Sensitivity Analysis on Least-Squares American Options Pricing

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Thesis to obtain the Master of Science Degree in

Engineering Physics

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Examination Committee

Chairperson: Prof. Full Name
Supervisor: Prof. Full Name 1 (or 2)
Member of the Committee: Prof. Full Name 3

Month Year

To my parents

Acknowledgments

A few words about the university, financial support, research advisor, dissertation readers, faculty or other professors, lab mates, other friends and family...

Resumo

Inserir o resumo em Português aqui com o máximo de 250 palavras e acompanhado de 4 a 6 palavras-chave...

Palavras-chave: palavra-chave1, palavra-chave2,...

Abstract

Insert your abstract here with a maximum of 250 words, followed by 4 to 6 keywords...

Keywords: keyword1, keyword2,...

Contents

List of Tables

List of Figures

Nomenclature

Greek symbols

α Angle of attack.

β Angle of side-slip.

κ Thermal conductivity coefficient.

μ Molecular viscosity coefficient.

ρ Density.

Roman symbols

C_D Coefficient of drag.

C_L Coefficient of lift.

C_M Coefficient of moment.

p Pressure.

$\mathbf{u}$ Velocity vector.

u, v, w Velocity Cartesian components.

Subscripts

∞ Free-stream condition.

i, j, k Computational indexes.

n Normal component.

x, y, z Cartesian components.

ref Reference condition.

Superscripts

* Adjoint.

T Transpose.

Glossary

CFD Computational Fluid Dynamics is a branch of fluid mechanics that uses numerical methods and algorithms to solve problems that involve fluid flows.

CSM Computational Structural Mechanics is a branch of structure mechanics that uses numerical methods and algorithms to perform the analysis of structures and its components.

MDO Multi-Disciplinar Optimization is an engineering technique that uses optimization methods to solve design problems incorporating two or more disciplines.

Chapter 1

Introduction

1.1 Mathematical Finance

Mathematical finance, also known as quantitative finance, is a field of applied mathematics focused on the modeling of financial instruments. It is rather difficult to overestimate its importance since it is heavily used by investors and investment banks in everyday transactions. In recent decades, this field suffered a complete paradigm shift, following developments in computer science and new theoretical results that enabled investors to better price their assets. With the colossal sums traded daily in financial markets around the world, mathematical finance has become increasingly important and many resources are invested in the research and development of new and better theories and algorithms.

1.2 Derivatives

One of the subjects most studied by financial mathematicians is derivatives. In finance, a derivative is simply a contract whose value depends on other simpler financial instruments, known as underlying assets, such as stock prices or interest rates. Derivatives can virtually take any form desirable, so long as there are two parties interested in taking a part in it and no government regulations are broken.

The importance of derivatives has grown greatly in recent years. In fact, as of June 2017, derivatives were responsible for over $\$542$ trillion worth of trades, in the Over-the-Counter (OTC) market alone [1], as can be seen in Figure 1.1 (the OTC market refers to all deals signed outside of exchanges). This growth stalled after the 2008 global financial crisis due to new government regulations since derivatives were one of the reasons the crisis occurred at all [2].

1.3 Options

Of all kinds of derivatives, in this master thesis we will focus particularly on the most traded one: options [3]. As the name implies, an option contract grants its buyer the option to buy or sell its underlying asset within a given time frame, known as the expiration date.

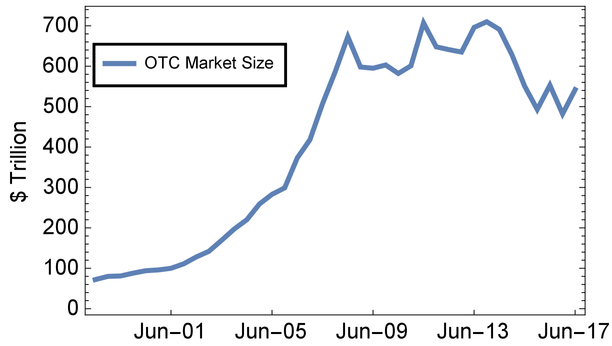

Figure 1.1: Size of OTC derivatives market since May 1996.

1.3.1 Call and Put Options

The two main types of options are calls and puts. They grant their buyer the right to buy and sell the underlying asset, respectively, for a fixed price, known as the strike price, in the future. It's important to emphasize the fact that an option grants its buyer the right to do something. If this action would lead to losses, a buyer can simply decide to let the expiration date pass, thus avoiding further losses. This is indeed the most attractive characteristic of options.

1.3.2 European and American Options

Options are also categorized by the period at which the buyer is allowed to exercise his right. The two most traded types are European and American options. With an European option, an investor is only able to exercise the contract at the expiration date. The value of the underlying asset up to that point in time is irrelevant. As for American options, the buyer can exercise the contract at any moment up to the expiration date. In this case, the investor is faced with an optimal-stopping problem: when is the best time to exercise the option? For this very reason, it should be clear that American options are significantly more difficult to study than their European counterparts.

1.3.3 Why Options are Important

Options are very useful tools to all investors. To hedgers (i.e. investors that want to limit their exposure to risk), options provide safety by fixing the future price of their underlying asset (e.g. if hedgers want to protect themselves against a potential price crash affecting one of their assets, they can buy put options on them. Now, even if the value their asset decreases significantly, their losses are contained because they can always exercise the options and sell the asset at the option's higher strike price).

To speculators (i.e. investors that want to take advantage of the uncertainty of future markets by speculating on their outcome), options grant access to much higher profits (e.g. if speculators strongly believe that the value of a given asset will greatly increase in the future, they can buy call options on that asset. If their prediction proves right, they can buy that asset for the option's lower strike price).

Due to all their advantages, and unlike some other types of derivatives, options have a cost. Finding the ideal price for an option is a fundamental concern to investors, since knowing its true value can give them a chance to take advantage of an under or overpriced option. Finding this price can be very difficult for some options, however. Though a lot of research has been done towards this goal, a great deal more is still required.

1.4 Topic Overview

Provide an overview of the topic to be studied...

1.5 Objectives

Explicitly state the objectives set to be achieved with this thesis...

1.6 Thesis Outline

Briefly explain the contents of the different chapters...

Chapter 2

Background

The payoff function of these two types of derivatives can then simply be deduced as

$$\begin{aligned}\text{Payoff}_{\text{call}}(t) &= (S(t) - K)^+;\\ \text{Payoff}_{\text{put}}(t) &= (K - S(t))^+,\end{aligned} \tag{2.1}$$

where K is the strike price and $S(t)$ is the asset price at the time of exercise, t. These functions are represented in Figure 2.1.

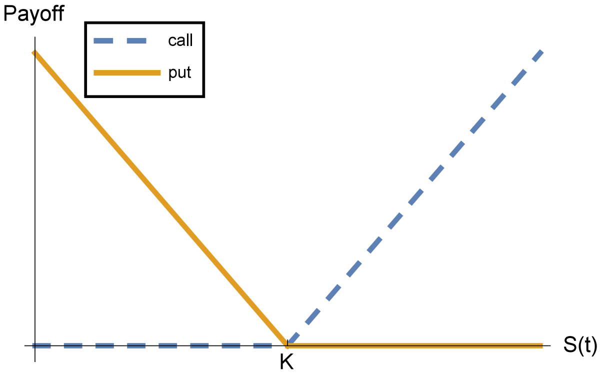

Figure 2.1: Payoff functions of *call* and *put options*.

Due to their high importance, options have been studied in detail in the past. Possibly the most important result in this field came from Fischer Black, Myron Scholes and Robert Merton, who developed a mathematical model to price European options [4] - the famous Black-Scholes-Merton model - still in use in present days [5]. The last two actually earned the 1997 Nobel prize in Economics for this development.

This model states that a European call or put option's price follows the partial differential equation (PDE)

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0, \tag{2.2}$$

where V is the price of the option, S is the price of the underlying stock, r is the risk-free interest rate and σ is the stock price volatility. Simply put, the volatility of a stock price is a measure of how uncertain the price movement is in the future. In other words, a high volatility will lead to great future fluctuations in the stock price. This parameter will be studied in more detail in the next sections.

One important assumption of this model is that stock prices follow a Geometric Brownian Motion (GBM), defined as

$$dS(t) = rS(t)dt + \sigma S(t)dW(t), \tag{2.3}$$

with $\{W(t),\ t > 0\}$ defining a one-dimensional Brownian motion.

Pricing European options is fairly straightforward - we simply need to solve the PDE in eq. (2.2) in a similar fashion to the initial value problem for the diffusion equation [6]. The results published in the original article by Black et al. state that call and put options can be valued as

$$\begin{aligned} C(S(t),t) &= N(d_1)S(t) - N(d_2)Ke^{-r(T-t)}; \\ P(S(t),t) &= -N(-d_1)S(t) + N(-d_2)Ke^{-r(T-t)}, \end{aligned} \tag{2.4}$$

where $N(\cdot)$ is the cumulative distribution function of the standard normal distribution, T is the maturity time, and d_1, d_2 are given by

$$\begin{aligned} d_1 &= \frac{1}{\sigma\sqrt{T-t}}\left[\ln\left(\frac{S_t}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)(T-t)\right]; \\ d_2 &= d_1 - \sigma\sqrt{T-t}. \end{aligned} \tag{2.5}$$

Because the stock price depends on a Brownian motion process, it follows that it is not differentiable. For this reason, it's impossible to exactly simulate such a process. An approximation is possible, however, using discrete jumps of length Δt and using the Brownian motion property $W(t) \sim \sqrt{t}N(0,1)$ [7], with $N(0,1)$ being a normal distribution with 0 expected value and 1 variance. We can then simply discretize eq. (2.3) into

$$S(t+\Delta t) = S(t) + rS(t)\Delta t + \sqrt{\Delta t}\sigma S(t)N(0,1), \tag{2.6}$$

where Δt corresponds to a given time step. An example of this discretization is illustrated in Figure 2.2 with the realization of three sample paths.

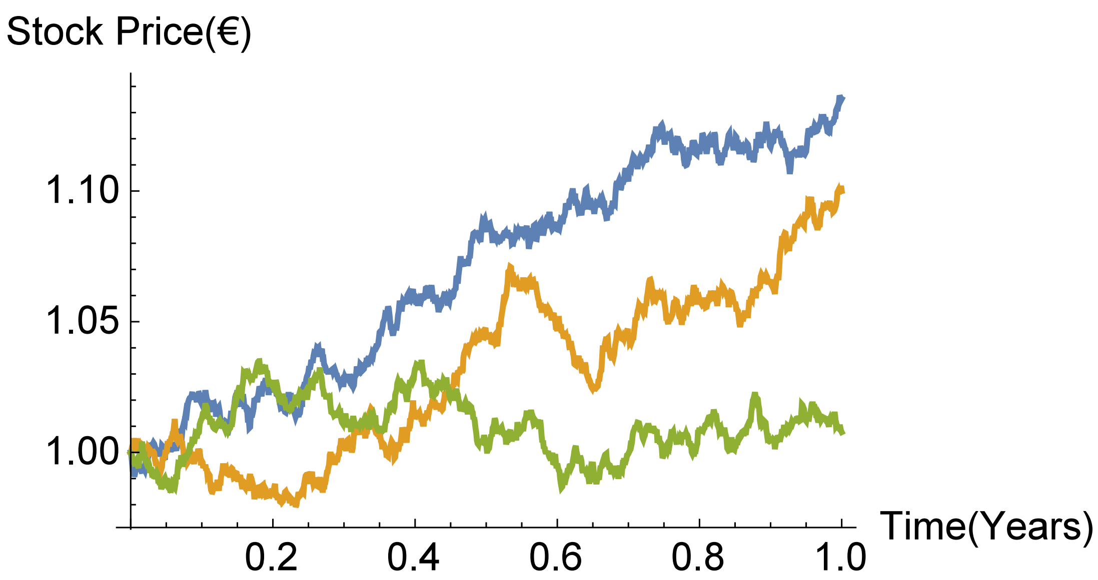

Figure 2.2: Example of three GBM processes, using the parameters $r = 0.06\,\text{yr}^{-1}$, $\sigma = 0.05$, $S(0) = €1$ and time steps $\Delta t = 10^{-3}$yr.

By simulating a large number of paths, some underlying tendencies might become apparent, which will prove useful in option pricing.

American options, however, pose a much greater challenge. Unlike European options, no analytic pricing model currently exists for this type of derivatives. Several numerical models have been proposed in the past in an attempt to solve this problem [3, 8], such as the Longstaff-Schwartz algorithm [9], which we shall approach in later sections of the present thesis.

!!!!!Put-call parity!!!! !!!!!American Puts=European Puts!!!!

Insert your chapter material here...

2.1 Theoretical Overview

2.2 Put-Call parity

Some overview of the underlying theory about the topic...

2.3 Theoretical Model 1

The research should be supported with a comprehensive list of references. These should appear whenever necessary, in the limit, from the first to the last chapter.

A reference can be cited in any of the following ways:

- Citation mode #1 - [10]
- Citation mode #2 - Jameson et al. [10]

- Citation mode #3 - [10]
- Citation mode #4 - Jameson, Pierce, and Martinelli [10]
- Citation mode #5 - [10]
- Citation mode #6 - Jameson et al. 10
- Citation mode #7 - 10
- Citation mode #8 - Jameson et al.
- Citation mode #9 - 1998
- Citation mode #10 - [1998]

Several citations can be made simultaneously as [11, 12].

This is often the default bibliography style adopted (numbers following the citation order), according to the options:
`\usepackage{natbib}` in file `Thesis_Preamble.tex`,
`\bibliographystyle{abbrvnat}` in file `Thesis.tex`.

Notice however that this style can be changed from numerical citation order to authors' last name with the options:
`\usepackage[numbers]{natbib}` in file `Thesis_Preamble.tex`,
`\bibliographystyle{abbrvunsrtnat}` in file `Thesis.tex`.

Multiple citations are compressed when using the `sort&compress` option when loading the `natbib` package as `\usepackage[numbers,sort&compress]{natbib}` in file `Thesis_Preamble.tex`, resulting in citations like [13–16].

2.4 Theoretical Model 2

Other models...

Chapter 3

Implementation

Insert your chapter material here...

3.1 Numerical Model

Description of the numerical implementation of the models explained in Chapter 2...

3.2 Verification and Validation

Basic test cases to compare the implemented model against other numerical tools (verification) and experimental data (validation)...

Chapter 4

Results

Insert your chapter material here...

4.1 Problem Description

Description of the baseline problem...

4.2 Baseline Solution

Analysis of the baseline solution...

4.3 Enhanced Solution

Quest for the optimal solution...

4.3.1 Figures

Insert your section material and possibly a few figures...

Make sure all figures presented are referenced in the text!

Images

Figure 4.1: Caption for figure.

(a) Airbus A320 (b) Bombardier CRJ200

Figure 4.2: Some aircrafts.

Make reference to Figures 4.1 and 4.2.

By default, the supported file types are *.png,.pdf,.jpg,.mps,.jpeg,.PNG,.PDF,.JPG,.JPEG*.

See `http://mactex-wiki.tug.org/wiki/index.php/Graphics_inclusion` for adding support to other extensions.

Drawings

Insert your subsection material and for instance a few drawings...

The schematic illustrated in Fig. 4.3 can represent some sort of algorithm.

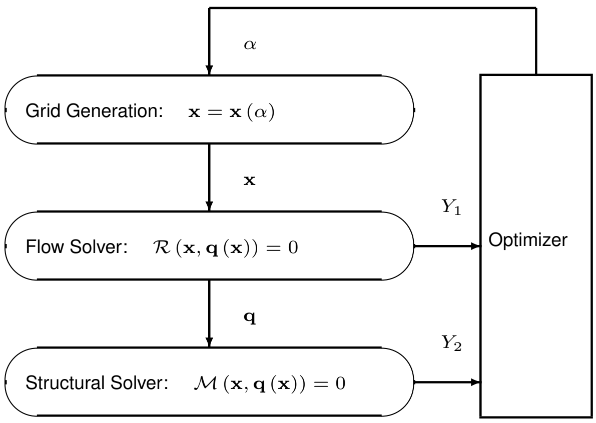

Figure 4.3: Schematic of some algorithm.

4.3.2 Equations

Equations can be inserted in different ways.

The simplest way is in a separate line like this

$$\frac{\mathrm{d}q_{ijk}}{\mathrm{d}t} + \mathcal{R}_{ijk}(\mathbf{q}) = 0\,. \tag{4.1}$$

If the equation is to be embedded in the text. One can do it like this $\partial\mathcal{R}/\partial\mathbf{q} = 0$.

It may also be split in different lines like this

$$
\begin{aligned}
\text{Minimize} \quad & Y(\alpha, \mathbf{q}(\alpha)) \\
\text{w.r.t.} \quad & \alpha\,, \\
\text{subject to} \quad & \mathcal{R}(\alpha, \mathbf{q}(\alpha)) = 0 \\
& C(\alpha, \mathbf{q}(\alpha)) = 0\,.
\end{aligned}
\tag{4.2}
$$

It is also possible to use subequations. Equations 4.3a, 4.3b and 4.3c form the Naver–Stokes equations 4.3.

$$\frac{\partial\rho}{\partial t} + \frac{\partial}{\partial x_j}\left(\rho u_j\right) = 0\,, \tag{4.3a}$$

$$\frac{\partial}{\partial t}\left(\rho u_i\right) + \frac{\partial}{\partial x_j}\left(\rho u_i u_j + p\delta_{ij} - \tau_{ji}\right) = 0, \quad i = 1, 2, 3\,, \tag{4.3b}$$

$$\frac{\partial}{\partial t}\left(\rho E\right) + \frac{\partial}{\partial x_j}\left(\rho E u_j + p u_j - u_i\tau_{ij} + q_j\right) = 0\,. \tag{4.3c}$$

4.3.3 Tables

Insert your subsection material and for instance a few tables...

Make sure all tables presented are referenced in the text!

Follow some guidelines when making tables:

- Avoid vertical lines
- Avoid “boxing up” cells, usually 3 horizontal lines are enough: above, below, and after heading
- Avoid double horizontal lines
- Add enough space between rows

Model	C_L	C_D	C_{My}
Euler	0.083	0.021	-0.110
Navier–Stokes	0.078	0.023	-0.101

Table 4.1: Table caption.

Make reference to Table 4.1.

Tables 4.2 and 4.3 are examples of tables with merging columns:

An example with merging rows can be seen in Tab.4.4.

If the table has too many columns, it can be scaled to fit the text widht, as in Tab.4.5.

	Virtual memory [MB]	
	Euler	Navier–Stokes
Wing only	1,000	2,000
Aircraft	5,000	10,000
(ratio)	$5.0\times$	$5.0\times$

Table 4.2: Memory usage comparison (in MB).

	$w = 2$			$w = 4$		
	$t = 0$	$t = 1$	$t = 2$	$t = 0$	$t = 1$	$t = 2$
$dir = 1$						
c	0.07	0.16	0.29	0.36	0.71	3.18
c	-0.86	50.04	5.93	-9.07	29.09	46.21
c	14.27	-50.96	-14.27	12.22	-63.54	-381.09
$dir = 0$						
c	0.03	1.24	0.21	0.35	-0.27	2.14
c	-17.90	-37.11	8.85	-30.73	-9.59	-3.00
c	105.55	23.11	-94.73	100.24	41.27	-25.73

Table 4.3: Another table caption.

ABC	header			
	1.1	2.2	3.3	4.4
IJK	group		0.5	0.6
			0.7	1.2

Table 4.4: Yet another table caption.

Variable	a	b	c	d	e	f	g	h	i	j
Test 1	10,000	20,000	30,000	40,000	50,000	60,000	70,000	80,000	90,000	100,000
Test 2	20,000	40,000	60,000	80,000	100,000	120,000	140,000	160,000	180,000	200,000

Table 4.5: Very wide table.

4.3.4 Mixing

If necessary, a figure and a table can be put side-by-side as in Fig.4.4

Legend		
A	B	C
0	0	0
0	1	0
1	0	0
1	1	1

Figure 4.4: Figure and table side-by-side.

Chapter 5

Conclusions

Insert your chapter material here...

5.1 Achievements

The major achievements of the present work...

5.2 Future Work

A few ideas for future work...

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Appendix A

Vector calculus

In case an appendix if deemed necessary, the document cannot exceed a total of 100 pages...

Some definitions and vector identities are listed in the section below.

A.1 Vector identities

$$\nabla \times (\nabla \phi) = 0 \tag{A.1}$$

$$\nabla \cdot (\nabla \times \mathbf{u}) = 0 \tag{A.2}$$

Appendix B

Technical Datasheets

It is possible to add PDF files to the document, such as technical sheets of some equipment used in the work.

B.1 Some Datasheet

SUNPOWER

C60 SOLAR CELL

MONO CRYSTALLINE SILICON

SUNPOWER

C60 SOLAR CELL

MONO CRYSTALLINE SILICON

BENEFITS

Maximum Light Capture

SunPower's all-back contact cell design moves gridlines to the back of the cell, leaving the entire front surface exposed to sunlight, enabling up to 10% more sunlight capture than conventional cells.

Superior Temperature Performance

Due to lower temperature coefficients and lower normal cell operating temperatures, our cells generate more energy at higher temperatures compared to standard c-Si solar cells.

No Light-Induced Degradation

SunPower n-type solar cells don't lose 3% of their initial power once exposed to sunlight as they are not subject to light-induced degradation like conventional p-type c-Si cells.

Broad Spectral Response

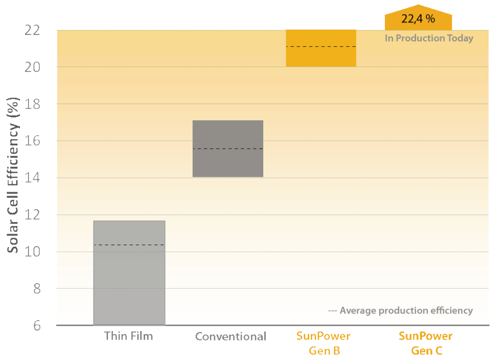
SunPower cells capture more light from the blue and infrared parts of the spectrum, enabling higher performance in overcast and low-light conditions.

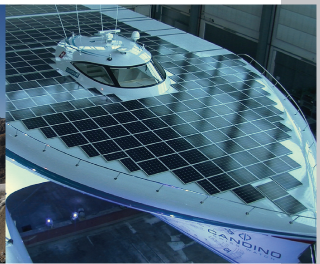
Broad Range Of Application

SunPower cells provide reliable performance in a broad range of applications for years to come.

The SunPower™ C60 solar cell with proprietary Maxeon™ cell technology delivers today's highest efficiency and performance. The anti-reflective coating and the reduced voltage-temperature coefficients provide outstanding energy delivery per peak power watt. Our innovative all-back contact design moves gridlines to the back of the cell, which not only generates more power, but also presents a more attractive cell design compared to conventional cells.

SunPower's High Efficiency Advantage





C60 SOLAR CELL

Electrical Characteristics of Typical Cell at Standard Test Conditions (STC)

STC: 1000W/m², AM 1.5g and cell temp 25°C

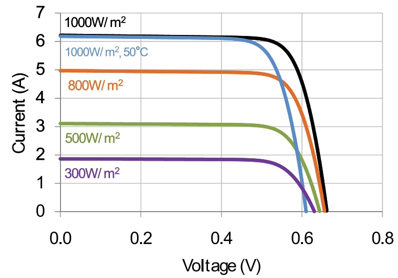
Bin	Pmpp (Wp)	Eff. (%)	Vmpp (V)	Impp (A)	Voc (V)	Isc (A)
G	3.34	21.8	0.574	5.83	0.682	6.24
H	3.38	22.1	0.577	5.87	0.684	6.26
I	3.40	22.3	0.581	5.90	0.686	6.27
J	3.42	22.5	0.582	5.93	0.687	6.28

All Electrical Characteristics parameters are nominal
Unlaminated Cell Temperature Coefficients
Voltage: -1.8 mV / °C   Power: -0.32% / °C

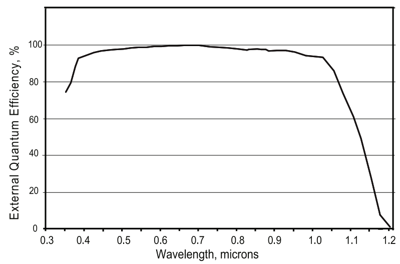
Positive Electrical Ground

Modules and systems produced using these cells must be configured as "positive ground systems".

TYPICAL I-V CURVE



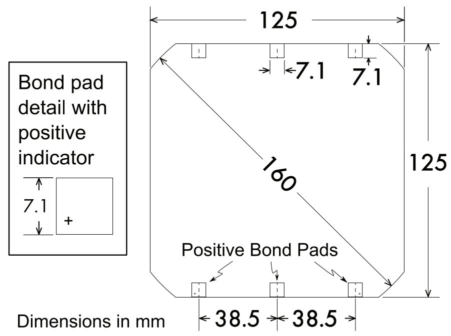
SPECTRAL RESPONSE



Physical Characteristics

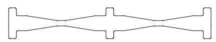
Construction:	All back contact
Dimensions:	125mm x 125mm (nominal)
Thickness:	165μm ± 40μm
Diameter:	160mm (nominal)

Cell and Bond Pad Dimensions



Bond pad area dimensions are 7.1mm x 7.1mm
Positive pole bond pad side has "+" indicator on leftmost and rightmost bond pads.

Interconnect Tab and Process Recommendations



Tin plated copper interconnect. Compatible with lead free process.

Packaging

Cells are packed in boxes of 1,200 each; grouped in shrink-wrapped stacks of 150 with interleaving. Twelve boxes are packed in a water-resistant "Master Carton" containing 14,400 cells suitable for air transport.

Interconnect tabs are packaged in boxes of 1,200 each.

About SunPower

SunPower designs, manufactures, and delivers high-performance solar electric technology worldwide. Our high-efficiency solar cells generate up to 50 percent more power than conventional solar cells. Our high-performance solar panels, roof tiles, and trackers deliver significantly more energy than competing systems.

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