Positive Modal Logic

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Abstract. We give a set of postulates for the minimal normal modal logic $\mathbf{K}_+$ without negation or any kind of implication. The connectives are simply $\wedge, \vee, \Box, \Diamond$. The postulates (and theorems) are all deducibility statements $\varphi \vdash \psi$. The only postulates that might not be obvious are

$$\Diamond\varphi \wedge \Box\psi \vdash \Diamond(\varphi \wedge \psi) \quad \Box(\varphi \vee \psi) \vdash \Box\varphi \vee \Diamond\psi.$$

It is shown that $\mathbf{K}_+$ is complete with respect to the usual Kripke-style semantics. The proof is by way of a Henkin-style construction, with "possible worlds" being taken to be prime theories. The construction has the somewhat unusual feature of using at an intermediate stage disjoint pairs consisting of a theory and a "counter-theory", the counter-theory replacing the role of negation in the standard construction. Extension to other modal logics is discussed, as well as a representation theorem for the corresponding modal algebras. We also discuss proof-theoretic arguments.

1. Introduction

There seems to be a lacuna in the modal logic literature regarding *positive* modal logic, by which we here mean modal logic with no negation (and no implication, whether material or strict).[1] To be explicit, the only connectives are $\wedge, \vee, \Box, \Diamond$. The question is, what set of postulates characterizes the definition of these connectives in the usual Kripke semantics? To begin with we address the question for the minimal positive normal modal logic $\mathbf{K}_+$ where no special conditions are put on the accessibility relation.

The postulates can easily be described algebraically as modal logic based on a distributive lattice instead of the usual Boolean algebra. More explicitly, a *positive modal algebra* (or "$\mathbf{K}_+$ algebra") is a structure $(D, \leq, \wedge, \vee, \Box, \Diamond)$,where $(D, \leq, \wedge, \vee)$ is a distributive lattice, $\Box$ distributes over $\wedge$, $\Diamond$ distributes over $\vee$, and

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[1]Perhaps this should be called *absolutely positive modal logic,* the idea being that all connectives are isotone with respect to the deducibility order (implication is antitone in its antecedent position). But we use the shorter phrase for simplicity.

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$$\Diamond x \wedge \Box y \leq \Diamond(x \wedge y) \qquad \Box(x \vee y) \leq \Box x \vee \Diamond y. \tag{1}$$

The proof we give of completeness is in the style of Lemmon-Scott (1977), but replacing maximally consistent theories with maximal "theory/counter-theory" pairs. As we shall describe, it can be straightforwardly modified to give a representation of positive modal algebras along the lines of Lemmon (1966). In sec. 9 we briefly discuss another proof using proof-theoretic methods.

2. The Positive Minimal Normal Modal Logic $\mathbf{K}_+$

Presenting this as a logistic system, sentences are defined in the usual inductive fashion from atomic sentences using just the connectives $\wedge, \vee, \Box, \Diamond$. Note that there are no theorems such as $\varphi \vee \neg\varphi$, or $\varphi \to \varphi$. So we formulate the system as a binary consequence system, whose formal objects are pairs of sentences (φ, ψ),to be read as "φ has ψ as a consequence". These are called *consequence pairs.* We write $\varphi \vdash \psi$ to indicate that the pair (φ, ψ) is derivable in the system, and $\varphi \dashv\vdash \psi$ as an abbreviation of $\varphi \vdash \psi$ and $\psi \vdash \varphi$. We postulate certain consequence pairs as derivable, and the rules show how to go from derivable consequence pairs as premises to a derivable consequence pair as a conclusion.

Postulates and Rules

$$\varphi \vdash \varphi \qquad \text{(Reflexitivity)}$$

$$\frac{\varphi \vdash \psi \qquad \psi \vdash \chi}{\varphi \vdash \chi} \quad \text{(Transitivity)}$$

$$\varphi \wedge \psi \vdash \varphi \qquad \varphi \wedge \psi \vdash \psi \quad \text{(Conjunction Elimination)}$$

$$\frac{\chi \vdash \varphi \qquad \chi \vdash \psi}{\chi \vdash \varphi \wedge \psi} \quad \text{(Conjunction Introduction)}$$

$$\varphi \vdash \varphi \vee \psi \qquad \psi \vdash \varphi \vee \psi \quad \text{(Disjunction Introduction)}$$

$$\frac{\varphi \vdash \chi \qquad \psi \vdash \chi}{\varphi \vee \psi \vdash \chi} \quad \text{(Disjunction Elimination)}$$

$$\varphi \wedge (\psi \vee \chi) \vdash (\varphi \wedge \psi) \vee (\varphi \wedge \chi) \quad \text{(Distribution)}$$

$$\frac{\varphi \vdash \psi}{\Box\varphi \vdash \Box\psi} \qquad \frac{\varphi \vdash \psi}{\Diamond\varphi \vdash \Diamond\psi} \quad \text{(Becker's Rules).}$$

$$\Box(\varphi \wedge \psi) \dashv\vdash \Box\varphi \wedge \Box\psi \qquad \Diamond(\varphi \vee \psi) \dashv\vdash \Diamond\varphi \vee \Diamond\psi \quad \text{(Linearity)}$$

$$\Diamond\varphi \wedge \Box\psi \vdash \Diamond(\varphi \wedge \psi) \qquad \Box(\varphi \vee \psi) \vdash \Box\varphi \vee \Diamond\psi \qquad (\Box\text{ -}\Diamond \text{ Interaction})$$

From these, other familiar principles can be derived, and in particular we will have to use the following, which has a routine derivation using Distribution:

$$\frac{\varphi \wedge \chi \vdash \psi \qquad \varphi \vdash \psi \vee \chi}{\varphi \vdash \psi} \quad (\text{Cut}).$$

3. Theories and Counter-Theories

By a *theory* we mean a set of sentences T closed under deducibility and conjunction, *i.e.*, if $\varphi \vdash \psi$ and $\varphi \in T$, then $\psi \in T$, and if $\varphi, \psi \in T$, then $\varphi \wedge \psi \in T$. The less familiar notion of a *counter-theory* F is defined dually, requiring both that if $\varphi \vdash \psi$ and $\psi \in F$, then $\varphi \in F$, and that if $\varphi, \psi \in F$, then $\varphi \vee \psi \in F$.[2]

It is easy to see from Disjunction Introduction that all theories have the following property:

$$\text{if } \varphi \in T \text{ or } \psi \in T, \text{ then } \varphi \vee \psi \in T. \tag{2}$$

By a *prime theory* is meant one that satisfies the converse of (2) as well.

Similarly in virtue of Conjunction Elimination all counter-theories satisfy:

$$\text{if } \varphi \in F \text{ or } \psi \in F, \text{ then } \varphi \wedge \psi \in F \tag{3}$$

A *prime counter-theory* is one that also satisfies the converse of (3).

Given a set of sentences Γ (thought of as axioms), by *the theory determined by* Γ (in symbols $Th(\Gamma)$) we mean the intersection of all theories $T \supseteq \Gamma$. Given a theory T and a sentence ψ, by $T + \psi$ we mean $Th(T \cup \{\psi\})$. Similarly given a set Δ of sentences (thought of as "counter-axioms"), we define the counter-theory $CTh(\Delta)$, and from this define the counter-theory $F + \psi$.

The proofs of the next three lemmas are left to the reader.

LEMMA 3.1. *$\chi \in Th(\Gamma)$ iff $\exists \varphi_1, ..., \varphi_m \in \Gamma$ such that $\varphi_1 \wedge ... \wedge \varphi_m \vdash \chi$; and $\chi \in CTh(\Delta)$ iff $\exists \varphi_1, ..., \varphi_m \in \Delta$ such that $\chi \vdash \varphi_1 \vee ... \vee \varphi_m$.*

[2]The notion of a "counter-theory" was introduced by Curry (1963), though he never explicitly gave it this name (he did talk of "counteraxioms"). Halmos (1962) might also be mentioned for connecting ideals with falsity. Readers with an algebraic background will see the analogy of theories with filters, and of counter-theories with ideals.

LEMMA 3.2. *For a non-empty theory $T, \chi \in T + \psi$ iff $\exists \varphi \in T$ such that $\varphi \wedge \psi \vdash \chi$; and dually for a non-empty counter-theory F, $\chi \in F + \psi$ iff $\exists \varphi \in F$ such that $\chi \vdash \psi \vee \varphi$.*

LEMMA 3.3. *The complement of a prime theory (relative to the set of sentences) is a prime counter-theory and vice versa.*

For purposes of a future construction we will also need to look at *theory pairs* $\alpha = (\alpha_1, \alpha_2)$, where α_1 is a theory and α_2 is a counter-theory.[3] Although this is not really needed until sec. 6, we will require of all theory pairs that they be *proper* in the sense that both α_1 and α_2 are non-empty. This assures that neither exhausts the whole set of sentences. When $\alpha_1 \cap \alpha_2 = \emptyset$ we shall call α *disjoint.* We define an order on theory pairs such that $\alpha \leq \beta$ iff $\alpha_1 \subseteq \beta_1$ and $\alpha_2 \subseteq \beta_2$. By a *maximally disjoint theory pair* is meant a theory pair α that is disjoint and for any other theory pair β, if $\alpha \leq \beta$ then β is not disjoint. For a *prime theory pair* $\alpha = (\alpha_1, \alpha_2)$, we require that α_1 is a prime theory and α_2 is its complement relative to the set of all sentences (hence α_2 is a prime counter-theory).

REMARK. Disjoint theory pairs play the role ordinarily played by consistent theories, but, given the absence of negation, the latter clearly are not available to us. Also clearly maximally disjoint theory pairs play the role usually played by maximally consistent theories. Lemma 3.8. below is the analog for theory pairs of the usual "Lindenbaum's Lemma" which says that every consistent theory can be extended to a maximally consistent theory.

We now prove a series of lemmas.

LEMMA 3.4. *A theory pair $\alpha = (\alpha_1, \alpha_2)$ fails to be disjoint iff there are sentences $\varphi \in \alpha_1$, $\psi \in \alpha_2$, such that*

$$\varphi \vdash \psi. \tag{4}$$

PROOF. For left-to-right, assume that $\varphi \in \alpha_1 \cap \alpha_2$. Then $\varphi \vdash \varphi$ (Reflexivity). For right-to-left, since we are supposing that $\varphi \vdash \psi$, then $\psi \in \alpha_1$. ■

[3]Theory pairs have been used on other occasions under a variety of names for a variety of purposes. Cf. Gabbay (1974), Dunn (1976), Dunn (1986) (cf. the "Belnap Extension Lemma"), Anderson, Belnap and Dunn (1993), Urquhart (1978), and Allwein and Dunn (1993) for starters. The construction of Belnap and Gabbay can be easily adapted to what might be called "positive first-order logic," which suggests that the results of the present paper could be extended to as least some first-order modal logics.

LEMMA 3.5. *Let $\alpha = (\alpha_1, \alpha_2)$ be a disjoint theory pair. Then for an arbitrary sentence χ, either $(\alpha_1 + \chi, \alpha_2)$ or $(\alpha_1, \alpha_2 + \chi)$ is a disjoint theory pair.*

PROOF. Suppose that both $(\alpha_1 + \chi, \alpha_2)$ and $(\alpha_1, \alpha_2 + \chi)$ are not disjoint. But then it is easy to establish that then there are sentences $\varphi' \in \alpha_1$, $\psi' \in \alpha_2$, such that

$$\varphi' \wedge \chi \vdash \psi'. \tag{5}$$

There are also sentences $\varphi'' \in \alpha_1$, $\psi'' \in \alpha_2$, such that

$$\varphi'' \vdash \chi \vee \psi''. \tag{6}$$

Setting $\varphi = \varphi' \wedge \varphi''$, $\psi = \psi' \vee \psi''$, it follows from the definitions of *theory* and *counter-theory* that $\varphi \in \alpha_1$ and $\psi \in \alpha_2$. We have from (5) and (6), by Conjunction Elimination and Disjunction Introduction, that

$$\varphi, \chi \vdash \psi, \tag{7}$$

$$\varphi \vdash \chi \vee \psi. \tag{8}$$

But then by Cut we have from (7) and (8),

$$\varphi \vdash \psi,$$

which (by Lemma 3.4.) contradicts the assumed disjointness of $\alpha = (\alpha_1, \alpha_2)$. ∎

LEMMA 3.6. *Let $\alpha = (\alpha_1, \alpha_2)$ be a maximally disjoint theory pair. Then for each sentence φ,either $\varphi \in \alpha_1$ or $\varphi \in \alpha_2$.*

PROOF. Suppose $\alpha = (\alpha_1, \alpha_2)$ is a maximally disjoint theory pair but that $\varphi \notin \alpha_1$ and $\varphi \notin \alpha_2$. We know from Lemma 3.5. that either $(\alpha_1 + \varphi, \alpha_2)$ or $(\alpha_1, \alpha_2 + \varphi)$ is a disjoint theory pair. Then one of these pairs properly extends (α_1, α_2), contradicting our assumption that it is *maximally* disjoint. ∎

LEMMA 3.7. *α is a maximally disjoint theory pair iff α is a prime theory pair.*

PROOF. Right-to-left is immediate, since α_1 and α_2 are complements. For left-to-right, let us assume that α is maximally disjoint and that $\varphi \vee \psi \in \alpha_1$, but that $\varphi, \psi \notin \alpha_1$. Then by Lemma 3.6. $\varphi, \psi \in \alpha_2$, and so $\varphi \vee \psi \in \alpha_2$, contradicting the disjointness of α. Primeness of the counter-theory α_2 is argued dually. ■

LEMMA 3.8. *(Lindenbaum's Lemma for Theory Pairs) Let α be a disjoint theory pair. Then $\exists \beta \geq \alpha$ such that β is a maximally disjoint theory pair.*

PROOF. Enumerate the set of sentences $\chi_1, \chi_2, \ldots$. The idea is that in turn we add a sentence either to the left side of the pair $\alpha = (\alpha_1, \alpha_2)$ if this retains disjointness, or otherwise to the right (which we can show still retains disjointness). More formally, we inductively define a sequence of pairs:

$\alpha^0 = (\alpha_1^0, \alpha_2^0) = (\alpha_1, \alpha_2)$;

$\alpha^{i+1} = (\alpha_1^i \cup \{\chi_i\}, \alpha_2^i)$ if disjoint, and otherwise $\alpha^{i+1} = (\alpha_1^i, \alpha_2^i \cup \{\chi_i\})$.

We know that this construction guarantees disjointness at each stage. The initial stage α^0 is given as disjoint, and Lemma 3.5. tells us that if α^i is disjoint then so is α^{i+1}.

We finish by setting $\beta_1 = \bigcup_{i \in \omega} \alpha_1^i, \beta_2 = \bigcup_{i \in \omega} \alpha_2^i$. It is easy to see that $\beta = (\beta_1, \beta_2)$ is the desired maximally disjoint pair. Thus every sentence has been put either into either β_1 or β_2. And β_1 and β_2 are disjoint, since otherwise some stage $\alpha^i = (\alpha_1^i, \alpha_2^i)$ would have failed to be disjoint. We still need to establish that β_1 and β_2 are respectively a theory and a counter-theory, but this is safely left to the reader. ■

4. Frames and Models

We sketch the usual Kripke semantics for some normal modal logics. A *frame* is a structure $\mathcal{F} = (U, R)$, where U is a non-empty set and R is a binary relation on U. Depending on whether one is a philosopher or a computer scientist, U is thought of as a set of "possible worlds" or "states". R is read as "relative possibility" or "accessibility". A *model* $\mathcal{M}$ is then a structure $(U, R, \models)$, where (U, R) is a frame and $\models$ is a relation ("satisfaction") between elements α of U and sentences φ, subject to the following conditions:

$$\alpha \models \varphi \wedge \psi \text{ iff } \alpha \models \varphi \text{ and } \alpha \models \psi, \tag{9}$$

$$\alpha \models \varphi \vee \psi \text{ iff } \alpha \models \varphi \text{ or } \alpha \models \psi, \tag{10}$$

$$\alpha \models \Box\varphi \text{ iff } \forall\beta(\text{ if } \alpha R \beta \text{ then } \beta \models \varphi), \tag{11}$$

$$\alpha \models \Diamond\varphi \text{ iff } \exists\beta(\alpha R\beta \text{ and } \beta \models \varphi). \tag{12}$$

Given a consequence pair (φ, ψ), we say that it is *valid in a model* $\mathcal{M} = (U, R, \models)$ (in symbols $\varphi \models_{\mathcal{M}} \psi$) iff for every $\alpha \in U$, if $\alpha \models \varphi$ then $\alpha \models \psi$. A consequence pair (φ, ψ) is *valid in a frame* $\mathcal{F} = (U, R)$ (in symbols $\varphi \models_{\mathcal{F}} \psi$) iff (φ, ψ) is valid in every model $(U, R, \models)$ (with that same frame). Finally, a consequence pair (φ, ψ) is *valid (simpliciter)* just when it is valid in every frame (equivalently in every model). When this happens we write $\varphi \models \psi$.

THEOREM 4.1. *(Soundness) If $\varphi \vdash \psi$ then $\varphi \models \psi$.*

PROOF is by a routine induction on derivations, left mostly to the reader. We will though examine $\Box$-$\Diamond$ Interaction to make sure that the reader sees its role in saying that $\Box$ and $\Diamond$ are defined using the same accessibility relation (and in the same direction, unlike say **G** and **P** in tense logic, one meaning "always *will* be", and the other meaning "sometime *was*"). Thus let us show $\Diamond\varphi \wedge \Box\psi \models \Diamond(\varphi \wedge \psi)$. Assume $\alpha \models \Diamond\varphi \wedge \Box\psi$. Then $\alpha \models \Diamond\varphi$ and so there is a β such that $\alpha R\beta$ and $\beta \models \varphi$. But since $\alpha \models \Box\psi$, we must also have $\beta \models \psi$, and so $\beta \models \varphi \wedge \psi$ and thus $\alpha \models \Diamond(\varphi \wedge \psi)$. We leave to the reader the verification of the dual part of $\Box$-$\Diamond$ Interaction, but we give the famous hint of Church to "try the contrapositive," which in this case means assuming $\alpha \not\models \Box\varphi \vee \Diamond\psi$ and showing $\alpha \not\models \Box(\varphi \vee \psi)$. ■

REMARK. It is instructive to consider frames with two accessibility relations $R^{\Box}$ and $R^{\Diamond}$, with (11) restated so that the satisfaction condition for $\Box$ is defined using $R^{\Box}$, and (12) similarly restated using $R^{\Diamond}$. It is easy to see then that $\Diamond\varphi \wedge \Box\psi \models \Diamond(\varphi \wedge \psi)$ corresponds to the condition that $R^{\Diamond} \subseteq R^{\Box}$ in the sense that a frame validates the consequence iff it satisfies the condition. Similarly $\Box(\varphi \vee \psi) \models \Box\varphi \vee \Diamond\psi$ corresponds to the condition that $R^{\Box} \subseteq R^{\Diamond}$.

5. Canonical Model and Completeness

The *(normal) canonical frame* (U^c, R^c) is defined as follows. U^c can be taken to be the set of all prime theories. It is however more convenient to equivalently take the elements of U^c to be pairs $\alpha = (\alpha_1, \alpha_2)$, where α_1 is a prime theory and α_2 is its complement, a prime counter-theory (cf. Lemma 3.3.). We call these *prime theory pairs*. We define two relations $R^c_{\Box}$ and $R^c_{\Diamond}$, the first a relation on theories and the second a relation on counter-theories.

$$\alpha_1 R^c_{\Box} \beta_1 \text{ iff, for all sentences } \varphi, \text{ if } \Box\varphi \in \alpha_1 \text{ then } \varphi \in \beta_1. \tag{13}$$

$$\alpha_2 R^c_{\Diamond} \beta_2 \text{ iff for all sentences } \varphi, \text{ if } \Diamond\varphi \in \alpha_2 \text{ then } \varphi \in \beta_2. \tag{14}$$

We then define a relation R^c on theory pairs such that

$$\alpha R^c \beta \text{ iff, both } \alpha_1 R^c_{\Box} \beta_1 \text{ and } \alpha_2 R^c_{\Diamond} \beta_2. \tag{15}$$

REMARK. Usually for modal logic with negation, $\Box$ is taken as the primitive connective and $\Diamond$ is defined from it by the definition $\Diamond = \neg\Box\neg$. The canonical accessibility relation is defined on maximally consistent theories using just the definition (13). Alternatively $\Diamond$ can be taken as primitive and the canonical accessibility relation can be defined as $\forall\varphi(\text{if } \varphi \in \beta_1, \text{ then } \Diamond\varphi \in \alpha_1)$. Note that when α and β are maximally disjoint, $\alpha_2 R^c_{\Diamond} \beta_2$ reduces to this more customary definition. Note also that these two customary definitions are actually equivalent given the presence of negation (since maximally consistent theories are also negation complete). But since we have no negation we have to combine them, so $\alpha R^c \beta$ is really the conjunction of the two customary alternative definitions.[4]

Moving to the *canonical model*, for $\alpha \in U^c$, we define $\alpha \models^c \varphi$ iff $\varphi \in \alpha_1$. We must now show that this satisfies the conditions (9)–(12). It follows routinely from the definition of a prime theory that $\models^c$ satisfies (9) and (10), so we concentrate on (11) and (12). We must show the following:

LEMMA 5.1. *In the canonical model,*

$$\Box\varphi \in \alpha_1 \textit{ iff } \forall\beta(\textit{ if } \alpha R^c \beta \textit{ then } \varphi \in \beta_1), \tag{16}$$

$$\Diamond\varphi \in \alpha_1 \textit{ iff } \exists\beta(\alpha R^c \beta \textit{ and } \varphi \in \beta_1). \tag{17}$$

PROOF. (17) is somewhat more direct than (16) and so we show it first. Right-to-left is easy. Thus suppose $\alpha R^c \beta$ and $\varphi \in \beta_1$. Assume for the sake of contradiction that $\Diamond\varphi \notin \alpha_1$. Then $\Diamond\varphi \in \alpha_2$. But then since $\alpha_2 R^c_{\Diamond} \beta_2$, we have $\varphi \in \beta_2$, contradicting that $\beta_1 \cap \beta_2 = \emptyset$.

We now turn to the more difficult left-to-right. Suppose that $\Diamond\varphi \in \alpha_1$. We define a theory pair $\beta' = (\beta'_1, \beta'_2)$, which will not be maximally disjoint, but which we can at least show to be disjoint. Given a set of sentences X, $\Box^{-1}X = \{\varphi : \Box\varphi \in X\}$ and $\Diamond^{-1}X = \{\varphi : \Diamond\varphi \in X\}$. We then set $\beta'_1 = \Box^{-1}\alpha_1 \cup \{\varphi\}$, $\beta'_2 = \Diamond^{-1}\alpha_2$. It is obvious from the definition that $\alpha R^c \beta'$

[4]Chellas (1980) contains both customary definitions of the canonical accessibility relation and shows their equivalence (cf. his Definition 5.9 and Theorems 5.10 and 4.29). Note that what we are calling a "canonical model" corresponds to what he terms a "proper canonical standard model".

and $\varphi \in \beta_1'$, and it is further obvious that $\alpha R^c \beta$ and $\varphi \in \beta_1$ for any $\beta \geq \beta'$. So if we can show that β' is disjoint we can use Lemma 3.8. to extend it to a maximally disjoint pair β and we are through.

Suppose to the contrary that β' is not disjoint, *i.e.,* some sentence $\chi \in \beta_1' \cap \beta_2'$. It is easy to see then that there are sentences $\varphi_1, ..., \varphi_m \in \beta_1', \psi_1, ..., \psi_n \in \beta_2'$ such that

$$\varphi \wedge \varphi_1 \wedge ... \wedge \varphi_m \vdash \psi_1 \vee ... \vee \psi_n, \tag{18}$$

where $\Box\varphi_1, ..., \Box\varphi_m \in \alpha_1$ and $\Diamond\psi_1, ..., \Diamond\psi_n \in \alpha_2$. But then $\Box\varphi_1 \wedge ... \wedge \Box\varphi_m \in \alpha_1$, and so by Linearity, $\Box(\varphi_1 \wedge ... \wedge \varphi_m) \in \alpha_1$. Since also $\Diamond\varphi \in \alpha_1$,we have $\Diamond\varphi \wedge \Box(\varphi_1 \wedge ... \wedge \varphi_m) \in \alpha_1$. But as an instance of $\Box$-$\Diamond$ Interaction, we have

$$\Diamond\varphi \wedge \Box(\varphi_1 \wedge ... \wedge \varphi_m) \vdash \Diamond(\varphi \wedge \varphi_1 \wedge ... \wedge \varphi_m). \tag{19}$$

By Becker's Rule for $\Diamond$ we have from (18):

$$\Diamond(\varphi \wedge \varphi_1 \wedge ... \wedge \varphi_m) \vdash \Diamond(\psi_1 \vee ... \vee \psi_n). \tag{20}$$

And so using (19) and (20) we have $\Diamond(\psi_1 \vee ... \vee \psi_n) \in \alpha_1$. But then by Linearity we get

$$\Diamond\psi_1 \vee ... \vee \Diamond\psi_n \in \alpha_1. \tag{21}$$

Since α_1 is prime, we would have some $\Diamond\psi_i \in \alpha_1$. But since also $\Diamond\psi_i \in \alpha_2$, this contradicts the fact that $\alpha_1 \cap \alpha_2 = \emptyset$.

We now address the proof of (16). Let us first contrapose it, obtaining:

$$\Box\varphi \notin \alpha_1 \text{ iff } \exists\beta(\alpha R^c \beta \text{ and } \varphi \notin \alpha_1), \tag{22}$$

and then rephrase this to obtain:

$$\Box\varphi \in \alpha_2 \text{ iff } \exists\beta(\alpha R^c \beta \text{ and } \varphi \in \alpha_2). \tag{23}$$

(23) is in a form analogous to (17), and so we merely sketch its proof.

Thus the proof of (23) from right to left is completely analogous. From left to right, assume that $\Box\varphi \in \alpha_2$,and define $\beta_1' = \Box^{-1}\alpha_1$, $\beta_2' = \Diamond^{-1}\alpha_2 \cup \{\varphi\}$. We show that β' is disjoint, so that we can again extend β' to a maximally disjoint pair.

Again we suppose to the contrary that β' is not disjoint, and again it is easy to see that there are sentences $\varphi_1, ..., \varphi_m \in \beta_1', \psi_1, ..., \psi_n \in \beta_2'$ such that

$$\varphi_1 \wedge ... \wedge \varphi_m \vdash \psi_1 \vee ... \vee \psi_n \vee \varphi, \tag{24}$$

where $\Box\varphi_1, \ldots, \Box\varphi_m \in \alpha_1$ and $\Diamond\psi_1, \ldots, \Diamond\psi_n \in \alpha_2$. Using Linearity and Becker's Rule for $\Box$,we get $\Box(\psi_1 \vee \ldots \vee \psi_n \vee \varphi) \in \alpha_1$.

But as an instance of $\Box$-$\Diamond$ Interaction, we have

$$\Box(\psi_1 \vee \ldots \vee \psi_n \vee \varphi) \vdash \Diamond(\psi_1 \vee \ldots \vee \psi_n) \vee \Box\varphi. \tag{25}$$

But then by Linearity for $\Diamond$ we have

$$\Diamond\psi_1 \vee \ldots \vee \Diamond\psi_n \vee \Box\varphi \in \alpha_1. \tag{26}$$

Since α_1 is prime, we would have either some $\Diamond\psi_i \in \alpha_1$ or else $\Box\varphi \in \alpha_1$. The first alternative is impossible, and so is the second, since each contradicts the fact that $\alpha_1 \cap \alpha_2 = \emptyset$. ■

We are now in a position to prove:

THEOREM 5.2. *(Completeness) If* $\varphi \models \psi$ *then* $\varphi \vdash \psi$.

PROOF: We prove the contrapositive. Suppose $\varphi \nvdash \psi$. Consider the theory pair $(Th(\{\varphi\}), CTh(\{\psi\}))$. This is disjoint, for otherwise there is a $\chi \in Th(\{\varphi\}) \cap CTh(\{\psi\}$, and from Lemma 3.1. it follows that $\varphi \vdash \chi$ and $\chi \vdash \psi$, and so by Transitivity $\varphi \vdash \psi$ (contrary to our contrapositive hypothesis). Since the theory pair $(Th(\{\varphi\}), CTh(\{\psi\}))$ is disjoint, by Lemma 3.8. it can be extended to a maximally disjoint theory pair (α_1, α_2). Clearly $\varphi \in \alpha_1, \psi \in \alpha_2$. Then in the canonical model $\alpha \models \varphi$ and $\alpha \not\models \psi$, and so $\varphi \not\models \psi$. ■

6. Adding Constants $\top$ and $\bot$

If one wishes one can add the constant sentences $\top$ and $\bot$ to the base of atomic sentences (corresponding to the lattice 1 and 0), abbreviating $\top \vdash \varphi$ as $\vdash \varphi$ so as to give the effect of φ's being a theorem.[5] For the proof theory one takes the postulates and rules of $\mathbf{K}_+$, and additionally postulates the derivability of Top and Necessitation for $\top$, and Bottom and Possibilization for $\bot$:

$$\varphi \vdash \top \text{ (Top)}, \qquad \bot \vdash \varphi \text{ (Bottom)}, \tag{27}$$

$$\top \vdash \Box\top \text{ (Necessitation)}, \qquad \Diamond\bot \vdash \bot \text{ (Possibilization)}. \tag{28}$$

[5]Though it can easily be seen from the model theory below that all theorems are either $\top$ or contain $\top$.

These give the effect of the usual algebraic postulates $\Box 1 = 1, \Diamond 0 = 0$. Let us label these additions $\mathbf{K}_+^{\top\bot}$.

For the model theory, we simply fix it so that $\top$ is true in every state and $\bot$ is false at every state:

$$\forall\alpha \in U, \alpha \models \top \text{ and } \alpha \not\models \bot. \tag{29}$$

Soundness for $\mathbf{K}_+^{\top\bot}$ is now easy to check. For completeness we now need to remind ourselves of a requirement on the canonical frame that was not necessary before. In defining U^c its members, the prime theory pairs (α_1, α_2), were required to be *proper* in the sense that neither α_1 nor α_2 is the whole set of sentences. The reader can check that none of the reasoning of sec. 5 is affected by having this requirement, or by not having it. The following is an immediate consequence of (27):

PROPOSITION 6.1. *A theory pair* (α_1, α_2) *is proper iff* $\bot \notin \alpha_1$ *and* $\top \notin \alpha_2$.

COROLLARY 6.2. $\forall\alpha \in U^c, \alpha \models^c \top$ *and* $\alpha \not\models^c \bot$.

PROOF. Follows from the above proposition by way of the following two equivalences:

$$\alpha \models^c \top \text{ iff } \top \in \alpha_1 \text{ iff } \top \notin \alpha_2; \tag{30}$$

$$\alpha \not\models^c \bot \text{ iff } \bot \notin \alpha_1. \tag{31}$$

These equivalences follow from the definition of the canonical model and the fact that α_1 and α_2 are complements of each other. ■

An anonymous referee has raised an interesting question, to wit, what happens if we drop the postulates of Necessitation and Possibilization. In the presence of negation this is the difference between *normal* modal logics and so-called *regular* modal logics, as described by Chellas (1980). Let us then mean by $\mathbf{R}_+^{\top\bot}$ the result of adding to the system $\mathbf{K}_+$ just the postulates Top and Bottom.

Chellas gives a model-theory for regular modal logics by in effect adding to a frame a set $N \subseteq U$ of "normal" states, at the same time changing the valuation rules (11) and (12) for the modal connectives so that they are stated in the following "non-standard" way:

$$\alpha \models \Box\varphi \text{ iff } \alpha \in N \text{ and } \forall\beta(\text{ if } \alpha R\beta \text{ then } \beta \models \varphi), \tag{32}$$

$$\alpha \models \Diamond\varphi \text{ iff } \alpha \notin N \text{ or } \exists\beta(\alpha R\beta \text{ and } \beta \models \varphi). \tag{33}$$

It is clear that the intent is to fix it so that there can be a state (non-normal) α such that $\alpha \not\models \Box\top$, and a state (again non-normal) β such that $\beta \models \Diamond\bot$.

For the *regular* canonical model we can define $N^c = \{\alpha \in U^c : \Box\top \in \alpha_1 \text{ or } \Diamond\bot \in \alpha_2\}$. For completeness we need then establish the following:

LEMMA 6.3. *In the regular canonical model,*

$$\Box\varphi \in \alpha_1 \textit{ iff } \alpha \in N^c \textit{ and } \forall\beta(\textit{ if } \alpha R^c\beta \textit{ then } \beta \in \alpha_1), \tag{34}$$

$$\Diamond\varphi \in \alpha_1 \textit{ iff } \alpha \notin N^c \textit{ or } \exists\beta(\alpha R^c\beta \textit{ and } \varphi \in \beta_1). \tag{35}$$

PROOF. Let us first address the clause (34). The only difference between (34) and (16) is that (34) contains the additional conjunct $\alpha \in N^c$ on its right hand side. So it suffices to show that

$$\text{if } \Box\varphi \in \alpha_1 \text{ then } \alpha \in N^c. \tag{36}$$

Since $\varphi \vdash \top$ (Top), it follows from Becker's Rules that $\Box\varphi \vdash \Box\top$. And so if $\Box\varphi \in \alpha_1$, then $\Box\top \in \alpha_1$, and so $\Box\top \in \alpha_1$ or $\Diamond\bot \in \alpha_2$. By definition $\alpha \in N^c$.

Let us now turn to (35). Here it is best to rephrase (35) as:

$$\Diamond\varphi \in \alpha_2 \text{ iff } \alpha \in N^c \text{ and } \forall\beta(\text{ if } \alpha R^c\beta \text{ then } \varphi \in \beta_2), \tag{37}$$

and to similarly rephrase (17) as

$$\Diamond\varphi \in \alpha_2 \text{ iff } \forall\beta(\text{ if } \alpha R^c\beta \text{ then } \varphi \in \beta_2). \tag{38}$$

These rephrasings are essentially contrapositions of their originals, given that the members of U^c are exhaustive disjoint pairs (Lemma 3.6.). Clearly now to establish (37), it suffices to show:

$$\text{if } \Diamond\varphi \in \alpha_2 \text{ then } \alpha \in N^c, \tag{39}$$

i.e., $\Diamond\varphi \in \alpha_2$ implies $\Box\top \in \alpha_1$ or $\Diamond\bot \in \alpha_2$. From $\bot \vdash \varphi$ (Bottom) and Becker's Rules, we obtain $\Diamond\bot \vdash \Diamond\varphi$, and so $\Diamond\bot \in \alpha_2$, and so $\alpha \in N^c$ as desired. ■

The reader might be wondering what happens if we add only one of the constants $\top$ and $\bot$ to the system $\mathbf{K}_+$. Thus there are systems $\mathbf{K}_+^\top$ and $\mathbf{K}_+^\bot$ obtained by adding one or the other of the constants, with the appropriate pair of postulate schemes Top and Necessitation, Bottom and Possibilization.

The reader can easily modify the soundness and completeness results for $\mathbf{K}_{+}^{\top\bot}$ so they apply to these simpler systems.

The process is much less straightforward with the "simpler" systems $\mathbf{R}_{+}^{\top}$ (add $\top$ to $\mathbf{K}_{+}$ with the postulate scheme Top), and $\mathbf{R}_{+}^{\bot}$ (add $\bot$ to $\mathbf{K}_{+}$ with the postulate scheme Bottom). Thus for $\mathbf{R}_{+}^{\top}$ one has to "mix and match" by combining the non-standard semantical clause (32) for $\Box$ with the standard clause (12) for $\Diamond$, at the same time modifying the definition of the set N^c of "normal" canonical states to be the set of those proper prime theory pairs α such that $\Box\top \in \alpha_1$. And for $\mathbf{R}_{+}^{\bot}$ one has to reverse these choices, using (11) for $\Box$ and (33) for $\Diamond$, and defining a normal state α to be such that $\Diamond\bot \in \alpha_2$. Since the semantical clauses change from one system to the other, the relation of the systems to each other is left something of a mystery. In particular it is unclear whether $\mathbf{R}_{+}^{\top\bot}$ is a conservative extension of the systems $\mathbf{R}_{+}^{\top}$ and $\mathbf{R}_{+}^{\bot}$.

An anonymous referee has raised the question whether $\mathbf{K}_{+}$ is not only the smallest positive *normal* modal logic, but also the smallest positive *regular* modal logic. The first is clear since the semantical clauses for $\Box$ and $\Diamond$ do not change in adding negation to $\mathbf{K}_{+}$ so as to obtain the system $\mathbf{K}$. The answer to the second question seems unclear, given that there is a change in the semantical clauses for $\Box$ and $\Diamond$ in going from the model theory for $\mathbf{K}_{+}$ to the model theory for $\mathbf{R}$.

7. Extension to Other Modal Logics

We have been considering the positive normal modal logic $\mathbf{K}_{+}$, which is minimal in the sense that there are no special conditions on the accessibility relation. But as is well-known, in the standard systems (with negation) one gets various normal modal logics other than $\mathbf{K}$ by imposing various conditions on the accessibility relation R, *e.g.,* for $\mathbf{T}$ the postulate scheme $\Box\varphi \supset \varphi$ is added to $\mathbf{K}$ and the relation R is required to be reflexive.

It is fortunate that the usual normal modal logics all arise by the addition of postulate schemes of the form $\varphi \supset \psi$,where φ and ψ are "positive" sentence schemes, i.e., sentence schemes containing as connectives only $\wedge, \vee, \Box, \Diamond$ and of course the metavariables φ, ψ, etc.[6] For $\varphi \supset \psi$ can simply be replaced with two schemes, the "positive" postulate scheme $\varphi \vdash \psi$ and its "dual" (precisely defined below), so as to obtain the corresponding positive modal logic.

Thus one forms the positive modal logic $\mathbf{T}_{+}$ by adding to $\mathbf{K}_{+}$ the postulate schemes $\Box\varphi \vdash \varphi$ and $\varphi \vdash \Diamond\varphi$. One needs to show that the canonical

[6] Compare e.g. Chellas (1980).

accessibility relation R^c is reflexive. For this one needs to show that $\Box\varphi \in \alpha_1$ implies $\varphi \in \alpha_1$, and also that $\Diamond\varphi \in \alpha_2$ implies $\varphi \in \alpha_2$. But given that α_2 is just the complement of α_1, this last reduces (by contraposition) to $\varphi \in \alpha_1$ implies $\Diamond\varphi \in \alpha_1$. Thus R^c is just the conjunction of the usual choices one has for defining the canonical accessibility relation, one in terms of $\Box$, the other in terms of $\Diamond$. Now the usual argument that R^c is reflexive, when defined in terms of $\Box$, is that if $\Box\varphi \in \alpha_1$, then since $\Box\varphi \vdash \varphi$ and α_1 is a theory, that $\varphi \in \alpha_1$. The same argument works whether negation is present or not, but we still have to show that dually $\varphi \in \alpha_1$ implies $\Diamond\varphi \in \alpha_1$. But again this follows in a similar fashion from the dual postulate scheme $\varphi \vdash \Diamond\varphi$.

Let us try to record the formal lessons to be learned from the above example. We first define, as usual, the *dual* φ' of a sentence scheme φ as the result of interchanging $\wedge$ with $\vee$, and $\Box$ with $\Diamond$. Given a postulate scheme $\varphi \vdash \psi$, its *dual* is $\psi' \vdash \varphi'$. Let us suppose a modal logic **S** is obtained from **K** by adding additional modal axioms of the form $\varphi \supset \psi$. Let us mean by $\mathbf{S}_+$ the result of adding to $\mathbf{K}_+$ for each such axiom both the postulate scheme $\varphi \vdash \psi$ and its dual $\psi' \vdash \varphi'$. It seems clear that if there is a Henkin-style argument that shows **S** complete by showing that the canonical accessibility relation satisfies certain, say first-order, conditions, that the very same argument (and its dual) should show that the canonical accessibility relation R^c for $\mathbf{S}_+$ satisfies those same conditions, and thus shows $\mathbf{S}_+$ complete.[7]

We will not take the trouble to make the above observation rigorous, but it seems clear that it is correct when φ and ψ are just atomic formulas with modal prefixes (strings of modal operators) in front of them, as is the case with familiar normal modal logics such as **B**, **S4**, **S5**, **KB**, **K4**, **K5**, etc.[8] It would seem to be okay too when φ and ψ contain conjunction and disjunction as well, given that our theories are prime, since the usual arguments only use this and not the further information that the theory is complete, which notion involves negation.

There is a remaining question as to what happens if one operator satisfies a postulate scheme, *e.g.*, $\Box\varphi \vdash \Box\Box\varphi$, but the other does not satisfy the dual postulate scheme, *e.g.*, $\Diamond\Diamond\varphi \vdash \Diamond\varphi$ (and is not otherwise derivable). As was pointed out by a referee such a system must be incomplete, for the dual of a postulate scheme is valid if the postulate scheme is valid.

[7] Actually one also needs to check that the additional postulates are valid in frames whose accessibility relations satisfy the required conditions, but in the cases we are discussing this is well known.

[8] Cf. e.g. Chellas (1980) for a lengthy list.

8. Representation of Positive Modal Algebras

The proof techniques above can be easily adapted to give a representation of positive modal algebras along the lines of Lemmon (1966). Let $\mathcal{F} = (U, R)$ be a frame. By a *concrete positive modal algebra on* $\mathcal{F}$ is meant a collection of subsets of U, closed under $\cap$, $\cup$, and the following operations: $\Box A = \{\chi : \forall\alpha(\text{if } \chi R\alpha \text{ then } \alpha \in A)$, $\Diamond A = \{\chi : \exists\alpha(\chi R\alpha \text{ and } \alpha \in A)$.

THEOREM 8.1. *Let $\mathfrak{M}$ be a positive modal algebra. Then there is a frame $\mathcal{F}$ so that $\mathfrak{M}$ is isomorphic to a concrete positive modal algebra on $\mathcal{F}$.*

PROOF. Define the isomorphism $h(a)$ = the set of prime filters of $\mathfrak{M}$ that contain a. It is well-known from Stone that this gives a representation of the underlying distributive lattice. But one can modify the completeness proof above so as to represent not just $\wedge$ and $\vee$, but the modal operators as well. The main idea is to simply reinterpret $\varphi \vdash \psi$ as an inequality $a \leq b$ in the distributive lattice. The proof of the completeness theorem then becomes a proof of the representation theorem, with only one modification, which has to do with the proof of Lindenbaum's Lemma (Lemma 3.8.). Our proof of it assumed that we can enumerate the sentences. But there may be non-countably many elements in $\mathfrak{M}$, so one has to use some initial segment of the ordinals to "enumerate" them. The inductive step is the same as before, and at limit ordinals one takes the union of all the previous stages.[9] ■

REMARK. The representation theorem above is just for $\mathbf{K}_+$ algebras. Like completeness theorem it is modeled on, it can be routinely modified for $\mathbf{T}_+$ algebras, $\mathbf{B}_+$ algebras, $\mathbf{S4}_+$ algebras, etc.

9. An Alternative Proof Using a Gentzen System

The following comes from a conversation with Arnon Avron after the paper was otherwise finished. Clearly any mistakes should be attributed to me, though the ideas are his. The usual Gentzen system for the system $\mathbf{K}$ is obtained by adding the following rule to those Gentzen gave for classical logic:

$$\frac{\Gamma \vdash \varphi}{\Box\Gamma \vdash \Box\varphi}$$

[9] Alternatively, one can use Zorn's Lemma to show that every disjoint filter/ideal pair can be extended to a maximal such pair. Cf. Urquhart (1978).

where Γ is a set of sentences and $\Box\Gamma$ is the result of prefixing $\Box$ to each sentence in the set. Ordinarily negation ($\neg$) is present and no separate rule is needed for $\Diamond$ since it can always be defined as $\neg\Box\neg$. But Avron suggests dropping the rule for negation and instead having the two rules:

$$\frac{\Gamma \vdash \varphi, \Delta}{\Box\Gamma \vdash \Box\varphi, \Diamond\Delta} \tag{40}$$

$$\frac{\Gamma, \varphi \vdash \Delta}{\Box\Gamma, \Diamond\varphi \vdash \Diamond\Delta} \tag{41}$$

where again Δ is a set of sentences and $\Diamond$ is the result of prefixing $\Diamond$ to each member of Δ.[10]

One can still prove the Cut Elimination Theorem, and one can then use the construction using saturated sequents from Avron (1984) to obtain a Kripke-model. Since Avron pointed this out to me I have come to realize that one can also obtain a proof of the completeness theorem from Fitting (1983).[11]

The methods of Avron or Fitting can be adapted as well to the positive fragments of extensions of **K** that have appropriate Gentzen (or tableaux) type presentations. Thus Fitting (1983) explicitly treats (**K, K4, T, S4, D, D4,** cf. his p. 90). But it is far from clear that there are similar Gentzen (or tableaux) style treatments for all of the various logics that can still be addressed by the Henkin-methods described above (e.g., **S5, K5**, and the bulk of the other systems that are axiomatized by either the $G^{k,l,m,n}$ or the $KG^{k,l,m,n}$ schemes in Chellas (1980)).

Besides the interest in these additional logics, what other interest is there in the Henkin-style arguments of the present paper, given that the positive fragments of most of the better known modal logics can in fact be isolated by Gentzen methods? There is of course a general interest in the Henkin-style methods, since they have been so widely-used in the study of the completeness of various logics, including modal logics. There is the further fact that the Henkin-style argument can be adapted to give a representation result for various classes of positive modal algebras (cf. sec. 8 above). There is also the possibility of extending it to modal logics based on non-distributive lattices, using the underlying lattice representation of Urquhart (1978), which uses filter/ideal pairs that are not necessarily exhaustive.[12]

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[10] Avron points out that the first rule can be found in M. Ryan and M Sadler (1992) (but they do have negation and so do not need the second rule).

[11] Fitting's rules L_2 and M_2 (p. 84) are essentially the rules (40) and (41) suggested by Avron, once $U\#$ and $V\flat$ are interpreted as Fitting would clearly intend for **K**, so $U\# = \{\varphi : \Box\varphi \in U\}$ and $V\flat = \{\varphi : \Diamond\varphi \in U\}$ (cf. p. 90).

[12] Cf. also Allwein and Dunn (1993).

References

[1] A. R. Anderson, N. D. Belnap, Jr., and J. M. Dunn (1992), *Entailment: The Logic of Relevance and Necessity*, vol. II, Princeton (Princeton University Press).

[2] G. Allwein and J. M. Dunn (1993), Kripke Models for Linear Logic, *The Journal for Symbolic Logic*, **58**, 514-545.

[3] A. Avron (1984), On Modal Systems having Arithmetical Interpretations, *The Journal of Symbolic Logic*, **49**, 935-942.

[4] B. F. Chellas (1980), *Modal Logic: An Introduction*, Cambridge (Cambridge University Press).

[5] J. M. Dunn (1976), Quantification and RM, *Studia Logica 38*, 315-322.

[6] J. M. Dunn (1986), Relevance Logic and Entailment, *Handbook of Philosophical Logic*, vol. III: Alternatives to Classical Logic, eds. D. Gabbay and F. Guenther, Dordrecht (D. Reidel Publishing Co.), 117-229.

[7] H. B. Curry (1963), *Foundations of Mathematical Logic*, New York (McGraw-Hill Inc.).

[8] M. Fitting (1983), *Proof Methods for Modal and Intuitionistic Logics*, Dordrecht (D. Reidel Publishing Co.).

[9] D. M. Gabbay (1974), On 2nd Order Intuitionistic Propositional Calculus with Full Comprehension, *Archiv für mathematische Logik und Grundlagenforschung* **16**, 177-186.

[10] P. R. Halmos (1962), *Algebraic Logic,* New York (Chelsea Publishing Co.).

[11] E. J. Lemmon (1966), Algebraic Semantics for Modal Logics I , II, *The Journal for symbolic Logic* **32**, 46-65, 191-218.

[12] E. J. Lemmon and D. S. Scott (1977), *The 'Lemmon Notes': An Introduction to Modal Logic*, ed. K. Segerberg, Oxford (Blackwell).

[13] M. Ryan and M. Sadler (1992), Valuation Systems and Consequence Relations, in *Handbook of Logic in Computer Science*, vol. **1** Background: Mathematics Structures, eds. S. Abramsky, Dov M. Gabbay, and T. S. E. Maibaum, Oxford (Clarendon Press), 1-78.

[14] A. I. F. Urquhart (1978), A Topological Representation Theorem for Lattices, *Algebra Universalis* **8**, 45-58.

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