MATH 425 Fall 2024 Homework 3

Miguel Antonio Logarta

Due: Tuesday Oct 5, at 4:00 PM, 2024

1a) Let $\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_m \end{pmatrix}$ and $\mathbf{w} = \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{pmatrix}$

Then $\mathbf{v}\mathbf{w}^T = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_m \end{pmatrix} \begin{pmatrix} w_1 & w_2 & \dots & w_n \end{pmatrix} = \begin{pmatrix} v_1w_1 & v_1w_2 & \dots & v_1w_n \\ v_2w_1 & v_2w_2 & \dots & v_2w_n \\ \vdots & \vdots & \ddots & \vdots \\ v_mw_1 & v_mw_2 & \dots & v_mw_n \end{pmatrix}$

We can see that every row after the first can be cancelled out using Gaussian Elimination

$$R_1 - \frac{v_1}{v_2} R_2 \to R_2$$

$$R_1 - \frac{v_1}{v_3} R_3 \to R_3$$

$$R_1 - \frac{v_1}{v_m} R_m \to R_m$$

We end up with a matrix like the following...

$$\begin{pmatrix} v_1w_1 & v_1w_2 & \dots & v_1w_n \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix}$$

Which is a matrix of rank 1.

1b)

2) Yes, vector $\begin{pmatrix} 3 \\ 0 \\ -1 \\ -2 \end{pmatrix}$ is a linear combination of $\begin{pmatrix} 1 \\ 2 \\ 0 \\ 1 \end{pmatrix}$, $\begin{pmatrix} 0 \\ -1 \\ 3 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 2 \\ 0 \\ 1 \\ -1 \end{pmatrix}$.

To find if the vector is a linear combination, we first create an augmented matrix that combines all the column vectors.

$$V = \left(\begin{array}{ccc|c} 1 & 0 & 2 & 3 \\ 2 & -1 & 0 & 0 \\ 0 & 3 & 1 & -1 \\ 1 & 0 & -1 & -2 \end{array}\right)$$

Using MATLAB, we then call rref(V) to get the reduced row echelon form of the matrix

$$rref(V) = \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right)$$

We can clearly see from the result of rref(V) that there is no zero row. This means that there is at least one solution for our original vector, making it a valid linear combination from our original question.

3a) Yes, however the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ only span a plane in $\mathbb{R}^3$, not the entirety of $\mathbb{R}^3$. This is because if we combine these column vectors into a matrix, then take the reduced row echelon form, we get

$$\begin{pmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Because of the zero row, we find that the vectors only span a plane in $\mathbb{R}^3$.

3b) No, vectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, and $\mathbf{v}_4$ are not linearly independent. If we use Definition 2.18 from the textbook: vectors $\mathbf{v}_1 + \cdots + \mathbf{v}_k$ are linearly dependent if there are scalars $c_1 + \cdots + c_k$ not all zero such that $c_1\mathbf{v}_1 + \ldots c_k\mathbf{v}_k = \mathbf{0}$, we find that the rref in 3a shows us that we have two free variables c_3 and c_4 which can be any value. Because of that, we have many solutions to our equation making the vectors linearly dependent.

3c) No, vectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, and $\mathbf{v}_4$ do not form a basis in $\mathbb{R}^3$. To form a basis in $\mathbb{R}^3$, the vectors have to span $\mathbb{R}^3$, but have to also be linearly independent. Since the vectors span $\mathbb{R}^3$, but aren't linearly independent, they do not form a basis in $\mathbb{R}^3$.
We can choose a subset by discarding our linearly dependent vectors. In this case, $\mathbf{v}_3$ and $\mathbf{v}_4$ will be discarded while $\mathbf{v}_1$ and $\mathbf{v}_2$ will be used to form our basis. We get a basis in $\mathbb{R}^2$.

3d) The dimension of the span of $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ is 2. From 3c, we found that the vectors form a basis in $\mathbb{R}^2$. After eliminating all the linearly dependent vectors, we end up with two linearly independent vectors that span a plane in $\mathbb{R}^3$ which is in 2 dimensions.