Sensitivity Analysis on Least-Squares American Options Pricing

Miguel Ângelo Maia Ribeiro[1]
[1] Departamento de Física, Instituto Superior Técnico, Universidade de Lisboa, Lisboa, Portugal

I. INTRODUCTION

The stock market has suffered a complete paradigm shift in the past decades. Recent developments in computer science and mathematical finance have greatly enhanced our abilities to predict and take advantage of stock price changes. It shouldn't come as much of a surprise that there has been an ever growing desire from investors to take advantage of these new developments to increase potential profits.

With the colossal sums handled daily in the stock market, even a small improvement on the predictive abilities of a given forecasting algorithm can lead to significant increases in profits for investors. In such a highly competitive subject, it should be clear that a great amount of resources should be devoted to the research and development of these algorithms. An investor that does not follow this strategy is bound to lose major profits when compared with his better prepared counterparts.

Due to the developments in stock price forecasting, our knowledge of derivatives has also greatly increased. A derivative is simply a contract whose value depends on other simpler financial instruments, like stocks or interest rates. Derivatives can virtually take any form desirable, so long as there are two parties interested in taking a part in it. In this work we will focus on the most common type of derivatives - options.

The derivatives market has become increasingly important in recent times. In fact, as of June 2017, derivatives are responsible for over \$542 trillion worth of trades, in the Over-the-Counter (OTC) market alone, as can be seen on FIG. 1 (the OTC market refers to all deals signed outside of derivatives exchanges). Though the market size peaked in 2013 with over \$710 trillion, it seems to have shrinked in the last decade, which might be attributed to the global financial crisis of 2007.

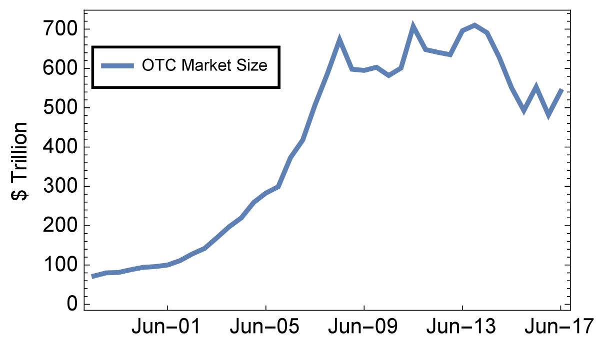

FIG. 1: Size of OTC derivatives market.
Source: stats.bis.org/statx/srs/table/d5.1 (on 16/11/2017)

A. Call and Put Options

Among the many types of derivatives, the most commonly traded are options, of which there are two main types - calls and puts.

In simple terms, a *call option* grants the investor the right to buy the underlying asset (e.g. stock) for a fixed price, known as the *strike price*, by a certain date, known as the *expiration date*.

On the other hand, a *put option* grants the investor the right to sell the underlying asset for the strike price, by the expiration date.

If at the time of exercise, the stock price is above (resp. below) the strike price, the owner of a call (resp. put) option should exercise his right to buy (resp. sell) the stocks, earning the difference between the stock price and the strike price. The payoff function of these two types of derivatives can then simply be deduced as

$$\begin{aligned} \text{Payoff}_{\text{call}}(t) &= (S(t) - K)^+, \\ \text{Payoff}_{\text{put}}(t) &= (K - S(t))^+; \end{aligned} \tag{1}$$

where K is the strike price and $S(t)$ is the stock price at the time of exercise, t.

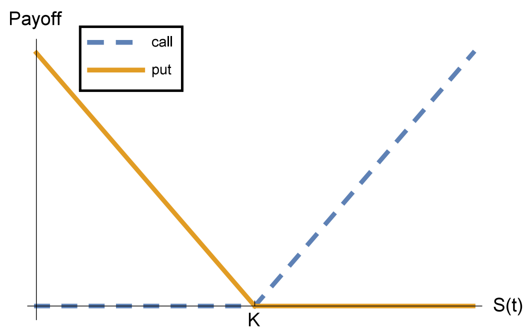

FIG. 2: Payoff functions of *call* and *put options*.

It's important to note that an option gives the holder the right to buy/sell the underlying asset but he is not obliged to do so. If exercising the option would lead to losses, the investor can simply let the option expire.

Options have several advantages that make them especially appealing to investors. To hedgers (i.e. investors that want to limit their exposure to risk), options provide safety by fixing the future price of a stock. If a hedger is afraid of a stock price crash in the future in one of the stocks he holds, by buying put options he ensures that his losses are contained because he can always sell the stocks he owns for the fixed strike price, even if the price crashes. To speculators (i.e. investors that want to take advantage of the uncertainty of future stock prices by betting on their outcome), options grant access to much larger profits when the market forecast proves true, with much smaller initial investments.

Because of all their advantages, unlike some other types of derivatives options have a cost. Finding the ideal price for an option is quite difficult, however. This pricing problem is a fundamental concern to investors, because under or overpricing an option could have severe consequences and lead to major losses. The price of an option and what influences it will be the main focus of this work.

B. European and American Options

Both call and put options can be further separated into several categories. Among these, European and American options are by far the most commonly traded. The holder of an *European option* can only exert his right to buy/sell (call/put) the underlying assets, also known as *exercising*, at the specified expiration date. On the other hand, an *American option* enables its owner to exercise it at any time up to its expiration date.

Due to their high importance, options have been studied in detail in the past. Possibly the most important result in this field came from Robert Merton and Myron Scholes who earned the 1997 Nobel prize in Economics for developing a mathematical model to price European options - the famous Black-Scholes formula - still used nowadays, though with some modifications. Their result states that an European call or put option's price follows the partial differential equation (PDE)

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial V^2}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0, \tag{2}$$

where V is the price of the option, S is the price of the underlying stock, r is the risk-free interest rate and *sigma* is the stock price volatility. This model assumes furthermore that stock prices follow a Geometric Brownian Motion (GBM), which can be defined as

$$dS(t) = rSdt + \sigma SdW(t), \tag{3}$$

with W(t) defining a Brownian motion.

Pricing American options, however, poses a much greater challenge. Because they can be exercised at any time, the underlying assets must be closely monitored so as to attempt optimal stopping. Furthermore, unlike European options, no analytic pricing model currently exists for this type of derivatives. Several numerical models have been proposed in the past in an attempt to price these options. Among these, Longstaff and Schwartz developed one of the most popular, based on Monte Carlo simulation with least squares regression to calculate the option price. Their results will be heavily used in this work.

Despite their complexity, American options are nonetheless very much used by investors. In fact, most of the options currently traded in exchanges are American. Thus, it is absolutely critical to understand which variables influence this type of derivatives and by what amount. With this knowledge, one can better prepare oneself to market changes and even mitigate potential risks.

C. Sensitivity Analysis

Many factors influence the final price of an option. Some factors are absolutely defined, such as the option strike price or the expiration date, while others have an associated uncertainty, like the stock price volatility or the interest rate.

To fully understand how models with uncertain inputs work, a sensitivity analysis is required. In short, sensitivity analysis studies which factors have a greater effect on the final model output. There are many types of sensitivity analysis that go from simple scatter plots to more complex variogram-based methods. One particularly useful type of analysis is the variance-based sensitivity analysis.

Initially created by Ilya Sobol in 1990 and later further developed by several people, among which the work by Andrea Saltelli is paramount, variance-based sensitivity analysis enables the knowledge of by how much the final model output's variance would decrease if we could completely nullify the uncertainty of a given input. In option pricing this is particularly useful. If an investor knows where the greatest source of uncertainty of the option price comes from, he can invest greater sums in the mitigation of that uncertainty.

Due to its usefulness, this particular type of sensitivity analysis and its application to option pricing will be the main focus of this work.

II. OBJECTIVES

The main objective of this thesis is to perform a sensitivity analysis on the price of American options.

We will begin by replicating the model developed by Longstaff and Schwartz to price American options. Their method is based upon a Monte Carlo generation of stock price paths with a set of market variables such as stock price volatility and interest rate (using the Black-Scholes stochastic differential equation). To these results, a least squares regression is then applied iteratively to generate an optimal stopping decision matrix with all the stock price paths generated.

Having replicated this algorithm, we shall then apply a variation-based sensitivity analysis, as developed by Sobol and Saltelli. This analysis outputs a weight for each of the variables used proportional to their influence in the variance of the final option price. Thus, if a variable has a large weight, its variance has a large impact in the variance of the option price.

Some other aspects of this sensitivity analysis will also be considered, such as how each variable's weight changes with time.

As a next step, we shall modify the Longstaff-Schwartz method to more closely resemble real-world stocks. We might, for example, implement the GARCH(1,1) model for the stock price volatility to account for daily changes on its value. We could also try to implement interest rate models for the same reason. Some more recent studies of the Longstaff-Schwartz algorithm suggest that a more effective method to price American options would consist of replacing the least squares regression by some other numerical method, related to optimal stopping. A further study of which models better suit our needs is still necessary, however, and perfecting the initial algorithm will also be a major section of this thesis.

Finally, we will try to apply our model to real-world stock prices, publicly available, and perform this sensitivity analysis using this data.

III. STATE OF THE ART

IV. COMMENTED BIBLIOGRAPHY

A. Miscellaneous Topics

John Hull's book is a great source for most option related information. In this book, Hull explains most option market mechanics and some recent results in this field. In what concerns the present work, the most important topic explored in the book is related to the GARCH(1,1) model.

- Hull, J. (2012). *Options, futures, and other derivatives.* Boston: Prentice Hall.

B. American Option Pricing

Longstaff and Schwartz's paper on the pricing of American options will be heavily used throughout this work. Most results will be based on some variation of their algorithm. Their paper is not only important for the results it presents but also for the examples it provides that enable a clear and deep understanding of the algorithm.

- Longstaff, F. A., & Schwartz, E. S. (2001, January 1). Valuing American Options by Simulation: A Simple Least-Squares Approach. *The Review of Financial Studies, 14*(1), 113-147.

This book presents some useful simulations of many option based results. Though Chloe presents no new results in the option pricing field, some of the simulations developed in this thesis may be based, up to some extent, in the ones presented in Chloe's book.

- Choe, G. H. (2016). *Stochastic Analysis for Finance with Simulations.* Universitext. Springer International Publishing.

C. Sensitivity Analysis

Sobol
Saltelli

V. TIMETABLE