, \\
& -\frac{2k \langle p_2 q_1 \rangle}{m_2^2} + \frac{2\gamma_R \langle p_2^2 \rangle}{m_2^3} = \frac{2D_R}{m_2^2}, \\
& -\frac{(k_L + k) \langle q_1 q_2 \rangle}{m_1} + \frac{k \langle q_2^2 \rangle}{m_1} + \frac{\gamma_L \langle p_2 q_1 \rangle}{m_1 m_2} + \frac{\langle p_1 p_2 \rangle}{m_1 m_2} = 0, \\
& \frac{(k_L + k) \langle p_2 q_1 \rangle}{m_1 m_2} - \frac{(k_R + k) \langle p_2 q_1 \rangle}{m_2^2} + \frac{\gamma_L \langle p_1 p_2 \rangle}{m_1^2 m_2} + \frac{\gamma_R \langle p_1 p_2 \rangle}{m_1 m_2^2} = 0, \\
& -\frac{(k_L + k) \langle q_1^2 \rangle}{m_1} + \frac{k \langle q_1 q_2 \rangle}{m_1} + \frac{\langle p_1^2 \rangle}{m_1^2} = 0, \\
& -\frac{(k_R + k) \langle q_2^2 \rangle}{m_2} + \frac{k \langle q_1 q_2 \rangle}{m_2} + \frac{\langle p_2^2 \rangle}{m_2^2} = 0, \\
& -\frac{(k_R + k) \langle q_1 q_2 \rangle}{m_2} + \frac{k \langle q_1^2 \rangle}{m_2} - \frac{\gamma_R \langle p_2 q_1 \rangle}{m_2^2} + \frac{\langle p_1 p_2 \rangle}{m_1 m_2} = 0.
\end{aligned}$$

Appendix F

Complete expressions for the Spectral Density Matrix

In Section 6.6 we used the characteristic polynomial $P_{\mathbb{A}}(\lambda)$ of the dynamical matrix \mathbb{A} for the calculation of the spectral density matrix. $P_{\mathbb{A}}(\lambda)$ is defined as

$$\begin{aligned}
 P_{\mathbb{A}}(\lambda) &\equiv \det(\mathbb{A} - \lambda) \\
 &= \lambda^4 \\
 &+ \lambda^3 \left(\frac{\gamma_L}{m_1} + \frac{\gamma_R}{m_2} \right) \\
 &+ \lambda^2 \frac{(\gamma_L \gamma_R + m_2(k + k_L) + m_1(k + k_R))}{m_1 m_2} \\
 &+ \lambda \frac{(\gamma_R(k + k_L) + \gamma_L(k + k_R))}{m_1 m_2} \\
 &+ \frac{k(k_L + k_R) + k_L k_R}{m_1 m_2}.
 \end{aligned} \tag{F.1}$$

We also used the polynomials $\mathbb{S}_L(\lambda) = \sum_{n=0}^6 \lambda^n \mathbb{S}_{L,n}$ and $\mathbb{S}_R(\lambda) = \sum_{n=0}^6 \lambda^n \mathbb{S}_{R,n}$. There are 14 different polynomial coefficients, which are 4×4 matrices. This is the full list of coefficients,

$$\mathbb{S}_{L,0} = \begin{pmatrix} (k + k_R)^2 & k(k + k_R) & 0 & 0 \\ k(k + k_R) & k^2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\begin{aligned}
\mathbb{S}_{R,0} &= \begin{pmatrix} k^2 & k(k+k_L) & 0 & 0 \\ k(k+k_L) & (k+k_L)^2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \\
\mathbb{S}_{L,1} &= \begin{pmatrix} 0 & k\gamma_R & -(k+k_R)^2 & -k(k+k_R) \\ -k\gamma_R & 0 & -k(k+k_R) & -k^2 \\ (k+k_R)^2 & k(k+k_R) & 0 & 0 \\ k(k+k_R) & k^2 & 0 & 0 \end{pmatrix}, \\
\mathbb{S}_{R,1} &= \begin{pmatrix} 0 & -k\gamma_L & -k^2 & -k(k+k_L) \\ k\gamma_L & 0 & -k(k+k_L) & -(k+k_L)^2 \\ k^2 & k(k+k_L) & 0 & 0 \\ k(k+k_L) & (k+k_L)^2 & 0 & 0 \end{pmatrix}, \\
\mathbb{S}_{L,2} &= \begin{pmatrix} 2(k+k_R)m_2 - \gamma_R^2 & km_2 & 0 & -k\gamma_R \\ km_2 & 0 & k\gamma_R & 0 \\ 0 & k\gamma_R & -(k+k_R)^2 & -k(k+k_R) \\ -k\gamma_R & 0 & -k(k+k_R) & -k^2 \end{pmatrix}, \\
\mathbb{S}_{R,2} &= \begin{pmatrix} 0 & km_1 & 0 & k\gamma_L \\ km_1 & 2(k+k_L)m_1 - \gamma_L^2 & -k\gamma_L & 0 \\ 0 & -k\gamma_L & -k^2 & -k(k+k_L) \\ k\gamma_L & 0 & -k(k+k_L) & -(k+k_L)^2 \end{pmatrix}, \\
\mathbb{S}_{L,3} &= \begin{pmatrix} 0 & 0 & \gamma_R^2 - 2(k+k_R)m_2 & -km_2 \\ 0 & 0 & -km_2 & 0 \\ 2(k+k_R)m_2 - \gamma_R^2 & km_2 & 0 & -k\gamma_R \\ km_2 & 0 & k\gamma_R & 0 \end{pmatrix}, \\
\mathbb{S}_{R,3} &= \begin{pmatrix} 0 & 0 & 0 & -km_1 \\ 0 & 0 & -km_1 & \gamma_L^2 - 2(k+k_L)m_1 \\ 0 & km_1 & 0 & k\gamma_L \\ km_1 & 2(k+k_L)m_1 - \gamma_L^2 & -k\gamma_L & 0 \end{pmatrix}, \\
\mathbb{S}_{L,4} &= \begin{pmatrix} m_2^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \gamma_R^2 - 2(k+k_R)m_2 & -km_2 \\ 0 & 0 & -km_2 & 0 \end{pmatrix},
\end{aligned}$$

$$\begin{aligned}
\$_{R,4} &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & m_1^2 & 0 & 0 \\ 0 & 0 & 0 & -km_1 \\ 0 & 0 & -km_1 & \gamma_L^2 - 2(k + k_L)m_1 \end{pmatrix}, \\
\$_{L,5} &= \begin{pmatrix} 0 & 0 & -m_2^2 & 0 \\ 0 & 0 & 0 & 0 \\ m_2^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \\
\$_{R,5} &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -m_1^2 \\ 0 & 0 & 0 & 0 \\ 0 & m_1^2 & 0 & 0 \end{pmatrix}, \\
\$_{L,6} &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -m_2^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \\
\$_{R,6} &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -m_1^2 \end{pmatrix}. \tag{F.2}
\end{aligned}$$

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