Fusion of sequent modal logic systems labelled with truth values

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Abstract

Fusion is a well-known form of combining normal modal logics endowed with a Hilbert calculi and a Kripke semantics. Herein, fusion is studied over logic systems using sequent calculi labelled with truth values and with a semantics based on a two-sorted algebra allowing, in particular, the representation of general Kripke structures. A wide variety of logics, including non-classical logics like, for instance, modal logics and intuitionistic logic can be presented by logic systems of this kind. A categorical approach of fusion is defined in the context of these logic systems. Preservation of soundness and completeness by fusion is studied. Soundness is preserved without further requirements, completeness is preserved under mild assumptions.

Keywords: fusion of modal logic, labelled deduction, sequent calculi, general Kripke semantics.

1 Introduction

Fusion is a well-known form of combining normal modal logics endowed with a Hilbert calculi and a Kripke semantics [11] that produces a new modal logic system from two others sharing a propositional basis. The resulting system has the shared basis and the modalities from both composed logics. It is a restricted form of fibring ([24, 25]) and appeared first in the work of Thomason [22]. Herein, fusion of modal logics possibly endowed with general Kripke semantics is addressed. The logics are endowed with labelled sequent calculi.

Modal sequent calculi labelled with truth values were introduced in [15, 16]. The semantics for these calculi is presented by a two-sorted algebra which allows the natural representation of general Kripke structures. General Kripke semantics [25] is important since it is possible to prove strong completeness results over this semantics for modal logics that are not strongly complete with respect to the standard Kripke semantics. An example of such logic is GL, the normal modal logic with the Löb axiom $\Box(\Box\xi \supset \xi) \supset \Box\xi$ which is not strongly complete with respect to any class of standard Kripke structures but is strongly complete for general Kripke structures. So in our setting it is possible to provide a strongly complete calculi also for these logics. Moreover, each modal logic strongly complete with respect to the standard Kripke semantics is naturally strongly complete with respect to the generalized semantics.

When deduction systems of different kinds are compared, usually it is concluded, in terms of use in practice, that Hilbert-style systems are not so intuitive as Gentzen-style systems such as natural deduction, sequent calculi and tableaux systems. However, the development of Gentzen systems for non-classical logics often requires more effort than the elaboration of a Hilbert calculus. For a modal calculus, as an example, one has to start from scratch. A solution to this is the employment of labelling techniques. These techniques can provide a general environment for presenting different logics in a uniform way. Worlds as labels

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was already found in the pioneer work of Prior [17]. Subsequently, the use of labelling was employed in presentations for several logics [5, 10, 18, 23] like, for instance, modal logic [3], finite-valued logic [2], relevance logic [20] and conditional logic [6]. Usually labelling in modal logic is used with worlds being the labels and assertions being of the form $w:\varphi$, where w is a world and φ is an usual modal formula. Together with expressions to reason about the accessibility relation they constitute what is called in [7, 21] a configuration. But in general in this approach it is not possible to distinguish global and local reasoning. The use of truth-values instead of worlds solves this problem.

The use of truth values as labels can provide analytical labelled calculi for a wide class of normal modal systems, sharing a common core of rules, which can be useful for automation. Moreover the use of truth values as labels is relevant to the theory and application of combination of deduction systems since this requires a sufficiently general notion of labelled deduction [19]. So, an interesting direction of research not developed in [16] is the study of combination of modal logic systems where the calculus is with sequents with formulas labelled with truth values.

Fusion will be defined here as a categorical operation. To the reader not used to category theory, it is recommended to see [14] for the basic concepts needed in this work. The categorical approach brings as advantages the possibility to define several ‘‘kinds of fusion’’ taking different sets of shared connectives instead of only the propositional ones. Using the fusion as a categorical operation it can be possible to calculate fusion of logics not previously considered, the only requirement is that the logic can be presented in our setting.

The paper begins, in Section 2, with the basics about logic systems labelled with truth values. In Section 3, it is presented the algebraic view of fusion of modal systems, based on the definition of fusion presented in [8]. In Section 4, fusion is defined in a categorical way as a pushout in an appropriate category. For that, logic system morphisms are defined and some of their properties investigated. Finally, in Section 5, the preservation of soundness and completeness by fusion is investigated. At the end, some conclusions and suggestions of future work are described.

2 Preliminaries

In this section it is presented the principal definitions and properties of logic systems labelled with truth values. First, the language used to construct terms, formulas and assertions is described, as well as the notion of substitution. Then, sequents, derivations and sequent calculi are defined. After that, the semantics, which is given by a two-sorted algebra, is presented, as well as the notion of logic system. Finally, some examples and applications of the concepts developed during the section are provided.

2.1 Language

The objective of this subsection is to define the basic assertions of the language. It contains expressions of the form $\theta\leq\varphi$ expressing that the truth value associated to the term θ is less than or equal to the denotation of formula φ. In modal logics endowed with a Kripke semantics truth values can be seen as sets of worlds.

The language is built over a signature where connectives, operators and sets of variables are specified. There are three kinds of variables: truth value unbound variables (usually represented as $\mathbf{x}$, element of X), truth value bound variables (usually represented as $\mathbf{y}$, element of Y) and formula unbound variables (usually represented as $\mathbf{z}$, element of Z).

Definition 2.1 A *signature* is a tuple $\Sigma = \langle C, O, X, Y, Z\rangle$ where $C = \{C_k : k \in \mathbb{N}\}$ and $O = \{O_k : k \in \mathbb{N}\}$ such that each C_k and O_k is a countable set and $\bot, \top \in O_0$, and X, Y, Z are countable sets. All these sets are pairwise disjoints. ∎

The elements of each C_k are known as (formula) constructors or connectives of arity k. Those of each O_k are known as (truth value) operators of arity k.

Definition 2.2 A *signature* $\Sigma_M = \langle C, O, X, Y, Z\rangle$ *for a modal logic with a modality* is such that:

- $C_0 = \{\mathbf{t}, \mathbf{f}\} \cup \{\mathbf{p}_i : i \in \mathbb{N}\}$;
- $C_1 = \{\neg, \Box\}$;
- $C_2 = \{\wedge, \vee, \supset\}$;
- $C_k = \varnothing$ for $k \geq 3$;
- $O_0 = \{\top, \bot\}$;
- $O_1 = \{\mathbf{I}, \mathbf{N}\}$;
- $O_2 = \{\mathbf{lb}\}$;
- $O_k = \varnothing$ for $k \geq 3$;
- X, Y and Z are disjoint countable sets of variables. ∎

The formula constructors are the usual ones in modal logic. Operators are used namely to reason about the accessibility relation and to relate terms. In the case of the signature presented, $\mathbf{N}$ is the neighbor operator representing the collection of neighbors according to the accessibility relation associated to the denotation of a term; $\mathbf{lb}(t_1, t_2)$ represents a lower bound of the terms t_1 and t_2; and $\mathbf{I}(t)$ represents an atomic truth value contained in the denotation of the term t.

Definition 2.3 The signature Σ_P for propositional logic is Σ_M without the connective $\Box$ and the operator $\mathbf{N}$. ∎

The next three definitions show how to construct formulas, terms and assertions. Formulas and terms follow the same construction process, as expected. For that, assume given the following three sets $\{\xi_i : i \in \mathbb{N}\}$, $\{\tau_i : i \in \mathbb{N}\}$ and $\{\Gamma_i : i \in \mathbb{N}\}$, whose elements are called meta-variables of formulas, terms and bags of assertions, respectively.

Definition 2.4 The set $F(\Sigma)$ of *(schema) simple formulas* over Σ is inductively defined as follows: (i) $\xi_i \in F(\Sigma)$ for every $i \in \mathbb{N}$; (ii) $\mathbf{z} \in F(\Sigma)$ for every $\mathbf{z} \in Z$; and (iii) $c(\varphi_1, ..., \varphi_k) \in F(\Sigma)$ whenever $c \in C_k$ and $\varphi_1, ..., \varphi_k \in F(\Sigma)$. The set $gF(\Sigma)$ of *ground simple formulas* is composed of the elements in $F(\Sigma)$ without meta-variables and $cF(\Sigma)$, the *closed simple formulas*, is the set of elements of $gF(\Sigma)$ without variables. ∎

For instance, $\xi_1 \vee \mathbf{z}_4$, $\Box\xi_2 \supset \xi_1$, $\neg\mathbf{p}_2 \wedge \mathbf{z}_1$ are some formulas that can be constructed using the signature of Definition 2.2.

As mentioned before, terms are constructed similarly to formulas. For the labelled logic systems considered in this work, terms are seen as labels: they represent truth values.

Definition 2.5 The set $T(\Sigma)$ of *(schema) terms* over Σ is inductively defined as follows: (i) $\tau_i \in T(\Sigma)$ for every $i \in \mathbb{N}$; (ii) $\mathbf{x} \in T(\Sigma)$ for every $\mathbf{x} \in X$; (iii) $\mathbf{y} \in T(\Sigma)$ for every $\mathbf{y} \in Y$;

(iv) $o(\theta_1,...,\theta_k)\in T(\Sigma)$ whenever $o\in O_k$ and $\theta_1,...,\theta_k\in T(\Sigma)$; and (v) $\#\varphi\in T(\Sigma)$ whenever $\varphi\in F(\Sigma)$. The set $gT(\Sigma)$ of *ground terms* is composed of the elements in $T(\Sigma)$ without meta-variables and $cT(\Sigma)$, the *closed terms*, is the set of elements of $gT(\Sigma)$ without variables. ■

For instance, τ_3, $\mathbf{lb}(\mathbf{y}_1,\tau_1)$, $\#\xi_1$ are some terms that can be constructed in the context of the signature of Definition 2.2. The intended purpose of $\#\varphi$ is to represent syntactically the truth value associated to the formula φ.

The reasoning in the logic systems considered herein is based on assertions. There are basically two types of assertions: about terms and their relationship and about formulas and their relationship with terms.

Definition 2.6 The set $A(\Sigma)$ of *(schema) assertions* over Σ is composed of expressions of the following six forms: (i) $\Omega\theta$ and $\mho\theta$ (positive and negative truth value indivisibility assertion, respectively) with $\theta\in T(\Sigma)$; (ii) $\theta\sqsubseteq\theta'$ and $\theta\not\sqsubseteq\theta'$ (positive and negative truth value comparison assertion, respectively) with $\theta,\theta'\in T(\Sigma)$; and (iii) $\theta\leq\varphi$ and $\theta\not\leq\varphi$ (positive and negative labelled formula, respectively) with $\theta\in T(\Sigma)$ and $\varphi\in F(\Sigma)$. The set $gA(\Sigma)$ of *ground assertions* is composed of the elements in $A(\Sigma)$ without meta-variables and $cA(\Sigma)$, the *closed assertions*, is the set of elements of $gA(\Sigma)$ without variables. ■

It is interesting to note at this point that the semantics of terms is given by algebras of truth values which, when induced by Kripke structures, may be such that the denotation of a term is a set of worlds. In this case this approach assigns a *range* of possible worlds to a formula instead of a single world (as is commonly used in labelled modal systems). The intended meaning of $\Omega\theta$ is to express that the truth value is atomic (there is no truth value strictly smaller than it besides falsum) and $\mho\theta$ is its conjugate. The notion of *conjugate* $\bar{\delta}$ is defined as follows: (i) $\overline{\Omega\theta}$ is $\mho\theta$; (ii) $\overline{\mho\theta}$ is $\Omega\theta$; (iii) $\overline{\theta\sqsubseteq\theta'}$ is $\theta\not\sqsubseteq\theta'$; (iv) $\overline{\theta\not\sqsubseteq\theta'}$ is $\theta\sqsubseteq\theta'$; (v) $\overline{\theta\leq\varphi}$ is $\theta\not\leq\varphi$; and (vi) $\overline{\theta\not\leq\varphi}$ is $\theta\leq\varphi$.

Some assertions that can be constructed using the signature of Definition 2.2 are $\Omega\mathbf{y}_1$ (the variable $\mathbf{y}_1$ represents an atomic truth value), $\bot\sqsubseteq\mathbf{y}_2$ (falsum is less or equal to the truth value associated to the variable $\mathbf{y}_2$), $\top\not\leq\xi_1\wedge\xi_2$ (the formula $\xi_1\wedge\xi_2$ is not associated to the truth value true).

In the following, $\mathcal{B}_f(U)$ denotes the set of all finite bags of elements in the set U.

Substitutions are maps over meta-variables as is described in the definition below. They are used in derivations to assign concrete values to meta-variables.

Definition 2.7 A *(schema) substitution* over Σ is a map σ such that, for all $i\in\mathbb{N}$: (i) $\sigma(\xi_i)\in F(\Sigma)$; (ii) $\sigma(\tau_i)\in T(\Sigma)$; and (iii) $\sigma(\Gamma_i)\in\mathcal{B}_f(A(\Sigma)\cup\{\Gamma_i : i\in\mathbb{N}\})$. The set of (schema) substitutions over Σ is denoted by $Sbs(\Sigma)$. A *ground substitution* over Σ is a schema substitution ρ such that, for all $i\in\mathbb{N}$: (i) $\rho(\xi_i)\in gF(\Sigma)$; (ii) $\rho(\tau_i)\in gT(\Sigma)$; and (iii) $\rho(\Gamma_i)\in\mathcal{B}_f(gA(\Sigma))$. The set of ground substitutions over Σ is denoted by $gSbs(\Sigma)$. ■

2.2 Calculi

This subsection starts by introducing the notion of sequent.

Definition 2.8 A *sequent* over a signature Σ is a pair $s=\langle\Delta_1,\Delta_2\rangle$, written $\Delta_1\rightarrow\Delta_2$, where $\Delta_1,\Delta_2\in\mathcal{B}_f(A(\Sigma)\cup\{\Gamma_i : i\in\mathbb{N}\})$. ■

TABLE 1. Specific rules for connectives

Lf	$\dfrac{\tau_1 \sqsubseteq \bot, \Gamma_1 \to \Gamma_2}{\tau_1 \leq \mathbf{f}, \Gamma_1 \to \Gamma_2}$	**Rf**	$\dfrac{\Gamma_1 \to \Gamma_2, \tau_1 \sqsubseteq \bot}{\Gamma_1 \to \Gamma_2, \tau_1 \leq \mathbf{f}}$
Lt	$\dfrac{\tau_1 \sqsubseteq \top, \Gamma_1 \to \Gamma_2}{\tau_1 \leq \mathbf{t}, \Gamma_1 \to \Gamma_2}$	**Rt**	$\dfrac{\Gamma_1 \to \Gamma_2, \tau_1 \sqsubseteq \top}{\Gamma_1 \to \Gamma_2, \tau_1 \leq \mathbf{t}}$
L$\wedge$	$\dfrac{\tau_1 \leq \xi_1, \tau_1 \leq \xi_2, \Gamma_1 \to \Gamma_2}{\tau_1 \leq (\xi_1 \wedge \xi_2), \Gamma_1 \to \Gamma_2}$	R$\wedge$	$\dfrac{\Gamma_1 \to \Gamma_2, \tau_1 \leq \xi_1 \quad \Gamma_1 \to \Gamma_2, \tau_1 \leq \xi_2}{\Gamma_1 \to \Gamma_2, \tau_1 \leq (\xi_1 \wedge \xi_2)}$
L$\neg$	$\dfrac{\Omega\tau_1, \Gamma_1 \to \Gamma_2, \tau_1 \leq \xi_1}{\Omega\tau_1, \tau_1 \leq (\neg\xi_1), \Gamma_1 \to \Gamma_2}$	R$\neg$	$\dfrac{\Omega\tau_1, \tau_1 \leq \xi_1, \Gamma_1 \to \Gamma_2}{\Omega\tau_1, \Gamma_1 \to \Gamma_2, \tau_1 \leq (\neg\xi_1)}$
L$\supset$	$\dfrac{\Omega\tau_1, \Gamma_1 \to \Gamma_2, \tau_1 \leq \xi_1 \quad \Omega\tau_1, \tau_1 \leq \xi_2, \Gamma_1 \to \Gamma_2}{\Omega\tau_1, \tau_1 \leq (\xi_1 \supset \xi_2), \Gamma_1 \to \Gamma_2}$	R$\supset$	$\dfrac{\Omega\tau_1, \tau_1 \leq \xi_1, \Gamma_1 \to \Gamma_2, \tau_1 \leq \xi_2}{\Omega\tau_1, \Gamma_1 \to \Gamma_2, \tau_1 \leq (\xi_1 \supset \xi_2)}$
L$\vee$	$\dfrac{\Omega\tau_1, \tau_1 \leq \xi_1, \Gamma_1 \to \Gamma_2 \quad \Omega\tau_1, \tau_1 \leq \xi_2, \Gamma_1 \to \Gamma_2}{\Omega\tau_1, \tau_1 \leq (\xi_1 \vee \xi_2), \Gamma_1 \to \Gamma_2}$	R$\vee$	$\dfrac{\Omega\tau_1, \Gamma_1 \to \Gamma_2, \tau_1 \leq \xi_1, \tau_1 \leq \xi_2}{\Omega\tau_1, \Gamma_1 \to \Gamma_2, \tau_1 \leq (\xi_1 \vee \xi_2)}$
L$\Box$	$\dfrac{\mathbf{N}(\tau_1) \leq \xi_1, \Gamma_1 \to \Gamma_2}{\tau_1 \leq (\Box\xi_1), \Gamma_1 \to \Gamma_2}$	R$\Box$	$\dfrac{\Gamma_1 \to \Gamma_2, \mathbf{N}(\tau_1) \leq \xi_1}{\Gamma_1 \to \Gamma_2, \tau_1 \leq (\Box\xi_1)}$

It is common the application of rules in derivations to be restricted by constraints. Here, constraints are presented in the form of provisos, that is, restrictions on substitutions.

Definition 2.9 A *(local) proviso* over Σ is a map $\pi : gSbs(\Sigma) \to \{0,1\}$. The unit proviso **up** is as follows: $\mathbf{up}(\rho) = 1$ for every $\rho \in gSbs(\Sigma)$. The zero proviso **zp** is as follows: $\mathbf{zp}(\rho) = 0$ for every $\rho \in gSbs(\Sigma)$. ∎

Given two provisos π and π', their intersection is the proviso $(\pi \cap \pi')$ such that $(\pi \cap \pi')(\rho) = \pi(\rho) \times \pi'(\rho)$. The expression $\pi \subseteq \pi'$ denotes that $\pi(\rho) \leq \pi'(\rho)$ for each ground substitution ρ.

In this work the following two provisos are needed:

- $(\tau_k : \mathbf{y})(\rho) = 1$ iff $\rho(\tau_k) \in Y$, and
- $(\tau_k \notin \Delta)(\rho)$ iff $\rho(\tau_k)$ does not occur in $\Delta\rho$.

The first proviso only allows ground substitutions where τ_k is replaced by a truth value bound variable. The other proviso accepts a ground substitution ρ only if τ_k is replaced by a ground term not occurring in $\Delta\rho$.

Definition 2.10 A *rule* over a signature Σ is a triple $r = \langle \{s_1, \ldots, s_p\}, s, \pi \rangle$ written $\dfrac{s_1 \quad \ldots \quad s_p}{s} \lhd \pi$ where $s_1, \ldots, s_p, s$ are sequents over Σ and π is a proviso over Σ. ∎

For (normal) modal logics several kinds of rules are needed: rules for connectives, rules for operators, structural rules, etc. For the sake of illustration see Table 1 for some specific rules for connectives. For instance in the rules for $\Box$ it can be seen the intrinsic relation between the **N** operator and the modality.

Table 2 shows some specific rules for the operators of a (normal) labelled modal logic. For example the rules for **N** state that the neighborhood of a truth value is induced by the neighborhoods of the atomic truth values contained in it. Rules **I** and Ω**I** impose that $\mathbf{I}(t)$ represents an atomic truth value contained in the denotation of t, as long as t is not $\bot$. Rules for **lb** establish that $\mathbf{lb}(t_1, t_2)$ is a lower bound of t_1 and t_2.

The unit proviso **up** is usually omitted. A rule is said to be *endowed with a persistent proviso* if its proviso does not change value when the context of the rule is enriched with closed assertions. Examples of rules with this kind of provisos are given below. The "fresh bound variable" proviso, with the form $\tau_2 : \mathbf{y}; \tau_2 \notin \tau_1, \Gamma_1, \Gamma_2$, is persistent in every rule. Indeed,

TABLE 2. Specific rules for operators

$$\mathrm{L}\mathbf{N}\Omega \quad \frac{\tau_3 \sqsubseteq \tau_1, \Omega\tau_3, \Omega\tau_2, \Gamma_1 \to \Gamma_2, \tau_2 \not\sqsubseteq \mathbf{N}(\tau_3)}{\Omega\tau_2, \Gamma_1 \to \Gamma_2, \tau_2 \not\sqsubseteq \mathbf{N}(\tau_1)} \lhd \tau_3 : \mathbf{y}, \tau_3 \notin \tau_1, \tau_2, \Gamma_1, \Gamma_2$$

$$\mathrm{R}\mathbf{N}\Omega \quad \frac{\Omega\tau_2, \Gamma_1 \to \Gamma_2, \Omega\tau_3 \quad \Omega\tau_3, \Omega\tau_2, \Gamma_1 \to \Gamma_2, \tau_3 \sqsubseteq \tau_1 \quad \Omega\tau_3, \Omega\tau_2, \Gamma_1 \to \Gamma_2, \tau_2 \sqsubseteq \mathbf{N}(\tau_3)}{\Omega\tau_2, \Gamma_1 \to \Gamma_2, \tau_2 \sqsubseteq \mathbf{N}(\tau_1)}$$

$$\mathbf{I} \quad \frac{}{\Gamma_1 \to \Gamma_2, \mathbf{I}(\tau_1) \sqsubseteq \tau_1} \qquad \Omega\mathbf{I} \quad \frac{\Gamma_1 \to \Gamma_2, \tau_1 \not\sqsubseteq \bot}{\Gamma_1 \to \Gamma_2, \Omega\mathbf{I}(\tau_1)}$$

$$\mathbf{lb}1 \quad \frac{}{\Gamma_1 \to \Gamma_2, \mathbf{lb}(\tau_1, \tau_2) \sqsubseteq \tau_1} \qquad \mathbf{lb}2 \quad \frac{}{\Gamma_1 \to \Gamma_2, \mathbf{lb}(\tau_1, \tau_2) \sqsubseteq \tau_2}$$

for every ground substitution ρ, it holds $(\tau_2 : \mathbf{y} \cap \tau_2 \notin \tau_1, \Gamma_1, \Gamma_2)(\rho) = (\tau_2 : \mathbf{y} \cap \tau_2 \notin \tau_1, \Gamma_1, \Gamma_2, \delta)(\rho)$ as long as δ is a closed expression. Clearly, every rule endowed with the unit proviso **up** is endowed with a persistent proviso.

Definition 2.11 A *(sequent) calculus* is a pair $\mathcal{C} = \langle \Sigma, \mathcal{R} \rangle$ where Σ is a signature and $\mathcal{R}$ is a finite set of rules over Σ. ■

The next notion of this subsection is the notion of derivation. It is a sequence of steps where each step is either an axiom or a hypothesis or results from the application of some rule.

Definition 2.12 Given a sequent calculus $\mathcal{C}$, a sequent s' is said to be *derived* from a set S of sequents with proviso π, written $S \vdash s' \lhd \pi$, if there is a *derivation sequence* $(d_1, \pi_1) \ldots, (d_n, \pi_n)$ such that: d_1 is s' and $\pi \subseteq \pi_1$; and for every $i = 1, \ldots, n$:

1. either $d_i \in S$ and π_i is **up**;
2. or there is an assertion that occurs in both sides of d_i and π_i is **up**;
3. or there are $r \in \mathcal{R}, \rho \in Sbs(\Sigma), p \in \mathbb{N}$ and $i_1, \ldots, i_p \in \{i+1, \ldots, n\}$ such that $r\rho = \dfrac{d_{i_1} \quad \ldots \quad d_{i_p}}{d_i} \lhd \pi'$ and $\pi_i = \pi' \cap \pi_{i_1} \cap \ldots \cap \pi_{i_p}$. ■

The use of truth value as labels permits to distinguish between local and global reasoning. A formula φ is globally derived from $\psi_1, \ldots, \psi_k$ provided that the sequent $\top \leq \psi_1, \ldots, \top \leq \psi_k \to \top \leq \varphi$ is derived. On the other hand, φ is locally derived from $\psi_1, \ldots, \psi_k$ provided that for any atomic element $\mathbf{y}_1$, $\Omega\mathbf{y}_1, \mathbf{y}_1 \leq \psi_1, \ldots, \mathbf{y}_1 \leq \psi_k \to \mathbf{y}_1 \leq \varphi$ is derived. Notice that in the proof of a local or a global consequence the derivation does not use hypothesis.

Definition 2.13 In a sequent calculus $\langle \Sigma, \mathcal{R} \rangle$ a formula φ is *globally derived* from formulas $\psi_1, \ldots, \psi_k$, denoted by $\psi_1, \ldots, \psi_k \vdash^g_{\mathcal{R}} \varphi$ iff $\vdash_{\mathcal{R}} \top \leq \psi_1, \ldots, \top \leq \psi_k \to \top \leq \varphi$. A formula φ is *locally derived* from formulas $\psi_1, \ldots, \psi_k$, denoted by $\psi_1, \ldots, \psi_k \vdash^l_{\mathcal{R}} \varphi$ iff $\vdash_{\mathcal{R}} \Omega\mathbf{y}_1, \mathbf{y}_1 \leq \psi_1, \ldots, \mathbf{y}_1 \leq \psi_k \to \mathbf{y}_1 \leq \varphi$.

2.3 Semantics

Semantics is based on a two-sorted algebra: a sort for truth values and another for denotations of formulas. In [16] it is shown that it is possible to move between general Kripke structures and these algebras (one of these ways is illustrated in Example 2.25).

Definition 2.14 Let Σ be a signature. A Σ-*algebra* is a triple $\mathbb{A} = \langle F, T, \cdot_{\mathbb{A}} \rangle$ where F and T are sets and $\cdot_{\mathbb{A}}$ is a map such that:

- $c_{\mathbb{A}} : F^k \to F$ for each $c \in C_k$;

- $o_{\mathbb{A}} : T^k \to T$ for each $o \in O_k$,
- $\#_{\mathbb{A}} : F \to T$;
- $\Omega_{\mathbb{A}} \subseteq T$;
- $\sqsubseteq_{\mathbb{A}} \subseteq T \times T$;
- $\leq_{\mathbb{A}} \subseteq T \times F$. ■

Given a Σ-algebra $\mathbb{A} = \langle F, T, \cdot_{\mathbb{A}} \rangle$, the set $\{\langle F, T \rangle\}$ is denoted by $V(\mathbb{A})$, and whenever $\mathcal{A}$ is a class of Σ-algebras, $V_{\mathcal{A}}$ denotes the set $\cup_{\mathbb{A} \in \mathcal{A}} V(\mathbb{A})$.

The notion of reduct of an algebra by a signature will be useful along the work, specially when defining fusion algebraically.

Definition 2.15 Given a Σ-algebra $\mathbb{A} = \langle F, T, \cdot_{\mathbb{A}} \rangle$ and a signature Σ' such that $\Sigma' \subseteq \Sigma$, the *reduct of* $\mathbb{A}$ *by* Σ' is the Σ'-algebra $\mathbb{A}|_{\Sigma'} = \langle F, T, \cdot_{\mathbb{A}|_{\Sigma'}} \rangle$ such that:

- $c_{\mathbb{A}|_{\Sigma'}} = c_{\mathbb{A}}$ *for all* $c \in \Sigma'$,
- $o_{\mathbb{A}|_{\Sigma'}} = o_{\mathbb{A}}$ *for all* $o \in \Sigma'$,
- $\Omega_{\mathbb{A}|_{\Sigma'}} = \Omega_{\mathbb{A}}$, $\sqsubseteq_{\mathbb{A}|_{\Sigma'}} = \sqsubseteq_{\mathbb{A}}$, *and* $\leq_{\mathbb{A}|_{\Sigma'}} = \leq_{\mathbb{A}}$. ■

In order to define the denotation of formulas and terms, the notion of assignment over a Σ-algebra is needed. Basically it specifies the meaning of the variables in the Σ-algebra.

Definition 2.16 An *unbound variable assignment* over $\mathbb{A}$ is a function α that maps each element of X to an element of T and each element of Z to an element of F. A *bound variable assignment* over $\mathbb{A}$ is a map β from Y to T. ■

The definition of denotation is made in two parts: the denotation of formulas (where only unbound variable assignments may be used) and the denotation of terms (where both kinds of assignments are needed).

Definition 2.17 The *denotation* of ground simple formulas at a Σ-algebra $\mathbb{A}$ for an unbound variable assignment α, is inductively defined in the following way:

- $[\![\mathbf{z}]\!]_{\mathbb{A}\alpha} = \alpha(z)$;
- $[\![c(\varphi_1, ..., \varphi_k)]\!]_{\mathbb{A}\alpha} = c_{\mathbb{A}}([\![\varphi_1]\!]_{\mathbb{A}\alpha}, ..., [\![\varphi_k]\!]_{\mathbb{A}\alpha})$.

The *denotation* at $\mathbb{A}$ for assignments α, β over $\mathbb{A}$ of ground terms is inductively defined in the following way:

- $[\![\mathbf{x}]\!]_{\mathbb{A}\alpha\beta} = \alpha(x)$;
- $[\![\mathbf{y}]\!]_{\mathbb{A}\alpha\beta} = \beta(y)$;
- $[\![o(\theta_1, ..., \theta_k)]\!]_{\mathbb{A}\alpha\beta} = o_{\mathbb{A}}([\![\theta_1]\!]_{\mathbb{A}\alpha\beta}, ..., [\![\theta_k]\!]_{\mathbb{A}\alpha\beta})$;
- $[\![\#\varphi]\!]_{\mathbb{A}\alpha\beta} = \#_{\mathbb{A}}([\![\varphi]\!]_{\mathbb{A}\alpha})$. ■

The definition of satisfaction of assertions and sequents can be given capitalizing on the denotation of terms and formulas presented above.

Definition 2.18 The *satisfaction* by $\mathbb{A}$ for α and β of ground assertions and sequents is defined as follows:

- $\mathbb{A}\alpha\beta \Vdash \Omega\theta$ iff $[\![\theta]\!]_{\mathbb{A}\alpha\beta} \in \Omega_{\mathbb{A}}$;
- $\mathbb{A}\alpha\beta \Vdash \mho\theta$ iff $[\![\theta]\!]_{\mathbb{A}\alpha\beta} \notin \Omega_{\mathbb{A}}$;
- $\mathbb{A}\alpha\beta \Vdash \theta \sqsubseteq \theta'$ iff $\langle [\![\theta]\!]_{\mathbb{A}\alpha\beta}, [\![\theta']\!]_{\mathbb{A}\alpha\beta} \rangle \in \sqsubseteq_{\mathbb{A}}$;
- $\mathbb{A}\alpha\beta \Vdash \theta \not\sqsubseteq \theta'$ iff $\langle [\![\theta]\!]_{\mathbb{A}\alpha\beta}, [\![\theta']\!]_{\mathbb{A}\alpha\beta} \rangle \notin \sqsubseteq_{\mathbb{A}}$;

- $\mathbb{A}\alpha\beta \Vdash \theta \leq \varphi$ iff $\langle [\![\theta]\!]_{\mathbb{A}\alpha\beta}, [\![\varphi]\!]_{\mathbb{A}\alpha} \rangle \in \leq_{\mathbb{A}}$;
- $\mathbb{A}\alpha\beta \Vdash \theta \nleq \varphi$ iff $\langle [\![\theta]\!]_{\mathbb{A}\alpha\beta}, [\![\varphi]\!]_{\mathbb{A}\alpha} \rangle \notin \leq_{\mathbb{A}}$;
- $\mathbb{A}\alpha\beta \Vdash \Delta' \to \Delta''$ iff $\mathbb{A}\alpha\beta \Vdash \delta$ for some $\delta \in \Delta'' \cup \overline{\Delta'}$. ■

Definition 2.19 A Σ-algebra $\mathbb{A}$ and an unbound variable assignment α over it *satisfies an ground formula* φ, denoted by $\mathbb{A}\alpha \Vdash \varphi$ whenever $\mathbb{A}\alpha\beta \Vdash \varphi$ for every bound variable assignment β over $\mathbb{A}$.

Entailment is now defined over a class of algebras.

Definition 2.20 Given a class $\mathcal{A}$ of Σ-algebras, a ground sequent s is $\mathcal{A}$-*entailed* by the ground sequents $s_1, \ldots, s_p$, written $s_1, \ldots, s_p \vDash_{\mathcal{A}} s$, iff, for each $\mathbb{A} \in \mathcal{A}$ and unbound variable assignment α over $\mathbb{A}$, $\mathbb{A}\alpha \Vdash s$ whenever $\mathbb{A}\alpha \Vdash s_i$ for every bound variable assignment β over $\mathbb{A}$ and every $i = 1, \ldots, p$. ■

The notion of entailment is easily extended to (schema) sequents possibly with provisos. A sequent s is $\mathcal{A}$-entailed by the sequents $s_1, \ldots, s_p$ with proviso π, written $s_1, \ldots, s_p \vDash_{\mathcal{A}} s \lhd \pi$, iff $s_1\rho, \ldots, s_p\rho \vDash_{\mathcal{A}} s\rho$ for every ground substitution ρ over Σ such that $\pi(\rho) = 1$.

Definition 2.21 A class of algebras $\mathcal{A}$ is called *appropriate* for the rule $r = \langle \{s_1, \ldots, s_p\}, s, \pi \rangle$ if $s_1, \ldots, s_p \vDash_{\mathcal{A}} s \lhd \pi$. And an algebra $\mathbb{A}$ is said to be appropriate for a rule if so is the class $\{\mathbb{A}\}$. The class of algebras is *full* for a calculus $\langle \Sigma, \mathcal{R} \rangle$ if it is the class of all Σ-algebras that are appropriate for all rules in $\mathcal{R}$. ■

2.4 Logic systems

The definition of logic system is now presented. A logic system puts together the different components of a logic: the syntax, the deductive system and the semantics.

Definition 2.22 A *logic system* is a triple $\langle \Sigma, \mathcal{R}, \mathcal{A} \rangle$ where $\langle \Sigma, \mathcal{R} \rangle$ is a sequent calculus and $\mathcal{A}$ is a class of Σ-algebras. ■

A logic system is said to be:

- *sound* if $\mathcal{A}$ is appropriate for each rule in $\mathcal{R}$;
- *full* if $\mathcal{A}$ is the class of all Σ-algebras that are appropriate for each rule in $\mathcal{R}$;
- *complete* if $s_1, \ldots, s_p \vdash_{\mathcal{R}} s$ whenever $s_1, \ldots, s_p \vDash_{\mathcal{A}} s$ for any closed sequents $s, s_1, \ldots, s_p$.

2.5 Applications and examples

To finish this section the notions developed in the preceding subsections are illustrated. The subsection starts by defining what is a labelled calculus for a normal modal logic, and then passes through examples of derivation and satisfaction until presenting examples of modal logic systems for deontic logic, knowledge logic and Löb provability logic.

The labelled sequent calculi developed herein for modal logic is composed by three kinds of rules: structural rules, rules about order (order rules) and rules for constructors.

The structural rules are composed of weakening and contraction rules, as presented in Table 3; conjugation rules, as presented in Table 4; and two cut rules (presented in Table 5): one for assertions with $\sqsubseteq$ and the other for assertions with $\leq$.

TABLE 3. Weakening and contraction rules

LwΩ	$\dfrac{\Gamma_1 \to \Gamma_2}{\Omega\tau_1, \Gamma_1 \to \Gamma_2}$	RwΩ	$\dfrac{\Gamma_1 \to \Gamma_2}{\Gamma_1 \to \Gamma_2, \Omega\tau_1}$
LwT	$\dfrac{\Gamma_1 \to \Gamma_2}{\tau_1 \sqsubseteq \tau_2, \Gamma_1 \to \Gamma_2}$	RwT	$\dfrac{\Gamma_1 \to \Gamma_2}{\Gamma_1 \to \Gamma_2, \tau_1 \sqsubseteq \tau_2}$
LwF	$\dfrac{\Gamma_1 \to \Gamma_2}{\tau_1 \leq \xi_1, \Gamma_1 \to \Gamma_2}$	RwF	$\dfrac{\Gamma_1 \to \Gamma_2}{\Gamma_1 \to \Gamma_2, \tau_1 \leq \xi_1}$
LcT	$\dfrac{\tau_1 \sqsubseteq \tau_2, \tau_1 \sqsubseteq \tau_2, \Gamma_1 \to \Gamma_2}{\tau_1 \sqsubseteq \tau_2, \Gamma_1 \to \Gamma_2}$	RcT	$\dfrac{\Gamma_1 \to \Gamma_2, \tau_1 \sqsubseteq \tau_2, \tau_1 \sqsubseteq \tau_2}{\Gamma_1 \to \Gamma_2, \tau_1 \sqsubseteq \tau_2}$
LcF	$\dfrac{\tau_1 \leq \xi_1, \tau_1 \leq \xi_1, \Gamma_1 \to \Gamma_2}{\tau_1 \leq \xi_1, \Gamma_1 \to \Gamma_2}$	RcF	$\dfrac{\Gamma_1 \to \Gamma_2, \tau_1 \leq \xi_1, \tau_1 \leq \xi_1}{\Gamma_1 \to \Gamma_2, \tau_1 \leq \xi_1}$

TABLE 4. Conjugation rules

LxiΩ	$\dfrac{\Gamma_1 \to \Gamma_2, \Omega\tau_1}{\mho\tau_1, \Gamma_1 \to \Gamma_2}$	RxiΩ	$\dfrac{\Omega\tau_1, \Gamma_1 \to \Gamma_2}{\Gamma_1 \to \Gamma_2, \mho\tau_1}$
LxiT	$\dfrac{\Gamma_1 \to \Gamma_2 \tau_1 \sqsubseteq \tau_2}{\tau_1 \not\sqsubseteq \tau_2, \Gamma_1 \to \Gamma_2}$	RxiT	$\dfrac{\tau_1 \sqsubseteq \tau_2, \Gamma_1 \to \Gamma_2}{\Gamma_1 \to \Gamma_2, \tau_1 \not\sqsubseteq \tau_2}$
LxiF	$\dfrac{\Gamma_1 \to \Gamma_2, \tau_1 \leq \xi_1}{\tau_1 \not\leq \xi_1, \Gamma_1 \to \Gamma_2}$	RxiF	$\dfrac{\tau_1 \leq \xi_1, \Gamma_1 \to \Gamma_2}{\Gamma_1 \to \Gamma_2, \tau_1 \not\leq \xi_1}$
LxeΩ	$\dfrac{\Gamma_1 \to \Gamma_2, \mho\tau_1}{\Omega\tau_1, \Gamma_1 \to \Gamma_2}$	RxeΩ	$\dfrac{\mho\tau_1, \Gamma_1 \to \Gamma_2}{\Gamma_1 \to \Gamma_2, \Omega\tau_1}$
LxeT	$\dfrac{\Gamma_1 \to \Gamma_2, \tau_1 \not\sqsubseteq \tau_2}{\tau_1 \sqsubseteq \tau_2, \Gamma_1 \to \Gamma_2}$	RxeT	$\dfrac{\tau_1 \not\sqsubseteq \tau_2, \Gamma_1 \to \Gamma_2}{\Gamma_1 \to \Gamma_2, \tau_1 \sqsubseteq \tau_2}$
LxeF	$\dfrac{\Gamma_1 \to \Gamma_2, \tau_1 \not\leq \xi_1}{\tau_1 \leq \xi_1, \Gamma_1 \to \Gamma_2}$	RxeF	$\dfrac{\tau_1 \not\leq \xi_1, \Gamma_1 \to \Gamma_2}{\Gamma_1 \to \Gamma_2, \tau_1 \leq \xi_1}$

TABLE 5. Cut rules

cutT	$\dfrac{\Gamma_1 \to \Gamma_2 \tau_1 \sqsubseteq \tau_2 \quad \tau_1 \sqsubseteq \tau_2, \Gamma_1 \to \Gamma_2}{\Gamma_1 \to \Gamma_2}$	cutF	$\dfrac{\Gamma_1 \to \Gamma_2, \tau_1 \leq \xi_1 \quad \tau_1 \leq \xi_1, \Gamma_1 \to \Gamma_2}{\Gamma_1 \to \Gamma_2}$

TABLE 6. Order rules for basic assertions

$$\Omega\top \quad \dfrac{\Omega\tau_1, \Gamma_1 \to \Gamma_2, \tau_1 \leq \xi_1}{\Gamma_1 \to \Gamma_2, \top \leq \xi_1} \; \lhd \tau_1 : \mathbf{y}_1, \tau_1 \notin \Gamma_1, \Gamma_2$$

$$\Omega \quad \dfrac{\Gamma_1 \to \Gamma_2, \tau_1 \not\sqsubseteq \bot \quad \Omega\tau_2, \tau_2 \sqsubseteq \tau_1, \Gamma_1 \to \Gamma_2, \tau_1 \sqsubseteq \tau_2}{\Gamma_1 \to \Gamma_2, \Omega\tau_1} \; \lhd \tau_2 : \mathbf{y}, \tau_2 \notin \tau_1, \Gamma_1, \Gamma_2$$

L#	$\dfrac{\tau_1 \leq \xi_1, \Gamma_1 \to \Gamma_2}{\tau_1 \sqsubseteq \#\xi_1, \Gamma_1 \to \Gamma_2}$	R#	$\dfrac{\Gamma_1 \to \Gamma_2, \tau_1 \leq \xi_1}{\Gamma_1 \to \Gamma_2, \tau_1 \sqsubseteq \#\xi_1}$
⊥T	$\dfrac{}{\Gamma_1 \to \Gamma_2, \bot \sqsubseteq \tau_1}$	⊥F	$\dfrac{}{\Gamma_1 \to \Gamma_2, \bot \leq \xi_1}$
Ω⊥	$\dfrac{\Gamma_1 \to \Gamma_2, \Omega\tau_1}{\Gamma_1 \to \Gamma_2, \tau \not\sqsubseteq \bot}$	⊤	$\dfrac{}{\Gamma_1 \to \Gamma_2, \tau_1 \sqsubseteq \top}$

The intuition behind the rules in Table 6 for basic assertions should be obvious. It is interesting to notice how it is expressed by a rule that the meaning of Ωt should be an atomic truth value.

Rules about the relation between the truth values are also needed. It is easy to understand their motivation when considering the case of truth values being subsets of worlds in a Kripke structure. These rules are presented in Table 7.

Finally, order rules for $\sqsubseteq$ and $\leq$ are presented in Table 8. It is worthwhile to explain the rules RgenF and LgenF. The rule RgenF indicates that if the denotation of τ_2 is less than or equal to the denotation of ξ_1 for all atomic truth values assigned to τ_2 included in the denotation of τ_1, then the denotation of τ_1 is less than or equal to the denotation of ξ_1.

TABLE 7. Order rules about truth values

cons	$\dfrac{}{\Gamma_1\to\Gamma_2,\top\not\sqsubseteq\bot}$	ref	$\dfrac{}{\Gamma_1\to\Gamma_2,\tau_1\sqsubseteq\tau_1}$
transT	$\dfrac{\Gamma_1\to\Gamma_2,\tau_1\sqsubseteq\tau_2\quad\Gamma_1\to\Gamma_2,\tau_2\sqsubseteq\tau_3}{\Gamma_1\to\Gamma_2,\tau_1\sqsubseteq\tau_3}$	transF	$\dfrac{\Gamma_1\to\Gamma_2,\tau_1\sqsubseteq\tau_2\quad\Gamma_1\to\Gamma_2,\tau_2\leq\xi_1}{\Gamma_1\to\Gamma_2,\tau_1\leq\xi_1}$
Lasym	$\dfrac{\Omega\tau_1,\Omega\tau_2,\tau_1\sqsubseteq\tau_2,\Gamma_1\to\Gamma_2}{\Omega\tau_1,\Omega\tau_2,\tau_2\sqsubseteq\tau_1,\Gamma_1\to\Gamma_2}$	Rasym	$\dfrac{\Omega\tau_1,\Omega\tau_2,\Gamma_1\to\Gamma_2,\tau_1\sqsubseteq\tau_1}{\Omega\tau_1,\Omega\tau_2,\Gamma_1\to\Gamma_2,\tau_2\sqsubseteq\tau_1}$

TABLE 8. Order rules

LgenT	$\dfrac{\Omega\tau_2,\tau_2\sqsubseteq\tau_3,\Gamma_1\to\Gamma_2\quad\Omega\tau_2,\Gamma_1\to\Gamma_2,\tau_2\sqsubseteq\tau_1\quad\tau_1\sqsubseteq\tau_3,\Gamma_1\to\Gamma_2,\Omega\tau_2}{\tau_1\sqsubseteq\tau_3,\Gamma_1\to\Gamma_2}$
RgenT	$\dfrac{\Omega\tau_2,\tau_2\sqsubseteq\tau_1,\Gamma_1\to\Gamma_2,\tau_2\sqsubseteq\tau_3}{\Gamma_1\to\Gamma_2,\tau_1\sqsubseteq\tau_3}\ \lhd\tau_2:\mathbf{y},\tau_2\notin\tau_1,\tau_3,\Gamma_1,\Gamma_2$
LgenF	$\dfrac{\Omega\tau_2,\tau_2\leq\xi_1,\Gamma_1\to\Gamma_2\quad\Omega\tau_2,\Gamma_1\to\Gamma_2,\tau_2\sqsubseteq\tau_1\quad\tau_1\leq\xi_1,\Gamma_1\to\Gamma_2,\Omega\tau_2}{\tau_1\leq\xi_1,\Gamma_1\to\Gamma_2}$
RgenF	$\dfrac{\Omega\tau_2,\tau_2\sqsubseteq\tau_1,\Gamma_1\to\Gamma_2,\tau_2\leq\xi_1}{\Gamma_1\to\Gamma_2,\tau_1\leq\xi_1}\ \lhd\tau_2:\mathbf{y},\tau_2\notin\tau_1,\Gamma_1,\Gamma_2$

Observe that it is imposed that τ_2 is fresh so that the universal quantifier does not capture other variables namely those in τ_1,Γ_1 and Γ_2. The rule LgenF can be interpreted as follows: assuming that $\tau_1\leq\xi_1$ and Γ_1 hold, in order to conclude γ_2 for some $\gamma_2\in\Gamma_2$ it is enough to show that there is an atomic element τ_2, less than or equal to τ_1 such that from $\tau_2\leq\xi_1$ it is possible to conclude γ_2.

Definition 2.23 A *sequent calculus for a normal modal logic* with a modality is the pair $\langle\Sigma_M,\mathcal{R}_M\rangle$ where Σ_M is the signature in Definition 2.2 and $\mathcal{R}_M$ is the set constituted by the rules in Tables 1, 2, 3, 4, 5, 6, 7 and 8. ■

It is now presented an example of a derivation in the context of a sequent calculus for a normal modal logic.

Example 2.24 In the context of the sequent calculus introduced in Definition 2.23, the following holds: $\Omega\mathbf{y}_1,\mathbf{y}_1\leq\psi_1,\mathbf{y}_1\leq\psi_2\to\mathbf{y}_1\leq\varphi\vdash_{\mathcal{R}}\to\top\leq(\psi_1\wedge\psi_2)\supset\varphi$.

1.		$\to$	$\top\leq(\psi_1\wedge\psi_2)\supset\varphi$	RgenF 2
2.	$\Omega\mathbf{y}_1,\mathbf{y}_1\sqsubseteq\top$	$\to$	$\mathbf{y}_1\leq(\psi_1\wedge\psi_2)\supset\varphi$	R$\supset$ 3
3.	$\Omega\mathbf{y}_1,\mathbf{y}_1\sqsubseteq\top$, $\mathbf{y}_1\leq\psi_1\wedge\psi_2$	$\to$	$\mathbf{y}_1\leq\varphi$	L$\wedge$ 4
4.	$\Omega\mathbf{y}_1,\mathbf{y}_1\sqsubseteq\top$, $\mathbf{y}_1\leq\psi_1,\mathbf{y}_1\leq\psi_2$	$\to$	$\mathbf{y}_1\leq\varphi$	Lw 5
5.	$\Omega\mathbf{y}_1,\mathbf{y}_1\leq\psi_1,\mathbf{y}_1\leq\psi_2$	$\to$	$\mathbf{y}_1\leq\varphi$	hyp

■

One of the most common ways to express the semantics of modal logics is by Kripke structures. Among Kripke structures, general Kripke structures deserve a particularly importance since they provide a complete semantics to some modal logic not complete with respect to the standard Kripke semantics. These general Kripke structures can be represented in our setting as showed in the next example.

Example 2.25 (Σ-algebra induced by a general Kripke structure) A general Kripke structure is a tuple $K=\langle W,\mathcal{B},\rightsquigarrow,V\rangle$ where W is the set of possible worlds, $\mathcal{B}\subseteq 2^W$ is the set of admissible values, $\rightsquigarrow\subseteq W^2$ is the accessible relation corresponding to the modal connective

$\Box$ and $V:\Pi\to\mathcal{B}$ is the valuation map. From any general Kripke structure K, it can be defined an algebra $\mathbb{A}_K=\langle F,T,\cdot_{\mathbb{A}_K}\rangle$ where:

- $F=\mathcal{B}$;
- $T=2^W$;
- $\#_{\mathbb{A}_K}=\lambda b.b$;
- $a\in\Omega_{\mathbb{A}_K}$ *iff a is a singleton*;
- $\langle a,a'\rangle\in\sqsubseteq_{\mathbb{A}_K}$ *iff* $a\subseteq a'$;
- $\langle a,b\rangle\in\leq_{\mathbb{A}_K}$ *iff* $a\subseteq b$;
- $\perp_{\mathbb{A}_K}=\varnothing$;
- $\top_{\mathbb{A}_K}=W$;
- $\mathbf{I}_{\mathbb{A}_K}=\lambda a.\iota(a)$;
- $\mathbf{N}_{\mathbb{A}_K}=\lambda a.\{w'\in W:\text{ exists } w\in a \text{ such that } w\rightsquigarrow w'\}$;
- $\mathbf{lb}_{\mathbb{A}_K}=\lambda aa'.a\cap a'$;
- $\mathbf{f}_{\mathbb{A}_K}=\varnothing$;
- $\mathbf{t}_{\mathbb{A}_K}=W$;
- $\mathbf{p}_{i\mathbb{A}_K}=V(\mathbf{p}_i)$;
- $\neg_{\mathbb{A}_K}=\lambda b.W\setminus b$;
- $\Box_{\mathbb{A}_K}=\lambda b.\{w\in W:\mathbf{N}_{\mathbb{A}_K}(\{w\})\subseteq b\}$;
- $\wedge_{\mathbb{A}_K}=\lambda bb'.b\cap b'$;
- $\vee_{\mathbb{A}_K}=\lambda bb'.b\cup b'$;
- $\supset_{\mathbb{A}_K}=\lambda bb'.(W\setminus b)\cup b'$;

with $\iota:T\to F$ being a choice function. ■

The next example shows that the axiom for reflexivity is satisfied by the $\boldsymbol{\Sigma}$-algebra induced by a general Kripke structure that is *not* reflexive. Actually, the axiom characterizes reflexivity only among standard frames. But it is possible to characterize general frames with a reflexive accessibility relation using rule T that appears in Table 9.

Example 2.26 Consider the general Kripke structure $\langle W,\mathcal{B},\rightsquigarrow,V\rangle$ where $W=\{w_1,w_2\}$, $\rightsquigarrow=\{\langle w_1,w_2\rangle,\langle w_2,w_1\rangle\}$, $\mathcal{B}=\{\varnothing,W\}$ and V such that $\mathbf{p}_1=\varnothing$ and $\mathbf{p}_2=W$. Let $\mathbb{A}$ be the $\boldsymbol{\Sigma}$-algebra induced by this general Kripke structure according to Example 2.25. Thus, $\mathbb{A}\Vdash\top\leq(\Box\mathbf{p}_i)\supset\mathbf{p}_i$ for $i=1,2$ as showed below. In fact:

- $[\![\top]\!]_{\mathbb{A}}=W$;
- $[\![\mathbf{p}_1]\!]_{\mathbb{A}}=\varnothing$;
- $[\![\mathbf{p}_2]\!]_{\mathbb{A}}=W$;
- $\Box_{\mathbb{A}}(\varnothing)=\varnothing$;
- $\Box_{\mathbb{A}}(W)=W$.

Then $\mathbb{A}\Vdash\top\leq(\Box\mathbf{p}_1)\supset\mathbf{p}_1$ iff $\langle[\![\top]\!]_{\mathbb{A}},[\![(\Box\mathbf{p}_1)\supset\mathbf{p}_1]\!]_{\mathbb{A}}\rangle\in\leq_{\mathbb{A}}$ iff $\langle W,\supset_{\mathbb{A}}([\![\Box\mathbf{p}_1]\!]_{\mathbb{A}},\ [\![\mathbf{p}_1]\!]_{\mathbb{A}})\rangle\in\leq_{\mathbb{A}}$ iff $\langle W,\supset_{\mathbb{A}}(\Box_{\mathbb{A}}([\![\mathbf{p}_1]\!]_{\mathbb{A}}),[\![\mathbf{p}_1]\!]_{\mathbb{A}})\rangle\in\leq_{\mathbb{A}}$ iff $\langle W,\supset_{\mathbb{A}}(\Box_{\mathbb{A}}(\varnothing),\varnothing)\rangle\in\leq_{\mathbb{A}}$ iff $\langle W,\supset_{\mathbb{A}}(\varnothing,\varnothing)\rangle\in\leq_{\mathbb{A}}$ iff $\langle W,W\rangle\in\leq_{\mathbb{A}}$.

And, for $\mathbf{p}_2$: $\mathbb{A}\Vdash\top\leq(\Box\mathbf{p}_2)\supset\mathbf{p}_2$ iff $\langle[\![\top]\!]_{\mathbb{A}},[\![(\Box\mathbf{p}_2)\supset\mathbf{p}_2]\!]_{\mathbb{A}}\rangle\in\leq_{\mathbb{A}}$ iff $\langle W,\supset_{\mathbb{A}}([\![\Box\mathbf{p}_2]\!]_{\mathbb{A}},\ [\![\mathbf{p}_2]\!]_{\mathbb{A}})\rangle\in\leq_{\mathbb{A}}$ iff $\langle W,\supset_{\mathbb{A}}(\Box_{\mathbb{A}}([\![\mathbf{p}_2]\!]_{\mathbb{A}}),[\![\mathbf{p}_2]\!]_{\mathbb{A}})\rangle\in\leq_{\mathbb{A}}$ iff $\langle W,\supset_{\mathbb{A}}(\Box_{\mathbb{A}}(W),W)\rangle\in\leq_{\mathbb{A}}$ iff $\langle W,\supset_{\mathbb{A}}(W,W)\rangle\in\leq_{\mathbb{A}}$ iff $\langle W,W\rangle\in\leq_{\mathbb{A}}$. ■

It is now presented the definition of modal logic system, which is used in the next section to define fusion.

Definition 2.27 A *modal logic system* is a logic system $\langle\Sigma,\mathcal{R},\mathcal{A}\rangle$ where the signature Σ is as Definition 2.2, the rules are at least those described in Definition 2.23 and $\mathcal{A}$ is a class of Σ-algebras as in Definition 2.14. ■

Observe that in a modal logic system the following consequences hold:

- $\varphi \vdash^g_{\mathcal{R}} \Box\varphi$ and
- $\vdash^g_{\mathcal{R}} \Box(\varphi \supset \psi) \supset (\Box\varphi \supset \Box\psi)$.

As practical examples of modal logic systems, a deontic logic system, a Löb provability logic system and a knowledge logic system are presented below.

Example 2.28 Consider the modal logic system $\mathcal{D} = \langle\Sigma_D,\mathcal{R}_D,\mathcal{A}_D\rangle$ for a deontic logic system, see [13], where:

- Σ_D is a modal signature as in Definition 2.2 where $\Box\varphi$ is intended to mean that φ is obligatory;
- $\mathcal{R}_D$ is the set of rules as in Definition 2.23 plus the rule D:

$$\frac{}{\Omega\tau_1,\Gamma_1 \to \Gamma_2,\mathbf{N}(\tau_1) \not\sqsubseteq \bot}$$

  expressing that for any atomic truth value there is at least another truth value *acceptable* by it, i.e., everything obligatory associated to the atomic truth value holds in that other truth value;
- $\mathcal{A}_D$ is a class of Σ_D-algebras induced by general Kripke structures where the accessible relation is right-unbounded, see [4], full for $\langle\Sigma_D,\mathcal{R}_D\rangle$. ■

Deontic logic systems are concerned with what is obligatory at a certain point. It has been widely used in a multitude of fields ranging from computer science to philosophy or even in law. The logic system GL is concerned with reasoning about provability, more specifically about what can be expressed by arithmetical theories about their provability predicates, and was considered for instance, when dealing with the logical omniscience problem, in combination with typed theories and programming languages, and in dealing with reflection in artificial intelligence, automated deduction and verification.

Example 2.29 Consider the modal logic system $\mathcal{GL} = \langle\Sigma_{GL},\mathcal{R}_{GL},\mathcal{A}_{GL}\rangle$ corresponding to the modal Löb provability logic for a certain theory $\mathbf{T}$, see [1], where:

- Σ_{GL} is a modal signature as in Definition 2.2 where $\Box\varphi$ is intended to mean that φ is provable in $\mathbf{T}$;
- $\mathcal{R}_{GL}$ is the set of rules as in Definition 2.23 plus the rule W (see [16] for more information about this rule):

$$\frac{\Omega\tau_1,\Omega\tau_3,\tau_3 \sqsubseteq \mathbf{N}(\tau_1),\Gamma_1 \to \Gamma_2,\tau_3 \sqsubseteq \tau_2,\mathbf{N}(\tau_3) \not\sqsubseteq \tau_2}{\Omega\tau_1,\Gamma_1 \to \Gamma_2,\mathbf{N}(\tau_1) \sqsubseteq \tau_2} \lhd \tau_3 : \mathbf{y}, \tau_3 \notin \tau_1,\tau_2,\Gamma_1,\Gamma_2$$

- $\mathcal{A}_{GL}$ is a class of Σ_{GL}-algebras induced by general Kripke structures where the accessibility relation is transitive and conversely well-founded, full for $\langle\Sigma_{GL},\mathcal{R}_{GL}\rangle$. A relation is conversely well-founded if and only if there no infinite ascending sequences. ■

TABLE 9. Additional rules for the Knowledge Logic System $\mathcal{K}$

$$\frac{}{\Gamma_1 \to \Gamma_2, \tau_1 \sqsubseteq \mathbf{N}_i(\tau_1)}\ \mathrm{T}_i$$

$$\frac{}{\Gamma_1 \to \Gamma_2, \mathbf{N}_i(\mathbf{N}_i(\tau_1)) \sqsubseteq \mathbf{N}_i(\tau_1)}\ 4_i$$

$$\frac{\Omega\tau_1, \Omega\tau_2, \Omega\tau_3, \Gamma_1 \to \Gamma_2, \tau_1 \sqsubseteq \mathbf{N}_i(\tau_3) \quad \Omega\tau_1, \Omega\tau_2, \Omega\tau_3, \Gamma_1 \to \Gamma_2, \tau_2 \sqsubseteq \mathbf{N}_i(\tau_3) \quad \Omega\tau_1, \Omega\tau_2, \Gamma_1 \to \Gamma_2, \Omega\tau_3}{\Omega\tau_1, \Omega\tau_2, \Gamma_1 \to \Gamma_2, \tau_1 \sqsubseteq \mathbf{N}_i(\tau_2)}\ 5_i$$

A multi-modal logic system, for a knowledge logic system is also presented. This system can actually be viewed as an example of fusion of n modal logic systems, as it will be clear when fusion is introduced later on in the paper.

Example 2.30 Consider a knowledge logic system $\mathcal{K}$ for n agents where each modal formula $K_i\varphi$ is intended to mean that agent i knows φ, see [12]. That is, $\mathcal{K} = \langle \Sigma_K, \mathcal{R}_K, \mathcal{A}_K \rangle$ is such that:

- Σ_K is a modal signature as Definition 2.2 with $C_1 = \{\neg, K_1, \ldots, K_n\}$ and $O_1 = \{\mathbf{I}, \mathbf{N}_1, \ldots, \mathbf{N}_n\}$;
- $\mathcal{R}_K$ is the set of rules from Definition 2.23 with $L\Box, R\Box, L\mathbf{N}\Omega$ and $R\mathbf{N}\Omega$ renamed as $LK_i, RK_i, L\mathbf{N}_i\Omega$ and $R\mathbf{N}_i\Omega$ for $i = 1, \ldots, n$, together with the rules listed in Table 9. As it is clear from the inspection of the rules, T_i imposes that the associated accessibility relation is reflexive, 4_i that it is transitive, and the rule 5_i that the accessibility relation is Euclidean. Together they impose that the relation is an equivalence;
- $\mathcal{A}_K$ is a class of Σ_K-algebras induced by general Kripke structures where there are n accessibility relations which are equivalence relations, full for $\langle \Sigma_K, \mathcal{R}_K \rangle$. ∎

3 Algebraic account of fusion

In this section an algebraic account of fusion of modal logic systems based in [8] are described. In the algebraic account, propositional connectives, operators and rules may be shared between the modal logic systems but modalities are not shared. In Section 4, fusion of modal logic systems possibly sharing modalities is analyzed. Other types of combinations can be obtained when sharing different sets of connectives, operators and rules which is easily and naturally reached when working in the context of category theory.

Definition 3.1 Let $\mathcal{L}_1 = \langle \Sigma_1, \mathcal{R}_1, \mathcal{A}_1 \rangle$ and $\mathcal{L}_2 = \langle \Sigma_2, \mathcal{R}_2, \mathcal{A}_2 \rangle$ be two modal logic systems such that all the connectives are shared except the modalities. Moreover assume that the rules of these systems are the same except the rules for the modal connectives and the rules for the respective $\mathbf{N}$ operators.

The *fusion* of $\mathcal{L}_1$ and $\mathcal{L}_2$ is the logic system $\langle \Sigma, \mathcal{R}, \mathcal{A} \rangle$ where:

- Σ is the signature $\langle C, O, X, Y, Z \rangle$ where $C_0 = \{\mathbf{t}, \mathbf{f}\} \cup \{\mathbf{p}_i : i \in \mathbb{N}\}$; $C_1 = \{\neg\} \cup \{\Box'\} \cup \{\Box''\}$; $C_2 = \{\wedge, \vee, \supset\}$; $C_k = \varnothing$ for all $k \geq 3$; $O_0 = \{\top, \bot\}$; $O_1 = \{\mathbf{I}\} \cup \{\mathbf{N}'\} \cup \{\mathbf{N}''\}$; $O_2 = \{\mathbf{lb}\}$; $O_k = \varnothing$ for all $k \geq 3$;
- $\mathcal{R}$ keeps all the structural rules, the order rules and the logical rules for $\supset$, $\neg$, $\vee$, $\wedge$, $\mathbf{lb}$ and $\mathbf{I}$. Furthermore, each rule in $\mathcal{L}_1$ ($\mathcal{L}_2$) that involves the connective $\Box$ or the operator $\mathbf{N}$, renaming the connective $\Box$ to $\Box'$ ($\Box''$) and the operator $\mathbf{N}$ to $\mathbf{N}'$ ($\mathbf{N}''$), is in $\mathcal{R}$;
- $\mathcal{A}$ is the class of Σ-algebras $\{\mathbb{A}_{\mathbb{A}_1,\mathbb{A}_2} : \mathbb{A}_1 \in \mathcal{A}_1, \mathbb{A}_2 \in \mathcal{A}_2, \mathbb{A}_1|_{\Sigma_P} = \mathbb{A}_2|_{\Sigma_P}\}$ where $\mathbb{A}_{\mathbb{A}_1,\mathbb{A}_2}$ is such that $\mathbb{A}_{\mathbb{A}_1,\mathbb{A}_2}|_{\Sigma_P} = \mathbb{A}_1|_{\Sigma_P}$, $\Box'_{\mathbb{A}_{\mathbb{A}_1,\mathbb{A}_2}} = \Box_{\mathbb{A}_1}, \Box''_{\mathbb{A}_{\mathbb{A}_1,\mathbb{A}_2}} = \Box_{\mathbb{A}_2}, \mathbf{N}'_{\mathbb{A}_{\mathbb{A}_1,\mathbb{A}_2}} = \mathbf{N}_{\mathbb{A}_1}$ and $\mathbf{N}''_{\mathbb{A}_{\mathbb{A}_1,\mathbb{A}_2}} = \mathbf{N}_{\mathbb{A}_2}$,

where Σ_P is the signature with the shared propositional connectives and operators. ■

As illustration, it is now described the algebraic account of fusion of the deontic logic system and the Löb provability logic introduced in Example 2.28 and 2.29, respectively.

Example 3.2 The fusion of the logic system $\mathcal{D}$ presented in Example 2.28 and the logic system $\mathcal{GL}$ presented in Example 2.29 is such that:

- the signature is Σ as described in Definition 3.1, for simplicity, $\Box_D$ is used instead of $\Box'$ (the modality for $\mathcal{D}$) and $\Box_{GL}$ for $\Box''$ (the modality for $\mathcal{GL}$). Note that it allows expressions like $\Box_D\Box_{GL}\varphi$ with the intended meaning that it is obligatory that φ is provable in a certain theory $\mathbf{T}$;
- the set of rules is $\mathcal{R}$ as described in Definition 3.1 (actually, it is $\mathcal{R}_D \cup \mathcal{R}_{GL}$ with the appropriate substitutions in the names of $\Box$ and $\mathbf{N}$);
- the class $\mathcal{A}$ is constituted by the algebras inducted by generalized Kripke structures that have one accessibility relation which is right-unbounded and another accessibility relation which is transitive and right linear. ■

An example of a deduction in the context of the sequent calculus resulting from the fusion of $\mathcal{D}$ and $\mathcal{GL}$, described in Example 3.2, is now presented. Note that the formula involved uses modalities from the two logics.

Example 3.3 It is now presented a deduction for $\Box_D(\Box_{GL}(\xi_1 \supset \xi_2) \supset (\Box_{GL}\xi_1 \supset \Box_{GL}\xi_2))$ in the context of the logic system resulting for the fusion of $\mathcal{D}$ and $\mathcal{GL}$ as described in Example 3.2:

1.		$\rightarrow$	$\top \leq \Box_D(\Box_{GL}(\xi_1 \supset \xi_2) \supset (\Box_{GL}\xi_1 \supset \Box_{GL}\xi_2))$	$R\Box_D$ 2
2.		$\rightarrow$	$\mathbf{N}_D(\top) \leq \Box_{GL}(\xi_1 \supset \xi_2) \supset (\Box_{GL}\xi_1 \supset \Box_{GL}\xi_2))$	*transF* 3,4
3.		$\rightarrow$	$\mathbf{N}_D(\top) \sqsubseteq \top$	$\top$
4.		$\rightarrow$	$\top \leq \Box_{GL}(\xi_1 \supset \xi_2) \supset (\Box_{GL}\xi_1 \supset \Box_{GL}\xi_2))$	*RgenF* 5
5.	$\Omega\mathbf{y}_1, \mathbf{y}_1 \sqsubseteq \top$	$\rightarrow$	$\mathbf{y}_1 \leq \Box_{GL}(\xi_1 \supset \xi_2) \supset (\Box_{GL}\xi_1 \supset \Box_{GL}1\xi_2)$	$R\supset$ 6
6.	$\Omega\mathbf{y}_1, \mathbf{y}_1 \sqsubseteq \top,$ $\mathbf{y}_1 \leq \Box_{GL}(\xi_1 \supset \xi_2)$	$\rightarrow$	$\mathbf{y}_1 \leq \Box_{GL}\xi_1 \supset \Box_{GL}\xi_2$	$R\supset$ 7
7.	$\Omega\mathbf{y}_1, \mathbf{y}_1 \sqsubseteq \top, \mathbf{y}_1 \leq \Box_{GL}\xi_1$ $\mathbf{y}_1 \leq \Box_{GL}(\xi_1 \supset \xi_2)$	$\rightarrow$	$\mathbf{y}_1 \leq \Box_{GL}\xi_2$	$R\Box_{GL}$ 8
8.	$\Omega\mathbf{y}_1, \mathbf{y}_1 \sqsubseteq \top, \mathbf{y}_1 \leq \Box_{GL}\xi_1$ $\mathbf{y}_1 \leq \Box_{GL}(\xi_1 \supset \xi_2)$	$\rightarrow$	$\mathbf{N}_{GL}(\mathbf{y}_1) \leq \xi_2$	*RgenF*9
9.	$\Omega\mathbf{y}_1, \mathbf{y}_1 \sqsubseteq \top, \mathbf{y}_1 \leq \Box_{GL}\xi_1$ $\mathbf{y}_1 \leq \Box_{GL}(\xi_1 \supset \xi_2)$ $\Omega\mathbf{y}_2, \mathbf{y}_2 \sqsubseteq \mathbf{N}_{GL}(\mathbf{y}_1)$	$\rightarrow$	$\mathbf{y}_2 \leq \xi_2$	$L\Box_{GL}$ 10
10.	$\Omega\mathbf{y}_1, \mathbf{y}_1 \sqsubseteq \top, \mathbf{N}_{GL}(\mathbf{y}_1) \leq \xi_1$ $\mathbf{y}_1 \leq \Box_{GL}(\xi_1 \supset \xi_2)$ $\Omega\mathbf{y}_2, \mathbf{y}_2 \sqsubseteq \mathbf{N}_{GL}(\mathbf{y}_1)$	$\rightarrow$	$\mathbf{y}_2 \leq \xi_2$	*LgenF* 11,12,13
11.	$\Omega\mathbf{y}_1, \mathbf{y}_1 \sqsubseteq \top, \mathbf{y}_2 \leq \xi_1$ $\mathbf{y}_1 \leq \Box_{GL}(\xi_1 \supset \xi_2)$ $\Omega\mathbf{y}_2, \mathbf{y}_2 \sqsubseteq \mathbf{N}_{GL}(\mathbf{y}_1), \Omega\mathbf{y}_2$	$\rightarrow$	$\mathbf{y}_2 \leq \xi_2$	$L\Box_{GL}$ 14

12.	$\Omega\mathbf{y}_1, \mathbf{y}_1 \sqsubseteq \top$ $\mathbf{y}_1 \leq \Box_{GL}(\xi_1 \supset \xi_2)$ $\Omega\mathbf{y}_2, \mathbf{y}_2 \sqsubseteq \mathbf{N}_{GL}(\mathbf{y}_1)$	$\rightarrow$	$\mathbf{y}_2 \leq \xi_2, \mathbf{y}_2 \sqsubseteq \mathbf{N}_{GL}(\mathbf{y}_1)$	*ax*
13.	$\Omega\mathbf{y}_1, \mathbf{y}_1 \sqsubseteq \top, \mathbf{N}_{GL}(\mathbf{y}_1) \leq \xi_1$ $\mathbf{y}_1 \leq K_1(\xi_1 \supset \xi_2)$ $\Omega\mathbf{y}_2, \mathbf{y}_2 \sqsubseteq \mathbf{N}_{GL}(\mathbf{y}_1)$	$\rightarrow$	$\mathbf{y}_2 \leq \xi_2, \Omega\mathbf{y}_2$	*ax*
14.	$\Omega\mathbf{y}_1, \mathbf{y}_1 \sqsubseteq \top, \mathbf{y}_2 \leq \xi_1$ $\mathbf{N}_{GL}(\mathbf{y}_1) \leq \xi_1 \supset \xi_2$ $\Omega\mathbf{y}_2, \mathbf{y}_2 \sqsubseteq \mathbf{N}_{GL}(y_1), \Omega\mathbf{y}_2$	$\rightarrow$	$\mathbf{y}_2 \leq \xi_2$	*LgenF* 15,16,17
15.	$\Omega\mathbf{y}_1, \mathbf{y}_1 \sqsubseteq \top, \mathbf{y}_2 \leq \xi_1$ $\mathbf{N}_{GL}(\mathbf{y}_1) \leq \xi_1 \supset \xi_2$ $\Omega\mathbf{y}_2, \mathbf{y}_2 \sqsubseteq \mathbf{N}_{GL}(\mathbf{y}_1), \Omega\mathbf{y}_2$	$\rightarrow$	$\mathbf{y}_2 \leq \xi_2, \Omega\mathbf{y}_2$	*ax*
16.	$\Omega\mathbf{y}_1, \mathbf{y}_1 \sqsubseteq \top, \mathbf{y}_2 \leq \xi_1$ $\Omega\mathbf{y}_2, \mathbf{y}_2 \sqsubseteq \mathbf{N}_{GL}(\mathbf{y}_1)$ $\Omega\mathbf{y}_2, \Omega\mathbf{y}_2$	$\rightarrow$	$\mathbf{y}_2 \leq \xi_2, \mathbf{y}_2 \sqsubseteq \mathbf{N}_{GL}(\mathbf{y}_1)$	*ax*
17.	$\Omega\mathbf{y}_1, \mathbf{y}_1 \sqsubseteq \top, \mathbf{y}_2 \leq \xi_1$ $\Omega\mathbf{y}_2, \mathbf{y}_2 \leq \xi_1 \supset \xi_2$ $\Omega\mathbf{y}_2, \mathbf{y}_2 \sqsubseteq \mathbf{N}_{GL}(\mathbf{y}_1), \Omega\mathbf{y}_2$	$\rightarrow$	$\mathbf{y}_2 \leq \xi_2$	$L\supset$ 18,19
18.	$\Omega\mathbf{y}_1, \mathbf{y}_1 \sqsubseteq \top, \mathbf{y}_2 \leq \xi_1$ $\Omega\mathbf{y}_2, \Omega\mathbf{y}_2$ $\Omega\mathbf{y}_2, \mathbf{y}_2 \sqsubseteq \mathbf{N}_{GL}(\mathbf{y}_1)$	$\rightarrow$	$\mathbf{y}_2 \leq \xi_2, \mathbf{y}_2 \leq \xi_1$	*ax*
19.	$\Omega\mathbf{y}_1, \mathbf{y}_1 \sqsubseteq \top, \mathbf{y}_2 \leq \xi_1$ $\Omega\mathbf{y}_2, \mathbf{y}_2 \leq \xi_2$ $\Omega\mathbf{y}_2, \mathbf{y}_2 \sqsubseteq \mathbf{N}_{GL}(\mathbf{y}_1), \Omega\mathbf{y}_2$	$\rightarrow$	$\mathbf{y}_2 \leq \xi_2$	*ax*

■

4 Categorical view of fusion

The aim of this section is to define in categorical terms the fusion presented algebraically in the previous section. An advantage of this approach is to abstract fusion to a categorical construction which can be applied to other types of logic systems and not only to modal logic systems. Another advantage of using the categorical presentation of fusion is that it is possible to share modalities, besides the propositional connectives. In fact it is even more general: any set of connectives can be shared.

The section is organized as follows: first it is presented the notion of morphism between logic systems, since operations in category theory are essentially operations between morphisms (for instance, in [9] the authors define a category as a one-sorted algebra with just morphisms, but herein the most common approach with objects is also used). After that, fusion is showed to be a colimit.

4.1 Logic system morphisms

In order to define fusion as a categorical operation, logic system morphisms are now introduced. Basically they are a pair of maps: one between signatures and the other (a contravariant map) between classes of algebras preserving the sets of terms and formulas. First the map between signatures is defined.

Definition 4.1 Given signatures Σ and Σ', a *signature morphism* is a map $h: \Sigma \to \Sigma'$ such that, for each $n \in \mathbb{N}$:

- $h_{C_n}: C_n \to C'_n$;
- $h_{O_n}: O_n \to O'_n$, preserving $\top$ and $\bot$;
- $h_X: X \to X'$;
- $h_Y: Y \to Y'$;
- $h_Z: Z \to Z'$. ■

Signatures and signature morphisms constitute a category, named $\mathcal{S}ig$. It is easy to see that $\mathcal{S}ig$ is a subcategory of $\mathcal{S}et$ and has all limits and colimits. The map between algebras in a logic system morphism is contravariant essentially in order for entailment to be preserved, and that happens if all the models in the target logic have a correspondent by the morphism in the source logic.

Definition 4.2 A *logic system morphism* $m: \mathcal{L} \to \mathcal{L}'$ where $\mathcal{L} = \langle \Sigma, \mathcal{R}, \mathcal{A} \rangle$ and $\mathcal{L}' = \langle \Sigma', \mathcal{R}', \mathcal{A}' \rangle$ is a pair $m = \langle h, a \rangle$ such that $h: \Sigma \to \Sigma'$ is a signature morphism and for each $r = \langle \{s_1, ..., s_p\}, s, \pi \rangle \in \mathcal{R}$ there is a rule in $\mathcal{L}'$

$$\frac{h^*(s_1) \quad \dots \quad h^*(s_p)}{h^*(s)} \lhd h^*(\pi)$$

where h^* the extension of h to assertions, sequents and provisos. Moreover, $a: \mathcal{A}' \to \mathcal{A}$ is a map such that:

- $V(\mathbb{A}') = V(a(\mathbb{A}'))$
- $c_{a(\mathbb{A}')}(f_1, ..., f_k) = h(c)_{\mathbb{A}'}(f_1, ..., f_k)$;
- $o_{a(\mathbb{A}')}(t_1, ..., t_k) = h(o)_{\mathbb{A}'}(t_1, ..., t_k)$;
- $\#_{a(\mathbb{A}')} f = \#_{\mathbb{A}'} f$;
- $\sqsubseteq_{\mathbb{A}'} = \sqsubseteq_{a(\mathbb{A}')}$;
- $\leq_{\mathbb{A}'} = \leq_{a(\mathbb{A}')}$;
- $\Omega_{\mathbb{A}'} = \Omega_{a(\mathbb{A}')}$. ■

Proposition 4.3 Logic systems and logic systems morphisms constitute a category named $\mathcal{L}og$.

Some important preservation properties are valid in the context of $\mathcal{L}og$ as described below.

Proposition 4.4 (Preservation of derivations) Given $m: \mathcal{L} \to \mathcal{L}'$ if $S \vdash_{\mathcal{C}} s' \lhd \pi$ then $h^*(S) \vdash_{\mathcal{C}'} h^*(s') \lhd h^*(\pi)$.

Proof. The proof follows by induction on the size of a derivation $d = (d_1, \pi_1), \dots, (d_n, \pi_n)$ for $S \vdash_{\mathcal{C}} s' \lhd \pi$.

Base: $d = (d_1, \pi_1)$.

So $d_1 = s'$. Then:

- either $d_1 \in S$ and $\pi_1 = \mathbf{up}$ and so $h^*(d_1) \in h^*(S)$ and $h^*(\pi_1) = \mathbf{up}$;
- or there is an assertion that occurs in both sides of d_1 and $\pi_1 = \mathbf{up}$. Then $h^*(d_1)$ also has an assertion that occurs in both sides of the sequent and $h^*(\pi_1) = \mathbf{up}$;
- or there are $r \in \mathcal{R}, \rho \in Sbs(\Sigma)$ such that $r\rho = \dfrac{\qquad}{d_1} \lhd \pi_1$. Then, $h^*(r)$ is in $\mathcal{R}'$ and using the fact that $h^*(r\rho) = h^*(r)h^*(\rho)$, the substitution $h^*(\rho)$ results in $\vdash_{\mathcal{C}'} h^*(d_1) \lhd h^*(\pi_1)$.

Step: $d=(d_1,\pi_1),(d_2,\pi_2),\ldots,(d_n,\pi_n)$ is a derivation in $\mathcal{L}$.

The Induction Hypothesis is that $(d_2,\pi_2),\ldots,(d_n,\pi_n)$ has an "equivalent" derivation, named d_{Step}, in $\mathcal{L}'$. Then:

- either $d_1 \in S$ and $\pi_1 = \mathbf{up}$ and so $h^*(d_1) \in h^*(S)$ and $h^*(\pi_1) = \mathbf{up}$;
- or there is an assertion that occurs in both sides of d_1 and $\pi_1 = \mathbf{up}$. Then $h^*(d_1)$ also has an assertion that occurs in both sides of the sequent and $h^*(\pi_1) = \mathbf{up}$;
- or there are $r \in \mathcal{R}, \rho \in Sbs(\Sigma), p \in \mathbb{N}$ and $i_1,\ldots,i_p \in \{2,3,\ldots,n\}$ such that $r\rho = \dfrac{d_{i_1} \quad \ldots \quad d_{i_p}}{d_1} \lhd \pi.$ and $\pi_1 = \pi. \cap \pi_{i_1} \cap \ldots \cap \pi_{i_p}$. Then consider the rule in $\mathcal{L}'$

$$\frac{h^*(s_1) \quad \ldots \quad h^*(s_p)}{h^*(s)} \lhd h^*(\pi)$$

  and by applying the substitution $h^*(\rho)$ on that rule (knowing that $h^*(\sigma)(h^*(p)) = h^*(\sigma(p))$), it happens that, for $j=1,\ldots,p$, $h^*(\rho)(h^*(s_j)) \; = h^*(\rho s_j) = h^*(d_{i_j})$ and $h^*(\rho)(h^*(s)) = h^*(\rho s) = h^*(d_1)$.

So, the derivation in $\mathcal{L}'$ corresponding to d is $h^*(d_r)\, d_{Step}$. ∎

Given a morphism between logic systems and assignments over a destination algebra, it is possible to consider corresponding assignments over the source algebra in order to establish interesting preservation properties.

Definition 4.5 Given algebras $\mathbb{A}$ and $\mathbb{A}'$ and assignments α and β over $\mathbb{A}$, the assignments over $\mathbb{A}'$ such that the diagrams in Figure 1 commute, are denoted by α' and β'. ∎

The lemma that follows establishes the preservation of the denotation of a formula by a logic system morphism.

Lemma 4.6 Let φ be a formula, $\langle h, a\rangle : \mathcal{L} \to \mathcal{L}'$ a logic system morphism and α' and β' as the previous definition, then $[\![\varphi]\!]_{a(\mathbb{A}')\alpha\beta} = [\![h(\varphi)]\!]_{\mathbb{A}'\alpha'\beta'}$.

Proof. The proof follows by induction on the structure of φ.

- φ is $\mathbf{z}$.
  $[\![\mathbf{z}]\!]_{a(\mathbb{A}')\alpha\beta} = \alpha(\mathbf{z}) = \alpha'(h_Z(\mathbf{z})) = [\![h(\mathbf{z})]\!]_{\mathbb{A}'\alpha'\beta'}$.
- φ is $c(\varphi_1,\ldots,\varphi_k)$.
  $[\![\; c(\varphi_1,\ldots,\varphi_k) \;]\!]_{a(\mathbb{A}')\alpha\beta} = c_{a(\mathbb{A}')} \; ([\![\varphi_1 \;]\!]_{a(\mathbb{A}')\alpha\beta}, \; \ldots, [\![\varphi_k \;]\!]_{a(\mathbb{A}')\alpha\beta}) = h(c)_{\mathbb{A}'}$ $([\![h(\varphi_1)]\!]_{\mathbb{A}'\alpha'\beta'},\ldots,[\![h(\varphi_k)]\!]_{\mathbb{A}'\alpha'\beta'}) = [\![h(c(\varphi_1,\ldots,\varphi_k))]\!]_{\mathbb{A}'\alpha'\beta'}$. ∎

The next three lemmas show the preservation of the denotations of the three relations used in the satisfaction of assertions. They are used in the proof of Proposition 4.10.

$$\begin{tikzcd} X \arrow[r, "\alpha"] \arrow[d, "h_X"'] & T \\ X' \arrow[ur, "\alpha'"'] & \end{tikzcd} \qquad \begin{tikzcd} Y \arrow[r, "\beta"] \arrow[d, "h_Y"'] & T \\ Y' \arrow[ur, "\beta'"'] & \end{tikzcd} \qquad \begin{tikzcd} Z \arrow[r, "\alpha"] \arrow[d, "h_Z"'] & F \\ Z' \arrow[ur, "\alpha'"'] & \end{tikzcd}$$

Fig. 1. Commutative diagrams for α' and β'.

Lemma 4.7 Given $\mathbb{A}' \in \mathcal{A}'$, $[\![\theta]\!]_{a(\mathbb{A}')\alpha\beta} \in \Omega_{a(\mathbb{A}')}$ iff $[\![h(\theta)]\!]_{\mathbb{A}'\alpha'\beta'} \in \Omega_{\mathbb{A}'}$, where α' and β' are as in Definition 4.5.

Proof. The proof follows by induction on the structure of θ.
θ is $\mathbf{x} \in X$.

$$\begin{aligned}
[\![\mathbf{x}]\!]_{a(\mathbb{A}')\alpha\beta} \in \Omega_{a(\mathbb{A}')} \quad & \text{iff} \quad \alpha(\mathbf{x}) \in \Omega_{a(\mathbb{A}')} \\
& \text{iff} \quad \alpha(\mathbf{x}) \in \Omega_{\mathbb{A}'} \\
& \text{iff} \quad \alpha'(h_X(\mathbf{x})) \in \Omega_{\mathbb{A}'} \\
& \text{iff} \quad [\![h(\mathbf{x})]\!]_{\mathbb{A}'\alpha'\beta'} \in \Omega_{\mathbb{A}'}
\end{aligned}$$

θ is $\mathbf{y} \in Y$.

$$\begin{aligned}
[\![\mathbf{y}]\!]_{a(\mathbb{A}')\alpha\beta} \in \Omega_{a(\mathbb{A}')} \quad & \text{iff} \quad \beta(\mathbf{y}) \in \Omega_{a(\mathbb{A}')} \\
& \text{iff} \quad \beta(\mathbf{y}) \in \Omega_{\mathbb{A}'} \\
& \text{iff} \quad \beta'(h_Y(\mathbf{y})) \in \Omega_{\mathbb{A}'} \\
& \text{iff} \quad [\![h(\mathbf{y})]\!]_{\mathbb{A}'\alpha'\beta'} \in \Omega_{\mathbb{A}'}
\end{aligned}$$

θ is $o(\theta_1,\ldots,\theta_k)$.

$$\begin{aligned}
[\![o(\theta_1,\ldots,\theta_k)]\!]_{a(\mathbb{A}')\alpha\beta} \in \Omega_{a(\mathbb{A}')} \quad & \text{iff} \quad o_{a(\mathbb{A}')}([\![\theta_1]\!]_{a(\mathbb{A}')\alpha\beta},\ldots,[\![\theta_k]\!]_{a(\mathbb{A}')\alpha\beta}) \in \Omega_{a(\mathbb{A}')} \\
& \text{iff} \quad h(o)_{\mathbb{A}'}([\![\theta_1]\!]_{a(\mathbb{A}')\alpha\beta},\ldots,[\![\theta_k]\!]_{a(\mathbb{A}')\alpha\beta}) \in \Omega_{\mathbb{A}'} \\
& \text{iff} \quad h(o)_{\mathbb{A}'}([\![h(\theta_1)]\!]_{\mathbb{A}'\alpha'\beta'},\ldots,[\![h(\theta_k)]\!]_{\mathbb{A}'\alpha'\beta'}) \in \Omega_{\mathbb{A}'} \\
& \text{iff} \quad [\![h(o)(h(\theta_1),\ldots,h(\theta_k))]\!]_{\mathbb{A}'\alpha'\beta'} \in \Omega_{\mathbb{A}'} \\
& \text{iff} \quad [\![h(o(\theta_1,\ldots,\theta_k))]\!]_{\mathbb{A}'\alpha'\beta'} \in \Omega_{\mathbb{A}'}
\end{aligned}$$

θ is $\#\varphi$.

- φ is $\mathbf{z} \in Z$.

$$\begin{aligned}
[\![\#\mathbf{z}]\!]_{a(\mathbb{A}')\alpha\beta} \in \Omega_{a(\mathbb{A}')} \quad & \text{iff} \quad \#_{a(\mathbb{A}')}([\![\mathbf{z}]\!]_{a(\mathbb{A}')\alpha}) \in \Omega_{a(\mathbb{A}')} \\
& \text{iff} \quad \#_{a(\mathbb{A}')}(\alpha(\mathbf{z})) \in \Omega_{a(\mathbb{A}')} \\
& \text{iff} \quad \#_{\mathbb{A}'}(\alpha(\mathbf{z})) \in \Omega_{\mathbb{A}'} \\
& \text{iff} \quad \#_{\mathbb{A}'}\alpha'(h_Z(\mathbf{z})) \in \Omega_{\mathbb{A}'} \\
& \text{iff} \quad \#_{\mathbb{A}'}([\![h(\mathbf{z})]\!]_{\mathbb{A}'\alpha'}) \in \Omega_{\mathbb{A}'} \\
& \text{iff} \quad [\![h(\#\mathbf{z})]\!]_{\mathbb{A}'\alpha'\beta'} \in \Omega_{\mathbb{A}'}
\end{aligned}$$

- φ is $c(\varphi_1,\ldots,\varphi_k)$.
  $[\![\#(c(\varphi_1,\ldots,\varphi_k))]\!]_{a(\mathbb{A}')\alpha\beta} \in \Omega_{a(\mathbb{A}')}$

$$\begin{aligned}
& \text{iff} \quad \#_{a(\mathbb{A}')}([\![c(\varphi_1,\ldots,\varphi_k)]\!]_{a(\mathbb{A}')\alpha}) \in \Omega_{a(\mathbb{A}')} \\
& \text{iff} \quad \#_{a(\mathbb{A}')}(c_{a(\mathbb{A}')}([\![\varphi_1]\!]_{a(\mathbb{A}')\alpha},\ldots,[\![\varphi_k]\!]_{a(\mathbb{A}')\alpha})) \in \Omega_{a(\mathbb{A}')} \\
& \text{iff} \quad \#_{\mathbb{A}'}(c_{a(\mathbb{A}')}([\![\varphi_1]\!]_{a(\mathbb{A}')\alpha},\ldots,[\![\varphi_k]\!]_{a(\mathbb{A}')\alpha})) \in \Omega_{\mathbb{A}'}
\end{aligned}$$

$$\begin{array}{ll}
\text{iff} & \#_{\mathbb{A}'}(h(c)_{\mathbb{A}'}([\![\varphi_1]\!]_{a(\mathbb{A}')\alpha},\ldots,[\![\varphi_k]\!]_{a(\mathbb{A}')\alpha})) \in \Omega_{\mathbb{A}'}\\
\text{iff} & \#_{\mathbb{A}'}(h(c)_{\mathbb{A}'}([\![h(\varphi_1)]\!]_{\mathbb{A}'\alpha'},\ldots,[\![h(\varphi_k)]\!]_{\mathbb{A}'\alpha'})) \in \Omega_{\mathbb{A}'}\\
\text{iff} & \#_{\mathbb{A}'}([\![h(c)(h(\varphi_1),\ldots,h(\varphi_k))]\!]_{\mathbb{A}'\alpha'}) \in \Omega_{\mathbb{A}'}\\
\text{iff} & [\![\#h(c(\varphi_1,\ldots,\varphi_k))]\!]_{\mathbb{A}'\alpha'\beta'} \in \Omega_{\mathbb{A}'}\\
\text{iff} & [\![h(\#(c(\varphi_1,\ldots,\varphi_k)))]\!]_{\mathbb{A}'\alpha'\beta'} \in \Omega_{\mathbb{A}'}
\end{array} \quad \blacksquare$$

The previous lemma and the two that follow are used to show that the relations between terms and the relation between terms and formulas are preserved by morphisms.

Lemma 4.8 Given $\mathbb{A}' \in \mathcal{A}'$, $\langle [\![\theta]\!]_{a(\mathbb{A}')\alpha\beta}, [\![\theta']\!]_{a(\mathbb{A}')\alpha\beta}\rangle \in \sqsubseteq_{a(\mathbb{A}')}$ iff $\langle [\![h(\theta)]\!]_{\mathbb{A}'\alpha'\beta'},\ [\![h(\theta')]\!]_{\mathbb{A}'\alpha'\beta'}\rangle \in \sqsubseteq_{\mathbb{A}'}$, where α' and β' are as in Definition 4.5.

Proof. The proof follows by induction on the structure of θ and θ'. It is only considered the case when θ is $\mathbf{x}_1$ in X since the other cases are similar.

θ' is $\mathbf{x}_2 \in X$.

$$\begin{array}{rll}
\langle [\![\mathbf{x}_1]\!]_{a(\mathbb{A}')\alpha\beta}, [\![\mathbf{x}_2]\!]_{a(\mathbb{A}')\alpha\beta}\rangle \in \sqsubseteq_{a(\mathbb{A}')} & \text{iff} & \langle \alpha(\mathbf{x}_1), \alpha(\mathbf{x}_2)\rangle \in \sqsubseteq_{a(\mathbb{A}')}\\
& \text{iff} & \langle \alpha(\mathbf{x}_1), \alpha(\mathbf{x}_2)\rangle \in \sqsubseteq_{\mathbb{A}'}\\
& \text{iff} & \langle \alpha'(h_X(\mathbf{x}_1)), \alpha'(h_X(\mathbf{x}_2))\rangle \in \sqsubseteq_{\mathbb{A}'}\\
& \text{iff} & \langle [\![h(\mathbf{x}_1)]\!]_{\mathbb{A}'\alpha'\beta'},\ [\![h(\mathbf{x}_2)]\!]_{\mathbb{A}'\alpha'\beta'}\rangle \in \sqsubseteq_{\mathbb{A}'}
\end{array}$$

θ' is $\mathbf{y} \in Y$.

$$\begin{array}{rll}
\langle [\![\mathbf{x}_1]\!]_{a(\mathbb{A}')\alpha\beta}, [\![\mathbf{y}]\!]_{a(\mathbb{A}')\alpha\beta}\rangle \in \sqsubseteq_{a(\mathbb{A}')} & \text{iff} & \langle \alpha(\mathbf{x}_1), \beta(\mathbf{y})\rangle \in \sqsubseteq_{a(\mathbb{A}')}\\
& \text{iff} & \langle \alpha(\mathbf{x}_1), \beta(\mathbf{y})\rangle \in \sqsubseteq_{\mathbb{A}'}\\
& \text{iff} & \langle \alpha'(h_X(\mathbf{x}_1)), \beta'(h_Y(\mathbf{y}))\rangle \in \sqsubseteq_{\mathbb{A}'}\\
& \text{iff} & \langle [\![h(\mathbf{x}_1)]\!]_{\mathbb{A}'\alpha'\beta'}, [\![h(\mathbf{y})]\!]_{\mathbb{A}'\alpha'\beta'}\rangle \in \sqsubseteq_{\mathbb{A}'}
\end{array}$$

θ' is $o(\theta_1,\ldots,\theta_k)$.
$\langle [\![\mathbf{x}_1]\!]_{a(\mathbb{A}')\alpha\beta}, [\![o(\theta_1,\ldots,\theta_k)]\!]_{a(\mathbb{A}')\alpha\beta}\rangle \in \sqsubseteq_{a(\mathbb{A}')}$

$$\begin{array}{ll}
\text{iff} & \langle \alpha(\mathbf{x}_1), o_{a(\mathbb{A}')}([\![\theta_1]\!]_{a(\mathbb{A}')\alpha\beta},\ldots,[\![\theta_k]\!]_{a(\mathbb{A}')\alpha\beta})\rangle \in \sqsubseteq_{a(\mathbb{A}')}\\
\text{iff} & \langle \alpha(\mathbf{x}_1), o_{a(\mathbb{A}')}([\![\theta_1]\!]_{a(\mathbb{A}')\alpha\beta},\ldots,[\![\theta_k]\!]_{a(\mathbb{A}')\alpha\beta})\rangle \in \sqsubseteq_{\mathbb{A}'}\\
\text{iff} & \langle \alpha(\mathbf{x}_1), h(o)_{\mathbb{A}'}([\![\theta_1]\!]_{a(\mathbb{A}')\alpha\beta},\ldots,[\![\theta_k]\!]_{a(\mathbb{A}')\alpha\beta})\rangle \in \sqsubseteq_{\mathbb{A}'}\\
\text{iff} & \langle \alpha'(h_X(\mathbf{x}_1)), h(o)_{\mathbb{A}'}\ ([\![h(\theta_1)]\!]_{\mathbb{A}'\alpha'\beta'},\ldots,[\![h(\theta_k)]\!]_{\mathbb{A}'\alpha'\beta'})\rangle \in \sqsubseteq_{\mathbb{A}'}\\
\text{iff} & \langle [\![h(\mathbf{x}_1)]\!]_{\mathbb{A}'\alpha'\beta'}, [\![h(o(\theta_1,\ldots,\theta_k))]\!]_{\mathbb{A}'\alpha'\beta'}\rangle \in \sqsubseteq_{\mathbb{A}'}
\end{array}$$

θ' is $\#\varphi$.

- φ is $\mathbf{z} \in Z$.

$$\begin{array}{rll}
\langle [\![\mathbf{x}_1]\!]_{a(\mathbb{A}')\alpha\beta}, [\![\#\mathbf{z}]\!]_{a(\mathbb{A}')\alpha\beta}\rangle \in \sqsubseteq_{a(\mathbb{A}')} & \text{iff} & \langle \alpha(\mathbf{x}_1), \#_{a(\mathbb{A}')}([\![\mathbf{z}]\!]_{a(\mathbb{A}')\alpha}\rangle \in \sqsubseteq_{a(\mathbb{A}')}\\
& \text{iff} & \langle \alpha(\mathbf{x}_1), \#_{a(\mathbb{A}')}(\alpha(\mathbf{z}))\rangle \in \sqsubseteq_{a(\mathbb{A}')}\\
& \text{iff} & \langle \alpha(\mathbf{x}_1), \#_{\mathbb{A}'}(\alpha(\mathbf{z}))\rangle \in \sqsubseteq_{\mathbb{A}'}
\end{array}$$

$$\text{iff} \quad \langle \alpha'(h_X(\mathbf{x}_1)), \#_{\mathbb{A}'}(\alpha'(h_Z(\mathbf{z}_1)))\rangle \in \sqsubseteq_{\mathbb{A}'}$$
$$\text{iff} \quad \langle [\![h(\mathbf{x}_1)]\!]_{\mathbb{A}'\alpha'\beta'}, \#_{\mathbb{A}'}([\![h(\mathbf{z})]\!]_{\mathbb{A}'\alpha'\beta'})\rangle \in \sqsubseteq_{\mathbb{A}'}$$
$$\text{iff} \quad \langle [\![h(\mathbf{x}_1)]\!]_{\mathbb{A}'\alpha'\beta'}, [\![h(\#\mathbf{z})]\!]_{\mathbb{A}'\alpha'\beta'}\rangle \in \sqsubseteq_{\mathbb{A}'}$$

- φ is $c(\varphi_1,\ldots,\varphi_k)$.
  $\langle [\![\mathbf{x}_1]\!]_{a(\mathbb{A}')\alpha\beta}, [\![\#(c(\varphi_1,\ldots,\varphi_k))]\!]_{a(\mathbb{A}')\alpha\beta}\rangle \in \sqsubseteq_{a(\mathbb{A}')}$

$$\text{iff} \quad \langle \alpha(\mathbf{x}_1), \#_{a(\mathbb{A}')}([\![c(\varphi_1,\ldots,\varphi_k)]\!]_{a(\mathbb{A}')\alpha})\rangle \in \sqsubseteq_{a(\mathbb{A}')}$$
$$\text{iff} \quad \langle \alpha(\mathbf{x}_1), \#_{a(\mathbb{A}')}(c_{a(\mathbb{A}')}([\![\varphi_1]\!]_{a(\mathbb{A}')\alpha},\ldots,[\![\varphi_k]\!]_{a(\mathbb{A}')\alpha}))\rangle \in \sqsubseteq_{a(\mathbb{A}')}$$
$$\text{iff} \quad \langle \alpha(\mathbf{x}_1), \#_{\mathbb{A}'}(c_{a(\mathbb{A}')}([\![\varphi_1]\!]_{a(\mathbb{A}')\alpha},\ldots,[\![\varphi_k]\!]_{a(\mathbb{A}')\alpha}))\rangle \in \sqsubseteq_{\mathbb{A}'}$$
$$\text{iff} \quad \langle \alpha'(h_X(\mathbf{x}_1)), \#_{\mathbb{A}'}(h(c)_{\mathbb{A}'}([\![\varphi_1]\!]_{a(\mathbb{A}')\alpha},\ldots,[\![\varphi_k]\!]_{a(\mathbb{A}')\alpha}))\rangle \in \sqsubseteq_{\mathbb{A}'}$$
$$\text{iff} \quad \langle [\![h(\mathbf{x}_1)]\!]_{\mathbb{A}'\alpha'\beta'}, \#_{\mathbb{A}'}(h(c)_{\mathbb{A}'}([\![h(\varphi_1)]\!]_{\mathbb{A}'\alpha'},\ldots,[\![h(\varphi_k)]\!]_{\mathbb{A}'\alpha'}))\rangle \in \sqsubseteq_{\mathbb{A}'}$$
$$\text{iff} \quad \langle [\![h(\mathbf{x}_1)]\!]_{\mathbb{A}'\alpha'\beta'}, \#_{\mathbb{A}'}([\![h(c(\varphi_1,\ldots,\varphi_k))]\!]_{\mathbb{A}'\alpha'})\rangle \in \sqsubseteq_{\mathbb{A}'}$$
$$\text{iff} \quad \langle [\![h(\mathbf{x}_1)]\!]_{\mathbb{A}'\alpha'\beta'}, [\![\#h(c(\varphi_1,\ldots,\varphi_k))]\!]_{\mathbb{A}'\alpha'\beta'}\rangle \in \sqsubseteq_{\mathbb{A}'}$$
$$\text{iff} \quad \langle [\![h(\mathbf{x}_1)]\!]_{\mathbb{A}'\alpha'\beta'}, [\![h(\#(c(\varphi_1,\ldots,\varphi_k)))]\!]_{\mathbb{A}'\alpha'\beta'}\rangle \in \sqsubseteq_{\mathbb{A}'} \qquad \blacksquare$$

Lemma 4.9 Given $\mathbb{A}' \in \mathcal{A}'$, $\langle [\![\theta]\!]_{a(\mathbb{A}')\alpha\beta}, [\![\varphi]\!]_{a(\mathbb{A}')\alpha}\rangle \in \leq_{a(\mathbb{A}')}$ iff $\langle [\![h(\theta)]\!]_{\mathbb{A}'\alpha'\beta'}, [\![h(\varphi)]\!]_{\mathbb{A}'\alpha'}\rangle \in \leq_{\mathbb{A}'}$, where α' and β' are as in Definition 4.5.

Proof. The proof follows by structural induction on θ and in φ. It is only considered the case when θ is $\mathbf{x}$ in X since the proof of the other ones are similar to the proofs shown before.

- φ is $\mathbf{z} \in Z$.
  $\langle [\![\mathbf{x}]\!]_{a(\mathbb{A}')\alpha\beta}, [\![\mathbf{z}]\!]_{a(\mathbb{A}')\alpha}\rangle \in \leq_{a(\mathbb{A}')}$ iff $\langle \alpha(\mathbf{x}), \alpha(\mathbf{z})\rangle \in \leq_{a(\mathbb{A}')}$ iff $\langle \alpha(\mathbf{x}), \alpha(\mathbf{z})\rangle \in \leq_{\mathbb{A}'}$ iff $\langle \alpha'(h_X(\mathbf{x})), \alpha'(h_Z(\mathbf{z}))\rangle \in \leq_{\mathbb{A}'}$ iff $\langle [\![h(\mathbf{x})]\!]_{\mathbb{A}'\alpha'\beta'}, [\![h(\mathbf{z})]\!]_{\mathbb{A}'\alpha'}\rangle \in \leq_{\mathbb{A}'}$
- φ is $c(\varphi_1,\ldots,\varphi_k)$.
  $\langle [\![\mathbf{x}]\!]_{a(\mathbb{A}')\alpha\beta}, [\![c(\varphi_1,\ldots,\varphi_k)]\!]_{a(\mathbb{A}')\alpha}\rangle \in \leq_{a(\mathbb{A}')}$

$$\text{iff} \quad \langle \alpha(\mathbf{x}), c_{a(\mathbb{A}')}([\![\varphi_1]\!]_{a(\mathbb{A}')\alpha},\ldots,[\![\varphi_k]\!]_{a(\mathbb{A}')\alpha})\rangle \in \leq_{a(\mathbb{A}')}$$
$$\text{iff} \quad \langle \alpha(\mathbf{x}), h(c)_{\mathbb{A}'}([\![\varphi_1]\!]_{a(\mathbb{A}')\alpha},\ldots,[\![\varphi_k]\!]_{a(\mathbb{A}')\alpha})\rangle \in \leq_{\mathbb{A}'}$$
$$\text{iff} \quad \langle \alpha'(h_X(\mathbf{x})), h(c)_{\mathbb{A}'}([\![h(\varphi_1)]\!]_{\mathbb{A}'\alpha'},\ldots,[\![h(\varphi_k)]\!]_{\mathbb{A}'\alpha'})\rangle \in \leq_{\mathbb{A}'}$$
$$\text{iff} \quad \langle [\![h(\mathbf{x})]\!]_{\mathbb{A}'\alpha'\beta'}, [\![h(c)(h(\varphi_1),\ldots,h(\varphi_k))]\!]_{\mathbb{A}'\alpha'}\rangle \in \leq_{\mathbb{A}'}$$
$$\text{iff} \quad \langle [\![h(\mathbf{x})]\!]_{\mathbb{A}'\alpha'\beta'}, [\![h(c(\varphi_1,\ldots,\varphi_k))]\!]_{\mathbb{A}'\alpha'}\rangle \in \leq_{\mathbb{A}'} \qquad \blacksquare$$

The next proposition shows that logic system morphisms preserve satisfaction of assertions. That is, the semantics of the source logic system is kept by the morphism.

Proposition 4.10 (Preservation of satisfaction) Given $m: \mathcal{L} \to \mathcal{L}'$ and $\mathbb{A}' \in \mathcal{A}'$, $a(\mathbb{A}')\alpha\beta \Vdash_{\mathcal{L}} \gamma$ iff $\mathbb{A}'\alpha'\beta' \Vdash_{\mathcal{L}'} h(\gamma)$ where α' and β' are as in the Definition 4.5.

Proof. The proof is by induction on γ.

(i) $a(\mathbb{A}')\alpha\beta \Vdash \Omega\theta$ iff $[\![\theta]\!]_{a(\mathbb{A}')\alpha\beta} \in \Omega_{a(\mathbb{A}')}$ iff (by Lemma 4.7) $[\![h(\theta)]\!]_{\mathbb{A}'\alpha'\beta'} \in \Omega_{\mathbb{A}'}$ iff $\mathbb{A}'\alpha'\beta' \Vdash \Omega h(\theta)$ iff $\mathbb{A}'\alpha'\beta' \Vdash h(\Omega\theta)$.

(ii) $a(\mathbb{A}')\alpha\beta \Vdash \mho\theta$ iff $[\![\theta]\!]_{a(\mathbb{A}')\alpha\beta} \notin \Omega_{a(\mathbb{A}')}$ iff (by a variant of Lemma 4.7) $[\![h(\theta)]\!]_{\mathbb{A}'\alpha'\beta'} \notin \Omega_{\mathbb{A}'}$ iff $\mathbb{A}'\alpha'\beta' \Vdash \mho h(\theta)$ iff $\mathbb{A}'\alpha'\beta' \Vdash h(\mho\theta)$.

(iii) $a(\mathbb{A}')\alpha\beta \Vdash \theta \sqsubseteq \theta'$ iff $\langle [\![\theta]\!]_{a(\mathbb{A}')\alpha\beta}, [\![\theta']\!]_{a(\mathbb{A}')\alpha\beta}\rangle \in \sqsubseteq_{a(\mathbb{A}')}$ iff (by Lemma 4.8) $\langle [\![h(\theta)]\!]_{\mathbb{A}'\alpha'\beta'}, [\![h(\theta')]\!]_{\mathbb{A}'\alpha'\beta'}\rangle \in \sqsubseteq_{\mathbb{A}'}$ iff $\mathbb{A}'\alpha'\beta' \Vdash h(\theta) \sqsubseteq h(\theta')$ iff $\mathbb{A}'\alpha'\beta' \Vdash h(\theta \sqsubseteq \theta')$.

(iv) $a(\mathbb{A}')\alpha\beta \Vdash \theta \not\sqsubseteq \theta'$ iff $\langle [\![\theta]\!]_{a(\mathbb{A}')\alpha\beta}, [\![\theta]\!]_{a(\mathbb{A}')\alpha\beta}\rangle \notin \sqsubseteq_{a(\mathbb{A}')}$ iff (by a variant of Lemma 4.8) $\langle [\![h(\theta)]\!]_{\mathbb{A}'\alpha'\beta'}, [\![h(\theta')]\!]_{\mathbb{A}'\alpha'\beta'}\rangle \notin \sqsubseteq_{\mathbb{A}'}$ iff $\mathbb{A}'\alpha'\beta' \Vdash h(\theta) \not\sqsubseteq h(\theta')$ iff $\mathbb{A}'\alpha'\beta' \Vdash h(\theta \not\sqsubseteq \theta')$.

(v) $a(\mathbb{A}')\alpha\beta \Vdash \theta \leq \varphi$ iff $\langle [\![\theta]\!]_{a(\mathbb{A}')\alpha\beta}, [\![\varphi]\!]_{a(\mathbb{A}')\alpha\beta}\rangle \in \leq_{a(\mathbb{A}')}$ iff (by Lemma 4.9) $\langle [\![h(\theta)]\!]_{\mathbb{A}'\alpha'\beta'}, [\![h(\varphi)]\!]_{\mathbb{A}'\alpha'\beta'}\rangle \in \leq_{\mathbb{A}'}$ iff $\mathbb{A}'\alpha'\beta' \Vdash h(\theta) \leq h(\varphi)$ iff $\mathbb{A}'\alpha'\beta' \Vdash h(\theta \leq \varphi)$.

(vi) $a(\mathbb{A}')\alpha\beta \Vdash \theta \not\leq \varphi$ iff $\langle [\![\theta]\!]_{a(\mathbb{A}')\alpha\beta}, [\![\varphi]\!]_{a(\mathbb{A}')\alpha\beta}\rangle \notin \leq_{a(\mathbb{A}')}$ iff (by a variant of Lemma 4.9) $\langle [\![h(\theta)]\!]_{\mathbb{A}'\alpha'\beta'}, [\![h(\varphi)]\!]_{\mathbb{A}'\alpha'\beta'}\rangle \notin \leq_{\mathbb{A}'}$ iff $\mathbb{A}'\alpha'\beta' \Vdash h(\theta) \not\leq h(\varphi)$ iff $\mathbb{A}'\alpha'\beta' \Vdash h(\theta \not\leq \varphi)$.

(vii) $a(\mathbb{A}')\alpha\beta \Vdash \Delta' \to \Delta''$ iff $a(\mathbb{A}')\alpha\beta \Vdash \delta$ for some $\delta \in \Delta'' \cup \overline{\Delta'}$. So δ is an assertion that is proved in (i)-(vi) and so $\mathbb{A}'\alpha'\beta' \Vdash h(\delta)$ with $h(\delta) \in h(\Delta'' \cup \overline{\Delta'})$, then $\mathbb{A}'\alpha'\beta' \Vdash h(\Delta' \to \Delta'')$. The other side is analogous. ∎

It is possible to establish results that do not seem obvious using the previous proposition as showed in the following corollary.

Corollary 4.11 Given a morphism $m: \mathcal{L} \to \mathcal{L}'$ and a rule $r = \langle \{\gamma_1, \ldots, \gamma_k\}, \gamma, \pi\rangle \in \mathcal{R}$, if $\mathbb{A}' \Vdash h^*(r)$ then $a(\mathbb{A}') \Vdash r$, for any $\mathbb{A}' \in \mathcal{A}'$.

Proof. Let α be an unbound variable assignment over $a(\mathbb{A}')$. Suppose that $a(\mathbb{A}')\alpha \Vdash \gamma_n$ for $n = 1, \ldots, k$. Then by Proposition 4.10 $\mathbb{A}'\alpha' \Vdash h^*(\gamma_n)$ for $n = 1, \ldots, k$. By hypothesis, $\mathbb{A}'\alpha' \Vdash h^*(\gamma)$ and so, applying again Proposition 4.10, $a(\mathbb{A}')\alpha \Vdash \gamma$. ∎

An interesting property is the preservation of the satisfaction of the rules of a logic system. The next proposition shows that the rules are preserved by logic system morphisms.

Proposition 4.12 Given a sound logic system $\mathcal{L}$ and a logic system morphism $m: \mathcal{L} \to \mathcal{L}'$, if $r = \langle \{\gamma_1, \ldots, \gamma_n\}, \varphi, \pi\rangle \in \mathcal{R}$ then $h(\gamma_1), \ldots, h(\gamma_n) \vDash_{\mathcal{L}'} h(\varphi)$.

Proof. Taking $r = \langle \{\gamma_1, \ldots, \gamma_n\}, \varphi, \pi\rangle \in \mathcal{R}$ then $\gamma_1, \ldots, \gamma_n \vdash_{\mathcal{L}} \varphi$. Since $\mathcal{L}$ is sound, then $\gamma_1, \ldots, \gamma_n \vDash_{\mathcal{L}} \varphi$ and so, by the definition of entailment, if $\mathbb{A}\alpha \Vdash_{\mathcal{L}} \gamma_i$ for $i = 1, \ldots, n$ then $\mathbb{A}\alpha \Vdash_{\mathcal{L}} \varphi$ for any algebra $\mathbb{A}$ in $\mathcal{A}$ and any unbound variable assignment α over $\mathbb{A}$.

Let $\mathbb{A}' \in \mathcal{A}'$ and α' be an unbound variable assignment over $\mathbb{A}'$. Suppose $\mathbb{A}'\alpha' \Vdash h(\gamma_i)$ for $i = 1, \ldots, n$. So, by Proposition 4.10, $a(\mathbb{A}')\alpha \Vdash \gamma_i$ for $i = 1, \ldots, n$ where $\alpha = \alpha' \circ h$. By satisfaction of r, $a(\mathbb{A}')\alpha \Vdash \varphi$ and so again by Proposition 4.10, $\mathbb{A}'\alpha' \Vdash h(\varphi)$. Thus, $h(\gamma_1), \ldots, h(\gamma_n) \vDash_{\mathcal{L}'} h(\varphi)$. ∎

4.2 Fusion as a pushout

In this subsection a categorical description of fusion is given by showing that it is a pushout in $\mathcal{L}og$. In a general way, fusion is obtained as a colimit as illustrated by the diagram in Figure 2 where the morphisms are inclusions. Herein the emphasis is on the case where both logic systems are modal.

For this purpose, the concept of reduct of an algebra *by a morphism* is introduced.

Definition 4.13 Given a Σ-algebra $\mathbb{A} = \langle F, T, \cdot_{\mathbb{A}}\rangle$ and a signature map $h: \Sigma' \to \Sigma$, the *reduct of* $\mathbb{A}$ *by* h is a Σ'-algebra $\mathbb{A}|_h = \langle F, T, \cdot_{\mathbb{A}|_h}\rangle$ where:

- for each connective $c \in \Sigma'$, $c_{\mathbb{A}|_h} = h(c)_{\mathbb{A}}$;

$$\begin{tikzcd}
& \textit{Common Logic System} \arrow[dl, hook] \arrow[dr, hook] & \\
\textit{Logic System A} \arrow[dr, hook] & & \textit{Logic System B} \arrow[dl, hook] \\
& \textit{Fusion} &
\end{tikzcd}$$

FIG. 2. General diagram of fusion.

$$\begin{tikzcd}
& \mathcal{L}_C \arrow[dl, hook, "inc_1"'] \arrow[dr, hook, "inc_2"] & \\
\mathcal{L}_1 & & \mathcal{L}_2
\end{tikzcd}$$

FIG. 3. Diagram for fusion.

- for each operator $o \in \Sigma'$, $o_{\mathbb{A}|_h} = h(o)_{\mathbb{A}}$;
- $\#_{\mathbb{A}|_h} = \#_{\mathbb{A}}$, $\Omega_{\mathbb{A}|_h} = \Omega_{\mathbb{A}}$, $\sqsubseteq_{\mathbb{A}|_h} = \sqsubseteq_{\mathbb{A}}$ and $\leq_{\mathbb{A}|_h} = \leq_{\mathbb{A}}$. ∎

The categorical definition of fusion is given by the next definition: it is the pushout in $\mathcal{L}og$ of the diagram of Figure 3.

Definition 4.14 Let $\mathcal{L}_1 = \langle \Sigma_1, \mathcal{R}_1, \mathcal{A}_1 \rangle$ and $\mathcal{L}_2 = \langle \Sigma_2, \mathcal{R}_2, \mathcal{A}_2 \rangle$ be two modal logic systems. The *fusion* of these logic systems is the object of the pushout in $\mathcal{L}og$ of the diagram of Figure 3, where $\mathcal{L}_C = \langle \Sigma_C, \mathcal{R}_C, \mathcal{A}_C \rangle$ is such that

- $\Sigma_C \supseteq \Sigma_P$ where Σ_P is the signature of propositional logic;
- $\mathcal{R}_C \supseteq \text{StructuralRules} \cup \text{OrderRules} \cup \{\mathbf{I}, \mathbf{\Omega I}, \mathbf{lb}1, \mathbf{lb}2, \text{L}\mathbf{f}, \text{R}\mathbf{f}, \text{L}\mathbf{t}, \text{R}\mathbf{t}, \text{L}\wedge, \text{R}\wedge, \text{L}\vee, \text{R}\vee, \text{L}\supset, \text{R}\supset, \text{L}\neg, \text{R}\neg\}$;
- $\mathcal{A}_C = \langle F, T, \cdot_{\mathbb{A}_C} \rangle$ is the class of all Σ_C-algebras;

and the morphisms $inc_i = \langle inc_i\_h, inc_i\_a \rangle$ with $i = 1,2$ are inclusions and $inc_i\_a(\mathbb{A}_i) = \mathbb{A}_i|_{inc_i\_h}$. ∎

The algebraic fusion of $\mathcal{L}_1$ and $\mathcal{L}_2$ mentioned in Section 3 coincides with the object of the pushout described in Definition 4.14 when the signature Σ_C coincides with Σ_P and similarly for $\mathcal{R}_C$.

The following example illustrates the fusion of two modal systems: one for deontic logic and the other for knowledge logic. One possible area of use of a system that reasons about knowledge and obligation is for example in multi-agent systems in computer science since it allows to talk about obligatory behaviors and knowledge of the agents.

Example 4.15 The fusion of the deontic modal system $\mathcal{D}$ of Example 2.28 with the knowledge modal system $\mathcal{K}$ of Example 2.30 is obtained by taking the morphisms $inc_1 : \mathcal{L}_C \to \mathcal{D}$ and $inc_2 : \mathcal{L}_C \to \mathcal{K}$, where $inc_1 = \langle inc_1\_h, inc_1\_a \rangle$ and $inc_2 = \langle inc_2\_h, inc_2\_a \rangle$, such that:

- Σ_C coincides with Σ_P where Σ_P is the signature of propositional logic and $\mathcal{R}_C$ is such that the second item in Definition 4.14 is an equality;

- $inc_1\_h : \Sigma_C \to \Sigma_D$ and $inc_2\_h : \Sigma_C \to \Sigma_K$ are the inclusions of Σ_C in Σ_D and Σ_K respectively,
- $inc_1\_a(\mathbb{A}_D) = \mathbb{A}_D|_{inc_1\_h}$ and $inc_2\_a(\mathbb{A}_K) = \mathbb{A}_K|_{inc_2\_h}$ for $\mathbb{A}_D \in \mathcal{A}_D$ and $\mathbb{A}_K \in \mathcal{A}_K$ (note that both reducts are Σ_C-algebras and so, are in $\mathcal{A}_C$).

The object of the pushout of $\langle inc_1, inc_2\rangle$ is the fusion of $\mathcal{D}$ and $\mathcal{K}$ and is such that:

- the signature is $\Sigma_C \cup \langle\{\Box_D, K_1,\dots,K_n\},\{\mathbf{N}_D,\mathbf{N}_1,\dots,\mathbf{N}_n\}, X, Y, Z\rangle$;
- the set of rules is $\mathcal{R}_C \cup \{L\mathbf{N}_D\Omega,\ L\mathbf{N}_i\Omega,\ R\mathbf{N}_D\Omega,\ R\mathbf{N}_i\Omega,\ L\Box_D,\ LK_i,\ R\Box_D,\ RK_i,\ D, T_i,\ 4_i\ 5_i\}$ for $i = 1,\dots,n$;
- the class of algebras is the class of algebras induced by general Kripke structures where there are n accessibility relations that are equivalence relations together with a relation that is right-unbounded.

Note that, in the context of fusion, formulas as $\Box_D K_1\varphi$ mean that it is obligatory that agent 1 knows φ. ■

It is now provided a more detailed characterization of the pushout, that is, a description of the components of the objects and of the morphisms that constitute the colimit of the diagram of Figure 3. Note that the object of the pushout is the fusion of the logic systems.

Proposition 4.16 The triple $\langle\mathcal{L}, m_1, m_2\rangle$ where $\mathcal{L} = \langle\Sigma,\mathcal{R},\mathcal{A}\rangle$ and $m_i = \langle m_i\_h, m_i\_a\rangle : \mathcal{L}_i \to \mathcal{L}$ for $i = 1,2$, are such that

- $\langle\Sigma, m_1\_h, m_2\_h\rangle$ is the pushout in $\mathcal{S}ig$ of the morphisms inc_1_h and inc_2_h introduced in Definition 4.14;
- $\mathcal{R} = m_1\_h^*(inc_1\_h^*(\mathcal{R}_C)) \cup m_1\_h^*(\mathcal{R}_1 \backslash \mathcal{R}_C) \cup m_2\_h^*(\mathcal{R}_2 \backslash \mathcal{R}_C)$;
- $\mathcal{A} = \{\mathbb{A}_1\_\mathbb{A}_2 = \langle F, T, \cdot_{\mathbb{A}_1\_\mathbb{A}_2}\rangle : inc_1\_a(\mathbb{A}_1) = inc_2\_a(\mathbb{A}_2) = \langle F, T, \cdot_{\mathbb{A}_C}\rangle$ and
  $m_1\_h(c)_{\mathbb{A}_1\_\mathbb{A}_2} = c_{\mathbb{A}_1}$ for $c \in C_1$, $m_2\_h(c)_{\mathbb{A}_1\_\mathbb{A}_2} = c_{\mathbb{A}_2}$ for $c \in C_2$;
  $m_1\_h(o)_{\mathbb{A}_1\_\mathbb{A}_2} = o_{\mathbb{A}_1}$ for $o \in O_1$, $m_2\_h(o)_{\mathbb{A}_1\_\mathbb{A}_2} = o_{\mathbb{A}_2}$ for $o \in O_2$;
  $\#_{\mathbb{A}_1\_\mathbb{A}_2} = \#_{\mathbb{A}_1}, \sqsubseteq_{\mathbb{A}_1\_\mathbb{A}_2} = \sqsubseteq_{\mathbb{A}_1}, \leq_{\mathbb{A}_1\_\mathbb{A}_2} = \leq_{\mathbb{A}_1}, \Omega_{\mathbb{A}_1\_\mathbb{A}_2} = \Omega_{\mathbb{A}_1}\}$;
- $m_i\_a(\mathbb{A}_1\_\mathbb{A}_2) = \mathbb{A}_i$ for $i = 1,2$

is the pushout of the diagram of Figure 3. So, $\mathcal{L}$ is the fusion of $\mathcal{L}_1$ and $\mathcal{L}_2$.

Proof. Suppose there is $\langle\mathcal{L}', l_1, l_2\rangle$ such that $l_1 \circ inc_1 = l_2 \circ inc_2$. It has to be shown that there is a unique morphism $u : \mathcal{L} \to \mathcal{L}'$ such that $u \circ m_1 = l_1$ and $u \circ m_2 = l_2$. Let $u\_h(inc_i\_h)) = l_i\_h \circ m_i\_h^{-1}$ and $u\_a(\mathbb{A}') = l_1\_a(\mathbb{A}')\_l_2\_a(\mathbb{A}')$.

Unicity of u: Suppose there is $n : \mathcal{L} \to \mathcal{L}'$ such that $n \circ m_1 = l_1$ and $n \circ m_2 = l_2$ and $n \neq u$. Then $n\_h = u\_h$ since $\langle\Sigma, m_1\_h, m_2\_h\rangle$ is the pushout in $\mathcal{S}ig$. Let $\mathbb{A}' \in \mathcal{A}'$ and $n\_a(\mathbb{A}') = \mathbb{A}_x\_\mathbb{A}_y$. So, $m_1\_a \circ n\_a(\mathbb{A}') = m_1\_a(\mathbb{A}_x\_\mathbb{A}_y) = \mathbb{A}_x = l_1\_a(\mathbb{A}')$ and $m_2\_a \circ n\_a(\mathbb{A}') = m_2\_a(\mathbb{A}_x\_\mathbb{A}_y) = \mathbb{A}_y = l_2\_a(\mathbb{A}')$. Thus, $n\_a = u\_a$. Therefore, $n = u$. ■

An interesting point about the categorical account of fusion is that is possible to change the shared connectives and operators by simply changing the "common logic system", $\mathcal{L}_C$, of Definition 4.14. In fact, the definition assumes at least the propositional logic system as the common core, so it is possible to consider extensions of that logic.

The possibility of sharing modalities is now illustrated with an example of the fusion of two well known normal modal logics: a system with a reflexive accessibility relation and other with a transitive accessibility relation.

Example 4.17 Let $\mathcal{L}_4=\langle\Sigma_M,\mathcal{R}_M\cup\{4\},\mathcal{A}_4\rangle$ be the modal logic system where 4 is the rule 4_i in Table 9 and $\mathcal{A}_4$ is a class of algebras induced by general Kripke structures such that the accessibility relation is transitive, and $\mathcal{L}_T=\langle\Sigma_M,\mathcal{R}_M\cup\{T\},\mathcal{A}_T\rangle$ be the modal logic system where T is the rule T_i in Table 9 and $\mathcal{A}_T$ is a class of algebras induced by general Kripke structures such that the accessibility relation is reflexive.

Take $\mathcal{L}_C=\langle\Sigma_M,\mathcal{R}_M,\mathcal{A}_M\rangle$, the "common logic system" in the fusion according to Definition 4.14, such that Σ_M is the signature described in Definition 2.2, $\mathcal{R}_M$ is the set of rules described in Definition 2.23, and $\mathcal{A}_M$ is a class of Σ_M-algebras full for $\langle\Sigma_M,\mathcal{R}_M\rangle$. That is, $\mathcal{L}_C$ is the normal modal logic system known in the literature as K.

The system resulting from the fusion of $\mathcal{L}_4$ and $\mathcal{L}_T$ sharing $\mathcal{L}_C$ is $\mathcal{L}_{S4}=\langle\Sigma_M,\mathcal{R}_M\cup\{4,T\},\mathcal{A}_{S4}\rangle$ where $\mathcal{A}_{S4}$ is the class of algebras induced by general Kripke structures such that the accessibility relation is transitive and reflexive, commonly known as the $\mathsf{S4}$ modal system. ■

5 Preservation of properties

The section starts by presenting results about preservation of soundness and completeness by fusion and ends with an illustration of the application of these results.

Theorem 5.1 (Soundness of the fusion) The fusion of sound logic systems is sound.

Proof. Consider the logic systems $\mathcal{L}_1$ and $\mathcal{L}_2$ and their fusion $\mathcal{L}$ using the logic system $\mathcal{L}_C$ for specifying the shared connectives as described in Definition 4.14.

The proof follows by applying Proposition 4.12 to each rule $r=\langle\{s_1,\dots,s_n\},\ s,\pi\rangle$ of $\mathcal{R}$. Several cases have to be considered:

- $r\in m_1\_h^*(\mathcal{R}_1\backslash\mathcal{R}_C)$, that is, it is a rule from $\mathcal{L}_1$. As $\mathcal{L}_1$ is sound, by Proposition 4.12 $m_1\_h^*(s_1),\dots,m_1\_h^*(s_n)\vDash_{\mathcal{L}} m_1\_h^*(s)\lhd\pi$. The reasoning is analogous if $r\in m_2\_h^*(\mathcal{R}_2\backslash\mathcal{R}_C)$;
- $r\in\mathcal{R}_C$. Then $r=m_1\_h^*(r')=m_2\_h^*(r'')$ for some $r'\in\mathcal{R}_1$ and $r''\in\mathcal{R}_2$, so it is the same case as before.

Therefore, each rule in $\mathcal{R}$ is sound entailed in $\mathcal{L}$. So the class of algebras $\mathcal{A}$ is appropriate for $\langle\Sigma,\mathcal{R}\rangle$. ■

The main goal is preservation of completeness. The idea is to show that completeness is preserved through the preservation of sufficient conditions for completeness.

Thus, first, sufficient conditions for a logic system to be complete are established: to be full and with rules endowed with persistent provisos. This result is proved by a Lindenbaum technique. For this purpose, it is now defined what is a consistent set and a syntactical algebra.

Definition 5.2 A set S of closed sequents is said to be *consistent* if for no closed assertion δ both $S\vdash\rightarrow\delta$ and $S\vdash\rightarrow\overline{\delta}$ hold. And it is said to be *maximal consistent* if for every closed assertion δ either $\rightarrow\delta\in S$ or $\rightarrow\overline{\delta}\in S$ but not both. ■

Definition 5.3 Given a sequent calculus $\mathcal{C}=\langle\Sigma,\mathcal{R}\rangle$ and a maximal consistent set S of closed sequents over Σ, the *syntactic algebra* induced by $\mathcal{C}$ and S is the following Σ-algebra:

$$\mathbb{A}(\mathcal{C},S)=\langle cgF(\Sigma),cgT(\Sigma),\cdot_{\mathbb{A}(\mathcal{C},S)}\rangle$$

where

- $c_{\mathbb{A}(\mathcal{C},S)} = \lambda f_1 \ldots f_k . c(f_1, \ldots, f_k)$;
- $o_{\mathbb{A}(\mathcal{C},S)} = \lambda t_1 \ldots t_k . o(t_1, \ldots, t_k)$;
- $\#_{\mathbb{A}(\mathcal{C},S)} = \lambda f . \# f$;
- $\tau \in \Omega_{\mathbb{A}(\mathcal{C},S)}$ iff $S \vdash_{\mathcal{L}} \to \Omega\tau$;
- $\langle \tau_1, \tau_2 \rangle \in \sqsubseteq_{\mathbb{A}(\mathcal{C},S)}$ iff $S \vdash_{\mathcal{C}} \to \tau_1 \sqsubseteq \tau_2$;
- $\langle \tau, \varphi \rangle \in \leq_{\mathbb{A}(\mathcal{C},S)}$ iff $S \vdash_{\mathcal{C}} \to \tau \leq \varphi$. ■

Let $\varphi \in gF(\Sigma)$ and $\theta \in gT(\Sigma)$. Given an unbounded variable assignment α and a bounded variable assignment β both over a syntactic algebra $\mathbb{A}(\mathcal{C},S)$, $\varphi\alpha$ denotes the closed simple formula obtained from φ by replacing each variable $z \in Z$ by $\alpha(z)$; and $\theta\alpha\beta$ denotes the closed term obtained from θ by replacing each variable $x \in X$ by $\alpha(x)$ and each variable $y \in Y$ by $\beta(y)$. This notation is extended to ground assertions and bags of ground assertions by identifying $\varphi\alpha\beta$ with $\varphi\alpha$.

Lemma 5.4 Let $\mathcal{C}$ be a structural calculus, S a maximal consistent set of closed sequents, α an unbound variable assignment and β a bound variable assignment (both over $\mathbb{A}(\mathcal{C},S)$), φ a ground simple formula, θ a ground term, δ a ground assertion and $\Delta' \to \Delta''$ a ground sequent. Then:

- $[\![\varphi]\!]_{\mathbb{A}(\mathcal{C},S)\alpha\beta} = \varphi\alpha$;
- $[\![\theta]\!]_{\mathbb{A}(\mathcal{C},S)\alpha\beta} = \theta\alpha\beta$;
- $\mathbb{A}(\mathcal{C},S)\alpha\beta \Vdash \delta$ *iff* $S \vdash_{\mathcal{C}} \to \delta\alpha\beta$;
- $\mathbb{A}(\mathcal{C},S)\alpha\beta \Vdash \Delta' \to \Delta''$ *iff* $S \vdash_{\mathcal{C}} \Delta'\alpha\beta \to \Delta''\alpha\beta$.

Proof. The first two items are proved from a straightforward induction on the complexity of φ and θ, respectively.

The third item follows by using the previous results and the fact that S is maximal consistent.

The last item follows from the third one, the metatheorem of conjugation and the metatheorem of contradiction. ■

Taken a sequent calculus and a class of syntactical algebras induced by it, it is now shown that these algebras are appropriate for this calculus.

Lemma 5.5 The class of all syntactic algebras induced by a sequent calculus is appropriate for it.

Proof. The proof follows by using Lemma 5.4 for each rule of the sequent calculus. ■

The next lemma is crucial to prove completeness. It makes possible to consider the consistent extension of a consistent set of closed sequents.

Lemma 5.6 Let $\mathcal{C}$ be a structural sequent calculus with rules endowed with persistent provisos. If S is a consistent set of closed sequents and $S \nvdash_{\mathcal{C}} \to v_1, \ldots, v_m$ for closed assertions $v_1, \ldots, v_m$ then the set $S \cup \{\to \overline{v_1}, \ldots, \to \overline{v_m}\}$ is still consistent.

Proof. Assuming that $S \cup \{\to \overline{v_1}, \ldots, \to \overline{v_m}\}$ is inconsistent and using the metatheorem of contradiction, the metatheorem of deduction, the metatheorem of conjugation and by right contraction, a contradiction is found. ■

Theorem 5.7 (Algebraic Completeness) Every full structural sequent logic system with rules endowed with persistent provisos is complete.

Proof. Consider the logic $\mathcal{L}=\langle\Sigma,\mathcal{R},\mathcal{A}\rangle$ and let $\mathcal{C}=\langle\Sigma,\mathcal{R}\rangle$. Assume that $S\nvdash_{\mathcal{R}}\Delta'\to\Delta''$ with $S\cup\{\Delta'\to\Delta''\}$ composed of closed sequents.

Given an enumeration v_n with $n\in\mathbb{N}$ of the set of closed assertions, start by extending S to a maximal consistent set $S^\bullet$ as follows:

- $S_0=S\cup\{\to\delta:\delta\in\overline{\Delta''}\cup\Delta'\}$;
- $S_{n+1}=\begin{cases} S\cup\{\to v_n\} & \text{provided that } S_n\vdash_{\mathcal{R}}\to v_n \\ S\cup\{\to\overline{v_n}\} & \text{otherwise}\end{cases}$;
- $S^\bullet=\cup_{n\in\mathbb{N}}S_n$.

Observe that $S^\bullet$ is still consistent thanks to Lemma 5.6. Furthermore, by construction, it is maximal consistent. Therefore, $S^\bullet\nvdash_{\mathcal{R}}\Delta'\to\Delta''$ because otherwise $S^\bullet\vdash_{\mathcal{R}}\delta$ for some $\delta\in\Delta''\cup\overline{\Delta'}$ and, hence, $S^\bullet$ would be inconsistent. Thus, by Lemma 5.4 applied to a closed sequent, $\mathbb{A}(\mathcal{C},S^\bullet)\nVdash\Delta'\to\Delta''$.

On the other hand, for every $s\in S$, $S\vdash_{\mathcal{R}}s$ holds, and thus, again thanks to Lemma 5.4, $\mathbb{A}(\mathcal{C},S^\bullet)\Vdash s$.

Since the logic is full and taking into account Lemma 5.5, $\mathbb{A}(\mathcal{C},S^\bullet)$ is in $\mathcal{A}$. Hence, $S\nvDash_{\mathcal{A}}\Delta'\to\Delta''$. ∎

Using the result for completeness given in Theorem 5.7, it is possible to show the preservation of completeness by the fusion under mild conditions.

Theorem 5.8 (Preservation of completeness by fusion) The fusion of two full logic systems with rules endowed with persistent provisos is complete.

Proof. From Theorem 5.7 it can be concluded that if a logic system is full with rules endowed with persistent provisos, then it is complete.

Let $\mathcal{L}_i=\langle\Sigma_i,\mathcal{R}_i,\mathcal{A}_i\rangle$ with $i=1,2$ be the component logic systems and $\mathcal{L}=\langle\Sigma,\mathcal{R},\mathcal{A}\rangle$ the fusion of them.

As the rules of $\mathcal{R}$ comes only from $\mathcal{R}_1$ or $\mathcal{R}_2$ and both have only rules endowed with persistent provisos, so it will have $\mathcal{R}$.

Taking an appropriate Σ-algebra $\mathbb{A}$ it must be shown that $\mathbb{A}\in\mathcal{A}$.

Take $\mathbb{A}|_{\Sigma_1}$, it is appropriate for the calculus $\langle\Sigma_1,\mathcal{R}_1\rangle$ since all rules $r=\langle\{s_1,\dots,s_p\},s,\pi\rangle$ that belongs to $\mathcal{R}_1$ are also in $\mathcal{R}$. Since $s_1,\dots,s_p\vDash_{\mathbb{A}}s\lhd\pi$ then $s_1,\dots,s_p\vDash_{\mathbb{A}|_{\Sigma_1}}$.

The same happens to $\mathbb{A}|_{\Sigma_2}$. So $\mathbb{A}=\mathbb{A}|_{\Sigma_1}\_\mathbb{A}|_{\Sigma_2}\in\mathcal{A}$. ∎

The paper finishes with a conclusion about soundness and completeness of the fusion of $\mathcal{K}$ and $\mathcal{D}$ logic systems presented in Example 4.15, and of the fusion of $\mathcal{D}$ and $\mathcal{GL}$ logic systems showed in Section 3.

Example 5.9 Consider the logic system resulting from the fusion of a knowledge logic system and a deontic logic system as described in Example 4.15. The knowledge and the deontic systems are sound, so the logic system resulting from the fusion is also sound. Moreover it is complete since all the rules of knowledge and deontic systems are endowed with persistent provisos and their classes of algebras are full.

Now consider the logic system resulting from the fusion of the deontic logic system and the provability logic system, presented in Example 3.2. Both component logic systems are sound, so their fusion is also sound. The logic system resulting from the fusion is complete since all the rules of deontic and Löb provability systems are endowed with persistent provisos and both classes of algebras are full. ■

6 Concluding remarks

Fusion was extended to sequent systems labelled with truth values and sufficient conditions for preservation of completeness and soundness were obtained. These conditions are fulfilled by a wide class of systems. The technique was applied to the fusion of a deontic logic system with a knowledge logic system with n agents and to the fusion of a Löb provability modal logic system with a deontic logic system. The extension of fusion to other types of logic systems, like systems for non-normal modal logics, deserves further investigation. In fact, our intuition is that these systems are naturally expressed in the context described herein.

Future work includes preservation by fusion of other logic properties (such as interpolation), the generalization of the logic system morphism to the case where algebras do not have the same values as well as the investigation of fusion in this context. Moreover applications to combination of intuitionist, relevance and many-valued logics are envisaged.

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