1. The function $f_1(x,\delta) = \cos(x+\delta) - \cos(x)$ can be transformed into another form, $f_2(x,\delta)$, using the trigonometric formula

$$\cos(\phi) - \cos(\psi) = -2\sin\left(\frac{\phi+\psi}{2}\right)\sin\left(\frac{\phi-\psi}{2}\right).$$

Thus, f_1 and f_2 have the same values, in exact arithmetic, for any given argument values x and δ.

a) Derive $f_2(x,\delta)$.

b) Write a MATLAB script which will calculate $g_1(x,\delta) = f_1(x,\delta)/\delta + \sin(x)$ and $g_2(x,\delta) = f_2(x,\delta)/\delta + \sin(x)$ for $x = 3$ and $\delta = 1.e-11$.

c) Explain the difference in the results of the two calculations.

2. The function $f_1(x_0,h) = \sin(x_0+h) - \sin(x_0)$ can be transformed into another form, $f_2(x_0,h)$, using the trigonometric formula

$$\sin(\phi) - \sin(\psi) = 2\cos\left(\frac{\phi+\psi}{2}\right)\sin\left(\frac{\phi-\psi}{2}\right).$$

Thus, f_1 and f_2 have the same values, in exact arithmetic, for any given argument values x_0 and h.

a) Derive $f_2(x_0,h)$.

b) Suggest a formula that avoids cancellation errors for computing the approximation $(f(x_0+h)-f(x_0))/h$ to the derivative of $f(x) = \sin(x)$ at $x = x_0$. Write a MATLAB program that implements your formula and computes an approximation of $f'(1.2)$, for $h = 1e-20, 1e-19, \ldots, 1$.

3. If you generate in MATLAB a row vector x containing all the floating point numbers in a given system, another row vector y of the same dimension as x containing $0's$, and then plot discrete values of y vs x using the symbol '+', you'll get a picture of sorts of the floating point number system. The relevant commands for such a display are

```
y = zeros(1,length(x));
plot (x,y,'+')
```

Produce such a plot for the system $(\beta, t, L, U) = (2, 3, -2, 3)$. (Do not assume the IEEE special conventions.) What do you observe? Also, calculate the rounding unit for this modest floating point system.

4. In the statistical treatment of data one often needs to compute the quantities

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i, \quad s^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2,$$

where $x_1, x_2, \ldots, x_n$ are the given data. Assume that n is large, say $n = 10,000$. It is easy to see that s^2 can also be written as

$$s^2 = \frac{1}{n}\sum_{i=1}^{n} x_i^2 \; - \; \bar{x}^2.$$

(a) Which of the two methods to calculate s^2 is cheaper in terms of flop counts? (Assume $\bar{x}$ has already been calculated and give the operation counts for these two options.)

(b) Which of the two methods is expected to give more accurate results for s^2 in general? (Justify briefly.)

(c) Write a little MATLAB script to check whether your answer to the previous question was correct.