MATH 425 Fall 2024 Homework 6

Miguel Antonio Logarta

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1a) If λ is an eigenvalue of A, then $A\mathbf{v} = \lambda\mathbf{v}$. To show that $c\lambda + d$ and is an eigenvalue of B where $B = cA + dI$, substite B then

$$B\mathbf{v} = (cA + dI)\mathbf{v}$$
$$B\mathbf{v} = cA\mathbf{v} + dI\mathbf{v}$$

Substitute $A\mathbf{v}$ with $\lambda\mathbf{v}$ since $A\mathbf{v} = \lambda\mathbf{v}$

$$B\mathbf{v} = c(A\mathbf{v}) + dI\mathbf{v}$$
$$B\mathbf{v} = c\lambda\mathbf{v} + dI\mathbf{v}$$
$$B\mathbf{v} = (c\lambda + dI)\mathbf{v}$$

We can see that matrix B operates on eigenvector $\mathbf{v}$ that results in $\mathbf{v}$ scaled by some value $(c\lambda + dI)$. From this, we observe that $c\lambda + d$ is indeed an eigenvalue of B.

1b) Using proof by induction, we first start out with the base case k=0, $A\mathbf{v} = \lambda\mathbf{v}$. Next, to prove that λ^2 is an eigenvalue of A^2...

$$A^2\mathbf{v} = AA\mathbf{v}$$

Subsitute $A\mathbf{v}$ with $\lambda\mathbf{v}$ since $A\mathbf{v} = \lambda\mathbf{v}$

$$AA\mathbf{v} = A(A\mathbf{v}) = A\lambda\mathbf{v}$$

Rearrange terms and subsite $A\mathbf{v}$ again

$$A\lambda\mathbf{v} = \lambda(A\mathbf{v}) = \lambda(\lambda\mathbf{v}) = \lambda^2\mathbf{v}$$

This shows that λ^2 is an eigenvalue of A^2. Proceeding with the k terms, we get:

$$k = 2,\ A^2\mathbf{v} = A(A\mathbf{v}) = A\lambda\mathbf{v} = \lambda(A\mathbf{v}) = \lambda^2\mathbf{v}$$
$$k = 3,\ A^3\mathbf{v} = A(AA\mathbf{v}) = A\lambda^2\mathbf{v} = \lambda^2(A\mathbf{v}) = \lambda^3\mathbf{v}$$
$$\vdots$$
$$k = k,\ A^k\mathbf{v} = A(A^{k-1}\mathbf{v}) = A(\lambda^{k-1}\mathbf{v}) = \lambda^{k-1}(A\mathbf{v}) = \lambda^{k-1}\lambda\mathbf{v} = \lambda^k\mathbf{v}$$

This shows that λ^k is an eigenvalue of A^k.

1c) $Av = \lambda v$. If we let $\lambda = 0$, we get the equation $Av = 0$. Since v is not a zero vector, A must have a non-trivial kernel (There is a non-zero vector in A that causes v to be in the nullspace) which makes A a singular matrix.

1d) $A = \begin{pmatrix} 1 & 1 & \dots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \dots & 1 \end{pmatrix}$

$$Av = \lambda v$$

$$\begin{pmatrix} 1 & 1 & \dots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \dots & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} = \begin{pmatrix} \lambda v_1 \\ \lambda v_2 \\ \vdots \\ \lambda v_n \end{pmatrix}$$

$$\begin{pmatrix} v_1 + v_2 + \cdots + v_n \\ v_1 + v_2 + \cdots + v_n \\ \vdots \\ v_1 + v_2 + \cdots + v_n \end{pmatrix} = \begin{pmatrix} \lambda v_1 \\ \lambda v_2 \\ \vdots \\ \lambda v_n \end{pmatrix}$$

$$\begin{pmatrix} v_1 + v_2 + \cdots + v_n \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} \lambda v_1 \\ \lambda v_2 - \lambda v_1 \\ \vdots \\ \lambda v_n - \lambda v_1 \end{pmatrix}$$

To find our first eigenvalue, we see that our matrix A does not have full rank, meaning that it is a singular matrix. A singular matrix has a zero eigenvalue $\lambda = 0$. Plugging that eigenvalue, we get $Av = 0v$, which results in our eigenvector being $v_1 + v_2 + \cdots + v_n = 0$

To find our other eigenvalue, we can observe that we have $n - 1$ free variables. If we set variables $v_2, \dots, v_n$ to be equal to λv_1, we get

$$v_1 + v_2 + \cdots + v_n = \lambda v_1$$

$$v_1 + (\lambda v_2) + \cdots + (\lambda v_n) = \lambda v_1$$

$$nv_1 = \lambda v_1$$

$$\lambda = n$$

The eigenvector corresponding to $\lambda = n$ is $\begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$

1e) Since A is a nonsingular matrix, it is invertible, we proceed with

$$A\mathbf{v} = \lambda \mathbf{v}$$

$$AA^{-1}\mathbf{v} = \lambda A^{-1}\mathbf{v}$$
$$\mathbf{v} = \lambda A^{-1}\mathbf{v}$$

Since λ is a scalar,

$$\frac{1}{\lambda}\mathbf{v} = A^{-1}\mathbf{v}$$
$$\lambda^{-1}\mathbf{v} = A^{-1}\mathbf{v}$$

Which shows that λ^{-1} is an eigenvalue of A^{-1}

2a) The rank of A is rank 1 because it is a product of **u** and **u** itself. A squishes all the vectors it operates on into a subspace that spans a line. To get our eigenvalues, we first substitue A

$$Av = uu^T = u(u^Tv)$$

If we pick v to equal u, we get

$$Au = u(u^Tu) = u$$

Since $u^Tu = 1$.

$$u = \lambda v = \lambda u$$
$$\lambda = 1$$

To get a zero eigenvalue, we can pick v to be an orthogonal vector to u. This causes $u^Tv = 0$

$$Av = u(u^Tv) = 0$$
$$\lambda = 0$$

In the end, our eigenvectors for A are $\lambda = 0$ and $\lambda = 1$

2b) To find the eigenvectors of our Householder matrix, we can first pick a unit vector u so that:

$$Hu = \lambda u$$
$$Hu = (I - 2uu^T)u = u - 2(uu^T)u = u - 2u(u^Tu) = \lambda u$$

Since $u^Tu = 1$,

$$Hu = u - 2u = -u = \lambda u$$
$$-u = \lambda u$$
$$\lambda = -1$$

If we pick a vector v that is orthogonal to u, we get:

$$Hv = \lambda v$$
$$Hv = (I - 2uu^T)v = v - 2(uu^T)v = v - 2u(u^Tv) = \lambda v$$

Since v is orthogonal to u, $u^Tv = 0$,

$$Hv = 0 = \lambda v$$
$$\lambda = 0$$

In the end, our eigenvectors for A are $\lambda = -1$ and $\lambda = 0$

2c) A projection matrix has a property in which it is idempotent meaning that $P^2 = P$.
To find the eigenvalues we substitute P in $P\mathbf{v} = \lambda\mathbf{v}$ with P^2 then:

$$P^2v = P(Pv) = P(\lambda v) = \lambda Pv$$

$$\lambda Pv = \lambda(Pv) = \lambda(\lambda v) = \lambda^2 v$$

Next:

$$P^2 = P$$

$$\lambda^2 v = \lambda v$$

$$\lambda^2 = \lambda$$

$$\lambda^2 - \lambda = 0$$

$$\lambda(1-\lambda) = 0$$

$$\lambda = 0, \lambda = 1$$

3) To find the eigenvalues of the matrix, we use the characteristic equation $|A - \lambda I| = 0$ and find the the determinant.

$$det \begin{pmatrix} -\lambda & c & -b \\ -c & -\lambda & a \\ b & -a & -\lambda \end{pmatrix} = 0$$

$$-\lambda \cdot det \begin{pmatrix} -\lambda & a \\ -a & -\lambda \end{pmatrix} - c \cdot det \begin{pmatrix} -c & a \\ -b & -\lambda \end{pmatrix} - b \cdot det \begin{pmatrix} -c & -\lambda \\ b & -a \end{pmatrix} = 0$$

$$-\lambda^3 - \lambda a^2 - \lambda b^2 - \lambda c^2 = 0$$

$$-\lambda(\lambda^2 + (a^2 + b^2 + c^2)) = 0$$

$$\lambda = 0, \lambda = \sqrt{a^2 + b^2 + c^2}, \lambda = -\sqrt{a^2 + b^2 + c^2}$$

Since A is a 3x3 matrix with 3 distinct eigenvalues, it has a multiplicity of 1. As a result, A is diagonalizable.

4) Using the spectral theorem, we can construct a real matrix using $A = S\Lambda S^{-1}$ where S contains our eigenvectors, while Λ is a diagonal matrix containing our eigenvalues. We obtain a real matrix A=

$$\begin{pmatrix} -1 & 2 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 3 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -2 \end{pmatrix} \begin{pmatrix} -1 & 2 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 3 \end{pmatrix}^T = \begin{pmatrix} 8 & -4 & 4 \\ -4 & 0 & -8 \\ 4 & -8 & -16 \end{pmatrix}$$