# Sensitivity Analysis on Least-Squares American Options Pricing

**Miguel Ângelo Maia Ribeiro**

Thesis to obtain the Master of Science Degree in

## Engineering Physics

Supervisors: Prof. Cláudia Rita Ribeiro Coelho Nunes Philippart
Prof. Rui Manuel Agostinho Dilão

### Examination Committee

Chairperson: Prof. Full Name
Supervisor: Prof. Full Name 1 (or 2)
Member of the Committee: Prof. Full Name 3

**Month Year**

To my parents and sister

# Acknowledgments

A few words about the university, financial support, research advisor, dissertation readers, faculty or other professors, lab mates, other friends and family...

# Resumo

Inserir o resumo em Português aqui com o máximo de 250 palavras e acompanhado de 4 a 6 palavras-chave...

**Palavras-chave:** palavra-chave1, palavra-chave2,...

# Abstract

Insert your abstract here with a maximum of 250 words, followed by 4 to 6 keywords...

**Keywords:** keyword1, keyword2,...

# Contents

Acknowledgments . . . . . . . . v
Resumo . . . . . . . . vii
Abstract . . . . . . . . ix
List of Tables . . . . . . . . xiii
List of Figures . . . . . . . . xv
Nomenclature . . . . . . . . xvii
Glossary . . . . . . . . xix

**1 Introduction 1**
1.1 Mathematical Finance . . . . . . . . 1
1.2 Derivatives . . . . . . . . 1
1.3 Options . . . . . . . . 1
1.3.1 Why Options are Important . . . . . . . . 2

**2 Background 3**
2.1 Call and Put Options . . . . . . . . 3
2.2 Black-Scholes-Merton Formulae . . . . . . . . 4
2.3 Volatility . . . . . . . . 5
2.3.1 Implied Volatility . . . . . . . . 5
2.3.2 Local Volatility . . . . . . . . 6
2.3.3 Stochastic Volatility . . . . . . . . 8

**3 Implementation 11**

**4 Results 13**

**5 Conclusions 15**

**Bibliography 17**

**A Dupire's Formula Derivation 19**
A.1 Topic Overview . . . . . . . . 21
A.2 Objectives . . . . . . . . 21
A.3 Thesis Outline . . . . . . . . 21

A.4 Theoretical Overview . . . . . . . . . . 21
A.5 Theoretical Model 1 . . . . . . . . . . 21
A.6 Theoretical Model 2 . . . . . . . . . . 21
A.7 Numerical Model . . . . . . . . . . 21
A.8 Verification and Validation . . . . . . . . . . 21
A.9 Problem Description . . . . . . . . . . 22
A.10 Baseline Solution . . . . . . . . . . 22
A.11 Enhanced Solution . . . . . . . . . . 22
    A.11.1 Figures . . . . . . . . . . 22
    A.11.2 Equations . . . . . . . . . . 22
    A.11.3 Tables . . . . . . . . . . 23
    A.11.4 Mixing . . . . . . . . . . 23
A.12 Achievements . . . . . . . . . . 23
A.13 Future Work . . . . . . . . . . 23
A.14 Vector identities . . . . . . . . . . 23

# List of Tables

# List of Figures

1.1 Size of OTC derivatives market since May 1996. . . . . . . . . . . . . . . . . . . . . . . . . . 2

2.1 Payoff functions of call and put options . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 3
2.2 Example of an implied volatility smile and skew. . . . . . . . . . . . . . . . . . . . . . . . . . . 6

# Nomenclature

## Greek symbols

| | |
|---|---|
| $\alpha$ | Angle of attack. |
| $\beta$ | Angle of side-slip. |
| $\kappa$ | Thermal conductivity coefficient. |
| $\mu$ | Molecular viscosity coefficient. |
| $\rho$ | Density. |

## Roman symbols

| | |
|---|---|
| $C_D$ | Coefficient of drag. |
| $C_L$ | Coefficient of lift. |
| $C_M$ | Coefficient of moment. |
| $p$ | Pressure. |
| $\mathbf{u}$ | Velocity vector. |
| $u, v, w$ | Velocity Cartesian components. |

## Subscripts

| | |
|---|---|
| $\infty$ | Free-stream condition. |
| $i, j, k$ | Computational indexes. |
| $n$ | Normal component. |
| $x, y, z$ | Cartesian components. |
| ref | Reference condition. |

## Superscripts

| | |
|---|---|
| * | Adjoint. |
| T | Transpose. |

# Glossary

| | |
|---|---|
| **CFD** | Computational Fluid Dynamics is a branch of fluid mechanics that uses numerical methods and algorithms to solve problems that involve fluid flows. |
| **CSM** | Computational Structural Mechanics is a branch of structure mechanics that uses numerical methods and algorithms to perform the analysis of structures and its components. |
| **MDO** | Multi-Disciplinar Optimization is an engineering technique that uses optimization methods to solve design problems incorporating two or more disciplines. |

# Chapter 1

# Introduction

## 1.1 Mathematical Finance

Mathematical finance, also known as quantitative finance, is a field of applied mathematics focused on the modeling of financial instruments. It is rather difficult to overestimate its importance since it is heavily used by investors and investment banks in everyday transactions. In recent decades, this field suffered a complete paradigm shift, following developments in computer science and new theoretical results that enabled investors to better price their assets. With the colossal sums traded daily in financial markets around the world, mathematical finance has become increasingly important and many resources are invested in the research and development of new and better theories and algorithms.

## 1.2 Derivatives

One of the subjects most studied by financial mathematicians is derivatives. In finance, a derivative is simply a contract whose value depends on other simpler financial instruments, known as *underlying assets*, such as stock prices or interest rates. They can virtually take any form desirable, so long as there are two parties interested in signing it and all government regulations are met.

The importance of derivatives has grown greatly in recent years. In fact, as of June 2017, derivatives were responsible for over $542 trillion worth of trades, in the Over-the-Counter (OTC) market alone [1], as can be seen in Figure 1.1 (the OTC market refers to all deals signed outside of exchanges). This growth stalled after the 2008 global financial crisis due to new government regulations, implemented because of the role of derivatives in the market crashes [2].

## 1.3 Options

Of all kinds of derivatives, in this master thesis we will focus particularly on the most traded one: *options* [3].

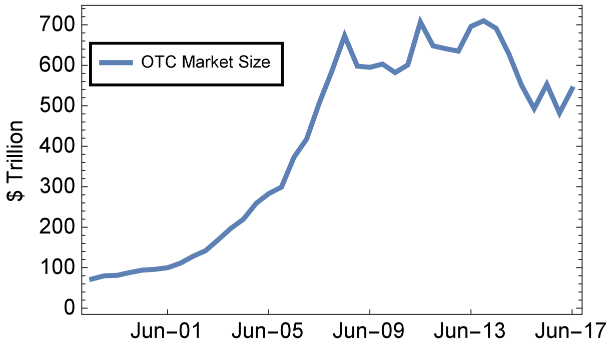


Figure 1.1: Size of OTC derivatives market since May 1996.

As the name implies, an option contract grants its buyer the *option* to buy (in the case of a *call* option) or sell (for *put* options) its underlying asset at a future date, known as the *maturity*, for a fixed price, known as the *strike price*.

The description above pertains only to the most traded type of option -*European* options. In this thesis, unless stated otherwise, all options will be assumed European. There are, however, other types that enable exercising at dates other than maturity. The most well known example is American options, that enable their buyer to exercise at any point in type until the expiration. Other types, commonly known as *exotic*, will be studied in more detail in the following sections.

It's important to emphasize the fact that an option grants its buyer the right to do something. If *exercising* the option would lead to further losses, its buyer can simply decide to let the expiration date pass. This is indeed the most attractive characteristic of options.

### 1.3.1 Why Options are Important

Options are very useful tools to all investors. To hedgers (i.e. investors that want to limit their exposure to risk), options provide safety by fixing the future price of their underlying asset (e.g. if hedgers want to protect themselves against a potential price crash affecting one of their assets, they can buy put options on them. Now, even if the value their asset decreases significantly, their losses are contained because they can always exercise the options and sell the asset at the option's higher strike price).

To speculators (i.e. investors that want to take advantage of the uncertainty of future markets by speculating on their outcome), options grant access to much higher profits (e.g. if speculators strongly believe that the value of a given asset will greatly increase in the future, they can buy call options on that asset. If their prediction proves right, they can buy that asset for the option's lower strike price).

Due to all their advantages, and unlike some other types of derivatives, options have a cost. Finding the ideal price for an option is a fundamental concern to investors, since knowing its true value can give them a chance to take advantage of an under or overpriced option. Finding this price can be very difficult for some options, however. Though a lot of research has been done towards this goal, a great deal more is still required.

# Chapter 2

# Background

## 2.1 Call and Put Options

As stated before, call and put options enable their buyer to buy and sell the underlying stock at the maturity for the fixed strike price. In the case of a call option, if at the maturity the price of its underlying asset is greater than the strike, an investor can buy it for the latter and immediately sell it for its higher market price. Thus, the payoff of the option would be the difference of these two values. On the other hand, if the price of the asset decreases past the strike at the maturity, the investor should let the option expire, since the asset is available for a price in the market price lower than the higher strike. In this case, the payoff of the option would be zero. The same reasoning can be made for put type options. The payoff function of these two types of options can then simply be deduced as

$$\begin{aligned} \text{Payoff}_{\text{call}} &= \max\left(S(T) - K, 0\right); \\ \text{Payoff}_{\text{put}} &= \max\left(K - S(T), 0\right), \end{aligned} \tag{2.1}$$

where $K$ is the strike price and $S(T)$ is the asset price, $S(t)$, at the maturity, $T$. These functions are represented in Figure 2.1.

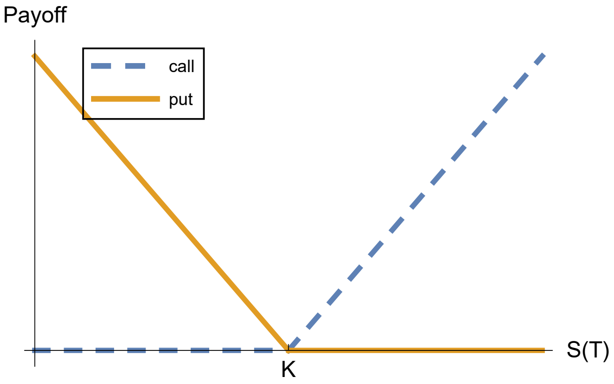


Figure 2.1: Payoff functions of *call* and *put options*.

## 2.2 Black-Scholes-Merton Formulae

Due to their high importance, options have been studied in great detail in the past. Probably the most important result in this field came from Fischer Black, Myron Scholes and Robert Merton, who developed a mathematical model to price options [4] - the famous Black-Scholes-Merton model - still in use in present days [5]. The last two actually earned the 1997 Nobel prize in Economics for this development.

This model states that the price of an European call or put option, whose underlying asset is a stock, follows the partial differential equation (PDE)

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0, \tag{2.2}$$

where $V$ is the price of the option, $S$ is the price of the underlying stock, $r$ is the risk-free interest rate and $\sigma$ is the stock price volatility.

The risk-free interest rate, $r$, is the interest an investor would receive from a risk-free investment. An investor should never invest in risky products whose expected return is lower than this interest, since there's the alternative of investing without risk. In general, this rate changes slightly with time, but Black *et al.* assumed, in their original model, that it remains constant throghout the option duration. Some later developments dealt with this shortcoming, but because option prices do not significantly depend on this value [5], in the remainder of this thesis we shall assume it is constant and known.

Finally, the volatility, $\sigma$, is a measure of uncertainty and will be studied in more detail in section 2.3.

One important assumption of this model is that stock prices follow a Geometric Brownian Motion (GBM), defined as

$$dS(t) = rS(t)dt + \sigma S(t)dW(t), \tag{2.3}$$

with $\{W(t),\ t > 0\}$ defining a one-dimensional Brownian motion.

Pricing options is fairly straightforward - we simply need to solve the PDE in eq. (2.2) in a similar fashion to the initial value problem for the diffusion equation [6]. The results published in the original article by Black *et al.* state that, at time $t$, call and put options can be valued as

$$\begin{aligned} C(S(t),t) &= N(d_1)S(t) - N(d_2)Ke^{-r(T-t)}; \\ P(S(t),t) &= -N(-d_1)S(t) + N(-d_2)Ke^{-r(T-t)}, \end{aligned} \tag{2.4}$$

where $N(\cdot)$ is the cumulative distribution function of the standard normal distribution, $T$ is the maturity time, and $d_1$, $d_2$ are given by

$$\begin{aligned} d_1 &= \frac{1}{\sigma\sqrt{T-t}}\left[\ln\left(\frac{S_t}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)(T-t)\right]; \\ d_2 &= d_1 - \sigma\sqrt{T-t}. \end{aligned} \tag{2.5}$$

From eq. (2.4) we can derive a relationship between $C(S,t)$ and $P(S,t)$, known as the *put-call parity*

$$C(S(t),t) = S(t) - Ke^{-r(T-t)} + P(S(t),t). \tag{2.6}$$

Because of this duality, we can always obtain the prices of put options from call options with the same underlying asset, maturity and strike. For this reason, unless otherwise stated, all options will be assumed calls in the following sections.

## 2.3 Volatility

Volatility is a measure of the uncertainty of future stock prices changes. In other words, a high volatility will lead to great future fluctuations in the stock price, whereas a stock with low volatility is more stable.

Of all the parameters in the Black-Scholes formula, (2.2), volatility is the only one we can't easily observe from market data. Furthermore, unlike the interest rate, volatility has a great impact on the behavior of stock prices and consequently on the price of options. These two factors make volatility one of the most studied subjects in option pricing.

It should be noted that there are several types of volatility, depending on what is being measured. Some of these types will be approached in the next subsections.

### 2.3.1 Implied Volatility

The *implied volatility* can be described as the value of stock price volatility that, when input into the Black-Scholes pricer in eq. (2.4), outputs a value equal to the market price of a given option. In other words, it would be the stock volatility that the seller/buyer of the option assumed when pricing it (given that the Black-Scholes model was used).

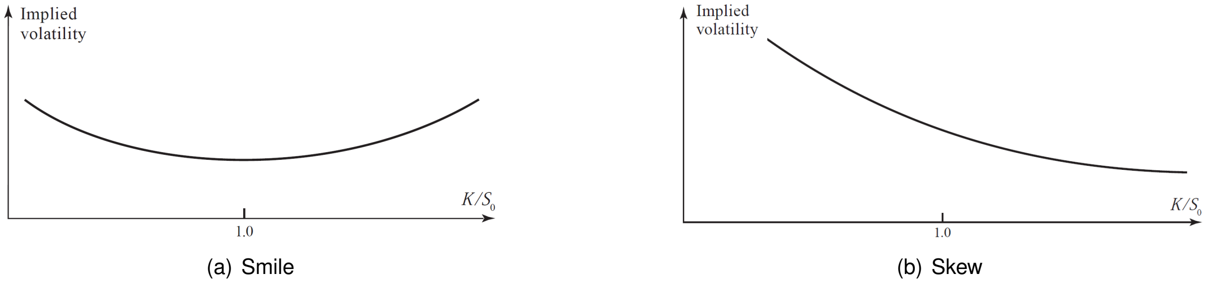
Because eq. (2.4) is not invertible, we need to use some numerical method (e.g Newton's method) to find the solution to the equation

$$C(\sigma_{imp}, S(t), t) - \overline{C} = 0, \tag{2.7}$$

where $C(\sigma_{imp}, \cdot)$ corresponds to the solution of eq. (2.4) using the implied volatility $\sigma_{imp}$ and $\overline{C}$ is the price of the option observed at the market.

We can obtain the implied volatility of an option from its price or the price from its implied volatility, because eq. (2.4) is monotonic in the volatility. This duality is so fundamental that exchanges sometimes sell their options providing only the implied volatility instead of the price [5].

One interesting property of implied volatility is that, in the real-world, it varies with the strike price and maturity. In principle, if investors really used the Black-Scholes model to price their options, two options with the same underlying asset should have the same implied volatility, despite their strike prices or maturities. However, when observing market data, this is not the case. The observed implied volatilities form two possible shapes in a scatter plot, known as *smile* and *skew*. An implied volatility smile shows greater $\sigma_{imp}$ for options with strikes different from the current stock price and the minimum where the strike equals the stock price. A skew, on the other hand, only shows greater $\sigma_{imp}$ in one of the directions (i.e. for strikes either higher or lower than the current stock price). You can observe both these phenomena in Figure 2.2. We can therefore conclude that options with strikes different from the current stock price are overpriced. The reason behind this odd market behavior is the simple demand-supply rule [5]. On the one hand, some investors are risk-averse and want to hedge their losses in case of a market crash (as explained in subsection 1.3.1). They don't mind paying a higher price for an option if this means they would be safe from crashes. For this reason, the prices of low strike (call) options increase which drives their implied volatility up. On the other hand, some other investors are risk-seekers and want to take advantage of possible sudden price increases, buying the stocks for lower prices. They don't mind paying higher prices for the options and this drives the prices of high strike options (and, consequently, their implied volatility) up.



(a) Smile

(b) Skew

Figure 2.2: Example of an implied volatility smile and skew.source=Hull

### 2.3.2 Local Volatility

In their original work, Black *et al.* assumed that volatility is constant throughout the whole contract. From market data, it can be clearly seen that this is not the case. There may be times where new information reaches the market and trading increases, driving volatility up. The opposite is also true.

The model in eq. (2.4) is clearly not enough to truly grasp real-world trading. We should have a model where volatility is dynamic, measuring the amount of randomness in the stock price at any given time. The geometric Brownian motion from eq. 2.3 would thus become

$$dS(t) = rS(t)dt + \sigma(t)S(t)dW(t),$$

where $\sigma(t)$ is now some function of time.

However, as we saw in subsection 2.3.1 for implied volatility, the market's view of volatility also varies with the strike price. A simple dynamic volatility model is thus unsufficient. The local volatility should then be a function of both time and strike price. We are left with a new GBM, given by

$$dS(t) = rS(t)dt + \sigma(K,t)S(t)dW(t), \tag{2.8}$$

where $\sigma(K,t)$ depends now on $K$ and $t$.

Finding the local volatility function is not very important when pricing typical European options, because we can simply use the implied volatility in eq. 2.3 and assume it remains constant. However, for other contracts, such as American, Asian or Barrier options (among others) where the option value depends on the path the stock price takes, this function is indeed crucial.

The local volatility function is very difficult to calibrate, because we can't directly measure the local volatility of a stock from market data. Some models have been proposed in an attempt to achieve this goal. The most well known model is known as Dupire's formula.

### Dupire's Formula

One of the most famous results in the modelling of the local volatility function was obtained by Dupire [7]. In his article, he derives a theoretical formula for this function, given by

$$\sigma(K,T) = \sqrt{\frac{\dfrac{\partial C}{\partial T} + rK\dfrac{\partial C}{\partial K}}{\dfrac{1}{2}K^2\dfrac{\partial^2 C}{\partial K^2}}}, \tag{2.9}$$

where $\sigma(K,T)$ is the volatility function for stock prices $K$ at time $T$ and $C = C(K,T)$ is the price of an European call option with strike price $K$ and maturity $T$. A brief demonstration of this formula can be found in appendix A.

As can be seen, we need to differentiate the option prices with respect to their strikes and maturities. To achieve this, we need to gather a large number of option prices from the market, for options with different maturities and strikes, and interpolate between them to obtain an option price surface (with $K$ and $T$ as variables). We then calculate the gradients in this surface and input them into eq. 2.9 to obtain the local volatility surface.

Even before implementation, four potential sources of error can be found. First, it should be noted that markets only trade options with very specific maturities (e.g. 1, 2, 4, 6-months maturity). For this reason, our data will be very sparse with respect to maturity and the interpolation obtained may not closely resemble reality. Furthermore, it can be seen that we are dividing by the second derivative with respect to the strike price. As we will see in the next section, the prices of options with strikes very far from the current stock price change at an approximately constant rate with respect to the strike. The second derivative in these regions would then be very close to or actually zero. Because we are dividing by this value, our volatilities may explode for very large or very small strikes. Then, there is the problem of noise. Because we are interpolating very sparse data, even small errors in the option market price will cause variations in the interpolation which can be very problematic if the second derivative changes, because we are dividing by it. Finally, some problems arise from the market itself. While investors use mathematical finance in their trades, the market is still governed by the demand-supply rule. If many investors want to buy the option and few want to sell it, the option price will increase, even if it means that the option will be overpriced. It may also happen that the market is not liquid enough (i.e. very few trades or even no trades at all occur for some options) which causes the option prices to not truly follow the market's perception of future price movements.

Dupire also developed an alternative formula based on the implied volatility surface rather than the option price's. The relation he obtained is as follows

$$\sigma(K,T) = \sqrt{\frac{\sigma_{imp}^2 + 2(T-t)\sigma_{imp}\frac{\partial \sigma_{imp}}{\partial T} + 2rK(T-t)\sigma_{imp}\frac{\partial \sigma_{imp}}{\partial K}}{\left(1 + Kd_1\sqrt{T-t}\frac{\partial \sigma_{imp}}{\partial K}\right)^2 + K^2(T-t)\sigma_{imp}\left(\frac{\partial^2 \sigma_{imp}}{\partial K^2} - d_1\left(\frac{\partial \sigma_{imp}}{\partial K}\right)^2\sqrt{T-t}\right)}}, \tag{2.10}$$

where $d_1$ is given by

$$d_1 = \frac{\log(S(t)/K) + \left(r + \frac{1}{2}\sigma_{imp}^2\right)(T-t)}{\sigma_{imp}\sqrt{T-t}}, \tag{2.11}$$

with $t$ representing the current time (usually $t = 0$), $T$ being the time at which the local volatility is being measured, and $S(t)$ the stock price at time $t$. We assume that $\sigma_{imp} = \sigma_{imp}(K,T)$ is the interpolated surface of the implied volatilities evaluated at time $T$, and price $K$. This relation might be more useful than eq. (2.9) because the implied volatility surface might be smoother than the option price's. We will compare the results from both formulas in the next chapters.

If the local volatility surface truly matched reality, it should remain constant in time, assuming the market's view of future price movements remained constant as well. However, this is not the case [5], so we can conclude that the model doesn't completely correspond to reality and for that reason it shouldn't be used blindly.

Despite all of its shortcomings, Dupire's formula is nonetheless very much used by traders to find the local volatility surfaces of assets and price exotic options afterwards. Due to its importance, we shall implement and study this model in more detail.

### 2.3.3 Stochastic Volatility

As stated before, the volatility is not constant, is not observable and is not predictable, despite our attempts to model it. This seems to indicate that volatility is itself a stochastic process. Some research has been done into this hypothesis, and some models have been developed so far.

As before, we assume that the stock price follows a geometric Brownian motion

$$dS = rSdt + \sigma SdW_1 \tag{2.12}$$

but we further hypothesize that the volatility follows

$$d\sigma = p(S,\sigma,t)dt + q(S,\sigma,t)dW_2 \tag{2.13}$$

where $p(S,\sigma,t)$ and $q(S,\sigma,t)$ are some functions that depend on the model used and $W_1$ and $W_2$ are two Brownian motion processes with a correlation of $\rho$. This correlation can be explained by the relationship between prices and volatilities. As an example, we can consider a stock that costs \$100 and changes by \$0.10 daily. We can estimate, even without calculations, that it is very stable and thus has a low volatility. On the other hand if another stock costs \$1 and changes by \$0.10 in a day, we can see that it is extremely volatile even though it changed by the same amount as the first. With this example, we can see that the volatility has some correlation with the stock price.

Choosing the right $p(S, \sigma, t)$ and $q(S, \sigma, t)$ is very important in the evolution of the volatility and each stochastic volatility model will depend on the choice these functions. Furthermore, each model will have parameters that we have to calibrate using past values of the stock price. In the next subsections, we will present two of the most used models.

### Heston Model

One of the most popular stochastic volatility models is known as *Heston model*. It was developed in 1993 by Steven Heston [8] and it states that our system satisfies the relations

$$dS = \mu S dt + \sqrt{\nu} S dW_1, \tag{2.14}$$

$$d\nu = \kappa(\overline{\nu} - \nu)dt + \xi\sqrt{\nu}dW_2, \tag{2.15}$$

with $\nu$ corresponding to the variance (i.e. $\nu = \sigma^2$) and where $W_1$ and $W_2$ have a correlation of $\rho$. Note that our stock price is now governed by a drift parameter $\mu$, unlike the usual risk-free measure drift of $r$, as seen in eq. (2.3). A transformation into the risk-free measure, using Gisarnov's theorem, is possible but the original model used the drift parameter $\mu$. The parameters $\kappa$, $\overline{\nu}$ and $\xi$ are, respectively, the mean-reversion rate (i.e. how fast the volatility converges to its mean value), the long-term variance (i.e. the mean value of variance) and the volatility of volatility (i.e. how erratic is the volatility).

Calibrating the parameters $\rho$, $\kappa$, $\overline{\nu}$ and $\xi$ is absolutely critical. A model with badly calibrated parameters would output wrong predictions, rendering it completely useless. This calibration requires a fair amount of past market data and is by far the most complex and computationally demanding section of this model. We will deal with it in the next section.

### SABR Model

One other very famous model of stochastic volatility was developed by Hagan *et al.* [9] and is known as *SABR*. It stands for "*stochastic-$\alpha\beta\rho$*" and in this model it is assumed that the option prices and volatilities follow

$$dF = \sigma F^{\beta} dW_1, \tag{2.16}$$

$$d\sigma = \nu\sigma dW_2, \tag{2.17}$$

where $\sigma(0) = \alpha$, $F(0) = f$ and the two Brownian motion processes $W_1$ and $W_2$ have a correlation of $\rho$. It should be noted that we now are using the forward measure, so in eq. (2.16) we use $F$, the *forward price*, instead of the usual spot price $S$ from eq. 2.3. These two quantities are related by $S(t) = e^{-r(T-t)}F(t)$, so we can easily obtain one from the other.

According to Hagan *et al.*, the volatility smile obtained from the local-volatility model, as developed by Dupire (eq.(2.9)), doesn't follow true market dynamics [9]. They demonstrate that when the price of the asset increases or decreases, the volatility smile shifts in the opposite direction. The minimum of the volatility smile would therefore be offset and no longer correspond to the local spot price. The dynamics obtained from the local-volatility model should then be actually worse than if we assumed a constant volatility. With SABR, the authors argue that this problem is solved and the smile shifts in the correct direction.

In the original article, the authors claim that $\beta$ can be fitted from historical market data, but usually investors actually choose this value arbitrarily, depending on the type of assets traded. Typical values of $\beta$ are $\beta = 1$ (stochastic lognormal model), usually used for foreign exchange options, $\beta = 0$ (stochastic normal model), typical for interest rate options where forwards $f$ can be negative and $\beta = 0.5$ (stochastic CIR model), also common for interest rate options.

With SABR, Hagan *et al.* show that we can price new European call and put options using the typical Black-Scholes pricing equations, as in eqs. (2.4) and (2.5), but replacing the implied volatility, $\sigma$, in (2.5) with $\sigma_{SABR}$ given by

$$\begin{aligned}\sigma_{SABR}(K,f) \sim & \frac{\alpha}{(fK)^{(1-\beta)/2}\left\{1+\frac{(1-\beta)^2}{24}\log^2 f/K+\frac{(1-\beta)^4}{1920}\log^4 f/K\right\}}\cdot\left(\frac{z}{x(z)}\right)\cdot \\ & \left\{1+\left(\frac{(1-\beta)^2}{24}\frac{\alpha^2}{(fK)^{1-\beta}}+\frac{1}{4}\frac{\rho\beta\nu\alpha}{(fK)^{(1-\beta)/2}}+\frac{2-3\rho^2}{24}\nu^2\right)T\right\},\end{aligned} \tag{2.18}$$

where $f = F(0) = S(0)e^{-rT}$, $z$ is given by

$$z = \frac{\nu}{\alpha}(fK)^{(1-\beta)/2}\log f/K, \tag{2.19}$$

and $x(z)$ is

$$x(z) = \log\left\{\frac{\sqrt{1-2\rho z+z^2}+z-\rho}{1-\rho}\right\}. \tag{2.20}$$

As with the Heston model, the main concern with saber is correctly calibrating the model parameters. This should take the most computational expense in the whole method.

# Chapter 3

# Implementation

To calibrate SABR, we can either follow the Monte-Carlo approach and simulate a large number of paths, assuming stochastic volatility with some starting parameters and minimizing the difference between model-generated prices and real-world option prices, or we can use eq. (2.18) to obtain the implied volatility and minimize the market implied volatility.

## Chapter 4

# Results

# Chapter 5

# Conclusions

# Bibliography

[1] Bank for International Settlements. Semiannual otc derivatives statistics. `http://stats.bis.org/statx/srs/table/d5.1`, November 2017.

[2] Financial Times. Otc derivatives shrink to lowest level since financial crisis. `https://www.ft.com/content/dbc08ae2-1247-11e6-91da-096d89bd2173`, May 2016.

[3] J. Hull. *Options, Futures, and Other Derivatives*. Boston: Prentice Hall, 2012.

[4] F. Black and M. Scholes. The pricing of options and corporate liabilities. *Journal of political economy*, 81(3):637–654, 1973.

[5] P. Wilmott. *Paul Wilmott Introduces Quantitative Finance*. The Wiley Finance Series. Wiley, 2013.

[6] R. Dilão, J. A. de Matos, and B. Ferreira. On the value of european options on a stock paying a discrete dividend. *Journal of Modelling in Management*, 4(3):235–248, 2009.

[7] B. Dupire. Pricing with a smile. *Risk Magazine*, pages 18–20, 1994.

[8] S. Heston. A closed-form solution for options with stochastic volatility with applications to bond and currency options. 6:327–43, 02 1993.

[9] H. Patrick et al. Managing smile risk. 1:84–108, 01 2002.

# Appendix A

# Dupire's Formula Derivation

Here is presented a brief demonstration of Dupire's formula, as shown in eq. (2.9).

In his article, Dupire begins by assuming that the stock price $S$ follows a dynamic transition probability density function $p(S(t), t, S'(t'), t')$. In other words, integrating this function would result in the probability of the stock price reaching a price $S'$ at a time $t'$ having started at $S$ at time $t$.

The present value of a call option, $C(S, t, K, T)$, can be deduced as its expected future payoff, discounted backwards in time, which results in

$$\begin{aligned} C(K,T) = e^{-r(T-t)}\mathbb{E}\left[\max\left(S'-K,0\right)\right] &= e^{-r(T-t)}\int_0^\infty \max\left(S'-K,0\right)p(S,t,S',T)dS' \\ &= e^{-r(T-t)}\int_K^\infty (S'-K)p(S,t,S',T)dS'. \end{aligned} \tag{A.1}$$

Deriving this result once with respect to the strike price $K$, we obtain

$$\frac{\partial C}{\partial K} = -e^{-r(T-t)}\int_K^\infty p(S,t,S',T)dS'. \tag{A.2}$$

Deriving again with respect to the same variable results in

$$\frac{\partial^2 C}{\partial K^2} = e^{-r(T-t)}p(S,t,S',T). \tag{A.3}$$

Due to its stochastic nature, the transition probability density function follows the Fokker-Planck equation, given by

$$\frac{\partial p}{\partial T} = \frac{1}{2}\sigma^2\frac{\partial^2(S^2p)}{\partial S^2} - r\frac{\partial(Sp)}{\partial S}. \tag{A.4}$$

with $\sigma$ our, still unknown, function of $S$ and $t$, evaluated at $t = T$.

From eq. A.1 we can easily derive

$$\frac{\partial C}{\partial T} = -rC + e^{-r(T-t)}\int_K^\infty (S'-K)\frac{\partial p}{\partial T}dS'. \tag{A.5}$$

Using eq. A.4, we can transform this relation into

$$\frac{\partial C}{\partial T} = -rC + e^{-r(T-t)} \int_K^\infty (S' - K) \left( \frac{1}{2}\sigma^2 \frac{\partial^2 (S'^2 p)}{\partial S'^2} - r \frac{\partial (S' p)}{\partial S'} \right) dS'. \qquad \text{(A.6)}$$

Integrating twice by parts and collecting all terms, we get

$$\frac{\partial C}{\partial T} = \frac{1}{2}\sigma^2 K^2 \frac{\partial^2 C}{\partial K^2} - rK \frac{\partial C}{\partial K}. \qquad \text{(A.7)}$$

Rearranging all terms, we are left with the Dupire's formula

$$\sigma = \sqrt{\frac{\frac{\partial C}{\partial T} + rK \frac{\partial C}{\partial K}}{\frac{1}{2} K^2 \frac{\partial^2 C}{\partial K^2}}}. \qquad \text{(A.8)}$$

## A.1 Topic Overview

Provide an overview of the topic to be studied...

## A.2 Objectives

Explicitly state the objectives set to be achieved with this thesis...

## A.3 Thesis Outline

Briefly explain the contents of the different chapters...

## A.4 Theoretical Overview

Some overview of the underlying theory about the topic...

## A.5 Theoretical Model 1

Multiple citations are compressed when using the `sort&compress` option when loading the `natbib` package as `\usepackage[numbers,sort&compress]{natbib}` in file `Thesis_Preamble.tex`, resulting in citations like [3, 5].

## A.6 Theoretical Model 2

Other models...

Insert your chapter material here...

## A.7 Numerical Model

Description of the numerical implementation of the models explained in Chapter 2...

## A.8 Verification and Validation

Basic test cases to compare the implemented model against other numerical tools (verification) and experimental data (validation)...

Insert your chapter material here...

## A.9 Problem Description

Description of the baseline problem...

## A.10 Baseline Solution

Analysis of the baseline solution...

## A.11 Enhanced Solution

Quest for the optimal solution...

### A.11.1 Figures

Insert your section material and possibly a few figures...

Make sure all figures presented are referenced in the text!

#### Images

Make reference to Figures **??** and **??**.

By default, the supported file types are *.png,.pdf,.jpg,.mps,.jpeg,.PNG,.PDF,.JPG,.JPEG*.

See `http://mactex-wiki.tug.org/wiki/index.php/Graphics_inclusion` for adding support to other extensions.

#### Drawings

Insert your subsection material and for instance a few drawings...

The schematic illustrated in Fig. **??** can represent some sort of algorithm.

### A.11.2 Equations

Equations can be inserted in different ways.

The simplest way is in a separate line like this

$$\frac{\mathrm{d}q_{ijk}}{\mathrm{d}t} + \mathcal{R}_{ijk}(\mathbf{q}) = 0\,. \tag{A.9}$$

If the equation is to be embedded in the text. One can do it like this $\partial\mathcal{R}/\partial\mathbf{q} = 0$.

It may also be split in different lines like this

### A.11.3 Tables

Insert your subsection material and for instance a few tables...

Make sure all tables presented are referenced in the text!

Follow some guidelines when making tables:

### A.11.4 Mixing

If necessary, a figure and a table can be put side-by-side as in Fig.**??**

Insert your chapter material here...

## A.12 Achievements

The major achievements of the present work...

## A.13 Future Work

A few ideas for future work...

In case an appendix if deemed necessary, the document cannot exceed a total of 100 pages...

Some definitions and vector identities are listed in the section below.

## A.14 Vector identities

$$\nabla \times (\nabla \phi) = 0 \tag{A.10}$$

$$\nabla \cdot (\nabla \times \mathbf{u}) = 0 \tag{A.11}$$