Sensitivity Analysis on Least-Squares American Options Pricing

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Derivatives have become increasingly important in recent decades, with the sums currently handled in these markets amounting to over \$542 trillion. For this reason, it is of the utmost importance to be able to accurately predict the prices of such contracts and be aware of which factors influence them the most. In this thesis we will study, in particular, the pricing of American options. These contracts are especially difficult to price due to the high uncertainty associated with optimal stopping. We shall use the procedure proposed by Longstaff and Schwartz to price this type of derivatives. We will, however, make some adjustments in order to better replicate real-world contracts. Afterwards, we shall perform a variance-based sensitivity analysis, as proposed by Sobol, which provides a measure of how each parameter's uncertainty affects the final option price variance. This analysis is particularly important in option pricing due to the sometimes-large uncertainty associated with the parameters used in the model, such as the stock price volatility.

I. INTRODUCTION

The stock market has suffered a complete paradigm shift in the past decades. Recent developments in computer science and mathematical finance have greatly enhanced our abilities to predict and take advantage of stock price changes. New forecasting algorithms not only enable us to better defend ourselves against unexpected unfavorable market shifts but even to take advantage of them, making such techniques highly desirable to investors. With the colossal sums handled daily in the stock market, even a small improvement on the predictive abilities of a given forecasting algorithm can lead to significant increases in profits for investors. In such a highly competitive subject, it should be clear that a lot of resources must be devoted to the research and development of these algorithms. An investor that does not follow this strategy is bound to lose major profits when compared to his better-prepared counterparts.

Along with stock prices, derivatives have also seen great algorithmic developments in recent times. A derivative is simply a contract whose value depends on other simpler financial instruments, such as stocks or interest rates. Derivatives can virtually take any form desirable, so long as there are two parties interested in taking a part in it. In this work, we will focus on the most common type of derivatives [5] - options.

The derivatives market has become increasingly important in recent times [5]. In fact, as of June 2017, derivatives were responsible for over \$542 trillion worth of trades, in the Over-the-Counter (OTC) market alone [1], as can be seen in FIG. 1 (the OTC market refers to all deals signed outside of exchanges). Though the market size peaked in 2013 with over \$710 trillion, it shrunk in the last decade, due to new regulations implemented after the 2007 global financial crisis [4].

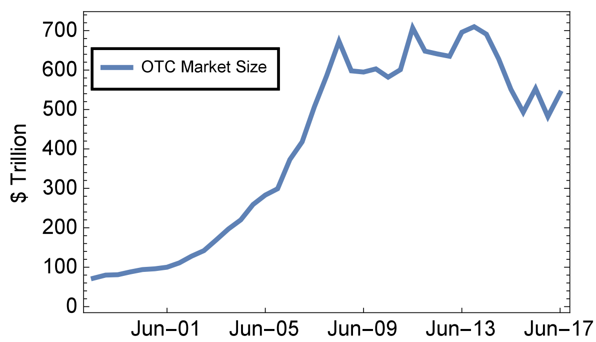

FIG. 1: Size of OTC derivatives market since May 1996.

A. Call and Put Options

Among the many types of derivatives, the most commonly traded are options [5], of which there are two main types - calls and puts.

In simple terms, a *call option* grants an investor the right to buy the underlying asset (e.g. stock) for a fixed price, known as the *strike price*, by a certain date, known as the *expiration date*. On the other hand, a *put option* grants an investor the right to sell the underlying asset for the strike price, by the expiration date.

If at the time of exercise, the asset price is above (resp. below) the strike price, we say the option is *in the money* and the owner of a call (resp. put) option should exert his right to buy (resp. sell) the underlying asset, also known as *exercising*, earning the difference between the asset price at the time of exercise and the fixed strike price. Otherwise, if the asset price is below (resp. above) the strike price, we say the option is *out of the money* and it should not be exercised, as this would lead to losses to its owner.

The payoff function of these two types of derivatives can then simply be deduced as

$$\begin{aligned}\text{Payoff}_{\text{call}}(t) &= (S(t) - K)^+;\\ \text{Payoff}_{\text{put}}(t) &= (K - S(t))^+,\end{aligned} \tag{1}$$

where K is the strike price and $S(t)$ is the asset price at the time of exercise, t. These functions are represented in FIG. 2.

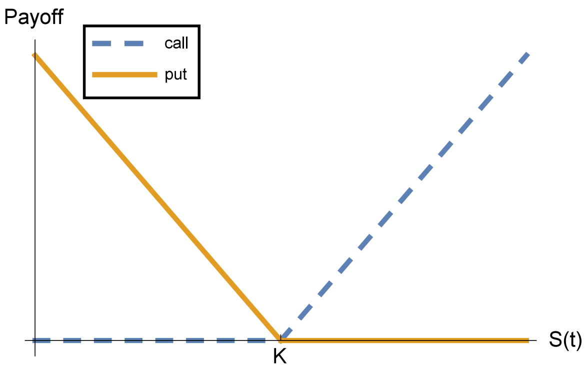

FIG. 2: Payoff functions of *call* and *put options*.

It's important to note that an option gives the holder the right to buy/sell the underlying asset but he is not obliged to do so. If exercising the option would lead to losses the investor can simply let the option expire. This is one of the most atractive characteristics of options.

In the present work, we shall focus on options whose underlying assets are stocks. Though some of the results presented might also be applied to other option types [5], these have to be analyzed carefully.

Options have several other advantages that make them especially appealing to investors. To hedgers (i.e. investors that want to limit their exposure to risk), options provide safety by fixing the future price of a stock i.e. if a hedger is afraid of a future stock price crash in one of the stocks he owns, by buying put options on that same stock, he ensures that his losses are contained because he can always sell the stocks for the fixed strike price, after the stock price crash. To speculators (i.e. investors that want to take advantage of the uncertainty of future stock prices by betting on their outcome), options grant access to much higher profits i.e. due to the options having a lower price than their underlying stock, a speculator can buy a much larger number of options than stocks with the same initial investment, magnifying the consequences of the outcome: if the speculator's initial prediction proves true, his profits will increase greatly whereas a wrong prediction will lead to a total loss of the initial investment.

Due to all their advantages, and unlike some other types of derivatives, options have a price. Finding the ideal price for an option is quite difficult, however, due to the great complexity of these financial instruments [5]. This is a fundamental concern to investors, since knowing the true value of an option can give them a chance to take advantage of under or overpriced options.

B. European and American Options

Both call and put options can be further separated into several categories. Among these, European and American options are by far the most commonly traded [5].

The holder of a *European option* can only exert his right to buy/sell (resp. call/put) the underlying assets at the specified expiration date. On the other hand, an *American option* enables its owner to exercise it at any time up to its expiration date.

Due to their high importance, options have been studied in detail in the past. Possibly the most important result in this field came from Fischer Black, Myron Scholes and Robert Merton, who developed a mathematical model to price European options [2] - the famous Black-Scholes-Merton model - still in use in present days [5]. The last two actually earned the 1997 Nobel prize in Economics for this development.

This model states that a European call or put option's price follows the partial differential equation (PDE)

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0, \tag{2}$$

where V is the price of the option, S is the price of the underlying stock, r is the risk-free interest rate and σ is the stock price volatility. Simply put, the volatility of a stock price is a measure of how uncertain the price movement is in the future. A high volatility will lead to great future fluctuations in the stock price.

This model assumes furthermore that stock prices follow a Geometric Brownian Motion (GBM), which can be defined as

$$dS(t) = rS(t)dt + \sigma S(t)dW(t), \tag{3}$$

with $\{W(t),\ t > 0\}$ defining a one-dimensional Brownian motion.

Because the stock price depends on a Brownian motion process, it follows that it is not differentiable. For this reason, it's impossible to exactly simulate such a process. An approximation is possible, however, using discrete jumps of length Δt and using the Brownian motion property $W(t) \sim \sqrt{t}N(0,1)$ [7], with $N(0,1)$ being a normal distribution with 0 expected value and 1 variance. We can then simply discretize eq. (3) into

$$S(t + \Delta t) = S(t) + rS(t)\Delta t + \sqrt{\Delta t}\sigma S(t)N(0,1), \tag{4}$$

where Δt corresponds to a given time step. An example of this discretization is represented in FIG. 3.

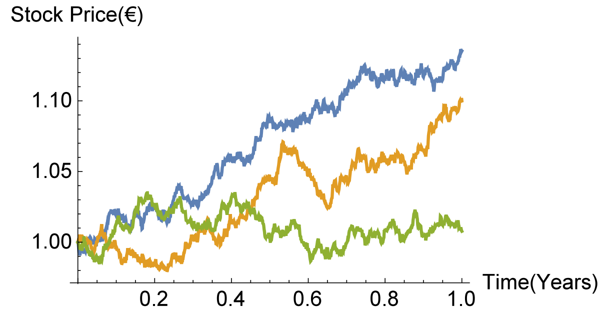

FIG. 3: Example of three GBM processes, using the parameters $r = 0.06\,\text{yr}^{-1}$, $\sigma = 0.05$, $S(0) =$ €1 and time steps $\Delta t = 10^{-3}$yr.

By simulating a large number of paths, some underlying tendencies might become apparent, which will prove useful in option pricing.

Pricing European options is fairly straightforward - we simply need to solve the PDE in eq. (2) in a similar fashion to the initial value problem for the diffusion equation [3]. American options, however, pose a much greater challenge. Unlike European options, no analytic pricing model currently exists for this type of derivatives. Several numerical models have been proposed in the past in an attempt to solve this problem [5], such as the Longstaff-Schwartz algorithm, which we shall approach in later sections of the present thesis.

Despite their complexity, American options are nonetheless the most currently traded options in exchanges [5]. Thus, it is absolutely critical to understand which variables influence this type of derivatives and by what amount. With this knowledge, one can better defend oneself against market changes and even mitigate potential risks.

C. Sensitivity Analysis

One of the main difficulties in the study of option pricing arises from the fact that the price of an option is influenced by many factors [5]. From the strike price to the expiration date, some factors are well defined in the option contract and the investor is well aware of their values. Others, like the stock price volatility or the risk-free interest rate, have an associated uncertainty and require a detailed market study. This input uncertainty will cause a variance in the final model output price.

To fully understand how models with uncertain inputs work, a sensitivity analysis is required. In short, sensitivity analyses indicate which factors have a greater impact on the final model output's variance. There are many types of sensitivity analyses that go from simple scatter plots [8] to more complex Monte-Carlo filtering [8]. Variance-based sensitivity analysis is particularly useful when dealing with options.

Initially created by Ilya Sobol in 1990 and later further developed by several people, among which the work by Andrea Saltelli is paramount, variance-based sensitivity analysis enables the knowledge of by how much the final model output's variance would decrease if we could completely nullify the uncertainty of a given input [8, 9].

The output variance has contributions from each of the uncertain inputs, as well as their combinations. We can therefore decompose the model output variance, $\text{Var}(Y)$, into each of its components as

$$\text{Var}(Y) = \sum_{i=1}^{d} V_i + \sum_{i<j}^{d} V_{ij} + \cdots + V_{1\ldots d}, \tag{5}$$

where d is the total number of uncertain input variables, V_i is the contribution of the i^{th} input variable to the total variance, V_{klm} is the combined contribution of the k^{th}, l^{th} and m^{th} variables, and so on. We can then define the *first order sensitivity index*, S_i, as

$$S_i = \frac{V_i}{\text{Var}(Y)}, \tag{6}$$

which simply gives the effect in the total output variance when varying only the i^{th} variable. Furthermore, we can define the *total effect index*, S_{Ti}, as

$$S_{Ti} = S_i + \sum_{j}^{d} S_{ij} + \cdots + S_{1\ldots d}, \tag{7}$$

which can be interpreted as the importance of the i^{th} variable to the total model output variance. By finding each of these indices we can better understand which variables contribute the most to our uncertainty in the final option price as well as their relationships.

The usefulness of this method in option pricing arises from the fact that if an investor knows where the greatest source of uncertainty of the option price is originated, he can invest greater resources in the mitigation of that uncertainty. This particular type of sensitivity analysis will be studied in more detail in the later sections of the present work.

II. OBJECTIVES

In this work, we shall attempt to perform a sensitivity analysis on the price of American options.

As a first step, we will begin by replicating the model developed by Longstaff and Schwartz to price American options.

Having replicated this algorithm, we shall then modify it to more closely resemble real-world stocks. The model assumes a constant stock price volatility throughout the simulations, which does not hold in the real-world stock market. For this reason, we might implement the GARCH(1,1) model to account for changes in the volatility. We could also attempt to implement new models for the risk-free interest rate since its value also changes with time. A further study of which models better suit our needs is still necessary, however, and perfecting the initial algorithm will also be a major section of this thesis.

With the enhanced Longstaff-Schwartz algorithm, we will then apply a variation-based sensitivity analysis, as developed by Sobol and Saltelli. Some other aspects of this sensitivity analysis will also be considered, such as how each variable's weight changes with time.

Finally, we will try to apply our model to real-world American options. We shall apply some algorithms, such as the implied volatility model, to obtain some of the required model inputs, like the stock price volatility. The remaining inputs shall be obtained from publicly available data, like the risk-free interest rate. We will then perform a sensitivity analysis on these results and study them.

III. STATE OF THE ART

A. American Option Pricing

As previously stated, Longstaff and Schwartz have presented a numerical algorithm to price American options [6].
They begin by simulating a very large number (e.g. 100,000) of stock price paths using the PDE in eq. (3). To do so, they partition the option time frame into many (e.g. 50) smaller intervals to simulate each stock price jump, similar to what was presented in FIG. 3. It should be noted that such an approximation no longer perfectly mimics the behavior of an American option since this type of option assumes a continuous exercising time frame. However, a sufficiently great number of partitions would approximate such an option as closely as desired.
With the paths simulated, they then check which of the simulated stock price paths are in the money at the time step immediately before expiration. For each of these, they perform a least squares regression - a possible choice of basis functions for the regression is the set of Laguerre polynomials - using the cash flows of immediate exercise against the discounted cash flows of the following time step as data. With this regression, they obtain a function that returns the expected cash flow of continuation i.e. how much cash flow is a given stock price path expected to generate if it were not immediately exercised. By comparing the expected cash flow of continuation with the cash flow of immediate exercise, we can check if, for a particular path, whether is optimal to immediately exercise the option or to hold on to the option and then pick the best decision. This algorithm is then iterated backward through every time step until the present. Thus, we obtain the expected cash flows of every simulated stock price which we simply need to average over all paths to obtain the American option price.

This algorithm is quite useful and has been shown to give results very similar to other more computationally demanding procedures, such as finite difference [6].

Despite its usefulness, Longstaff and Schwartz's algorithm only uses a simple GBM to describe each of the paths. For this reason, a constant stock price volatility is assumed throughout the simulations, which does not hold in real world stocks. For this reason, some modifications might be applied to the GBM used in the simulations performed prior to the least-squares section of the algorithm.

One such possible modification is the GARCH(1,1) model for the volatility [10] (short for Generalized Autoregressive Conditional Heteroscedasticity). It states that the volatility in each stock price path has to be updated after each time step using the formula

$$\sigma_n^2 = \gamma V_L + \alpha u_{n-1}^2 + \beta \sigma_{n-1}^2, \tag{8}$$

where γ, α and β are weights (summing to unity), V_L corresponds to the long-run average variance, σ_i corresponds to the volatility at time step i and u_i follows

$$u_i = \frac{S_i - S_{i-1}}{S_{i-1}}, \tag{9}$$

with S_j the stock price at time step j. This model assumes that the volatility increases when great changes in the daily stock price occur and that it decreases otherwise. Furthermore, the long-run average variance, V_L, can be deduced from past data and makes the volatilities tend to a constant value. The GARCH(1,1) is by far the most used of the GARCH(p, q) models, that depend on the p most recent values of u_n and on the q most recent estimates of σ_n [5].

The GARCH(1,1) model provides a better representation of volatilities than the simple assumption that they are constant. Due to its simplicity, it is also easily implemented with negligible computational impact in the calculation of the option prices.

Besides changing over time, the volatility of a stock price is also difficult to estimate. Some methods have been developed in an attempt to solve this problem. One such method is the *implied volatility*. In short, this model outputs the volatility that would be required for an option pricing model (e.g. Black-Scholes method) to return a price equal to the option's market price. We thus try to find the implied volatility, $\widetilde{\sigma}$, that is a solution to the problem

$$f(\widetilde{\sigma}, \cdot) - P = 0, \tag{10}$$

where $f(\widetilde{\sigma}, .)$ is the price output by the pricing model and P is the option's market price. Because this problem has no analytical solution, we must use a numerical method to solve it, such as Newton's method. With this result we can deduce the volatility of the underlying stock price from publicly available option prices.

B. Variance-Based Sensitivity Analysis

The algorithm developed by Ilya Sobol [9] to calculate the indices in eqs. (6) and (7) consists of simply generating two matrices, **A** and **B**, of size $N \times k$, with each of their N rows corresponding to a random extraction from the known distributions of the k uncertain input variables. A set of k new matrices, $\{\mathbf{C}^{(i)},\ i = 1, \ldots, k\}$, is then generated, each of these matrices being equal to matrix $\mathbf{A}$ with its i$^{\text{th}}$ column replaced by matrix $\mathbf{B}$'s. Refer to the scheme below as an example of these matrices.

$$\mathbf{A} = \begin{bmatrix} 0.500 & 0.500 & 0.500 \\ 0.250 & 0.750 & 0.250 \\ 0.750 & 0.250 & 0.750 \\ 0.125 & 0.625 & 0.875 \end{bmatrix},\ \mathbf{B} = \begin{bmatrix} 0.500 & 0.500 & 0.500 \\ 0.750 & 0.250 & 0.750 \\ 0.250 & 0.750 & 0.250 \\ 0.875 & 0.625 & 0.125 \end{bmatrix}$$

$$\mathbf{C}^{(1)} = \begin{bmatrix} 0.500 & 0.500 & 0.500 \\ 0.750 & 0.750 & 0.250 \\ 0.250 & 0.250 & 0.750 \\ 0.875 & 0.625 & 0.875 \end{bmatrix}$$

$$\mathbf{C}^{(2)} = \begin{bmatrix} 0.500 & 0.500 & 0.500 \\ 0.250 & 0.250 & 0.250 \\ 0.750 & 0.750 & 0.750 \\ 0.125 & 0.625 & 0.875 \end{bmatrix}$$

$$\mathbf{C}^{(3)} = \begin{bmatrix} 0.500 & 0.500 & 0.500 \\ 0.250 & 0.750 & 0.750 \\ 0.750 & 0.250 & 0.250 \\ 0.125 & 0.625 & 0.125 \end{bmatrix}$$

Each of the rows of each of these matrices is then used as input for our pricing model - vectors $f(\mathbf{A})$, $f(\mathbf{B})$ and $\{f(\mathbf{C}^{(i)}),\ i = 1, \ldots, k\}$ are produced, with each of their entries corresponding to the output of the model using the respective matrix's row.

Having obtained all the vectors, we can use the formulas proposed by Saltelli [8] to calculate the indices

$$S_i = \frac{(1/N)\sum_{j=1}^{N} f(\mathbf{B})_j . f(\mathbf{C}^{(i)})_j - f_0^2}{(1/N)\sum_{j=1}^{N} \left(f(\mathbf{B})_j\right)^2 - f_0^2}; \tag{11}$$

$$S_{Ti} = 1 - \frac{(1/N)\sum_{j=1}^{N} f(\mathbf{A})_j . f(\mathbf{C}^{(i)})_j - f_0^2}{(1/N)\sum_{j=1}^{N} \left(f(\mathbf{B})_j\right)^2 - f_0^2}, \tag{12}$$

where $f(\mathbf{B})_j$, $f(\mathbf{C}^{(i)})_j$ and $f(\mathbf{A})_j$ are the j$^{\text{th}}$ entries of the vectors $f(\mathbf{B})$, $f(\mathbf{C}^{(i)})$ and $f(\mathbf{A})$, respectively, and f_0 is given by

$$f_0 = \frac{1}{N}\sum_{j=1}^{N} f(\mathbf{B})_j. \tag{13}$$

An empirical explanation of eq. (11) can be obtained by the relation between vectors $f(\mathbf{C}^{(i)})$ and $f(\mathbf{B})$. Because they share the i$^{\text{th}}$ variable as input, if this variable has a large influence on the output result, then vectors $f(\mathbf{C}^{(i)})$ and $f(\mathbf{B})$ would be very similar. For this reason, high (or low) values of $f(\mathbf{B})_j$ would be multiplied by high (or low) values of $f(\mathbf{C}^{(i)})_j$, we would have a large scalar product and therefore a large value for S_i. On the other hand, if this input variable has little effect on the output, $f(\mathbf{B})_j$ and $f(\mathbf{C}^{(i)})_j$ would be randomly associated and a small S_i would be obtained. As for eq. (12) a similar explanation can be found. Because $f(\mathbf{A})$ and $f(\mathbf{C}^{(i)})$ share all the inputs but the i$^{\text{th}}$, the scalar product between $f(\mathbf{A})_j$ and $f(\mathbf{C}^{(i)})_j$ would return the first order effect of all input variables but the shared i$^{\text{th}}$. Using the same line of thought as for eq (11), we are left with a value proportional to the influence of the i$^{\text{th}}$ variable in the output.

Having obtained the values for all variable indices, the results can now be interpreted and conclusions can be extracted from them.

IV. PRELIMINARY WORK

Due to its high importance in the development of the sensitivity analysis, the Longstaff-Schwartz algorithm has already been implemented. The results obtained closely resemble the ones presented in the original article [6], as can be seen on TAB. 1.

$S(0)$(€)	σ	T(yr)	Original price(€)	Replicated price(€)
36	0.2	1	4.472	4.455
36	0.2	2	4.821	4.784
36	0.4	1	7.091	6.688
36	0.4	2	8.488	7.867
40	0.2	1	2.313	2.285
40	0.2	2	2.879	2.810
40	0.4	1	5.308	5.013
40	0.4	2	6.921	6.386

TAB. 1: Comparison between the prices provided by Longstaff and Schwartz in their original article and the results obtained from the replicated algorithm, developed for this work. We assumed an American put option with strike price $K = €40$ and an interest rate $r = 0.06\,\text{yr}^{-1}$. A number of 100,000 paths were simulated, each with 50 time steps.

V. COMMENTED BIBLIOGRAPHY

A. Miscellaneous Topics

John Hull's book is a great source for most option related information. In it, Hull covers most option market mechanics as well as some other recent results in this field in a clear and intuitive way. In what concerns the present work, the most important state-of-the-art topic explored in the book is related to the GARCH(1,1) model.

- J. Hull. *Options, Futures, and Other Derivatives.* Boston: Prentice Hall, 2012.

B. American Option Pricing

Longstaff and Schwartz's paper on the pricing of American options will be heavily used throughout this work. Most results will be based on some variation of their algorithm. Their paper is not only important for the results it presents but also for the examples it provides that enable a clear and deep understanding of the algorithm.

- F. Longstaff and E. Schwartz. *Valuing American Options by Simulation: a Simple Least-Squares Approach.* The review of financial studies, 14(1):113–147, 2001.

Choe's book presents some useful algorithms in the derivatives field. Though Choe presents no new results in the option pricing field, some of the simulations developed in this thesis may be based, up to some extent, in the ones presented in the book.

- G. H. Choe. *Stochastic Analysis for Finance with Simulations.* Springer International Publishing, 2016.

C. Sensitivity Analysis

Saltelli's book provides a deep insight into sensitivity analysis. In particular, it gives a great explanation of the variance-based sensitivity analysis used in the present work as well as all the necessary background.

- A. Saltelli et al. *Global Sensitivity Analysis: The Primer.* John Wiley, 2008.

VI. TIMETABLE

1. Time series analysis of expired American put options. (1 month)
2. Development of the algorithms for the variance-based sensitivity analysis with respect to the volatility, interest rate and initial stock price. (2.5 months)
3. Improvement of the Longstaff-Schwartz algorithm. (1.5 months)
   - Research of recent developments in the stock price simulation field.
   - Implementation of the GARCH(1,1) model for the volatility.
   - Implementation of the implied volatility model.
   - Implementation of new methods found.
4. Writing of the thesis. (1 month)

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