A. Lewis-Smith
P. Oliva
E. Robinson

Kripke Semantics for Intuitionistic Łukasiewicz Logic

Abstract. This paper proposes a generalization of the Kripke semantics of intuitionistic logic **IL** appropriate for *intuitionistic Łukasiewicz logic* **IŁL** —a logic in the intersection between **IL** and (classical) Łukasiewicz logic. This generalised Kripke semantics is based on the *poset sum* construction, used in Bova and Montagna (Theoret Comput Sci 410(12):1143–1158, 2009). to show the decidability (and PSPACE completeness) of the quasiequational theory of commutative, integral and bounded GBL algebras. The main idea is that $w \Vdash \psi$—which for **IL** is a relation between worlds w and formulas ψ, and can be seen as a function taking values in the booleans $(w \Vdash \psi) \in \mathbb{B}$—becomes a function taking values in the unit interval $(w \Vdash \psi) \in [0,1]$. An appropriate monotonicity restriction (which we call *sloping functions*) needs to be put on such functions in order to ensure soundness and completeness of the semantics.

Keywords: Łukasiewicz logic, Intuitionistic Łukasiewicz logic, Kripke semantics, GBL algebras.

1. Introduction

In [2], Bova and Montagna study the computational complexity of the propositional logic $\mathbf{GBL}_{\mathrm{ewf}}$—a common fragment of intuitionistic logic and Łukasiewicz logic. We will refer to this logic as *intuitionistic Łukasiewicz logic* **IŁL**. The original name $\mathbf{GBL}_{\mathrm{ewf}}$ derives from the fact that this logic has a sound and complete algebraic semantics based on commutative (exchange), integral (weakening) and bounded (ex-falsum) GBL algebras. Bova and Montagna have shown that the consequence problem for **IŁL** is PSPACE complete, by showing that the equational and the quasiequational theories of commutative, integral and bounded GBL algebras are PSPACE complete.

Their decision procedure relies on the construction of a particular commutative, integral and bounded GBL algebra (called *poset sum*) from any given poset. The elements of the poset sum are particular monotone functions assigning fuzzy values (in $[0,1]$) to each element of the poset. In what

follows we demonstrate how this can be seen as a novel generalization of Kripke semantics adequate for intuitionistic Łukasiewicz logic **IŁL**.

IŁL, viewed as a fragment of Łukasiewicz logic, acts as the nexus of several important nominally 'fuzzy' logics, such as basic logic (BL), Gödel logic, and the product logic. The Hilbert-style presentation of **IŁL** coincides with Hajek's BL [6] minus pre-linearity, and therefore behaves as the constructive (or intuitionistic) kernel of BL. As such, **IŁL** can also be viewed as a generalisation of intuitionistic propositional logic. GBL structures, introduced by Montagna and Jipsen in [7], can be regarded as a generalization of Heyting algebras, the standard algebraic model for intuitionistic logic. In general GBL lacks the commutativity present in BL and Heyting algebras, although finite GBL algebras are commutative (see [7]).

We pursue this connection between **IŁL** and intuitionistic logic in detail, but with respect to a generalization of Kripke semantics [8]. This suggests a new way to view many-valued logics in terms of relational semantics, as opposed to algebraic semantics. This development may be welcome considering the prevalence of algebraic approaches in the area has distinguished it from the broader field of non-classical and substructural logics, many of which can be characterised both algebraically and in terms of some variant of relational semantics.

In Section 1.1, we will start by revisiting the standard Kripke semantics for propositional intuitionistic logic (with a view to generalise it) and introducing the logic **IŁL**. Section 2 introduces the various algebras that we will use in the paper, and discusses the algebraic semantics of **IŁL**. In Section 3, we present the generalisation of Kripke semantics, which we shall call *Bova–Montagna*[1] *semantics*, for **IŁL**. In Section 3.1 we outline how this semantics is a natural generalization of the Kripke semantics for intuitionistic logic. Sections 3.3 and 3.4 prove the soundness and completeness of the semantics. Although there are several generalizations of Kripke semantics in the literature, to our knowledge this particular generalization is novel. Section 4 contains a discussion, from a categorical point of view, of the relationship between algebraic semantics and Kripke semantics.

We anticipate that the methods presented here for constructing Kripke semantics for sub-structural logics can be adapted to other logics extending **IŁL** and might even suggest alternative proof methods for logics in the

---

[1]The semantics we present has been extracted from Bova and Montagna's definition of poset sums ([2], Definition 2), which itself is based on the work of Jipsen and Montagna on ordinal sum constructions [7].

vicinity of BL and GBL e.g. new tableaux defined out of accessibility relations of a Kripke model, labelled systems making specific use of the forcing definition, or even extending Negri's approach in [13] for 'sequents of rules' out of poset sums. This is discussed further in the final section.

1.1. Intuitionistic Logic IL

The formulas of intuitionistic logic are inductively defined from atomic formulas (we use $p, q, \ldots$ for propositional variables), including $\bot$, and the binary connectives $\psi \wedge \chi$, $\psi \vee \chi$ and $\psi \to \chi$. We will refer to this language as $\mathcal{L}$. Let us recall the standard Kripke semantics for intuitionistic logic.

DEFINITION 1.1. (Kripke structure, [8]) A *Kripke structure* consists of a pair $\mathcal{K} = \langle \mathcal{W}, \Vdash^K \rangle$, where $\mathcal{W} = \langle W, \succeq \rangle$ is a partial order, and $w \Vdash^K p$ is a binary relation between worlds $w \in W$ and propositional variables p satisfying the following conditions:

(M) If $w \Vdash^K p$ and $v \succeq w$ then $v \Vdash^K p$

($\bot$) $\neg(w \Vdash^K \bot)$

DEFINITION 1.2. (Kripke semantics for $\mathcal{L}$, [8]) Given a Kripke structure

$$\mathcal{K} = \langle \mathcal{W}, \Vdash^K \rangle$$

we extend the relation $\Vdash^K$ to a relation between worlds and arbitrary $\mathcal{L}$-formulas as

$$w \Vdash^K \psi \wedge \chi := w \Vdash^K \psi \text{ and } w \Vdash^K \chi$$
$$w \Vdash^K \psi \vee \chi := w \Vdash^K \psi \text{ or } w \Vdash^K \chi$$
$$w \Vdash^K \psi \to \chi := v \Vdash^K \psi \text{ implies } v \Vdash^K \chi, \text{ for all } v \succeq w$$

Using the above, one obtains an essential (and well-known) property characterising the satisfaction of formulas in intuitionistic logic.

PROPOSITION 1.3. *The monotonicity property (M) holds for all $\mathcal{L}$-formulas ϕ, i.e.*

$$\text{if } w \Vdash^K \phi \text{ and } v \succeq w \text{ then } v \Vdash^K \phi$$

1.2. Intuitionistic Łukasiewicz Logic IŁL

The formulas of Łukasiewicz logic are inductively defined from atomic formulas, including $\bot$, and the binary connectives $\psi \wedge \chi$, $\psi \vee \chi$, $\psi \otimes \chi$ and $\psi \to \chi$. We will refer to this language as $\mathcal{L}_\otimes$, since it extends the language $\mathcal{L}$ of intuitionistic logic with a second form of conjunction $\psi \otimes \chi$.

$$\frac{}{\phi \vdash \phi} \text{ Ax} \qquad \frac{\Gamma \vdash \psi}{\Gamma, \phi \vdash \psi} \text{ W} \qquad \frac{\Gamma, \phi, \psi, \Delta \vdash \chi}{\Gamma, \psi, \phi, \Delta \vdash \chi} \text{ Ex}$$

$$\frac{\Gamma, \phi \vdash \psi}{\Gamma \vdash \phi \to \psi} \to \text{I} \qquad\qquad \frac{\Gamma \vdash \phi \to \psi \qquad \Delta \vdash \phi}{\Gamma, \Delta \vdash \psi} \to \text{E}$$

$$\frac{\Gamma \vdash \phi \qquad \Delta \vdash \psi}{\Gamma, \Delta \vdash \phi \otimes \psi} \otimes \text{I} \qquad\qquad \frac{\Gamma, \phi, \psi \vdash \chi \qquad \Delta \vdash \phi \otimes \psi}{\Gamma, \Delta \vdash \chi} \otimes \text{E}$$

$$\frac{\Gamma \vdash \phi \qquad \Gamma \vdash \psi}{\Gamma \vdash \phi \wedge \psi} \wedge \text{I} \qquad\qquad \frac{\Gamma \vdash \phi_1 \wedge \phi_2}{\Gamma \vdash \phi_i} \wedge \text{E} \ \ (i \in \{1,2\})$$

$$\frac{\Gamma \vdash \phi_i}{\Gamma \vdash \phi_1 \vee \phi_2} \vee \text{I} \ \ (i \in \{1,2\}) \qquad \frac{\Gamma \vdash \phi \vee \psi \qquad \Delta, \phi \vdash \chi \qquad \Delta, \psi \vdash \chi}{\Gamma, \Delta \vdash \chi} \vee \text{E}$$

$$\frac{\Gamma, \phi, \phi \to \psi \vdash \chi}{\Gamma, \psi, \psi \to \phi \vdash \chi} \text{ DIV} \qquad\qquad \frac{\Gamma \vdash \bot}{\Gamma \vdash \phi} \bot \text{E}$$

Figure 1. Intuitionistic Łukasiewicz logic **IŁL**

Figure 1 gives a natural deduction system for intuitionistic (proposi-
tional) Łukasiewicz logic **IŁL**. When we write a sequent $\Gamma \vdash \phi$ we are al-
ways assuming Γ to be a finite sequence of formulas. Note that we have the
structural rules of weakening and exchange, but not contraction. Hence, the
number of occurrences of a formula in Γ matters, and one could think of the
contexts Γ as multisets. In particular, the rule $\to$ I removes one occurrence
of ϕ from the context Γ, ϕ, concluding $\phi \to \psi$ from the smaller context Γ.
This makes **IŁL** a form of affine logic.

    IŁL has a deduction theorem: the connective $\to$ internalises the conse-
quence relation $\vdash$, and $\otimes$ internalises the comma in the sequent:

PROPOSITION 1.4. *The following hold in any calculus with rules (Ax,$\to$ I,
$\to$ E, $\otimes$ I, $\otimes$ E):*

1. *$\Gamma, \psi \vdash \chi$ iff $\Gamma \vdash \psi \to \chi$.*
2. *$\Gamma, \phi, \psi \vdash \chi$ iff $\Gamma, \phi \otimes \psi \vdash \chi$.*

    Since **IŁL** has the exchange rule, we can extend this to $\phi_1, \ldots, \phi_n \vdash \psi$ iff
$\phi_{\pi_1} \otimes \ldots \otimes \phi_{\pi_n} \vdash \chi$ iff $\vdash (\phi_{\pi_1} \otimes \ldots \otimes \phi_{\pi_n}) \to \chi$ iff $\vdash \phi_{\pi_1} \to \ldots \to \phi_{\pi_n} \to \chi$,
where π is any permutation of $\{1, \ldots, n\}$.

    The natural deduction system **IŁL** is inspired by, and, as we will see in
Proposition 1.5, corresponds to, the Hilbert-style system **GBL**$_\text{ewf}$ of [2]

(A1) $\phi \rightarrow \phi$

(A2) $(\phi \rightarrow \psi) \rightarrow ((\psi \rightarrow \chi) \rightarrow (\phi \rightarrow \chi))$

(A3) $(\phi \otimes \psi) \rightarrow (\psi \otimes \phi)$

(A4) $(\phi \otimes \psi) \rightarrow \psi$

(A5) $(\phi \rightarrow (\psi \rightarrow \chi)) \rightarrow ((\phi \otimes \psi) \rightarrow \chi)$

(A6) $((\phi \otimes \psi) \rightarrow \chi)) \rightarrow (\phi \rightarrow (\psi \rightarrow \chi))$

(A7) $(\phi \otimes (\phi \rightarrow \psi)) \rightarrow (\phi \wedge \psi)$

(A8) $(\phi \wedge \psi) \rightarrow (\phi \otimes (\phi \rightarrow \psi))$

(A9) $(\phi \wedge \psi) \rightarrow (\psi \wedge \phi)$

(A10) $\phi \rightarrow (\phi \vee \psi)$

(A11) $\psi \rightarrow (\phi \vee \psi)$

(A12) $((\phi \rightarrow \psi) \wedge (\chi \rightarrow \psi)) \rightarrow ((\phi \vee \chi) \rightarrow \psi)$

(A13) $\bot \rightarrow \phi$

(R1) $\phi, \phi \rightarrow \psi \vdash_{\mathbf{GBL}_{\mathrm{ewf}}} \psi$

When we wish to stress the precise system in which a sequent $\Gamma \vdash \phi$ is derivable we use the system as a subscript of the provability sign, e.g. $\Gamma \vdash_{\mathbf{IŁL}} \phi$.

PROPOSITION 1.5. *The natural deduction system* **IŁL** *(Figure 1) has the same derivable formulas as the Hilbert-style system* $\mathbf{GBL}_{ewf}$ *of [2], and hence corresponds to it in the following sense*[2]

$$\psi_1, \ldots, \psi_n \vdash_{\mathbf{IŁL}} \phi \quad \text{iff} \quad \vdash_{\mathbf{GBL}_{ewf}} \psi_1 \rightarrow \ldots \rightarrow \psi_n \rightarrow \phi$$

PROOF. Left-to-right: The result follows by a simple induction on the structure of the natural deduction proof once we have established that each instance of a natural deduction rule translates to a theorem of $\mathbf{GBL}_{\mathrm{ewf}}$. We translate each sequent $\phi_1, \ldots, \phi_n \vdash \chi$ to the formula $[\phi_1, \ldots, \phi_n \vdash \chi] = \phi_1 \rightarrow \ldots \rightarrow \phi_n \rightarrow \chi$, and each rule

$$\frac{\Theta_1 \ldots \Theta_m}{\Psi}$$

to $[\Theta_1] \rightarrow \ldots \rightarrow [\Theta_m] \rightarrow [\Psi]$. For example, (Ax) translates to $\phi \rightarrow \phi$ (A1) and ($\rightarrow$I) to $(\chi_1 \rightarrow \ldots \chi_n \rightarrow \phi \rightarrow \psi) \rightarrow \chi_1 \rightarrow \ldots \chi_n \rightarrow \phi \rightarrow \psi$, which is also a form of (A1).

---

[2]Note that we use $\psi \otimes \chi$ where Bova and Montagna in [2] use $\psi \odot \chi$.

The analysis of many of the other rules is simplified if we introduce a relation between formulae:

$$\phi \leq \psi \text{ iff } \vdash_{\mathbf{GBL}_{\mathrm{ewf}}} \phi \to \psi$$

Omitting mention of use of (R1), (A1) says that this relation is reflexive, and (A2) that it is transitive. We therefore view it as generating a partial order on its equivalence classes. (A2) also implies that $\to$ is antitone in its first argument, and (A3), (A5) and (A6) now imply that the relation is monotone in its last argument, and that $\phi \to \psi \to \chi$ is equivalent to $\phi \otimes \psi \to \chi$, $\psi \otimes \phi \to \chi$ and $\psi \to \phi \to \chi$.

We can now derive the remaining rules fairly easily. For instance, consider the rule (DIV). Using the deduction theorem for **IŁL** from Proposition 1.4 we can assume that Γ is a single formula θ. We have to derive

$$(\theta \to \phi \to (\phi \to \psi) \to \chi) \to (\theta \to \psi \to (\psi \to \phi) \to \chi)$$

in $\mathbf{GBL}_{\mathrm{ewf}}$. We work with the provability ordering on formulae just introduced. $\theta \to \phi \to (\phi \to \psi) \to \chi$ is equivalent to $\theta \to (\phi \otimes (\phi \to \psi)) \to \xi$, and by (A8) plus monotonicity, implies (in fact is equivalent to) $\theta \to (\phi \wedge \psi) \to \xi$. The commutativity of $\wedge$ (A9) allows us to swap ϕ and ψ, i.e. $\theta \to (\psi \wedge \phi) \to \xi$. We now reverse the steps, using (A7), we obtain $\theta \to (\psi \otimes (\psi \to \phi)) \to \xi$. Finally, uncurrying (A6), gives us $\theta \to \psi \to (\psi \to \phi) \to \xi$ as desired.

Right-to-left: This follows by induction on the $\mathbf{GBL}_{\mathrm{ewf}}$ derivation of $\psi_1 \to \ldots \to \psi_n \to \phi$ once we have shown that each of the axioms of $\mathbf{GBL}_{\mathrm{ewf}}$ is a theorem of **IŁL**. The only non-trivial case is (A8), which states $(\phi \wedge \psi) \to \phi \otimes (\phi \to \psi)$, and requires an application of DIV. We can show that $\phi \wedge \psi \vdash \phi \otimes (\phi \to \psi)$ is derivable in **IŁL** as follows. First we show from ($\wedge$ E) that $\phi \to \phi \wedge \psi \vdash \phi \to \psi$:

$$
\cfrac{
  \cfrac{
    \cfrac{}{\phi \to \phi \wedge \psi \vdash \phi \to \phi \wedge \psi}\ \text{Ax} \qquad \cfrac{}{\phi \vdash \phi}\ \text{Ax}
  }{\phi \to \phi \wedge \psi, \phi \vdash \phi \wedge \psi}\ \to \text{E}
  \qquad
  \cfrac{
    \cfrac{
      \cfrac{\cfrac{}{\phi \wedge \psi \vdash \phi \wedge \psi}\ \text{Ax}}{\phi \wedge \psi \vdash \psi}\ \wedge \text{E}
    }{\vdash \phi \wedge \psi \to \psi}\ \to \text{I}
  }{}\ \to \text{E}
}{
  \cfrac{\phi \to \phi \wedge \psi, \phi \vdash \psi}{\phi \to \phi \wedge \psi \vdash \phi \to \psi}\ \to \text{I}
}
$$

We then use (DIV):

$$
\cfrac{
  \cfrac{
    \cfrac{\cfrac{}{\phi \wedge \psi \vdash \phi \wedge \psi}\ \text{Ax}}{\phi \wedge \psi \vdash \phi}\ \wedge \text{E}
  }{\vdash \phi \wedge \psi \to \phi}\ \to \text{I}
  \qquad
  \cfrac{
    \cfrac{
      \cfrac{
        \cfrac{}{\phi \vdash \phi}\ \text{Ax} \qquad \cfrac{\cdots}{\phi \to \phi \wedge \psi \vdash \phi \to \psi}\ \to \text{I}
      }{\phi, \phi \to \phi \wedge \psi \vdash \phi \otimes (\phi \to \psi)}\ \otimes \text{I}
    }{\phi \wedge \psi, \phi \wedge \psi \to \phi \vdash \phi \otimes (\phi \to \psi)}\ \text{DIV}
  }{\phi \wedge \psi \vdash (\phi \wedge \psi \to \phi) \to \phi \otimes (\phi \to \psi)}\ \to \text{I}
}{
  \phi \wedge \psi \vdash \phi \otimes (\phi \to \psi)
}\ \to \text{E}
$$

∎

REMARK 1.6. We stress at this point that Proposition 1.5 does not imply that the notions of logical consequence in **IŁL** and **GBL**$_{\text{ewf}}$ coincide, but rather that these two proof systems have the same set of derivable formulas. We have noted that **IŁL** satisfies the deduction theorem in the form $\Gamma, \phi \vdash_{\textbf{IŁL}} \psi$ iff $\Gamma \vdash_{\textbf{IŁL}} \phi \to \psi$. But in the standard notion of consequence for Hilbert-style systems, $\Gamma \vdash \phi$ is interpreted as "ϕ is derivable from axioms (A1)-(A13) + Γ using the cut rule (R1)". In this case formulas in Γ can be used multiple times to derive ϕ. This is reflected in the failure of the deduction theorem for **GBL**$_{\text{ewf}}$ ($\phi \otimes \phi$ is a consequence of ϕ, but we do not have $\vdash_{\textbf{GBL}_{\text{ewf}}} \phi \to \phi \otimes \phi$). Multiple uses of a hypothesis is not allowed in $\Gamma \vdash_{\textbf{IŁL}} \phi$ as **IŁL** lacks contraction.

2. Algebraic Semantics for GBL$_{\text{ewf}}$ and IŁL

Before we introduce our Kripke semantics for **IŁL**, we remind the reader about some standard algebraic semantics, for **GBL**$_{\text{ewf}}$ and **IŁL**.

In total we will consider four different classes of algebras: lattice-ordered monoids, residuated lattices, GBL algebras, and MV algebras.

DEFINITION 2.1. (Commutative Lattice-ordered monoid) A structure $\mathcal{A} = \langle A, \wedge, \vee, \otimes, 1 \rangle$ is a *commutative lattice-ordered monoid* if

- $\langle A, \wedge, \vee \rangle$ is a lattice
- $\langle A, \otimes, 1 \rangle$ is a commutative monoid
- $\otimes$ is monotonic increasing with respect to the lattice order on A.

There are a number of slightly different definitions of this concept in the literature, varying with the exact relationship required between the lattice and the monoid structure. Our definition is weak. The most common definition has that $\otimes$ distributes over $\vee$, and some definitions have that $\otimes$ distributes over both $\vee$ and $\wedge$.

DEFINITION 2.2. (Residuated lattice) A structure $\mathcal{A} = \langle A, \wedge, \vee, \otimes, 1, \to \rangle$ is called a *residuated lattice* if

- $\langle A, \wedge, \vee, \otimes, 1 \rangle$ is a commutative lattice-ordered monoid
- $x \otimes y \leq z$ if and only if $x \leq y \to z$

Standard definitions of residuated lattice do not include the fact that $\otimes$ preserves the lattice order. This, however follows from the residuation property: Suppose $x' \leq x$, then since $x \otimes y \leq x \otimes y$, $x \leq y \to (x \otimes y)$. Therefore $x' \leq y \to (x \otimes y)$ and hence $x' \otimes y \leq x \otimes y$.

From the categorical perspective, residuation is an adjunction. Any partial order can be viewed as a category in which the hom-sets have at most one element. The objects are the elements of the partial order and there is a morphism $a \to b$ if and only if $a \leq b$. In this case the residuation property says exactly that $(\ ) \otimes y \vdash y \to (\ )$.

DEFINITION 2.3. (GBL-algebra) A *GBL algebra* is a residuated lattice which satisfies the *divisibility property*[3]: if $x \leq y$ then $y \otimes (y \to x) = x$. This is equivalent to requiring that the residuated lattice satisfies the equation:

$$x \otimes (x \to y) = y \otimes (y \to x)$$

A GBL algebra is said to be *commutative* if $\otimes$ is a commutative operation. A GBL algebra is said to be *integral* if 1 is the top element of the lattice, i.e. $x \leq 1$ for all $x \in A$. In this case we also denote 1 by $\top$.
A GBL algebra is said to be *bounded* if the lattice has a bottom element $\bot$, i.e. $\bot \leq x$ for all $x \in A$.
From now on let us abbreviate "commutative, integral and bounded GBL algebras" by simply writing GBL_{ewf} algebras—where ewf abbreviate exchange, weakening and falsum.

GBL_{ewf} algebras provide an algebraic semantics for both $\mathbf{GBL}_{\text{ewf}}$ and $\mathbf{I\!L\!L}$. For $\mathbf{GBL}_{\text{ewf}}$, this is mentioned in various papers of Montagna et al, e.g. [2] says "$\mathbf{GBL}_{\text{ewf}}$ is strongly algebraizable ... Its equivalent algebraic semantics is the variety of commutative, integral and bounded GBL algebras". Let us formulate this semantics of $\mathbf{GBL}_{\text{ewf}}$, and describe the appropriate semantics for provability in $\mathbf{I\!L\!L}$.

DEFINITION 2.4. (Algebraic semantics for $\mathcal{L}_{\otimes}$) Given a GBL_{ewf} algebra $\mathcal{A} = \langle A, \wedge, \vee, \otimes, \top, \bot, \to \rangle$ and a mapping $h \colon \text{Atom} \to A$ from propositional variables to elements of $\mathcal{A}$ we can extend that mapping to mapping $[\![\phi]\!]_h^{\mathcal{A}} \in A$ on all formulas ϕ as

$$\begin{aligned}
[\![p]\!]_h^{\mathcal{A}} &:= h(p) \\
[\![\bot]\!]_h^{\mathcal{A}} &:= \bot \\
[\![\phi \wedge \psi]\!]_h^{\mathcal{A}} &:= [\![\phi]\!]_h^{\mathcal{A}} \wedge [\![\psi]\!]_h^{\mathcal{A}} \\
[\![\phi \vee \psi]\!]_h^{\mathcal{A}} &:= [\![\phi]\!]_h^{\mathcal{A}} \vee [\![\psi]\!]_h^{\mathcal{A}} \\
[\![\phi \otimes \psi]\!]_h^{\mathcal{A}} &:= [\![\phi]\!]_h^{\mathcal{A}} \otimes [\![\psi]\!]_h^{\mathcal{A}} \\
[\![\phi \to \psi]\!]_h^{\mathcal{A}} &:= [\![\phi]\!]_h^{\mathcal{A}} \to [\![\psi]\!]_h^{\mathcal{A}}
\end{aligned}$$

---

[3]Note that since $y \to x \leq y \to x$, it is always the case that $y \otimes (y \to x) \leq x$ (this is the counit of the adjunction defining residuation). The name "divisibility" property makes sense if one interprets $x \otimes y$ as multiplication $x \times y$, and $y \to x$ as division $\frac{x}{y}$. This is saying that if $0 \leq x \leq y \leq 1$ then $y \times \frac{x}{y} = x$. Note that if $y = 0$ then $x = 0$ as well.

A sequent $\phi_1, \ldots, \phi_n \vdash \psi$ is then said to be **GBL**$_{\text{ewf}}$-valid in $\mathcal{A}$, if for any mapping $h\colon \text{Atom} \to A$

$$\llbracket \phi_1 \rrbracket_h^{\mathcal{A}} = \top \text{ and } \ldots \text{ and } \llbracket \phi_n \rrbracket_h^{\mathcal{A}} = \top \quad \text{implies} \quad \llbracket \psi \rrbracket_h^{\mathcal{A}} = \top$$

A sequent $\phi_1, \ldots, \phi_n \vdash \psi$ is said to be **GBL**$_{\text{ewf}}$-valid if it is **GBL**$_{\text{ewf}}$-valid in all GBL$_{\text{ewf}}$ algebras.

A sequent $\phi_1, \ldots, \phi_n \vdash \psi$ is said to be **IŁL**-valid in $\mathcal{A}$, if for any $h\colon \text{Atom} \to A$

$$\llbracket \phi_1 \rrbracket_h^{\mathcal{A}} \otimes \ldots \otimes \llbracket \phi_n \rrbracket_h^{\mathcal{A}} \leq \llbracket \psi \rrbracket_h^{\mathcal{A}}$$

with the understanding that if the context $\Gamma = \phi_1, \ldots, \phi_n$ is empty ($n = 0$) then $\llbracket \phi_1 \rrbracket_h^{\mathcal{A}} \otimes \ldots \otimes \llbracket \phi_n \rrbracket_h^{\mathcal{A}} = \top$.

A sequent is said to be **IŁL**-valid if it is **IŁL**-valid in all GBL$_{\text{ewf}}$ algebras.

PROPOSITION 2.5. *A sequent $\Gamma \vdash \psi$ is **IŁL**-valid iff it is provable in **IŁL**.*

We note again that, even though the sequents of **GBL**$_{\text{ewf}}$ and **IŁL** are given different interpretations, these interpretations coincide for theorems, i.e. for sequents with empty context $\vdash \psi$.

DEFINITION 2.6. (MV algebra) A bounded GBL$_{\text{ewf}}$ algebra is called an *MV algebra* if the negation map ($\neg x = x \to \bot$) is an involution, i.e. $(x \to \bot) \to \bot = x$, for all x.

MV algebras, provide an algebraic semantics for (classical) Łukasiewicz logic. Here we are interested in a particular MV algebra which we will use in our Kripke semantics for **IŁL**:

DEFINITION 2.7. (Standard MV-chain) For $x \in [0, 1]$, let $\overline{x} := 1 - x$. The *standard MV-chain*, denoted $[0, 1]_{\text{MV}}$, is the *MV* algebra defined as follows: The domain of $[0, 1]_{\text{MV}}$ is the unit interval $[0, 1]$, with the constants and binary operations defined as

$$
\begin{aligned}
\top &:= 1 \\
\bot &:= 0 \\
x \wedge y &:= \min\{x, y\} \\
x \vee y &:= \max\{x, y\} \\
x \otimes y &:= \max\{0, \overline{\overline{x} + \overline{y}}\} \\
x \to y &:= \min\{1, \overline{y} - \overline{x}\}
\end{aligned}
$$

NOTE 1. $x \otimes y$ is equivalent to $\max\{0, x + y - 1\}$, and $x \to y$ is equivalent to $\min\{1, y - x + 1\}$.

LEMMA 2.8. *Recall that we are using the abbreviation $\overline{x} := 1 - x$. The following hold in the standard MV-chain $[0, 1]_{\text{MV}}$*

(i)   *For all $n \geq 2$, $x_1 \otimes \ldots \otimes x_n = \max\{0, \overline{\overline{x_1} + \ldots + \overline{x_n}}\}$.*

(ii) *$x_1 \otimes \ldots \otimes x_n \leq x_i$, for $i \in \{1, \ldots, n\}$*

(iii) *if $x \leq y \vee z$ and $u \otimes y \leq v$ and $u \otimes z \leq v$ then $u \otimes x \leq v$.*

(iv) *If $x \leq y$ and $u \leq y \rightarrow z$ then $x \otimes u \leq z$.*

(v)   *If $x \leq y$ and $z \leq w$ then $x \otimes z \leq y \otimes w$.*

(vi) *If $x \leq y$ and $v \otimes y \leq z$ then $v \otimes x \leq z$.*

(vii) *$x \otimes (x \rightarrow y) = y \otimes (y \rightarrow x)$.*

PROOF. We prove (i) in detail, the other properties follow easily from the fact that $[0,1]_{MV}$ is an MV algebra. By induction on n.
Basis: $n = 2$. By Definition 2.7.
Induction Step: Assume the result holds for $n > 2$, we show

$$x_1 \otimes \ldots \otimes x_n \otimes x_{n+1} = \max\{0, \overline{\overline{x_1} + \ldots + \overline{x_n} + \overline{x_{n+1}}}\}$$

Indeed

$$
\begin{aligned}
x_1 \otimes \ldots \otimes x_n \otimes x_{n+1} &= (x_1 \otimes \ldots \otimes x_n) \otimes x_{n+1} \\
&\overset{\text{D}2.7}{=} \max\{0, \overline{\overline{x_1 \otimes \ldots \otimes x_n} + \overline{x_{n+1}}}\} \\
&\overset{\text{IH}}{=} \max\{0, \overline{\overline{\max\{0, \overline{\overline{x_1} + \ldots + \overline{x_n}}\}} + \overline{x_{n+1}}}\} \\
&\overset{(*)}{=} \max\{0, \overline{\min\{1, \overline{x_1} + \ldots + \overline{x_n}\} + \overline{x_{n+1}}}\} \\
&\overset{(\dagger)}{=} \max\{0, \overline{\overline{x_1} + \ldots + \overline{x_n} + \overline{x_{n+1}}}\}
\end{aligned}
$$

using that

$(*)$ $\overline{\max\{0, \overline{a}\}} = \min\{1, a\}$

$(\dagger)$ $\max\{0, \overline{\min\{1, a\} + b}\} = \max\{0, \overline{a + b}\}$.   ∎

3.   Kripke Semantics for IŁL

The Kripke semantics for **IŁL** that we propose is based on the poset sums construction of [2] (see Section 3.2 for more details). We first need to define a particular class of functions a partial order $\mathcal{W} = \langle W, \succeq \rangle$ to the standard MV-chain:

DEFINITION 3.1. (Sloping functions) Let $\mathcal{W} = \langle W, \succeq \rangle$ be a partial order, and let $v \succ w := v \succeq w$ and $v \neq w$. A function $f \colon W \rightarrow [0,1]$ is said to be a *sloping function* if $f(w) > \bot$ implies $\forall v \succ w(f(v) = \top)$.

The above implies that if $f \colon W \rightarrow [0,1]$ is a sloping function and $f(w) < \top$ then $\forall v \prec w(f(v) = \bot)$. That is, along any increasing chain $w_1 \prec w_2 \prec$

$\ldots \prec w_n$, there can only be at most one point i such that $\bot < f(w_i) < \top$, and for $j < i$ we must have $f(w_j) = \bot$, and for $j > i$ we must have $f(w_j) = \top$.

LEMMA 3.2. *If* $f\colon W \to [0,1]$ *and* $g\colon W \to [0,1]$ *are sloping functions, then the following functions are also sloping functions:*

$$(f \wedge g)(w) := \min\{fw, gw\}$$
$$(f \vee g)(w) := \max\{fw, gw\}$$
$$(f \otimes g)(w) := \max\{0, \overline{\overline{fw} + \overline{gw}}\}$$

PROOF. Let f, g be sloping functions. Let us consider each case:

- $f \wedge g$. Assume $(f \wedge g)(w) > \bot$, i.e. $\min\{fw, gw\} > \bot$. This implies that we have both $fw > \bot$ and $gw > \bot$. But since f and g are assumed to be sloping functions, we get that $\forall v \succ w(f(v) = \top)$ and $\forall v \succ w(g(v) = \top)$, from which it follows that $\forall v \succ w(\min\{f(v), g(v)\} = \top)$.

- $f \vee g$. Assume $(f \vee g)(w) > \bot$ i.e. $\max\{fw, gw\} > \bot$. This implies that we have at least one of $fw > \bot$ or $gw > \bot$. In case $fw > \bot$, f is a sloping function by hypothesis, so we have $\forall v \succ w(f(v) = \top)$ from which it follows $\forall v \succ w(\max\{f(v), g(v)\} = \top)$. The case of $gw > \bot$ is similar.

- $f \otimes g$. Assume $(f \otimes g)(w) > \bot$ i.e. $\max\{0, \overline{\overline{fw} + \overline{gw}}\} > 0$. This means $\max\{0, \overline{\overline{fw} + \overline{gw}}\} = \max\{0, f(w) + g(w) - 1\} > 0$; and hence $f(w) + g(w) - 1 > 0$. This implies that neither $f(w) = \bot$ nor $g(w) = \bot$, i.e. we have both $f(w) > \bot$ and $g(w) > \bot$. Since both $f(w), g(w)$ are sloping functions by hypothesis $\forall v \succ w(f(v) = \top)$ and $\forall v \succ w(g(v) = \top)$. So $\forall v \succ w \max\{0, f(v) + g(v) - 1\} = \max\{0, \top + \top - 1\} = \max\{0, \top + 0\} = \top$, as desired. ∎

DEFINITION 3.3. A *Bova–Montagna structure* (or BM-structure) is a pair $\mathcal{M} = \langle \mathcal{W}, \Vdash^{\mathrm{BM}} \rangle$ where $\mathcal{W} = \langle W, \succeq \rangle$ is a poset, and $\Vdash^{\mathrm{BM}}$ is an infix operator (on worlds and propositional variables) taking values in $[0,1]_{\mathrm{MV}}$, i.e. $(w \Vdash^{\mathrm{BM}} p) \in [0,1]_{\mathrm{MV}}$, such that for any propositional variable p the function $\lambda w.(w \Vdash^{\mathrm{BM}} p)\colon W \to [0,1]$ is a sloping function.

DEFINITION 3.4. Let $\lfloor \cdot \rfloor$ be the usual "floor" operation on the standard MV-chain $[0,1]_{\mathrm{MV}}$, corresponding to the case distinction

$$\lfloor x \rfloor := \begin{cases} \top & \text{if } x = \top \\ \bot & \text{if } x < \top \end{cases}$$

which is known as the "Monteiro–Baaz Δ-operator". Given a (not necessarily sloping) function $f \colon W \to [0,1]$ and a $w \in W$, let us write $\lfloor \mathbf{inf} \rfloor_{v \succeq w}$ for the following construction:

$$\lfloor \mathbf{inf} \rfloor_{v \succeq w} f(v) \;:=\; \min\{f(w), \inf_{v \succ w} \lfloor f(v) \rfloor\}$$

where $\inf_{v \succ w} \lfloor f(v) \rfloor$ is the infimum of the set $\{\lfloor f(v) \rfloor \;:\; v \succ w\} \subseteq [0,1]$.

LEMMA 3.5. *This definition of* $\lfloor \mathbf{inf} \rfloor_{v \succeq w}$ *can also be equivalently written as*

$$\lfloor \mathbf{inf} \rfloor_{v \succeq w} f(v) := \begin{cases} f(w) & \text{if } \forall v \succ w(f(v) = \top) \\ \bot & \text{if } \exists v \succ w(f(v) < \top) \end{cases}$$

and for any $f \colon W \to [0,1]$ *the function* $\lambda w.\lfloor \mathbf{inf} \rfloor_{v \succeq w} f(v)$ *is a sloping function.*

PROOF. First let us show that this is an equivalent definition. Consider two cases:

Case 1. $\forall v \succ w(f(v) = \top)$. In this case $\inf_{v \succ w} \lfloor f(v) \rfloor = \top$ and hence

$$\lfloor \mathbf{inf} \rfloor_{v \succeq w} f(v) \;=\; \min\{f(w), \top\} \;=\; f(w)$$

Case 2. $\exists v \succ w(f(v) < \top)$. In this case $\inf_{v \succ w} \lfloor f(v) \rfloor = \bot$

$$\lfloor \mathbf{inf} \rfloor_{v \succeq w} f(v) \;=\; \min\{f(w), \bot\} \;=\; \bot$$

In order to see that $\lambda w.\lfloor \mathbf{inf} \rfloor_{v \succeq w} f(v)$ is a sloping function, assume that for some w we have $\lfloor \mathbf{inf} \rfloor_{v \succeq w} f(v) > \bot$, and let $w' \succ w$. By definition we have that $\forall v \succ w(f(v) = \top)$, and hence $f(w') = \top$ and $\forall v \succ w'(f(v) = \top)$, which implies $\lfloor \mathbf{inf} \rfloor_{v \succeq w'} f(v) = \top$. ∎

DEFINITION 3.6. (Kripke Semantics for $\mathcal{L}_{\otimes}$) Given a BM-structure

$$\mathcal{M} = \langle \mathcal{W}, \Vdash^{\mathrm{BM}} \rangle$$

the valuation function $w \Vdash^{\mathrm{BM}} p$ on propositional variables p can be extended to all $\mathcal{L}_{\otimes}$-formulas as:

$$\begin{aligned}
w &\Vdash^{\mathrm{BM}} \bot & &:= \bot \\
w &\Vdash^{\mathrm{BM}} \phi \wedge \psi & &:= (w \Vdash^{\mathrm{BM}} \phi) \wedge (w \Vdash^{\mathrm{BM}} \psi) \\
w &\Vdash^{\mathrm{BM}} \phi \vee \psi & &:= (w \Vdash^{\mathrm{BM}} \phi) \vee (w \Vdash^{\mathrm{BM}} \psi) \\
w &\Vdash^{\mathrm{BM}} \phi \otimes \psi & &:= (w \Vdash^{\mathrm{BM}} \phi) \otimes (w \Vdash^{\mathrm{BM}} \psi) \\
w &\Vdash^{\mathrm{BM}} \phi \to \psi & &:= \lfloor \mathbf{inf} \rfloor_{v \succeq w}((v \Vdash^{\mathrm{BM}} \phi) \to (v \Vdash^{\mathrm{BM}} \psi))
\end{aligned}$$

where the operations on the right-hand side are the operations on the standard MV-chain $[0,1]_{\mathrm{MV}}$, and $\lfloor \mathbf{inf} \rfloor_{v \succeq w}$ as in Definition 3.4.

LEMMA 3.7. *For any formula* ϕ *the function* $\lambda w.(w \Vdash^{\mathrm{BM}} \phi) \colon W \to [0,1]$ *is a sloping function.*

PROOF. By induction on the complexity of the formula ϕ. The cases for $\psi \vee \xi, \psi \wedge \xi$ and $\psi \otimes \xi$ follow directly from Lemma 3.2. The case for $\psi \rightarrow \xi$ follows from Lemma 3.5. ∎

We can now generalise the monotonicity property of intuitionistic logic to intuitionistic Łukasiewicz logic **IŁL**:

COROLLARY 3.8. (Monotonicity) *The following (generalised) monotonicity property holds for all $\mathcal{L}_\otimes$-formulas ϕ, i.e.*

$$\text{if } v \succeq w \text{ then } (v \Vdash^K \phi) \geq (w \Vdash^K \phi)$$

PROOF. This follows from the observation that sloping functions are in particular monotone functions. ∎

DEFINITION 3.9. Let $\Gamma = \psi_1, \ldots, \psi_n$. Consider the following definitions:

- We say that a sequent $\Gamma \vdash \phi$ holds in a BM-structure $\mathcal{M}$ (written $\Gamma \Vdash^{BM}_{\mathcal{M}} \phi$) if for all $w \in W$ we have

$$(w \Vdash^{BM} \psi_1 \otimes \ldots \otimes \psi_n) \leq (w \Vdash^{BM} \phi)$$

- A sequent $\Gamma \vdash \phi$ is said to be valid under the Kripke semantics for $\mathcal{L}_\otimes$ (written $\Gamma \Vdash^{BM} \phi$) if $\Gamma \Vdash^{BM}_{\mathcal{M}} \phi$ for all BM-structures $\mathcal{M}$.

We will prove that this semantics is sound and complete for **IŁL**, i.e. a sequent $\Gamma \vdash \phi$ is provable in **IŁL** iff it is valid in all BM-structures. But first let us show that the semantics presented above is a direct generalisation of Kripke's original semantics.

3.1. BM-Structures Generalise Kripke Structures

Bova–Montagna structures generalise Kripke structures, i.e. Kripke structures are a particular case of BM-structures, when the valuations $w \models p \in [0,1]_{MV}$ are always in the finite set $\{\top, \bot\}$. These can then be identified with the Booleans. Therefore, any Kripke structure $\mathcal{K} = \langle \mathcal{W}, \Vdash^K \rangle$ can be seen as a BM-structure $\mathcal{M} = \langle \mathcal{W}, \Vdash^{BM} \rangle$, by defining

$$w \Vdash^{BM} p = \begin{cases} \top & \text{if } w \Vdash^K p \\ \bot & \text{if } w \nVdash^K p \end{cases}$$

for all $w \in W$ and propositional variables p. Recall that $\mathcal{L} \subset \mathcal{L}_\otimes$, so any $\mathcal{L}$-formula is also an $\mathcal{L}_\otimes$-formula.

THEOREM 3.10. *For any Kripke structure $\mathcal{K} = \langle \mathcal{W}, \Vdash^K \rangle$ and corresponding BM-structure $\mathcal{M} = \langle \mathcal{W}, \Vdash^{BM} \rangle$ we have that*

$$w \Vdash^{\mathrm{K}} \phi \quad \mathit{iff} \quad (w \Vdash^{\mathrm{BM}} \phi) = \top$$

for all $\mathcal{L}$-formula ϕ.

PROOF. It is easy to check that, when restricted to Kripke structures, we have $(v \Vdash^{\mathrm{BM}} \phi) \in \{\top, \bot\}$ for all formulas ϕ. Hence, the result above can be proven by a simple induction on the complexity of the formula ϕ.

Basis: If ϕ is an atomic formulas the result is immediate.

Induction step: Suppose the result holds for all sub-formulas of ϕ:

Case 1. $\phi = \psi \wedge \chi$. We have:

$$
\begin{aligned}
w \Vdash^{\mathrm{K}} \psi \wedge \chi \ &\equiv\ (w \Vdash^{\mathrm{K}} \psi) \text{ and } (w \Vdash^{\mathrm{K}} \chi) \\
&\overset{(\mathrm{IH})}{\Leftrightarrow}\ (w \Vdash^{\mathrm{BM}} \psi) = \top \text{ and } (w \Vdash^{\mathrm{BM}} \chi) = \top \\
&\Leftrightarrow\ \min\{w \Vdash^{\mathrm{BM}} \psi, w \Vdash^{\mathrm{BM}} \chi\} = \top \\
&\equiv\ (w \Vdash^{\mathrm{BM}} \psi \wedge \chi) = \top
\end{aligned}
$$

Case 2. $\phi = \psi \vee \chi$. We have:

$$
\begin{aligned}
w \Vdash^{\mathrm{K}} \psi \vee \chi \ &\equiv\ (w \Vdash^{\mathrm{K}} \psi) \text{ or } (w \Vdash^{\mathrm{K}} \chi) \\
&\overset{(\mathrm{IH})}{\Leftrightarrow}\ (w \Vdash^{\mathrm{BM}} \psi) = \top \text{ or } (w \Vdash^{\mathrm{BM}} \chi) = \top \\
&\Leftrightarrow\ \max\{w \Vdash^{\mathrm{BM}} \psi, w \Vdash^{\mathrm{BM}} \chi\} = \top \\
&\equiv\ (w \Vdash^{\mathrm{BM}} \psi \vee \chi) = \top
\end{aligned}
$$

Case 3. $\phi = \psi \to \chi$. We have

(i) $(v \Vdash^{\mathrm{BM}} \psi) = \top$ implies $(v \Vdash^{\mathrm{BM}} \chi) = \top$ iff $(v \Vdash^{\mathrm{BM}} \psi) \to (v \Vdash^{\mathrm{BM}} \chi) = \top$

(ii) $\lfloor (v \Vdash^{\mathrm{BM}} \psi) \to (v \Vdash^{\mathrm{BM}} \chi) \rfloor = (v \Vdash^{\mathrm{BM}} \psi) \to (v \Vdash^{\mathrm{BM}} \chi)$, i.e. the "floor operation" is unnecessary, and $\lfloor \inf \rfloor_{v \succeq w}$ becomes the standard $\inf_{v \succeq w}$ operation.

Therefore:

$$
\begin{aligned}
w \Vdash^{\mathrm{K}} \psi \to \chi \ &\equiv\ \forall v \succeq w((v \Vdash^{\mathrm{K}} \psi) \text{ implies } (v \Vdash^{\mathrm{K}} \chi)) \\
&\overset{(\mathrm{IH})}{\Leftrightarrow}\ \forall v \succeq w((v \Vdash^{\mathrm{BM}} \psi) = \top \text{ implies } (v \Vdash^{\mathrm{BM}} \chi) = \top) \\
&\overset{(i)}{\Leftrightarrow}\ \forall v \succeq w((v \Vdash^{\mathrm{BM}} \psi) \to (v \Vdash^{\mathrm{BM}} \chi) = \top) \\
&\overset{(ii)}{\Leftrightarrow}\ \lfloor \inf \rfloor_{v \succeq w}((v \Vdash^{\mathrm{BM}} \psi) \to (v \Vdash^{\mathrm{BM}} \chi)) = \top \\
&\equiv\ (w \Vdash^{\mathrm{BM}} \psi \to \chi) = \top
\end{aligned}
$$

which concludes the proof.   ■

3.2.   BM-Structures and Poset Sums

Recall that a *poset sum* (cf. [2, Def. 2] and [7]) is defined over a poset $\mathcal{W} = \langle W, \succeq \rangle$, as the algebra $\mathbf{A}_{\mathcal{W}}$ of signature $\mathcal{L}_{\otimes}$ whose elements are sloping functions $f \colon W \to [0, 1]$ and operations are defined as

$$
\begin{aligned}
(\bot)(w) &:= \bot \\
(f_1 \wedge f_2)(w) &:= \min\{f_1 w, f_2 w\} \\
(f_1 \vee f_2)(w) &:= \max\{f_1 w, f_2 w\} \\
(f_1 \otimes f_2)(w) &:= \max\{0, \overline{\overline{f_1 w} + \overline{f_2 w}}\} \\
(f_1 \rightarrow f_2)(w) &:= \begin{cases} f_1(w) \rightarrow f_2(w) & \text{if } \forall v \succ w(f_1(v) \leq f_2(v)) \\ \bot & \text{if } \exists v \succ w(f_1(v) > f_2(v)) \end{cases}
\end{aligned}
$$

Since f_1 and f_2 are sloping functions, we have that

$$
\forall v \succ w(f_1(v) \leq f_2(v)) \quad \Leftrightarrow \quad \forall v \succ w((f_1(v) \rightarrow f_2(v)) = \top)
$$

Therefore, this last clause of the definition can be simplified to

$$
(f_1 \rightarrow f_2)(w) := \lfloor \mathbf{inf} \rfloor_{v \succeq w}(f_1(v) \rightarrow f_2(v))
$$

DEFINITION 3.11. (Poset sum semantics for $\mathcal{L}_\otimes$) Let $\mathcal{W} = \langle W, \succeq \rangle$ be a fixed poset, and $\mathbf{A}_\mathcal{W}$ be the poset sum described above. Given $h\colon \text{Atom} \rightarrow \mathbf{A}_\mathcal{W}$ an assignment of atomic formulas to elements of $\mathbf{A}_\mathcal{W}$, any formula ϕ can be mapped to an element $[\![\phi]\!]_h \in \mathbf{A}_\mathcal{W}$ as follows:

$$
\begin{aligned}
[\![p]\!]_h &:= h(p) \quad \text{(for atomic formulas } p) \\
[\![\bot]\!]_h &:= \bot \\
[\![\phi \wedge \psi]\!]_h &:= [\![\phi]\!]_h \wedge [\![\psi]\!]_h \\
[\![\phi \vee \psi]\!]_h &:= [\![\phi]\!]_h \vee [\![\psi]\!]_h \\
[\![\phi \otimes \psi]\!]_h &:= [\![\phi]\!]_h \otimes [\![\psi]\!]_h \\
[\![\phi \rightarrow \psi]\!]_h &:= [\![\phi]\!]_h \rightarrow [\![\psi]\!]_h
\end{aligned}
$$

A formula ϕ is said to be *valid in* $\mathbf{A}_\mathcal{W}$ *under* h if for every $w \in W$

$$
[\![\phi]\!]_h^{\mathbf{A}_\mathcal{W}}(w) = \top
$$

(which is 1 in $[0,1]_{\mathrm{MV}}$). A formula ϕ is said to be *valid in* $\mathbf{A}_\mathcal{W}$ if it is valid in $\mathbf{A}_\mathcal{W}$ under h for any possible mapping $h\colon \text{Atom} \rightarrow \mathbf{A}_\mathcal{W}$.

Let us conclude this section by observing that given a poset sum $\mathbf{A}_\mathcal{W}$ (for a poset $\mathcal{W} = \langle W, \succeq \rangle$) and a mapping $h\colon \text{Atom} \rightarrow \mathbf{A}_\mathcal{W}$ of atomic formulas to elements of $\mathbf{A}_\mathcal{W}$, we can obtain a BM structure $\mathcal{M}^{\mathbf{A}_\mathcal{W}} = \langle \mathcal{W}, \Vdash_h^{\mathrm{BM}} \rangle$, by taking

$$
w \Vdash_h^{\mathrm{BM}} p \ := \ h(p)(w)
$$

recalling that $h(p)\colon W \rightarrow [0,1]$ is a sloping function.

PROPOSITION 3.12. *Let* $\mathbf{A}_\mathcal{W}$ *be the poset sum over* $\mathcal{W}$, *and* $h\colon \text{Atom} \rightarrow \mathbf{A}_\mathcal{W}$ *be a fixed mapping of atomic formulas to elements of* $\mathcal{W}$. *Let* $\mathcal{M}^{\mathbf{A}_\mathcal{W}}$ *be the BM-structure defined above. Then, for any formula* ϕ

$$w \Vdash_h^{\mathrm{BM}} \phi \;\; = \;\; \llbracket \phi \rrbracket_h^{\mathbf{A}_{\mathcal{W}}}(w)$$

PROOF. The above can be shown by a straightforward induction on the complexity of ϕ. ∎

Therefore, one can always transform an interpretation of $\mathcal{L}_\otimes$ formulas in a poset sum $\mathbf{A}_{\mathcal{W}}$ into a Kripke semantics (on the Kripke frame $\mathcal{W}$) for $\mathcal{L}_\otimes$ formulas.

3.3. Soundness

Let us now prove the soundness of the Kripke semantics for **IŁL**.

THEOREM 3.13. *(Soundness) If* $\Gamma \vdash_{\mathbf{IŁL}} \phi$ *then* $\Gamma \Vdash^{\mathrm{BM}} \phi$.

PROOF. By induction on the derivation of $\Gamma \vdash \phi$. Assume $\Gamma = \psi_1, \ldots, \psi_n$ and let $\otimes\Gamma := \psi_1 \otimes \ldots \otimes \psi_n$. Fix a BM-structure $\mathcal{M} = \langle \mathcal{W}, \Vdash^{\mathrm{BM}} \rangle$ with $\mathcal{W} = \langle W, \succeq \rangle$, and let $w \in W$.

(Axiom) $\Gamma, \phi \vdash \phi$. By Definition 3.9, we need to show:

$$w \Vdash^{\mathrm{BM}} (\otimes\Gamma) \otimes \phi \;\; \overset{(\mathrm{D}.3.6)}{\equiv} \;\; (w \Vdash^{\mathrm{BM}} \psi_1) \otimes \ldots (w \Vdash^{\mathrm{BM}} \psi_n) \otimes (w \Vdash^{\mathrm{BM}} \phi)$$
$$\overset{(\mathrm{L}.2.8(ii))}{\leq} \;\; w \Vdash^{\mathrm{BM}} \phi$$

($\wedge$I) By IH we have $(w \Vdash^{\mathrm{BM}} \otimes\Gamma) \leq (w \Vdash^{\mathrm{BM}} \phi)$ and $(w \Vdash^{\mathrm{BM}} \otimes\Gamma) \leq (w \Vdash^{\mathrm{BM}} \psi)$. Hence

$$(w \Vdash^{\mathrm{BM}} \otimes\Gamma) \;\; \leq \;\; \min\{w \Vdash^{\mathrm{BM}} \phi, w \Vdash^{\mathrm{BM}} \psi\} \;\; \equiv \;\; w \Vdash^{\mathrm{BM}} \phi \wedge \psi$$

($\wedge$E) By IH we have $(w \Vdash^{\mathrm{BM}} \otimes\Gamma) \leq (w \Vdash^{\mathrm{BM}} \phi \wedge \psi)$, i.e.

$$(w \Vdash^{\mathrm{BM}} \otimes\Gamma) \leq \min\{w \Vdash^{\mathrm{BM}} \phi, w \Vdash^{\mathrm{BM}} \psi\}$$

This implies both $(w \Vdash^{\mathrm{BM}} \otimes\Gamma) \leq (w \Vdash^{\mathrm{BM}} \phi)$ and $(w \Vdash^{\mathrm{BM}} \otimes\Gamma) \leq (w \Vdash^{\mathrm{BM}} \psi)$.

($\vee$I) By IH we have $(w \Vdash^{\mathrm{BM}} \otimes\Gamma) \leq (w \Vdash^{\mathrm{BM}} \phi)$. Therefore

$$(w \Vdash^{\mathrm{BM}} \otimes\Gamma) \;\; \leq \;\; \max\{w \Vdash^{\mathrm{BM}} \phi, w \Vdash^{\mathrm{BM}} \psi\} \;\; \equiv \;\; w \Vdash^{\mathrm{BM}} \phi \vee \psi$$

($\vee$E) By IH we have

- $w \Vdash^{\mathrm{BM}} \otimes\Gamma \leq \max\{w \Vdash^{\mathrm{BM}} \phi, w \Vdash^{\mathrm{BM}} \psi\}$
- $(w \Vdash^{\mathrm{BM}} (\otimes\Delta) \otimes \phi) \leq (w \Vdash^{\mathrm{BM}} \chi)$
- $(w \Vdash^{\mathrm{BM}} (\otimes\Delta) \otimes \psi) \leq (w \Vdash^{\mathrm{BM}} \chi)$

By Lemma 2.8 (*iii*), these imply $(w \Vdash^{\mathrm{BM}} (\otimes\Gamma) \otimes (\otimes\Delta)) \leq w \Vdash^{\mathrm{BM}} \chi$.
(→I) By IH we have $(w \Vdash^{\mathrm{BM}} (\otimes\Gamma) \otimes \phi) \leq (w \Vdash^{\mathrm{BM}} \psi)$, for all $w \in W$. By the adjointness property we get

$$(w \Vdash^{\mathrm{BM}} \otimes\Gamma) \leq (w \Vdash^{\mathrm{BM}} \phi) \rightarrow (w \Vdash^{\mathrm{BM}} \psi)$$

for all $w \in W$. Fix $w \in W$, and let us consider two cases. First, if for some $v \succ w$ we have $(v \Vdash^{\mathrm{BM}} \phi) \rightarrow (v \Vdash^{\mathrm{BM}} \psi) < \top$, then we must have that $(v \Vdash^{\mathrm{BM}} \otimes\Gamma) < \top$, and hence $(w \Vdash^{\mathrm{BM}} \otimes\Gamma) = \bot$, and trivially

$$(w \Vdash^{\mathrm{BM}} \otimes\Gamma) \leq \lfloor \mathbf{inf} \rfloor_{v \succeq w} ((v \Vdash^{\mathrm{BM}} \phi) \rightarrow (v \Vdash^{\mathrm{BM}} \psi))$$

If on the other hand, $(v \Vdash^{\mathrm{BM}} \phi) \rightarrow (v \Vdash^{\mathrm{BM}} \psi) = \top$ for all $v \succ w$, then

$$\lfloor \mathbf{inf} \rfloor_{v \succeq w} ((v \Vdash^{\mathrm{BM}} \phi) \rightarrow (v \Vdash^{\mathrm{BM}} \psi)) = (w \Vdash^{\mathrm{BM}} \phi) \rightarrow (w \Vdash^{\mathrm{BM}} \psi)$$

and we indeed have $(w \Vdash^{\mathrm{BM}} \otimes\Gamma) \leq (w \Vdash^{\mathrm{BM}} \phi) \rightarrow (w \Vdash^{\mathrm{BM}} \psi)$.
(→E) By IH we have

- $(w \Vdash^{\mathrm{BM}} \otimes\Gamma) \leq w \Vdash^{\mathrm{BM}} \phi$
- $(w \Vdash^{\mathrm{BM}} \otimes\Delta) \leq \lfloor \mathbf{inf} \rfloor_{v \succeq w} ((v \Vdash^{\mathrm{BM}} \phi) \rightarrow (v \Vdash^{\mathrm{BM}} \psi))$

We again consider two cases. First, if for some $v \succ w$ we have $(v \Vdash^{\mathrm{BM}} \phi) \rightarrow (v \Vdash^{\mathrm{BM}} \psi) < \top$, then

$$\lfloor \mathbf{inf} \rfloor_{v \succeq w} ((v \Vdash^{\mathrm{BM}} \phi) \rightarrow (v \Vdash^{\mathrm{BM}} \psi)) = \bot$$

and hence $(w \Vdash^{\mathrm{BM}} \otimes\Delta) = \bot$ and $(w \Vdash^{\mathrm{BM}} (\otimes\Gamma) \otimes (\otimes\Delta)) \leq w \Vdash^{\mathrm{BM}} \psi$. If on the other hand, $(v \Vdash^{\mathrm{BM}} \phi) \rightarrow (v \Vdash^{\mathrm{BM}} \psi) = \top$ for all $v \succ w$, then

$$\lfloor \mathbf{inf} \rfloor_{v \succeq w} ((v \Vdash^{\mathrm{BM}} \phi) \rightarrow (v \Vdash^{\mathrm{BM}} \psi)) = (w \Vdash^{\mathrm{BM}} \phi) \rightarrow (w \Vdash^{\mathrm{BM}} \psi)$$

so that our assumption is $(w \Vdash^{\mathrm{BM}} \otimes\Delta) \leq (w \Vdash^{\mathrm{BM}} \phi) \rightarrow (w \Vdash^{\mathrm{BM}} \psi)$. By Lemma 2.8 (*iv*) we obtain $(w \Vdash^{\mathrm{BM}} (\otimes\Gamma) \otimes (\otimes\Delta)) \leq w \Vdash^{\mathrm{BM}} \psi$.
(⊥E) By IH we have $(w \Vdash^{\mathrm{BM}} \otimes\Gamma) \leq w \Vdash^{\mathrm{BM}} \bot$. Since $(w \Vdash^{\mathrm{BM}} \bot) = 0$, we have that $(w \Vdash^{\mathrm{BM}} \otimes\Gamma) = 0$, which implies $(w \Vdash^{\mathrm{BM}} \otimes\Gamma) \leq (w \Vdash^{\mathrm{BM}} \phi)$, for any ϕ.
(⊗I) By IH $(w \Vdash^{\mathrm{BM}} \otimes\Gamma) \leq w \Vdash^{\mathrm{BM}} \phi$ and $(w \Vdash^{\mathrm{BM}} \otimes\Delta) \leq w \Vdash^{\mathrm{BM}} \psi$. By Lemma 2.8 (*v*) we have

$$(w \Vdash^{\mathrm{BM}} \otimes\Gamma) \otimes (w \Vdash^{\mathrm{BM}} \otimes\Delta) \leq (w \Vdash^{\mathrm{BM}} \phi) \otimes (w \Vdash^{\mathrm{BM}} \psi)$$

and hence

$$(w \Vdash^{\mathrm{BM}} (\otimes \Gamma) \otimes (\otimes \Delta)) \leq (w \Vdash^{\mathrm{BM}} \phi \otimes \psi)$$

($\otimes$E) By IH we have

- $(w \Vdash^{\mathrm{BM}} \otimes \Gamma) \leq (w \Vdash^{\mathrm{BM}} \phi) \otimes (w \Vdash^{\mathrm{BM}} \psi)$
- $(w \Vdash^{\mathrm{BM}} \otimes \Delta) \otimes (w \Vdash^{\mathrm{BM}} \phi) \otimes (w \Vdash^{\mathrm{BM}} \psi) \leq w \Vdash^{\mathrm{BM}} \chi$

By Lemma 2.8 (vi), we have

$$(w \Vdash^{\mathrm{BM}} \otimes \Gamma) \otimes (w \Vdash^{\mathrm{BM}} \otimes \Delta) \leq w \Vdash^{\mathrm{BM}} \chi$$

i.e. $(w \Vdash^{\mathrm{BM}} (\otimes \Gamma) \otimes (\otimes \Delta)) \leq w \Vdash^{\mathrm{BM}} \chi$.

(DIV) It is sufficient to show that

$$w \Vdash^{\mathrm{BM}} (\phi \to \psi) \otimes \phi \quad \leq \quad w \Vdash^{\mathrm{BM}} (\psi \to \phi) \otimes \psi$$

i.e.

$$(w \Vdash^{\mathrm{BM}} \phi \to \psi) \otimes (w \Vdash^{\mathrm{BM}} \phi) \ \leq \ (w \Vdash^{\mathrm{BM}} \psi \to \phi) \otimes (w \Vdash^{\mathrm{BM}} \psi)$$

We consider two cases:

Case 1. $w \Vdash^{\mathrm{BM}} \phi = \bot$. In this case the result is immediate.

Case 2. $w \Vdash^{\mathrm{BM}} \phi > \bot$. This implies that $\forall v \succ w(w \Vdash^{\mathrm{BM}} \phi = \top)$, and hence $\forall v \succ w((w \Vdash^{\mathrm{BM}} \psi \to w \Vdash^{\mathrm{BM}} \phi) = \top)$, so

$$w \Vdash^{\mathrm{BM}} \psi \to \phi \quad = \quad (w \Vdash^{\mathrm{BM}} \psi) \to (w \Vdash^{\mathrm{BM}} \phi)$$

Since

$$w \Vdash^{\mathrm{BM}} \phi \to \psi, \phi \ \leq \ ((w \Vdash^{\mathrm{BM}} \phi) \to (w \Vdash^{\mathrm{BM}} \psi)) \otimes (w \Vdash^{\mathrm{BM}} \phi)$$

it remains to show that

$$\begin{aligned} ((w \Vdash^{\mathrm{BM}} \phi) &\to (w \Vdash^{\mathrm{BM}} \psi)) \otimes (w \Vdash^{\mathrm{BM}} \phi) \\ &\leq ((w \Vdash^{\mathrm{BM}} \psi) \to (w \Vdash^{\mathrm{BM}} \phi)) \otimes (w \Vdash^{\mathrm{BM}} \psi) \end{aligned}$$

which follows from Lemma 2.8 (vii).   ∎

3.4.   Completeness

We conclude this section by arguing that the Kripke semantics above is also complete, referring to Proposition 3.12 which relates poset sums and BM-structures, and the completeness results of Jipsen and Montagna [7] and Bova and Montagna [2] for poset sums and GBL_{ewf} algebras.

THEOREM 3.14. (Completeness) *If* $\Gamma \Vdash^{BM} \phi$ *then* $\Gamma \vdash_{I\!L\!L} \phi$.

PROOF. Let $\Gamma \equiv \psi_1, \ldots, \psi_n$. Suppose

$$\Gamma \not\vdash_{I\!L\!L} \phi$$

By Proposition 1.5 it follows that

$$\not\vdash_{GBL_{ewf}} \psi_1 \to \ldots \to \psi_n \to \phi$$

By the algebraic completeness result for GBL_{ewf} algebras with respect to the proof system $\mathbf{GBL_{ewf}}$ (cf. Section 2), it follows that for some GBL_{ewf} algebra $\mathcal{G}$ and some mapping $h \colon \mathrm{Atom} \to G$ from propositional variables to elements of $\mathcal{G}$, we have

$$[\![\psi_1 \to \ldots \to \psi_n \to \phi]\!]_h^{\mathcal{G}} \neq \top$$

By ([2, Theorem 1]—see also [7]) there exists a finite poset $\mathcal{W} = \langle W, \succeq \rangle$ and an assignment $h' \colon \mathrm{Atom} \to [0, 1]$ of atomic formulas to elements of the poset sum $\mathbf{A}_{\mathcal{W}}$, such that for some $w \in W$

$$[\![\psi_1 \to \ldots \to \psi_n \to \phi]\!]_{h'}^{\mathbf{A}_{\mathcal{W}}}(w) \neq \top$$

By Proposition 3.12, we have a BM-structure $\mathcal{M}^{\mathbf{A}_{\mathcal{W}}}$ such that for some $w \in W$

$$(w \Vdash_{h'}^{BM} \psi_1 \to \ldots \to \psi_n \to \phi) \; \neq \; \top$$

and hence

$$(w \Vdash_{h'}^{BM} \psi_1 \otimes \ldots \otimes \psi_n) \not\preceq (w \Vdash^{BM} \phi)$$

and $\psi_1, \ldots, \psi_n \not\Vdash^{BM} \phi$. ∎

NOTE 2. It would be interesting to be able to prove this completeness result directly, by constructing a BM-structure directly from the logic (term model), as is done for intuitionistic logic. However, we have not been able to find such direct proof, and hence have appealed to Bova and Montagna results on the completeness of $\mathbf{GBL_{ewf}}$ for poset sums.

4. Algebraic and Kripke Models

Part of the thesis of this paper is that the Bova–Montagna semantics for $\mathbf{GBL}_{\mathrm{ewf}}$ can be viewed as a generalisation of the Kripke semantics for intuitionistic logic. In this section we discuss how this fits into a general framework alongside the algebraic semantics and models, by showing how the Kripke semantics construction above is in fact an instance of a more general construction on $\mathrm{GBL}_{\mathrm{ewf}}$ algebras (cf. Theorem 4.10).

Many logics have algebraic semantics. Propositions of the logic form an algebra, with logical consequence giving a partial order and logical equivalence giving equality. Connectives are interpreted by algebraic operations and the algebras themselves form a variety characterised by an equational theory. The logics we consider have algebras that are lattices, and inequalities can be reduced to equations via equivalences such as

$$a \le b \text{ iff } a \wedge b = a$$

Examples of this include Boolean algebras for classical logic, Heyting algebras for intuitionistic logic, MV algebras for Lukasiewicz logic and $\mathrm{GBL}_{\mathrm{ewf}}$ algebras for $\mathbf{GBL}_{\mathrm{ewf}}$. In each case the algebras are sound and complete for the logic.

In Section 1.1 we characterised Kripke models as being given by a Kripke structure. The key point of this is that we have a binary relation between worlds w and propositional variables p. We can recast this as a function $(\cdot) \Vdash^{\mathrm{K}} p : W \to \{0, 1\}$, and then generalise it to $(\cdot) \Vdash^{\mathrm{K}} p : W \to A$, where A is a suitable algebra. We will not consider the possibility that A depends on $w \in W$. This gives a theory in which propositions are modelled as elements of A^W. Since algebras of a given kind form a variety, A^W is an algebra of the same kind as A, and the interpretation of all the operations is pointwise.

However, in the standard Kripke theory there is a partial order on W, and the functions that interpret propositions must preserve that partial order. We want to turn the set of monotone increasing functions $W \to A$ into an algebra. If the operators in the algebra are themselves monotone increasing (isotone), then this is not a problem. The set of isotone operators is a subalgebra of A^W. In particular:

LEMMA 4.1. *Let $\mathcal{A} = \langle A, \wedge, \vee, \otimes, 1 \rangle$ be a lattice-ordered monoid. Then for any partial order $\mathcal{W} = \langle W, \succeq \rangle$, the set of isotone functions $Inc(\mathcal{W}, \mathcal{A})$ is also a lattice-ordered monoid, in which the operations are calculated pointwise.*

Constructs like residuation break this construction as they give rise to operators that are not isotone. Residuation $x \to y$ is antitone in its first variable x, while isotone in its second variable y. But we can nevertheless obtain a residuated lattice. The residuations are not calculated pointwise. They are calculated as suitable reflections in the categorical sense. When we view partial orders as categories, then functors are exactly isotone functions, and natural transformations exactly the standard order relation between functions.

LEMMA 4.2. *If* $\mathcal{A} = \langle A, \wedge, \vee \rangle$ *is a complete lattice and* $\mathcal{W} = \langle W, \succeq \rangle$ *a partial order, then the inclusion* $Inc(\mathcal{W}, \mathcal{A}) \to \mathcal{A}^{\mathcal{W}}$ *has a right adjoint* r. *In other words,* $Inc(\mathcal{W}, \mathcal{A})$ *is a reflective subcategory of* $\mathcal{A}^{\mathcal{W}}$.

PROOF. If $f : W \to A$ then $(rf)(w) = \bigvee_{w' \succeq w} fw'$. ∎

In particular, if $A = \{\bot, \top\}$, the standard booleans, then this means

$$(rf)(w) = \top \quad \text{iff} \quad \forall w' \succeq w \, (f(w') = \top).$$

LEMMA 4.3. *If* $\mathcal{A} = \langle A, \wedge, \vee, \otimes, 1, \to \rangle$ *is a complete residuated lattice, and* $\mathcal{W} = \langle W, \succeq \rangle$ *is a partial order, then* $Inc(\mathcal{W}, \mathcal{A})$ *is also a residuated lattice.*

PROOF. We have already seen that $Inc(\mathcal{W}, \mathcal{A})$ is a lattice-ordered monoid. It is therefore sufficient to show that it is residuated. We recall that the definition of residuation says that $y \to (\ )$ is the right adjoint of $(\ ) \otimes y$. Given monotone $g : W \to A$, there is an adjunction between $Inc(\mathcal{W}, \mathcal{A})$ and $\mathcal{A}^{\mathcal{W}}$

$$\text{Inc}(W, A) \underset{r}{\overset{i}{\underset{\perp}{\rightleftarrows}}} A^W \underset{g \to (\ )}{\overset{(\ ) \otimes g}{\underset{\perp}{\rightleftarrows}}} A^W$$

in which the left adjoint is $\lambda f.(\lambda w. fw \otimes gw)$ and the right adjoint is

$$\lambda H. r(\lambda w. gw \to Hw)$$

(we apply r to the functor calculated pointwise). The left adjoint factors through the inclusion $i : Inc(\mathcal{W}, \mathcal{A}) \to \mathcal{A}^{\mathcal{W}}$, and hence we can restrict this adjunction to the one we require on $Inc(\mathcal{W}, \mathcal{A})$. ∎

In the case that $\otimes = \wedge$ we have the standard Kripke definition of implication for models of intuitionistic logic. One should note, however, that while the pointwise interpretation preserves all equations between terms, this new interpretation does not. An example is $\neg\neg x = x$, where $\neg x$ is defined as $x \to \bot$. This is true in $\{\bot, \top\}$ but false in non-trivial Kripke models.

EXAMPLE 4.4. Let $W = \{u, v\}$, with $v \succ u$, and let $\mathcal{A}$ be the boolean lattice $A = \{\top, \bot\}$. Take $f : W \to \{\bot, \top\}$, where $fu = \bot$ and $fv = \top$. Pointwise in $\mathcal{A}$ we indeed have $\neg\neg x = x$. But $\neg f : W \to A$ is

$$(\neg f)(w) = \forall w' \succeq w \, (f(w') = \bot)$$

Hence, $(\neg f)(u) = (\neg f)(v) = \bot$ so that $(\neg\neg f)(u) = \top \neq f(u)$.

One would expect, and even wish for, the Kripke construction to invalidate classical logic—in the form of double negation elimination. However, the new interpretation also breaks the divisibility property:

EXAMPLE 4.5. Monotone functions do not necessarily have the divisibility property. Take $W = \{u, v\}$ as in the previous example, and consider monotone functions to the standard MV-chain. Take $f(u) = f(v) = 1/4$ and $g(u) = 1/2$, $g(v) = 3/4$. Then $f \leq g$ pointwise and

$$g(v) \to f(v) = \min(1, 1/4 - 3/4 + 1) = 1/2$$

while

$$g(u) \to f(u) = \min(1, 1/4 - 1/2 + 1) = 3/4$$

so $(g \to f)(w) = \inf_{w' \succeq w}(gw \to fw)$ is the constant $1/2$ function. Therefore

$$(g \otimes (g \to f))(u) = \max(0, 1/2 + 1/2 - 1) = 0$$

but $(f \otimes (f \to g))(u) = f(u) \otimes (f \to g)(u) = 1/4 \otimes 1 = 1/4$.

Since isotone functions do not have the divisibility property, we have to look at a different class of functions if we are to construct a GBL$_{\text{ewf}}$ algebra. One suitable class is the class of sloping functions.

LEMMA 4.6. *Let $\mathcal{W}$ be a partial order and $\mathcal{A}$ a complete bounded lattice. Let $(\ ) \cdot (\ )$ be a binary operator on $\mathcal{A}$ such that*

$$\begin{array}{lll} x \cdot \top = \top & & x \cdot \bot = \bot \\ \top \cdot x = \top & \quad or \quad & \bot \cdot x = \bot \\ \bot \cdot \bot = \bot & & \top \cdot \top = \top \end{array}$$

Let f and g be two sloping functions $W \to A$, then $(f \cdot g)(w) = fw \cdot gw$ is also a sloping function $W \to A$.

PROOF. Let us first assume $(\ ) \cdot (\ )$ has the three properties on the left. Suppose $(f \cdot g)(w)$ is neither $\top$ nor $\bot$. Then neither fw nor gw is $\top$ and at least one of them is not $\bot$, say fw. Then, by the assumption that f is a sloping function, for all $w' \succ w$, $fw' = \top$, and hence $(f \cdot g)(w') = fw' \cdot gw' = \top$. The argument is similar when we assume that $(\ ) \cdot (\ )$ has the three properties on the right. ■

In any lattice, the meet $\wedge$ and the join $\vee$ have one of these properties, and in a commutative bounded lattice-ordered monoid so does $\otimes$. We can now construct bounded lattice-ordered monoids as before.

LEMMA 4.7. *Let $\mathcal{A} = \langle A, \wedge, \vee, \otimes, 1 = \top, \bot \rangle$ be a commutative bounded lattice-ordered monoid. Then for any partial order $\mathcal{W} = \langle W, \succeq \rangle$, the set of sloping functions $Step(\mathcal{W}, \mathcal{A})$ is also a commutative bounded lattice-ordered monoid, and the operations are calculated pointwise.*

Once again, residuation gives rise to operators that are not isotone and hence not sloping functions. As before we will use the reflection of the sloping functions in the lattice of all functions.

LEMMA 4.8. *Let $\mathcal{W}$ be a partial order and $\mathcal{A}$ a bounded lattice. Then the lattice of sloping functions $W \to A$ is a reflective subcategory of the lattice of all functions (or isotone functions), where the reflection is given by $\lfloor \mathbf{inf} \rfloor$, where*

$$(\lfloor \mathbf{inf} \rfloor f)(w) = \begin{cases} fw & \text{if } \forall w' \succ w(fw' = \top) \\ \bot & \text{if } \exists w' \succ w(fw' < \top) \end{cases}$$

This now allows us to construct a bounded residuated lattice.

LEMMA 4.9. *If $\mathcal{A} = \langle A, \wedge, \vee, \otimes, 1 = \top, \bot, \to \rangle$ is a bounded integral residuated lattice and $\mathcal{W}$ is a partial order then $Step(\mathcal{W}, \mathcal{A})$ is also a bounded integral residuated lattice.*

In contrast to the isotone functions, the sloping functions are divisible.

THEOREM 4.10. *If $\mathcal{A} = \langle A, \wedge, \vee, \otimes, 1 = \top, \bot, \to \rangle$ is a GBL_{ewf} algebra and $\mathcal{W}$ is a partial order then $Step(\mathcal{W}, \mathcal{A})$ is also a GBL_{ewf} algebra.*

PROOF. Given the above lemmas, it remains for us to check the divisibility property. Suppose $f \leq g$ for sloping functions $f, g \colon W \to A$, and let $w \in W$. We consider three cases:

1. If $f(w) = \top$, then $g(w) = \top$, and, moreover, for all $w' \succ w$, $f(w') = g(w') = \top$. Hence $(g \otimes (g \to f))(w) = \top = f(w)$.

2. If $f(w) = \bot$, since $g \otimes (g \to f) \leq f$, we have $(g \otimes (g \to f))(w) = \bot = f(w)$.

3. If $\bot < f(w) < \top$, then for all $w' \succ w$, $f(w') = g(w') = \top$. In this case $(g \to f)(w) = g(w) \to f(w)$. Hence

   $$(g \otimes (g \to f))(w) = g(w) \otimes (g(w) \to f(w)) = f(w)$$

   since $\mathcal{A}$ itself satisfies the divisibility property. ∎

It is interesting to note that even if $\mathcal{A}$ is a *involutive* GBL$_{\text{ewf}}$ algebra, it might be that Step$(\mathcal{W}, \mathcal{A})$ is not an *involutive* GBL$_{\text{ewf}}$ algebra. In particular, if we take A to be the standard MV-chain, then this is the Bova–Montagna construction.

5. Conclusion and Future Work

In the foregoing we have shown a certain generalization of Kripke semantics based on the notion of poset sums defined in [2,7] is adequate for intuitionistic Łukasiewicz logic **IŁL**. Along the way, we have shown that the semantics presented herein really does generalise Kripke semantics for intuitionistic logic in various respects, e.g. monotonicity of valuations as in the case of intuitionistic logic under Kripke models, with the domain of our semantics being that of $[0,1]_{\text{MV}}$; the agreement between Kripke models and our own with respect to the language of tensorless formulae.

From here there are several directions one could take. Identifying analytic calculi for basic logic (**BL**) or generalised basic logic (**GBL**) considerably motivates the present work. We conjecture that use can be made of [13] to internalise our semantics so that we might obtain an analytic system. A key issue is whether Negri and Von Plato's approach can work for generalisations of Kripke semantics. Fortunately, in [14] the authors have shown their approach is robust enough to accommodate Routley–Meyer semantics for relevance logics such as **R**. This is a good sign, as the Routley–Meyer semantics generalizes Kripke semantics and is a popular, unifying semantics for many substructural logics (even if the motivation behind such semantics is still poorly understood). Perhaps as a worst-case scenario, one might translate our semantics into Routley–Meyer's and attempt to internalise the semantics into the resulting sequent rules. This is less preferable to the direct route of internalising poset sums into appropriate sequent rules.

Although indirect, there is some precedent for analysing many-valued logics via the Routley–Meyer framework: Urquhart's **C**, for example, has both an algebraic semantics and a Routley–Meyer semantics [12]. Importantly, both are adequate for **C**. Moreover, Urquhart's **C** forms the tensorless fragment of **BL** and admits an analytic hypersequent calculus presentation (see [3,4]). Given these latter insights and the family resemblance between **BL**, **GBL**$_{\text{ewf}}$, Łukasiewicz logic and **C**—all many-valued logics lacking full-contraction, featuring well-explored algebraic semantics adequate for the intended systems—this strongly suggests that Von Plato and Negri's approach can work for many-valued logics in this family, for *some* appropriate generalisation of Kripke semantics.

Another research direction is to derive tableaux methods for **IŁL** from Bova–Montagna's poset sums as a point of departure. Presently, there's a dearth of existent tableaux for infinitely many-valued logics. **BL** and **GBL**$_{\text{ewf}}$ are no exception to this trend, with **BL** seeing progress recently. We know of two such approaches with respect to **BL**. For example, using the fact that **BL** is the logic of continuous t-norms [5,6], Agnieszka Kułacka in [10] and [9] introduces a semantic tableau calculus for **BL** that is sound and complete with respect to continuous t-norms, and demonstrates the refutational procedure and search for countermodels on select examples. These are, in a way, preceded by Olivetti's own tableau calculi [15] and connect with the relational hypersequents approach in [11] whose rules are based on that of Łukasiewicz t-norms, although Kułacka's approach in [9,10] is considerably more general (as it is a calculus adequate for *any* continuous t-norm). Preceding Kułacka's work, Bova and Montagna's [1] features a calculus for **BL** not only analytic but also invertible, and show that the tautology problem for **BL** is **coNP**-complete.

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

References

[1] BOVA, S., and F. MONTAGNA, Proof search in Hájek's Basic Logic. *ACM Trans. Comput. Log.* 9:21:1–21:26, 2008.

[2] BOVA, S., and F. MONTAGNA, The consequence relation in the logic of commutative GBL-algebras is PSPACE-complete. *Theoretical Computer Science* 410(12):1143–1158, 2009.

[3] CIABATTONI, A., On Urquhart's C logic. In *30th IEEE International Symposium on Multiple-Valued Logic, ISMVL 2000, Portland, Oregon, USA, May 23-25, 2000, Proceedings* IEEE Computer Society, 2000, pp. 113–120.

[4] CIABATTONI, A., and C. G. FERMÜLLER, Hypersequents as a uniform framework for Urquhart's C, MTL and related logics. In *31st IEEE International Symposium on Multiple-Valued Logic, ISMVL 2001, Warsaw, Poland, May 22-24, 2001, Proceedings* IEEE Computer Society, 2001, pp. 227–232.

[5] CIGNOLI, R., F. ESTEVA, L. GODO, and A. TORRENS. Basic fuzzy logic is the logic of continuous t-norms and their residua. *Soft Computing* 4(2):106–112, 2000.

[6] HÁJEK, P., Basic fuzzy logic and BL-algebras. *Soft Computing* 2(3):124–128, 1998.

[7] JIPSEN, P., and F. MONTAGNA, On the structure of generalized BL-algebras. *Algebra Universalis* 55(2):227–238, 2006.

[8] KRIPKE, S. A., Semantical analysis of intuitionistic logic I. In J.N. Crossley and M.A.E. Dummett, (eds.), *Formal Systems and Recursive Functions*, volume 40 of *Studies in Logic and the Foundations of Mathematics*, Elsevier, 1965, pp. 92–130.

[9] KUŁACKA, A., Tableau calculus for basic fuzzy logic BL. In A. Laurent, O. Strauss, A. Bouchon-Meunier, and R. R. Yager, (eds.), *Information Processing and Management of Uncertainty in Knowledge-Based Systems*. Springer International Publishing, Cham, 2014, pp. 325–334.

[10] KUŁACKA, A., *Propositional Fuzzy Logics: Tableaux and Strong Completeness*. PhD thesis, Imperial College London, London, UK, 2017.

[11] KUŁACKA, A., D. PATTINSON, and L. SCHRÖDER, Syntactic labelled tableaux for Łukasiewicz fuzzy ALC. In *IJCAI 2013, Proceedings of the 23rd International Joint Conference on Artificial Intelligence, Beijing, China, August 3-9*, 2013, pp. 962–968.

[12] MÉNDEZ, J. M., and F. SALTO, Urquhart's C with intuitionistic negation: Dummett's LC without the contraction axiom. *Notre Dame J. Formal Logic* 36(3):407–413, 07 1995.

[13] NEGRI, S., Proof analysis in modal logic. *Journal of Philosophical Logic* 34(5-6):507–544, 2005.

[14] NEGRI, S., and J. VON PLATO, *Proof Analysis: A Contribution to Hilbert's Last Problem*. Cambridge University Press, 2011.

[15] OLIVETTI, N., Tableaux for Łukasiewicz infinite-valued logic. *Studia Logica* 73(1):81–111, 2003.

A. Lewis-Smith, P. Oliva, E. Robinson
School of Electronic Engineering and Computer Science
Queen Mary University of London
Mile End Road
London E1 4NS
UK
a.lewis-smith@qmul.ac.uk

P. Oliva
p.oliva@qmul.ac.uk

E. Robinson
e.p.robinson@qmul.ac.uk