MATH 425 Fall 2024 Homework 7

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1) Since we know that A is nonsingular, that means that A is invertible. We can solve the system of linear equations $A\mathbf{x} = \mathbf{b}$ by multiplying both sides by A^{-1} to get $\mathbf{x} = A^{-1}\mathbf{b}$.
The SVD of A is the equation $A = P\Sigma Q^T$. To get A^{-1} we take the inverse of the SVD:
$$A^{-1} = (P\Sigma Q^T)^{-1}$$
$$A^{-1} = (Q^T)^{-1}\Sigma^{-1}P^{-1}$$
From the SVD, we know that the matrices P and Q and orthogonal. The tranpose of an orthogonal matrix is the same as its inverse so $P^T = P^{-1}$ and $Q^T = Q^{-1}$
$$A^{-1} = (Q^{-1})^{-1}\Sigma^{-1}P^T$$
$$A^{-1} = Q\Sigma^{-1}P^T$$
The diagonal matrix Σ^{-1} containing our singular values looks like $\begin{pmatrix} \frac{1}{\sigma_1} & & & \\ & \frac{1}{\sigma_2} & & \\ & & \ddots & \\ & & & \frac{1}{\sigma_n} \end{pmatrix}$.
A being nonsingular is important, because it means it has full rank. When A has full rank it means that A has no eigenvalues that are equal to zero. And since $\sigma_i = \sqrt{\lambda_i}$, if there exists $\lambda = 0$, then that would make Σ^{-1} impossible since we would be dividing by 0.

2) They are related because the singular values of A $(\sigma_1, \sigma_2, ..., \sigma_r,)$ are reciprocated in A^{-1} $(\frac{1}{\sigma_1}, \frac{1}{\sigma_2}, ..., \frac{1}{\sigma_n})$. Since A is nonsingular, it forces the singular values $\sigma_i > 0$, meaning the eigenvalues of the associated gram matrix A^TA to be greater than 0.

3a)
$$tr(BC) = tr(\begin{pmatrix} b_{11} & ... & b_{1r} \\ \vdots & \ddots & \vdots \\ b_{p1} & ... & b_{pr} \end{pmatrix} \begin{pmatrix} c_{11} & ... & c_{1p} \\ \vdots & \ddots & \vdots \\ c_{r1} & ... & c_{rp} \end{pmatrix})$$
$$= (b_{11}c_{11} + b_{12}c_{21} + ... + b_{1r}c_{r1}) +$$
$$(b_{21}c_{12} + b_{22}c_{22} + ... + b_{2r}c_{r2}) + ... +$$

$$(b_{p1}c_{1p} + ... + b_{pr}c_{rp})$$
$$= \sum_{j=1}^{r}\sum_{i=1}^{p} b_{pr}c_{rp}$$

Swapping out B and C will achieve the same result since it gives us the same terms

$$tr(CB) = \sum_{i=1}^{p}\sum_{j=1}^{r} c_{rp}b_{pr} = \sum_{j=1}^{r}\sum_{i=1}^{p} b_{pr}c_{rp}$$

3b) $||A||^2 = trace(AA^T) = trace(A^TA)$, and $||A||\sqrt{\sum_{j=1}^{n}\sum_{i=1}^{m} a_{ij}^2}$.

$$trace(AA^T) = trace(\begin{pmatrix} a_{11} & ... & a_{1j} \\ \vdots & \ddots & \vdots \\ a_{i1} & ... & a_{ij} \end{pmatrix}\begin{pmatrix} a_{11} & ... & a_{i1} \\ \vdots & \ddots & \vdots \\ a_{1j} & ... & a_{ji} \end{pmatrix})$$

$$= (a_{11}a_11 + a_{12}a_12 + ...) + (a_{21}a_21 + a_{22}a_22 + ...)$$
$$= \sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}a_{ij}$$
$$= \sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}^2 = ||A||^2$$

Since we proved in 2) that $tr(BC) = tr(CB)$, we can also say that $tr(AA^T) = tr(A^TA)$

3c) Proof that $||UA|| = ||A||$

We know that $\sqrt{\sum_{j=1}^{n}\sum_{i=1}^{m} a_{ij}^2}$, $tr(AA^T) = tr(A^TA) = ||A||^2$, and that U is orthogonal meaning $U^T = U^{-1}$ and $U^TU = 1$

$$||A|| = \sqrt{tr(AA^T)} = \sqrt{tr(A^TA)}$$

$$||UA|| = \sqrt{tr((UA)^T(UA))} = \sqrt{tr(A^TU^TUA)} =$$

$$\sqrt{tr(A^TA)} = \sqrt{\sum_{j=1}^{n}\sum_{i=1}^{m} a_{ij}^2} = ||A||$$

3d) We know that $||A|| = \sqrt{tr(AA^T)}$ and $A = P\Sigma Q^T$

$$||A||^2 = tr(AA^T) = tr(P\Sigma Q^T(P\Sigma Q^T)^T) = tr(P\Sigma Q^TQ\Sigma^TP^T)$$
$$= tr(P\Sigma\Sigma^TP^T)$$

Using $tr(BC) = tr(CB)$, we can swap the matrices around

$$= tr((P\Sigma)(\Sigma^T P^T))$$

$$= tr((\Sigma^T P^T)(P\Sigma))$$

$$= tr(\Sigma^T P^T P\Sigma) = tr(\Sigma^T \Sigma) = tr(\Sigma\Sigma^T) = \sum_{i=1}^{r} \sigma_r^2 = ||A||^2$$

Taking the square root gives us A

$$||A|| = \sqrt{\sum_{i=1}^{r} \sigma_r^2} = \sqrt{\sigma_1^2 + \sigma_2^2 + \cdots + \sigma_r^2}$$