Basic Concepts in Modal Logic[1]

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Preface

These notes were composed while teaching a class at Stanford and studying the work of Brian Chellas (*Modal Logic: An Introduction*, Cambridge: Cambridge University Press, 1980), Robert Goldblatt (*Logics of Time and Computation*, Stanford: CSLI, 1987), George Hughes and Max Cresswell (*An Introduction to Modal Logic*, London: Methuen, 1968; *A Companion to Modal Logic*, London: Methuen, 1984), and E. J. Lemmon (*An Introduction to Modal Logic*, Oxford: Blackwell, 1977). The Chellas text influenced me the most, though the order of presentation is inspired more by Goldblatt.[2]

My goal was to write a text for dedicated undergraduates with no previous experience in modal logic. The text had to meet the following desiderata: (1) the level of difficulty should depend on how much the student tries to prove on his or her own—it should be an easy text for those who look up all the proofs in the appendix, yet more difficult for those who try to prove everything themselves; (2) philosophers (i.e., colleagues) with a basic training in logic should be able to work through the text on their own; (3) graduate students should find it useful in preparing for a graduate course in modal logic; (4) the text should prepare people for reading advanced texts in modal logic, such as Goldblatt, Chellas, Hughes and Cresswell, and van Benthem, and in particular, it should help the student to see what motivated the choices in these texts; (5) it should link the two conceptions of logic, namely, the conception of a logic as an axiom system (in which the set of theorems is constructed from the bottom up through proof sequences) and the conception of a logic as a set containing initial 'axioms' and closed under 'rules of inference' (in which the set of theorems is constructed from the top down, by carving out the logic from the set of all formulas as the smallest set closed under the rules); finally, (6) the pace for the presentation of the completeness theorems should be moderate—the text should be intermediate between Goldblatt and Chellas in this regard (in Goldblatt, the completeness proofs come too quickly for the undergraduate, whereas in Chellas, too many unrelated

[2]Three other texts worthy of mention are: K. Segerberg, *An Essay in Classical Modal Logic*, Philosophy Society and Department of Philosophy, University of Uppsala, Vol. 13, 1971; and R. Bull and K. Segerberg, 'Basic Modal Logic', in *Handbook of Philosophical Logic: II*, D. Gabbay and F. Günthner (eds.), Dordrecht: Reidel, 1984l; and Johan van Benthem, *A Manual of Intensional Logic*, 2nd edition, Stanford, CA: Center for the Study of Language and Information Publications, 1988.

facts are proved before completeness is presented).

My plan is to fill in Chapter 5 on quantified modal logic. At present this chapter has only been sketched. It begins with the simplest quantified modal logic, which combines classical quantification theory and the classical modal axioms (and adds the Barcan formula). This logic is then compared with the system in Kripke's 'Semantical Considerations on Modal Logic'. There are interesting observations to make concerning the two systems: (1) a comparison of the formulas valid in the simplest QML that are invalid in Kripke's system, (2) a consideration of the metaphysical presuppositions that led Kripke to set up his system the way he did, and finally, (3) a description of the techniques Kripke uses for excluding the 'offending' formulas. Until Chapter 5 is completed, the work in the coauthored paper 'In Defense of the Simplest Quantified Modal Logic' (with Bernard Linsky) explains the approach I shall take in filling in the details. The citation for this paper can be found toward the end of Chapter 5.

Given that usefulness was a primary goal, I followed the standard procedure of dropping the distinguished worlds from models and defining *truth in a model* as truth at every world in the model. However, I think this is a philosophically objectionable procedure and definition, and in the final version of the text, this may change. In the meantime, the work in my paper 'Logical and Analytic Truths that are not Necessary' explains my philosophical objections to developing modal logic without a distinguished actual world. The citation for this paper also appears at the end of Chapter 5.

The class I taught while writing this text (Philosophy 169/Spring 1990) was supposed to be accessible to philosophy majors with only an intermediate background in logic. I tried to make the class accessible to undergraduates at Stanford who have had only Philosophy 159 (Basic Concepts in Mathematical Logic). Philosophy 160a (Model Theory) was not presupposed. As it turned out, most of the students had had Philosophy 160a. But even so, they didn't find the results repetitive, since they all take place in the new setting of modal languages. Of course, the presentation of the material was probably somewhat slow-paced for the graduate students who were sitting in, but the majority found the pace about right. There are fifteen sections in Chapters 2, 3, and 4, and these can be covered in as little as 10 and as many as 15 weeks. I usually covered about a section (§) of the text in a lecture of about an hour and fifteen minutes (we met twice a week). Of course, some sections go more quickly, others more slowly. As I see it, the job of the instructor using these notes is to illustrate the definitions and theorems with lots of diagrams and to prove the most interesting and/or difficult theorems.

I would like to acknowledge my indebtedness to Bernard Linsky, who not only helped me to see what motivated the choices made in these logic texts and to understand numerous subtleties therein but who also carefully read the successive drafts. I am also indebted to Kees van Deemter, Chris Menzel, Nathan Tawil, Greg O'Hair, and Peter Apostoli. Finally, I am indebted to the Center for the Study of Language and Information, which has provided me with office space and and various other kinds of support over the past years.

Chapter One: Introduction

Modal logic is the study of modal propositions and the logical relationships that they bear to one another. The most well-known modal propositions are propositions about what is necessarily the case and what is possibly the case. For example, the following are all modal propositions:

> It is possible that it will rain tomorrow.
>
> It is possible for humans to travel to Mars.
>
> It is not possible that: every person is mortal, Socrates is a person, and Socrates is not mortal.
>
> It is necessary that either it is raining here now or it is not raining here now.
>
> A proposition p is not possible if and only if the negation of p is necessary.

The operators *it is possible that* and *it is necessary that* are called 'modal' operators, because they specify a way or mode in which the rest of the proposition can be said to be true. There are other modal operators, however. For example, *it once was the case that*, *it will once be the case that*, and *it ought to be the case that*.

Our investigation is grounded in judgments to the effect that certain modal propositions logically imply others. For example, the proposition *it is necessary that* p logically implies the proposition that *it is possible that* p, but not vice versa. These judgments simply reflect our intuitive understanding of the modal propositions involved, for to understand a proposition is, in part, to grasp what it logically implies. In the recent tradition in logic, the judgment that one proposition logically implies another has been analyzed in terms of one of the following two logical relationships: (a) the model-theoretic logical consequence relation, and (b) the proof-theoretic derivability relation. In this text, we shall define and study these relations, and their connections, in a precise way.

§1: A Brief History of Modal Logic

Modal logic was first discussed in a systematic way by Aristotle in *De Interpretatione*. Aristotle noticed not simply that necessity implies possibility (and not vice versa), but that the notions of necessity and possibility were interdefinable. The proposition *p is possible* may be defined as: *not-p is not necessary*. Similarly, the proposition *p is necessary* may be defined as: *not-p is not possible*. Aristotle also pointed out that from the separate facts that p is possible and that q is possible, it does not follow that the conjunctive proposition *p and q* is possible. Similarly, it does not follow from the fact that a disjunction is necessary that that the disjuncts are necessary, i.e., it does not follow from *necessarily, p or q* that *necessarily p or necessarily q*. For example, it is necessary that either it is raining or it is not raining. But it doesn't follow from this either that it is necessary that it is raining, or that it is necessary that it is not raining. This simple point of modal logic has been verified by recent techniques in modal logic, in which the proposition *necessarily, p* has been analyzed as: *p is true in all possible worlds*. Using this analysis, it is easy to see that from the fact that the proposition *p or not-p* is true in all possible worlds, it does not follow either that p is true in all worlds or that *not-p* is true in all worlds. And more generally, it does not follow from the fact that the proposition *p or q* is true in all possible worlds either that p is true in all worlds or that q is true in all worlds.

Aristotle also seems to have noted that the following modal propositions are both true:

If it is necessary that if-p-then-q, then if p is possible, so is q

If it is necessary that if-p-then-q, then if p is necessary, so is q

Philosophers after Aristotle added other interesting observations to this catalog of implications. Contributions were made by the Megarians, the Stoics, Ockham, and Pseudo-Scotus, among others. Interested readers may consult 'the Lemmon notes' for a more detailed discussion of these contributions.[3]

Work in modal logic after the Scholastics stagnated, with the exception of Leibniz's suggestion there are other possible worlds besides the actual world. Interest in modal logic resumed in the twentieth century though, when C. I. Lewis began the search for an axiom system to characterize 'strict implication'.[4] He constructed several different systems which, he

[3]See Lemmon, E., *An Introduction to Modal Logic*, in collaboration with D. Scott, Oxford: Blackwell, 1977.

[4]See C. I. Lewis, 'Implication and the Algebra of Logic', *Mind* (1912) **12**: 522–31; *A Survey of Symbolic Logic*, Berkeley: University of California Press, 1918; and C. Lewis and C. Langford, *Symbolic Logic*, New York: The Century Company, 1932.

thought, directly characterized the logical consequence relation. Today, it is best to think of his work as an axiomatization of the binary modal operation of implication. Consider the following relation:

> p implies $q =_{df}$ Necessarily, if p then q

Lewis defined five systems in the attempt to axiomatize the *implication* relation: *S1* – *S5*. Two of these systems, *S4* and *S5* are still in use today. They are often discussed as candidates for the right logic of necessity and possibility, and we will study them in more detail in what follows. In addition to Lewis, both Ernst Mally and G. Henrik von Wright were instrumental in developing *deontic* systems of modal logic, involving the modal propositions *it ought to be the case that p*.[5] This work, however, was not model-theoretic in character.

The model-theoretic study of the logical consequence relation in modal logic began with R. Carnap.[6] Instead of considering modal propositions, Carnap considered *modal sentences* and evaluated such sentences in *state descriptions*. State descriptions are sets of simple (atomic) sentences, and an simple sentence 'p' is true with respect to a state-description S iff 'p' $\in S$. Carnap was then able to define truth for all the complex sentences of his modal language; for example, he defined: (a) '*not-p*' is true in S iff 'p' $\notin S$, (b) '*if p, then q*' is true in S iff either 'p' $\notin S$ or 'q' $\in S$, and so on for conjunctive and disjunctive sentences. Then, with respect to a collection **M** of state-descriptions, Carnap essentially defined:

> The sentence '*Necessarily p*' is true in S if and only if for every state-description S' in **M**, the sentence 'p' is true in S'

So, for example, if given a set of state descriptions **M**, a sentence such as 'Necessarily, Bill is happy' is true in a state description S if and only if the sentence 'Bill is happy' is a member of every other state description in **M**. Unfortunately, Carnap's definition yields the result that iterations of the modal prefix 'necessarily' have no effect. (*Exercise*: Using Carnap's definition, show that the sentence '*necessarily necessarily p*' is true in a state-description S if and only if the sentence '*necessarily p*' is true in S.)

[5] See E. Mally, *Grundgesetze des Sollens: Elemente der Logik des Willens*, Graz: Lenscher and Lugensky, 1926; and G. H. von Wright, *An Essay in Modal Logic*, Amsterdam: North Holland, 1951. These systems are described in D. Føllesdal and R. Hilpinen, 'Deontic Logic: An Introduction', in Hilpinen [1971], 1–35 [1971].

[6] See R. Carnap, *Introduction to Semantics*, Cambridge, MA: Harvard, 1942; *Meaning and Necessity*, Chicago: University of Chicago Press, 1947.

The problem with Carnap's definition is that it fails to define the truth of a modal sentence at a state-description S in terms of a condition on S. As it stands, the state description S in the definiendum never appears in the definiens, and so Carnap's definition places a 'vacuous' condition on S in his definition.

In the second half of this century, Arthur Prior intuitively saw that the following were the correct truth conditions for the sentence 'it was once the case that p':

> 'it was once the case that p' is true at a time t if and only if p is true at some time t' earlier than t.

Notice that the time t at which the tensed sentence 'it was once the case that p' is said to be true appears in the truth conditions. So the truth conditions for the modal sentence at time t are not vacuous with respect to t. Notice also that in the truth conditions, a relation of temporal precedence ('earlier than') is used.[7] The introduction of this relation gave Prior flexibility to define various other tense operators.

§2: Kripke's Formulation of Modal Logic

The innovations in modal logic that we shall study in this text were developed by S. Kripke, though they were anticipated in the work of S. Kanger and J. Hintikka.[8] For the most part, modal logicians have followed the framework developed in Kripke's work. Kripke introduced a domain of possible worlds and regarded the modal prefix 'it is necesary that' as a quantifier over worlds. However, Kripke did *not* define truth for modal sentences as follows:

> 'Necessarily p' is true at world w if and only if 'p' is true at every possible world.

[7]See A. N. Prior, *Time and Modality*, Westport, CT: Greenwood Press, 1957.

[8]See S. Kripke, 'A Completeness Theorem in Modal Logic', *Journal of Symbolic Logic* **24** (1959): 1–14; 'Semantical Considerations on Modal Logic', *Acta Philosophica Fennica* **16** (1963): 83-94; S. Kanger, *Provability in Logic*, Dissertation, University of Stockholm, 1957; 'A Note on Quantification and Modalities', *Theoria* **23** (1957): 131–4; and J. Hintikka, *Quantifiers in Deontic Logic*, Societas Scientiarum Fennica, Commentationes humanarum litterarum, **23** (1957):4, Helsingfors; 'Modality and Quantification', *Theoria* **27** (1961): 119–28; *Knowledge and Belief: An Introduction to the Logic of the Two Notions*, Ithaca: Cornell University Press, 1962.

Such a definition would have repeated Carnap's error, for it would have defined the truth of a modal sentence at a world w in terms of a condition that is vacuous on w. Such a definition collapses the truth conditions of 'necessarily p' and 'necessarily necessarily p', among other things. Instead, Kripke introduced an *accessibility* relation on the possible worlds and this accessibility relation played a role in the definition of truth for modal sentences. Kripke's definition was:

> 'Necessarily p' is true at a world w if and only if 'p' is true at every world w' accessible from w.

The idea here is that not every world is modally accessible from a given world w. A world w can access a world w' (or, conversely, w' is accessible from w) just in case every proposition that is true at w' is possibly true at w. If there are propositions that are true at w' but which aren't possibly true at w, then that must be because w' represents a state of affairs that is not possible from the point of view of w. So a sentence 'necessarily p' is true at world w so long as 'p' is true at all the worlds that are possible from the point of view of w.

This idea of using an accessibility relation on possible worlds opened up the study of modal logic. In what follows, we learn that this accessibility relation must have certain properties (such as reflexivity, symmetry, transitivity) if certain modal sentences are to be (logically) true. In the remainder of this section, we describe the traditional conception of modal logic as it is now embodied in the basic texts written in the past thirty-five years. These works usually begin with an inductive definition of a *language* containing certain 'proposition letters' ($p, q, r, \ldots$) as atomic sentences. Complex sentences are then defined and these take the form $\neg\varphi$ ('it is not the case that φ'), $\varphi \rightarrow \psi$ ('if φ, then ψ'), and $\Box\varphi$ ('necessarily φ'), where φ and ψ are any sentence (not necessarily atomic). Other sentences may be defined in terms of these basic sentences.

The next step is to define models or interpretations *for the language*. A model $\mathbf{M}$ for the language is typically defined to be a triple $\langle \mathbf{W}, \mathbf{R}, \mathbf{V} \rangle$, where $\mathbf{W}$ is a nonempty set of possible worlds, $\mathbf{R}$ the accessibility relation, and $\mathbf{V}$ a valuation function that assigns to each atomic sentence p a set of worlds $\mathbf{V}(p)$. These models allow one to define the model-theoretic notions of truth, logical truth, and logical consequence. Whereas truth and logical truth are model-theoretic, or semantic, properties of the sentences of the language, logical consequence is a model-theoretic relation among sentences. A sentence is said to be logically true, or valid, just in case it is true in all models, and it is said to be valid with respect to a class $\mathbf{C}$ of models just in case it is valid in every model in the class.

The proof theory proceeds along similar lines. Rules of inference relate certain sentences to others, indicating which sentences can be inferred from others. A logic Σ is defined to be a set of sentences (which may contain some 'axioms' and) which is closed under the rules of inference that define that logic. A theorem of a logic is simply a sentence that is a member of Σ. A logic Σ is said to be sound with respect to a class of models $\mathbf{C}$ just in case every sentence φ that is a theorem of Σ is valid with respect to the class $\mathbf{C}$. And a logic Σ is said to be complete with respect to a class $\mathbf{C}$ of models just in case every sentence φ that is valid with respect $\mathbf{C}$ is a theorem of Σ. Such is the traditional conception of modal logic and we shall follow these definitions here.

Chapter Two: The Language

1) Our first task is to define a class of very general modal languages each of which is relativized to a set of atomic formulas. To do this, we let the set Ω be any non-empty set of *atomic* formulas, with a typical member of Ω being p_i (where i is some natural number). Ω may be finite (in which case, for some n, $\Omega = \{p_1, p_2, \ldots, p_n\}$) or infinite (in which case, $\Omega = \{p_i | i \geq 1\} = \{p_1, p_2, p_3, \ldots\}$). The main requirement is that the members of Ω can be enumerated. We shall use the variables p, q and r to range over the elements of Ω.

2) For any given set Ω, we define by induction *the set of formulas based on* Ω as the smallest set $Fml(\Omega)$ satisfying the following conditions:

.1) $p \in Fml(\Omega)$, for every $p \in \Omega$
.2) $\bot \in Fml(\Omega)$
.3) If $\varphi \in Fml(\Omega)$, then $(\neg\varphi) \in Fml(\Omega)$
.4) If $\varphi, \psi \in Fml(\Omega)$, then $(\varphi \to \psi) \in Fml(\Omega)$
.5) If $\varphi \in Fml(\Omega)$, then $(\Box\varphi) \in Fml(\Omega)$

3) Finally, we define *the modal language based on* Ω (in symbols: Λ_Ω) = $Fml(\Omega)$. It is sometimes useful to be able to discuss the subformulas of a given formula φ. We therefore define ψ *is a subformula of* φ as follows:

.1) φ is a subformula of φ.
.2) If $\varphi = \neg\psi$, $\psi \to \chi$, or $\Box\psi$, then ψ (χ) is a subformula of φ.
.3) If ψ is a subformula of χ and χ is a subformula of φ, then ψ is a subformula of φ.

Remark: We read the formula $\bot$ as 'the falsum', $\neg\varphi$ as 'it is not the case that φ', $\varphi \to \psi$ as 'if φ, then ψ', and $\Box\varphi$ as 'necessarily, φ'. In general, we use the variables $\varphi, \psi, \chi, \theta$ to range over the formulas in Λ_Ω. We drop the parentheses in formulas when there is little potential for ambiguity, and we employ the convention that $\to$ dominates both $\neg$ and $\Box$. So, for example, the formula $\neg p \to q$ is to be understood as $(\neg p) \to q$, and the formula $\Box p \to q$ is to be understood as $(\Box p) \to q$. Finally, we define the truth functional connectives & ('and'), $\vee$ ('or'), and $\leftrightarrow$ ('if and only if') in the usual way, and we define $\Diamond\varphi$ ('possibly φ') in the usual way as $\neg\Box\neg\varphi$. Again we drop parentheses with the convention that the order of dominance is: $\leftrightarrow$ dominates $\to$, $\to$ dominates & and $\vee$, and these last two dominate $\neg$, $\Box$, and $\Diamond$. So, for example, the formula $p \mathbin{\&} \Diamond p \rightarrow q$ is to be understood as $(p \mathbin{\&} \Diamond p) \rightarrow q$.

Note that we could do either without the formula $\perp$ or without formulas of the form $\neg\varphi$. The formula $\perp$ will be interpreted as a contradiction. We could have taken any formula φ and defined $\perp$ as $\varphi \mathbin{\&} \neg\varphi$. Alternatively, we could have defined $\neg\varphi$ as $\varphi \rightarrow \perp$. These equivalences are frequently used in developments of propositional logic. It is sometimes convenient to have both $\perp$ and $\neg\varphi$ as primitives of the language when proving metatheoretical facts, and that is why we include them both as primitive. And when it is convenient to do so, we shall sometimes assume that formulas of the form $\varphi \mathbin{\&} \psi$ and $\Diamond\varphi$ are primitive as well.

4) We define a *schema* to be a set of sentences all having the same form. For example, we take the schema $\Box\varphi \rightarrow \varphi$ to be: $\{\Box\varphi \rightarrow \varphi | \varphi \in \Lambda_\Omega\}$. So the instances of this schema are just the members of this set. Likewise for other schemata. Typically, we shall label schemata using an upper case Roman letter. For example, the schema $\Box\varphi \rightarrow \varphi$ is labeled 'T'. However, it has been the custom to label certain schemata with numbers. For example, the schema $\Box\varphi \rightarrow \Box\Box\varphi$ is labeled '4'. In what follows, we reserve the upper case Roman letter 'S' as a variable to range over schemata.

Chapter Three: Semantics and Model Theory

§1: Models, Truth, and Validity

5) A *standard model* $\mathbf{M}$ for a set of atomic formulas Ω shall be any triple $\langle \mathbf{W}, \mathbf{R}, \mathbf{V} \rangle$ satisfying the following conditions:

.1) $\mathbf{W}$ is a non-empty set,

.2) $\mathbf{R}$ is a binary relation on $\mathbf{W}$, i.e., $\mathbf{R} \subseteq (\mathbf{W} \times \mathbf{W})$,

.3) $\mathbf{V}$ is a function that assigns to each $p \in \Omega$ a subset $\mathbf{V}(p)$ of $\mathbf{W}$; i.e., $\mathbf{V} : \Omega \rightarrow \mathcal{P}(\mathbf{W})$ (where $\mathcal{P}(\mathbf{W})$ is the power set of $\mathbf{W}$).

Remark 1: For precise identification, it is best to refer to the first member of a particular model $\mathbf{M}$ as $\mathbf{W_M}$, to the second member of $\mathbf{M}$ as $\mathbf{R_M}$, and the third member of $\mathbf{M}$ as $\mathbf{V_M}$. For any given model $\mathbf{M}$, we call $\mathbf{W_M}$ *the set of worlds in* $\mathbf{M}$, $\mathbf{R_M}$ *the accessibility relation for* $\mathbf{M}$, and $\mathbf{V_M}$ *the valuation function for* $\mathbf{M}$. Since all of the models we shall be studying are standard models, we generally omit reference to the fact that they are standard. Note that the notion of a model $\mathbf{M}$ is defined relative to a set of atomic formulas Ω. The model itself assigns a set of worlds only to atomic formulas in Ω. In general, the context usually makes it clear which set of atomic formulas we are dealing with, and so we typically omit mention of the set to which $\mathbf{M}$ is relative.

Remark 2: How should we think of the accessibility relation $\mathbf{R}$? One intuitive way (using notions we have not yet defined) is to suppose that $\mathbf{Rww'}$ (i.e., $\mathbf{w}$ *has access to* $\mathbf{w'}$, or $\mathbf{w'}$ *is accessible from* $\mathbf{w}$) iff every proposition p true at $\mathbf{w'}$ is possibly true at $\mathbf{w}$. The idea is that what goes on at $\mathbf{w'}$ is a genuine possibility from the standpoint of $\mathbf{w}$ and so the propositions true at $\mathbf{w'}$ are possible at $\mathbf{w}$. This idea suggests another way of thinking about accessibility. We can think of the set of all worlds in $\mathbf{W}$ as the set of all worlds that are possible in the eyes of God. But from the point of view of the inhabitants of a given world $\mathbf{w} \in \mathbf{W}$, not all worlds $\mathbf{w'}$ may be possible. That is, there may be truths of $\mathbf{w'}$ which are not possible from the point of view of $\mathbf{w}$. The accessibility relation, therefore, makes it explicit as to which worlds are genuine possible worlds from the point of view of a given world $\mathbf{w}$, namely all the worlds $\mathbf{w'}$ such that $\mathbf{Rww'}$. Intuitively, then, whenever $\mathbf{Rww'}$, if φ is true at $\mathbf{w'}$ then

$\Diamond\varphi$ is true at $\mathbf{w}$, and just as importantly, if $\Box\varphi$ is true at $\mathbf{w}$, φ is true at $\mathbf{w}'$. Our definition of *truth at* below will capture these intuitions.

Example: Let $\Omega = \{p, q\}$. Then here is an example of a model $\mathbf{M}$ for Ω. Let $\mathbf{W_M} = \{w_1, w_2, w_3\}$. Let $\mathbf{R_M} = \{\langle w_1, w_2\rangle, \langle w_1, w_3\rangle\}$. Let $\mathbf{V_M}(p) = \{w_1, w_2\}$, and $\mathbf{V_M}(q) = \{w_2\}$. We can now draw a picture (note that p has been placed in the circle defining $\mathbf{w}$ whenever $\mathbf{w} \in \mathbf{V_M}(p)$).

Remark 3: The indexing of $\mathbf{W_M}$, $\mathbf{R_M}$, and $\mathbf{V_M}$ when discussing a particular model $\mathbf{M}$ is sometimes cumbersome. Since it is usually clear in the context of discussing a particular model $\mathbf{M}$ that $\mathbf{W}$, $\mathbf{R}$, and $\mathbf{V}$ are part of $\mathbf{M}$, we shall often suppress their index.

6) We now define φ is *true at* world $\mathbf{w}$ *in* model $\mathbf{M}$ (in symbols: $\models^{\mathbf{M}}_{\mathbf{w}} \varphi$) as follows (suppressing indices):

.1) $\models^{\mathbf{M}}_{\mathbf{w}} p$ iff $\mathbf{w} \in \mathbf{V}(p)$

.2) $\not\models^{\mathbf{M}}_{\mathbf{w}} \bot$ (i.e., not $\models^{\mathbf{M}}_{\mathbf{w}} \bot$)

.3) $\models^{\mathbf{M}}_{\mathbf{w}} \neg\psi$ iff $\not\models^{\mathbf{M}}_{\mathbf{w}} \psi$

.4) $\models^{\mathbf{M}}_{\mathbf{w}} \psi \to \chi$ iff either $\not\models^{\mathbf{M}}_{\mathbf{w}} \psi$ or $\models^{\mathbf{M}}_{\mathbf{w}} \chi$

.5) $\models^{\mathbf{M}}_{\mathbf{w}} \Box\psi$ iff for every $\mathbf{w}' \in \mathbf{W}$, if $\mathbf{Rww}'$, then $\models^{\mathbf{M}}_{\mathbf{w}'} \psi$

We say φ is *false at* $\mathbf{w}$ *in* $\mathbf{M}$ iff $\not\models^{\mathbf{M}}_{\mathbf{w}} \varphi$.

Example: Let us show that the truth conditions of $\Box(p \to q)$ at a world $\mathbf{w}$ are not the same as the truth conditions of $p \to \Box q$ at $\mathbf{w}$. We can do this by describing a model and a world where the former is true but the latter is not. Note that we need a model for the set $\Omega = \{p, q\}$. The previous example was actually chosen for our present purpose, so consider $\mathbf{M}$ and $\mathbf{w}_1$ as specified in the previous example. First, let us see whether $\models^{\mathbf{M}}_{\mathbf{w}_1} \Box(p \to q)$. By (6.5), $\models^{\mathbf{M}}_{\mathbf{w}_1} \Box(p \to q)$ iff for every $\mathbf{w}' \in \mathbf{W}$, if $\mathbf{Rw}_1\mathbf{w}'$, then $\models^{\mathbf{M}}_{\mathbf{w}'} p \to q$. Since $\mathbf{Rw}_1\mathbf{w}_2$ and $\mathbf{Rw}_1\mathbf{w}_3$, we have to check both $\mathbf{w}_2$ and $\mathbf{w}_3$ to see whether $p \to q$ is true there. Well, since $\mathbf{w}_2 \in \mathbf{V}(q)$, it follows by (6.1) that $\models^{\mathbf{M}}_{\mathbf{w}_2} q$, and then by (6.4) that $\models^{\mathbf{M}}_{\mathbf{w}_2} p \to q$. Moreover, since $\mathbf{w}_3 \notin \mathbf{V}(p)$, $\not\models^{\mathbf{M}}_{\mathbf{w}_3} p$ (6.1), and so $\models^{\mathbf{M}}_{\mathbf{w}_3} p \to q$ (6.4). So $p \to q$ is true at all the worlds $\mathbf{R}$-related to $\mathbf{w}_1$. Hence, $\models^{\mathbf{M}}_{\mathbf{w}_1} \Box(p \to q)$.

Now let us see whether $\models^{\mathbf{M}}_{\mathbf{w}_1} p \to \Box q$. By (6.4), $\models^{\mathbf{M}}_{\mathbf{w}_1} p \to \Box q$ iff either $\not\models^{\mathbf{M}}_{\mathbf{w}_1} p$ or $\models^{\mathbf{M}}_{\mathbf{w}_1} \Box q$. That is, iff either $\not\models^{\mathbf{M}}_{\mathbf{w}_1} p$ or for every $\mathbf{w}' \in \mathbf{W}$, if $\mathbf{Rw}_1\mathbf{w}'$, then $\models^{\mathbf{M}}_{\mathbf{w}'} q$ (6.5). But, in the example, $\mathbf{w}_1 \in \mathbf{V}(p)$, and so $\models^{\mathbf{M}}_{\mathbf{w}_1} p$

(6.1). Hence it is not the case that $\not\models^{\mathbf{M}}_{\mathbf{w}_1} p$. So, for every $\mathbf{w}' \in \mathbf{W}$, if $\mathbf{R}\mathbf{w}_1\mathbf{w}'$, then $\models^{\mathbf{M}}_{\mathbf{w}'} q$. Let us check the example to see whether this is true. Since both $\mathbf{R}\mathbf{w}_1\mathbf{w}_2$ and $\mathbf{R}\mathbf{w}_1\mathbf{w}_3$, we have to check both $\mathbf{w}_2$ and $\mathbf{w}_3$ to see whether q is true there. Well, $\models^{\mathbf{M}}_{\mathbf{w}_2} q$, since $\mathbf{w}_2 \in \mathbf{V}(q)$ (6.1). But $\not\models^{\mathbf{M}}_{\mathbf{w}_3} q$, since $\mathbf{w}_3 \notin \mathbf{V}(q)$ (6.1). Consequently, $\not\models^{\mathbf{M}}_{\mathbf{w}_1} p \rightarrow \Box q$.

So we have seen a model $\mathbf{M}$ and world $\mathbf{w}$ such that $\Box(p \rightarrow q)$ is true at $\mathbf{w}$ in $\mathbf{M}$, but $p \rightarrow \Box q$ is not true at $\mathbf{w}$ in $\mathbf{M}$. This shows that the truth conditions of these two formulas are distinct.

Exercise 1: Though we have shown that $\Box(p \rightarrow q)$ can be true while $p \rightarrow \Box q$ false at a world, we don't yet know that the truth conditions of these two formulas are completely independent of one another unless we exhibit a model $\mathbf{M}$ and world $\mathbf{w}$ where $p \rightarrow \Box q$ is true and $\Box(p \rightarrow q)$ is false. Develop such model.

Remark: Note that in the previous example and exercise, we need only develop a model for the set $\Omega = \{p, q\}$ to describe a world where $p \rightarrow \Box q$ is false. We can ignore models for other sets of atomic formulas, and so ignore what $\mathbf{V}$ assigns to any other atomic formula. This should explain why we didn't require that the language be based on the infinite set $\Omega = \{p_1, p_2, \ldots\}$. Had we required that Ω be infinite, then in specifying (falsifying) models for a given formula φ, we would always have to include a catch-all condition indicating what $\mathbf{V}$ assigns to the infinite number of atomic formulas that don't appear in φ.

Exercise 2: Suppose that $\Diamond\varphi$ is a primitive formula of the language. Then the recursive clause for $\Diamond\varphi$ in the definition of $\models^{\mathbf{M}}_{\mathbf{w}}$ is :

$$\models^{\mathbf{M}}_{\mathbf{w}} \Diamond\psi \text{ iff there is a world } \mathbf{w}' \in \mathbf{W} \text{ such that } \mathbf{R}\mathbf{w}\mathbf{w}' \text{ and } \models^{\mathbf{M}}_{\mathbf{w}'} \psi.$$

Prove that: $\models^{\mathbf{M}}_{\mathbf{w}} \Diamond\varphi$ iff $\models^{\mathbf{M}}_{\mathbf{w}} \neg\Box\neg\varphi$.

Exercise 3; Formulate the clause in the definition of $\models^{\mathbf{M}}_{\mathbf{w}}$ which is needed for languages in which '&' is primitive.

Exercise 4: Suppose that $\mathbf{w}$ and $\mathbf{w}'$ agree on all the atomic subformulas in φ and that, in a given model $\mathbf{M}$, for every world $\mathbf{u}$, $\mathbf{R}\mathbf{w}\mathbf{u}$ iff $\mathbf{R}\mathbf{w}'\mathbf{u}$ (i.e., $\{\mathbf{u} \mid \mathbf{R}\mathbf{w}\mathbf{u}\} = \{\mathbf{v} \mid \mathbf{R}\mathbf{w}'\mathbf{v}\}$). Prove that $\models^{\mathbf{M}}_{\mathbf{w}} \varphi$ iff $\models^{\mathbf{M}}_{\mathbf{w}'} \varphi$.

7) We define φ is *true in* model $\mathbf{M}$ (in symbols: $\models^{\mathbf{M}} \varphi$) as follows:

$$\models^{\mathbf{M}} \varphi =_{df} \text{for every } \mathbf{w} \in \mathbf{W}, \models^{\mathbf{M}}_{\mathbf{w}} \varphi$$

We say that a schema S is *true in* $\mathbf{M}$ iff every instance of S is true in $\mathbf{M}$.

Example: Look again at the particular model $\mathbf{M}$ specified in the above example. We know already that $\models^{\mathbf{M}}_{\mathbf{w}_1} \Box(p \rightarrow q)$. To see whether $\models^{\mathbf{M}} \Box(p \rightarrow q)$ (i.e., to see whether $\Box(p \rightarrow q)$ is true in $\mathbf{M}$), we need to check to see whether $\Box(p \rightarrow q)$ is true at both $\mathbf{w}_2$ and $\mathbf{w}_3$. But this is indeed the case, for consider whether $\models^{\mathbf{M}}_{\mathbf{w}_2} \Box(p \rightarrow q)$. That is, by (6.5), consider whether for every $\mathbf{w}' \in \mathbf{W}$, $\models^{\mathbf{M}}_{\mathbf{w}'} p \rightarrow q$. In fact, there is no $\mathbf{w}'$ such that $\mathbf{R}\mathbf{w}_2\mathbf{w}'$. So, by the failure of the antecedent, indeed, every $\mathbf{w}' \in \mathbf{W}_{\mathbf{M}}$, $\models^{\mathbf{M}}_{\mathbf{w}'} p \rightarrow q$. We find the same situation with respect to $\mathbf{w}_3$. So here is an example of a model $\mathbf{M}$ and formula φ ($= \Box(p \rightarrow q)$) such that $\models^{\mathbf{M}} \varphi$.

Exercise 1: Develop a model $\mathbf{M}$ such that $\models^{\mathbf{M}} \Box(p \rightarrow q)$ but not because of any vacuous satisfaction of the definition of truth.

Remark: Note that whereas $\models^{\mathbf{M}}_{\mathbf{w}} \varphi$ if and only if $\not\models^{\mathbf{M}}_{\mathbf{w}} \neg\varphi$ (by 6.3), it is not the case that $\models^{\mathbf{M}} \varphi$ if and only if $\not\models^{\mathbf{M}} \neg\varphi$, though the biconditional does hold in the left-right direction. To see this, suppose that $\models^{\mathbf{M}} \varphi$. Then every world $\mathbf{w} \in \mathbf{W}$ is such that $\models^{\mathbf{M}}_{\mathbf{w}} \varphi$, and so by (6.3), every world $\mathbf{w} \in \mathbf{W}$ is such that $\not\models^{\mathbf{M}}_{\mathbf{w}} \neg\varphi$. But we know that $\mathbf{W}_{\mathbf{M}}$ is nonempty (by 5.1). So there is a world $\mathbf{w} \in \mathbf{W}$ such that $\not\models^{\mathbf{M}}_{\mathbf{w}} \neg\varphi$, i.e., not every world $\mathbf{w} \in \mathbf{W}$ is such that $\models^{\mathbf{M}}_{\mathbf{w}} \neg\varphi$, i.e., $\not\models^{\mathbf{M}} \neg\varphi$. So, by our conditional proof, if $\models^{\mathbf{M}} \varphi$, then $\not\models^{\mathbf{M}} \neg\varphi$.

However, to see that the converse does not hold, we produce a model which constitutes a counterexample. Let $\mathbf{W}_{\mathbf{M}} = \{\mathbf{w}_1, \mathbf{w}_2\}$. Let $\mathbf{R}_{\mathbf{M}}$ be $\{\langle \mathbf{w}_1, \mathbf{w}_2 \rangle\}$ (though it could be empty). And let $\mathbf{V}_{\mathbf{M}}(p) = \{\mathbf{w}_1\}$. Here is the picture:

Note that $\mathbf{w}_1 \in \mathbf{V}(p)$, and so $\models^{\mathbf{M}}_{\mathbf{w}_1} p$. Thus, $\not\models^{\mathbf{M}}_{\mathbf{w}_1} \neg p$. So there is a world $\mathbf{w} \in \mathbf{W}$ such that $\not\models^{\mathbf{M}}_{\mathbf{w}} \neg p$. But this just means that not every world $\mathbf{w} \in \mathbf{W}$ is such that $\models^{\mathbf{M}}_{\mathbf{w}} \neg p$, i.e., $\not\models^{\mathbf{M}} \neg p$. However, since $\mathbf{w}_2 \notin \mathbf{V}(p)$, $\not\models^{\mathbf{M}}_{\mathbf{w}_2} p$, and so there is a world $\mathbf{w} \in \mathbf{W}$ such that $\not\models^{\mathbf{M}}_{\mathbf{w}} p$. Consequently, not every world $\mathbf{w} \in \mathbf{W}$ is such that $\models^{\mathbf{M}}_{\mathbf{w}} p$. So $\not\models^{\mathbf{M}} p$. Thus, we have a model in which $\not\models^{\mathbf{M}} \neg p$ and $\not\models^{\mathbf{M}} p$, which shows that it is not the case that if $\not\models^{\mathbf{M}} \neg p$, then $\models^{\mathbf{M}} p$.

A similar remark should be made in the case of $\varphi \rightarrow \psi$. Note that $\models^{\mathbf{M}}_{\mathbf{w}} \varphi \rightarrow \psi$ if and only if the conditional, if $\models^{\mathbf{M}}_{\mathbf{w}} \varphi$ then $\models^{\mathbf{M}}_{\mathbf{w}} \psi$, holds. However, it is not the case that $\models^{\mathbf{M}} \varphi \rightarrow \psi$ if and only if the conditional, if $\models^{\mathbf{M}} \varphi$ then $\models^{\mathbf{M}} \psi$, holds. Again, $\models^{\mathbf{M}} \varphi \rightarrow \psi$ does imply that if $\models^{\mathbf{M}} \varphi$ then $\models^{\mathbf{M}} \psi$, but the conditional if $\models^{\mathbf{M}} \varphi$ then $\models^{\mathbf{M}} \psi$ does *not* imply that $\models^{\mathbf{M}} \varphi \rightarrow \psi$.

Exercise 2: (a) Prove that $\models^{\mathbf{M}} \varphi \to \psi$ implies that if $\models^{\mathbf{M}} \varphi$ then $\models^{\mathbf{M}} \psi$. (b) Develop a model that shows that the converse does not hold.

8) Finally, we now define φ is *valid* (in symbols: $\models \varphi$) as follows:

$\models \varphi$ iff for every standard model $\mathbf{M}$, $\models^{\mathbf{M}} \varphi$

By convention, we say that a schema S is *valid* iff every instance of S is valid.

Example 1: We show that $\models \Box p \to \Box(q \to p)$.

Exercise 1: (a) Suppose that $\Diamond\varphi$ is a primitive formula of the language, with the truth conditions specified in (6), *Exercise 2*. Show that $\models \Box\varphi \leftrightarrow \neg\Diamond\neg\varphi$. (b) Show that $\models \Box\neg\bot$, and $\models \Box\neg(\varphi \mathbin{\&} \neg\varphi)$.

Example 2: We show that $\models \Box p \to (\Box q \to \Box p)$.

Exercise 2: Show that $\models \Box p \leftrightarrow \neg\neg\Box p$.

Exercise 3: Show that $\models \bot \leftrightarrow (\varphi \mathbin{\&} \neg\varphi)$.

Remark 1: Note the difference between the formulas in *Example 1* and *Exercise 1*, on the one hand, and in *Example 2* and *Exercises 2* and *3*, on the other. Whereas the formulas in the first example and exercise are ordinary modal formulas that are valid, the formulas in *Example 2* and *Exercise 2* and *3* are not only valid, they are instances of tautologies in propositional logic. The validity of these latter formulas does not depend on the truth conditions for modal operators. Rather, it depends solely on the truth conditions for the formula $\bot$ and the propositional connectives $\neg$ and $\to$. Note that in *Example 2* and *Exercises 2* and *3*, we never appeal to the modal clauses in the proof of validity. The reader should check a few other modal formulas that have the form of a propositional tautology to see whether they are valid.

Remark 2: Here is what we plan to do in the next five sections. In §2, we work our way towards a proof that the tautologies *as a class* are valid, since the evidence suggests that they are. In §3, we study in detail the realm of valid and invalid schemata. In §4, we next look at some invalid schemata that nevertheless prove to be valid with respect to certain interesting classes of models in the sense that the schemata are true in all the models of the class. In §5, we shall examine why it is that from the point of view of a certain class $\mathbf{C}$ of models, certain schemata are invalid. Finally, in §6, we investigate some truth and validity preserving relationships among the formulas of our language.

§2: Tautologies are Valid

We turn, then, to the first of our tasks, which is to prove that every tautology is valid. It is important to do this not only for the obvious reason that propositional tautologies had better be valid, but for a less obvious one as well. We may think of the set of tautologies (taken as axioms) and the rule of inference Modus Ponens as constituting *propositional logic*. This propositional logic will constitute the basis of all modal logics (and indeed, we will suppose that it just is the weakest modal logic, for a modal logic will be defined to be any (possibly null) extension of the axioms and rules of propositional logic). So if we can show that all the tautologies are valid, and (in §6) that the rule Modus Ponens preserves validity, we can show that the propositional basis of modal logic is sound, i.e., that every theorem derivable from the set of tautologies using Modus Ponens is valid.

The problem we face first is that we want to distinguish the tautologies in some way from the rest of the formulas that are valid. We can't just use the notion 'true at every world in every model', for that is just the notion of validity. So to prove that the tautologies, as a class, are valid, we have to distinguish them in some way from the other valid formulas. The basic idea we want to capture is that the tautologies have the same form as tautologies in propositional logic. For example, we want to treat the formula $\Box p$ in $\Box p \to (\neg q \to \Box p)$ as a kind of atomic formula. If we treat $\Box p$ as the atomic formula r, then $\Box p \to (\neg q \to \Box p)$ would begin to look like the tautology $r \to (\neg q \to r)$ in propositional logic. So let us think of $\Box p$ as a quasi-atomic subformula of $\Box p \to (\neg q \to \Box p)$, and then consider all the ways of assigning truth-values to all the quasi-atomic subformulas of the language. We can extend each such 'basic' assignment to a 'total' assignment of truth values to all the formulas in the language, and if φ comes out true in all the total assignments, then φ is a tautology. So we need to define the notions of quasi-atomic formula, and basic assignment, and then, finally, total assignment, before we can define the notion of a tautology.

9) If given a language Λ_Ω, we define *the set of quasi-atomic formulas in* Λ_Ω (in symbols: Ω^*) as follows:

$$\Omega^* = \{\varphi \,|\, \varphi = p \text{ (for some } p \in \Omega) \text{ or } \varphi = \Box\psi \text{ (for some } \psi \in \Lambda_\Omega)\}$$

Let p^* be a variable ranging over the members of Ω^*.

Example: If we begin with the set $\Omega = \{p, q\}$ of atomic formulas, then the following are elements of Ω^*: $p, q, \Box p, \Box q, \Box\bot, \Box\Box p, \Box\Box q, \Box\Box\bot, \ldots, \Box\neg p, \Box\neg q, \Box\Box\neg p, \Box\Box\neg\bot, \ldots, \Box(p \to p), \Box(\bot \to \bot), \Box\Box(p \to p), \ldots, \Box(p \to q), \Box\Box(p \to q), \ldots, \Box(\neg p \to p), \ldots$. The important thing to see here is that, in addition to genuine atomic formulas, any complex formula beginning with a $\Box$ is quasi-atomic. Note that there will be only a finite number of quasi-atomic formulas in any given φ.

10) We next define a *basic assignment (of truth values)* to be any function $\mathbf{f}^*$ defined on Ω^* which is such that, for any $p^* \in \Omega^*$, $\mathbf{f}^*(p^*) \in \{T, F\}$.

Remark: Note that from the set $\Omega^* \cup \{\bot\}$ of quasi-atomic formulas, we can generate every formula of the language Λ_Ω by using the connectives $\neg$ and $\to$. That means that all we have to do to extend $\mathbf{f}^*$ to all the formulas of the language is to extend it to formulas of the form $\bot$, $\neg\psi$, and $\psi \to \chi$. Thus, the following recursive definition does give us a total assignment of truth values to all the formulas $\varphi \in \Lambda_\Omega$.

11) We define a *(total) assignment* to be any function $\mathbf{f}$ defined on Λ_Ω which meets the following conditions:

.1) for some $\mathbf{f}^*$, $\mathbf{f}(p^*) = \mathbf{f}^*(p^*)$, for every $p^* \in \Omega^*$ (i.e., $\mathbf{f}$ agrees with some basic assignment $\mathbf{f}^*$ of all the quasi-atomic subformulas in Ω^*)

.2) $\mathbf{f}(\bot) = F$

.3) $\mathbf{f}(\neg\psi) = \begin{cases} T, \text{ if } \mathbf{f}(\psi) = F \\ F, \text{ otherwise} \end{cases}$

.4) $\mathbf{f}(\psi \to \chi) = \begin{cases} T \text{ iff either } \mathbf{f}\,(\psi) = F \text{ or } \mathbf{f}(\chi) = T \\ F \text{ otherwise} \end{cases}$

Whenever $\mathbf{f}(p^*) = \mathbf{f}^*(p^*)$, we say that that $\mathbf{f}$ *extends* or is *based on* $\mathbf{f}^*$, and that $\mathbf{f}^*$ *extends to* $\mathbf{f}$. It now follows that if $\mathbf{f}$ and $\mathbf{f}'$ are both based on $\mathbf{f}^*$, then for every φ, $\mathbf{f}(\varphi) = \mathbf{f}'(\varphi)$:

12) *Theorem*: If $\mathbf{f}$ and $\mathbf{f}'$ are based on the same $\mathbf{f}^*$, then, for any φ, $\mathbf{f}(\varphi) = \mathbf{f}'(\varphi)$.

13) We may now say that a formula φ is a *tautology* iff every assignment $\mathbf{f}$ is such that $\mathbf{f}(\varphi) = T$.

Example: Let us show that $\varphi = \Box p \to (\neg q \to \Box p)$ is a tautology (this particular φ is an instance of the tautology $\psi \to (\chi \to \psi)$). To show that every $\mathbf{f}$ is such that $\mathbf{f}(\varphi) = T$, pick an arbitrary $\mathbf{f}$. Since, $\Box p$ is a quasi-atomic formula, either $\mathbf{f}(\Box p) = T$ or $\mathbf{f}(\Box p) = F$ (since $\mathbf{f}$ agrees with some $\mathbf{f}^*$). If the latter, then by (11.4), $\mathbf{f}(\varphi) = T$. If the former, then by (11.4), $\mathbf{f}(\neg q \to \Box p) = T$. So again by (11.4), $\mathbf{f}(\varphi) = T$.

Exercise 1: (a) Show that $\neg\bot$ is a tautology. (b) Show that $(\varphi \to \bot) \leftrightarrow \neg\varphi$ is a tautology.

Remark: These examples and exercises show that our definition of a *tautology* allows us to prove that certain formulas are tautologies. The definition is reasonably simple and serves us well in subsequent work. But, for arbitrary φ, there is no mechanical way of finding arguments such as the one in the above *Example* that establish that φ is a tautology if indeed it is. Moreover, you can't mechanically use the definition to show, for a given tautology, that indeed it is a tautology, since you can't check every assignment $\mathbf{f}$. Even if we start with a language based on the set $\Omega = \{p_1\}$, it would take a very long time to even specify a basic assignment of the quasi-atomic subformulas (since, as we have seen, Ω^* will be an infinite set). So the definition of 'tautology' *per se* doesn't offer a mechanical procedure to discover, for a given φ, whether or not φ is a tautology, since strictly speaking, you would have to check an infinite number of assignments (none of which you can even specify completely).

But we know from work in propositional logic that the truth table method gives us a mechanical procedure by which we can discover whether or not a given φ is a tautology. Have we lost anything in the move to modal logic? Actually, we haven't, for there is a way to construct such a decision procedure that tests for tautologyhood. Such a procedure will be described in the *Digression* that follows (disinterested readers, or readers who don't wish to interrupt the train of development of the concepts, may skip directly ahead to (14)).

Digression: It is best to intuitively demonstrate our procedure by example first, and then make it precise it afterwards. Suppose you want to test whether φ in the example immediately above is a tautology. Note that the following are subformulas of φ: p, q, $\neg q$, $\Box p$, $\neg q \to \Box p$, and φ itself. Now of all of these subformulas, only five are relevant to the truth functional analysis of φ: q, $\neg q$, $\Box p$, $q \to \Box p$, and φ itself. Let us call these the truth-functionally relevant (TFR) subformulas of φ. Note that p is not a TFR-subformula of our particular φ, because the truth value of the subformula of φ in which it is contained, namely $\Box p$, does *not* depend on

p; so p is not relevant to the truth functional analysis of φ. And out of the subformulas that are relevant, only two are quasi-atomic: q and $\Box p$. Thus, φ's truth value, from a truth functional point of view, depends just on the value of q and $\Box p$. So, really, all of the basic assignment functions relevant to the truth functional evaluation of φ fall into the following four classes:

$$F_1^* = \{\mathbf{f}^* | \mathbf{f}^*(q) = T \text{ and } \mathbf{f}^*(\Box p) = T\}$$

$$F_2^* = \{\mathbf{f}^* | \mathbf{f}^*(q) = T \text{ and } \mathbf{f}^*(\Box p) = F\}$$

$$F_3^* = \{\mathbf{f}^* | \mathbf{f}^*(q) = F \text{ and } \mathbf{f}^*(\Box p) = T\}$$

$$F_4^* = \{\mathbf{f}^* | \mathbf{f}^*(q) = F \text{ and } \mathbf{f}^*(\Box p) = F\}$$

In general, if there are n quasi-atomic TFR-subformulas in φ, there are 2^n different classes of relevant basic assignment functions. Each class F_i^* defines a row in a truth table, and each of the quasi-atomic TFR-subformulas defines a column. To complete the truth table, we define a new column for $\neg q$, a column for $\neg q \to \Box p$, and finally, a column for φ itself. That is, we define a new column for each of the other TFR-subformulas of φ. Now the value in the final column (headed by φ) on row F_i^* represents the class F_i of all assignments $\mathbf{f}$ that (1) agree with a member $\mathbf{f}^*$ of F_i^* on the quasi-atomic TFR-subformulas in φ and (2) make a final truth functional assignment to φ by extending $\mathbf{f}^*$ in a way that satisfies the definition of $\mathbf{f}$ in (11). If we have set things up properly, each of the assignments $\mathbf{f} \in F_i$ should agree on the truth value of φ (since each is based on a basic assignment in F_i^*, all of which agree on the relevant quasi atomics in φ). Consequently, if for every i, each member $\mathbf{f}$ of F_i is such that $\mathbf{f}(\varphi) = T$ (i.e., if the value T appears in every row of the final column of the truth table), then φ is a tautology. This is our mechanical procedure for checking whether φ is a tautology. The reader should check that T does appear in every row under the column headed by φ in the above example.

Of course, this intuitive description of a decision procedure depends on our having a precise way to delineate of the truth-functionally relevant subformulas of φ, and on a proof that whenever $\mathbf{f}$ and $\mathbf{f}'$ agree on the relevant quasi-atomic formulas in φ, then they agree on φ. The latter shall be an exercise. For the former, consider the following definition of *truth-functionally relevant subformula* should work:

1. φ is a TFR-subformula of φ.
2. If $\varphi = \neg\psi$, or $\psi \to \chi$, then ψ, χ are TFR-subformulas of φ
3. If ψ is a TFR-subformula of χ and χ is a TFR-subformula of φ, then ψ is a TFR-subformula of φ.

Note that there is no clause for $\varphi = \Box\psi$, since in this case, ψ would not be a TFR-subformula of φ.

Here, then, is our decision procedure for determining whether φ is a tautology:

1. Determine the set of TFR-subformulas of φ (this will be a finite set, and the set can be determined by applying the definition of TFR-subformula a finite number of times).

2. Isolate from this class the formulas that are quasi-atomic (this will also be a finite set).

3. Begin the construction of a truth table, with each quasi-atomic TRF-subformula heading a column (if there are n quasi-atomic truth functionally relevant subformulas, there will be 2^n rows in the truth table).

4. Extend the truth table to all the other TFR-subformulas in φ (with φ heading the final column), filling in the truth table in the usual way.

5. If every row under the column marked φ is the value T, then φ is a tautology.

There is a way of checking this whole procedure. And that is, after isolating the quasi-atomic TRF-subformulas of φ, find all the ones *that begin with a* $\Box$. Replace each such formula in φ with a new atomic formula not in φ, and the result should be a tautology in propositional logic. For example, the quasi-atomic TFR-subformulas of $\varphi = \Box p \to (\neg q \to \Box p)$ are q and $\Box p$. Replace $\Box p$ in φ with a new atomic formula not already in φ, say r. The result is: $r \to (\neg q \to r)$, and the reader may now employ the usual decision procedures of propositional logic to verify that this is indeed a tautology of propositional logic.

Exercise 2: Show that if $\mathbf{f}$ and $\mathbf{f}'$ agree on the TFR-subformulas of φ, then $\mathbf{f}(\varphi) = \mathbf{f}'(\varphi)$.

14) We now identify a particular kind of basic assignment; they are defined to have a special property which is inherited by any (total) assignment based on them and which plays an important role in the proof that every tautology is valid. Each language Λ_Ω, model $\mathbf{M}$, and world $\mathbf{w} \in \mathbf{W_M}$ determines a unique basic assignment function $\mathbf{f}^*_\mathbf{w}$ as follows:

for every $p^* \in \Omega^*$, $\mathbf{f}^*_\mathbf{w}(p^*) = T$ iff $\models^\mathbf{M}_\mathbf{w} p^*$

We call $\mathbf{f}^*_\mathbf{w}$ the basic assignment *determined by* $\mathbf{M}$ *and* $\mathbf{w}$. Note that corresponding to $\mathbf{f}^*_\mathbf{w}$, there is a (total) assignment $\mathbf{f_w}$ (based on $\mathbf{f}^*_\mathbf{w}$) of every $\varphi \in \Lambda_\Omega$. We call $\mathbf{f_w}$ the total assignment determined by $\mathbf{M}$ and $\mathbf{w}$.

15) *Lemma*: For any Λ_Ω, $\mathbf{M}$, $\mathbf{w} \in \mathbf{W_M}$, and φ, $\mathbf{f_w}(\varphi) = T$ iff $\models^\mathbf{M}_\mathbf{w} \varphi$.

Proof: By induction on Λ_Ω.

16) *Theorem*: $\models \varphi$, for every tautology φ.

Proof: Appeal to (15).

Alternative §2: Tautologies are Valid (following Enderton)

In some developments of propositional logic (Enderton's, for example), the notion of tautology is: φ is a tautology iff φ is true in all the extensions of basic assignments of its atomic subformulas.[9] The difference here is that instead of being defined for all the atomic formulas in the language, basic assignments $\mathbf{f}^*$ are defined relative to arbitrary sets of atomic formulas. The basic assignments *for* a given formula φ will be functions that assign truth values to every member of the set of atomic subformulas in φ. An extended assignment $\mathbf{f}$ is then defined relative to a basic assignment $\mathbf{f}^*$, and extends $\mathbf{f}^*$ to all the formulas that can be constructed out of the set of atomic formulas over which $\mathbf{f}^*$ is defined. So, for a given formula φ, $\mathbf{f}$ extends a given basic assignment $\mathbf{f}^*$ by being defined on all the formulas that can be constructed out of the set of atomic subformulas in φ. Such $\mathbf{f}$s will therefore be defined on all of the subformulas in φ, including φ itself. The definition of a tautology, then, is: φ is a tautology iff for every basic assignment $\mathbf{f}^*$ (of the atomic subformulas in φ), the extended assignment $\mathbf{f}$ (based on $\mathbf{f}^*$) assigns φ the value T.

[9]See Herbert Enderton, *A Mathematical Introduction to Logic*, New York: Academic Press, 1972.

One advantage of doing things this way is that for any given formula φ, there will be only a finite number of basic assignments, since there will always be a finite number of atomic subformulas in φ. Whenever there are n atomic subformulas of φ, there will be 2^n basic assignment functions. Thus, our decision procedure for determining whether an arbitrary φ is a tautology will simply be: check all the basic assignments $\mathbf{f}^*$ to see whether $\mathbf{f}$ assigns φ the value T.

In this section, we redevelop the definitions of the previous section for those readers who prefer Enderton's definition of tautology. The twist is that we have to define basic assignments relative to a given set of *quasi*-atomic formulas. So for any given φ, the basic assignments $\mathbf{f}^*$ will be defined on the set of quasi-atomic subformulas in φ. Then we extend those basic assignments to total assignments defined on all the formulas constructible from such sets of quasi-atomics (these will therefore be defined for the subformulas of φ and φ itself). To accomplish all of this, we need to define the notions of subformula, quasi-atomic formula, and basic truth assignment to a set of quasi-atomic formulas, and then, finally, extended assignment, before we can define the notion of a tautology. Readers who are not familiar with Enderton's method, or who have little interest in seeing how the method is adapted to our modal setting, should simply skip ahead to §3.

8.5) We begin with the notion of subformula. Given the definition of subformula in (3), we define, for each $\varphi \in \Lambda_\Omega$, *the set of subformulas of* φ (in symbols: $Sub(\varphi)$) inductively as follows:

$$Sub(\varphi) =_{df} \{\psi | \, \psi \text{ is a subformula of } \varphi\}$$

Example: Consider the tautology $\varphi = \Box p \rightarrow (\neg q \rightarrow \Box p)$, which is an instance of the tautology $\psi \rightarrow (\chi \rightarrow \psi)$. By (3.1), φ is a subformula of φ, so φ is a member of $Sub(\varphi)$. By (3.2), $\Box p$ and $\neg q \rightarrow \Box p$ are members of $Sub(\varphi)$. By (3.2), p is a subformula of $\Box p$, and so by (3.3), p is a member of $Sub(\varphi)$. Finally, by (3.2), q is a subformula of $\neg q$, and so by (3.3), q is a member of $Sub(\varphi)$. So $Sub(\varphi) = \{p, q, \neg q, \Box p, (\neg q \rightarrow \Box p), \varphi\}$. Note that for any formula φ, $Sub(\varphi)$ is a finite set, since there are only a finite number of steps in the construction of φ from its basic atomic constituents.

9) If given a language Λ_Ω, we define *the set of quasi-atomic formulas in* Λ_Ω (in symbols: Ω^*) as follows:

$$\Omega^* = \{\varphi \,|\, \varphi = p \text{ (for some } p \in \Omega) \text{ or } \varphi = \Box\psi \text{ (for some } \psi \in \Lambda_\Omega)\}$$

Let p^* be a variable ranging over the members of Ω^*.

Example: If we begin with the set $\Omega = \{p, q\}$ of atomic formulas, then the following are elements of Ω^*: $p, q, \Box p, \Box q, \Box\bot, \Box\Box p, \Box\Box q, \Box\Box\bot, \ldots, \Box\neg p, \Box\neg q, \Box\Box\neg p, \Box\Box\neg\bot, \ldots, \Box(p \to p), \Box(\bot \to \bot), \Box\Box(p \to p), \ldots, \Box(p \to q), \Box\Box(p \to q), \ldots, \Box(\neg p \to p), \ldots$. The important thing to see here is that every complex formula beginning with a $\Box$ is quasi-atomic.

9.5 We now define the *the set of quasi-atomic subformulas in* φ (in symbols: Ω^*_φ) as:

$$\Omega^*_\varphi = \Omega^* \cap Sub(\varphi)$$

Example 1: If $\varphi = \Box p \to (\neg q \to \Box p)$, then $\Omega^*_\varphi = \{p, q, \Box p\}$. Note that since $Sub(\varphi)$ is finite, so is Ω^*_φ.

Example 2: If $\varphi = \Box\varphi \to \bot$, then $\Omega^*_\varphi = \{\Box\varphi\}$. Though $\bot$ is a subformula of φ, it is not quasi-atomic.

10) Next we define a basic assignment for a set of quasi-atomic formulas. If given any set Γ^* of quasi-atomic formulas (i.e., if given any subset Γ^* of Ω^*), we say that $\mathbf{f}^*$ is a basic assignment function *for* Γ^* iff $\mathbf{f}^*$ maps every $p^* \in \Gamma^*$ to a member of $\{T, F\}$. Note that if Γ^* has n members, there are 2^n basic assignment functions for Γ^*.

Exercise: Consider φ in the above example. Describe the basic assignment functions for the set Ω^*_φ.

Remark: Note that from the set $\Omega^*_\varphi \cup \{\bot\}$, we can always regenerate φ by using the connectives $\neg$ and $\to$. For example, let $\varphi = \Box p \to (\neg q \to \Box p)$. Then the set Ω^*_φ of quasi-atomic formulas in φ is, as we saw above, $\{p, q, \Box p\}$. But from q we can generate $\neg q$, and from $\neg q$ and $\Box p$ we can generate $(\neg q \to \Box p)$, and from this latter formula, we can generate $\Box p \to (\neg q \to \Box p) = \varphi$. The point of considering this is that we now want to extend basic assignments of the set Ω^*_φ to assignment functions that cover all of the subformulas of φ, including φ itself. So to define such extended assignment functions, we need to extend basic assignments of Ω^*_φ to the set of formulas generated from $\Omega^*_\varphi \cup \{\bot\}$ using $\neg$ and $\to$, for φ itself will then be in the domain of such a function. So let us introduce notation to denote this set.

10.5) If given a set of quasi-atomic formulas Γ^*, we can generate a set of formulas from $\Gamma^* \cup \{\bot\}$ by using the connectives $\neg$ and $\rightarrow$. Let $Fml^*(\Gamma^* \cup \{\bot\})$ denote the set of formulas generated from $\Gamma^* \cup \{\bot\}$ using $\neg$ and $\rightarrow$.

Remark: Note that when $\Gamma^* = \Omega^*$ (that is, when the set of quasi-atomic formulas Γ^* is the entire set of quasi-atomic formulas for our language Λ_Ω), then $Fml^*(\Gamma^* \cup \{\bot\}) = Fml(\Omega)$ (i.e., $= \Lambda_\Omega$). In other words, from $\Omega^* \cup \{\bot\}$, we can generate every formula in our language by using the connectives $\neg$ and $\rightarrow$.

11) If given a basic assignment $\mathbf{f}^*$ for a set Γ^*, we define *the extended assignment function* $\mathbf{f}$ *of* $\mathbf{f}^*$ to be the function defined on the set of formulas generated from $\Gamma^* \cup \{\bot\}$ using $\neg$ and $\rightarrow$ (i.e., defined on $Fml^*(\Gamma^* \cup \{\bot\})$) that meets the following conditions:

.1) $\mathbf{f}(p^*) = \mathbf{f}^*(p^*)$, for every $p^* \in \Gamma^*$ (i.e., $\mathbf{f}$ agrees with $\mathbf{f}^*$ on the quasi-atomic subformulas in Γ^*)

.2) $\mathbf{f}(\bot) = F$

.3) $\mathbf{f}(\neg\psi) = \begin{cases} T, \text{ if } \mathbf{f}(\psi) = F \\ F, \text{ otherwise} \end{cases}$

.4) $\mathbf{f}(\psi \rightarrow \chi) = \begin{cases} T \text{ iff either } \mathbf{f}(\psi) = F \text{ or } \mathbf{f}(\chi) = T \\ F \text{ otherwise} \end{cases}$

Note that when $\mathbf{f}^*$ is a assignment for the set Ω^*_φ of quasi-atomic formulas in φ, then φ is in the domain of $\mathbf{f}$, since given the *Remark* in (10), φ is in $Fml^*(\Omega^*_\varphi \cup \{\bot\})$.

Remark: Clearly, there is an extended assignment $\mathbf{f}$ for every $\mathbf{f}^*$. In addition, however, for $\mathbf{f}$ to be well-defined, we need to show that there is a unique extended assignment $\mathbf{f}$ for a given $\mathbf{f}^*$:

12 *Theorem*: Let $\mathbf{f}^*$ be a assignment of Γ^*. Then if $\mathbf{f}$ and $\mathbf{f}'$ both extend $\mathbf{f}^*$ to all the formulas in $Fml^*(\Gamma^* \cup \{\bot\})$, then $\mathbf{f} = \mathbf{f}'$.

Remark: In what follows, we correlate the variables $\mathbf{f}^*$ and $\mathbf{f}$, and we sometimes say that $\mathbf{f}$ *extends* $\mathbf{f}^*$.

13) Finally, we may say: φ is a tautology iff for every basic assignment $\mathbf{f}^*$ of Ω^*_φ, $\mathbf{f}(\varphi) = T$ (i.e., iff for every basic assignment of the set of quasi-atomic subformulas of φ, the extended assignment $\mathbf{f}$ assigns φ the value T).

Example: Let us show that $\varphi = \Box p \to (\neg q \to \Box p)$ is a tautology. To show that every $\mathbf{f}^*$ is such that $\mathbf{f}(\varphi) = T$, pick an arbitrary $\mathbf{f}^*$. Since, $\Box p$ is a quasi-atomic formula, either $\mathbf{f}^*(\Box p) = T$ or $\mathbf{f}^*(\Box p) = F$. If the latter, then by (11.4), $\mathbf{f}(\varphi) = T$. If the former, then by (11.4), $\mathbf{f}(\neg q \to \Box p) = T$. So again by (11.4), $\mathbf{f}(\varphi) = T$.

Remark: Not only does our definition allow us to prove that a given formula φ is a tautology, it gives us a decision procedure for determining, for an arbitrary φ, whether or not φ is a tautology. The set of quasi-atomic subformulas of φ (Ω^*_φ) is finite. Suppose it has n members. Then we have only to check 2^n basic assignments $\mathbf{f}^*$ and determine, in each case, whether $\mathbf{f}$ assigns φ the value T. So our modal logic has not lost any of the special status that propositional logic has with regard to the tautologies. Indeed, there is a simple way to show that tautologies in our modal language correspond with tautologies in propositional language. An example shows the relationship. Again let $\varphi = \Box p \to (\neg q \to \Box p)$. Recall the set of quasi-atomics in φ is $\{p, q, \Box p\}$. Now replace each quasi-atomic subformula of φ beginning with a $\Box$ by a new propositional letter that is not a subformula of φ, say r. The result is: $r \to (\neg q \to r)$, and the reader may now use the decision procedures of propositional logic to verify that this is a tautology in any propositional language that generates formulas from the set $\{q, r\}$ by using the connectives $\neg$ and $\to$.

14) We now identify, relative to each model $\mathbf{M}$ and world $\mathbf{w}$, a particular basic assignment; it is defined to have a special property which is inherited by any extended based on it and which plays an important role in the proof that every tautology is valid. For any given $\varphi \in \Lambda_\Omega$, each model $\mathbf{M}$ and world $\mathbf{w} \in \mathbf{W}$ defines a unique basic assignment function $\mathbf{f}^*_{\mathbf{w}}$ of the set Ω^*_φ of quasi-atomic subformulas in φ as follows:

for every $p^* \in \Omega^*_\varphi$, $\mathbf{f}^*_{\mathbf{w}}(p^*) = T$ iff $\models^{\mathbf{M}}_{\mathbf{w}} p^*$

We call $\mathbf{f}^*_{\mathbf{w}}$ the *basic assignment* of Ω^*_φ *determined by* $\mathbf{M}$ *and* $\mathbf{w}$. Note that given $\mathbf{f}^*_{\mathbf{w}}$, we have defined a unique extended assignment $\mathbf{f}_{\mathbf{w}}$ which assigns φ a truth value.

15) *Lemma*: For any Λ_Ω, $\mathbf{M}$, $\mathbf{w} \in \mathbf{W_M}$, φ, and $\mathbf{f}^*_{\mathbf{w}}$ of Ω^*_φ, $\mathbf{f}_{\mathbf{w}}(\varphi) = T$ iff $\models^{\mathbf{M}}_{\mathbf{w}} \varphi$.

16) *Theorem*: $\models \varphi$, for every tautology φ.

Proof: Appeal to (15).

§3: Validities and Invalidities

We now look at a variety of non-tautologous, but nevertheless valid, schemata. Valid schemata bear an interesting relationship to the schema K ($= \Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$), which we prove to be valid in (17). The axiom schema K plays a role in defining the weakest *normal modal logic* K. A normal modal logic is a modal logic which has the tautologies and instances of K as axioms, and which has as theorems all of the formulas derivable from these axioms by using Modus Ponens and the Rule of Necessitation (i.e., the rule: from φ, infer $\Box\varphi$). Note that we distinguish the axiom K, written in Roman, from the logic K, written in italic; we abide by this convention of writing names of logics in italics throughout. Now when we prove in later chapters that the logic K is *complete*, we show that every valid formula is a theorem of K. That means that all of the instances of the other schemata that we prove to be valid in this section will be theorems of K. In later chapters, we also prove that the logic K is *sound*, that is, that every theorem of K is valid. We do this in part by showing that the axiom K is valid and that the Rule of Necessitation preserves validity. This, together with our demonstration that the tautologies are valid and that Modus Ponens preserves validity, guarantees the soundness of K, for there are no other theorems of K besides the tautologies, instances of the axiom K, and the formulas provable from these axioms by Modus Ponens or the Rule of Necessitation.

17) *Theorem*: $\models \Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$.

18) *Theorem*: The following schemata are valid:

$$\Box(\varphi \to \psi) \to (\Diamond\varphi \to \Diamond\psi)$$
$$\Diamond(\varphi \to \psi) \to (\Box\varphi \to \Diamond\psi)$$
$$\Box(\varphi \mathbin{\&} \psi) \leftrightarrow (\Box\varphi \mathbin{\&} \Box\psi)$$
$$\Box\varphi \to \Box(\psi \to \varphi)$$
$$\neg\Diamond\varphi \to \Box(\varphi \to \psi)$$
$$\neg\Diamond\bot$$
$$(\Diamond\varphi \vee \Diamond\psi) \to \Diamond(\varphi \vee \psi)$$
$$\Diamond(\varphi \vee \psi) \to (\Diamond\varphi \vee \Diamond\psi)$$

19) Now that we have looked at a wide sample of valid schemata, let us look at a sample of invalid ones. To show that a formula φ is invalid, we construct a model $\mathbf{M}$ and world $\mathbf{w}$ where $\not\models^{\mathbf{M}}_{\mathbf{w}} \varphi$. Such models are called *falsifying models*. To show that schema S is invalid, we build a falsifying model for an instance of S. The easiest way to construct a falsifying model $\mathbf{M}$ for φ is to build a picture of $\mathbf{M}$. Draw a large rectangle to represent the set of worlds $\mathbf{W_M}$. Then draw a circle to represent the world $\mathbf{w}$ where φ is going to be false. If the formula we plan to falsify is the conditional $\varphi \to \psi$, we suppose that φ is true at $\mathbf{w}$ (by sticking it in the circle) and that ψ is false at $\mathbf{w}$ (by sticking $\neg\psi$ into the circle). If the formula we plan to falsify is $\Box\varphi$, we suppose $\neg\Box\varphi$ is true at $\mathbf{w}$ (sticking $\neg\Box\varphi$ into the circle). After doing this, we fill out the details of the model, by adding ψ-worlds accessible from $\mathbf{w}$ whenever $\Diamond\psi$ is true at $\mathbf{w}$ (and $\neg\psi$-worlds when $\neg\Box\psi$, i.e., $\Diamond\neg\psi$, is true at $\mathbf{w}$). To add ψ-worlds accessible to $\mathbf{w}$ to our picture, we draw another circle, label it (say as '$\mathbf{w}'$'), draw an arrow from $\mathbf{w}$ to $\mathbf{w}'$ (to represent the accessibility relationship), and insert ψ into $\mathbf{w}'$. Be sure to add χ at every accessible world introduced whenever $\Box\chi$ is true at $\mathbf{w}$. If all we have at $\mathbf{w}$ is a formula of the form $\Box\chi$, it is best to add at least one $\mathbf{R}$-related world where φ is true. We proceed in this fashion until we have reached the atomic subformulas of φ. Of course, it is essential that we do this in enough detail to ensure that we have not constructed an incoherent description of a model, i.e., a model such that for some world $\mathbf{w}$, both p and $\neg p$ are true at $\mathbf{w}$. So, if we can build a coherent picture in which $\neg\varphi$, we know that there is a model and world where φ is false. So φ is not valid. Once we have developed a picture that convinces us that a formula φ is not valid, we can always decode our picture into a formal description of a model $\mathbf{M}$, and give a formal proof that there is a world $\mathbf{w} \in \mathbf{W}$ such that $\not\models^{\mathbf{M}}_{\mathbf{w}} \varphi$.

Example 1: We build a falsifying model for an instance of D ($= \Box\varphi \to \Diamond\varphi$) with a single world and the empty accessibility relation.

Remark 1: Notice that if we were to add any other world $\mathbf{w}'$ and allow $\mathbf{w}$ to access it, the model would become incoherent, for we would have to add p to $\mathbf{w}'$ (since $\Box p$ is true at $\mathbf{w}$) and $\neg p$ to $\mathbf{w}'$ (since $\neg\Diamond p$, i.e., $\Box\neg p$, is true at $\mathbf{w}$).

Example 2: We build a falsifying model for an instance of T (with $\Box p$ and $\neg p$ true at $\mathbf{w}$ and p true at accessible world $\mathbf{w}'$).

Remark 2: Notice that the picture would become incoherent were $\mathbf{w}$ accessible from $\mathbf{w}'$.

20) *Theorem*: The following schemata are not valid.

$\Box(\varphi \lor \psi) \rightarrow (\Box\varphi \lor \Box\psi)$
$(\Diamond\varphi \;\&\; \Diamond\psi) \rightarrow \Diamond(\varphi \;\&\; \psi)$
$\varphi \rightarrow \Box\Diamond\varphi$ ('B')
$\Box\varphi \rightarrow \Box\Box\varphi$ ('4')
$\Diamond\varphi \rightarrow \Box\Diamond\varphi$ ('5')
$\Diamond\Box\varphi \rightarrow \Box\Diamond\varphi$ ('G')

21) *Exercise:* Determine whether the following are invalid by trying to construct a falsifying model. Note that if your attempts to produce a falsifying model always end in incoherent pictures, it may be because φ is valid. Prove that φ is invalid, if it is invalid, or valid, if it is valid:

$\varphi \rightarrow \Box\varphi$
$(\neg\Diamond\varphi \;\&\; \Diamond\psi) \rightarrow \Diamond(\neg\varphi \;\&\; \psi)$
$\Box\Diamond\varphi \rightarrow \varphi$
$(\Box\varphi \;\&\; \Diamond\psi) \rightarrow \Diamond(\varphi \;\&\; \psi)$
$\Box\Box\varphi \rightarrow \Box\varphi$
$\Diamond(\varphi \rightarrow \psi) \lor \Box(\psi \rightarrow \varphi)$
$\Box\Diamond\varphi \rightarrow \Diamond\varphi$
$(\Diamond\varphi \rightarrow \Box\psi) \rightarrow (\Box\varphi \rightarrow \Box\psi)$
$\Diamond\varphi \rightarrow \Box\varphi$
$\Box\Diamond\varphi \rightarrow \Diamond\Box\varphi$

§4: Validity with respect to a Class of Models and Validity on Frames and Classes of Frames

In this section, we look at some invalid schemata that nevertheless prove to be valid with respect to a certain class of models, in the sense of being true in all the models of a certain class. Some of these formula will be valid with respect to models in which the set of worlds has a certain size. However, our principle focus shall be on formulas true in all models in which the accessibility relation meets a certain interesting condition. The schema 4 proves to be valid with respect to the class of models having a transitive accessibility relation. Results of this kind are important for our work in later chapters. Once we have seen that a schema S is valid with respect to a certain interesting class **C** of models, we will be in a position to show that the normal modal logic based on S (i.e., having S as an axiom schema) is sound with respect to **C**, i.e., that every theorem of the logic based on S is valid with respect to **C**. For example, we shall prove that every theorem of the normal modal logic *K4* is valid with respect to the class of transitive models (and therefore is sound with respect to this class). The modal logic *K4* has the tautologies, instances of K, and instances of the 4 schema as axioms, and has as theorems all of the formulas derivable from these by Modus Ponens and the Rule of Necessitation. Still later on, when we prove that a normal modal logic is complete with respect to a class of models $\mathbf{C}$, we show that all the formulas valid with respect to $\mathbf{C}$ are theorems of the logic. For example, when we show that *K4* is complete with respect to the class of transitive models, we show that the formulas which are valid in the class of transitive models are theorems of *K4*.

22) Let us now define φ *is valid with respect to* a class $\mathbf{C}$ of standard models (in symbols: $\mathbf{C} \models \varphi$) as follows:

$$\mathbf{C} \models \varphi =_{df} \text{for every } \mathbf{M} \in \mathbf{C}, \models^{\mathbf{M}} \varphi$$

We say that a schema S is valid with respect to $\mathbf{C}$ iff all of the instances of S are valid with respect to $\mathbf{C}$.

Remark: Clearly, any formula that is valid *simpliciter* is valid in every class of models (i.e., if $\models \varphi$, then $\mathbf{C} \models \varphi$, for any class $\mathbf{C}$). So the tautologies and other valid formula we have studied so far are valid with respect to every class $\mathbf{C}$. However, many of the invalid formulas we've studied prove to be true in all the models of a certain interesting class. We say 'interesting' class because every non-valid non-contradiction is true in at least some trivial class of models, namely, the class of models in which it is true. But there are some non-valid non-contradictions that are valid with respect to the class of all models meeting a certain non-trivial condition.

Example 1: We show that $\varphi \rightarrow \Box\varphi$ is valid with respect to the class of models having a single world in $\mathbf{W}$. Since we know, for every $\mathbf{M}$, that $\mathbf{W_M}$ must have at least one world, we may define the class $\mathbf{C}_1$ of single world models as follows: $\mathbf{M} \in \mathbf{C}_1$ iff for every $\mathbf{w}, \mathbf{w}' \in \mathbf{W_M}$, $\mathbf{w} = \mathbf{w}'$. We now show that $\mathbf{C}_1 \models \varphi \rightarrow \Box\varphi$. Pick an arbitrary $\mathbf{M} \in \mathbf{C}_1$ and $\mathbf{w} \in \mathbf{W_M}$. Either $\models^{\mathbf{M}}_{\mathbf{w}} \varphi$ or $\not\models^{\mathbf{M}}_{\mathbf{w}} \varphi$. If the latter, then $\models^{\mathbf{M}}_{\mathbf{w}} \varphi \rightarrow \Box\varphi$. If the former, then suppose that $\mathbf{Rww}'$, for some arbitrary $\mathbf{w}'$. Since $\mathbf{M} \in \mathbf{C}_1$, we know that $\mathbf{w} = \mathbf{w}'$. So we know that $\models^{\mathbf{M}}_{\mathbf{w}'} \varphi$. Consequently, by conditional proof, if $\mathbf{Rww}'$, then $\models^{\mathbf{M}}_{\mathbf{w}'} \varphi$, and since $\mathbf{w}'$ was arbitrary, we know that for every $\mathbf{w}'$, if $\mathbf{Rww}'$, then $\models^{\mathbf{M}}_{\mathbf{w}'} \varphi$. So $\models^{\mathbf{M}}_{\mathbf{w}} \Box\varphi$, by (6.5). So by (6.4),

$\models^{\mathbf{M}}_{\mathbf{w}} \varphi \to \Box\varphi$. So, by disjunctive syllogism, it follows in either case that $\models^{\mathbf{M}}_{\mathbf{w}} \varphi \to \Box\varphi$. So since $\mathbf{M}$ and $\mathbf{w}$ were arbitrarily chosen, $\mathbf{C}_1 \models \varphi \to \Box\varphi$.

Example 2: We show that $\psi \to \Box\varphi$ and $\Diamond\varphi \to \psi$ are valid with respect to the class of models in which the accessibility relation is empty, i.e., in which no worlds are R-related to each other. In such models, for any world $\mathbf{w}$, $\models^{\mathbf{M}}_{\mathbf{w}} \Box\varphi$, since it is vacuously true that for every $\mathbf{w}'$, if $\mathbf{Rww}'$ then $\models^{\mathbf{M}}_{\mathbf{w}'} \varphi$. And so, for any world $\mathbf{w}$, we always find that $\models^{\mathbf{M}}_{\mathbf{w}} \psi \to \Box\varphi$, for any formula ψ. Moreover, in models with an empty accessibility relation, it is never the case that there is a $\mathbf{w}'$ such that both $\mathbf{Rww}'$ and $\models^{\mathbf{M}}_{\mathbf{w}'} \varphi$. So, $\not\models^{\mathbf{M}}_{\mathbf{w}} \Diamond\varphi$, for any world $\mathbf{w}$. Thus, for any world, $\models^{\mathbf{M}}_{\mathbf{w}} \Diamond\varphi \to \psi$, for any formula ψ. So $\Diamond\varphi \to \psi$ is valid with respect to models in which the accessibility relation is empty.

Remark 2: The two examples we just looked at show us how invalid formulas can be valid with respect to a class of models, where the models in the class satisfy a certain somewhat interesting condition. In the next subsection, we look at formulas that are valid with respect to a classes of models satisfying even more interesting conditions. For example, we discover that instances of the schema T ($= \Box\varphi \to \varphi$) (which we already know are invalid) are valid with respect to the class of models in which the accessibility relation is reflexive. Of course, by redefining the notion of a model so that all models are stipulated to have reflexive accessibility relations, it would follow that the T schema is valid *simpliciter*. But instead of doing this, we just use the relative definition of validity.

23) Consider the following list of properties of a binary relation R:[10]

P$_1$) $\forall u \exists v Ruv$ ('serial')

P$_2$) $\forall u Ruu$ ('reflexive')

P$_3$) $\forall u \forall v(Ruv \to Rvu)$ ('symmetric')

P$_4$) $\forall u \forall v \forall w(Ruv \,\&\, Rvw \to Ruw)$ ('transitive')

P$_5$) $\forall u \forall v \forall w(Ruv \,\&\, Ruw \to Rvw)$ ('euclidean')

P$_6$) $\forall u \forall v \forall w(Ruv \,\&\, Ruw \to v = w)$ ('partly functional')

P$_7$) $\forall u \exists! v Ruv$ ('functional')

[10]This follows Goldblatt [1987], pp. 12-13.

P$_8$) $\forall u \forall v(Ruv \rightarrow \exists w(Ruw \,\&\, Rwv))$ ('weakly dense')

P$_9$) $\forall u \forall v \forall w(Ruv \& Ruw \rightarrow Rvw \vee v{=}w \vee Rwv)$ ('weakly connected')

P$_{10}$) $\forall u \forall v \forall w(Ruv \,\&\, Ruw \rightarrow \exists t(Rvt \,\&\, Rwt))$ ('weakly directed')

Consider next a corresponding list of schemata:

S$_1$) $\Box\varphi \rightarrow \Diamond\varphi$ ('D')

S$_2$) $\Box\varphi \rightarrow \varphi$ ('T')

S$_3$) $\varphi \rightarrow \Box\Diamond\varphi$ ('B')

S$_4$) $\Box\varphi \rightarrow \Box\Box\varphi$ ('4')

S$_5$) $\Diamond\varphi \rightarrow \Box\Diamond\varphi$ ('5')

S$_6$) $\Diamond\varphi \rightarrow \Box\varphi$

S$_7$) $\Diamond\varphi \leftrightarrow \Box\varphi$

S$_8$) $\Box\Box\varphi \rightarrow \Box\varphi$

S$_9$) $\Box[(\varphi \,\&\, \Box\varphi) \rightarrow \psi] \ \vee \ \Box[(\psi \,\&\, \Box\psi) \rightarrow \varphi]$ ('L')

S$_{10}$) $\Diamond\Box\varphi \rightarrow \Box\Diamond\varphi$ ('G')

Now consider the following theorem:

Theorem: For any model $\mathbf{M}$, if $\mathbf{R_M}$ satisfies P$_i$, then $\models^{\mathbf{M}}$S$_i$ $(1 \leq i \leq 10)$.

Remark 1: Note that this theorem in effect says that S$_i$ is valid with respect to the class of P$_i$-models (i.e., models $\mathbf{M}$ in which $\mathbf{R_M}$ is P$_i$). Consider what the theorem says about P$_2$ and S$_2$, for example: for every model $\mathbf{M}$, if $\mathbf{R_M}$ is reflexive, then $\models^{\mathbf{M}} \Box\varphi \rightarrow \varphi$. In other words, the T schema is *valid with respect to* the class of reflexive models.

Remark 2: It is an interesting fact that the 'converse' of this theorem is false. Consider what the 'converse' says in the case of P$_2$ and S$_2$: for any model $\mathbf{M}$, if $\models^{\mathbf{M}} \Box\varphi \rightarrow \varphi$, then $\mathbf{R_M}$ is reflexive. To see that this is false requires some argument. We construct a particular model $\mathbf{M_1}$ (for the language $\Lambda_\Omega = \{p\}$) in which *all* the instances of the T schema are true but where $\mathbf{R_{M_1}}$ is not reflexive. $\mathbf{M}_1$ has the following components: $\mathbf{W} = \{\mathbf{w}_1, \mathbf{w}_2\}$; $\mathbf{R} = \{\langle \mathbf{w}_1, \mathbf{w}_2\rangle, \langle \mathbf{w}_2, \mathbf{w}_1\rangle\}$; and $\mathbf{V}(p) = \{\mathbf{w}_1, \mathbf{w}_2\}$. To see that all of the instances of the T schema are true in this model, we have to first argue that the following is a fact about $\mathbf{M}_1$:

Fact: $\models^{\mathbf{M}_1}_{\mathbf{w}_1} \varphi$ iff $\models^{\mathbf{M}_1}_{\mathbf{w}_2} \varphi$.

Proof: By induction on the construction of φ.

Now we use this *Fact* to argue for the following two claims, for any φ: (a) that $\models^{\mathbf{M}_1}_{\mathbf{w}_1} \Box\varphi \to \varphi$, and (b) that $\models^{\mathbf{M}_1} \Box\varphi \to \varphi$.

Proof of (a): Clearly, either (i) $\models^{\mathbf{M}_1}_{\mathbf{w}_1} \Box\varphi$ or (ii) $\not\models^{\mathbf{M}_1}_{\mathbf{w}_1} \Box\varphi$. Assume (i). Then by our *Fact*, $\models^{\mathbf{M}_1}_{\mathbf{w}_2} \Box\varphi$. So for any $\mathbf{w}'$, if $\mathbf{Rw}_2\mathbf{w}'$, then $\models^{\mathbf{M}_1}_{\mathbf{w}'} \varphi$. But $\mathbf{Rw}_2\mathbf{w}_1$. So $\models^{\mathbf{M}_1}_{\mathbf{w}_1} \varphi$. So by conditional proof, if $\models^{\mathbf{M}_1}_{\mathbf{w}_1} \Box\varphi$, then $\models^{\mathbf{M}_1}_{\mathbf{w}_1} \varphi$, which means, by the *Remark* in (7), that $\models^{\mathbf{M}_1}_{\mathbf{w}_1} \Box\varphi \to \varphi$. Now assume (ii). Then, by antecedent failure, $\models^{\mathbf{M}_1}_{\mathbf{w}_1} \Box\varphi \to \varphi$. So, in either case, we have $\models^{\mathbf{M}_1}_{\mathbf{w}_1} \Box\varphi \to \varphi$.

Proof of (b): By (a) and the *Fact*, we know $\models^{\mathbf{M}_1}_{\mathbf{w}_2} \Box\varphi \to \varphi$. So since $\Box\varphi \to \varphi$ is true in both $\mathbf{w}_1$ and $\mathbf{w}_2$, we have $\models^{\mathbf{M}_1} \Box\varphi \to \varphi$.

Since it is clear that $\mathbf{R}_{\mathbf{M}_1}$ is not reflexive, we have established that every instance of the T schema is true in $\mathbf{M}_1$, but $\mathbf{R}_{\mathbf{M}_1}$ is not reflexive. This counterexample shows that the 'converse' of the present theorem is false.

Exercise: Show that the 'converse' of this theorem is false in the case of the schema 4 and transitivity; i.e., find a model $\mathbf{M}$ in which every instance of the 4 schema is true but in which $\mathbf{R}_{\mathbf{M}}$ is not transitive.

Remark 3: It is interesting that if instead of focusing on classes of models, we focus on the underlying structure of a model, we can produce an interesting and true 'converse' to our theorem. The underlying structure of a given model is called a frame.

24) A *frame* $\mathbf{F}$ is any pair $\langle \mathbf{W}, \mathbf{R} \rangle$, where $\mathbf{W}$ is a non-empty set of worlds and $\mathbf{R}$ is an accessibility relation on $\mathbf{W}$. Again, for precise identification, we refer to the set of worlds in frame $\mathbf{F}$ as $\mathbf{W}_{\mathbf{F}}$, and refer to the accessibility relation of $\mathbf{F}$ as $\mathbf{R}_{\mathbf{F}}$. The only difference between frames and models is that frames do not have valuation functions $\mathbf{V}$ that assign sets of worlds to the atomic formulas of the language. Frames constitute the purely structural component of models. We say that the model $\mathbf{M}$ is *based on* the frame $\mathbf{F}$ iff both $\mathbf{W}_{\mathbf{M}} = \mathbf{W}_{\mathbf{F}}$ and $\mathbf{R}_{\mathbf{M}} = \mathbf{R}_{\mathbf{F}}$. We may now define another sense of validity that is relative to a frame: φ is *valid on* the frame $\mathbf{F}$ (in symbols: $\mathbf{F} \models \varphi$) iff for every model $\mathbf{M}$ based on $\mathbf{F}$, $\models^{\mathbf{M}} \varphi$. We say that a schema is valid on frame $\mathbf{F}$ iff every instance of the schema is valid in every model based on $\mathbf{F}$. Clearly, for any given frame $\mathbf{F}$, any formula that is valid *simpliciter* is valid on $\mathbf{F}$ (i.e., if $\models \varphi$, then $\mathbf{F} \models \varphi$, for any $\mathbf{F}$). But consider now the following claim, which looks similar to the 'converse' of the theorem in (23), and which does hold for the S_i and P_i in (23):

25) *Theorem*: For any frame $\mathbf{F}$, if $\mathbf{F} \models \mathrm{S}_i$, then $\mathbf{R_F}$ satisfies P_i.

Remark: In other words, if schema S_i is valid on the frame $\mathbf{F}$, then the accessibility relation of $\mathbf{F}$ has the property P_i. Consider what this asserts with respect to S_3 and P_3: for any frame $\mathbf{F}$, if $\mathbf{F} \models \Box\varphi \to \Box\Box\varphi$, then $\mathbf{R_F}$ is transitive.

Question: Why is it that the claim:
For any model $\mathbf{M}$, if $\models^{\mathbf{M}} \mathrm{S}_i$, then $\mathbf{R_M}$ satisfies P_i,
is false, whereas the claim:
For any frame $\mathbf{F}$, if $\mathbf{F} \models \mathrm{S}_i$, then $\mathbf{R_F}$ satisfies P_i,
is true?

Remark: When we showed in (23) that the T schema is valid wrt the class of reflexive models, we were showing something about the abstract structure of the models in the class. The particular valuation function $\mathbf{V}$ to atomic formulas doesn't really make a difference! Rather it is the structure, namely the set of worlds plus accessibility relation, that is responsible for the validity of the schema. What this 'converse' theorem tells us is that the validity of the schema relative to just the structure (frame), guarantees that the accessibility relation of the structure has the corresponding property. These facts should convince you that the following holds for the P_i and S_i in (23):

26) *Theorem*: For any frame $\mathbf{F}$, $\mathbf{R_F}$ satisfies P_i iff $\mathbf{F} \models \mathrm{S}_i$ $(1 \leq i \leq 10)$.

27) There is one more relativized notion of validity which proves to be useful. And that is the notion of validity with respect to a class of frames. Let $\mathbf{C_F}$ be a class of frames. Then φ is *valid with respect to the class of frames* $\mathbf{C_F}$ just in case φ is valid on every frame $\mathbf{F}$ in $\mathbf{C_F}$, i.e., just in case for every $\mathbf{F} \in \mathbf{C_F}$, $\mathbf{F} \models \varphi$.

Remark: Note that for φ to be valid with respect to the class of frames $\mathbf{C_F}$, φ must be true in every model $\mathbf{M}$ based on any frame in $\mathbf{C_F}$. Clearly, then, any formula that is valid *simpliciter* is valid with respect to every class of frames (i.e., if $\models \varphi$, then $\mathbf{C_F} \models \varphi$, for any $\mathbf{C_F}$). Note also that using this notion of validity, we may read the theorem in (26) in the left-right direction as: S_i is valid with respect to the class of all P_i-frames (a

P_i-frame is a frame $\mathbf{F}$ in which $\mathbf{R_F}$ satisfies P_i). Of course, if φ is valid with respect to the class of all P_i-frames, then φ is valid with respect to the class of all P_i-models. The notion of a frame, and of a class of frames, have been developed in important recent work in modal logic. We introduce them now so that the reader will start to become familiar with the rather powerful notion of validity with respect to a class of frames.

§5: Validity and Invalidity with respect to a Class

In this section, we develop our intuitions about the kinds of schemata that are *invalid* in certain classes of models (or frames). It is instructive to see why, for example, the schema 4 is invalid with respect to the class of reflexive models. These exercises help us to visualize the relationships between modal schemata and interesting classes of models and frames, and thus give us a deeper understanding of modality.

28) Some facts:

.1) The schemata B, 4, and 5 are not valid with respect to the class of reflexive models (frames)

.2) The schema 4 is not valid with respect to the class of symmetrical models (frames).

.3) The schema 5 is not valid with respect to the class of transitive models, nor in the class of symmetric models (frames).

.4) The schemata 4 and 5 are not valid with respect to the class of reflexive symmetric models (frames).

.5) The schemata B and 5 and not valid with respect to the class of reflexive transitive models (frames).

.6) The schemata T and B are not valid with respect to the class of serial transitive euclidean models (frames).

.7) The schema T is not valid with respect to the class of serial symmetric models (frames).

.8) The schema 4 is not valid with respect to the class of serial euclidean models (frames).

.9) The schema D is not valid with respect to the class of symmetric transitive models (frames).

Remark 1: There is a another important reason for proving these facts besides that of developing our intuitions. And that is they play an important role in establishing the independence of modal logics, i.e., in establishing that there are theorems of logic Σ that are not theorems of logic Σ'. We shall not spend time in the present work investigating such questions about the independence of logics, but simply prepare the reader for such a study, indicating in general how these facts play a role. For example we know that the schema T is valid with respect to reflexive models. In the next chapter we shall consider the modal logic KT based on the axioms K and T. And in the final chapter, we shall prove that the logic KT is sound with respect to the class of reflexive models, i.e., that every theorem φ of KT is valid with respect to the class of reflexive models, i.e., that if φ is *not* valid with respect to the class of reflexive models, then φ is not a theorem of KT. But by (28.1), the schema 4 is not valid with respect to the class of reflexive models. So, by the soundness of KT, the schema 4 is not a theorem of KT. This means that any modal logic that contains 4 as a theorem will be a distinct logic, and moreover, that KT is not an an extension of any logic Σ containing 4, since there are theorems of Σ not in KT. Similarly, the schemas B and 5 will not be theorems of KT, since neither of these is valid in the class of reflexive models.

Consider, as a second example, (28.2). The fact that the schema 4 is not valid in the class of symmetric models can be used to show that 4 is not a theorem of the modal logic KB (the modal logic based on the axioms K and B), since once it is shown that KB is sound with respect to the class of symmetric models, it follows that any schema not valid in the class of symmetric models is not a theorem of KB.

Remark 2: From (28.4), we discovered that the schemata 4 and 5 were both invalid with respect to the class of reflexive symmetric models. Note that we can produce a single model in which both 4 and 5 are false. In such models, we need only show that 4 is false at one world and that 5 is false at another world. To do this, it suffices to show that an instance of 4, say $\Box p \to \Box\Box p$, is false at one world, whereas an instance of 5, say $\Diamond q \to \Box\Diamond q$ (or even $\Diamond\neg p \to \Box\Diamond\neg p$), is false at another world. Here is a model that works:

Exercise: Develop a reflexive transitive model that falsifies both B and 5, and develop a serial transitive euclidean model that falsifies both T and B.

29) Some facts about relations:

.1) If relation R is reflexive, R is serial.

.2) A symmetric relation R is transitive iff it is euclidean.

.3) A relation R is reflexive, symmetric, and transitive iff R is reflexive and euclidean iff R is serial, symmetric, and transitive iff R is serial, symmetric, and euclidean.

.4) If a relation R is symmetrical or euclidean, then R is weakly directed.

.5) If a relation R is euclidean, it is weakly connected.

.6) If a relation R is functional, it is serial.

Remark: The reason for studying facts of this kind is that they help us to show that a given logic Σ is an extension of another logic Σ', once the soundness of Σ and the completeness of Σ' are both established. Take (29.1), for example. The fact that every reflexive relation is serial implies that the class of reflexive models is a subset of the class of serial models. So any formula φ valid in the class of all serial models is valid in the class of reflexive models, i.e., (a) if **C**-serial$\models \varphi$, then **C**-refl$\models \varphi$. In later chapters, we discover (b) that the modal logic KD is sound with respect to the class of all serial models in the sense that the theorems of KD are all valid with respect to the class of serial models, and (c) that the modal logic KT is complete with respect to the class of reflexive models in the sense that the formulas valid with respect to the class of reflexive models are all theorems of KT. In other words, we prove (b) if $\vdash_{KD} \varphi$, then **C**-serial$\models \varphi$ (here the symbols '$\vdash_{KD} \varphi$' mean that φ is a theorem of KD), and (c) if **C**-refl$\models \varphi$, then $\vdash_{KT} \varphi$. So, putting (b), (a), and (c) together, it follows that if $\vdash_{KD} \varphi$, then $\vdash_{KT} \varphi$ (i.e., that every theorem of KD is a theorem of KT). This means that the logic KT is an extension of the logic KD. So facts about the accessibility relation R of the present kind will eventually help us to establish interesting relationships about modal systems.

Exercise: Find other entailments between the properties of relations P_1 – P_{10} defined in (23).

§6: Preserving Validity and Truth

We conclude this chapter with some results that prepare us for the following chapter on logic and proof theory. These results have to do with validity- and truth-preserving relationships among certain formulas. These relationships ground the rules of inference defined in the next chapter. It is appropriate to demonstrate now that the relationships that serve to justify the rules of inference are indeed validity and truth preserving, for this shows that the rules of inference based on these relationships themselves preserve validity and truth. Recall that to say that a modal logic Σ is sound (wrt a class $\mathbf{C}$ of models) is to say that every theorem of Σ is valid (with respect to $\mathbf{C}$). Recall also that the theorems of a logic are the formulas derivable from its axioms using its rules of inference. We've already seen how some of the formulas that will be taken as axioms prove to be valid with respect to certain classes of models. So by showing that the rules of inference associated with (normal) modal logics preserve validity and truth, we show that such logics never allow us to derive invalidities from validities already in it, nor falsehoods from non-logical truths we might want to add to the logic.

There are four rules of inference that will play a significant role in Chapters 3 and 4. In this section, we describe these rules only informally, relying on the readers past experience with such rules to understand what role they play in logic. The four rules we shall study may be described as follows:

> Modus Ponens: From $\varphi \to \psi$ and φ, infer ψ.
>
> Rule of Propositional Logic (RPL): From $\varphi_1, \ldots, \varphi_n$, infer ψ, whenever ψ is a tautological consequence of $\varphi_1, \ldots, \varphi_n$ (the notion of 'tautological consequence' will be defined shortly).
>
> Rule of Necessitation (RN): From φ, infer $\Box\varphi$.
>
> Rule K (RK): From $\varphi_1 \to \ldots \to \varphi_n \to \psi$, infer $\Box\varphi_1 \to \ldots \to \Box\varphi_n \to \Box\psi$. (Here the formula $\varphi_1 \to \ldots \to \varphi_n \to \psi$ is an abbreviation of the formula $\varphi_1 \to (\ldots \to (\varphi_n \to \psi) \ldots)$. In addition, the formula $\Box\varphi_1 \to \ldots \to \Box\varphi_n \to \Box\psi$ is an abbreviation of the formula $\Box\varphi_1 \to (\ldots \to (\Box\varphi_n \to \Box\psi) \ldots)$.)

We now examine whether these rules preserve validity and truth.

30) Consider the relationship among the formulas φ, $\varphi \rightarrow \psi$, and ψ established by the following theorem:

> *Theorem*: If $\models \varphi \rightarrow \psi$ and $\models \varphi$, then $\models \psi$.

Remark: This relationship among formulas grounds the rule of inference Modus Ponens (MP). MP allows us to regard ψ as a theorem of a modal logic whenever $\varphi \rightarrow \psi$ and φ are theorems of that logic. The present (meta-)theorem tells us that MP preserves validity. Note that in proving this theorem, we in effect show that Modus Ponens also preserves truth in a model and truth at a world in a model. That is, we have shown both: (a) if $\models^{\mathbf{M}} \varphi \rightarrow \psi$ and $\models^{\mathbf{M}} \varphi$, then $\models^{\mathbf{M}} \psi$, and (b) if $\models^{\mathbf{M}}_{\mathbf{w}} \varphi \rightarrow \psi$ and $\models^{\mathbf{M}}_{\mathbf{w}} \varphi$, then $\models^{\mathbf{M}}_{\mathbf{w}} \psi$.

Exercise: Show that MP preserves validity with respect to a class of models, validity *on* a frame, and validity with respect to a class of frames. That is, show: (a) for any class of standard models $\mathbf{C}$, if $\mathbf{C} \models \varphi \rightarrow \psi$ and $\mathbf{C} \models \varphi$, then $\mathbf{C} \models \psi$, (b) for any frame $\mathbf{F}$, if $\mathbf{F} \models \varphi \rightarrow \psi$ and $\mathbf{F} \models \varphi$, then $\mathbf{F} \models \psi$, and (c) for any class of frames $\mathbf{C_F}$, if $\mathbf{C_F} \models \varphi \rightarrow \psi$ and $\mathbf{C_F} \models \varphi$, then $\mathbf{C_F} \models \psi$.

Remark: Note that we can form a 'corresponding conditional' by conjoining the hypotheses of the rule into an antecedent and the conclusion of the rule into the conclusion of a condtional. The corresponding conditional for MP is: $((\varphi \rightarrow \psi) \ \& \ \varphi) \rightarrow \psi$. *Question*: Is the corresponding conditional for the rule MP valid?

31) The following definitions will prepare the ground for showing that a very special rule of inference is valid. We say that ψ is a *tautological consequence* of formulas $\varphi_1, \ldots, \varphi_n$ iff for every assignment function $\mathbf{f}$, if $\mathbf{f}(\varphi_1) = T$ and $\ldots$ and $\mathbf{f}(\varphi_n) = T$, then $\mathbf{f}(\psi) = T$.[11] By convention, if $n = 0$, then ψ is a tautologous consequence of the empty set of formulas $\varphi_1, \ldots, \varphi_n$ just in case every $\mathbf{f}$ is such that $\mathbf{f}(\psi) = T$. The following are straightforward consequences of these definitions:

[11] If you have read Alternative §2 and prefer the Enderton method of defining assignments, then you should define tautological consequence a little differently. Let Γ be a finite of sentences, say $\{\varphi_1, \ldots, \varphi_n\}$. Then, we define *the set of quasi-atomic subformulas in* Γ (in symbols: Ω^*_Γ) as:

$$\Omega^*_\Gamma = \Omega^*_{\varphi_1} \cup \ldots \cup \Omega^*_{\varphi_n}.$$

Now we may say: ψ is a tautological consequence of $\varphi_1, \ldots, \varphi_n$ iff for every basic assignment $\mathbf{f}^*$ of the set $\Omega^*_{\Gamma \cup \{\psi\}}$, if $\mathbf{f}(\varphi_1) = T$ and $\ldots$ and $\mathbf{f}(\varphi_n) = T$, then $\mathbf{f}(\psi) = T$.

.1) ψ is a tautological consequence of φ iff $\varphi \to \psi$ is a tautology.

.2) ψ is a tautological consequence of $\varphi_1, \ldots, \varphi_n$ iff $\varphi_1 \to \ldots \to \varphi_n \to \psi$ is a tautology.

.3) If $\varphi_1, \ldots, \varphi_n$ are all tautologies, and ψ is a tautological consequence of $\varphi_1, \ldots, \varphi_n$, then ψ is a tautology.

.4) φ is a tautology iff φ is a tautological consequence of the empty set of formulas $\varphi_1, \ldots, \varphi_n$ when $n = 0$.

.5) $\varphi \to (\psi \to \chi)$ and $(\varphi \mathbin{\&} \psi) \to \chi$ are tautological consequences of each other, and in general, $\varphi_1 \to \ldots \to \varphi_n \to \psi$ and $(\varphi_1 \& \ldots \& \varphi_n) \to \psi$ are tautological consequences of each other.

.6) $\bot$ and $\varphi \mathbin{\&} \neg\varphi$ are tautological consequences of each other.

.7) ψ is a tautological consequence of $\varphi_1, \ldots, \varphi_n$, and $\varphi_1 \to \ldots \to \varphi_n \to \psi$.

32) *Theorem*: Let ψ be a tautological consequence of $\varphi_1, \ldots, \varphi_n$. Then, if $\models \varphi_1$ and $\ldots$ and $\models \varphi_n$, then $\models \psi$.

Proof: Use (31.2), (16), and apply (30) n-times.

Remark: This relationship among formulas grounds the Rule of Propositional Logic (RPL). This rule will allow us to conclude, when ψ is a tautological consequence of $\varphi_1, \ldots, \varphi_n$, that ψ is a theorem of a modal logic whenever $\varphi_1, \ldots, \varphi_n$ are theorems of the logic. The present (meta-)theorem tells us that RPL preserves validity.

Exercise 1: Show that truth in a model and truth at a world in a model are both preserved whenever ψ is a tautological consequence of $\varphi_1, \ldots, \varphi_n$.

Exercise 2: Show that validity with respect to a class of models, validity *on* a frame, and validity with respect to a class of frames is preserved whenever ψ is a tautological consequence of $\varphi_1, \ldots, \varphi_n$. That is, show: (a) for any class of standard models $\mathbf{C}$, if $\mathbf{C} \models \varphi_1$ and $\ldots$ and $\mathbf{C} \models \varphi_n$, then $\mathbf{C} \models \psi$, whenever ψ is a tautological consequence of φ, (b) for any frame $\mathbf{F}$, if $\mathbf{F} \models \varphi_1$, and $\ldots$ and $\mathbf{F} \models \varphi_n$, then $\mathbf{F} \models \psi$, whenever ψ is a tautological consequence of φ; and (c) for any class of frames $\mathbf{C_F}$, if $\mathbf{C_F} \models \varphi_1$ and $\ldots$ and $\mathbf{C_F} \models \varphi_n$, then $\mathbf{C_F} \models \psi$, whenever ψ is a tautological consequence of φ.

Remark: The corresponding conditional for RPL is: $(\varphi_1 \,\&\, \ldots \,\&\, \varphi_n) \to \psi$, where ψ is a tautological consequence of $\varphi_1, \ldots, \varphi_n$. *Question*: Is the corresponding conditional for RPL valid, i.e., is $\models (\varphi_1 \,\&\, \ldots \,\&\, \varphi_n) \to \psi$, whenever ψ is a tautological consequence of $\varphi_1, \ldots, \varphi_n$?

33) Consider the following relationship between φ and $\Box\varphi$:

Theorem: If $\models \varphi$, then $\models \Box\varphi$

Remark: This relationship grounds the Rule of Necessitation (RN). It proves to be important to the definition of 'normal' modal logics. RN allows us to suppose that $\Box\varphi$ is a theorem of a normal modal logic whenever φ is a theorem of the logic. The present (meta-)theorem tells us that RN preserves validity.

Exercise 1: Show that this rule preserves truth in a model but *not* truth at a world in a model.

Remark: Note that the corresponding conditional for RN, $\varphi \to \Box\varphi$, is not valid (as we showed in (20)). However, the corresponding conditional of the two previous rules we've examined are valid. For example, the corresponding conditional for MP, $((\varphi \to \psi) \,\&\, \varphi) \to \psi$, is valid. Moreover, the corresponding conditional for RPL, $(\varphi_1 \,\&\, \ldots \,\&\, \varphi_n) \to \psi$ (when ψ is a tautological consequence of $\varphi_1, \ldots, \varphi_n$), is valid . *Question*: What is the difference among these rules that accounts for the different properties of their corresponding conditionals?

Exercise 2: Show that RN rule preserves validity with respect to a class of models, validity *on* a frame, and validity with respect to a class of frames.

34) Consider one final relationship among formulas:

Theorem: If $\models \varphi_1 \to \ldots \to \varphi_n \to \psi$, then
$\models \Box\varphi_1 \to \ldots \to \Box\varphi_n \to \Box\psi$

Proof: By induction on n. In the base case, use (33). In the inductive case, use (17).

Remark: This relationship grounds the Rule RK. It is a rule that characterizes normal modal logics, and it allows us to suppose that $\Box\varphi_1 \to \ldots \to \Box\varphi_n \to \Box\psi$ is a theorem of a normal modal logic whenever $\varphi_1 \to \ldots \to \varphi_n \to \psi$ is a theorem of the logic. The present (meta-)theorem tells us that RK preserves validity.

Exercise 1: Show that rule RK preserves truth in a model, but *not* truth at a world in a model. *Question*: Is the corresponding conditional for this rule valid? Why not?

Exercise 2: Show that rule RK preserves validity with respect to a class of models, validity *on* a frame, and validity with respect to a class of frames. That is, show: (a) for any class of models $\mathbf{C}$, if $\mathbf{C} \models \varphi_1 \rightarrow \ldots \rightarrow \varphi_n \rightarrow \psi$, then $\mathbf{C} \models \Box\varphi_1 \rightarrow \ldots \rightarrow \Box\varphi_n \rightarrow \Box\psi$. (b) for any frame $\mathbf{F}$, if $\mathbf{F} \models \varphi_1 \rightarrow \ldots \rightarrow \varphi_n \rightarrow \psi$, then $\mathbf{F} \models \Box\varphi_1 \rightarrow \ldots \rightarrow \Box\varphi_n \rightarrow \Box\psi$; and (c) for any class of frames $\mathbf{C_F}$, if $\mathbf{C_F} \models \varphi_1 \rightarrow \ldots \rightarrow \varphi_n \rightarrow \psi$, then $\mathbf{C_F} \models \Box\varphi_1 \rightarrow \ldots \rightarrow \Box\varphi_n \rightarrow \Box\psi$.

Chapter Four: Logic and Proof Theory

We now work our way towards a definition of a logic and its theorems. Given the anticipatory remarks we have been making in the previous chapter, the reader may expect the following traditional definitions:

> A *logic* is a set of axioms plus some rules of inference. A *proof* is any sequence of formulas which is such that every member of the sequence either (a) is an axiom, or (b) follows from previous members of the sequence by a rule of inference. A *theorem* is any formula φ for which there is a proof. The symbol '$\vdash \varphi$' is employed to denote that φ is a theorem.

This conception of logic, which we refer to in what follows as "the conception of a logic as an axiom system," has no room for unaxiomatizable logics.[12]

However, in the present work, we do *not* adopt this conception of a logic. The recent work in modal logic employs a slightly different conception, one that allows for the possibility of unaxiomatizable logics. For some readers, the notion we develop may seem unfamiliar, but since it does appears most often in the recent works of modal logicians, we introduce it here, with the hope that our discussion will prove to be valuable to the reader when he or she approaches more advanced works in the field. The first remarkable feature of present conception concerns rules of inference, which we look at somewhat differently. Instead of having the form 'from $\varphi_1, \ldots, \varphi_n$, infer ψ', a rule R is simply taken to be a relation between the hypotheses of the rule and the conclusion. We can use this relation to talk about sets of formulas that contain the conclusion of rule R whenever they contain the hypotheses of R. We call such sets 'closed under' the rule R. So rules of inference are relations that allow us to define certain sets. This brings us to the second remarkable difference about the present conception. A *logic* is conceived to be just a set of formulas, and the set is usually identified by stipulating: (1) that it contains some initial elements (the 'axioms'), and (2) that it is closed under certain rules of

[12]On this conception, it is standard to require that there be an effective method for determining whether a formula is an axiom, and that for each rule, there be an effective method for deciding when sentences are related by the rule as hypotheses and conclusion. Give these requirements, then, the theorems of a logic may be effectively enumerated.

inference. The theorems of a logic Γ will be defined simply as the members of Γ. Consequently, the present conception doesn't require that a logic be axiomatizable. However, it turns out that we shall be interested primarily in the logics that are axiomatizable, and so one might wonder, why use this conception of logic?

The principal advantage of this conception is that it allows us to describe, in a much more perspicuous way, the frequently encountered situation in which there are several distinct axiom systems each having the same set of theorems. For example, there are numerous ways of axiomatizing propositional logic, each of which has the same set of theorems (namely, the set of tautologies). On the conception of logic as an axiom system, we would have to say that each of these axiomatizations constitutes a different logic. But on the present conception, we in effect identify the logic with the set of its theorems. Thus, we may say that there is only one set, namely, the set of tautologies, that constitutes propositional logic. And similarly with other logics for which there are distinct axiomatizations.

§1: Rules of Inference

35) Let us say that a *rule of inference* R is any relation defined by pair sequences of the form $\langle\{\varphi_1, \ldots, \varphi_n\}, \psi\rangle$ $(n \geq 0)$, where the members of $\{\varphi_1, \ldots, \varphi_n\}$ are said to be the *hypotheses* of R, and ψ the *conclusion* of R. For example, the rule Modus Ponens is the relation defined by pair sequences of the form:

$$\langle\{\varphi \rightarrow \psi, \varphi\}, \psi\rangle.$$

The Rule of Necessitation is the relation defined by pair sequences of the form:

$$\langle\{\varphi\}, \Box\varphi\rangle.$$

Hereafter, we shall more simply designate rules as having the following form: $\varphi_1, \ldots, \varphi_n/\psi$. So we shall hereafter designate the four rules of inference previously introduced as follows:

MP: $\varphi \rightarrow \psi,\ \varphi/\psi$

RPL: $\varphi_1, \ldots, \varphi_n/\psi$ $(n \geq 0)$, where ψ is a tautological consequence of $\varphi_1, \ldots, \varphi_n$.

RN: $\varphi/\Box\varphi$

RK: $\varphi_1 \to \ldots \to \varphi_n \to \psi/\Box\varphi_1 \to \ldots \to \Box\varphi_n \to \Box\psi$

A set of formulas Γ is said to be *closed under* rule R just in case Γ contains the conclusion of R whenever it contains the hypotheses of R (or just contains the conclusion of R when $n = 0$ and there are no hypotheses). So, for example, Γ is closed under Modus Ponens just in case: $\psi \in \Gamma$ whenever both $\varphi \to \psi \in \Gamma$ and $\varphi \in \Gamma$. Γ is closed under RPL just in case: if $\varphi_1, \ldots, \varphi_n \in \Gamma$, then $\psi \in \Gamma$, whenever ψ is a tautological consequence of $\varphi_1, \ldots, \varphi_n$. So to prove Γ is closed under RPL, we typically assume (a) that $\varphi_1, \ldots, \varphi_n \in \Gamma$ and (b) that ψ is a tautological consequence of $\varphi_1, \ldots, \varphi_n$, and then show that $\psi \in \Gamma$.

Remark 1: Traditionally, rules of inference are construed in a somewhat dynamic way. They permit us to infer formulas from other formulas. This is the way we looked at rules in §6 of Chapter 2. This traditional understanding goes naturally with the conception of a logic as an axiom system, for the notion of a proof is basic to that conception. Thus rules of inference allow us to build proofs of the theorems from the axioms, and on this conception, the set of theorems is constructed "from the bottom up".

But now we are looking at rules a little differently. We are looking at them as relations between sets of formulas and other formulas. Modus Ponens, for example, relates sets of the form $\{\varphi \to \psi,\ \varphi\}$ to the formula ψ. This is all that is meant by saying that ψ is the consequence of φ and $\varphi \to \psi$ by MP. RPL, for example, relates sets of the form $\{\varphi_1, \ldots, \varphi_n\}$ to those formulas ψ which are tautological consequences of $\varphi_1, \ldots, \varphi_n$. In the next section, we define a logic, in general, to be any set which contains certain initial axioms and which is closed under certain rules of inference. This views the set of theorems (i.e., the logic itself) as constructed "from the top down". The logic is carved out from the set of all formulas. Later in this chapter, particular logics will be defined as the smallest set containing the instances of certain schemata and closed under certain rules. For example, the logic K is later identified as the smallest set which contains the tautologies, the instances of the K schema, and which is closed under the rules MP and RN. This, too, carves out the logic K "from the top down", for we may think of this as arriving at the set K by paring down all the sets which contain the tautologies, contain the K axiom, and are closed under MP and RN, until we reach the *smallest* one.

Remark 2: Note that RPL has a special feature that distinguishes it from the other rules: the conclusion ψ of RPL need not be related in any formal way to the hypotheses $\varphi_1, \ldots, \varphi_n$. In particular, the conclusion does not have to be a subformula of one of the hypotheses, nor do subformulas of the conclusion have to appear in the hypotheses. For example, the sequence $\bot/\varphi \,\&\, \neg\varphi$ constitutes an instance of RPL, since the conclusion is a tautological consequence of the hypothesis. This is unlike the other rules, such as MP and RN. In MP, one of the subformulas of a hypothesis appears as the conclusion. In RN, a subformula of the conclusion appears as a hypothesis. These facts mean that, unlike the other rules, RPL may be applied even in the case when there are no hypotheses, as long as the conclusion is a tautological consequence of the empty set of hypotheses. This in fact may happen, though it can't happen in the case of MP, RN, or RK, since such rules cannot be applied when there are no hypotheses.

Remark 3: On the present conception of logic, we may think of a *propositional* logic as any set that contains all the tautologies and which is closed under MP. Using this definition, lots of sets will qualify as propositional logics. But in what follows, we let propositional logic *per se* be the *smallest* set containing all tautologies and closed under MP. This is just the set of tautologies (we prove this below). Our next task is to define the notion of a modal logic, and the notion we want is this: a modal logic is any extension (possibly even the null extension) of a propositional logic. Since 'contains all tautologies and is closed under MP' is a perfectly adequate definition of a propositional logic, we simply use this as the definition of a modal logic. So once we define a modal logic as any set containing all the tautologies and closed under MP, it should turn out that the set of tautologies is the weakest modal logic, i.e., it should turn out that every modal logic is an extension of the set of tautologies. We turn now to the definitions and theorems that yield these consequences.

§2: Modal Logics and Theoremhood

36) We say that a set of sentences is a *modal logic* just in case it contains every tautology and is closed under the rule MP.

37) *Theorem*: PL $(= \{\varphi \mid \varphi \text{ is a tautology}\})$ is a modal logic.

38) *Theorem*: Γ is a modal logic iff Γ is closed under RPL.

Remark 2: This theorem tells us that the sets that both contain every tautology and are closed under MP are precisely the sets closed under RPL. This means not only that RPL is a 'derived' rule (in the sense that every modal logic 'obeys' or is closed under this rule), but also that 'closure under RPL' constitutes an equivalent definition of a modal logic. We shall exploit these facts on many occasions in what follows, for we can now establish that a set Γ is a modal logic simply by showing that Γ is closed under RPL. This often proves to be more efficient than showing that Γ contains all the tautologies and is closed under MP.[13]

39) In what follows we use the variable Σ to range over sets of sentences that qualify as modal logics. If Σ is a modal logic, then we say φ is a *theorem of* Σ (in symbols: $\vdash_\Sigma \varphi$) iff $\varphi \in \Sigma$. In terms of this definition, we can make our talk of one logic being the extension of another logic more precise. Whenever Σ and Σ' are modal logics, we say that Σ' is a Σ-*logic* (or an *extension of* the logic Σ) just in case for every φ, if $\varphi \in \Sigma$, then $\varphi \in \Sigma'$. In other words, Σ' is a Σ-logic just in case it contains every theorem of Σ.

40) *Theorem*: (.1) If $\vdash_{PL} \varphi$, then $\vdash_\Sigma \varphi$. (.2) Every modal logic Σ is a *PL*-logic. (.3) *PL* is the smallest modal logic.

Remark: At first it may seem odd to think of *PL* as a modal logic. But the notion of modal logic we were after is: any extension, including the null extension, of propositional logic. These theorems show that we have captured this idea. It therefore does little harm to our understanding of a modal logic to suppose that *PL* is one and it simplifies everything if we do so. There is one nice feature of this understanding of *PL*. And that is the soundness and completeness of the *propositional basis* of each modal logic is built right in. To say that the propositional logic *PL* is sound is to say that every theorem is a tautology, and to say that it is complete is to say that every tautology is a theorem. Clearly, *PL* is both sound and complete—its theorems (i.e., members) just are all and only the tautologies. So since *PL* is the embodiment of propositional logic and forms the basis of every modal logic, every modal logic has a sound and complete propositional basis.

[13]Chellas, in [1980], just uses 'closure under RPL' as the definition of a modal logic. While this definition works perfectly well for the study of propositional modal logics (which is what is covered in Chellas' book), it is not as useful for the case of predicate modal logics. Since we shall investigate basic modal predicate logic in the present text, we use the more traditional definition.

41) *Theorem*: The following are modal logics:

.1) $\{\varphi | \models^{\mathbf{M}}_{\mathbf{w}} \varphi\}$, for every **M** and **w**

.2) $\{\varphi | \models^{\mathbf{M}} \varphi\}$, for every **M**

.3) $\{\varphi | \models \varphi\}$

.4) $\{\varphi | \mathbf{C} \models \varphi\}$, for every class of models **C**

.5) $\{\varphi | \mathbf{F} \models \varphi\}$, for every frame **F**

.6) $\{\varphi | \mathbf{C_F} \models \varphi\}$, for every class of frames $\mathbf{C_F}$

.7) $\{\varphi | \varphi \in \Lambda_\Omega\}$, for each Λ_Ω

Remark 1: In light of (38), we can prove these facts either by showing that the sets in question contain every tautology and are closed under MP, or by showing that the sets in question are closed under RPL.

Remark 2: It is important to see next that some modal logics can be axiomatized. To axiomatize a logic Σ, it is not enough to find axioms and rules of inference from which the members of Σ can be generated. For every logic Σ is trivially axiomatized by the set Σ and the Rule of Reiteration: φ/φ. An axiomatization should have two important features: (a) there should be an effective method of determining whether a given formula is an axiom, and (b) there should be, for each rule of inference, an effective method of determining whether formulas are related by the rule as hypothesis and conclusion. Feature (a) requires that the set of axioms be 'decidable', and that is why $\{\varphi | \models^{\mathbf{M}}_{\mathbf{w}} \varphi\}$ (41.1) typically fails to be axiomatized by the trivial axiomatization consisting of itself and the Rule of Reiteration. There is no effective method of determining whether a formula φ is an element of the set of formulas true at **w** in **M** (would that there be such an effective method!). Feature (b) requires that the rules can be effectively applied. Typically, this requirement is satisfied by rules in which the conclusion has a certain form that is related to the form of the hypotheses.

42) Suppose that Γ is a decidable set, and that rules $\mathrm{R}_1, \mathrm{R}_2, \ldots$ can be effectively applied. Then we may say that the logic Σ is *axiomatized by* the set Γ (of axioms) *and* the rules of inference $\mathrm{R}_1, \mathrm{R}_2, \ldots$ just in case: $\varphi \in \Sigma$ iff there is a proof of φ from Γ (using the rules $\mathrm{R}_1, \mathrm{R}_2, \ldots$). A *proof* of φ *from* Γ (*using* the rules $\mathrm{R}_1, \mathrm{R}_2, \ldots$) is any sequence of formulas

$\langle\varphi_1,\ldots,\varphi_n\rangle$, with $\varphi = \varphi_n$, such that each member of the sequence φ_i $(1 \leq i \leq n)$ either (a) is a member of Γ, or (b) is the conclusion, by one of the rules $(\mathrm{R}_1, \mathrm{R}_2, \ldots)$ of previous members of the sequence. Note, for example, that to say that φ_i is a conclusion of previous members of the sequence by the rule MP is, in precise terms, to say: $\exists j,k < i$ such that $\varphi_k = (\varphi_j \to \varphi_i)$. To say, for example, that φ_i is a conclusion of previous members of the sequence by the rule RPL is, in precise terms, to say: $\exists j_1,\ldots,j_k$, with $1 \leq j_1 \leq j_k < i$, such that φ_i is the conclusion by RPL of $\varphi_{j_1},\ldots,\varphi_{j_k}$. Similarly precise statements can be formulated for the other rules of inference we have discussed so far. Note also that since we have identified theoremhood with membership, the definition of *axiomatized by* guarantees that the notion of theoremhood associated with the present conception of logic is equivalent to the notion of theoremhood associated the conception of a logic as an axiom system.

Remark: Clearly, if given a finite list of axioms, or a finite list of schemata (the instances of which are taken as axioms), then there is an effective method for determining whether a formula is an axiom. Moreover, for each of the rules of inference discussed so far, there is an effective method for determining whether formulas are related as hypotheses and conclusion. This is clearly true in the case of MP, RN, and RK, but it is true even in the case of RPL. Our work developing a decision procedure that tests whether a formula is a tautology can easily be turned into a decision procedure that determines whether a formula ψ is a tautological consequence of $\varphi_1,\ldots,\varphi_n$. So RPL can be effectively applied as well.

43) *Theorem*: (.1) *PL* is axiomatized by the set of tautologies and the rule MP. (.2) *PL* is axiomatized by the empty set $\emptyset$ of axioms and the rule RPL.

Exercise: Find other axiomatizations of propositional logic in other logic texts, and examine how it is established that these axiomatizations generate all and only tautologies as theorems.

§3: Deducibility

Traditionally, deducibility is linked with axiomatizability via the notion of a proof. Under the conception of a logic as an axiom system, it is standard to introduce the notion of deducibility before the notion of theoremhood, since the former is a slightly more general notion. On that conception, one typically sees: φ is deducible in the logic Σ from a set of sentences Γ (in symbols: $\Gamma \vdash_\Sigma \varphi$) iff there is a proof of φ from $\Sigma \cup \Gamma$. Then the notion of a theorem would be introduced as a special case: φ is a theorem of logic Σ iff φ is derivable in Σ from the empty set $\emptyset$ (i.e., $\vdash_\Sigma \varphi$ iff $\emptyset \vdash_\Sigma \varphi$). From these definitions, it is possible to prove that the deducibility relation has all sorts of interesting properties.

However, on the present conception of logic, we employ a different notion of deducibility. It proves to be equivalent to the traditional notion in the case of axiomatizable logics. Consider the following definition:

44) A formula φ is *deducible (derivable) from* a set of sentences Γ in a modal logic Σ (in symbols: $\Gamma \vdash_\Sigma \varphi$) iff there are formulas $\varphi_1, \ldots, \varphi_n \in \Gamma$ such that Σ contains the theorem $\varphi_1 \to \ldots \to \varphi_n \to \varphi$. Formally:

$$\Gamma \vdash_\Sigma \varphi =_{df} \exists \varphi_1, \ldots, \varphi_n \in \Gamma \ (n \geq 0) \text{ such that } \vdash_\Sigma \varphi_1 \to \ldots \to \varphi_n \to \varphi$$

Remember here that the formula $\varphi_1 \to \ldots \to \varphi_n \to \varphi$ is shorthand for the formula $\varphi_1 \to (\ldots(\varphi_n \to \varphi)\ldots)$. Note also that this latter formula is tautologically equivalent to the defined notation: $(\varphi_1 \,\&\, \ldots \,\&\, \varphi_n) \to \varphi$. The biconditional having these two formulas as the conditions is therefore a tautology, and so an element of every logic Σ. Thus MP guarantees that the latter is in Σ iff the former is. So it is an immediate consequence of our definitions that $\Gamma \vdash_\Sigma \varphi$ iff $\exists \varphi_1, \ldots, \varphi_n \in \Gamma$ $(n \geq 0)$ such that $\vdash_\Sigma (\varphi_1 \,\&\, \ldots \,\&\, \varphi_n) \to \varphi$.

In what follows, we write $\Gamma \not\vdash_\Sigma \varphi$ whenever it is not the case that $\Gamma \vdash_\Sigma \varphi$.

Remark 1: Let us think of Γ as a non-logical theory. Then $\Gamma \vdash_\Sigma \varphi$ essentially says that φ is a derivable consequence of the theory Γ if given Σ as the underlying logic. For example, one might hold both that $\Box p \to \neg\Diamond q$ and $\Box p$ for non-logical reasons. Our definition should capture the intuition that $\neg\Diamond q$ is deducible from these two hypotheses in any modal logic Σ. To see that our definition does capture this intuition, we let Γ_1 be $\{\Box p \to \neg\Diamond q, \Box p\}$ and we show $\Gamma_1 \vdash_\Sigma \neg\Diamond q$. By definition, we need to show that there are formulas $\varphi_1, \ldots, \varphi_n \in \Gamma_1$ such that $\varphi_1 \to \ldots \to \varphi_n \to \neg\Diamond q$ is a theorem of Σ. To see that there are such formulas, note that $(\Box p \to \neg\Diamond q)$ $\to$ $(\Box p \to \neg\Diamond q)$ is a tautology. So $\vdash_{PL} (\Box p \to \neg\Diamond q) \to (\Box p \to \neg\Diamond q)$. So by (40), $\vdash_\Sigma (\Box p \to \neg\Diamond q) \to (\Box p \to \neg\Diamond q)$. So there are formulas φ_1 $(= \Box p \to \neg\Diamond q)$ and φ_2 $(= \Box p)$ in Γ_1 such that $\Sigma \vdash_\Sigma \varphi_1 \to (\varphi_2 \to \neg\Diamond q)$. Thus, by (44), $\Gamma_1 \vdash_\Sigma \neg\Diamond q$.

Remark 2: Deducibility is defined relative to a logic Σ. It would serve well to give an example of a set of formulas Γ and a formula φ such that φ is not derivable from Γ relative to logic Σ but is derivable relative to logic Σ'. Here is such an example. Suppose that your theory consists of the following two propositions: $\Box(p \to q)$ and $\Box p$. Let Γ_2 be $\{\Box(p \to q), \Box p\}$. Now in propositional logic, one may not validly infer $\Box q$ from Γ_2. This is captured by our definition, since $\Box(p \to q) \to (\Box p \to \Box q)$ is not a tautology and so not a theorem of *PL*. In addition, none of the following are tautologies: $\Box p \to \Box q$, and $\Box(p \to q) \to \Box q$, and $\Box q$. So none of these is a theorem of *PL*. But now we have considered all the combinations (even the empty combination) of formulas in Γ_2, and we've discovered that there are no formulas $\varphi_1, \ldots, \varphi_n \in \Gamma_2$ such that $\vdash_{PL} \varphi_1 \to \ldots \to \varphi_n \to \Box q$. So $\Gamma_2 \nvdash_{PL} \Box q$, confirming our intuitions.

However, in the modal logic K (which we define formally in a later section), the instances of the axiom K are theorems. So $\vdash_K \Box(p \to q) \to (\Box p \to \Box q)$. Clearly, then, there are formulas $\varphi_1, \varphi_2 \in \Gamma_2$ such that $\vdash_K \varphi_1 \to (\varphi_2 \to \Box q)$. So by the definition of deducibility (relative to the logic K), we have $\Gamma_2 \vdash_K \Box q$, which is what we should expect. Relative to a logic containing instances of the schema K, we should be able to derive $\Box q$ from $\Box(p \to q)$ and $\Box p$.

45) *Lemma*: If $\Gamma \vdash_\Sigma \varphi$, and ψ is a tautological consequence of φ, then $\Gamma \vdash_\Sigma \psi$.

46) *Theorem*: The definitions of theorem and deducibility have the following consequences (some of which may be more easily proved by using the previous lemma):[14]

.1) $\vdash_\Sigma \varphi$ iff $\emptyset \vdash_\Sigma \varphi$
.2) $\vdash_\Sigma \varphi$ iff for every Γ, $\Gamma \vdash_\Sigma \varphi$
.3) If $\Gamma \vdash_{PL} \varphi$, then $\Gamma \vdash_\Sigma \varphi$
.4) When Σ' is a Σ-logic, if $\Gamma \vdash_\Sigma \varphi$, then $\Gamma \vdash_{\Sigma'} \varphi$
.5) If $\varphi \in \Gamma$, then $\Gamma \vdash_\Sigma \varphi$
.6) If $\Gamma \vdash_\Sigma \psi$ and $\{\psi\} \vdash_\Sigma \varphi$, then $\Gamma \vdash_\Sigma \varphi$
.7) If $\Gamma \vdash_\Sigma \varphi$ and $\Gamma \subseteq \Delta$, then $\Delta \vdash_\Sigma \varphi$

[14] For the most part, this follows Chellas [1980], p. 47, and to a lesser extent, Lemmon [1977], p. 17. However, we think it important to add the lemma in (45) prior to the introduction of these facts, for the lemma follows just as directly from the definition and is slightly more general.

.8) $\Gamma \vdash_\Sigma \varphi$ iff there is a finite subset Δ of Γ such that $\Delta \vdash_\Sigma \varphi$
.9) $\Gamma \vdash_\Sigma \varphi \rightarrow \psi$ iff $\Gamma \cup \{\varphi\} \vdash_\Sigma \psi$

Remark: (46.1) tells us that the theorems of Σ are precisely those formulas derivable in Σ from the empty set of sentences; (46.2) says that the theorems of Σ are precisely those formulas derivable in Σ from every set of formulas; (46.3) means the deducibility relation in propositional logic is preserved in all modal logics; (46.4) says the deducibility relation in Σ is preserved by every extension of Σ; (46.5) says the members of a set Γ are all derivable from Γ; (46.6) asserts a kind of transitivity of the deducibility relation; (46.7) says any formula derivable from a set is derivable from any of its supersets; (46.8) says derivability is compact in the sense that derivability from a set Γ always implies derivability from a finite subset of Γ; and finally (46.9) is a 'deduction theorem', namely, that $\varphi \rightarrow \psi$ is derivable from a set Γ iff ψ is derivable from the enlarged set $\Gamma \cup \{\varphi\}$ (this follows from (45)).

47) *Generalized Lemma*: If $\Gamma \vdash_\Sigma \varphi_1$ and ... and $\Gamma \vdash_\Sigma \varphi_n$, and ψ is a tautological consequence of $\varphi_1, \ldots, \varphi_n$, then $\Gamma \vdash_\Sigma \psi$.

48) *Facts about Deducibility* (most of which are immediate consequences of (45) and (47)):[15]

.1) $\{\bot\} \vdash_\Sigma \varphi$ (for all φ)
.2) $\{\varphi, \neg\varphi\} \vdash_\Sigma \bot$
.3) $\{\neg\varphi \rightarrow \bot\} \vdash_\Sigma \varphi$
.4) $\Gamma \vdash_\Sigma \varphi \,\&\, \psi$ iff $\Gamma \vdash_\Sigma \varphi$ and $\Gamma \vdash_\Sigma \psi$
.5) If $\Gamma \vdash_\Sigma \varphi$ or $\Gamma \vdash_\Sigma \psi$, then $\Gamma \vdash_\Sigma \varphi \vee \psi$
.6) If $\Gamma \vdash_\Sigma \varphi \vee \psi$ and $\Gamma \vdash_\Sigma \neg\varphi$, then $\Gamma \vdash_\Sigma \psi$
.7) If $\Gamma \vdash_\Sigma \varphi \rightarrow \psi$ and $\Gamma \vdash_\Sigma \varphi$, then $\Gamma \vdash_\Sigma \psi$
.8) If $\Gamma \vdash_\Sigma \neg(\varphi \rightarrow \psi)$, then $\Gamma \vdash_\Sigma \varphi$ and $\Gamma \vdash_\Sigma \neg\psi$
.9) If $\Gamma \vdash_\Sigma \psi$, then $\Gamma \vdash_\Sigma \varphi \rightarrow \psi$
.10) If $\Gamma \vdash_\Sigma \neg\varphi$, then $\Gamma \vdash_\Sigma \varphi \rightarrow \psi$
.11) If $\Gamma \vdash_\Sigma \varphi \rightarrow \psi$ and $\Gamma \vdash_\Sigma \psi \rightarrow \chi$, then $\Gamma \vdash_\Sigma \varphi \rightarrow \chi$
.12) $\Gamma \vdash_\Sigma (\varphi \,\&\, \psi) \rightarrow \chi$ iff $\Gamma \vdash_\Sigma \varphi \rightarrow (\psi \rightarrow \chi)$
.13) $\Gamma \vdash_\Sigma (\varphi_1 \,\&\, \ldots \,\&\, \varphi_n) \rightarrow \psi$ iff $\Gamma \vdash_\Sigma \varphi_1 \rightarrow \ldots \rightarrow \varphi_n \rightarrow \psi$

[15] For the most part, this follows Chellas, [1980], p. 50. However, we've introduced here the generalized lemma as the easiest way to prove these facts.

49) *Theorem*: Let Γ be a set of sentences and Σ a modal logic. Then, $\Gamma \vdash_\Sigma \varphi$ iff there is a finite sequence $\varphi_1, \ldots, \varphi_m$ (with $\varphi_m = \varphi$) every member of which either (a) is a member of $\Sigma \cup \Gamma$, or (b) follows from previous members by MP.

Remark: This theorem establishes that our definition of *deducibility*, which bypasses the notion of a proof altogether, is equivalent to the more traditional definition in terms of the notion of a proof. It also explains why the properties of our deducibility relation, demonstrated in (46) and (48), correspond exactly to the properties had by the deducibility relation defined in the conception of a logic as an axiom system.

Exercise 1: Let us say that a set of sentences Γ is *deductively closed*$_\Sigma$ just in case every formula that is deducible$_\Sigma$ from Γ is already in Γ. Formally:

$$\Gamma \text{ is } \textit{deductively closed}_\Sigma =_{df} \text{for every } \varphi, \text{ if } \Gamma \vdash_\Sigma \varphi, \text{ then } \varphi \in \Gamma$$

Prove that for any set Γ, Γ is deductively closed$_\Sigma$ iff Γ is a Σ-logic.

Exercise 2: (a) Prove that if $\models^{\mathbf{M}}_{\mathbf{w}} \Sigma \cup \Gamma$ (i.e., $\models^{\mathbf{M}}_{\mathbf{w}} \psi$, for $\psi \in \Sigma \cup \Gamma$) and $\Gamma \vdash_\Sigma \varphi$, then $\models^{\mathbf{M}}_{\mathbf{w}} \varphi$. (b) Prove that $\{\varphi \,|\, \Gamma \vdash_\Sigma \varphi\}$ is the smallest modal logic containing $\Sigma \cup \Gamma$ (i.e., prove that $\{\varphi \,|\, \Gamma \vdash_\Sigma \varphi\}$ is a modal logic, that $\{\varphi \,|\, \Gamma \vdash_\Sigma \varphi\}$ contains $\Sigma \cup \Gamma$, and that if modal logic Σ' contains $\Sigma \cup \Gamma$, then $\{\varphi \,|\, \Gamma \vdash_\Sigma \varphi\} \subseteq \Sigma'$).

§4: Consistent and Maximal-Consistent Sets of Formulas

Some readers may have already encountered the idea that a set of formulas Γ is consistent (relative to logic Σ) just in case there is no formula φ such that both φ and $\neg\varphi$ are deducible from Γ (in Σ). But the following theorem shows this to be equivalent to saying that a set Γ is consistent (relative to Σ) just in case the falsum is not derivable from Γ (in Σ)

50) *Theorem*: $\Gamma \vdash_\Sigma \bot$ iff there is a formula φ such that $\Gamma \vdash_\Sigma \varphi \,\&\, \neg\varphi$.

Proof: By (45), given that $\bot$ and $\varphi \,\&\, \neg\varphi$ are tautological consequences of each other.

Remark: Given this equivalence, we adopt non-derivability of the falsum as our definition of consistency:

51) A set of formulas Γ is *consistent*$_\Sigma$ (in symbols: $\mathrm{Con}_\Sigma(\Gamma)$) iff the falsum ($\bot$) is not deducible from Γ in Σ. Formally,

$\text{Con}_\Sigma(\Gamma) =_{df} \Gamma \nvdash_\Sigma \bot$.

We shall write $\text{C}\not{o}\text{n}_\Sigma(\Gamma)$ whenever it is not the case that $\text{Con}_\Sigma(\Gamma)$.

Example 1: Here is an example of a set Γ that is consistent relative to a logic Σ but not consistent relative to logic Σ'. A variant of the example in (44) *Remark 2* works well. Consider the set $\Gamma_3 = \{\Box(p \to q), \Box p, \neg\Box q\}$. Now relative to the logic *PL*, Γ_3 is consistent. If our underlying logic consists of the tautologies and the rule MP, then no contradictions can be derived$_{PL}$ from Γ_3. For example, the formula $\Box(p \to q) \to (\Box p \to \Box q)$ is not a tautology and so not a theorem of *PL*. So from the two members of Γ_3, $\Box(p \to q)$ and $\Box p$, we can not derive$_{PL}$ $\Box q$. Otherwise, Γ_3 would be inconsistent$_{PL}$. So far, then, Γ_3 looks like it is consistent$_{PL}$. But we have to show that we cannot derive$_{PL}$ the negation of any member of Γ_3 from any combination of formulas in Γ_3. In fact, as the reader may verify, there is no member of Γ_3 having a negation that can be derived$_{PL}$ from some combination of members of Γ_3.

However, relative to the logic K, Γ_3 is not consistent. From $\Box(p \to q)$ and $\Box p$, one may derive$_K$ $\Box q$, as we saw in *Remark 2* of (44). And since $\neg\Box q \in \Gamma_3$, we know by (46.5) that $\Gamma_3 \vdash_K \neg\Box q$. Moreover, by (46.5), $\Gamma_3 \vdash_K \Box q$. So, by (48.4), $\Gamma_3 \vdash_K \Box q \;\&\; \neg\Box q$, and so by (50), $\Gamma_3 \vdash_K \bot$. Hence, by the definition of consistency$_\Sigma$, Γ_3 is not consistent$_K$.

Example 2: Here is an example of a set that is inconsistent relative to any logic containing the instances of the 5 schema: $\Diamond\varphi \to \Box\Diamond\varphi$. The logic *K5* is such a logic (it also contains instances of the K schema; *K5* will be precisely defined in §4, but for now, the fact that it contains the instances of the 5 schema is all that is important). Consider, then the set Γ_4: $\{\Diamond p, \Box\Diamond p \to q, \neg q\}$. We can prove that $\text{C}\not{o}\text{n}_{K5}(\Gamma_4)$:

(a) First we show that $\Gamma_4 \vdash_{K5} \Box\Diamond p$: Since $\Diamond p \in \Gamma_4$, we know by (46.5) that $\Gamma_4 \vdash_{K5} \Diamond p$. Since *K5* contains the instances of $\Diamond\varphi \to \Box\Diamond\varphi$, $\Diamond p \to \Box\Diamond p \in K5$. So $\vdash_{K5} \Diamond p \to \Box\Diamond p$. But by (46.2), $\Gamma_4 \vdash_{K5} \Diamond p \to \Box\Diamond p$. So by (48.7), $\Gamma_4 \vdash_{K5} \Box\Diamond p$.

(b) Next we show that $\Gamma_4 \vdash_{K5} q$: Since $\Box\Diamond p \to q \in \Gamma_4$, it follows that $\Gamma_4 \vdash_{K5} \Box\Diamond p \to q$. But by (a), $\Gamma_4 \vdash_{K5} \Box\Diamond p$. So again by (48.7), $\Gamma_4 \vdash_{K5} q$.

(c) Finally we show $\text{C}\not{o}\text{n}_{K5}(\Gamma_4)$: Since $\neg q \in \Gamma_4$, we know that $\Gamma_4 \vdash_{K5} \neg q$. But by (b), we also know that $\Gamma_4 \vdash_{K5} q$. So by (48.4), $\Gamma_4 \vdash_{K5} q \;\&\; \neg q$. Thus, by (50), $\Gamma_4 \vdash_{K5} \bot$. Hence, $\text{C}\not{o}\text{n}_{K5}(\Gamma_4)$.

52) *Theorem*: The definition of *consistent*$_\Sigma$ has the following consequences:

.1) If $\text{Con}_\Sigma(\Gamma)$, then $\bot \notin \Gamma$ and $(\varphi \,\&\, \neg\varphi) \notin \Gamma$
.2) $\text{Con}_\Sigma(\Gamma)$ iff there is a φ such that $\Gamma \nvdash_\Sigma \varphi$
.3) If $\text{C{\o}n}_{PL}(\Gamma)$, then $\text{C{\o}n}_\Sigma(\Gamma)$
.4) If $\text{C{\o}n}_\Sigma(\Gamma)$ and Σ' is a Σ-logic, then $\text{C{\o}n}_{\Sigma'}(\Gamma)$
.5) If $\text{Con}_\Sigma(\Gamma)$ and $\Delta \subseteq \Gamma$, then $\text{Con}_\Sigma(\Delta)$
.6) If $\text{C{\o}n}_\Sigma(\Delta)$ and $\Delta \subseteq \Gamma$, then $\text{C{\o}n}_\Sigma(\Gamma)$
.7) $\text{Con}_\Sigma(\Gamma)$ iff for every finite subset Δ of Γ, $\text{Con}_\Sigma(\Delta)$
.8) $\Gamma \vdash_\Sigma \varphi$ iff $\text{C{\o}n}_\Sigma(\Gamma \cup \{\neg\varphi\})$
.9) $\text{Con}_\Sigma(\Gamma \cup \{\varphi\})$ iff $\Gamma \nvdash_\Sigma \neg\varphi$
.10) If $\text{Con}_\Sigma(\Gamma)$, then for any formula φ, either $\text{Con}_\Sigma(\Gamma \cup \{\varphi\})$ or $\text{Con}_\Sigma(\Gamma \cup \{\neg\varphi\})$

Remark: Explain what these signify.

53) A set of formulas Γ is *maximal* (in symbols: $\text{Max}(\Gamma)$) iff for every φ, either $\varphi \in \Gamma$ or $\neg\varphi \in \Gamma$. A set of formulas Γ is *maximal-consistent*$_\Sigma$ (in symbols: $\text{MaxCon}_\Sigma(\Gamma)$) iff both $\text{Max}(\Gamma)$ and $\text{Con}_\Sigma(\Gamma)$.

Remark: Note that maximal-consistent sets are relativized to a logic. Some sets are maximal-consistent relative to logic Σ but not maximal-consistent relative to logic Σ'. For example.

54) *Theorem*: Suppose $\text{MaxCon}_\Sigma(\Gamma)$. Then:[16]

.1) $\varphi \in \Gamma$ iff $\Gamma \vdash_\Sigma \varphi$
.2) $\Sigma \subseteq \Gamma$
.3) $\bot \notin \Gamma$
.4) $\neg\varphi \in \Gamma$ iff $\varphi \notin \Gamma$
.5) $\varphi \,\&\, \psi \in \Gamma$ iff both $\varphi \in \Gamma$ and $\psi \in \Gamma$
.6) $\varphi \vee \psi \in \Gamma$ iff either $\varphi \in \Gamma$ or $\psi \in \Gamma$
.7) $\varphi \rightarrow \psi \in \Gamma$ iff if $\varphi \in \Gamma$ then $\psi \in \Gamma$
.8) $\varphi \leftrightarrow \psi \in \Gamma$ iff $\varphi \in \Gamma$ if and only if $\psi \in \Gamma$
.9) Γ is a Σ-logic

Remark: Note that in the right-left direction, (.1) asserts that maximal-consistent sets are deductively closed.

55) *Theorem* (Lindenbaum's Lemma): If $\text{Con}_\Sigma(\Gamma)$, then there is a $\text{MaxCon}_\Sigma(\Delta)$ such that $\Gamma \subseteq \Delta$.

[16] This follows Chellas [1980], p.53.

Proof: Let $\varphi_0, \varphi_1, \varphi_2, \ldots$ be a listing of the sentences in Λ_Ω. We define an infinite sequence of *sets of sentences* $\Delta_0, \Delta_1, \Delta_2, \ldots$, and then the general union Δ of the members of the sequence:

$$\Delta_0 = \Gamma$$

$$\Delta_{n+1} = \begin{cases} \Delta_n \cup \{\varphi_n\}, \text{ if } \Delta_n \vdash_\Sigma \varphi_n \\ \Delta_n \cup \{\neg\varphi_n\}, \text{ otherwise} \end{cases}$$

$$\Delta = \bigcup_{n \geq 0} \Delta_n$$

In other words, Δ_0 is the set Γ, and Δ_{n+1} is constructed by adding φ_n (the n^{th} formula in the list) to Δ_n if φ_n is derivable$_\Sigma$ from Δ_n; otherwise, Δ_{n+1} is constructed by adding $\neg\varphi_n$ to Δ_n. We show first that $\Gamma \subseteq \Delta$ and then $\text{MaxCon}_\Sigma(\Delta)$.

Lemma 1: $\Delta_n \subseteq \Delta$, for $n \geq 0$.

Lemma 2: $\Gamma \subseteq \Delta$

To show $\text{MaxCon}_\Sigma(\Delta)$, we prove first:

Lemma 3: $\text{Max}(\Delta)$

Proof: We show, for arbitrary n, that either $\varphi_n \in \Delta$ or $\neg\varphi_n \in \Delta$. Since either $\Delta_n \vdash_\Sigma \varphi_n$ or $\Delta_n \nvdash_\Sigma \varphi_n$, we know Δ_{n+1} is equal either to $\Delta_n \cup \{\varphi_n\}$ or to $\Delta_n \cup \{\neg\varphi_n\}$. So either $\varphi_n \in \Delta_{n+1}$ or $\neg\varphi_n \in \Delta_{n+1}$. So by *Lemma 1*, either $\varphi_n \in \Delta$ or $\neg\varphi_n \in \Delta$.

It remains only to show $\text{Con}_\Sigma(\Delta)$. Our strategy will be to show (a) that all the Δ_ns are consistent, and (b) that if Δ were inconsistent, the inconsistency would show up in one of the Δ_ns, contradicting (a). We show next that each of the Δ_ns is consistent$_\Sigma$:

Lemma 4: $\text{Con}_\Sigma(\Delta_n)$, for $n \geq 0$

Proof: By induction on the Δ_ns.

(a) Base case: By hypothesis, $\text{Con}_\Sigma(\Gamma)$, and since $\Gamma = \Delta_0$, we have $\text{Con}_\Sigma(\Delta_0)$.

(b) Inductive case: Either $\Delta_{n+1} = \Delta_n \cup \{\varphi_n\}$ or $\Delta_{n+1} = \Delta_n \cup \{\neg\varphi_n\}$ (but not both). (1) Suppose $\Delta_{n+1} = \Delta_n \cup \{\varphi_n\}$. So by construction, $\Delta_n \vdash_\Sigma \varphi_n$. Hence, by (52.8), $\text{C\o n}_\Sigma(\Delta_n \cup \{\neg\varphi_n\})$. But by the inductive hypothesis, $\mathrm{Con}_{\Sigma}(\Delta_n)$. So by (52.10), $\mathrm{Con}_{\Sigma}(\Delta_n \cup \{\varphi_n\})$, i.e., $\mathrm{Con}_{\Sigma}(\Delta_{n+1})$. (2) Suppose $\Delta_{n+1} = \Delta_n \cup \{\neg\varphi_n\}$. Then, by construction, $\Delta_n \nvdash_{\Sigma} \varphi_n$. So, by (52.8), $\mathrm{Con}_{\Sigma}(\Delta_n \cup \{\neg\varphi_n\})$, i.e., $\mathrm{Con}_{\Sigma}(\Delta_{n+1})$.

If we are to argue that an inconsistency in Δ would show up in one of the Δ_ns, it suffices to show, by (52.7), that every finite subset Δ' of Δ is a subset of one of the Δ_ns, for by (52.6), the inconsistency would be inherited by the Δ_n of which Δ' is a subset, contradicting the lemma we just proved. We prove that every finite subset of Δ is a subset of one of the Δ_ns as the last of the following three lemmas, the first two of which are required for the proof:

Lemma 5: $\Delta_k \subseteq \Delta_n$, for $0 \leq k \leq n$.

Lemma 6: if $\varphi_k \in \Delta$, then $\varphi_k \in \Delta_{k+1}$, for $k \geq 0$

Lemma 7: For every finite subset Δ' of Δ, $\exists n \geq 0$ such that $\Delta' \subseteq \Delta_n$

Proof: Suppose Δ' is a finite subset of Δ. Let φ_n be the sentence in Δ' with the largest index n. We show that $\Delta' \subseteq \Delta_{n+1}$. Assume $\varphi \in \Delta'$. Then $\varphi = \varphi_k$, for some $k \leq n$. Since $\Delta' \subseteq \Delta$, we know that φ $(=\varphi_k)$ must be in Δ. So by *Lemma 6*, $\varphi \in \Delta_{k+1}$. But by *Lemma 5*, $\Delta_{k+1} \subseteq \Delta_{n+1}$. So $\varphi \in \Delta_{n+1}$.

Finally, we argue:

Lemma 8: $\mathrm{Con}_{\Sigma}(\Delta)$

Proof: Suppose $\mathrm{C\emptyset n}_{\Sigma}(\Delta)$. Then, by (52.7), there is a finite subset Δ' of Δ such that $\mathrm{C\emptyset n}_{\Sigma}(\Delta')$. But by *Lemma 7*, there exists an n such that $\Delta' \subseteq \Delta_n$. So by (52.6), there exists an n such that $\mathrm{C\emptyset n}_{\Sigma}(\Delta_n)$, contradicting *Lemma 4*.

56) *Corollary 1*: $\Gamma \vdash_{\Sigma} \varphi$ iff φ is an element of every $\mathrm{MaxCon}_{\Sigma}(\Delta)$ such that $\Gamma \subseteq \Delta$.

57) *Corollary 2*: $\vdash_{\Sigma} \varphi$ iff for every $\mathrm{MaxCon}_{\Sigma}(\Delta)$, $\varphi \in \Delta$

Proof: Let Γ in (56) be $\emptyset$ and use (46.1).

Exercise: Some modal logicians use the following, somewhat different notion of a maximal-consistent$_{\Sigma}$ set: Γ is maximal-consistent$_{\Sigma}$ iff (a)

$\text{Con}_\Sigma(\Gamma)$ and (b) for every φ, if $\text{Con}_\Sigma(\Gamma \cup \{\varphi\})$, then $\varphi \in \Gamma$. The idea here is that a maximal-consistent set is a consistent set Γ such that the addition of one formula φ not already in Γ would result in an inconsistent set. Show that this notion is equivalent to the notion that we have employed.

§5: Normal Logics

So far, we have been concerned with the properties of modal logics, possibly of the weakest kind. Modal logics are defined so as to capture, at the very least, all of the propositionally correct forms of reasoning. As such, they contain all the tautologies, and contain the tautological consequences of any combination of formulas they contain. We turn next to a class of modal logics defined so as to capture the most basic, modally correct, forms of reasoning. These logics contain not only the tautologies and propositionally correct consequences of formulas they contain, but also both all the other valid formulas and all of the modally correct consequences of any combination of formulas they contain. These are the *normal* modal logics, the weakest of which is the logic K. The logic K contains the tautologies, the instances of the axiom K, and is closed under MP and the Rule of Necessitation. Since this will be the weakest normal modal logic, we use 'contains K and closed under RN' as our definition of a normal modal logic (see below). We show in the next chapter that K is sound (its theorems are all valid) and complete (all valid formulas are theorems of K).

58) Recall that the rule RN is the sequence $\varphi / \Box\varphi$. We say that a modal logic Σ is *normal* iff every instance of the schema K is an element of Σ and Σ is closed under RN.

59) *Theorem*: A modal logic Σ is normal iff Σ is closed under the following rule RK: $\varphi_1 \to \ldots \to \varphi_n \to \psi \ / \ \Box\varphi_1 \to \ldots \to \Box\varphi_n \to \Box\psi$.

Remark: This theorem shows that instead of defining normal modal logics as containing K and being closed under RN, we could have defined them more simply as being closed under RK. From our work in (33) and (34), we know that both of these rules preserve validity (in every class of models) and truth in a model. However, in this case, we have chosen to use the axiom K and rule RN because it may seem more intuitive to readers accustomed to the conception of a logic as an axiom system and because each piece of the notation $KS_1 \ldots S_n$, which we introduce below to designate the normal logic containing the schemata $\mathrm{K}, \mathrm{S}_1, \ldots, \mathrm{S}_n$, is linked to a schema that identifies the logic (so in the case when $n = 0$, the normal logic K is identified by the axiom K).

60) *Theorem*: If Σ is normal, then:

.1) $\vdash_\Sigma \Box\varphi \leftrightarrow \neg\Diamond\neg\varphi$.

.2) $\vdash_\Sigma \neg\Diamond\bot$

.3) $\vdash_\Sigma \Box\varphi$, for any tautology φ

Remark: (.1) highlights the fact that propositional logic alone is not sufficient for deriving the equivalence $\Box\varphi \leftrightarrow \neg\Diamond\neg\varphi$ from the definition $\Diamond\varphi \leftrightarrow \neg\Box\neg\varphi$. The added machinery that comes with normal logics is essential for deriving the interdefinability of the $\Box$ and $\Diamond$. That is, the interdefinability of the $\Box$ and $\Diamond$ requires *modal* inferences. This may come as a surprise. Moreover, it is a consequence that if $\Diamond\varphi$ is taken as a primitive formula of the language rather than defined as $\neg\Box\neg\varphi$, then in the definition of a normal logic we have to stipulate that normal logics, in addition to containing K and being closed under RN, contain all the instances of the valid schema $\Diamond\varphi \leftrightarrow \neg\Box\neg\varphi$, for otherwise we could not preserve the interdefinability of the $\Box$ and $\Diamond$ in normal systems.

Exercise 1: Prove that if Σ is a normal logic, then Σ contains the following theorems and is closed under the following rules:[17]

$\Diamond\neg\varphi \leftrightarrow \neg\Box\varphi$
$\varphi \to \psi / \Box\varphi \to \Box\psi$ ('RM')
$\varphi \leftrightarrow \psi / \Box\varphi \leftrightarrow \Box\psi$ ('RE')
$\varphi \to (\psi \to \chi) / \Box\varphi \to (\Box\psi \to \Box\chi)$ ('RR')
$\Box(\varphi \,\&\, \psi) \to (\Box\varphi \,\&\, \Box\psi)$ ('M')
$(\Box\varphi \,\&\, \Box\psi) \to \Box(\varphi \,\&\, \psi)$ ('C')
$\Box(\varphi \,\&\, \psi) \leftrightarrow (\Box\varphi \,\&\, \Box\psi)$ ('R')
$(\varphi_1 \,\&\, \ldots \,\&\, \varphi_n) \to \varphi / (\Box\varphi_1 \,\&\, \ldots \,\&\, \Box\varphi_n) \to \Box\varphi$ ('RK&')
$\varphi \to (\varphi_1 \vee \ldots \vee \varphi_n) / \Diamond\varphi \to (\Diamond\varphi_1 \vee \ldots \vee \Diamond\varphi_n)$ ('RK$\vee$')
$\varphi \to \psi / \Diamond\varphi \to \Diamond\psi$ ('RM$\Diamond$')
$\varphi \leftrightarrow \psi / \Diamond\varphi \leftrightarrow \Diamond\psi$ ('RE$\Diamond$')
$\Diamond(\varphi \vee \psi) \leftrightarrow (\Diamond\varphi \vee \Diamond\psi)$ ('$\vee\Diamond$')
$\Diamond(\varphi \,\&\, \psi) \to (\Diamond\varphi \,\&\, \Diamond\psi)$ ('&$\Diamond$')
$\Diamond(\varphi_1 \,\&\, \ldots \,\&\, \varphi_n) \to (\Diamond\varphi_1 \,\&\, \ldots \,\&\, \Diamond\varphi_n)$ ('&$_n\Diamond$')

[17]Many of the labels on these schemata and rules follow Chellas [1980], pp. 114–19.

Exercise 2: Prove that (a) Σ is closed under RK iff Σ is closed under RR and RN, (b) if Σ is normal, then Σ is closed under RK$\vee$, and (c) if Σ is closed under RK$\vee$ and contains the instances of the schema $\Box\varphi \leftrightarrow \neg\Diamond\neg\varphi$, then Σ is normal.

Exercise 3: (a) Show that (for any Σ) if $\text{MaxCon}_\Sigma(\Gamma)$, then Γ is a modal logic but not a necessarily a normal modal logic. (b) In particular, for $\text{MaxCon}_K(\Gamma)$, explain whether or not Γ must satisfy any of the conditions that define normal logics.

Exercise 4: Let $\varphi[\psi'/\psi]$ be the result of replacing zero or more occurrences of ψ in φ by ψ'. Then every normal modal logic has the Rule of Replacement: $\psi \leftrightarrow \psi'/\varphi \leftrightarrow \varphi[\psi'/\psi]$.

61) *Theorem*: The following are examples of normal modal logics:

.1) $\{\varphi \,|\, \models^{\mathbf{M}} \varphi\}$, for every model $\mathbf{M}$ (this is an example of a normal modal logic that is typically not axiomatizable)
.2) $\{\varphi \,|\, \models \varphi\}$
.3) $\{\varphi \,|\, \mathbf{C} \models \varphi\}$, for every class of models $\mathbf{C}$
.4) $\{\varphi \,|\, \mathbf{F} \models \varphi\}$, for every frame $\mathbf{F}$
.5) $\{\varphi \,|\, \mathbf{C_F} \models \varphi\}$, for any class of frames $\mathbf{C_F}$.
.6) $\{\varphi \,|\, \varphi \in \Lambda_\Omega\}$

62) We define the system K to be the smallest normal logic, i.e.,

$$K = \bigcap\{\Sigma \,|\, \Sigma \text{ is a normal modal logic}\}$$

Remark: This definition carves out the system K "from the top down." That is, we pare down all the sets which contain the tautologies, the instances of the K axiom, and which are closed under MP and RN until we reach the smallest one. As the *smallest* normal logic, K is a subset of every normal logic. Thus, the *only* way a formula can qualify as a theorem of K is by being a tautology, an instance of K, or by being the conclusion by MP or RN of formulas in K. For suppose φ is a theorem of K but neither a tautology, instance of K nor the conclusion by MP or RN of formulas in K. Then the set $K - \{\varphi\}$ would qualify as a normal logic (since it still has all the tautologies, instances of K, and is closed under MP and RN), yet would be a proper subset of K, contradicting the fact that K is the *smallest* normal modal logic. Thus, K contains nothing more than what it has to contain by meeting the definition of a normal logic. Consequently, to prove that *all* the theorems of K have a certain property F, it suffices to prove (inductively) that the tautologies and instances of K have F and that property F is preserved by the rules of inference MP and RN.

Note, also, that our definition implies that K is a subset of each of the examples of modal logics in (61).

63) *Theorem*: K is axiomatized by the tautologies, the axiom schema K, and the rules MP and RN.

Exercise: (a) Show that K is axiomatized by the tautologies and the rules MP and RK; (b) Show that K is axiomatized by the axiom schema K and the rules RPL and RN; (c) Show that K is axiomatized by the empty set $\emptyset$ of axioms and the rules RPL and RK.

64) Following Lemmon [1977], we define the logic $KS_1 \dots S_n$ as the smallest normal logic containing (the instances of) the schemata $\mathrm{S}_1, \dots, \mathrm{S}_n$. Set theoretically, we define:

$$KS_1 \dots S_n =_{df} \bigcap\{\Sigma \,|\, \Sigma \text{ is normal and } \mathrm{S}_1 \cup \dots \cup \mathrm{S}_n \subseteq \Sigma\}$$

However, there are some names of normal modal logics which have already become established in the literature and which do not follow this notation. Here are some examples, identified in terms of our defined notation:

$T = KT$
$B = KB$
$S4 = KT4$
$S5 = KT5$
$K4.3 = K4L$
$S4.3 = KT4L$

Remark: As the smallest normal logic containing $\mathrm{S}_1, \dots, \mathrm{S}_n$, $KS_1 \dots S_n$ is a subset of every normal logic containing the schemata $\mathrm{S}_1, \dots, \mathrm{S}_n$. Thus, the *only* way a formula can qualify as a theorem of $KS_1 \dots S_n$ is by being a tautology, an instance of $\mathrm{K}, \mathrm{S}_1, \dots, \mathrm{S}_n$, or by being the conclusion by MP or RN of formulas already in $KS_1 \dots S_n$. For otherwise, suppose φ is a theorem of $KS_1 \dots S_n$ but neither a tautology, an instance of $\mathrm{K}, \mathrm{S}_1, \dots,$ S_n, nor the conclusion by MP or RN of formulas in $KS_1 \dots S_n$. Then the set $KS_1 \dots S_n - \{\varphi\}$ would qualify as a normal logic containing $\mathrm{K}, \mathrm{S}_1, \dots,$ S_n (since it still has all the tautologies, instances of $\mathrm{K}, \mathrm{S}_1, \dots, \mathrm{S}_n$, and is closed under MP and RN), yet would be a proper subset of $KS_1 \ldots S_n$, contradicting the fact that $KS_1 \ldots S_n$ is the *smallest* normal modal logic. Consequently, to prove that the theorems of $KS_1 \ldots S_n$ have property F, it suffices to prove (inductively) (a) that the tautologies and instances of $\mathrm{K}, \mathrm{S}_1, \ldots, \mathrm{S}_n$ have F and (b) that property F is preserved by the rules of inference MP and RN.

65) *Theorem*: $KS_1 \ldots S_n$ is axiomatized by the tautologies, the axiom schemata K, S_1, ..., S_n, and the rules MP and RN.

§6: Normal Logics and Maximal-Consistent Sets

In this last section of Chapter 3, we work our way towards the proof that maximal-consistent sets relative to normal logics have some rather interesting properties. These properties play an important role in the completeness theorems that we prove in the next chapter. In these proofs of completeness, we develop a general method of constructing models for consistent normal logics. The 'worlds' of the model for a logic Σ will be the sets of sentences Γ that are maximal-consistent relative to Σ. In such a model, it will turn out that a formula φ is true at the 'world' Γ just in case φ is a member of Γ. Moreover, since a maximal-consistent set Γ has as elements precisely the formulas derivable from it, when Γ plays the role of a world, it will have the interesting property that the formulas true at Γ will be precisely the ones derivable from Γ. In particular, when a maximal-consistent Γ plays the role of a world in the models we shall construct, any consequence of Γ is already 'true' at Γ.

When maximal-consistent sets play the role of worlds, an appropriate accessibility relation $\mathbf{R}$ has to be defined. A proper definition of accessibility will have to ensure that when $\Box\varphi$ is a member of Γ, and Γ is $\mathbf{R}$-related to Δ, then φ will be a member of Δ. And, moreover, a proper definition of the accessibility relation will have to ensure that whenever a maximal-consistent set Γ is $\mathbf{R}$-related to a maximal-consistent set Δ, then any sentence φ that is a member of (i.e., true at) Δ is such that $\Diamond\varphi$ is a member of (i.e., true at) Γ. In this section, then, we show that maximal-consistent sets of sentences in normal logics can be related to each other in precisely these ways. We begin by proving some lemmas concerning ordinary sets Γ (not necessarily maximal-consistent) in normal logics.

66) *Lemma*: Suppose Σ is normal. Then if $\Gamma \vdash_\Sigma \varphi$, then $\{\Box\psi \,|\, \psi \in \Gamma\} \vdash_\Sigma \Box\varphi$.

Remark: This theorem shows that the deducibility relation in normal logics behaves in a certain interesting way, namely, that whenever φ is deducible from a set Γ (in normal logic Σ), then the necessitation of φ is deducible (in Σ) from the set of the necessitations of the members of Γ.

67) *Lemma*: Suppose Σ is normal. Then if $\{\psi | \Box\psi \in \Gamma\} \vdash_\Sigma \varphi$, then $\Gamma \vdash_\Sigma \Box\varphi$

Remark: This lemma is an equivalent statement of the preceding one. It says that if a formula φ is derivable (in a normal Σ) from the set of sentences having necessitations in Γ, then the necessitation of φ is derivable (in Σ) from Γ. It is instructive to understand why these two lemmas are equivalent.

68) *Theorem*: Suppose Σ is normal and that $\mathrm{MaxCon}_\Sigma(\Gamma)$. Then, $\Box\varphi \in \Gamma$ iff φ is a member of every $\mathrm{MaxCon}_\Sigma(\Delta)$ such that $\{\psi | \Box\psi \in \Gamma\} \subseteq \Delta$.

Remark: The real force of the present theorem is this. Think of Γ and Δ as 'worlds', and think of the formulas that are members of these sets as true at that world. Then, the clause $\{\psi | \Box\psi \in \Gamma\} \subseteq \Delta$ says that ψ is 'true at' Δ whenever $\Box\psi$ is 'true at' Γ. Intuitively, this means Γ bears the accessibility relation $\mathbf{R}$ to Δ. Given this understanding, then, this theorem is just an analogue, constructed out of syntactic entities, of (6.5), i.e., of the conditions that must be satisfied if $\Box\varphi$ is to be true at a world. For when we think of Γ as a world bearing $\mathbf{R}$ to Δ, then this theorem just tells us that a formula $\Box\varphi$ is true at Γ iff φ is true at every $\mathbf{R}$-related world Δ.

Exercise: Find a proof of this theorem in the right-left direction using (66) instead of (67).

69) *Lemma*: Suppose Σ is normal, $\mathrm{MaxCon}_\Sigma(\Gamma)$, and $\mathrm{MaxCon}_\Sigma(\Delta)$. Then, [for every φ, if $\Box\varphi \in \Gamma$, then $\varphi \in \Delta$] iff [for every φ, if $\varphi \in \Delta$, then $\Diamond\varphi \in \Gamma$]

Remark: Here is a slightly more efficient way to state the present lemma: $\{\varphi \,|\, \Box\varphi \in \Gamma\} \subseteq \Delta$ iff $\{\Diamond\varphi \,|\, \varphi \in \Delta\} \subseteq \Gamma$. To see what this says, think of Γ and Δ as 'worlds' and membership in as 'truth at'. Then, the lemma says that [Γ and Δ are $\mathbf{R}$-related in the sense that φ is true in (i.e., a member of) Δ whenever $\Box\varphi$ is true in (i.e., a member of) Γ] iff [Γ and Δ are also $\mathbf{R}$-related in the sense that $\Diamond\varphi$ is true in Γ whenever φ is true in Δ]. Since normal logics are modally well-behaved in the sense that they contain the usual biconditionals interdefining the $\Box$ and $\Diamond$, whenever two maximal consistent sets instantiate the **R** relation with respect to one modality, they automatically instantiate the **R** relation with respect to the other modality.

70) *Theorem*: Suppose that Σ is normal and that $\mathrm{MaxCon}_\Sigma(\Gamma)$. Then $\Diamond\varphi \in \Gamma$ iff there is a $\mathrm{MaxCon}_\Sigma(\Delta)$ such that both (a) $\{\Diamond\psi | \, \psi \in \Delta\} \subseteq \Gamma$ and (b) $\varphi \in \Delta$.

Remark: The present theorem does for the $\Diamond$ what (68) does for the $\Box$. Think of Γ and Δ as worlds. Suppose further that the accessibility relation **R** holds when formulas true at Δ are possibly true at Γ. Then this theorem is just an analogue, constructed out of syntactic entities, of *Remark 2* in (6.5), i.e., of the conditions that must be satisfied if $\Diamond\varphi$ is to be true at a world. For when we think of Γ as a world bearing **R** to Δ, then this theorem just tells us that a formula $\Diamond\varphi$ is true at (i.e., a member of) Γ iff φ is true at some **R**-related world Δ.

Chapter Five:
Soundness and Completeness

In this chapter, we assemble the results of the previous two chapters so that we may show that certain normal modal logics are sound and complete with respect to certain classes of models and classes of frames.

§1: Soundness

71) A logic Σ is *sound with respect to a class of models* $\mathbf{C}$ iff every theorem of Σ is valid with respect to $\mathbf{C}$. That is,

$$\Sigma \textit{ is sound with respect to } \mathbf{C} =_{df} \text{for every } \varphi, \text{ if } \vdash_\Sigma \varphi, \text{ then } \mathbf{C} \models \varphi$$

Remark: In what follows we prove, for $\Sigma = K$ and $\Sigma = K\mathrm{S}_1 \ldots \mathrm{S}_n$, that Σ is sound with respect to a certain class $\mathbf{C}$ of models. Our argument shall consist of two claims: (a) that the tautologies and schemata identifying Σ are valid with respect to $\mathbf{C}$, and (b) that the rules of inference MP and RN preserve validity with respect to $\mathbf{C}$. There are two different reasons why this argument establishes that Σ is sound with respect to $\mathbf{C}$. We discuss them in turn.

(.1) Recall that the logic K is the smallest normal logic. In the *Remark* in (63), we established that the *only* way a formula can qualify as a theorem of K is by being a tautology, an instance of K, or by being the conclusion by MP or RN of formulas in K. Consequently, to prove that *all* the theorems of K have a certain property F, it suffices to prove that the tautologies and instances of the schema K have F and that property F is preserved by the rules of inference MP and RN. In particular, if we want to show that the theorems of the logic K are all valid with respect to the class of all models, then we show that the tautologies and instances of the schema K are valid with respect to this class, and that MP and RN preserve validity with respect to this class.

Similar remarks apply to $KS_1 \ldots S_n$. In the *Remark* in (65), we established that the *only* way a formula can qualify as a theorem of $KS_1 \ldots S_n$ is by being a tautology, an instance of $\mathrm{K}, \mathrm{S}_1, \ldots, \mathrm{S}_n$, or by being the conclusion by MP or RN of formulas already in $KS_1 \ldots S_n$. So to prove that the theorems of $KS_1 \ldots S_n$ have property F, it suffices to prove (inductively) (a) that the tautologies and instances of $\mathrm{K}, \mathrm{S}_1, \ldots, \mathrm{S}_n$ have F and (b) that property F is preserved by the rules of inference MP and RN.

In particular, if we want to show that the theorems of $KS_1 \dots S_n$ are all valid with respect to a class of models $\mathbf{C}$, then we show that the tautologies and instances of the schemata $\mathrm{K}, \mathrm{S}_1, \dots, \mathrm{S}_n$ are valid with respect to $\mathbf{C}$, and that MP and RN preserve validity with respect to $\mathbf{C}$.

(.2) The other reason why this two-part argument for the soundness of K and $KS_1 \dots S_n$ is sufficient derives from the fact that these logics are axiomatizable. Consider the following lemma:

> *Lemma*: If (i) Σ is axiomatized by the set Γ and the rules $\mathrm{R}_1, \mathrm{R}_2, \dots,$ and (ii) the members of Γ are valid with respect to $\mathbf{C}$, and (iii) the rules preserve validity with respect to $\mathbf{C}$, then Σ is sound with respect to $\mathbf{C}$.

Thus, since we know in particular that K is axiomatized by the tautologies, the schema K, and the rules MP and RN, to establish the soundness of Σ with respect to the class of all models, we need only argue (a) that the tautologies and K are valid with respect to the class of all models and (b) that MP and RN preserve validity with respect to the class of all models. Similarly for the logic $KS_1 \dots S_n$. Given that $KS_1 \dots S_n$ is axiomatized by the tautologies, $\mathrm{K}, \mathrm{S}_1, \dots, \mathrm{S}_n$, and the rules MP and RN, we need only argue (a) that the tautologies and $\mathrm{K}, \mathrm{S}_1, \dots, \mathrm{S}_n$ are valid with respect to the class $\mathbf{C}$ of models, and (b) the rules MP and RN preserve validity with respect to $\mathbf{C}$, to show that the logic $KS_1 \dots S_n$ is sound with respect to $\mathbf{C}$.

72) *Theorem*: K is sound with respect to the class of all models.

Proof: Given the lengthy *Remark* in (71), this follows from the facts that (a) the tautologies and instances of K are valid, by (16) and (17), and so valid with respect to the class of all models, and (b) that MP and RN preserve validity with respect to any class of models, by (30) and (33).

Remark: There is now in the literature an even more ingenious proof of the soundness of K. The argument requires little commentary, since it is so simple. Assume $\vdash_K \varphi$, i.e., $\varphi \in K$ (to show $\models \varphi$). Since K is the smallest normal modal logic, K is a subset of every normal modal logic. In particular, by (61), $\{\psi \mid \models \psi\}$ is a normal modal logic. So K is a subset of this set. Hence $\varphi \in \{\psi \mid \models \psi\}$, i.e., $\models \varphi$. Since φ is valid, it is valid with respect to the class of all models.

73) *Theorem*: Let $\mathbf{C}$P designate the class of all models $\mathbf{M}$ such that $\mathbf{R_M}$ has property P. Suppose that $\mathrm{S}_1, \dots, \mathrm{S}_n$ are schemata which are valid, respectively, in the classes of standard models $\mathbf{C}P_1, \ldots, \mathbf{C}P_n$. Let $\mathbf{C}P_1 \ldots P_n$ designate the class of models $\mathbf{C}P_1 \cap \ldots \cap \mathbf{C}P_n$. Then the modal logic $KS_1 \ldots S_n$ is sound with respect to $\mathbf{C}P_1 \ldots P_n$; i.e., if $\vdash_{KS_1 \ldots S_n} \varphi$, then $\mathbf{C}P_1 \ldots P_n \models \varphi$.

Proof: Given the lengthy *Remark* in (71), it suffices to argue as follows: (a) If φ is a tautology or an instance of K, φ is valid, and so valid with respect to $\mathbf{C}P_1 \ldots P_n$. If φ is an instance of S_i ($1 \leq i \leq n$), then by hypothesis, φ is valid with respect to $\mathbf{C}P_i$. But if so, then φ is valid with respect to the class $\mathbf{C}P_1 \cap \ldots \cap \mathbf{C}P_i \cap \ldots \cap \mathbf{C}P_n$, since this is a subset of $\mathbf{C}P_i$. So, for each i, the instances of S_i are valid with respect to the class $\mathbf{C}P_1 \ldots P_n$. And so, in general, the instances of the axioms $K, S_1, \ldots, S_n$ are valid with respect to the class of models $\mathbf{C}P_1 \ldots P_n$. (b) MP and RN both preserve validity with respect to any class of models, and so preserve validity with respect to $\mathbf{C}P_1 \ldots P_n$.

Remark 1: Now reconsider the facts proved in (23). These facts establish that certain schemata are valid with respect to certain classes of models. So, for example, we know that T is valid with respect to the class of reflexive models, and 5 is valid with respect to the class of euclidean models. We have as an instance of the present theorem, therefore, that the logic *KT5* is sound with respect to the class of all reflexive, euclidean models, i.e., if $\vdash_{KT5} \varphi$, then $\mathbf{C}$-refl,eucl $\models \varphi$. Similarly, for any other combination of the schemata in (23).

Remark 2: Again, we may argue that $KS_1 \ldots S_n$ is sound with respect to $\mathbf{C}$ in a somewhat more ingenious manner. By hypothesis, the instances of S_i are valid with respect to the class of models $\mathbf{C}P_i$ ($1 \leq i \leq n$). So the instances of S_i are valid with respect to the class of models $\mathbf{C}P_1 \ldots P_n$, since this is a subset of $\mathbf{C}P_i$. Consequently, the instances of $S_1, \ldots,$ S_n are elements of $\{\psi \mid \mathbf{C}P_1 \ldots P_n \models \psi\}$. But by (61.4), this set is a normal logic, and so it is a normal logic containing (the instances of) $S_1, \ldots, S_n$. But by definition, $KS_1 \ldots S_n$ is the smallest logic containing $S_1, \ldots, S_n$. So $KS_1 \ldots S_n$ must be a subset of $\{\psi \mid \mathbf{C}P_1 \ldots P_n \models \psi\}$. That is, if $\varphi \in KS_1 \ldots S_n$, then $\varphi \in \{\psi \mid \mathbf{C}P_1 \ldots P_n \models \psi\}$. In other words, if $\vdash_{KS_1 \ldots S_n} \varphi$, then $\mathbf{C}P_1 \ldots P_n \models \varphi$. Thus, $KS_1 \ldots S_n$ is sound with respect to $\mathbf{C}P_1 \ldots P_n$.

Exercise: Reconsider the *Remark 1* in (28) in light of the present proof of soundness for various systems. Show that the schemata B, 4, and 5 are not theorems of KT. Show that the schema 4 is not a theorem of the logic KB. What other facts can you prove from the results in (28) given our soundness results?

74) Let us say that Σ is *sound with respect to a class of frames* $\mathbf{C_F}$ iff every theorem of Σ is valid with respect to $\mathbf{C_F}$. That is,

$$\Sigma \textit{ is sound with respect to } \mathbf{C_F} =_{df} \text{for every } \varphi, \text{ if } \vdash_\Sigma \varphi, \text{ then } \mathbf{C_F} \models \varphi$$

Remark: Note that for a logic Σ to be sound with respect to a class of frames, the theorems of Σ must be true in every model based on any frame in the class.

75) *Theorem*: K is sound with respect to the class of all frames.

Proof: Use reasoning analogous to that used in the *Remark* in (72).

76) *Theorem*: Let the class of P-frames (in symbols: $\mathbf{C_F}$P) be the class of all frames $\mathbf{F}$ in which $\mathbf{R_F}$ has property P. Then $KS_1 \ldots S_n$ is sound with respect to the class of all $\mathrm{P}_1 \ldots \mathrm{P}_n$-frames, i.e., if $\vdash_{KS_1 \ldots S_n} \varphi$, then $\mathbf{C_F}\mathrm{P}_1 \ldots \mathrm{P}_n \models \varphi$.

Proof: Use reasoning that generalizes on that used in the previous theorem.

§2: Completeness

77) A logic Σ is *complete with respect to* a class $\mathbf{C}$ of models iff every formula valid with respect to $\mathbf{C}$ is a theorem of Σ. Formally:

> Σ is complete with respect to $\mathbf{C} =_{df}$ for every φ, if $\mathbf{C} \models \varphi$, then $\vdash_\Sigma \varphi$

For example, to say that the logic *K4* is complete with respect to the class of transitive models is to say that every formula valid in the class of transitive models is a theorem of *K4*; i.e., for every φ, if $\mathbf{C}$-trans $\models \varphi$, then $\vdash_{K4} \varphi$.

Extended Remark: The favored way of establishing that Σ is complete relative to $\mathbf{C}$ is by proving the 'contrapositive' of the definition, that is, by proving that if φ is not a theorem of Σ, then φ is not valid with respect to $\mathbf{C}$. This is a helpful way of picturing and understanding the definition. Somewhat more formally, the 'contrapositive' amounts to:

(A) For every φ, if $\nvdash_\Sigma \varphi$, then $\exists \mathbf{M} \in \mathbf{C}$ such that $\not\models^{\mathbf{M}} \varphi$.

So to prove completeness, one might look for a general way of constructing, for an arbitrary non-theorem φ of Σ, a model $\mathbf{M} \in \mathbf{C}$ which falsifies φ.

In fact, however, it turns out that in developing proofs that certain consistent normal logics are complete, logicians have discovered a general way of proving something even stronger. They have discovered a way of constructing a unique model $\mathbf{M} \in \mathbf{C}$ which falsifies *every* non-theorem of Σ! This construction technique yields a very general method for proving:

(B) $\exists \mathbf{M} \in \mathbf{C}$ such that for every φ, if $\nvdash_\Sigma \varphi$, then $\not\models^{\mathbf{M}} \varphi$

The model $\mathbf{M}$ which falsifies all the non-theorems of Σ is called *the canonical model of* Σ (in symbols: $\mathbf{M}^\Sigma$). For example, from our soundness results, we know that the instances of the schema 4 ($= \Box\varphi \to \Box\Box\varphi$) are not theorems of KT (since 4 is not valid with respect to the class of reflexive frames). So the canonical model for KT, $\mathbf{M}^{KT}$, will contain worlds that falsify the instances of the 4 schema. These will be worlds at which $\Box\varphi$ is true, but at which $\Box\Box\varphi$ is not true.

Note that (B) entails (A), but not vice versa. Clearly, if there is a model which falsifies every non-theorem of Σ, then for any non-theorem of Σ, there is a model which falsifies it. Thus, by constructing the canonical model $\mathbf{M}$ for Σ, showing that $\mathbf{M} \in \mathbf{C}$ and that $\mathbf{M}$ falsifies every non-theorem of Σ, we prove (A), which, by the reasoning in the previous paragraph, establishes that Σ is complete with respect to $\mathbf{C}$. This, then, is the general strategy we shall pursue in proving completeness.

The canonical model $\mathbf{M}^\Sigma$ for a logic Σ is able to do its job of falsifying every non-theorem because it has some very special features. The most important feature that $\mathbf{M}^\Sigma$ has is that its worlds are just all the MaxCon_Σ sets. By defining the accessibility relation and valuation function of $\mathbf{M}^\Sigma$ in the right way, we shall be able to show that truth at a world, i.e., at a $\text{MaxCon}_\Sigma(\Gamma)$, in the canonical model just is membership in Γ. This leaves $\mathbf{M}^\Sigma$ with another very special feature. Given Corollary 2 to Lindenbaum's Lemma, we know that the formulas true in all the MaxCon_Σ sets are precisely the theorems of Σ. So, given our remarks about defining truth, the theorems of Σ will be true in $\mathbf{M}^\Sigma$, since they are true at (i.e., members of) every world (i.e., MaxCon_Σ set). Thus, $\mathbf{M}^\Sigma$ will have the special feature of *determining* Σ in the sense that all and only the theorems of Σ are true in $\mathbf{M}^\Sigma$. Formally:

$$\mathbf{M}^\Sigma \textit{ determines } \Sigma =_{df} \models^{\mathbf{M}^\Sigma} \varphi \text{ if and only if } \vdash_\Sigma \varphi, \text{ for every } \varphi$$

Clearly, if $\mathbf{M}^{\Sigma}$ has this feature, then every non-theorem of Σ is false at some world in $\mathbf{M}^{\Sigma}$. So if we can prove that $\mathbf{M}^{\Sigma}$ determines Σ and that $\mathbf{M}^{\Sigma}$ is in the class of models $\mathbf{C}$, then *a fortiori*, we have established (B). So the actual proof of completeness divides up into two parts: (a) a proof that $\mathbf{M}^{\Sigma}$ determines Σ, and (b) a proof that $\mathbf{M}^{\Sigma} \in \mathbf{C}$. Part (a) is completely general, and is proved only once, for arbitrary $\mathbf{M}^{\Sigma}$ and Σ. Part (b) is proved individually for each particular logic Σ and class $\mathbf{C}$. For example, to show that *K4* is complete with respect to the class of transitive models, we show: (a) that the canonical model $\mathbf{M}^{K4}$ determines *K4*, and (b) that $\mathbf{M}^{K4}$ is an element of the class of transitive models, i.e., that $\mathbf{R}^{K4}$ (i.e, the accessibility relation of $\mathbf{M}^{K4}$) is transitive. Part (a) of the proof is an automatic consequence of the very general result that every canonical model determines its corresponding logic. Part (b) of the proof, however, requires that we argue that the particular accessibility relation $\mathbf{R}^{K4}$ of the canonical model is transitive. This shows that $\mathbf{M}^{K}$ is in the class of transitive models $\mathbf{C}$-trans. This is a required step if we are to show that *K4* is complete with respect to $\mathbf{C}$-trans. For other logics Σ, we have to argue in part (b) of the completeness proof that $\mathbf{R}^{\Sigma}$ (i.e., the accessibility relation of $\mathbf{M}^{\Sigma}$) has the relevant property P. This shows that $\mathbf{M}^{\Sigma}$ is in the class of all P-models $\mathbf{C}$P. This step is needed if we are to show that Σ is complete with respect to $\mathbf{C}$P. For example, for part (b) of the proof that *K5* is complete with respect to the class of euclidean models, we have to argue that the accessibility relation of $\mathbf{R}^{K5}$ is euclidean. This shows that $\mathbf{M}^{K5}$ is in the class of euclidean models $\mathbf{C}$-eucl, which given (a), shows that *K5* is complete with respect to $\mathbf{C}$-eucl. And so forth.

The proofs of part (b) for the various logics are based on another special feature of canonical models: the proof-theoretic properties of logic Σ have an effect on the maximal-consistent$_{\Sigma}$ sets that serve as the 'worlds' of $\mathbf{M}^{\Sigma}$. To see how, note for example, that the 'worlds' of the canonical model $\mathbf{M}^{K4}$ for the logic *K4* will be the sets of sentences that are maximal-consistent relative to *K4*. By (57), it follows that since $\Box\varphi \to \Box\Box\varphi$ is a theorem of *K4*, each instance of this schema is a member of all of the maximal-consistent$_{K4}$ sets of sentences. So all of the worlds in the canonical model $\mathbf{M}^{K4}$ will contain all of the instances of the 4 schema. This fact affects the accessibility relation of $\mathbf{M}^{K4}$, which is defined so as to preserve the theorems concerning normal logics and maximal consistent sets which were proved in §6 of Chapter 4. It can be shown that if all the 'worlds' in $\mathbf{M}^{K4}$ contain the instances of the 4 schema, then the accessibility relation $\mathbf{R}^{K4}$ of the canonical model will have to be transitive (this is proved in (82)). Similarly for other logics and canonical models. For example, it follows from the fact that the instances of the 5 schema are members of all the maximal-consistent$_{K5}$ sets that the accessibility relation of $\mathbf{M}^{K5}$ is euclidean. This, then, is how we prove part (b) of the completeness proofs, i.e., that $\mathbf{M}^{\Sigma}$ is in the relevant class of models $\mathbf{C}$. We prove it by showing that $\mathbf{R}^{\Sigma}$ has the relevant property, and the fact that it does derives from the effect the proof-theoretic properties of the logic Σ have on maximal-consistent sets relative to Σ. The resulting group of maximal-consistent sets, in turn, have an effect on the properties of $\mathbf{R}^{\Sigma}$.

78) *The canonical model of* a consistent normal logic Σ is the model $\mathbf{M}^{\Sigma}$ ($= \langle \mathbf{W}^{\Sigma}, \mathbf{R}^{\Sigma}, \mathbf{V}^{\Sigma} \rangle$) that satisfies the following conditions:

.1) $\mathbf{W}^{\Sigma} = \{\Gamma \mid \text{MaxCon}_{\Sigma}(\Gamma)\}$

.2) $\mathbf{R}^{\Sigma}\mathbf{w}\mathbf{w}'$ iff $\{\varphi \mid \Box\varphi \in \mathbf{w}\} \subseteq \mathbf{w}'$

.3) $\mathbf{V}^{\Sigma}(p) = \{\mathbf{w} \in \mathbf{W}^{\Sigma} \mid p \in \mathbf{w}\}$

Note that if Σ is not consistent, then there are no MaxCon$_{\Sigma}$ sets and so $\mathbf{M}^{\Sigma}$ does not exist.

Remark 1: Clause (.1) tells us that in the canonical model $\mathbf{M}^{\Sigma}$, the worlds in $\mathbf{W}^{\Sigma}$ are precisely the sets of formulas which are maximal-consistent relative to Σ. Clause (.2) requires that the accessibility relation $\mathbf{R}^{\Sigma}$ holds between $\mathbf{w}$ and $\mathbf{w}'$ just in case $\varphi \in \mathbf{w}'$ whenever $\Box\varphi \in \mathbf{w}$. Note that this definition of $\mathbf{R}$ allows us to appeal to the theorems concerning normal logics and maximal-consistent sets in §6 of Chapter 4. So by (69), clause (.2) is equivalent to saying that $\Diamond\varphi \in \mathbf{w}$ whenever $\varphi \in \mathbf{w}'$. Finally, clause (.3) says that the valuation function $\mathbf{V}^{\Sigma}$ is defined so that an atomic formula p is true at all the worlds *of which it is an element*. This will prove to be the basis for arguing that, for all $\mathbf{w}$, φ is true at $\mathbf{w}$ in $\mathbf{M}^{\Sigma}$ iff $\varphi \in \mathbf{w}$. And given (57), this in turn will be the basis for arguing that φ is true in $\mathbf{M}^{\Sigma}$ iff φ is a theorem of Σ (i.e., that $\mathbf{M}^{\Sigma}$ determines Σ).

Remark 2: Here is an intuitive picture of the facts proved in what follows. Note that by (57), if φ is not a theorem of Σ, then there is a maximal-consistent$_{\Sigma}$ set that fails to contain φ. So by (54.4), there is a maximal-consistent$_{\Sigma}$ set that contains $\neg\varphi$. So, since our maximal-consistent$_{\Sigma}$ sets serve as the worlds in $\mathbf{M}^{\Sigma}$, the negation of each non-theorem of Σ gets embedded in some world in $\mathbf{W}^\Sigma$. Intuitively, this means that each non-theorem of Σ will be false in some world. Thus, $\mathbf{M}^\Sigma$ is rich enough to contain a falsifying world for each non-theorem. For example, the negation of each instance of the T schema will get embedded in some maximal-consistent$_K$ set in $\mathbf{W}^K$ (since the instances of the T schema are non-theorems of K). Consider, for example, the instance $\Box p \rightarrow p$. Our definitions require that there be a world $\mathbf{w}$ in $\mathbf{W}^K$ containing $\neg(\Box p \rightarrow p)$, and therefore, by the properties of maximal-consistency, containing $\Box p \ \& \ \neg p$, $\Box p$, and $\neg p$ (any such 'world' is consistent relative to K). Moreover, by the definition of the accessibility relation, any world $\mathbf{w}'$ in $\mathbf{W}^K$ such that $\mathbf{Rww}'$ will contain p, since $\mathbf{w}$ contains $\Box p$. In this manner, the canonical model $\mathbf{M}^\Sigma$ will contain a world that falsifies each non-theorem of Σ.

We turn, then, to the theorems that prove that this picture is an accurate one.

79) *Lemma*: Let $\mathbf{M}^\Sigma$ be the canonical model for Σ. Then, for every $\mathbf{w} \in \mathbf{W}^\Sigma$, $\models^{\mathbf{M}^\Sigma}_{\mathbf{w}} \varphi$ iff $\varphi \in \mathbf{w}$.

80) *Theorem*: $\models^{\mathbf{M}^\Sigma} \varphi$ iff $\vdash_\Sigma \varphi$, for every consistent, normal modal logic Σ.

Remark 1: It is interesting to note that $\mathbf{M}^\Sigma$ will typically contain isolated groups of worlds, i.e., groups γ and δ of maximal-consistent sets such that the members of γ may be $\mathbf{R}$-related in some way to each other, and the members of δ may be $\mathbf{R}$-related in some way to each other, but no maximal-consistent set in γ is $\mathbf{R}$-related to any maximal-consistent set in δ, and vice versa. Here is an argument that shows why this must occur. Note that in *KT5* (i.e., *S5*) for example, the formula $\neg p$ is not a theorem (why?). So there is a world $\mathbf{w}$ in $\mathbf{W}^{S5}$ such that $p \in \mathbf{w}$ (i.e., $\models^{\mathbf{M}^{S5}}_{\mathbf{w}} p$). Consequently, $\Diamond p$ is true in any world that accesses $\mathbf{w}$. Let γ be the group of worlds that access $\mathbf{w}$. Not every world can be in γ, for otherwise every world would contain $\Diamond p$ and so $\Diamond p$ would have to be a theorem of *S5*. But $\Diamond p$ is not a theorem of *S5* (why?). So there must be some world $\mathbf{w}'$ that contains $\neg\Diamond p$, and thus, $\Box\neg p$. But for $\Box\neg p$ to be true at $\mathbf{w}'$, $\neg p$ must be true at worlds that $\mathbf{w}'$ can access. Call the group of worlds that $\mathbf{w}'$ can access δ. *Claim*: There can be no relationship between any members of γ and δ. *Argument*: Let $\mathbf{w}_1$ be an arbitrary member of γ and $\mathbf{w}_2$ be an arbitrary member of δ. Suppose that $\mathbf{Rw}_1\mathbf{w}_2$. Then by hypothesis $\mathbf{Rw}_1\mathbf{w}$. But $\mathbf{M}^{S5}$ is reflexive, euclidean, and so by the euclidean property,

$\mathbf{Rw_2w}$. But by (29.3), reflexive euclidean models are transitive. So by transitivity, and the fact that both $\mathbf{Rw'w_2}$ (by hypothesis) and $\mathbf{Rw_2w}$, it follows that $\mathbf{Rw'w}$. Since $\Box\neg p$ is true at $\mathbf{w}'$, it follows that $\neg p$ is true at $\mathbf{w}$, which contradicts our initial assumption that $p \in \mathbf{w}$, i.e., that p is true in $\mathbf{w}$. Analogous reasoning establishes that $\neg\mathbf{Rw_2w_1}$.

Remark 2: We've now established *in general* that the canonical model $\mathbf{M}^\Sigma$ determines Σ, for any consistent, normal logic Σ. Recall that to prove a logic Σ complete with respect to a class $\mathbf{C}$ of models, our strategy is show (a) that $\mathbf{M}^\Sigma$ determines Σ, and (b) that $\mathbf{M}^\Sigma \in \mathbf{C}$. Together, (a) and (b) establish that $\exists\mathbf{M} \in \mathbf{C}$ such that if $\not\vdash_\Sigma \varphi$ then $\mathbf{C} \not\models \varphi$, which in turn establishes that if $\mathbf{C} \models \varphi$ then $\vdash_\Sigma \varphi$ (i.e., that Σ is complete with respect to $\mathbf{C}$). It remains therefore to show part (b) for a each logic Σ. That is, we need to show that $\mathbf{R}^\Sigma$ has the property that qualifies $\mathbf{M}$ as a member of the relevant class $\mathbf{C}$. In the case of the logic K, we need to show very little, for K is complete with respect to the class of *all* models. For K at least, we need only show that $\exists\mathbf{M}$ in the class of all models such that $\mathbf{M}$ determines K. This is immediate, since the canonical model $\mathbf{M}^K$ is in the class of all models.

81) *Theorem*: K is complete with respect to the class $\mathbf{C}$ of all models.

Proof: By (80), $\models^{\mathbf{M}^K} \varphi$ iff $\vdash_K \varphi$. Hence, $\exists\mathbf{M} \in \mathbf{C}$ such that if $\not\vdash_K \varphi$, then $\not\models^{\mathbf{M}} \varphi$. *A fortiori*, if $\not\vdash_K \varphi$, then $\exists\mathbf{M} \in \mathbf{C}$ such that $\not\models^{\mathbf{M}} \varphi$. Thus, if $\not\vdash_K \varphi$, then φ is not valid with respect to $\mathbf{C}$, and so K is complete with respect to the class of all models.

Remark: We now turn to the penultimate step in the proofs of completeness. In virtue of our earlier remarks, we know that to prove a logic Σ is complete with respect to a class of models $\mathbf{C}$, it is now sufficient to show that the proper canonical model $\mathbf{M}^\Sigma$ is an element of $\mathbf{C}$. To do this, we show that if a logic Σ contains the instances of the axiom schema S_i, for the schemata in (23), then the accessibility relation $\mathbf{R}^\Sigma$ of the canonical model $\mathbf{M}^\Sigma$ satisfies the corresponding property P_i.

82) *Lemma*: Let P_i and S_i be as in (23). Let Σ be normal. Then if Σ contains S_i, $\mathbf{R}^\Sigma$ satisfies P_i.

83) *Theorem*: $KS_1 \ldots S_n$ is complete with respect to the class $\mathbf{C}\mathrm{P}_1 \ldots \mathrm{P}_n$ of models. That is, if $\mathbf{C}\mathrm{P}_1 \ldots \mathrm{P}_n \models \varphi$, then $\vdash_{KS_1 \ldots S_n} \varphi$.

Remark: Here are two instances of this theorem: (a) The logic *KT4* is complete with respect to the class of reflexive, transitive models, and

(b) the logic $KTB4$ is complete with respect to the class of reflexive, symmetric, and transitive models.

84) *Exercise*: Reconsider the facts established in (29) and the *Remark* about those facts. Prove that KT is an extension of KD. Prove that $KB4$ is the same logic as $KB5$. Prove that $KTB4 = K5 = KDB4 = KDB5$. What other facts about modal logics can be established on the basis of (29) in light of the completeness results?

Chapter Six: Basic Quantified Modal Logic

§1: The Simplest Quantified Modal Logic

85) *Vocabulary*: The primitive non-logical vocabulary of our language is a set $\mathcal{C}$ of constants consisting of object constants and relation constants. Specifically, $\mathcal{C} = \mathcal{OC} \cup \mathcal{RC}$, where:

.1) $\mathcal{OC}$ is the set of object constants: $\{a_1, a_2, \ldots\}$

.2) $\mathcal{RC}$ is the set of relation constants: $\{P^n_1, P^n_2, \ldots\}$ $(n \geq 0)$

We use $a, b, \ldots$ as typical members of $\mathcal{OC}$, and $P^n, Q^n, R^n \ldots$ as typical members of $\mathcal{RC}$. In addition to this non-logical vocabulary, we have the following logical vocabulary:

.3) The set $\mathcal{V}$ of (object) variables: $\{x_1, x_2, \ldots\}$

(We use $x, y, z, \ldots$ as typical members of $\mathcal{V}$.)

.4) The truth functional connectives $\neg$ and $\rightarrow$, the quantifier $\forall$, and the modal operator $\Box$. For languages with identity, we add the identity sign $=$.

86) *Terms*: We define the set $\mathcal{T}$ of *terms* to be the set $\mathcal{OC} \cup \mathcal{V}$.
87) *Formulas*: The set $Fml(\mathcal{C})$ of *formulas* based on the non-logical vocabulary $\mathcal{C}$ is the smallest set satisfying the following conditions:

.1) If $P^n \in \mathcal{RC}$, and $\tau_1, \ldots, \tau_n \in \mathcal{T}$, then $P^n\tau_1 \ldots \tau_n \in Fml(\mathcal{C})$ $(n \geq 0)$.

.2) If $\varphi, \psi \in Fml(\mathcal{C})$, and $x \in \mathcal{V}$, then $(\neg\varphi)$, $(\varphi \rightarrow \psi)$, $(\forall x\varphi)$, and $(\Box\varphi) \in Fml(\mathcal{C})$.

For languages with identity, we add:

.3) If $\tau, \tau' \in \mathcal{T}$, then $(\tau = \tau') \in Fml(\mathcal{C})$

88) *Models*: A *model* $\mathbf{M}$ for the non-logical vocabulary $\mathcal{C}$ is any quadruple $\langle \mathbf{W}, \mathbf{D}, \mathbf{R}, \mathbf{V} \rangle$ satisfying the following conditions:

.1) $\mathbf{W}$ is a non-empty set of *worlds*,

.2) $\mathbf{D}$ is a non-empty domain of *objects*,

.3) $\mathbf{R}$ is a binary *accessibility* relation on $\mathbf{W}$, i.e., $\mathbf{R} \subseteq \mathbf{W} \times \mathbf{W}$,

.4) $\mathbf{V}$ is a *valuation* function that has the set $\mathcal{C}$ as its domain and meets the following conditions:

a) If $a \in \mathcal{OC}$, $\mathbf{V}(a) \in \mathbf{D}$,

b) If $P^n \in \mathcal{RC}$ and $n = 0$, $\mathbf{V}(P^n) \in \mathcal{P}(\mathbf{W})$,

c) If $P^n \in \mathcal{RC}$ and $n \geq 1$, $\mathbf{V}(P^n) \in \{\mathbf{g} \,|\, \mathbf{g} : \mathbf{W} \to \mathcal{P}(\mathbf{D}^n)\}$.

89) $\mathbf{M}$-*Assignments to variables*: An $\mathbf{M}$-assignment to the variables is any function $\mathbf{f}$ such that $\mathbf{f} : \mathcal{V} \to \mathbf{D}$. If $\mathbf{f}$ and $\mathbf{f}'$ are both assignments, we write $\mathbf{f}' \overset{x}{=} \mathbf{f}$ whenever $\mathbf{f}'$ is an assignment identical to $\mathbf{f}$ except perhaps for what it assigns to x.

90) *Denotations of terms*: The denotation of term τ with respect to model $\mathbf{M}$ and $\mathbf{M}$-assignment $\mathbf{f}$ (in symbols: $\mathbf{d}_{\mathbf{M},\mathbf{f}}(\tau)$) is defined as follows:

.1) If $\tau \in \mathcal{OC}$, $\mathbf{d}_{\mathbf{M},\mathbf{f}}(\tau) = \mathbf{V}(\tau)$

.2) If $\tau \in \mathcal{V}$, then $\mathbf{d}_{\mathbf{M},\mathbf{f}}(\tau) = \mathbf{f}(\tau)$

91) *Satisfaction*: We define: $\mathbf{f}$ *satisfies*$_{\mathbf{M}}$ formula φ *with respect to* world $\mathbf{w}$ as follows:

.1) $(n \geq 1)$ $\mathbf{f}$ satisfies$_{\mathbf{M}}$ $P^n\tau_1 \ldots \tau_n$ wrt $\mathbf{w}$ iff

$$\langle \mathbf{d}_{\mathbf{M},\mathbf{f}}(\tau_1), \ldots, \mathbf{d}_{\mathbf{M},\mathbf{f}}(\tau_n) \rangle \in [\mathbf{V}(P^n)](\mathbf{w})$$

$(n = 0)$ $\mathbf{f}$ satisfies$_{\mathbf{M}}$ P^0 wrt $\mathbf{w}$ iff $\mathbf{w} \in \mathbf{V}(P^0)$

.2) $\mathbf{f}$ satisfies$_{\mathbf{M}}$ $\neg\psi$ wrt $\mathbf{w}$ iff $\mathbf{f}$ fails to satisfy$_{\mathbf{M}}$ ψ wrt $\mathbf{w}$

.3) $\mathbf{f}$ satisfies$_{\mathbf{M}}$ $\psi \to \chi$ wrt $\mathbf{w}$ iff

either $\mathbf{f}$ fails to satisfy$_{\mathbf{M}}$ wrt $\mathbf{w}$ or $\mathbf{f}$ satisfies$_{\mathbf{M}}$ χ wrt $\mathbf{w}$

.4) $\mathbf{f}$ satisfies$_{\mathbf{M}}$ $\forall x\psi$ wrt $\mathbf{w}$ iff

for every $\mathbf{f}'$, if $\mathbf{f}' \overset{x}{=} \mathbf{f}$, then $\mathbf{f}'$ satisfies$_{\mathbf{M}}$ ψ wrt $\mathbf{w}$

.5) $\mathbf{f}$ satisfies$_{\mathbf{M}}$ $\Box\psi$ wrt $\mathbf{w}$ iff

for every $\mathbf{w}'$, if $\mathbf{R}\mathbf{w}\mathbf{w}'$, then $\mathbf{f}$ satisfies$_{\mathbf{M}}$ ψ wrt $\mathbf{w}'$.

For languages with identity, we add:

.6) $\mathbf{f}$ satisfies$_{\mathbf{M}}$ $\tau = \tau'$ wrt $\mathbf{w}$ iff $\mathbf{d}_{\mathbf{M},\mathbf{f}}(\tau) = \mathbf{d}_{\mathbf{M},\mathbf{f}}(\tau')$

92) *Truth at a World*: We say φ is *true*$_{\mathbf{M}}$ *at* world $\mathbf{w}$ (in symbols: $\models^{\mathbf{M}}_{\mathbf{w}} \varphi$) iff every assignment $\mathbf{f}$ satisfies$_{\mathbf{M}}$ φ with respect to $\mathbf{w}$.

93) *Truth*: We say that φ is *true*$_{\mathbf{M}}$ (in symbols: $\models^{\mathbf{M}} \varphi$) iff for every world $\mathbf{w}$, $\models^{\mathbf{M}}_{\mathbf{w}} \varphi$ (i.e., iff for every world $\mathbf{w}$, φ is true$_{\mathbf{M}}$ at $\mathbf{w}$)

94) *Validity*: We say φ is *valid* (in symbols: $\models \varphi$) iff for every model $\mathbf{M}$, $\models^{\mathbf{M}} \varphi$ (i.e., iff for every $\mathbf{M}$, φ is true$_{\mathbf{M}}$).

95) *Logical Consequence*: φ is a logical consequence of a set Γ with respect to a class of models $\mathbf{C}$ (in symbols: $\Gamma \models_{\mathbf{C}} \varphi$) iff $\forall \mathbf{M} \in \mathbf{C}, \forall \mathbf{w} \in \mathbf{M}$, if $\models^{\mathbf{M}}_{\mathbf{w}} \Gamma$, then $\models^{\mathbf{M}}_{\mathbf{w}} \varphi$.

96) *Logic*: A set Γ is a *quantified modal logic* iff

.1) Γ is closed under RPL

.2) Γ contains the instances of the following schemata:

a) $\forall x\varphi \rightarrow \varphi^{\tau}_{x}$, provided τ is substitutable for x in φ.[18]

b) $\forall x(\varphi \rightarrow \psi) \rightarrow (\varphi \rightarrow \forall x\psi)$, where x is any variable not free in φ

.3) Γ is closed under the Rule of Generalization: $\varphi/\forall x\varphi$

If the language has identity, then we say that Γ is a quantified modal logic *with identity* in case Γ satisfies the above definition with the following two additions to clause (.2):

c) $x = x$, for any variable x

d) $x = y \rightarrow (\varphi(x, x) \leftrightarrow \varphi(x, y))$, where $\varphi(x, y)$ is the result of replacing some, but not necessarily all, occurrences of x by y in $\varphi(x, x)$.

A set Γ is a *normal* quantified modal logic (with identity) if in addition to being a quantified modal logic (with identity), Γ contains all the instances of the schema K, all the instances of the Barcan formula ($= \forall x\Box\varphi \rightarrow \Box\forall x\varphi$), and is closed under RN.

[18] The symbol φ^{τ}_{x} stands for the result of substituting the term τ for free occurrences of the variable x everywhere in φ. τ is substitutable for x in φ provided no variable y in τ is captured by a quantifier $\forall y$ in φ^{τ}_{x}.

§2: Kripke's Semantical Considerations on Modal Logic

97) *Kripke's System*:

See Kripke's paper, 'Semantical Considerations on Modal Logic', *Acta Philosophica Fennica* **16** (1963): 83-94

98) *Formulas Valid in the Simplest QML That Are Invalid in Kripke's System*:

$\forall x Px \rightarrow Py$

$\forall x Px \rightarrow \exists x Px$

$\forall x \Box Px \rightarrow \Box \forall x Px$

$\Box \forall x Px \rightarrow \forall x \Box Px$

$\exists y\, y = x$

$\Box(Fx \rightarrow Fx)$

99) *Reasons Philosophers Cite for Excluding These Formulas*:

100) *Techniques for Excluding These Formulas*:

Defining the Existence Predicate as Quantification

Variable Domains

Free Logic

Generality Interpretation — Defining Validity only for Closed Formulas

101) *Reasons for Preferring the Simpler Quantified Modal Logic:*

See the paper, 'In Defense of the Simplest Quantified Modal Logic", coauthored by Bernard Linsky and Edward N. Zalta, in *Philosophical Perspectives (8): Philosophy of Logic and Language*, J. Tomberlin (ed.), Atascadero: Ridgeview, 1994

§3: Modal Logic and a Distinguished Actual World

102) *Reasons for Formulating Modal Logic with a Distinguished Actual World.*

> See the paper, 'Logical and Analytic Truths That Are Not Necessary', by Edward N. Zalta, in the *Journal of Philosophy*, **85**/2 (February 1988): 57–74

Appendix: Proofs of Theorems and Exercises

Chapter Three

Proof of Exercise 2 in (7):

(a) Assume $\models^{\mathbf{M}} \varphi \to \psi$. So for every $\mathbf{w}$, $\models^{\mathbf{M}}_{\mathbf{w}} \varphi \to \psi$. Assume now that $\models^{\mathbf{M}} \varphi$. So for every $\mathbf{w}$, $\models^{\mathbf{M}}_{\mathbf{w}} \varphi$. Pick an arbitrary $\mathbf{w}'$. So $\models^{\mathbf{M}}_{\mathbf{w}'} \varphi \to \psi$ and $\models^{\mathbf{M}}_{\mathbf{w}'} \varphi$. Hence, by (6.4), $\models^{\mathbf{M}}_{\mathbf{w}'} \psi$. So, since $\mathbf{w}'$ was arbitrary, for every world $\mathbf{w}$, $\models^{\mathbf{M}}_{\mathbf{w}} \psi$, i.e., $\models^{\mathbf{M}} \psi$. So, by our conditional proof, it follows from $\models^{\mathbf{M}} \varphi \to \psi$ that if $\models^{\mathbf{M}} \varphi$ then $\models^{\mathbf{M}} \psi$.

(b) Consider the following $\mathbf{M}$: let $\mathbf{W} = \{\mathbf{w}_1, \mathbf{w}_2\}$; $\mathbf{R}$ be empty; and $\mathbf{V}(p) = \{\mathbf{w}_1\}$ and $\mathbf{V_M}(q) = \{\,\}$. Since $\not\models^{\mathbf{M}}_{\mathbf{w}_2} p$, we know that not every world $\mathbf{w}$ is such that $\models^{\mathbf{M}}_{\mathbf{w}} p$. Hence, $\not\models^{\mathbf{M}} p$, and thus by antecedent failure, if $\models^{\mathbf{M}} p$ then $\models^{\mathbf{M}} q$. However, $\models^{\mathbf{M}}_{\mathbf{w}_1} p$ and $\not\models^{\mathbf{M}}_{\mathbf{w}_1} q$, and $\not\models^{\mathbf{M}}_{\mathbf{w}_1} p \to q$. So not every $\mathbf{w}$ is such that $\models^{\mathbf{M}}_{\mathbf{w}} p \to q$. Thus, $\not\models^{\mathbf{M}} p \to q$. This means we have a model in which the conditional if $\models^{\mathbf{M}} \varphi$ then $\models^{\mathbf{M}} \psi$ holds, but for which $\not\models^{\mathbf{M}} \varphi \to \psi$, demonstrating that the conditional, if $\models^{\mathbf{M}} \varphi$ then $\models^{\mathbf{M}} \psi$, does *not* imply $\models^{\mathbf{M}} \varphi \to \psi$.

Proof of (12): By induction on Λ_Ω (though the induction proceeds by considering, in the base case, the quasi-atomic formulas p^*, and then, in the inductive cases, the formulas $\bot$, $\neg\psi$, and $\psi \to \chi$). Base case: Suppose that $\varphi = p^*$. If $\mathbf{f}$ is based on $\mathbf{f}^*$, then for every $q \in \Omega^*$, $\mathbf{f}(q^*) = \mathbf{f}^*(q^*)$. So $\mathbf{f}(p^*) = \mathbf{f}^*(p^*)$. Similarly, $\mathbf{f}'(p^*) = \mathbf{f}^*(p^*)$. So $\mathbf{f}(p^*) = \mathbf{f}'(^*)$; i.e., $\mathbf{f}(\varphi) = \mathbf{f}'(\varphi)$.

Induction cases: (1) Suppose that $\varphi = \bot$. Then, by the definition of subformula, $\bot$ is a subformula of φ, since every formula is a subformula of itself. So by hypothesis, $\mathbf{f}(\bot) = \mathbf{f}'(\bot)$, and hence, $\mathbf{f}(\varphi) = \mathbf{f}'(\varphi)$.

(2) Suppose $\varphi = \neg\psi$. We may assume, as an inductive hypothesis, that the theorem holds for ψ, i.e., we may assume for the inductive hypothesis that $\mathbf{f}(\psi) = \mathbf{f}'(\psi)$. *Exercise*: Complete the proof that $\mathbf{f}(\varphi) = \mathbf{f}'(\varphi)$.

(3) Suppose that $\varphi = \psi \to \chi$. We may assume as inductive hypotheses both that $\mathbf{f}(\psi) = \mathbf{f}'(\psi)$ and that $\mathbf{f}(\chi) = \mathbf{f}'(\chi)$. *Exercise*: Finish the proof that $\mathbf{f}(\varphi) = \mathbf{f}'(\varphi)$.

Proof of Exercise in (13): By induction on Λ_Ω. Base case: Suppose that $\varphi = p^*$. By the definition of TFR-subformula, every formula is a subformula of itself. So p^* is a subformula of φ. But p^* is quasi-atomic, and so by the hypothesis of the theorem (which gives us that $\mathbf{f}$ and $\mathbf{f}'$

agree on the TFR-subformulas in φ), we know that $\mathbf{f}(p^*) = \mathbf{f}'(p^*)$. So $\mathbf{f}(\varphi) = \mathbf{f}'(\varphi)$.

Inductive cases: (1) Suppose that $\varphi = \bot$. Then, again by the definition of TFR-subformula, $\bot$ is a TFR-subformula of φ. So by hypothesis, $\mathbf{f}(\bot) = \mathbf{f}'(\bot)$, and hence, $\mathbf{f}(\varphi) = \mathbf{f}'(\varphi)$.

(2) $\varphi = \neg\psi$. (The proof of this is just like corresponding inductive clause in the proof of the theorem in (11).)

(3) $\varphi = \psi \to \chi$. (Again, the proof of this is just like the corresponding inductive clause in (11).)

Proof of (16): Suppose φ is a tautology. Then by (13), every total assignment $\mathbf{f}$ is such that $\mathbf{f}(\varphi) = T$. So for any model $\mathbf{M}$ and world $\mathbf{w}$, the total assignment $\mathbf{f_w}$ determined by $\mathbf{M}$ and $\mathbf{w}$ is such that $\mathbf{f_w}(\varphi) = T$. But by (15), $\mathbf{f_w}(\varphi) = T$ iff $\models^{\mathbf{M}}_{\mathbf{w}} \varphi$. So for every $\mathbf{M}$ and $\mathbf{w}$, $\models^{\mathbf{M}}_{\mathbf{w}} \varphi$. So φ is valid.

Proof of (12) (Alternative §2): Suppose we are given a set Γ^* of quasi-atomic formulas. Then we prove our theorem by induction on $\varphi \in Fml^*(\Gamma^* \cup \{\bot\})$. Base case: $\varphi = p^*$. Suppose that $\mathbf{f}$ and $\mathbf{f}'$ both extend $\mathbf{f}^*$. Since $\mathbf{f}$ extends $\mathbf{f}^*$, then $\mathbf{f}$ agrees with $\mathbf{f}^*$ on all the quasi-atomics in Γ^*. So $\mathbf{f}(p^*) = \mathbf{f}^*(p^*)$, since $p^* \in \Gamma^*$. Similarly, $\mathbf{f}'(p^*) = \mathbf{f}^*(p^*)$. So $\mathbf{f}(p^*) = \mathbf{f}'(p^*)$; i.e., $\mathbf{f}(\varphi) = \mathbf{f}'(\varphi)$.

Induction cases: (1) Suppose that $\varphi = \bot$. Then, by the definition of subformula, $\bot$ is a subformula of φ, since every formula is a subformula of itself. So by hypothesis, $\mathbf{f}(\bot) = \mathbf{f}'(\bot)$, and hence, $\mathbf{f}(\varphi) = \mathbf{f}'(\varphi)$.

(2) Suppose $\varphi = \neg\psi$. We may assume, as an inductive hypothesis, that the theorem holds for ψ, i.e., we may assume for the inductive hypothesis that $\mathbf{f}(\psi) = \mathbf{f}'(\psi)$. *Exercise*: Show that $\mathbf{f}(\varphi) = \mathbf{f}'(\varphi)$.

(3) Suppose that $\varphi = \psi \to \chi$. We may assume as inductive hypotheses both that $\mathbf{f}(\psi) = \mathbf{f}'(\psi)$ and that $\mathbf{f}(\chi) = \mathbf{f}'(\chi)$. *Exercise*: Show that $\mathbf{f}(\varphi) = \mathbf{f}'(\varphi)$.

Proof of (15) (Alternative §2): Fix Λ_Ω, $\mathbf{M}$, and $\mathbf{w}$. Now pick an arbitrary formula φ and consider the basic assignment $\mathbf{f}^*_{\mathbf{w}}$ of Ω^*_φ determined by $\mathbf{M}$ and $\mathbf{w}$. We first argue by induction on $Fml^*(\Omega^*_\varphi \cup \{\bot\})$ that for every $\psi \in Fml^*(\Omega^*_\varphi \cup \{\bot\})$, $\mathbf{f}^*_{\mathbf{w}}(\psi) = T$ iff $\models^{\mathbf{M}}_{\mathbf{w}} \psi$. Then we argue that $\mathbf{f}^*_{\mathbf{w}}(\varphi) = T$ iff $\models^{\mathbf{M}}_{\mathbf{w}} \varphi$ on the grounds that $\varphi \in Fml^*(\Omega^*_\varphi \cup \{\bot\})$.

> *Induction on* $Fml^*(\Omega^*_\varphi \cup \{\bot\})$. Base case: $\psi = p^*$. Since $\mathbf{f_w}$ agrees with $\mathbf{f}^*_{\mathbf{w}}$ on all the quasi-atomic formulas in Ω^*_φ, $\mathbf{f_w}(p^*) = \mathbf{f}^*_{\mathbf{w}}(p^*)$. But by definition, $\mathbf{f}^*_{\mathbf{w}}(p^*) = T$ iff $\models^{\mathbf{M}}_{\mathbf{w}} (p^*)$. So $\mathbf{f_w}(p^*) = T$ iff $\models^{\mathbf{M}}_{\mathbf{w}} (p^*)$; i.e., $\mathbf{f_w}(\psi) = T$ iff $\models^{\mathbf{M}}_{\mathbf{w}} \psi$.

Inductive cases: (a) $\psi = \bot$. To prove our biconditional, we prove conditionals in both directions. ($\Rightarrow$) By the definition of extended assignments, $\mathbf{f_w}(\bot) = F$. So if $\mathbf{f_w}(\bot) = T$, then $\models^{\mathbf{M}}_{\mathbf{w}} \bot$ (by antecedent failure). ($\Leftarrow$) By the definition of truth at a world in a model, $\not\models^{\mathbf{M}}_{\mathbf{w}} \bot$. So if $\models^{\mathbf{M}}_{\mathbf{w}} \bot$, then $\mathbf{f_w}(\bot) = T$ (again by antecedent failure). So assembling our two conditionals, $\mathbf{f_w}(\bot) = T$ iff $\models^{\mathbf{M}}_{\mathbf{w}} \bot$, i.e., $\mathbf{f_w}(\psi) = T$ iff $\models^{\mathbf{M}}_{\mathbf{w}} \psi$.

(b) $\psi = \neg\chi$. (*Exercise*).

(c) $\psi = \chi \rightarrow \theta$. (*Exercise*)

Consequently, $\mathbf{f}^*_{\mathbf{w}}(\varphi) = T$ iff $\models^{\mathbf{M}}_{\mathbf{w}} \varphi$, since $\varphi \in \mathit{Fml}^*(\Omega^*_\varphi \cup \{\bot\})$.

Proof of (16) (Alternative §2): Suppose φ is a tautology. Then by (13), for every $\mathbf{f}^*$ of the quasi-atomic subformulas in φ, $\mathbf{f}(\varphi) = T$. But for every $\mathbf{M}$ and $\mathbf{w}$, there is a basic assignment function $\mathbf{f}^*_{\mathbf{w}}$, and so $\mathbf{f}^*_{\mathbf{w}}$ must determine an extended assignment $\mathbf{f_w}$ which is such that $\mathbf{f_w}(\varphi) = T$. In other words, for every model $\mathbf{M}$ and world $\mathbf{w}$, $\mathbf{f_w}(\varphi) = T$. But by (14), $\mathbf{f_w}(\varphi) = T$ iff $\models^{\mathbf{M}}_{\mathbf{w}} \varphi$. So, for every model $\mathbf{M}$ and world $\mathbf{w}$, $\models^{\mathbf{M}}_{\mathbf{w}} \varphi$. So φ is a valid.

Proof of (32): Suppose ψ is a tautological consequence of $\varphi_1, \ldots, \varphi_n$. Then by (31.2), $\varphi_1 \rightarrow \ldots \rightarrow \varphi_n \rightarrow \psi$ is a tautology. Hence, by (16), $\varphi_1 \rightarrow \ldots \rightarrow \varphi_n \rightarrow \psi$ is valid. And by hypothesis, each of $\varphi_1, \ldots, \varphi_n$ is valid. So by n applications of the fact (30) that Modus Ponens preserves validity, it follows that ψ is also valid.

Chapter Four

Proof of (38): ($\Rightarrow$) Suppose Γ is a modal logic, i.e., that Γ contains every tautology and is closed under Modus Ponens. To show that Γ is closed under RPL, suppose that that $\varphi_1, \ldots, \varphi_n \in \Gamma$ and ψ is a tautological consequence of $\varphi_1, \ldots, \varphi_n$. Since ψ is a tautological consequence of $\varphi_1, \ldots, \varphi_n$, $\varphi_1 \rightarrow \ldots \rightarrow \varphi_n \rightarrow \psi$ is a tautology, by (31.2). So $\varphi_1 \rightarrow \ldots \rightarrow \varphi_n \rightarrow \psi$ is an element of Γ. But since Γ is closed under MP, n applications of this rule yields that $\psi \in \Gamma$.

($\Leftarrow$) Suppose Γ is closed under RPL. (a) Suppose φ is a tautology. Then by (31.4), φ is a tautological consequence of $\varphi_1, \ldots, \varphi_n$ when $n = 0$. But when $n = 0$, $\varphi_1, \ldots, \varphi_n$ are all in Γ. So since Γ contains all the tautological consequences of any $\varphi_1, \ldots, \varphi_n$ whenever $\varphi_1, \ldots, \varphi_n \in \Gamma$, it contains φ. (b) Suppose, next that φ, $\varphi \to \psi \in \Gamma$. By (31.1), ψ is a tautological consequence of φ and $\varphi \to \psi$. So $\psi \in \Gamma$, since Γ is closed under RPL. Thus, Γ is closed under Modus Ponens.

Proof of (43.2): Clearly, there is an effective method for telling whether $\varphi \in \emptyset$. And given the previous *Remark*, there is an effective method for telling whether a formula φ is a tautological consequence of $\varphi_1, \ldots, \varphi_n$. So by (42), we have to show that $\varphi \in PL$ iff there is a sequence of formulas $\langle\varphi_1, \ldots, \varphi_n\rangle$, with $\varphi = \varphi_n$, such that each member of the sequence is either (a) a member of the empty set $\emptyset$ of axioms, or (b) is the conclusion by RPL of previous members of the sequence.

($\Rightarrow$) Suppose $\varphi \in PL$. Then φ is a tautology. To show that there is a sequence meeting the conditions of the theorem, we have only to show that there is a sequence $\langle\varphi_1, \ldots, \varphi_n\rangle$ (with $\varphi_n = \varphi$) every member of which is the conclusion by RPL of previous members (since none of the members of the sequence are ever in the empty set $\emptyset$ of axioms). Let the sequence be just φ itself. So to show that φ is the conclusion by RPL of the previous members in the sequence, we must establish that φ is a tautological consequence of the empty set $\emptyset$ of hypotheses. But φ is tautology, and so by (31.4), it is such a consequence.

($\leftarrow$) Assume that there is a sequence of formulas $\langle\varphi_1, \ldots, \varphi_n\rangle$ (with $\varphi_n = \varphi$) meeting the conditions of the theorem. Thus, each φ_i in the sequence must be a conclusion by RPL of previous members in the sequence. We want to show that $\varphi \in PL$. We prove this by induction on the length of the sequence. Suppose that the sequence has length 1. Then there are no previous members of the sequence. By hypothesis, then, φ is a conclusion by RPL of the empty set $\emptyset$. So φ must be a tautological consequence of $\emptyset$, and so by (31.4), φ is a tautology. Hence $\varphi \in PL$. Inductive case: Suppose the sequence has length $i < n$. So there is a sequence $\langle\varphi_1, \ldots, \varphi_i(=\varphi)\rangle$ such that each member is the conclusion by RPL of previous members of the sequence. So, in particular, φ_i must be a tautological consequence of previous members $\varphi_{j_1}, \ldots, \varphi_{j_k}$ $(1 \leq j_1 \leq j_k < i)$. So consider the sequences: $\langle\varphi_1, \ldots, \varphi_{j_1}\rangle, \langle\varphi_1, \ldots, \varphi_{j_2}\rangle, \ldots, \langle\varphi_1, \ldots, \varphi_{j_k}\rangle$. Now since each of these sequences has a length less than n, we may apply the inductive hypothesis to each sequence, concluding overall that $\varphi_{j_1}, \ldots, \varphi_{j_k}$ are members of PL, and so all are tautologies. But then φ_i is a tautological consequence of tautologies, and so by (31.3), φ_i, i.e., φ, is itself a tautology, and so a member of PL.

Proof of (45): Suppose that $\Gamma \vdash_\Sigma \varphi$ and that ψ is a tautological consequence of φ. Since $\Gamma \vdash_\Sigma \varphi$, there are $\varphi_1, \ldots, \varphi_n \in \Gamma$ such that $\vdash_\Sigma \varphi_1 \to \ldots \to \varphi_n \to \varphi$. Now since ψ is a tautological consequence of φ, $\varphi \to \psi$ is a tautology (by (31.1)). So $\vdash_\Sigma \varphi \to \psi$, since Σ contains every tautology. But the following is also a tautology:

$$[\varphi_1 \to \ldots \to \varphi_n \to \varphi] \to [(\varphi \to \psi) \to [\varphi_1 \to \ldots \to \varphi_n \to \psi]]$$

So this displayed formula is an element of Σ. But the antecedent of this displayed formula, and the antecedent of its consequent, are both elements of Σ, and since Σ is closed under MP, the consequent of the consequent must be an element of Σ as well. So $\vdash_\Sigma \varphi_1 \to \ldots \to \varphi_n \to \psi$, and thus $\exists \varphi_1, \ldots, \varphi_n \in \Gamma$ such that $\vdash_\Sigma \varphi_1 \to \ldots \to \varphi_n \to \psi$. Hence, $\Gamma \vdash_\Sigma \psi$.

Proof of (47): Suppose $\Gamma \vdash_\Sigma \varphi_1$ and ... and $\Gamma \vdash_\Sigma \varphi_n$, and ψ is a tautological consequence of $\varphi_1, \ldots, \varphi_n$. Since $\Gamma \vdash_\Sigma \varphi_1$, there are $\chi^1_1, \ldots, \chi^1_m \in \Gamma$ such that $\vdash_\Sigma \chi^1_1 \to \ldots \to \chi^1_m \to \varphi_1$; ... ; and since $\Gamma \vdash_\Sigma \varphi_n$, there are $\chi^n_1, \ldots, \chi^n_m \in \Gamma$ such that $\vdash_\Sigma \chi^n_1 \to \ldots \to \chi^n_m \to \varphi_n$. Now since ψ is a tautological consequence of $\varphi_1, \ldots, \varphi_n$, $\varphi_1 \to \ldots \to \varphi_n \to \psi$ is a tautology (by (31.2)). So $\vdash_\Sigma \varphi_1 \to \ldots \to \varphi_n \to \psi$. Thus far, we know the following are theorems of Σ:

$\mathbf{A}_1$: $\chi^1_1 \to \ldots \to \chi^1_m \to \varphi_1$
$\vdots$
$\mathbf{A}_n$: $\chi^n_1 \to \ldots \to \chi^n_m \to \varphi_n$
$\mathbf{B}$: $\varphi_1 \to \ldots \to \varphi_n \to \psi$

But the following is a tautology:

$$\begin{aligned}&\mathbf{A}_1 \to \ldots \to \mathbf{A}_n \to \mathbf{B} \to \\ &\quad \chi^1_1 \to \ldots \to \chi^1_m \to \ldots \to \chi^n_1 \to \ldots \to \chi^n_m \to \psi\end{aligned}$$

Since Σ is closed under MP, and $\mathbf{A}_1, \ldots, \mathbf{A}_n$, and $\mathbf{B}$ are elements of Σ, $n+1$ applications of MP yields the following as a theorem of Σ:

$$\chi^1_1 \to \ldots \to \chi^1_m \to \ldots \to \chi^n_1 \to \ldots \to \chi^n_m \to \psi$$

Since this is a theorem of Σ, and all of the χ^i_js are elements of Γ, it follows by definition that $\Gamma \vdash_\Sigma \psi$.

Proof of Exercise 1 in (49): ($\Rightarrow$) Assume that Γ is deductively-closed$_\Sigma$. We want to show: (a) that Γ is a modal logic, and (b) that $\Sigma \subseteq \Gamma$. To show (a), we need to show both (i) that Γ contains every tautology, and (ii) that Γ is closed under MP. To show (i), suppose that φ is a tautology. Then by (40), $\vdash_\Sigma \varphi$. And by (46.2), $\Gamma \vdash_\Sigma \varphi$. So by the fact that Γ is deductively closed, $\varphi \in \Gamma$. To show (ii), suppose that $\varphi \to \psi$ and φ and in Γ. Then by (46.5), both $\Gamma \vdash_\Sigma \varphi \to \psi$ and $\Gamma \vdash_\Sigma \varphi$. So by (48.7), $\Gamma \vdash_\Sigma \psi$. And by the deductive closure of Γ, $\psi \in \Gamma$.[19] To show (b), assume $\varphi \in \Sigma$, i.e., $\vdash_\Sigma \varphi$. By (46.2), φ is derivable$_\Sigma$ from every set of sentences. So $\Gamma \vdash_\Sigma \varphi$. Since Γ is deductively-closed$_\Sigma$, $\varphi \in \Gamma$.

($\Leftarrow$) Assume that Γ is a Σ-logic. So Γ contains every tautology, Γ is closed under MP, and $\Sigma \subseteq \Gamma$. Assume $\Gamma \vdash_\Sigma \varphi$ (to show $\varphi \in \Gamma$). Then $\exists \varphi_1, \ldots, \varphi_n \in \Gamma$ such that $\vdash_\Sigma \varphi_1 \to \ldots \to \varphi_n \to \varphi$. So, $\varphi_1 \to \ldots \to \varphi_n \to \varphi \in \Sigma$. But since $\Sigma \subseteq \Gamma$, $\varphi_1 \to \ldots \to \varphi_n \to \varphi \in \Gamma$. But since $\varphi_1, \ldots, \varphi_n \in \Gamma$, and Γ is closed under MP, $\varphi \in \Gamma$.

Proof of Exercise 2 in (49): (a) Suppose that $\Gamma \vdash_\Sigma \varphi$ and $\models^{\mathbf{M}}_{\mathbf{w}} \Sigma \cup \Gamma$. So $\exists \varphi_1, \ldots, \varphi_n \in \Gamma$ such that $\vdash_\Sigma \varphi_1 \to \ldots \to \varphi_n \to \varphi$. Since $\varphi_1 \to \ldots \to \varphi_n \to \varphi \in \Sigma$, $\varphi_1 \to \ldots \to \varphi_n \to \varphi \in \Sigma \cup \Gamma$. So $\models^{\mathbf{M}}_{\mathbf{w}} \varphi_1 \to \ldots \to \varphi_n \to \varphi$, by hypothesis. But since $\varphi_1, \ldots, \varphi_n \in \Gamma$, $\varphi_1, \ldots, \varphi_n \in \Sigma \cup \Gamma$. So $\models^{\mathbf{M}}_{\mathbf{w}} \varphi_1$ and $\ldots$ and $\models^{\mathbf{M}}_{\mathbf{w}} \varphi_n$. So by n applications of the fact that MP preserves truth at a world in a model (see the *Remark* in (30)), $\models^{\mathbf{M}}_{\mathbf{w}} \varphi$.

(b) To prove (i) that $\{\varphi \,|\, \Gamma \vdash_\Sigma \varphi\}$ is a modal logic, assume that ψ is a tautology (to show $\psi \in \{\varphi \,|\, \Gamma \vdash_\Sigma \varphi\}$). So by (40.1), $\psi \in \Sigma$, and by (46.2), $\Gamma \vdash_\Sigma \psi$. So $\psi \in \{\varphi \,|\, \Gamma \vdash_\Sigma \varphi\}$, and thus, this set contains every tautology. To see that $\{\varphi \,|\, \Gamma \vdash_\Sigma \varphi\}$ is closed under MP, assume that it contains both $\psi \to \chi$ and ψ. So $\Gamma \vdash_\Sigma \psi \to \chi$ and $\Gamma \vdash_\Sigma \psi$. By (48.7), $\Gamma \vdash_\Sigma \psi$, and so $\psi \in \{\varphi \,|\, \Gamma \vdash_\Sigma \varphi\}$. Thus, this set is closed under MP, and so it is a modal logic. To prove (ii) that $\{\varphi \,|\, \Gamma \vdash_\Sigma \varphi\}$ contains $\Sigma \cup \Gamma$, assume that $\psi \in \Sigma \cup \Gamma$ (to show $\Gamma \vdash_\Sigma \psi$). Then either $\psi \in \Sigma$ or $\psi \in \Gamma$. If the former, then $\vdash_\Sigma \psi$, and so by (46.2), $\Gamma \vdash_\Sigma \psi$. If the latter, then by (46.5), $\Gamma \vdash_\Sigma \psi$. To prove (iii) that $\{\varphi \,|\, \Gamma \vdash_\Sigma \varphi\}$ is a subset of any logic Σ' containing $\Sigma \cup \Gamma$, suppose that Σ' is a logic containing $\Sigma \cup \Gamma$. Suppose that $\psi \in \{\varphi \,|\, \Gamma \vdash_\Sigma \varphi\}$ (to show $\psi \in \Sigma'$). So $\Gamma \vdash_\Sigma \psi$. So $\exists \psi_1, \ldots \psi_n \in \Gamma$ such that $\vdash_\Sigma \psi_1 \to \ldots \to \psi_n \to \psi$. Since $\psi_1, \ldots, \psi_n \in \Gamma$ and $\psi_1 \to \ldots \to \psi_n \to \psi \in \Sigma$, we know that $\psi_1, \ldots,$ ψ_n, and $\psi_1 \to \ldots \to \psi_n \to \psi$ are all in $\Sigma \cup \Gamma$. But then $\psi_1, \ldots, \psi_n$, and $\psi_1 \to \ldots \to \psi_n \to \psi$ are all in Σ'. So since Σ' is a modal logic, it is closed

[19] An alternate proof of (a) is to show Γ is closed under RPL: assume that $\varphi_1, \ldots, \varphi_n \in \Gamma$, and that ψ is a tautological consequence of $\varphi_1, \ldots, \varphi_n$. Then, by (46.5), $\Gamma \vdash_\Sigma \varphi_1$, $\ldots$, $\Gamma \vdash_\Sigma \varphi_n$. So by (47), $\Gamma \vdash_\Sigma \psi$. But, by hypothesis, Γ is deductively-closed$_\Sigma$. So $\psi \in \Gamma$.

under MP. Hence, $\psi \in \Sigma'$.

Proof of (56): ($\Rightarrow$) Assume $\Gamma \vdash_\Sigma \varphi$ and that both $\mathrm{MaxCon}_\Sigma(\Delta)$ and $\Gamma \subseteq \Delta$. Then by (46.7), $\Delta \vdash_\Sigma \varphi$. So by (54.1), $\varphi \in \Delta$. ($\Leftarrow$) Assume that φ is an element of every $\mathrm{MaxCon}_\Sigma(\Delta)$ of which Γ is a subset. For reductio, suppose that $\Gamma \not\vdash_\Sigma \varphi$. Then, by (52.8), $\mathrm{Con}_\Sigma(\Gamma \cup \{\neg\varphi\})$. So by (55), there is a $\mathrm{MaxCon}_\Sigma(\Delta')$ of which $\Gamma \cup \{\neg\varphi\}$ is a subset. By this last fact, both $\Gamma \subseteq \Delta'$ and $\{\neg\varphi\} \subseteq \Delta'$, and so $\neg\varphi \in \Delta'$. But since φ is a member of every $\mathrm{MaxCon}_\Sigma(\Delta)$ of which Γ is a subset, $\varphi \in \Delta'$, contradicting (54.4) for $\mathrm{MaxCon}_\Sigma(\Delta)$.

Proof of Exercise in (57): Suppose throughout that $\mathrm{Con}_\Sigma(\Gamma)$. ($\Rightarrow$) Assume that for every φ, if $\mathrm{Con}_\Sigma(\Gamma \cup \{\varphi\})$, then $\varphi \in \Gamma$. To show that $\mathrm{Max}(\Gamma)$, we show that if $\psi \notin \Gamma$, then $\neg\psi \in \Gamma$. So assume $\psi \notin \Gamma$. Then, by our assumption, $\mathrm{C\o n}_\Sigma(\Gamma \cup \{\psi\})$. So by (52.9), $\Gamma \vdash_\Sigma \neg\psi$. Assume, for reductio, that $\neg\psi \notin \Gamma$. Then, again by assumption, $\mathrm{C\o n}_\Sigma(\Gamma \cup \{\neg\psi\})$. So by (52.8), $\Gamma \vdash_\Sigma \psi$. So both $\Gamma \vdash_\Sigma \neg\psi$ and $\Gamma \vdash_\Sigma \psi$, contradicting the consistency of Γ. So $\neg\psi \in \Gamma$ ($\Leftarrow$) Assume $\mathrm{Max}(\Gamma)$. Assume $\mathrm{Con}_\Sigma(\Gamma \cup \{\psi\})$. So by (52.9), $\Gamma \not\vdash_\Sigma \neg\psi$. Now suppose for reductio that $\psi \notin \Gamma$. Then, since $\mathrm{Max}(\Gamma)$, $\neg\psi \in \Gamma$. So by (46.5), $\Gamma \vdash_\Sigma \neg\psi$, which is a contradiction. So $\psi \in \Gamma$.

Proof of (59): ($\Rightarrow$) Suppose Σ is a normal modal logic. Since it is a modal logic, it contains every tautology and is closed under MP, and since it is normal, it contains the instances of the axiom K and is closed under RN. We want to show that if $\vdash_\Sigma \varphi_1 \to \ldots \to \varphi_n \to \varphi$, then $\vdash_\Sigma \Box\varphi_1 \to \ldots \to \Box\varphi_n \to \Box\varphi$. We prove this by induction: (a) Base case: When $n = 0$, then we have to show that if $\vdash_\Sigma \varphi$, then $\vdash_\Sigma \Box\varphi$. But this follows immediately, since Σ is closed under RN. (b) Assume $\vdash_\Sigma \varphi_1 \to \ldots \to \varphi_n \to \varphi$. Now our inductive hypothesis is: if $\vdash_\Sigma \varphi_1 \to \ldots \to \varphi_{n-1} \to \psi$, then $\vdash_\Sigma \Box\varphi_1 \to \ldots \to \Box\varphi_{n-1} \to \Box\psi$. So by our inductive hypothesis, $\vdash_\Sigma \Box\varphi_1 \to \ldots \to \Box\varphi_{n-1} \to \Box(\varphi_n \to \varphi)$. But since Σ contains all the instances of the axiom K, we know $\vdash_\Sigma \Box(\varphi_n \to \varphi) \to (\Box\varphi_n \to \Box\varphi)$. So given Σ contains all tautologies and is closed under MP, $\vdash_\Sigma \Box\varphi_1 \to \ldots \to \Box\varphi_{n-1} \to (\Box\varphi_n \to \Box\varphi)$, i.e., $\vdash_\Sigma \Box\varphi_1 \to \ldots \to \Box\varphi_n \to \Box\varphi$. ($\Leftarrow$) Suppose Σ is closed under RK. To show Σ is normal, we show (a) that it contains the (instances of the) schema K and (b) is closed under RN. (a) Since $(\varphi \to \psi) \to \varphi \to \psi$ is a tautology, $\vdash_\Sigma (\varphi \to \psi) \to \varphi \to \psi$. So by RK, $\vdash_\Sigma \Box(\varphi \to \psi) \to \Box\varphi \to \Box\psi$, i.e., $\vdash_\Sigma \Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$. (b) RN is the special case of RK where $n = 0$.

Proof of (60.1): Suppose Σ is normal. To simplify the proof, we note that since Σ is a logic, it is closed under RPL. By the definition of $\Diamond\psi$ in the *Remark* in (3), we know that Σ contains $\Diamond\psi \leftrightarrow \neg\Box\neg\psi$ (since Σ contains all tautologies and this just abbreviates the tautology $\neg\Box\neg\psi \leftrightarrow \neg\Box\neg\psi$). Since Σ contains all the tautologies of this form, it contains $\Diamond\neg\varphi \leftrightarrow \neg\Box\neg\neg\varphi$ (substituting $\neg\varphi$ for ψ). Thus, by closure under RPL, Σ will contain the following biconditional obtained by negating both sides: $\neg\Diamond\neg\varphi \leftrightarrow \neg\neg\Box\neg\neg\varphi$. But since $\neg\neg\Box\neg\neg\varphi \leftrightarrow \Box\neg\neg\varphi$ is a tautology, closure under RPL also requires that Σ contain $\neg\Diamond\neg\varphi \leftrightarrow \Box\neg\neg\varphi$. However, in order to show that Σ contains $\neg\Diamond\neg\varphi \leftrightarrow \Box\varphi$, we have to show that Σ contains $\Box\neg\neg\varphi \leftrightarrow \Box\varphi$. But to do this, we need the fact that Σ is closed under RK, for given $\varphi \rightarrow \neg\neg\varphi$ and $\neg\neg\varphi \rightarrow \varphi$ are tautologies and hence members of Σ, it follows by RK that both $\Box\varphi \rightarrow \Box\neg\neg\varphi$ and $\Box\neg\neg\varphi \rightarrow \Box\varphi$ are members of Σ. And so the biconditional $\Box\neg\neg\varphi \leftrightarrow \Box\varphi$ will be a member of Σ. Thus, by closure under RPL, Σ contains $\neg\Diamond\neg\varphi \leftrightarrow \Box\varphi$.

Proof of Exercise 4 in (60): Let Σ be normal and suppose that $\vdash_\Sigma \psi \leftrightarrow \psi'$. We show that $\vdash_\Sigma \varphi[\psi'/\psi]$. Suppose that φ and ψ are the same sentence. Then $\varphi[\psi'/\psi]$ is either φ (when zero occurrences of ψ are replaced by ψ') or ψ' (when φ, i.e., ψ, is replaced by ψ'). In either case, $\vdash_\Sigma \varphi[\psi'/\psi]$. So assume that φ and ψ are distinct. Then we complete the proof by induction on the complexity of φ. Base case: Suppose $\varphi = p$.

Inductive cases: (a) Suppose $\varphi = \bot$.

(b) Suppose that $\varphi = \chi \rightarrow \theta$. For inductive hypotheses, we may assume $\vdash_\Sigma \chi \leftrightarrow \chi[\psi'/\psi]$ and $\vdash_\Sigma \theta \leftrightarrow \theta[\psi'/\psi]$.

(c) Suppose that $\varphi = \Box\chi$. For the inductive hypothesis, we assume $\vdash_\Sigma \chi \leftrightarrow \chi[\psi'/\psi]$. Use RE and then the fact that $\Box(\chi[\psi'/\psi]) = (\Box\chi)[\psi'/\psi]$.

Proof of (63): Clearly, there is an effective way of telling whether a formula is a tautology or an instance of the K schema. Moreover, the rules MP and RN can be applied effectively. So given the definitions in (42), we have to show that $\varphi \in K$ iff there is a proof of φ from the tautologies and the K axiom using the rules MP and RN, i.e., we have to show that $\vdash_K \varphi$ iff there is a sequence of formulas $\varphi_1, \ldots, \varphi_n$ $(=\varphi)$ such that every member of the sequence either (a) is a tautology or an instance of K, or (b) follows from previous members by MP or RN.

Proof of (65): Since there is an effective method for telling whether a formula is a tautology or an instance of $\mathrm{K}, \mathrm{S}_1, \ldots, \mathrm{S}_n$, and the rules MP and RN can be applied effectively, it suffices to show, by the definitions in (42), that $\varphi \in K\mathrm{S}_1 \ldots \mathrm{S}_n$ iff there is a proof of φ from the tautologies and axioms $\mathrm{K}, \mathrm{S}_1, \ldots, \mathrm{S}_n$ using the rules of inference MP and RN; i.e., it suffices to show that $\vdash_{K\mathrm{S}_1 \ldots \mathrm{S}_n} \varphi$ iff there is a sequence of formulas $\langle \varphi_1, \ldots, \varphi_n \rangle$ (with $\varphi = \varphi_n$) every member of which either (a) is a tautology or an instance of K, or S_1, or $\ldots$, or S_n, or (b) follows from previous members of the sequence by MP or RN.

Proof of (66): Suppose that Σ is normal and that $\Gamma \vdash_\Sigma \varphi$. Then $\exists \varphi_1, \ldots, \varphi_n \in \Gamma$ such that $\vdash_\Sigma \varphi_1 \to \ldots \to \varphi_n \to \varphi$. Since Σ is normal, it follows by (59) that $\vdash_\Sigma \Box\varphi_1 \to \ldots \to \Box\varphi_n \to \Box\varphi$. But since $\varphi_1, \ldots, \varphi_n \in \Gamma$, $\Box\varphi_1, \ldots, \Box\varphi_n \in \{\Box\psi \,|\, \psi \in \Gamma\}$. So, by the definition of deducibility, $\{\Box\psi \,|\, \psi \in \Gamma\} \vdash_\Sigma \Box\varphi$.

Proof of (68): ($\Rightarrow$) Assume $\Box\varphi \in \Gamma$, $\mathrm{MaxCon}_\Sigma(\Delta)$ and $\{\psi \,|\, \Box\psi \in \Gamma\} \subseteq \Delta$, for arbitrary Δ. Then, by definition, $\varphi \in \{\psi \,|\, \Box\psi \in \Gamma\}$. So, by hypothesis, $\varphi \in \Delta'$. ($\Leftarrow$) We use (67), though a proof can be constructed using (66) as well. Assume that every $\mathrm{MaxCon}_\Sigma(\Delta)$ such that $\{\psi \,|\, \Box\psi \in \Gamma\} \subseteq \Delta$ is also such that $\varphi \in \Delta$. Then by (56), $\{\psi \,|\, \Box\psi \in \Gamma\} \vdash_\Sigma \varphi$. So by (67), $\Gamma \vdash_\Sigma \Box\varphi$. And by (54.1), $\Box\varphi \in \Gamma$, since Γ is MaxCon_Σ.

Proof of the Exercise in (68) : ($\Leftarrow$) Assume that every $\mathrm{MaxCon}_\Sigma(\Delta)$ such that $\{\psi \,|\, \Box\psi \in \Gamma\} \subseteq \Delta$ is also such that $\varphi \in \Delta$. Then by (56), $\{\psi \,|\, \Box\psi \in \Gamma\} \vdash_\Sigma \varphi$. So by (66), $\{\Box\chi \,|\, \chi \in \{\psi \,|\, \Box\psi \in \Gamma\}\} \vdash_\Sigma \Box\varphi$, i.e., $\{\Box\chi \,|\, \Box\chi \in \Gamma\} \vdash_\Sigma \Box\varphi$. Since $\{\Box\chi \,|\, \Box\chi \in \Gamma\} \subseteq \Gamma$, it follows by (46.7) that $\Gamma \vdash_\Sigma \Box\varphi$. So by (54.1), $\Box\varphi \in \Gamma$.

Proof of (69): ($\Rightarrow$) Assume that for every φ, if $\Box\varphi \in \Gamma$, then $\varphi \in \Delta$. Assume also that $\psi \in \Delta$ (to show $\Diamond\psi \in \Gamma$). If $\psi \in \Delta$, then $\neg\psi \notin \Delta$, by (54.4). So by hypothesis, $\Box\neg\psi \notin \Gamma$. So by the definition of $\Diamond\psi$, $\Diamond\psi \in \Gamma$. ($\Leftarrow$) Assume that for every φ, if $\varphi \in \Delta$, then $\Diamond\varphi \in \Gamma$. Assume that $\Box\psi \in \Gamma$ (to show $\psi \in \Delta$). Since Σ is normal, Σ contains $\Box\psi \leftrightarrow \neg\Diamond\neg\psi$, by (60). Since $\mathrm{MaxCon}_\Sigma\Gamma$, $\Sigma \subseteq \Gamma$, by (54.2). Consequently, Γ must also contain $\Box\psi \leftrightarrow \neg\Diamond\neg\psi$. So since Γ contains $\Box\psi$ by assumption, we know $\neg\Diamond\neg\psi \in \Gamma$, by (54.8), and so $\Diamond\neg\psi \notin \Gamma$, by (54.4). So by hypothesis, $\neg\psi \notin \Delta$. But then $\psi \in \Delta$, by (54.4).

Proof of (70): By (54.4), we know that $\Diamond\varphi \in \Gamma$ (i.e., $\neg\Box\neg\varphi \in \Gamma$) iff $\Box\neg\varphi \notin \Gamma$. But by (68), $\Box\neg\varphi \notin \Gamma$ iff there is a $\mathrm{MaxCon}_\Sigma(\Delta)$ such that both (i) $\{\psi \,|\, \Box\psi \in \Gamma\} \subseteq \Delta$ and (ii) $\neg\varphi \notin \Delta$. But by the *Remark* in (69), we know that clause (i) is equivalent to clause (a) in the statement of the theorem. And by (54.4), we know that clause (ii) is equivalent to clause

(b) in the statement of the theorem. Thus, substituting equivalents, we may infer $\Diamond\varphi \in \Gamma$ iff there is a $\mathrm{MaxCon}_{\Sigma}(\Delta)$ such that both $\{\Diamond\psi \,|\, \psi \in \Delta\} \subseteq \Gamma$ and $\varphi \in \Delta$.

Chapter Five

Proof of Lemma in (71.2): Assume (i), (ii), and (iii). Assume further that φ is a theorem of Σ (to show that φ is valid with respect to $\mathbf{C}$). Since Σ is axiomatized by Γ and the rules $\mathrm{R}_1, \mathrm{R}_2, \ldots$, it follows by (42) that there is a proof of φ from Γ using the rules, i.e., that there is a sequence of formulas $\varphi_1, \ldots, \varphi_n$ (with $\varphi = \varphi_n$) every member of which is either a member of Γ or a conclusion by a rule of previous members in the sequence. We now argue by induction on the length of a proof that φ is valid with respect to $\mathbf{C}$.

Base case: Suppose that the proof of φ is a one member sequence, namely, φ itself. Then either φ is a member of Γ or follows from previous members of the sequence (of which there are none) by a rule. If φ is a member of Γ, then φ is valid with respect to $\mathbf{C}$, by hypothesis. If φ follows from the empty set of previous members by a rule of inference, then since the rules all preserve validity with respect to $\mathbf{C}$, φ is valid with respect to $\mathbf{C}$.

Inductive case: Suppose that the proof of φ is an n member sequence. Then φ $(= \varphi_n)$ must follow from previous members, say $\varphi_{j_1}, \ldots, \varphi_{j_k}$ $(1 \leq j_1 \leq j_k < n)$, by one of the rules. But consider the following sequence of sequences: $\langle \varphi_1, \ldots, \varphi_{j_1} \rangle, \langle \varphi_1, \ldots, \varphi_{j_2} \rangle, \ldots, \langle \varphi_1, \ldots, \varphi_{j_k} \rangle$. Now since each of these sequences has a length less than n, and so we may apply the inductive hypothesis to each sequence, concluding overall that $\mathbf{C} \models \varphi_{j_1}$, $\mathbf{C} \models \varphi_{j_2}$, ... , $\mathbf{C} \models \varphi_{j_k}$. And since φ follows from $\varphi_{j_1}, \ldots, \varphi_{j_k}$ by R, and R preserves validity with resepct to $\mathbf{C}$, it follows that $\mathbf{C} \models \varphi$.

Proof of (75): Let $\mathbf{C_F}$ be the class of all frames and note by (61) that $\{\psi \,|\, \mathbf{C_F} \models \psi\}$ is a normal logic. Since K is the smallest normal modal logic, K is a subset of $\{\psi \,|\, \mathbf{C_F} \models \psi\}$, for any class of frames $\mathbf{C_F}$. So the theorems (i.e., members) of K are all members of this such set; i.e., if $\vdash_K \varphi$, then $\mathbf{C_F} \models \varphi$. So, K is sound with respect to the class of all frames.

Proof of (79): By induction on Λ_Ω with respect to an arbitrarily chosen world $\mathbf{w}$. Base case: Suppose that $\varphi = p$. By (6.6), $\models^{\mathbf{M}^\Sigma}_{\mathbf{w}} p$ iff $\mathbf{w} \in \mathbf{V}^\Sigma(p)$.

But by (78.3), $\mathbf{w} \in \mathbf{V}^{\Sigma}(p)$ iff $\mathbf{w} \in \{\mathbf{w}' \in \mathbf{W}^{\Sigma} \mid p \in \mathbf{w}'\}$, i.e., iff $p \in \mathbf{w}$. So $\models^{\mathbf{M}^{\Sigma}}_{\mathbf{w}} p$ iff $p \in \mathbf{w}$. So $\models^{\mathbf{M}^{\Sigma}}_{\mathbf{w}} \varphi$ iff $\varphi \in \mathbf{w}$.

Inductive cases: (i) Suppose that $\varphi = \bot$. By (6.2), $\not\models^{\mathbf{M}^{\Sigma}}_{\mathbf{w}} \bot$. So by antecedent failure, if $\models^{\mathbf{M}^{\Sigma}}_{\mathbf{w}} \bot$, $\bot \in \mathbf{w}$. Moreover, by (54.3) and the fact that $\mathbf{w}$ is MaxCon$_{\Sigma}$ (78.1), $\bot \notin \mathbf{w}$. So by antecedent failure, if $\bot \in \mathbf{w}$, then $\models^{\mathbf{M}^{\Sigma}}_{\mathbf{w}} \bot$. Thus, $\models^{\mathbf{M}^{\Sigma}}_{\mathbf{w}} \bot$ iff $\bot \in \mathbf{w}$; i.e., $\models^{\mathbf{M}^{\Sigma}}_{\mathbf{w}} \varphi$ iff $\varphi \in \mathbf{w}$.

(ii) Suppose $\varphi = \neg\psi$. By (6.3), $\models^{\mathbf{M}^{\Sigma}}_{\mathbf{w}} \neg\psi$ iff $\not\models^{\mathbf{M}^{\Sigma}}_{\mathbf{w}} \psi$. But, by our inductive hypothesis, $\models^{\mathbf{M}^{\Sigma}}_{\mathbf{w}} \psi$ iff $\psi \in \mathbf{w}$. So $\models^{\mathbf{M}^{\Sigma}}_{\mathbf{w}} \neg\psi$ iff $\psi \notin \mathbf{w}$. Since $\mathbf{w}$ is MaxCon$_{\Sigma}$ (by (78.1)), $\psi \notin \mathbf{w}$ iff $\neg\psi \in \mathbf{w}$ (by (54.4)). So $\models^{\mathbf{M}^{\Sigma}}_{\mathbf{w}} \neg\psi$ iff $\neg\psi \in \mathbf{w}$, i.e., $\models^{\mathbf{M}^{\Sigma}}_{\mathbf{w}} \varphi$ iff $\varphi \in \mathbf{w}$.

(iii) Suppose $\varphi = \psi \to \chi$. By (6.4), $\models^{\mathbf{M}^{\Sigma}}_{\mathbf{w}} \psi \to \chi$ iff $\not\models^{\mathbf{M}^{\Sigma}}_{\mathbf{w}} \psi$ or $\models^{\mathbf{M}^{\Sigma}}_{\mathbf{w}} \chi$. But our inductive hypotheses are:

$\models^{\mathbf{M}^{\Sigma}}_{\mathbf{w}} \psi$ iff $\psi \in \mathbf{w}$
$\models^{\mathbf{M}^{\Sigma}}_{\mathbf{w}} \chi$ iff $\chi \in \mathbf{w}$

So $\models^{\mathbf{M}^{\Sigma}}_{\mathbf{w}} \psi \to \chi$ iff either $\psi \notin \mathbf{w}$ or $\chi \in \mathbf{w}$. But since $\mathbf{w}$ is MaxCon$_{\Sigma}$ (78.1), either $\psi \notin \mathbf{w}$ or $\chi \in \mathbf{w}$ iff $\psi \to \chi \in \mathbf{w}$, by (54.7). So $\models^{\mathbf{M}^{\Sigma}}_{\mathbf{w}} \psi \to \chi$ iff $\psi \to \chi \in \mathbf{w}$, i.e., $\models^{\mathbf{M}^{\Sigma}}_{\mathbf{w}} \varphi$ iff $\varphi \in \mathbf{w}$.

(iv) Suppose $\varphi = \Box\psi$. By (6.5), $\models^{\mathbf{M}^{\Sigma}}_{\mathbf{w}} \Box\psi$ iff for every $\mathbf{w}'$, if (a) $\mathbf{R}^{\Sigma}\mathbf{w}\mathbf{w}'$, then (b) $\models^{\mathbf{M}^{\Sigma}}_{\mathbf{w}'} \psi$. By (78.2), (a) holds iff $\{\chi \mid \Box\chi \in \mathbf{w}\} \subseteq \mathbf{w}'$. By our inductive hypothesis, (b) holds iff $\psi \in \mathbf{w}'$. So, assembling our results so far, $\models^{\mathbf{M}^{\Sigma}}_{\mathbf{w}} \Box\psi$ iff for every $\mathbf{w}'$, if $\{\chi \mid \Box\chi \in \mathbf{w}\} \subseteq \mathbf{w}'$, then $\psi \in \mathbf{w}'$, i.e, iff ψ is an element of every world $\mathbf{w}'$ such that $\{\chi \mid \Box\chi \in \mathbf{w}\} \subseteq \mathbf{w}'$. But since $\mathbf{w}$ and $\mathbf{w}'$ are MaxCon$_{\Sigma}$ and Σ is normal, it follows by (68) that this condition holds iff $\Box\psi \in \mathbf{w}$. Thus, $\models^{\mathbf{M}^{\Sigma}}_{\mathbf{w}} \Box\psi$ iff $\Box\psi \in \mathbf{w}$, i.e, $\models^{\mathbf{M}^{\Sigma}}_{\mathbf{w}} \varphi$ iff $\varphi \in \mathbf{w}$.

Proof of (80): By (79), for all $\mathbf{w} \in \mathbf{W}^{\Sigma}$, [$\models^{\mathbf{M}^{\Sigma}}_{\mathbf{w}} \varphi$ iff $\varphi \in \mathbf{w}$]. It follows that: [for every $\mathbf{w}$, $\models^{\mathbf{M}^{\Sigma}}_{\mathbf{w}} \varphi$] iff [for every $\mathbf{w}$, $\varphi \in \mathbf{w}$], i.e., that $\models^{\mathbf{M}^{\Sigma}} \varphi$ iff for every $\mathbf{w}$, $\varphi \in \mathbf{w}$. But by (78.1), every world $\mathbf{w}$ is MaxCon$_{\Sigma}$. So, $\models^{\mathbf{M}^{\Sigma}} \varphi$ iff for every MaxCon$_{\Sigma}(\Gamma)$, $\varphi \in \Gamma$. But by (57), if follows that $\models^{\mathbf{M}^{\Sigma}} \varphi$ iff $\vdash_{\Sigma} \varphi$.