

# **Sensitivity Analysis on Least-Squares American Options Pricing**

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Thesis to obtain the Master of Science Degree in

## **Engineering Physics**

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**Month Year**



To my parents and sister



## **Acknowledgments**

A few words about the university, financial support, research advisor, dissertation readers, faculty or other professors, lab mates, other friends and family...



## Resumo

Inserir o resumo em Português aqui com o máximo de 250 palavras e acompanhado de 4 a 6 palavras-chave...

**Palavras-chave:** palavra-chave1, palavra-chave2,...





## **Abstract**

Insert your abstract here with a maximum of 250 words, followed by 4 to 6 keywords...

**Keywords:** keyword1, keyword2,...



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# Nomenclature

## Greek symbols

$\alpha$	Angle of attack.
$\beta$	Angle of side-slip.
$\kappa$	Thermal conductivity coefficient.
$\mu$	Molecular viscosity coefficient.
$\rho$	Density.

## Roman symbols

$C_D$	Coefficient of drag.
$C_L$	Coefficient of lift.
$C_M$	Coefficient of moment.
$p$	Pressure.
$\mathbf{u}$	Velocity vector.
$u, v, w$	Velocity Cartesian components.

## Subscripts

$\infty$	Free-stream condition.
$i, j, k$	Computational indexes.
$n$	Normal component.
$x, y, z$	Cartesian components.
ref	Reference condition.

## Superscripts

*	Adjoint.
T	Transpose.



# Glossary

- CFD** Computational Fluid Dynamics is a branch of fluid mechanics that uses numerical methods and algorithms to solve problems that involve fluid flows.
- CSM** Computational Structural Mechanics is a branch of structure mechanics that uses numerical methods and algorithms to perform the analysis of structures and its components.
- MDO** Multi-Disciplinary Optimization is an engineering technique that uses optimization methods to solve design problems incorporating two or more disciplines.



# Chapter 1

## Introduction

### 1.1 Mathematical Finance

Mathematical finance, also known as quantitative finance, is a field of applied mathematics focused on the modeling of financial instruments. It is rather difficult to overestimate its importance since it is heavily used by investors and investment banks in everyday transactions. In recent decades, this field suffered a complete paradigm shift, following developments in computer science and new theoretical results that enabled investors to better price their assets. With the colossal sums traded daily in financial markets around the world, mathematical finance has become increasingly important and many resources are invested in the research and development of new and better theories and algorithms.

### 1.2 Derivatives

Derivatives are currently one of the subjects most studied by financial mathematicians. In finance, a *derivative* is simply a contract whose value depends on other simpler financial instruments, known as *underlying assets*, such as stock prices or interest rates. They can virtually take any form desirable, so long as there are two parties interested in signing it and all government regulations are met.

The importance of derivatives has grown greatly in recent years. In fact, as of June 2017, derivatives were responsible for over \$542 trillion worth of trades, in the Over-the-Counter (OTC) market alone [1], as can be seen in Figure 1.1 (the OTC market refers to all deals signed outside of exchanges). This growth stalled after the 2008 global financial crisis due to new government regulations, implemented because of the role of derivatives in the market crash [2].

### 1.3 Options

Of all kinds of derivatives, in this master thesis we will focus particularly on the most traded type [3]: *options*.

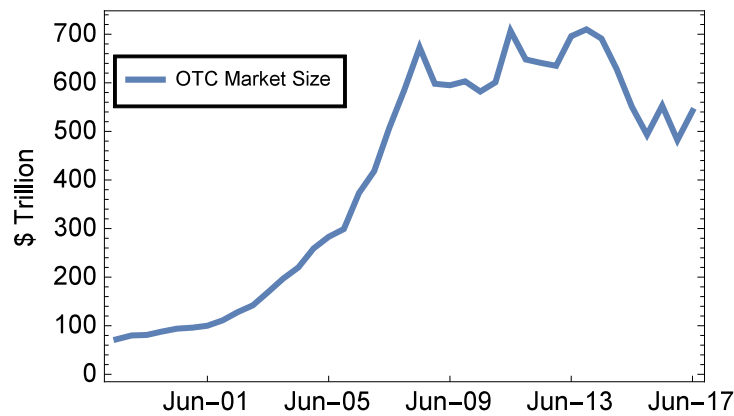


Figure 1.1: Size of OTC derivatives market since May 1996.

As the name implies, an *option* contract grants its buyer the option to buy (in the case of a *call* type option) or sell (for *put* options) its underlying asset at a future date, known as the *maturity*, for a fixed price, known as the *strike price*. In other words, when signing an option, buyers choose a price at which they want to buy/sell (call/put) some underlying asset and a future date to do this transaction. When this date arrives, if the transaction is favorable to the buyers, they exercise their right to execute it.

The description above pertains only to the most traded type of option - *European* options. In this thesis, unless otherwise stated, all options will be assumed European. There are, however, other types that enable exercising at dates other than maturity. The most well known such example is *American* options, contracts that enable their buyers to exercise their right to buy/sell the underlying asset at any point in time until the maturity date. Other types, commonly known as *Exotic*, will be studied in more detail in the following sections.

It's important to emphasize the fact that an option grants its buyer the right to do something. If *exercising* the option would lead to losses, the buyer can simply decide to let the maturity date pass, letting the option expire. This is indeed the most attractive characteristic of options.

### 1.3.1 Why Options are Important

Options are very useful tools to all investors.

To hedgers (i.e. investors that want to limit their exposure to risk), options provide safety by fixing a minimum future price on their underlying assets (e.g. if hedgers want to protect themselves against a potential price crash affecting one of their assets, they can buy put type options on it. With these, even if the value of their asset crashes, their losses will always be contained because they can exercise the options and sell the asset at the option's higher strike price).

Options are also very useful to speculators (i.e. investors that try to predict future market movements). The lower price of options (when compared to their underlying assets) grants this type of investors great leveraging capabilities and, with them, access to much higher profits if their predictions prove true. The opposite is also true and a wrong prediction can equally lead to greater losses.

Due to all their advantages, and unlike some other types of derivatives, options have a price. Finding the ideal price for an option is a fundamental concern to investors, because knowing the appropriate

value of an option can give them a chance to take advantage of under or overpriced options. Finding this price can be very difficult for some options, however. Though a lot of research has been done towards this goal, a great deal more is still required.





## Chapter 2

# Background

### 2.1 Call and Put Options

As stated before, call and put options enable their buyer to respectively buy and sell the underlying stock at the maturity for the fixed strike price. In the case of a call option, if at the maturity the market price of its underlying asset is greater than the strike, investors can exercise the option and buy the asset for the fixed strike price. Then they can immediately go to the market and sell the asset for its higher value. Thus, in this case, the payoff of the option would be the difference between these two prices. On the other hand, if at the maturity the price of the asset decreases past the strike, the investor should let the option expire, since the asset is available in the market for a price lower than the strike. In this case, the payoff of the option would be zero. The same reasoning can be made for put type options. The payoff function of these two types of options can then simply be deduced as

$$\begin{aligned}\text{Payoff}_{\text{call}} &= \max(S(T) - K, 0); \\ \text{Payoff}_{\text{put}} &= \max(K - S(T), 0),\end{aligned}\tag{2.1}$$

where  $K$  is the strike price and  $S(T)$  is the asset price,  $S(t)$ , at the maturity,  $T$ . These functions are represented in Figure 2.1.

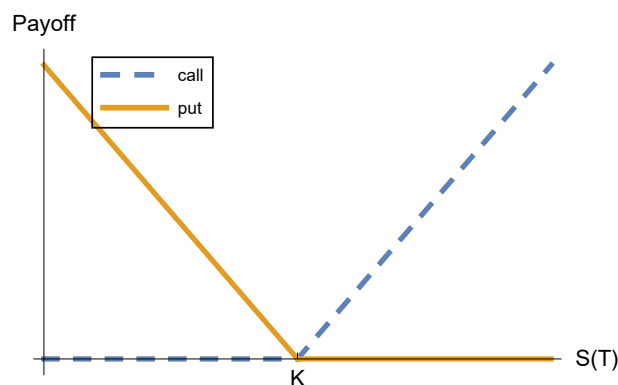


Figure 2.1: Payoff functions of *call* and *put* options.

With eqs.(2.1) in mind, it should be clear that the value of these two types of options corresponds to their expected future payoff discounted back to the present, which gives

$$\begin{aligned} C(K, T) &= e^{-rT} \mathbb{E} [\max (S(T) - K, 0)] = e^{-rT} \mathbb{E} [(S(T) - K) \mathbb{1}_{\{S(T) > K\}}] ; \\ P(K, T) &= e^{-rT} \mathbb{E} [\max (K - S(T), 0)] = e^{-rT} \mathbb{E} [(K - S(T)) \mathbb{1}_{\{S(T) < K\}}] , \end{aligned} \quad (2.2)$$

with  $C(K, T)$  and  $P(K, T)$  being the values of call and put options, respectively.  $\mathbb{E}[\cdot]$  and  $\mathbb{1}_{\{\cdot\}}$  correspond to the expected value and indicator functions, respectively.

## 2.2 Black-Scholes-Merton Formulae

Due to their high importance, options have been studied in great detail in the past. Probably the most important result in this field came from Fischer Black, Myron Scholes and Robert Merton, who developed a mathematical model to price options - the famous Black-Scholes-Merton model [4] - still in use in present days [5].

This model states that the price of an European call or put option, whose underlying asset is a stock, follows the partial differential equation (PDE)

$$\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} - rV = 0, \quad (2.3)$$

where  $V$  is the price of the option,  $S$  is the price of the underlying stock,  $r$  is the risk-free interest rate and  $\sigma$  is the stock price volatility.

**underlying asset=stocks**

The risk-free interest rate,  $r$ , is the interest an investor would receive from any risk-free investment (e.g. treasury bills). An investor should never invest in risky products whose expected return is lower than this interest, since there's the alternative of obtaining a higher payoff without the disadvantage of taking risks. In general, this rate changes slightly with time and is unknown, but Black *et al.*, in their original model (eq.(2.3)), assumed that it remains constant throughout the option's duration and that it is known. Some later developments dealt with this shortcoming, providing models to replicate the behavior of interest rates [6], but because option prices do not significantly depend on this value [5], in the remainder of this thesis we shall assume it is constant and known.

Finally, the volatility,  $\sigma$ , is a measure of uncertainty and will be studied in more detail in section 2.3.

One important assumption of the Black-Scholes-Merton model is that stock prices follow a stochastic process, known as Geometric Brownian Motion (GBM), which can be defined as

$$dS(t) = rS(t)dt + \sigma S(t)dW(t), \quad (2.4)$$

with  $\{W(t), t > 0\}$  defining a one-dimensional Brownian motion.

With this result, pricing options is fairly straightforward - we simply need to solve the PDE in eq.(2.3) in a similar fashion to the initial value problem for the diffusion equation [7]. The results published in the

original article by Black *et al.* state that, at time  $t$ , call and put options can be valued as

$$\begin{aligned} C(S(t), t) &= N(d_1)S(t) - N(d_2)Ke^{-r(T-t)}; \\ P(S(t), t) &= -N(-d_1)S(t) + N(-d_2)Ke^{-r(T-t)}, \end{aligned} \quad (2.5)$$

where  $N(\cdot)$  is the cumulative distribution function of the standard normal distribution and  $d_1, d_2$  are given by

$$\begin{aligned} d_1 &= \frac{1}{\sigma\sqrt{T-t}} \left[ \ln\left(\frac{S_t}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)(T-t) \right]; \\ d_2 &= d_1 - \sigma\sqrt{T-t}. \end{aligned} \quad (2.6)$$

From eq.(2.5) we can derive the relationship between  $C(S, t)$  and  $P(S, t)$ , known as the *put-call parity*

$$C(S(t), t) = S(t) - Ke^{-r(T-t)} + P(S(t), t). \quad (2.7)$$

Because of this duality, we can always obtain the prices of put options from the prices of call options with the same underlying asset, maturity and strike. For this reason, providing results for both option types is redundant and unless otherwise stated, all options will be assumed European calls in the following sections.

## 2.3 Volatility

Volatility is a measure of the uncertainty of future stock price movements. In other words, a high volatility will lead to great future fluctuations in the stock price, whereas a stock with low volatility is more stable.

put image of varying volatilities here

Of all the parameters in the Black-Scholes formula (eq.(2.3)), volatility is the only one we can't easily measure from market data. Furthermore, unlike the interest rate, volatility has a great impact on the behavior of stock prices and consequently on the price of options [5]. These two factors make volatility one of the most important subjects in all of mathematical finance and thus the focus of much research.

It should be noted that there are several types of volatility, depending on what is being measured. Some of these types will be approached in the next subsections.

### 2.3.1 Implied Volatility

*Implied volatility* can be described as the value of stock price volatility that, when input into the Black-Scholes pricer in eq.(2.5), outputs a value equal to the market price of a given option. In other words, it would be the stock price volatility that the seller/buyer of the option assumed when pricing it (given that the Black-Scholes model was used).

Because eq.(2.5) is not invertible, we need to use some numerical method (e.g. Newton's method) to find the value of implied volatility that matches the market and model prices. In other words, we must

find numerically the solution to the equation

$$C(\sigma_{imp}, S(t), t) - \bar{C} = 0, \quad (2.8)$$

where  $C(\sigma_{imp}, \cdot)$  corresponds to the result of eq.(2.5) using the implied volatility  $\sigma_{imp}$  and  $\bar{C}$  is the price of the option observed in the market.

We can obtain the implied volatility of an option from its price and vice versa, because eq.(2.5) is monotonic in the volatility. This duality is so fundamental that investors often disclose their options by providing their implied volatility instead of their price [8].

One interesting property of implied volatility is that, in the real-world, it depends on the strike price and the maturity. In the Black-Scholes world this should not occur. If investors really used the this model to price their options, two options with the same underlying asset should have the same implied volatility, regardless of their strike prices or maturities (i.e. the same underlying asset doesn't have two different volatilities at the same time). However, when observing real market data, this is indeed what is observed.

The implied volatilities' dependence on the strike price is a function that can take one of two forms, known as *smile* and *skew*. An implied volatility smile shows greater values of  $\sigma_{imp}$  for options with strikes farther from the current stock price. The minimum occurs where the strike equals the current stock price. A skew, on the other hand, only shows greater  $\sigma_{imp}$  in one of the directions (i.e. for strikes either bigger or smaller than the current stock price). Both phenomena are represented in Figure 2.2.

Because of their higher implied volatility, we can conclude that options with strikes different from the current stock price are overpriced. The reason behind this odd market behavior is related to the simple demand-supply rule [5]. On the one hand, some investors are risk-averse and want to hedge their losses in case of a market crash (as explained in subsection 1.3.1). They don't mind paying a higher price for an option if this means they would be relatively safe from potentially devastating market crashes. For this reason, the prices of (call) options with lower strikes increase, driving their implied volatility up. On the other hand, other investors are risk-seekers and want to take advantage of possible sudden price increases, buying the stocks for the lower strike prices. They don't mind paying higher prices for the options and this drives the prices of high strike options (and, consequently, their implied volatility) up. This fear-greed duality gives rise to the observed volatility smile.

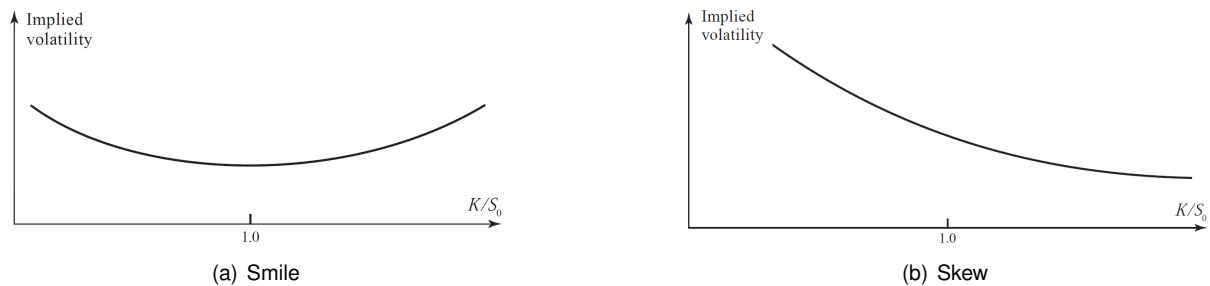


Figure 2.2: Example of an implied volatility smile and skew. [source=Hull](#)

The dependence of the implied volatility on the maturity date is more complex, but in general it

decreases with  $T$ .

It can also be shown that the implied volatility is the same for calls and puts [3].

### 2.3.2 Local Volatility

In their original work, Black *et al.* assumed that volatility is constant throughout the whole contract. From market data, it can be clearly seen that this is not the case. There may be times where new information reaches the market and trading increases, driving volatility up. On the other hand, if investors are waiting for some new information to reach the market (e.g. the results of an election) trading may stall, and volatilities go down.

The model in eq.(2.5) is therefore not enough to completely grasp real-world trading. We should have a model where volatility is dynamic, measuring the uncertainty on the stock price at any point in time. However, as we saw in subsection 2.3.1 for implied volatility, the market's view of volatility also changes with the strike price. A volatility model that merely depends on time is thus insufficient.

The volatility should therefore be a function of both time and strike price:  $\sigma(K, t)$ . We call this model *local volatility* and the geometric Brownian motion from eq. 2.4 is transformed into

$$dS(t) = rS(t)dt + \sigma(K, t)S(t)dW(t), \quad (2.9)$$

where  $\sigma(K, t)$  is some function of  $K$  and  $t$ .

Finding the local volatility function is not very important when pricing typical European options, because we can simply find the implied volatility from similar options in the market and assume a constant volatility throughout the option's duration. However, for other contracts, such as American, Asian or Barrier options (among others) where the option's value depends on the intermediate stock prices, this function is indeed crucial.

Because we can't directly measure the local volatility of a stock from market data, the local volatility function is very difficult to calibrate (i.e. finding the function parameters that best fit market data). Some models have been proposed in an attempt to achieve this goal. The most well known model is known as Dupire's formula.

#### Dupire's Formula

**notation problems...  $T=T'$ ,  $K=K'$**  One of the most famous results in the modelling of the local volatility function was obtained by Dupire [9]. In his article, he derives a theoretical formula for this function, given by

$$\sigma(K, T) = \sqrt{\frac{\frac{\partial C}{\partial T} + rK \frac{\partial C}{\partial K}}{\frac{1}{2}K^2 \frac{\partial^2 C}{\partial K^2}}}, \quad (2.10)$$

where  $\sigma(S, t)$  is the volatility function for stock prices  $S$  at time  $t$  and  $C = C(K, T)$  is the price of an European call option with strike price  $K$  and maturity  $T$ . A brief demonstration of this formula can be found in Appendix A.

As can be seen, we need to differentiate the option prices with respect to their strikes and maturities. To achieve this, we need to gather a large number of option prices from the market, for options with different maturities and strikes, and interpolate between them to obtain an option price surface (with  $K$  and  $T$  as variables). We then calculate the gradients of this interpolated surface and input them into eq. 2.10 to obtain the local volatility surface.

Even before implementation, four potential sources of error can be found.

- First, it should be noted that markets only trade options with very specific maturities (e.g. 1, 2, 4 and 6-months maturity). For this reason, our data will be very sparse with respect to maturity and the interpolation obtained may not closely resemble reality.
- Furthermore, it can be seen that we are dividing by the second derivative with respect to the strike price. **As we will see in the next section**, the prices of options with strikes very far from the current stock price change at an approximately constant rate with respect to the strike. The second derivative in these regions would then be very close to or even equal zero. Because this second derivative is in the denominator of eq.(2.10), our volatilities may explode for very large or very small strikes.
- Then, there is the problem of noise. Because we are interpolating very sparse data, even small fluctuations in the option market price will cause variations in the option price interpolation which can be very problematic if the second derivative changes, because the result is very sensitive to this value.
- Finally, some problems arise from the market itself. While most investors use some theoretical basis in their trades, the market is still governed by the demand-supply rule. If too many investors want to buy the option and few want to sell it, the option price will increase, even if it means that the option will be overpriced, and vice versa. It may also happen that the market is not liquid enough (i.e. very few trades or even no trades at all occur for some options) which causes the option prices to not truly follow the market's perception of future price movements.

Besides, it can be shown that if the local volatility surface truly matched reality, it should remain constant in time, assuming the market's view of future price movements remained constant as well [5]. However, this is not the case, so we can conclude that the model doesn't completely correspond to reality and for that reason it shouldn't be used blindly.

Some authors have also pointed out that the volatility smile obtained from Dupire's local-volatility model doesn't follow true market dynamics [10]. They demonstrate that when the price of the asset increases or decreases, the volatility smile shifts in the opposite direction. The volatility smile minimum would therefore be offset and no longer correspond to the local spot price. The dynamics obtained from the local-volatility model should then be actually worse than if we assumed a constant volatility.

Dupire also developed an alternative local-volatility formula based on the implied volatility surface

rather than the option price's as seen in eq.2.10. The relation obtained is as follows

$$\sigma(K, T) = \sqrt{\frac{\sigma_{imp}^2 + 2(T-t)\sigma_{imp}\frac{\partial\sigma_{imp}}{\partial T} + 2rK(T-t)\sigma_{imp}\frac{\partial\sigma_{imp}}{\partial K}}{\left(1 + Kd_1\sqrt{T-t}\frac{\partial\sigma_{imp}}{\partial K}\right)^2 + K^2(T-t)\sigma_{imp}\left(\frac{\partial^2\sigma_{imp}}{\partial K^2} - d_1\left(\frac{\partial\sigma_{imp}}{\partial K}\right)^2\sqrt{T-t}\right)}, \quad (2.11)$$

where  $d_1$  is given by

$$d_1 = \frac{\log(S(t)/K) + \left(r + \frac{1}{2}\sigma_{imp}^2\right)(T-t)}{\sigma_{imp}\sqrt{T-t}}, \quad (2.12)$$

with  $t$  representing the current time (usually  $t = 0$ ),  $T$  being the time at which the local volatility is being measured, and  $S(t)$  the stock price at time  $t$ . We assume that  $\sigma_{imp} = \sigma_{imp}(K, T)$  is the interpolated surface of the implied volatilities evaluated at time  $T$ , and price  $K$ . This relation might be more useful than eq.(2.10) because the implied volatility surface might be smoother than the option price's. We will compare the results from both formulas in the next chapters.

Despite all of its shortcomings, Dupire's formula is nonetheless very much used by traders to find the local volatility surfaces of assets and price exotic options afterwards. Due to its importance, we shall implement and study this model in more detail.

### 2.3.3 Stochastic Volatility

As stated before, the volatility is not constant, is not observable and is not predictable, despite our attempts to model it. This seems to indicate that volatility is itself a stochastic process. Some research has been done into this hypothesis, and many models have been developed to replicate real-world volatilities.

As before, we assume that the stock price follows a geometric Brownian motion

$$dS = rSdt + \sigma SdW_1 \quad (2.13)$$

but we further hypothesize that the volatility follows

$$d\sigma = p(S, \sigma, t)dt + q(S, \sigma, t)dW_2 \quad (2.14)$$

where  $p(S, \sigma, t)$  and  $q(S, \sigma, t)$  are some functions of the stock price and time. We also assume that  $W_1$  and  $W_2$  are two Brownian motion processes with a correlation of  $\rho$

$$dW_1dW_2 = \rho dt. \quad (2.15)$$

This correlation can be explained by the relationship between prices and volatilities. Usually, when prices decrease, trade goes up and with it, the volatility rises. The inverse is true when prices increase. This seems to indicate a negative correlation between stock prices and volatilities.

Choosing the right  $p(S, \sigma, t)$  and  $q(S, \sigma, t)$  is very important in the evolution of the volatility and all

stochastic volatility models will present some suggestion on the choice these functions. Furthermore, the functions  $p(S, \sigma, t)$  and  $q(S, \sigma, t)$  will have parameters that we have to calibrate using market data. We now present two of the most used models - *Heston* and *SABR*.

## Heston Model

One of the most popular stochastic volatility models is known as *Heston model*. It was developed in 1993 by Steven Heston [11] and it states that our stocks satisfy the relations

$$dS = \mu S dt + \sqrt{\nu} S dW_1, \quad (2.16)$$

$$d\nu = \kappa(\bar{\nu} - \nu)dt + \xi\sqrt{\nu}dW_2, \quad (2.17)$$

with  $\nu$  corresponding to the variance (i.e.  $\nu = \sigma^2$ ) and where  $W_1$  and  $W_2$  have a correlation of  $\rho$ . Note that our stock price is now governed by a drift parameter  $\mu$ , unlike the usual risk-free measure drift of  $r$ , as seen in eq. (2.4). A transformation into the risk-free measure, using Girsanov's theorem, is possible but the original model used the drift parameter  $\mu$ . The parameters  $\kappa$ ,  $\bar{\nu}$  and  $\xi$  are, respectively, the mean-reversion rate (i.e. how fast the volatility converges to its mean value), the long-term variance (i.e. the mean value of variance) and the volatility of volatility (i.e. how erratic is the volatility).

One of the main advantages that make the Heston model so popular is that there is a closed-form solution for the prices of options under this model given by

$$\begin{aligned} C_H(\theta; K, T) &= e^{-rT} \mathbb{E} [ (S(T) - K) \mathbb{1}_{\{S(T) > K\}} ] \\ &= e^{-rT} ( \mathbb{E} [ S(T) \mathbb{1}_{\{S(T) > K\}} ] - K \mathbb{E} [ \mathbb{1}_{\{S(T) > K\}} ] ) \\ &= S(0)P_1 - e^{-rT} K P_2, \end{aligned} \quad (2.18)$$

where  $C_H(\theta; K, T)$  corresponds to the theoretical call price of the Heston model, assuming a set of parameters  $\theta$ , strike  $K$  and maturity  $T$ . The variables  $P_1$  and  $P_2$  are given by

$$P_1 = \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \text{Re} \left( \frac{e^{-iu \log K}}{iu f} \phi(\theta; u - i, T) \right) du, \quad (2.19)$$

$$P_2 = \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \text{Re} \left( \frac{e^{-iu \log K}}{iu} \phi(\theta; u, T) \right) du, \quad (2.20)$$

where  $i$  is the imaginary unit,  $f$  is the forward price ( $f = S(0)e^{rT}$ ) and  $\phi(\theta; u, t)$  is the characteristic function of the logarithm of the stock price process (the characteristic function of a random variable is the Fourier transform of the probability density function of that variable).

Now we just have to find the appropriate characteristic function  $\phi(\theta; u, t)$  in order to evaluate the integrals in eqs.(2.19) and (2.20) and find the option price with eq.(2.18). In the original article, Heston derived this very formula [11] but it presented some discontinuities for large maturities and wasn't easily derivable. This led some other authors, such as Cui *et al.* [12] to slightly modify it, in order to solve these



shortcomings. The resulting characteristic function is given by

$$\phi(\theta; u, t) = \exp \left\{ iu (\log S_0 + rt) - \frac{t\kappa\bar{\nu}\rho iu}{\xi} - \nu_0 A + \frac{2\kappa\bar{\nu}}{\xi^2} D \right\}, \quad (2.21)$$

with  $A$  and  $D$  given by

$$A = \frac{A_1}{A_2}, \quad (2.22)$$

$$D = \log d + \frac{\kappa t}{2} - \log A_2, \quad (2.23)$$

where we have introduced the variables  $A_1$ ,  $A_2$  and  $d$

$$A_1 = (u^2 + iu) \sinh \frac{dt}{2}, \quad (2.24)$$

$$A_2 = d \cosh \frac{dt}{2} + (\kappa - \xi\rho iu) \sinh \frac{dt}{2}, \quad (2.25)$$

$$d = \sqrt{(\kappa - \xi\rho iu)^2 + \xi^2(u^2 + iu)}. \quad (2.26)$$

With this result, the calibration of the Heston model can be easily processed. We simply have to find the model parameters that minimize the difference between the model generated option prices,  $C_H(\theta; K, T)$ , and the market prices. This calibration will be explored in detail in the next sections.

## SABR Model

One other very famous model of stochastic volatility was developed by Hagan *et al.* [10] and is known as *SABR*. It stands for "*stochastic- $\alpha\beta\rho$* " and in this model it is assumed that the option prices and volatilities follow

$$dF = \sigma F^\beta dW_1, \quad (2.27)$$

$$d\sigma = \nu\sigma dW_2, \quad (2.28)$$

with  $F$  being the forward price (related to the spot price,  $S$ , by  $F(t) = S(t)e^{r(T-t)}$ ). We further define  $\alpha = \sigma(0)$  as the starting volatility,  $f = F(0) = S_0e^{rT}$  as the starting forward price. Finally, the two Brownian motion processes  $W_1$  and  $W_2$  have a correlation of  $\rho$ .

In the original article, the authors claim that  $\beta$  can be fitted from historical market data, but usually investors actually choose this value arbitrarily, depending on the type of assets traded. Typical values of  $\beta$  are  $\beta = 1$  (stochastic lognormal model), usually used for foreign exchange options,  $\beta = 0$  (stochastic normal model), typical for interest rate options where forwards  $f$  can be negative and  $\beta = 0.5$  (stochastic CIR model), also common for interest rate options.

One of the main reasons SABR is so popular is due to it presenting quasi-closed-form results that make its calibration much easier. With the corrections done by Oblój [13] on Hagan's original formula, it

can be shown that the implied volatilities of options priced under the SABR model are given by

$$\sigma_{SABR}(K, f, T) \approx \frac{1}{\left[1 + \frac{(1-\beta)^2}{24} \log^2\left(\frac{f}{K}\right) + \frac{(1-\beta)^4}{1920} \log^4\left(\frac{f}{K}\right)\right]} \cdot \left(\frac{\nu \log(f/K)}{x(z)}\right) \cdot \left\{1 + T \left[\frac{(1-\beta)^2}{24} \frac{\alpha^2}{(Kf)^{1-\beta}} + \frac{1}{4} \frac{\rho\beta\nu\alpha}{(Kf)^{(1-\beta)/2}} + \frac{2-3\rho^2}{24} \nu^2\right]\right\}, \quad (2.29)$$

with  $z$  and  $x(z)$  given by

$$z = \frac{\nu(f^{1-\beta} - K^{1-\beta})}{\alpha(1-\beta)}, \quad (2.30)$$

$$x(z) = \log \left\{ \frac{\sqrt{1-2\rho z + z^2} + z - \rho}{1-\rho} \right\}. \quad (2.31)$$

With this result, calibrating the SABR model is fairly straightforward. We simply need to minimize the difference between the implied volatilities obtained from the model using some set of parameters and those obtained from market data.

One of the main setbacks of the SABR model is the fact that this calibration only works for a set of options with the same maturity. The model behaves badly when we try to fit options with different maturities [10]. To solve this problem, Hagan *et al.* suggest a similar model known as *Dynamic SABR*. It is the same as the one in eqs.(2.27) and (2.28) but with time-dependent parameters  $\rho(t)$  and  $\nu(t)$ . This model can then be deduced as

$$dF = \sigma F^\beta dW_1, \quad (2.32)$$

$$d\sigma = \nu(t)\sigma dW_2, \quad (2.33)$$

with the correlation between  $W_1$  and  $W_2$  given now by  $\rho(t)$ .

To calibrate this new model, Hagan *et al.* derived again a quasi-closed-form solution for the implied volatility. Osajima later simplified this expression with asymptotic expansions. The resulting formula is given by

$$\sigma_{DynSABR}(K, f, T) = \frac{1}{\omega} \left(1 + A_1(T) \log\left(\frac{K}{f}\right) + A_2(T) \log^2\left(\frac{K}{f}\right) + B(T)T\right), \quad (2.34)$$

where  $\omega = f^{1-\beta}/\alpha$  and  $A_1(T)$ ,  $A_2(T)$  and  $B(T)$  are given by

$$A_1(T) = \frac{\beta-1}{2} + \frac{\eta_1(T)\omega}{2}, \quad (2.35)$$

$$A_2(T) = \frac{(1-\beta)^2}{12} + \frac{1-\beta-\eta_1(T)\omega}{4} + \frac{4\nu_1^2(T) + 3(\eta_2^2(T) - 3\eta_1^2(T))}{24} \omega^2, \quad (2.36)$$

$$B(T) = \frac{1}{\omega^2} \left( \frac{(1-\beta)^2}{24} + \frac{\omega\beta\eta_1(T)}{4} + \frac{2\nu_2^2(T) - 3\eta_2^2(T)}{24} \omega^2 \right), \quad (2.37)$$

with  $\nu_1^2(T)$ ,  $\nu_2^2(T)$ ,  $\eta_1(T)$  and  $\eta_2^2(T)$  as

$$\nu_1^2(T) = \frac{3}{T^3} \int_0^T (T-t)^2 \nu^2(t) dt, \quad (2.38)$$

$$\nu_2^2(T) = \frac{6}{T^3} \int_0^T (T-t)t\nu^2(t)dt, \quad (2.39)$$

$$\eta_1(T) = \frac{2}{T^2} \int_0^T (T-t)\nu(t)\rho(t)dt, \quad (2.40)$$

$$\eta_2^2(T) = \frac{12}{T^4} \int_0^T \int_0^t \left( \int_0^s \nu(u)\rho(u)du \right)^2 dsdt, \quad (2.41)$$

where  $\rho(t)$  and  $\nu(t)$  are the functions chosen to model the time dependent parameters. The choice of these functions will be determinant in the overall behavior of the model. It might happen that, for some chosen functions, one or more of the integrals in eqs.(2.38)-(2.41) is not analytically solvable. In these cases, the integral would have to be evaluated numerically.

One classical choice for the functions corresponds to [14]

$$\rho(t) = \rho_0 e^{-at}, \quad (2.42)$$

$$\nu(t) = \nu_0 e^{-bt}, \quad (2.43)$$

with  $\rho_0 \in [-1, 1]$ ,  $\nu_0 > 0$ ,  $a > 0$  and  $b > 0$ . In this particular case,  $\nu_1^2(T)$ ,  $\nu_2^2(T)$ ,  $\eta_1(T)$  and  $\eta_2^2(T)$  can be exactly derived as

$$\nu_1^2(T) = \frac{6\nu_0^2}{(2bT)^3} \left[ \left( \frac{(2bT)^2}{2} - 2bT + 1 \right) - e^{-2bT} \right], \quad (2.44)$$

$$\nu_2^2(T) = \frac{12\nu_0^2}{(2bT)^3} [e^{-2bT}(1 + bT) + bT - 1], \quad (2.45)$$

$$\eta_1(T) = \frac{2\nu_0\rho_0}{T^2(a+b)^2} [(a+b)T + e^{-(a+b)T} - 1], \quad (2.46)$$

$$\eta_2^2(T) = \frac{3\nu_0^2\rho_0^2}{T^4(a+b)^4} [e^{-2(a+b)T} - 8e^{-(a+b)T} + (7 + 2T(a+b)(-3 + (a+b)T))]. \quad (2.47)$$

Because we have more parameters to fit, the calibration of the Dynamic SABR will be slower than the original SABR. In the next sections we shall compare both models.



## Chapter 3

# Implementation

To calibrate SABR, we can either follow the Monte-Carlo approach and simulate a large number of paths, assuming stochastic volatility with some starting parameters and minimizing the difference between model-generated prices and real-world option prices, or we can use eq. xxx to obtain the implied volatility and minimize the market implied volatility.



## **Chapter 4**

# **Results**





## **Chapter 5**

## **Conclusions**



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## Appendix A

# Dupire's Formula Derivation

Here is presented a brief demonstration of Dupire's formula, as shown in eq. (2.10).

In his article, Dupire begins by assuming that the stock price  $S$  follows a dynamic transition probability density function  $p(S(t), t, S'(t'), t')$ . In other words, integrating this function would result in the probability of the stock price reaching a price  $S'$  at a time  $t'$  having started at  $S$  at time  $t$ .

The present value of a call option,  $C(S, t, K, T)$ , can be deduced as its expected future payoff, discounted backwards in time, which results in

$$\begin{aligned} C(K, T) &= e^{-r(T-t)} \mathbb{E} [\max(S' - K, 0)] = e^{-r(T-t)} \int_0^\infty \max(S' - K, 0) p(S, t, S', T) dS' \\ &= e^{-r(T-t)} \int_K^\infty (S' - K) p(S, t, S', T) dS'. \end{aligned} \quad (\text{A.1})$$

Deriving this result once with respect to the strike price  $K$ , we obtain

$$\frac{\partial C}{\partial K} = -e^{-r(T-t)} \int_K^\infty p(S, t, S', T) dS'. \quad (\text{A.2})$$

Deriving again with respect to the same variable results in

$$\frac{\partial^2 C}{\partial K^2} = e^{-r(T-t)} p(S, t, S', T). \quad (\text{A.3})$$

Due to its stochastic nature, the transition probability density function follows the Fokker-Planck equation, given by

$$\frac{\partial p}{\partial T} = \frac{1}{2} \sigma^2 \frac{\partial^2 (S^2 p)}{\partial S^2} - r \frac{\partial (S p)}{\partial S}. \quad (\text{A.4})$$

with  $\sigma$  our, still unknown, function of  $S$  and  $t$ , evaluated at  $t = T$ .

From eq. A.1 we can easily derive

$$\frac{\partial C}{\partial T} = -rC + e^{-r(T-t)} \int_K^\infty (S' - K) \frac{\partial p}{\partial T} dS'. \quad (\text{A.5})$$

Using eq. A.4, we can transform this relation into

$$\frac{\partial C}{\partial T} = -rC + e^{-r(T-t)} \int_K^\infty (S' - K) \left( \frac{1}{2} \sigma^2 \frac{\partial^2 (S'^2 p)}{\partial S'^2} - r \frac{\partial (S' p)}{\partial S'} \right) dS'. \quad (\text{A.6})$$

Integrating twice by parts and collecting all terms, we get

$$\frac{\partial C}{\partial T} = \frac{1}{2} \sigma^2 K^2 \frac{\partial^2 C}{\partial K^2} - rK \frac{\partial C}{\partial K}. \quad (\text{A.7})$$

Rearranging all terms, we are left with the Dupire's formula

$$\sigma = \sqrt{\frac{\frac{\partial C}{\partial T} + rK \frac{\partial C}{\partial K}}{\frac{1}{2} K^2 \frac{\partial^2 C}{\partial K^2}}}. \quad (\text{A.8})$$

## **A.1 Topic Overview**

Provide an overview of the topic to be studied...

## **A.2 Objectives**

Explicitly state the objectives set to be achieved with this thesis...

## **A.3 Thesis Outline**

Briefly explain the contents of the different chapters...

## **A.4 Theoretical Overview**

Some overview of the underlying theory about the topic...

## **A.5 Theoretical Model 1**

Multiple citations are compressed when using the `sort&compress` option when loading the `natbib` package as `\usepackage[numbers,sort&compress]{natbib}` in file `Thesis_Preamble.tex`, resulting in citations like [3, 8].

## **A.6 Theoretical Model 2**

Other models...

Insert your chapter material here...

## **A.7 Numerical Model**

Description of the numerical implementation of the models explained in Chapter 2...

## **A.8 Verification and Validation**

Basic test cases to compare the implemented model against other numerical tools (verification) and experimental data (validation)...

Insert your chapter material here...

## A.9 Problem Description

Description of the baseline problem...

## A.10 Baseline Solution

Analysis of the baseline solution...

## A.11 Enhanced Solution

Quest for the optimal solution...

### A.11.1 Figures

Insert your section material and possibly a few figures...

Make sure all figures presented are referenced in the text!

#### Images

Make reference to Figures.

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#### Drawings

Insert your subsection material and for instance a few drawings...

The schematic illustrated in Fig.can represent some sort of algorithm.

### A.11.2 Equations

Equations can be inserted in different ways.

The simplest way is in a separate line like this

$$\frac{dq_{ijk}}{dt} + \mathcal{R}_{ijk}(\mathbf{q}) = 0 . \quad (\text{A.9})$$

If the equation is to be embedded in the text. One can do it like this  $\partial\mathcal{R}/\partial\mathbf{q} = 0$ .

It may also be split in different lines like this



### **A.11.3 Tables**

Insert your subsection material and for instance a few tables...

Make sure all tables presented are referenced in the text!

Follow some guidelines when making tables:

### **A.11.4 Mixing**

If necessary, a figure and a table can be put side-by-side as in Fig.

Insert your chapter material here...

## **A.12 Achievements**

The major achievements of the present work...

## **A.13 Future Work**

A few ideas for future work...

In case an appendix is deemed necessary, the document cannot exceed a total of 100 pages...

Some definitions and vector identities are listed in the section below.

## **A.14 Vector identities**

$$\nabla \times (\nabla \phi) = 0 \tag{A.10}$$

$$\nabla \cdot (\nabla \times \mathbf{u}) = 0 \tag{A.11}$$