

MATH 425 Fall 2024 Homework 3

Miguel Antonio Logarta

Due: Tuesday Oct 5, at 4:00 PM, 2024

1a) Let $\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_m \end{pmatrix}$ and $\mathbf{w} = \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{pmatrix}$

Then $\mathbf{vw}^T = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_m \end{pmatrix} (w_1 \ w_2 \ \dots \ w_n) = \begin{pmatrix} v_1 w_1 & v_1 w_2 & \dots & v_1 w_n \\ v_2 w_1 & v_2 w_2 & \dots & v_2 w_n \\ \vdots & \vdots & \ddots & \vdots \\ v_m w_1 & v_m w_2 & \dots & v_m w_n \end{pmatrix}$

We can see that every row after the first can be cancelled out using Gaussian Elimination

$$R_1 - \frac{v_1}{v_2} R_2 \rightarrow R_2$$

$$R_1 - \frac{v_1}{v_3} R_3 \rightarrow R_3$$

$$R_1 - \frac{v_1}{v_m} R_m \rightarrow R_m$$

We end up with a matrix like the following...

$$\begin{pmatrix} v_1 w_1 & v_1 w_2 & \dots & v_1 w_n \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix}$$

Which is a matrix of rank 1.

1b)

2) Yes, vector $\begin{pmatrix} 3 \\ 0 \\ -1 \\ -2 \end{pmatrix}$ is a linear combination of $\begin{pmatrix} 1 \\ 2 \\ 0 \\ 1 \end{pmatrix}$, $\begin{pmatrix} 0 \\ -1 \\ 3 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 2 \\ 0 \\ 1 \\ -1 \end{pmatrix}$.

To find if the vector is a linear combination, we first create an augmented

matrix that combines all the column vectors.

$$V = \left(\begin{array}{ccc|c} 1 & 0 & 2 & 3 \\ 2 & -1 & 0 & 0 \\ 0 & 3 & 1 & -1 \\ 1 & 0 & -1 & -2 \end{array} \right)$$

Using MATLAB, we then call $\text{rref}(V)$ to get the reduced row echelon form of the matrix

$$\text{rref}(V) = \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array} \right)$$

We can clearly see from the result of $\text{rref}(V)$ that there is no zero row. This means that there is at least one solution for our original vector, making it a valid linear combination from our original question.

- 3a) Yes, however the vectors \mathbf{v}_1 , \mathbf{v}_2 , \mathbf{v}_3 only span a plane in \mathbb{R}^3 , not the entirety of \mathbb{R}^3 . This is because if we combine these column vectors into a matrix, then take the reduced row echelon form, we get

$$\left(\begin{array}{cccc} 1 & 0 & -1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right)$$

Because of the zero row, we find that the vectors only span a plane in \mathbb{R}^3 .

- 3b) No, vectors \mathbf{v}_1 , \mathbf{v}_2 , \mathbf{v}_3 , and \mathbf{v}_4 are not linearly independent. If we use Definition 2.18 from the textbook: vectors $\mathbf{v}_1 + \dots + \mathbf{v}_k$ are linearly dependent if there are scalars $c_1 + \dots + c_k$ not all zero such that $c_1\mathbf{v}_1 + \dots + c_k\mathbf{v}_k = \mathbf{0}$, we find that the rref in 3a shows us that we have two free variables c_3 and c_4 which can be any value. Because of that, we have many solutions to our equation making the vectors linearly dependent.

- 3c) No, vectors \mathbf{v}_1 , \mathbf{v}_2 , \mathbf{v}_3 , and \mathbf{v}_4 do not form a basis in \mathbb{R}^3 . To form a basis in \mathbb{R}^3 , the vectors have to span \mathbb{R}^3 , but have to also be linearly independent. Since the vectors span \mathbb{R}^3 , but aren't linearly independent, they do not form a basis in \mathbb{R}^3 .

We can choose a subset by discarding our linearly dependent vectors. In this case, \mathbf{v}_3 and \mathbf{v}_4 will be discarded while \mathbf{v}_1 and \mathbf{v}_2 will be used to form our basis. We get a basis in \mathbb{R}^2 .

- 3d) The dimension of the span of \mathbf{v}_1 , \mathbf{v}_2 , \mathbf{v}_3 is 2. From 3c, we found that the vectors form a basis in \mathbb{R}^2 . After eliminating all the linearly dependent vectors, we end up with two linearly independent vectors that span a plane in \mathbb{R}^3 which is in 2 dimensions.