How Not to Combine Modal Logics.

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How Not to Combine Modal Logics

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Abstract. This paper describes a failed attempt to decompose the fusion of two non-normal modal logics as a composition of iterated non-normal modalisations. That strategy had already succeeded in the transfer of soundness, completeness and decidability for normal modal logics, but we show here why it fails for the transference of some of these properties in the fusion non-normal modal logics. The possibility of the transference of these properties is not ruled out by this result.
We also show that the fusion of some of the more common non-normal modal logics transfers completeness, relying directly on the properties of each of the component modal logics.

1 Introduction

The *fusion* $\mathbf{M}_1 \otimes \mathbf{M}_2$ of two modal logics $\mathbf{M}_1$ and $\mathbf{M}_2$ is the smallest logic system that contains both $\mathbf{M}_1$ and $\mathbf{M}_2$. The two modal logics $\mathbf{M}_1$ and $\mathbf{M}_2$ are called the *components* of the fusion, and are assumed to be extensions of propositional classical logic that share the same set of propositional atoms $\mathcal{P} = \{p_0, p_1, \ldots\}$ and the boolean connectives. The modal symbols of $\mathbf{M}_1$ and $\mathbf{M}_2$ are assumed to be distinct; if one is fusing a logic with itself, the combined system has two distinct set of copies of the modal symbols of the original system.

In this setting, fusion is a form of combination of logics that can be seen as a restricted form of fibring logics [12, 18].

The study of fusion in particular, and of combination of logics in general, focuses on the *transference* of logical properties from the component logics to the combined system. Among the most studied properties are: soundness, completeness, decidability, compactness, finite model property, interpolation, etc.

The fusion of *normal* modal logics is a problem that has been extensively studied, and transference results have been shown for very general classes of modal logics. The notion of a normal modal logic is undisputed if the logic contains only one-place modalities. When dealing with modal logics with n-ary modalities, more than one notion of normality may be considered. In Section 2, we study how some of these different notions of normality interact with the fusion of normal modal or multimodal logic with n-ary modalities.

We then examine the fusion of non-normal modal logics. We note that when fusion is presented in an algebraic framework, the techniques for obtaining the transference of properties in the fusion of normal logics are not immediately applicable to non-normal modal logics.

This paper studies techniques for obtaining transference of logical properties in the fusion of non-normal modal logics based on Kripke-like semantics, in particular the transference of weak completeness. The main result is a negative one. In fact, we show that the usual strategy for obtaining transference of completeness in the normal case fails for the non-normal case. This does not mean that the transference is false, only that the technique used in the normal case cannot be directly applied to non-normal modal logics.

The rest of the paper proceeds as follows. Section 2 discussed the fusion of normal (multi)modal logics, presenting two distinct approaches to normality, a syntactic-algebraic approach and a semantical approach, based on Kripke relational structures. As the algebraic methods rely fundamentally in the notion of normality, Section 3 presents a strategy for considering fusion as an iterated modalisation; such a strategy relies on Kripke-like semantics called minimal models or neighbourhood semantics. Section 4 shows that such strategy fails to deliver the transference of completeness, by presenting a non-normal modal logic $\mathbf{M}_P$ whose fusion with itself contains a formula that is fusion inconsistent, but for which it is possible to build a model via iterated modalisations; the transference of soundness, however, is proved. Section 5 presents some special cases of fusion completeness for well-know non-normal modal logics and also describes a recent result in the literature on the general fusion decidability. The paper concludes in Section 6 with a discussion of possible paths to be explored in the study of the transference of completeness for general modal logics.

This paper extends on the talk "Fusions of Normal and Non-normal Modal Logics" presented at the Workshop on Combination of Logics: Theory and Applications (CombLog04), Department of Mathematics, IST, Lisbon, Portugal, July 28-30, 2004. It also contains some results of [4].

2 Fusions of Normal Modal Logics

In this section we review some of the works in the literature of combination of logics pertaining the fusion of normal modal logics.

We start with a few formal definitions. Suppose we define a logic as a pair $\mathbf{L} = (\mathcal{L}_\mathbf{L}, \vdash_\mathbf{L})$ where $\mathcal{L}_\mathbf{L}$ is the language and $\vdash_\mathbf{L} \subseteq 2^{\mathcal{L}_\mathbf{L}} \times \mathcal{L}_\mathbf{L}$ is a consequence relation. In this work, we consider only modal logics, so the language is given by a countable set of propositional letters $\mathcal{P} = \{p_0, p_1, \ldots\}$, the boolean connectives a set n-ary modalities, constructed with the usual formation rules; the consequence relation is assumed to extend classical logic. In this setting, the *fusion* of modal logics $\mathbf{M}_1 = (\mathcal{L}_{\mathbf{M}_1}, \vdash_{\mathbf{M}_1})$ and $\mathbf{M}_2 = (\mathcal{L}_{\mathbf{M}_2}, \vdash_{\mathbf{M}_2})$ is the logic $\mathbf{M}_1 \otimes \mathbf{M}_2 = (\mathcal{L}_{\mathbf{M}_1 \otimes \mathbf{M}_2}, \vdash_{\mathbf{M}_1 \otimes \mathbf{M}_2})$ where $\mathcal{L}_{\mathbf{M}_1 \otimes \mathbf{M}_2}$ is an extension of propositional logic constructed by the union of $\mathbf{M}_1$ and $\mathbf{M}_2$ modalities and $\vdash_{\mathbf{M}_1 \otimes \mathbf{M}_2}$ is the smallest consequence relation that contains both $\vdash_{\mathbf{M}_1}$ and $\vdash_{\mathbf{M}_2}$.

Modal logics are usually equipped with a semantic relation $\models$ with respect to which the consequence relation is sound and complete over a class $\mathcal{K}$ of models. In this setting, we consider the fusion of modal logics of the form $\mathbf{M}_1 = (\mathcal{L}_{\mathbf{M}_1}, \vdash_{\mathbf{M}_1}, \models_{\mathbf{M}_1}, \mathcal{K}_{\mathbf{M}_1})$ and $\mathbf{M}_2 = (\mathcal{L}_{\mathbf{M}_2}, \vdash_{\mathbf{M}_2}, \models_{\mathbf{M}_2}, \mathcal{K}_{\mathbf{M}_2})$ and we study whether there is a combined semantic relation $\models_{\mathbf{M}_1 \otimes \mathbf{M}_2}$ and a combined class of models $\mathcal{K}_{\mathbf{M}_1 \otimes \mathbf{M}_2}$ such that soundness and completeness (or any other logic property) *transfer* to the fusion $\mathbf{M}_1 \otimes \mathbf{M}_2 = (\mathcal{L}_{\mathbf{M}_1 \otimes \mathbf{M}_2}, \vdash_{\mathbf{M}_1 \otimes \mathbf{M}_2}, \models_{\mathbf{M}_1 \otimes \mathbf{M}_2}, \mathcal{K}_{\mathbf{M}_1 \otimes \mathbf{M}_2})$.

When the component modal logics $\mathbf{M}_1$ and $\mathbf{M}_2$ are presented in terms of axiomatisations, the consequence relation of the fusion $\mathbf{M}_1 \otimes \mathbf{M}_2$ is simply obtained by taking the union of the two axiomatisations (see definition below), occasionally renaming modal symbols if they belong to both systems. The study of this sort of independent axiomatisation is found in the literature since the work of Thomason [16], where the independent axiomatisation is shown to be a conservative extension of the two component modal systems.

A more systematic study of the transference of logical properties from the component logics to the combined system started with the works of Kracht and Wolter [15] and Fine and Schurz [6]. Both works considered only normal monomodal logics. The work of Kracht and Wolter dealt with the combination of only two modal logics, but considered the transference of a number of logical properties. The work of Fine and Schurz considered the fusion of any finite number of independent modal systems, but covered the transference of a restricted set of logical properties.

The next step in the study of the fusion of modal logics extended such results to the fusion of normal, multimodal logic systems, where each modality could be an n-ary modality, $n \geq 1$. The logics $\mathbf{M}_1$ and $\mathbf{M}_2$ share a set of propositional symbols $\mathcal{P}$, and each modality Δ_i is associated to a number of arguments $\text{arity}(\Delta_i) = n \geq 1$, so that if $A_1, \dots A_n$ are well formed formulas, so is $\Delta_i(A_1, \dots, A_n)$.

In this setting, a problem arises on how to define *normality*. In the case of monomodal logics, where each modality is a unary $\Box$, it suffices that the system satisfies one of the versions of the normality axiom, such as:

1. $\Box(p \to q) \to (\Box p \to \Box q)$; or
2. $\Box(p \wedge q) \leftrightarrow (\Box p \wedge \Box q)$; or
3. $\Diamond(p \vee q) \leftrightarrow (\Diamond p \vee \Diamond q)$

plus the normalisation inference rule $\frac{\vdash A}{\vdash \Box A}$ or its axiomatic correspondent, the validity of $\Box\top$.

However, when this notion of normality is extended to n-ary modalities, several distinct definitions of normality are possible, generating fusions with more or less expressivity.

2.1 The Syntactic Approach to Normality

The syntactic approach of normality has its origins in the study of algebraic semantics for modal logics. In the study of fusion, this is the approach used by

Wolter [17][1]. It generalises the notion of normality to n-ary multimodalities in the following way. If Δ is an n-ary modality, then for every argument position i, $1 \leq i \leq n$:

- $\vdash \neg\Delta(\ldots, \bot_i, \ldots)$;
- $\vdash \Delta(\ldots, A \vee B_i, \ldots) \leftrightarrow \Delta(\ldots, A_i, \ldots) \vee \Delta(\ldots, B_i, \ldots)$.

Note that Δ behaves like a diamond operator in version 3 of the definition of normality above. Such a definition of normality has a direct correspondence in the algebraic approach, and in [17] it was shown that soundness completeness and decidability transfer for normal modal systems that obey the restrictions above. This approach, however, relied firmly on the syntactical definition of normality and it did not suggest any obvious way to generalise the transference of logical properties to non-normal modalities.

It turns out that there are n-ary modalities in well known modal/temporal logics that have been considered normal throughout the literature since they were proposed, but that do not fit in the definition above. One case in hand is Kamp's temporal operators *until* (U) and *since* (S) [14]. These binary connectives have an existential semantic in one argument and a universal semantic in the other; the latter fails the syntactic definition above.

The algebraic approach was extended to the fusion of description logics [1], which employed a notion of *quasi-normality*. Under these new restrictions, the fusion of U and S temporal operators could be undertaken. The approach, however, remains not directly applicable to the study of transference of logic properties in the fusion on non-normal modal logics. So a different approach to normality could be investigated. So, despite the fact that a great number of very interesting works in the literature employ the algebraic approach with this syntactic notion of normality, this approach will not be explored here.

2.2 The (Kripke) Semantic Approach to Normality

The (Kripke) semantic approach to normality in the fusion of modal/temporal logics was presented in [10]; an earlier proposal in [7] dealt only with linear temporal logics.

In this setting, a modal logic is a 4-tuple $\mathbf{M} = (\mathcal{L}_{\mathbf{M}}, \vdash_{\mathbf{M}}, \models_{\mathbf{M}}, \mathcal{K}_{\mathbf{M}})$ where $\mathcal{L}_{\mathbf{M}}$ is the language, $\vdash_{\mathbf{M}}$ is the consequence relation, $\mathcal{K}_{\mathbf{M}}$ is the class of models according to which the semantic relation $\models_{\mathbf{M}}$ between formulas and models is defined. The logic $\mathbf{M}$ can be described as follows:

- The consequence relation contains a set of axioms, and the inference rules of Modus Ponens $\left(\frac{\vdash_{\mathbf{M}} A \to B \quad \vdash_{\mathbf{M}} A}{\vdash_{\mathbf{M}} B}\right)$, Substitutivity $\left(\frac{\vdash_{\mathbf{M}} A}{\vdash_{\mathbf{M}} A[p{:=}B]}\right)$ and Normalisation $\left(\frac{\vdash_{\mathbf{M}} A}{\vdash_{\mathbf{M}} \neg\Delta_i(\ldots,\neg A,\ldots)}\right)$.
- There is a class of models $\mathcal{K}_{\mathbf{M}}$ and a semantic relation $\models_{\mathbf{M}}$ such that for each model $\mathcal{M} \in \mathcal{K}_{\mathbf{M}}$ and each element $w \in \mathcal{M}$ and each formula A, one

[1] Since the publication of [17], the term *fusion* became widespread in the literature.

can check if the formula A holds in the model $\mathcal{M}$ at that point w, that is, $\mathcal{M}, w \models_{\mathbf{M}} A$.

Then the fusion $\mathbf{M}_1 \otimes \mathbf{M}_2 = (\mathcal{L}_{\mathbf{M}_1 \otimes \mathbf{M}_2}, \vdash_{\mathbf{M}_1 \otimes \mathbf{M}_2}, \models_{\mathbf{M}_1 \otimes \mathbf{M}_2}, \mathcal{K}_{\mathbf{M}_1 \otimes \mathbf{M}_2})$ of modal logics $\mathbf{M}_1$ and $\mathbf{M}_2$ will also define a language, a consequence relation, a class of models and a semantic relation, in the following way:

- The common propositional symbols are atomic formulas in the fusion.
- Each modality of the component systems is present in $\mathbf{M}_1 \otimes \mathbf{M}_2$, with the same syntactical formation rules.
- The consequence relation $\vdash_{\mathbf{M}_1 \otimes \mathbf{M}_2}$ is the union of the axiomatisations $\vdash_{\mathbf{M}_1}$ and $\vdash_{\mathbf{M}_2}$. Formally, an axiomatisation is a pair $(\mathcal{A}, \mathcal{I})$ where $\mathcal{A}$ is a finite set of axioms extending propositional classical logic, and $\mathcal{I}$ is a set of inference rules that contains (or admits) Modus Ponens and Uniform Substitution. In this case, the union of two axiomatisations $(\mathcal{A}_1, \mathcal{I}_1)$ and $(\mathcal{A}_2, \mathcal{I}_2)$ is simply $(\mathcal{A}_1 \cup \mathcal{A}_2, \mathcal{I}_1 \cup \mathcal{I}_2)$, so it also extends classical logic and admits Modus Ponens and Uniform Substitution.[2]
- The combined class of models $\mathcal{K}_{\mathbf{M}_1 \otimes \mathbf{M}_2}$ has models of the form

$$(W, R_1, \ldots R_{a_1}, S_1, \ldots S_{a_2})$$

  such that $(W, R_1, \ldots R_{a_1}) \in \mathcal{K}_{\mathbf{M}_1}$ and $(W, S_1, \ldots S_{a_2}) \in \mathcal{K}_{\mathbf{M}_2}$; the semantic relation $\models_{\mathbf{M}_1 \otimes \mathbf{M}_2}$ is defined using the rules of the component logics.

This approach to the study of fusion does not take in consideration the arity of the modalities. Instead, it focuses on the n-relational model in the underlying semantics. Suppose $(W, R_1, \ldots, R_n)$ is an acceptable model. The logic is normal according to the semantic approach if every R_i is associated with a definable one-place connective $\Box_i$ that is normal, that is, $\vdash \Box_i(p \to q) \to (\Box_i p \to \Box_i q)$ and $\frac{\vdash A}{\vdash \Box_i A}$ are valid in the logic for every i, $1 \leq i \leq n$.

Note that $\Box_i$ does not have to be a primitive symbol in the language; it suffices that it be *definable* from the primitive n-ary connectives. Over Kripke frames, the semantic approach to normality is strictly stronger than the syntactic one, in the sense that every logic that is normal in the syntactic approach is also normal according to the semantic approach, and there are modal logics such as Kamp's US-temporal logic that are normal in the latter approach but not in the former one.

In [10], it was shown that the fusion of multimodal logics according to the semantic approach does transfer the basic properties of soundness, completeness and decidability, including when modalities are n-ary.

Unlike the syntactic approach, which is algebraic, the semantic approach to the fusion of normal modal logics is based on Kripke semantics, and its proof strategy in the demonstration of transference could, in principle, be applied to the investigation of the fusion of non-normal modal logics. This possibility will be investigated next.

[2] If the consequence relation is presented in some other format, say, in terms of semantic tableaux, the definition of the combined consequence relation has to be appropriately given; in the case of tableaux, the combined consequence relation would possibly be given by the union of the connective rules.

3 Fusion via Modalisations: a Strategy

The proof strategy for showing the transference of logic properties in the fusion of two logics according to the semantic approach consisted in three steps:

1. The *external application* of a modal logic system $\mathbf{M}$ to a generic logic $\mathbf{L}$, generating $\mathbf{M}(\mathbf{L})$, in a process called *temporalisation* [8] or *modalisation* [9]. The modalisation is a form of combining logics weaker than the fusion, and one proves it transfers logic properties.
2. The study of finite *iterated modalisations* of two modal logic systems $\mathbf{M}_1$ and $\mathbf{M}_2$, generating $\mathbf{M}_1(\mathbf{M}_2(\mathbf{M}_1(\ldots)))$. The aim here is to show that an outer application of $\mathbf{M}_i$ does not conflict with an inner application of $\mathbf{M}_i$, $i = 1, 2$, such that the resulting system transfers logic properties.
3. The view of fusion as the union of iterated modalisations, such that every formula in a fusion $\mathbf{M}_1 \otimes \mathbf{M}_2$ can be seen as a formula in some iterated modalisation. The transfer of logic properties follows from this fact and some additional considerations.

This strategy has been used several times in the literature. It has first been applied in the fusion of linear temporal logics [8, 7], then generalised to the fusion of temporal logic over any class of flows of time [9, 10]. In [10] that strategy was used to show the transference of the basic logical properties of soundness, completeness and decidability in the fusion of any normal modal logics.

So far, all proofs of transference of (weak) completeness via fusion in the literature rely on some form of normality.

In the case of non-normal modal logics, the approach corresponding to Kripke semantics is based on *minimal models* [3], where a model is now given by (W, F, V), where $F : \mathcal{W} \to 2^{2^W}$ maps each point $w \in W$ to a set of sets of proposition, namely the set of propositions A such that $\Box A$ is true at w; function F is sometimes called the *neighbourhood mapping*, associating a world to a set of neighbourhoods. The function $V : \mathcal{L} \to 2^W$ is a valuation that associates a set of worlds to each formula according to the following restrictions:

- $V(\neg A) = W \setminus V(A)$.
- $V(A \wedge B) = V(A) \cap V(B)$.
- $V(\Box A) = \{w \in W | V(A) \in N(w)\}$.

We write $\mathcal{M}, w \models A$ iff $w \in V(A)$. Under this view, a formula is V-associated with the set of worlds in which it holds, and the function F associates a world $w \in W$ with a set of propositions that are *necessary* at w.

On the proof-theoretic side, no axioms are required to hold in general, the only obligatory inference rule is the rule of *congruence*, namely $\frac{\vdash A \leftrightarrow B}{\vdash \Box A \leftrightarrow \Box B}$.

3.1 Non-normal Modalisation

According to the proof strategy presented above, the first step in the investigation of the fusion of non-normal modal logics was the external application (modalisation) of a non-normal modal logic $\mathbf{M}$ to a generic logic $\mathbf{L}$, generating $\mathbf{M}(\mathbf{L})$. In [5], it was shown that the non-normal modalisation preserves soundness, completeness and decidability.

As a particular case, let $\mathbf{M}_1$ and $\mathbf{M}_2$ be two modal logics such that, in the modalisation process, we make $\mathbf{M} = \mathbf{M}_1$ and $\mathbf{L} = \mathbf{M}_2$. In this case, in the modalised language of $\mathbf{M}_1(\mathbf{M}_2)$ no modality of the external logic $\mathbf{M}_1$ can occur within the scope of the modalities of logic $\mathbf{M}_2$, so that the language of $\mathbf{M}_1(\mathbf{M}_2)$ is a proper sublanguage of the fusion $\mathbf{M}_1 \otimes \mathbf{M}_2$.

In general, to avoid double parsing of formulas in the modalised language of $\mathbf{M}(\mathbf{L})$, we consider the set of *monolithic formulas* of $\mathbf{L}$, $\mathrm{Mono}_{\mathbf{L}}$, as the formulas of $\mathbf{L}$ which are not boolean combinations of other formulas. The monolithic formulas are then the basic blocs to which the boolean connectives and the modalities of $\mathbf{M}$ are applied.

Let $\mathcal{K}_{\mathbf{M}}$ be a class of models of logic $\mathbf{M}$, usually defined by placing some restriction on the neighbourhood mapping F. Let $\mathcal{K}_{\mathbf{L}}$ be the class of models for valid formulas of $\mathbf{L}$; we specify some restrictions on the semantic relation $\models_L$ for the logic $\mathbf{L}$, whose class of models will be called $\mathcal{K}_L$. The class $\mathcal{K}_L$ must satisfy the following restriction: for each $\mathcal{M} \in \mathcal{K}_{\mathbf{L}}$ and $A \in \mathcal{L}_{\mathbf{L}}$ we have either $\mathcal{M} \models A$ or $\mathcal{M} \models \neg A$.

To satisfy this condition in the special case of $\mathbf{M}_1(\mathbf{M}_2)$, it is necessary to adapt the notion of a class of models. In this case, we consider an element of $\mathcal{K}_{\mathbf{M}_2}$ as a *pair* $\langle \mathcal{M}_{\mathbf{M}_2}, w\rangle$, where $\mathcal{M}_{\mathbf{M}_2} = (W', F', V')$ and $w \in W'$, such that either $\mathcal{M}_{\mathbf{M}_2}, w \models A$ or $\mathcal{M}_{\mathbf{M}_2}, w \models \neg A$.

Finally, we can define a minimal model for the modalised logic $\mathbf{M}(\mathbf{L})$ as a structure $\mathcal{M}_{\mathbf{M}(\mathbf{L})} = (W, F, g)$, where W and F are as above, and $g : W \to \mathcal{K}_{\mathbf{L}}$ associates to each $w \in W$ a model of $\mathbf{L}$. The satisfaction relation $\models$ is then defined recursively over the structure of modalised formulas:

i. $\mathcal{M}_{\mathbf{M}(\mathbf{L})}, w \models \alpha,\ \alpha \in \mathrm{Mono}_{\mathbf{L}}$ iff $g(w) = \mathcal{M}_{\mathbf{L}}$ and $\mathcal{M}_{\mathbf{L}} \models \alpha$ (denoted $g(w) \models \alpha$).
ii. $\mathcal{M}_{\mathbf{M}(\mathbf{L})}, w \models \neg\alpha$ iff $\mathcal{M}_{\mathbf{M}(\mathbf{L})}, w \not\models \alpha$.
iii. $\mathcal{M}_{\mathbf{M}(\mathbf{L})}, w \models (\alpha \wedge \beta)$ iff $\mathcal{M}_{\mathbf{M}(\mathbf{L})}, w \models \alpha$ and $\mathcal{M}_{\mathbf{M}(\mathbf{L})}, w \models \beta$.
iv. $\mathcal{M}_{\mathbf{M}(\mathbf{L})}, w \models \Box\alpha$ iff $\{w' \in W \mid \mathcal{M}_{\mathbf{M}(\mathbf{L})}, w' \models \alpha\} \in F(w)$.

In this setting, a class of modalised models $\mathcal{K}_{\mathbf{M}(\mathbf{L})}$ is obtained from $\mathcal{K}_{\mathbf{L}}$ and $\mathcal{K}_{\mathbf{M}}$ by placing over modalised models $\mathcal{M}_{\mathbf{M}(\mathbf{L})} = (W, F, g)$ the same restrictions over F that are placed on the class $\mathcal{K}_{\mathbf{M}}$.

The strategy for proving transference of completeness in [5] is illustrated by Figure 1.

The crucial step in the proof is the consistency preserving σ-translation, that maps a $\mathbf{M}(\mathbf{L})$-formula A into a $\mathbf{M}$-formula $\sigma(A)$ that is $\mathbf{M}$-consistent iff A is $\mathbf{M}(\mathbf{L})$-consistent. Then the completeness of $\mathbf{M}$ is used to obtain a model for $\sigma(A)$, and the completeness of the internal logic $\mathbf{L}$ is used to manipulate that model to obtain a model for $\mathbf{M}(\mathbf{L})$.

The details of the consistency preserving mapping σ are described in [5]. Here we limit ourselves to present the main results proven there.

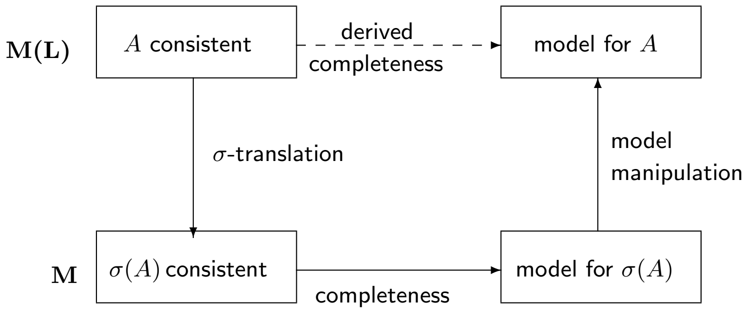

Fig. 1. Completeness proof strategy

Theorem 1 ([5]). *Soundness, completeness and decidability transfer via non-normal modalisations. That is:*

- *If* $\mathbf{L}$ *and* $\mathbf{M}$ *are sound, so is* $\mathbf{M}(\mathbf{L})$*.*
- *If* $\mathbf{L}$ *and* $\mathbf{M}$ *are complete logics, so is* $\mathbf{M}(\mathbf{L})$*.*
- *If the logics* $\mathbf{L}$ *and* $\mathbf{M}$ *are complete and decidable, so is* $\mathbf{M}(\mathbf{L})$*.*

4 The Failure of the Strategy

A notable fact occurs when one tries to prove the transference of logic properties in the case of iterated modalisations $\mathbf{M}_1(\mathbf{M}_2(\mathbf{M}_1(\ldots)))$. It turns out that non-normal modal logics behave in ways not expected in normal modal logics.

To show that, we present a non-normal modal logic $\mathbf{M}_P$, called a partition modal logic, and a class of models over which $\mathbf{M}_P$ is sound and complete. Then we present the fusion of $\mathbf{M}_P$ with itself, $\mathbf{M}_{P1} \otimes \mathbf{M}_{P2}$, and we present a formula A that is $\mathbf{M}_{P1} \otimes \mathbf{M}_{P2}$-inconsistent but for which there exists a $\mathbf{M}_{P1}(\mathbf{M}_{P2}(\mathbf{M}_{P1}))$-model via iterated modalisations. This shows that iterated non-normal modalisations cannot be used to represent the fusion of non-normal modal logics, as prescribed by our strategy.

4.1 Fusion of Non-Normal Modal Logics

Before we can show why the iterated modalisations cannot be used to obtain the transference of completeness in the fusion of non-normal modal logic, we have to formally describe how to construct this fusion.

The language of the fusion of two non-normal modal logics is obtained in the usual way. We are going to assume that we are combining two monomodal one-place modal logics, $\mathbf{M}_1$ with modality $\Box_1$ and $\mathbf{M}_2$ with modality $\Box_2$. We also assume that both $\mathbf{M}_1$ and $\mathbf{M}_2$ are presented through axiomatisations that extend classical logic and that contain modal axioms and the rules Modus Ponens, Substitution and Congruence, such that the axiom system for $\mathbf{M}_1 \otimes \mathbf{M}_2$ is simply the union of the two axiomatisations.

On the semantic side we suppose $\mathbf{M}_1$ is complete for the class of frames $\mathcal{K}_1$ and $\mathbf{M}_2$ is complete for the class of frames $\mathcal{K}_2$. The semantics of the fusion has an intrinsic complexity due to the difficulty in putting together distinct models to build a new one.

When considering relational Kripke structures of normal modal logics, there is a notion of a *connected component*, which allows one to combine two disjoint models into a single one, preserving all truth of all formulas at all points. A frame resulting from the fusion of normal Kripke frames is of the form (W, R_1, R_2), where (W, R_j) is a disjoint union of connected components (ie, frames) in class $\mathcal{K}_j$, $j = 1, 2$. The disjoint union of connected components generates a model for normal modal logics that satisfies, at all points, the same formulas as each separate component.

No such notion of connected component exists in non-normal modal logics over minimal models. The extension of a minimal model with a single point can drastically change the truth value of $\Box$-formulas evaluated on any of its points. To put together minimal models we introduce an operation called the *f-disjoint union* of a family of frames $\{(W_i, F_i)\}_{i\in I}$ as the frame (W, F), where $W = \bigcup_{i\in I} W_i$ is the (set) disjoint union of all W_i and, for every $w_i \in W_i$, $F(w_i)$ is given by:

$$F(w_i) = \{X \subseteq W | X \cap W_i \in F_i(w_i)\}.$$

The f-disjoint union of $\{(W_i, F_i)\}_{i\in I}$ is denoted by $\coprod_{i\in I}(W_i, F_i)$. A frame for $\mathbf{M}_1 \otimes \mathbf{M}_2$ is a triple (W, F_1, F_2), where each (W, F_j) is the f-disjoint union of several j-frames:

$$(W, F_j) = \coprod_{i\in I}(W_i, F_{i,j})$$

where each $(W_i, F_{i,j})$ is isomorphic to a frame in the class $\mathcal{K}_j$, for $j = 1, 2$.

The f-disjoint union is easily extended to models. Given a family of models $\{(W_i, F_i, V_i)\}\}_{i\in I}$, we define $\coprod_{i\in I}(W_i, F_i, V_i) = (W, F, V)$, where $(W, F) = \coprod_{i\in I}(W_i, F_i)$ and $V(p) = \bigcup_{i\in I} V_i(p)$. The following result actually shows that $V(A) = \bigcup_{i\in I} V_i(A)$ for every formula A.

Lemma 1. *Let $\{(W_i, F_i, V_i)\}\}_{i\in I}$ be a family of models and let $(W, F, V) = \coprod_{i\in I}(W_i, F_i, V_i)$. Given a formula A and $w_i \in W_i \subseteq W$, then*

$$(W, F, V), w_i \models A \textit{ iff } (W_i, F_i, V_i), w_i \models A.$$

Proof. Note that the condition to be proved is equivalent, by definition, to showing that $V(A) = \bigcup_{i\in I} V_i(A)$ for every formula A. We proceed by structural induction over A. If A is atomic the result follows from the definition of V. The inductive steps for boolean connectives $\wedge$ and $\neg$ are immediate.

Suppose that $(W, F, V), w_i \models \Box A$ which holds, by definition, if and only if $V(A) \in F(w_i)$. By the definition of the f-disjoint union, $F(w_i) = \{X \subseteq W | X \cap W_i \in F_i(w_i)\}$. Thus, $V(A) \in F(w_i)$ iff $V(A) \cap W_i \in F_i(w_i)$. The induction hypothesis gives us that $V(A) = \bigcup_{i\in I} V_i(A)$, so $V_i(A) = V(A) \cap W_i$ and $V(A) \in F(w_i)$ iff $V_i(A) \in F_i(w_i)$, which holds iff $(W_i, F_i, V_i), w_i \models \Box A$.

We are now in a position to define a model for the fusion as a 4-tuple $\mathcal{M} = (W, F_1, F_2, V)$, where each $(W, F_i) \in \mathcal{K}_i$ is a frame for a non-normal modal logic, and $V : \mathcal{P} \to 2^W$. For $w \in W$, the satisfaction relation $\mathcal{M}, w \models A$ is defined in the usual way, using F_1 for the modality $\Box_1$ and F_2 for the modality $\Box_2$. A class of structures for $\mathbf{M}_1 \otimes \mathbf{M}_2$ consists of all structures (W, F_1, F_2) where $(W, F_1) = \coprod_{i \in I}(W_i, F_{1,i})$ and $(W, F_2) = \coprod_{j \in J}(W_j, F_{2,j})$, where each $(W_i, F_{1,i})$ is isomorphic to a structure of $\mathbf{M}_1$ and $(W_j, F_{2,j})$ is isomorphic to a structure of $\mathbf{M}_2$.

As usual, we say that $\mathbf{M}_1 \otimes \mathbf{M}_2$ is *sound* with respect to the class of models $\mathcal{K}_{\mathbf{M}_1 \otimes \mathbf{M}_2}$ if $\vdash_{\mathbf{M}_1 \otimes \mathbf{M}_2} A$ implies that for every model $\mathcal{M} = (W, F, V) \in \mathcal{K}_{\mathbf{M}_1 \otimes \mathbf{M}_2}$ and for every $w \in W$, $\mathcal{M}, w \models A$. Conversely, $\mathbf{M}_1 \otimes \mathbf{M}_2$ is *complete* with respect to the class of models $\mathcal{K}_{\mathbf{M}_1 \otimes \mathbf{M}_2}$ whenever $\mathcal{M}, w \models A$ holds for every model $\mathcal{M} = (W, F, V) \in \mathcal{K}_{\mathbf{M}_1 \otimes \mathbf{M}_2}$ and for every $w \in W$, then $\vdash_{\mathbf{M}_1 \otimes \mathbf{M}_2} A$. The transference of soundness of $\mathbf{M}_1 \otimes \mathbf{M}_2$ is easily obtained.

Theorem 2. *If $\mathbf{M}_1$ and $\mathbf{M}_2$ are sound non-normal modal logics, so is the fusion $\mathbf{M}_1 \otimes \mathbf{M}_2$.*

Proof. By Lemma 1, all theorems of $\mathbf{M}_1$ and $\mathbf{M}_2$ are valid in $\mathbf{M}_1 \otimes \mathbf{M}_2$. The inference rules of Modus Ponens, Substitution and Congruence preserve the validity of formulas in non-normal minimal bimodal structures. As the axiomatic system of $\mathbf{M}_1 \otimes \mathbf{M}_2$ is the smallest axiomatic system that contains both the axiomatic systems of $\mathbf{M}_1$ and $\mathbf{M}_2$, the soundness of $\mathbf{M}_1 \otimes \mathbf{M}_2$ is thus obtained.

4.2 Non-normal Modal Partition Logic

We now present the Non-normal Modal Partition Logic, $\mathbf{M}_P$, containing the single axiom of *partition*.

$$(P) \quad \vdash_{\mathbf{M}_P} (\Box p \leftrightarrow p) \vee (\Box q \leftrightarrow \neg q)$$

The axiom P implies that in a given formula of $\mathbf{M}_P$, either all $\Box$'s are deleted or they are replaced by negation. In a normal modal logic we have $\Box\top \leftrightarrow \top$ so axiom P trivialises to $\Box p \leftrightarrow p$, which means that the modality can be eliminated.

Now consider the class $\mathcal{K}_P$ of minimal frames (W, F) such that for every $w \in W$, either:

1. $F(w) = \{X \subseteq W | w \in X\}$; or
2. $F(w) = \{X \subseteq W | w \notin X\}$.

In the class $\mathcal{K}_P$, for every $w \in W$, $F(w)$ contains either all sets containing w (case 1) or all sets not containing w (case 2). In this sense, every $w \in W$ creates a bipartition in the power set of W and associates $F(w)$ to one of these partitions.

Theorem 3. *The logic $\mathbf{M}_P$ is sound and complete with respect to the class $\mathcal{K}_P$.*

Proof. For soundness, consider a model $\mathcal{M} = (W, F, V) \in \mathcal{K}_P$ and let $w \in W$. If $F(w)$ is as in case 1, then for every formula A, $\mathcal{M}, w \models A \leftrightarrow \Box A$, so axiom $\mathbf{P}$ holds at w; similarly, if $F(w)$ is as in case 2, then for every formula A, $\mathcal{M}, w \models A \leftrightarrow \neg\Box A$, so axiom $\mathbf{P}$ also holds at w.

For completeness, first note that axiom $\mathbf{P}$ makes the modality behave either as a classic negation, or as a trivial symbol that can be deleted. Let φ be an $\mathbf{M}_P$-consistent formula. Let A_φ be the propositional formula obtained by deleting "$\Box$" from φ; similarly, let B_φ be the propositional formula obtained by replacing "$\Box$" by $\neg$ in φ.

First suppose that A_φ is consistent, then we construct a model for φ in the following way. Let $W = \{w\}$ be a singleton set and let V be a valuation that satisfies A_φ. Make $F(w) = \{\{w\}\}$, so $\mathcal{M}, w \models A \leftrightarrow \Box A$ for every formula A; by structural induction on φ, it follows that $\mathcal{M}, w \models A_\varphi \leftrightarrow \varphi$, and hence $\mathcal{M}, w \models \varphi$.

If B_φ is consistent, we obtain a model for φ in a totally analogous way, just making $F(w) = \{\varnothing\}$.

Finally, if A_φ and B_φ are inconsistent propositional formulas, we show that φ is $\mathbf{M}_P$-inconsistent. In fact

1. $\vdash_{\mathbf{M}_P} (\top \leftrightarrow \neg\Box\top) \vee (A \leftrightarrow \Box A)$	Substitute $p := A,\ q := \top$ in $\mathbf{P}$
2. $\vdash_{\mathbf{M}_P} \Box\top \rightarrow (A \leftrightarrow \Box A)$	classic logic on 1
3. $\vdash_{\mathbf{M}_P} (\top \leftrightarrow \Box\top) \vee (A \leftrightarrow \neg\Box A)$	Substitute $q := A,\ p := \top$ in $\mathbf{P}$
4. $\vdash_{\mathbf{M}_P} \neg\Box\top \rightarrow (A \leftrightarrow \neg\Box A)$	classic logic on 1

Now suppose $\Box\top$. From 2, using a structural induction, we obtain that $A_\varphi \leftrightarrow \varphi$ and since $\neg A_\varphi$ is a tautology, we have that

5. $\vdash_{\mathbf{M}_P} \Box\top \rightarrow \neg\varphi$

Suppose $\neg\Box\top$. From 4, using structural induction we obtain that $B_\varphi \leftrightarrow \varphi$ and hence

6. $\vdash_{\mathbf{M}_P} \neg\Box\top \rightarrow \neg\varphi$

From 5 and 6 we obtain $\vdash_{\mathbf{M}_P} \neg\varphi$, thus showing that φ is $\mathbf{M}_P$-inconsistent.

Lemma 2. *The class $\mathcal{K}_P$ is the largest (with respect to set inclusion) class of non-normal frames for which the logic $\mathbf{M}_P$ is sound.*

Proof. Let (W, F) be a non-normal frame, and suppose $w \in W$ is such that $F(w)$ satisfies neither case 1 nor case 2 above. We show we can build a model that falsifies $p \leftrightarrow \Box p$ if case 1 is falsified. If case 1 is falsified, there are two possibilities. Suppose there is $X \in F(w)$ such that $w \notin X$, thus falsifying case 1; in this case, take $V(p) = X$, so that $W, F, V, w \models \neg p \wedge \Box p$. Now suppose there is a $X \subseteq W$ with $w \in X$ but $X \notin F(w)$, also falsified case 1; make $V(p) = X$, so that $W, F, V, w \models p \wedge \neg\Box p$. In both cases $p \leftrightarrow \Box p$ is falsified. Note that we did not have to set a valuation for q in this process.

In a totally analogous way we show that if case 2 is falsified, than $q \leftrightarrow \neg\Box q$ is falsified, and such that the valuation of p is not mentioned in this process. So if a frame is outside $\mathcal{K}_P$, it is possible to find a valuation that simultaneously falsifies $p \leftrightarrow \Box p$ and $q \leftrightarrow \neg\Box q$. Therefore any frame that is outside $\mathcal{K}_P$ can falsify axiom **P**, and since $\mathcal{K}_P$ is sound with respect to axiom **P**, it is the maximal **P**-sound class of frames.

Lemma 3. *The logic* $\mathbf{M}_P$ *is decidable. Furthermore, the decision of* $\mathbf{M}_P$ *is Co-NP-complete.*

Proof. Let φ be a $\mathbf{M}_P$-formula, and consider $\psi = \neg\varphi$. Using the same construction as in the proof of Theorem 3, consider the formulas A_ψ and B_ψ, where the former is obtained by deleting all $\Box$'s and the latter is obtained by replacing each $\Box$ with a $\neg$. Theorem 3 shows that ψ is consistent iff A_ψ or B_ψ is consistent. So all we have to do is to apply a classic propositional SAT algorithm to A_ψ and B_ψ. If either A_ψ or B_ψ are satisfiable, a model can be built to ψ, so φ is not a theorem; otherwise, φ is a theorem.

Logic $\mathbf{M}_P$ is clearly Co-NP-hard for it contains propositional logic. To show that it is in Co-NP, just note that we have a NP-decision for the satisfiability of both A_ψ and B_ψ, so we have a Co-NP-decision for the validity of φ.

Beside these logical properties, there are some properties that distinguish $\mathbf{M}_P$. The most notable one is that $\Box$ and $\Diamond$ can be switched in $\mathbf{M}_P$.

Lemma 4. $\vdash_{\mathbf{M}_P} \Box p \leftrightarrow \Diamond p$.

Proof. Since we have completeness, it suffices to prove the result semantically. Consider a model $\mathcal{M} \in \mathcal{K}_P$. If $\mathcal{M}$ satisfies case 1, then $\Box p$ is equivalent to p, and $\neg\Box\neg p$ is equivalent to $\neg\neg p$. If $\mathcal{M}$ satisfies case 2, then $\Box p$ is equivalent to $\neg p$, and $\neg\Box\neg p$ is equivalent to $\neg\neg\neg p$. In both cases we have the equivalence of $\Box p$ and $\Diamond p$.

4.3 Fusion of Partition Logics

Now consider the fusion of $\mathbf{M}_P$ with itself, $\mathbf{M}_{P1} \otimes \mathbf{M}_{P2}$, with modalities $\Box_1$ and $\Box_2$, constructed according to the presentation in Section 4.1. In fact, such a fusion transfers all basic logic properties.

Theorem 4. $\mathbf{M}_{P1} \otimes \mathbf{M}_{P2}$ *is sound, complete and decidable.*

Proof. (Sketch) The proof is totally analogous to that of Theorem 3. The transference of soundness is dealt by Theorem 2. For completeness, we use the same strategy consisting of constructing propositional models, and analysing four cases (the truth values of $\Box_1\top$ and $\Box_2\top$) instead of just two cases (the truth values of $\Box\top$). We also consider four formulas, instead of only A_φ and B_φ, in which each of the modalities $\Box_1$ and $\Box_2$ is either deleted of substituted by negation. The rest of the proof is totally analogous.

For decidability, we also use the same technique as in Lemma 3, and transform a formula in 4 propositional formulas by either deleting all $\Box_1$ or substituting all $\Box_1$ by $\neg$, and similarly for $\Box_2$. If one of these propositional formulas is falsifiable, a model is constructed that falsifies the original formula; otherwise, all formulas are valid and the original formula is valid too. Details omitted.

The proof of decidability above implies that the decision problem for $\mathbf{M}_{P1} \otimes \mathbf{M}_{P2}$ is Co-NP-complete.

The fusion of two modal logics, being the smallest logic that contain both its components, is not expected to contain any form of interaction between the two components. Forms of interactions do arise in the *product* of two modal logics [11], that satisfy the commutativity of two modalities, namely,

$$\vdash \Box_1\Box_2 A \leftrightarrow \Box_2\Box_1 A$$

and the convergence axiom

$$\vdash \Diamond_1\Box_2 \rightarrow \Box_2\Diamond_1 A$$

Note, however, that these two axioms do not always characterise the product of two logics.

In the fusion of two non-trivial normal modal logics such interaction never occurs. However, those properties do hold in the fusion $\mathbf{M}_{P1} \otimes \mathbf{M}_{P2}$, as shown below.

Theorem 5 (Commutativity). *$\vdash \Box_1\Box_2 A \leftrightarrow \Box_2\Box_1 A$.*

Proof. The proof is syntactic. We assume $\Box_1\Box_2 A$ and analyse four cases, according to whether $\Box_1\Box_2 A$ implies the formulas A and $\Box_2 A$ or their negation.

Case 1: Assume $\Box_1\Box_2 A$, A and $\Box_2 A$. Using axiom $\mathbf{P}$ in $\mathbf{M}_{P1}$, make $q := \Box_2 A$ and $p := A$. We conclude that $A \leftrightarrow \Box_1 A$ (because we cannot have $\Box_1\Box_2 A \leftrightarrow \neg\Box_2 A$) and hence $\Box_1 A$. Using axiom $\mathbf{P}$ in $\mathbf{M}_{P2}$ we obtain $(\Box_2\Box_1 A \leftrightarrow \Box_1 A) \vee (\Box_2 A \leftrightarrow \neg A)$ and we conclude $(\Box_2\Box_1 A \leftrightarrow \Box_1 A)$, obtaining $\Box_2\Box_1 A$.

Case 2: Assume $\Box_1\Box_2 A$, A and $\neg\Box_2 A$. We have $\neg(\Box_1\Box_2 A \leftrightarrow \Box_2 A)$ so, by $\mathbf{P}$ in $\mathbf{M}_{P1}$, $\Box_1 A \leftrightarrow \neg A$ and, from the hypothesis, $\neg\Box_1 A$. We also have $\neg(\Box_2 A \leftrightarrow A)$ so, by $\mathbf{P}$ in $\mathbf{M}_{P2}$, we obtain $\Box_2\Box_1 A \leftrightarrow \neg\Box_1 A$ and hence $\Box_2\Box_1 A$.

Case 3: Assume $\Box_1\Box_2 A$, $\neg A$ and $\Box_2 A$. We have $\neg(\Box_1\Box_2 A \leftrightarrow \neg\Box_2 A)$, so $\Box_1 A \leftrightarrow A$ and, from the hypothesis, $\neg\Box_1 A$. We also have $\neg(\Box_2 A \leftrightarrow A)$, which yields $\Box_2\Box_1 A \leftrightarrow \neg\Box_1 A$ and thus $\Box_2\Box_1 A$.

Case 4: Assume $\Box_1\Box_2 A$, $\neg A$ and $\neg\Box_2 A$. We have $\neg(\Box_1\Box_2 A \leftrightarrow \Box_2 A)$, so $\Box_1 A \leftrightarrow \neg A$ and, from the hypothesis, $\Box_1 A$. We also have $\neg(\Box_2 A \leftrightarrow \neg A)$, which yields $\Box_2\Box_1 A \leftrightarrow \Box_1 A$ and thus $\Box_2\Box_1 A$.

From the four cases above, by classic logic, we obtain $\vdash \Box_1\Box_2 A \rightarrow \Box_2\Box_1 A$; the converse is totally analogous.

Theorem 6 (Confluence). *$\vdash \Diamond_1\Box_2 A \rightarrow \Box_2\Diamond_1 A$.*

Proof. Immediate from Theorem 5 and Lemma 4.

We call this phenomenon a *strong interaction* arising in the fusion of two non-normal modal logics. This brings us problems in considering a formula in a fusion as a formula in an iterated modalisation, which causes our proof strategy to fail.

To see that, consider the formula $A_0 = \Box_1\Box_2 p \wedge \neg\Box_2\Box_1 p$. From Theorem 5 we see that such a formula is $\mathbf{M}_{P1} \otimes \mathbf{M}_{P2}$-inconsistent. However, if we consider such formula as an iterated modalisation $\mathbf{M_{P}}_1(\mathbf{M_{P}}_2(\mathbf{M_{P}}_1))$, it is possible to build a model for it.

The construction of a model for A_0 in $\mathbf{M_{P}}_1(\mathbf{M_{P}}_2(\mathbf{M_{P}}_1))$ is done according to Figure 1. The σ-translation analyses the monolithic subformulas of A_0 to see if any boolean combination of those formulas is $\mathbf{M_{P}}_2(\mathbf{M_{P}}_1)$-inconsistent. If no such inconsistent combination arises, each monolithic subformula is mapped into a propositional letter. The monolithic subformulas of A_0 are $B_1 = \Box_2 p$ and $B_2 = \Box_2\Box_1 p$, and no boolean combination of these formulas is $\mathbf{M_{P}}_2(\mathbf{M_{P}}_1)$-inconsistent, so that $\sigma(A_0) = \Box_1 q_{B_1} \wedge \neg q_{B_2}$ is an $\mathbf{M}_{P1}$-consistent formula. So any $\mathbf{M}_{P1}$-model $\mathcal{M}_1$ for $\sigma(A_0)$ serves as a basis for a $\mathbf{M_{P}}_1(\mathbf{M_{P}}_2(\mathbf{M_{P}}_1))$-model for A_0. For each point $w \in \mathcal{M}_1$ we consider the set of $\mathbf{M_{P}}_2(\mathbf{M_{P}}_1)$ formulas $\Gamma_w = \{B_i | \mathcal{M}_1, w \models q_{B_i}\} \cup \{\neg B_i | \mathcal{M}_1, w \not\models q_{B_i}\}$. Since no boolean combination of B_i's is inconsistent, such a model always exists. Thus, recursively, a model for A_0 is built via iterated modalisations. We have shown that the $\mathbf{M}_{P1} \otimes \mathbf{M}_{P2}$-inconsistent A_0 has a $\mathbf{M_{P}}_1(\mathbf{M_{P}}_2(\mathbf{M_{P}}_1))$-model.

The fact that an inconsistent formula in the fusion has a model in the iterated modalisation contradicts the view of the strategy, which considers fusion as a set of iterated modalisations. Note, however, that the logic $\mathbf{M}_{P1} \otimes \mathbf{M}_{P2}$ is *not* a counterexample for the transference of logic properties in the fusion of two non-normal modal logics.

We have thus proved the following result.

Theorem 7. *There are non-normal modal logics whose fusion may not be reduced to an iterated modalisation.*

The result above does not rule out other forms of decomposition of fusions. In particular, it may be the case that a different composition, possibly stronger than modalisation, may be used to achieve that goal.

5 Other Special and General Results about Fusion

5.1 Some Examples of Transference of Completeness

Despite the fact that the strategy of viewing fusion as iterated modalisation did not succeed in proving the transference of completeness, that transference can be proved for the fusion of several non-normal modal logics directly from the properties of the component logics.

For that, let $\mathbf{K_{nn}}$ be the minimal non-normal modal logic, whose axiomatisation is given by all classical propositional tautologies plus the inference rules of Modus Ponens, Substitution and Congruence; its class of models is the class of all minimal models.

Consider the following axioms:

M $(\Box p \wedge \Box q) \to \Box(p \wedge q)$
C $\Box(p \wedge q) \to \Box p$
N $\Box\top$
D $\Box p \to \Diamond p$
T $\Box p \to p$
B $p \to \Box\Diamond p$
4 $\Box p \to \Box\Box p$
5 $\Diamond p \to \Box\Diamond p$

We then associate these axioms with restrictions over minimal structures (W, F), such that each axiom is sound and complete over minimal models respecting its corresponding restriction [3]:

M* For every $w \in W$, if $X, Y \in F(w)$ then $X \cap Y \in F(w)$.
C* For every $w \in W$, if $X \in F(w)$ and $X \subseteq Y \subseteq W$ then $Y \in F(w)$.
N* For every $w \in W$, $W \in F(w)$.
D* For every $w \in W$, if $X \in F(w)$ then $W \setminus X \notin F(w)$.
T* For every $w \in W$, if $X \in F(w)$ then $w \in X$.
B* For every $w \in W$, if $w \in X$ then $\{w' \in W | W \setminus X \notin F(w')\} \in F(w)$.
4* For every $w \in W$, if $X \in F(w)$ then $\{w' \in W | X \in F(w')\} \in F(w)$.
5* For every $w \in W$, if $X \notin F(w)$ then $\{w' \in W | X \notin F(w')\} \in F(w)$.

We have the following result concerning the fusion of non-normal modal logics.

Theorem 8. *The fusion* $\mathbf{M}_1 \otimes \mathbf{M}_2$ *is complete, where each* $\mathbf{M}_i$*,* $i = 1, 2$ *is one of the axiomatic systems below, with its corresponding semantics:*

1. $\mathbf{K_{nn}}$.
2. $\mathbf{K_{nn}}$ *extended with one or more of the normality axioms* **M**, **C** *and* **N**.
3. $\mathbf{K_{nn}}$ *extended with one of the axioms:* **D**, **T**, **B**, **4** *and* **5**.

Proof. (Sketch) The proof develops along the lines of completeness proofs for non-normal modal logics in [3], in which a canonical bi-relational minimal model is built for $\mathbf{M}_1 \otimes \mathbf{M}_2$. We omit the details of the proof, which can be found in [4].

5.2 Transference of Strong Completeness

Our discussion of completeness so far has dealt only with what is called *weak completeness*, a property that says that every consistent formula has a model. On the other hand, *strong completeness* holds if every (finite or infinite) consistent set of formulas has a model. Clearly, strong completeness implies weak completeness, but not the other way round. In fact, strong completeness is equivalent to weak completeness plus compactness, namely the property that says that if every finite subset of a set Γ of formulas has a model, then Γ has a model.

The transference of strong completeness has been proved for the fibring, a method for combining logics that has fusion as a particular case [18]. In fact, fusion is fibring of two logics that share precisely the classical boolean connectives, classical inference rules, and some form of classical model.

The proof of transference of strong completeness is obtained with a traditional Henkin construction for completeness. A consistent set of formulas is extended to a maximal consistent set. The strong completeness of each component logic is then used to build a canonical combined model.

This proof relies on the fact that a consistent infinite set of formulas has a model, a fact that obviously cannot be used in the transference of weak completeness.

5.3 Transference of Decidability

In a recent work, Baader, Ghilardi and Tinelli [2] have proved the transference of decidability in the fusion of any modal logics. That is, they have proved the following result.

Theorem 9 ([2]). *If $\mathbf{M}_1$ and $\mathbf{M}_2$ are decidable modal logics, so is the fusion $\mathbf{M}_1 \otimes \mathbf{M}_2$.*

The paper [2] actually studies the combination of first-order equational theories and shows that, when certain conditions are met, the union of two decidable first-order equational theories is also decidable. The proof uses non-trivial algebraic techniques to obtain such result.

The paper then shows a simple way of encoding modal axioms as first-order equational theories. Basically, propositional atoms become variables, boolean connectives and n-ary modalities become function symbols, such that any modal axiom $\vdash A$ is translated into an equation

$$\mathrm{term}(A) = 1$$

where $\mathrm{term}(A)$ is the first-order term obtained by the translation of axiom $\vdash A$, and 1 is the constant representing the top of a boolean algebra with operators. An equational axiomatisation of boolean algebra is added, so that any axiomatisable classical modal logic can be put into equational theory.

The paper then shows that any equational theory obtained from a modal axiomatisation obeys the restrictions that allow two decidable equational theories to be united preserving decidability. Because the fusion of axiomatisations is also the result of a union of theories, and as no extra requirement of normality is imposed to the modal theories, it follows that the fusion of decidable modal logics is also decidable.

The approach to fusion by Baader, Ghilardi and Tinelli is very original and interesting, and brings new techniques to the field. Unfortunately, since that work deals basically with first-order equational theory, it does not bring light on how to combine modal models, so as to obtain the transference of completeness.[3]

6 Conclusion

In this paper we have discussed strategies for proving the transference of completeness result in the fusion of non-normal modal logics. We have argued that the algebraic techniques based on a syntactic notion of normality in [17, 1] are not suitable to be directly applicable to the non-normal case, and we have shown that the semantical techniques for combining logics in [7, 10] are not applicable either.

6.1 What is the real difficulty?

So what is the real difficulty? Is this problem related to weak completeness only? In fact, the difficulty lies in the model-theoretic side of non-normal modal logics. Those structures are really hard to combine preserving the properties of the component structures. The usual disjoint union of two such structures does not produce an acceptable structure, which motivated us to define the f-disjoint union in Section 4.1. In that definition we see that to bring together two non-normal model, one needs to add a great number of sets to the neighbourhood of an element $F(w)$. It is this explosion that we have not been able to control with the iterated modalisation approach, as that approach maps one formula of a higher level into a single formula in the lower level. In the case there is access to infinitely many formulas, as in proofs of strong completeness, the problem may be overcome.

The results presented here leave us with three possibilities with respect to the study of the transference of completeness in the fusion of non-normal modal logics:

1. It may be the case that weak completeness is not transfered. In this case, a counterexample to be found has to involve at least one strictly non-normal modal logic, in which one of the axioms **M**, **N** or **C** fails. Incompleteness results for normal or non-normal modal logics tend to show that there is a formula that holds in all possible models that is not inferable from the set of axioms; see, for instance, [13].
2. Perhaps completeness is transfered via fusion, as the result in [2] seem to indicate. However, a new technique for showing such a result may be necessary, perhaps a modification of the techniques in [2].
3. The view that fusion can be expressed as iterated combinations may still prove to be correct. One basic step in the transference of completeness in

[3] This opinion was confirmed in a private conversation with Franz Baader.

modalisation is the existence of a consistency preserving mapping σ, as illustrated in Figure 1. We have shown in this paper that the mapping provided in [10] does not preserve consistency over iterated modalisation. But perhaps there exists another consistency preserving mapping that keeps that property over iterated modalities. An initial attempt in this direction was made in [4], where a modified mapping was defined not in terms of **L**-monolithic formulas, but in terms of all **L**-subformulas. That mapping, however, also failed to preserve consistency over iterated modalities.

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