Auxiliar 11.1

P_2). $d \geq 3$ $\exists c > 0$: $\forall u \in L^1(\mathbb{R}^d) \cap W^{1,2}(\mathbb{R}^d)$.

P.dq: $\|u\|_{L^2(\mathbb{R}^d)}^{1+2/d} \leq c \|u\|_{L^1}^{2/d} \|Du\|_{L^2(\mathbb{R}^d)}$.

Sol:

1ro: usar Hölder de la sgt forma:

si $\frac{1}{P_\lambda} = \frac{\lambda}{P_0} + \frac{1-\lambda}{P_1}$, $\lambda \in (0,1)$, entonces

$$1 = \frac{P_\lambda \cdot \lambda}{P_0} + \frac{P_\lambda(1-\lambda)}{P_1} = \frac{1}{p} + \frac{1}{q}$$

$$\int |u|^{P_\lambda} dx = \int |u|^{P_\lambda \cdot \lambda} |u|^{P_\lambda(1-\lambda)} dx$$

$$\leq \left(\int |u|^{\frac{P_\lambda \cdot \lambda}{P_\lambda \cdot \lambda} P_0} dx \right)^{\frac{P_\lambda \cdot \lambda}{P_0}} \left(\int |u|^{\frac{P_\lambda(1-\lambda)}{P_\lambda(1-\lambda)} P_1} \right)^{\frac{P_\lambda(1-\lambda)}{P_1}}$$

$$= \|u\|_{P_0}^{\lambda \cdot P_\lambda} \|u\|_{P_1}^{P_\lambda(1-\lambda)}$$

Así, tomando $P_\lambda = 2$, $P_0 = 2^*$, $P_1 = 1$

$\Rightarrow \frac{1}{2} = \frac{\lambda}{2^*} + \frac{(1-\lambda)}{1} \Rightarrow \lambda\left(1 - \frac{1}{2^*}\right) = \frac{1}{2}$

$\Rightarrow 2^* = \frac{2d}{d-2} \Rightarrow 1 - \frac{1}{2^*} = \frac{d+2}{2d} \quad \therefore \lambda = \frac{d}{d+2}$

$\lambda = \frac{d}{d+2}$ es tal que:

$$\|u\|_2 \leq \|u\|_{2^*}^{\frac{d}{d+2}} \|u\|_1^{2/d+2}$$

Por otra parte, como $u \in W^{1,2}(\mathbb{R}^d)$, por GNS

$$\|u\|_{2^*} \leq c \|Du\|_2$$

$$\therefore \|u\|_2 \leq c \|Du\|_2^{\frac{d}{d+2}} \|u\|_1^{2/d+2}$$

Entonces $\|u\|_2^{\frac{d+2}{2}} \leq c \|Du\|_2 \|u\|_1^{2/d}$