General recursive functions of natural numbers[1]).

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The substitution

$$(1) \qquad \varphi(x_1, \ldots, x_n) = \theta(\chi_1(x_1, \ldots, x_n), \ldots, \chi_m(x_1, \ldots, x_n)),$$

and the ordinary recursion with respect to one variable

$$(2) \qquad \begin{aligned} \varphi(0, x_2, \ldots, x_n) &= \psi(x_2, \ldots, x_n) \\ \varphi(y+1, x_2, \ldots, x_n) &= \chi(y, \varphi(y, x_2, \ldots, x_n), x_2, \ldots, x_n), \end{aligned}$$

where $\theta, \chi_1, \ldots, \chi_m, \psi, \chi$ are given functions of natural numbers, are examples of the definition of a function φ by equations which provide a step by step process for computing the value $\varphi(k_1, \ldots, k_n)$ for any given set $k_1, \ldots, k_n$ of natural numbers. It is known that there are other definitions of this sort, e. g. certain recursions with respect to two or more variables simultaneously, which cannot be reduced to a succession of substitutions and ordinary recursions[2]). Hence, a characterization of the notion of recursive definition in general, which would include all these cases, is desirable. A definition of general recursive function of natural numbers was suggested by Herbrand to Gödel, and was used by Gödel with an important modification in a series of lectures at Princeton in 1934. In this paper we offer several observations on general recursive functions, using essentially Gödel's form of the definition.

The definition will be stated in § 1. It consists in specifying the form of the equations and the nature of the steps admissible in the computation of the values, and in requiring that for each given set of arguments the computation yield a unique number as value. The operations on symbols which occur in the computation have a similarity to ordinary recursive operations on numbers. This similarity will be utilized, by the Gödel method of representing formulas by numbers, to prove that every (general) recursive function is expressible in the form $\psi(\varepsilon y[\varrho(x_1 \ldots, x_n, y) = 0])$ where ψ and ϱ are ordinary or „primitive"

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[1]) Presented to the American Mathematical Society, September 1935.

[2]) W. Ackermann, Zum Hilbertschen Aufbau der reellen Zahlen, Math. Annalen 99 (1928), S. 118—133; Rózsa Péter, Konstruktion nichtrekursiver Funktionen, Math. Annalen 111 (1935), S. 42—60.

recursive functions and $(x_1, \ldots, x_n)\,(E\,y)\,[\varrho\,(x_1, \ldots, x_n, y) = 0]$[3]). Also, it is seen directly that, for any recursive function $\varrho\,(x_1, \ldots, x_n, y)$, $\varepsilon\,y\,[\varrho\,(x_1, \ldots, x_n, y) = 0]$ is a recursive function, provided $(x_1, \ldots, x_n)\,(E\,y)$ $[\varrho\,(x_1, \ldots, x_n, y) = 0]$.

In § 2, the problem is raised, which systems of equations define recursive functions under the general definition. The systems which do cannot be recursively enumerated, if by a recursive enumeration is understood one such that the numbers ordered by the Gödel method to the systems of equations in the enumeration are a recursive sequence (i. e. the successive values of a recursive function of one variable), since from any recursive sequence of such numbers we can obtain the recursive definition of a new function by the familiar process of diagonalizing and adding 1. For the same reason, a recursive process of deciding which systems define recursive functions is unattainable, if by a recursive process is meant one such that there is a recursive function of the corresponding numbers whose value is 0 or 1 according to the result obtained. Since the condition under which a recursive function of n variables is defined can be expressed in the form $(x_1, \ldots, x_n)\,(E\,y)$ $[\varrho\,(x_1, \ldots, x_n, y) = 0]$, we are afforded an approach (somewhat different than Gödel's[4])) to the existence of undecidable number-theoretic propositions in formal logics satisfying certain general conditions. Roughly speaking, every such formal logic must contain undecidable propositions of the form $(x)\,(E\,y)\,[\varrho\,(x, y) = 0]$, where $\varrho\,(x, y)$ is a primitive recursive function, because otherwise the logic could be used to decide recursively which systems of equations define recursive functions, which we know in advance to be impossible. Every problem of the form, whether or not $(x)\,(E\,y)\,[\sigma\,(x, y) = 0]$, where $\sigma\,(x, y)$ is a recursive function, is included in the problem, which systems of equations define recursive functions of one variable.

Also, there are non-recursive functions definable using only one quantifier, thus: $\tau\,(x) = 0$ if $(y)\,[\varrho\,(x, y) = 0]$, $\tau\,(x) = 1$ otherwise, where $\varrho\,(x, y)$ is primitive recursive.

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[3]) In the "functions" which we consider, the arguments are understood to range over the natural numbers (i. e. non-negative integers) and the values to be natural numbers. Also, for abbreviation, we use propositional functions of natural numbers, calling them „relations" (alternatively "classes", when there is only one variable) and employing the following notations: $(x)\,A\,(x)$ [for all natural numbers, $A\,(x)$], $(E\,x)\,A\,(x)$ [there is a natural number x such that $A\,(x)$], $\varepsilon\,x\,[A\,(x)]$ [the least natural number x such that $A\,(x)$, or 0 if there is no such number], $-$ [not], $\vee$ [or], & [and], $\rightarrow$ [implies], $\equiv$ [is equivalent to].

[4]) Kurt Gödel, Über formal unentscheidbare Sätze der Principia Mathematica und verwandter Systeme I, Monatsh. für Math. u. Physik 38 (1931), S. 173—198.

§ 1.

The relation between primitive and general recursive functions.

A recursive function (relation) in the sense of Gödel[4] (S. 179—180) will now be called a *primitive recursive* function (relation). By using

$$
\begin{aligned}
&S(x) = x + 1 && \text{(the successor function),}\\
(3)\qquad &C(x) = 0 && \text{(the constant function 0),}\\
&U_i^n(x_1, \ldots, x_n) = x_i && \text{(identity functions)}
\end{aligned}
$$

as initial functions, the definition of primitive recursive function can be phrased thus:

Definition 1. A function is *primitive recursive* if it can be defined from the functions (3) by (zero or more) successive applications of schemas (1) and (2) ($m, n = 1, 2, \ldots;\ i = 1, \ldots, n$)[5].

In the study of general recursive functions, we treat the defining equations formally, as sequences of symbols. For abbreviation, we may omit to distinguish between the functions and numbers, and the symbols or sets of symbols which stand for them.

Now consider *expressions* consisting of finite sequences of the following symbols: 0 (the numeral 0), S (the successor function), $w_0, w_1, \ldots$ (numerical variables), $\varrho_0, \varrho_1, \ldots$ (variables for functions of $r_0, r_1, \ldots$ arguments, where $r_0, r_1, \ldots$ is a sequence of positive integers in which each occurs infinitely many times, say 1, 1, 2, 1, 2, 3, ...), (,), , , = (parentheses, comma, equality symbol). We define *term* thus: $0, w_0, w_1, \ldots$ are terms; if $a_1, a_2, \ldots$ are terms, $S(a_1)$, $\varrho_0(a_1, \ldots, a_{r_0})$, $\varrho_1(a_1, \ldots, a_{r_1})$, ... are terms. By *numeral* is meant one of the expressions 0, $S(0)$, $S(S(0))$, ... If a and b are terms (and if $\sigma_1, \ldots, \sigma_n$ are functional variables[6]) such that a least one of $\sigma_1, \ldots, \sigma_n$ occurs in a or b, but no functional variables other than $\sigma_1, \ldots, \sigma_n$ occur in a or b), $a = b$ will be called an *equation* (*in* $\sigma_1, \ldots, \sigma_n$). By a *system* of equations we mean a finite sequence of equations. $\mathrm{S}^{a_1 \ldots a_n}_{b_1 \ldots b_n} A \Big|$ shall denote the result of substituting b_i for a_i ($i = 1, \ldots, n$) throughout A (A itself, if $a_1, \ldots, a_n$

[5]) This form of the definition was introduced by Gödel to avoid the necessity of providing for omissions of arguments on the right in schemas (1) and (2). The operations in the construction of primitive recursive functions can be further restricted. See Rózsa Péter, Über den Zusammenhang der verschiedenen Begriffe der rekursiven Funktionen, Math. Annalen 110 (1934), S. 612—632.

[6]) That is, if $\sigma_1, \ldots, \sigma_n$ stand for $\varrho_{\alpha_1}, \ldots, \varrho_{\alpha_n}$ for some set of distinct numbers $\alpha_1, \ldots, \alpha_n$ (then we use s_i for r_{α_i}). Similarly, in R_1 below it is meant that $x_1, \ldots, x_n$ stand for $w_{\beta_1}, \ldots, w_{\beta_n}$ for some set of distinct numbers $\beta_1, \ldots, \beta_n$.

do not occur in A). $E \vdash_{\nu_1 \nu_2 \ldots} F$ shall denote that the expression F is derivable from the expressions E by (zero or more) applications of the operations $R_{\nu_1}, R_{\nu_2}, \ldots$.

We list the operations on expressions[7]):

R_1: *to replace A by* $\mathrm{S}_{k_1 \ldots k_n}^{x_1 \ldots x_n} A \Big|$, *where $x_1, \ldots, x_n$ are the numerical variables which occur in A, and $k_1, \ldots, k_n$ are numerals.*

R_2: *to pass from A and $\sigma(k_1, \ldots, k_s) = k$ to the result of substituting k for a particular occurrence of $\sigma(k_1, \ldots, k_s)$ in A, where $k_1, \ldots, k_s$, k are numerals.*

R_3: *to pass from A and $B = C$ to the result of substituting C for a particular occurrence of B in A.*

The Herbrand-Gödel definition of general recursive function of natural numbers can be formulated thus[8]):

Definition 2a. Given functional variables $\sigma_1, \ldots, \sigma_n$, let E_j^* denote the set of equations $\sigma_j(k_1, \ldots, k_{s_j}) = k$ where k is the "value" of $\sigma_j(k_1, \ldots, k_{s_j})$ as presently defined. The functions $\sigma_1, \ldots, \sigma_n$ are *defined recursively* by the system of equations $(E_1 \ldots E_n)$ if, for each i $(i=1, \ldots, n)$, E_i is a system of equations in $\sigma_1, \ldots, \sigma_i$, each of the form $\sigma_i(a_1, \ldots, a_{s_i}) = b$ where σ_i does not occur in $a_1, \ldots, a_{s_i}$, such that for each set of numerals $k_1, \ldots, k_{s_i}$ there is exactly one numeral k (called the *value* of $\sigma_i(k_1, \ldots, k_{s_i})$) for which $E_1^*, \ldots, E_{i-1}^*, E_i \vdash_{1,2} \sigma_i(k_1, \ldots, k_{s_i}) = k$. A function σ_n is *recursive* if there is an $(E_1 \ldots E_n)$ of this description.

We understand a function $\varphi(x_1, \ldots, x_m)$ to be recursive under this definition, if it is possible to define it by recursion equations of the type described, whether or not originally the function is so defined. More explicitly, a given function $\varphi(x_1, \ldots, x_m)$ is recursive under Def. 2a, if there exists an $(E_1 \ldots E_n)$ as described in Def. 2a in which σ_n may be regarded as representing φ. σ_n may be regarded as representing φ, if $s_n = m$ and whenever $k_1, \ldots, k_{s_n}$ are the numerals $S(\ldots x_1 \text{ times} \ldots S(0) \ldots)$, $\ldots, S(\ldots x_m \text{ times} \ldots S(0) \ldots)$, resp., the "value of $\sigma_n(k_1, \ldots, k_{s_n})$" under Def. 2a is the numeral $S(\ldots \varphi(x_1, \ldots, x_m) \text{ times} \ldots S(0) \ldots)$. A similar remark applies to Def. 2b below.

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[7]) In these operations we do not require that A and $B = C$ be equations and that σ be a functional variable, since $R_1 - R_3$ as stated when applied to equations generate equations. Thereby, our proof of IV is simplified.

[8]) In what follows, the word "recursive" (when not qualified by the adjective "primitive") will mean recursive under any one of the definitions 2a, 2b and 2c, except when the definition involved is mentioned explicitly (as is necessary in the course of establishing the theorems VI and IX on their equivalence).

We now show that Def. 2a is not more general than the following (which will later be proved equivalent to it):

Definition 2b. The functions $\sigma_1, \ldots, \sigma_n$ are *defined recursively* by E, if E is a system of equations in $\sigma_1, \ldots, \sigma_n$ such that for each i ($i = 1, \ldots, n$) and each set of numerals $k_1, \ldots, k_{s_i}$ there is exactly one numeral k (called the *value* of $\sigma_i(k_1, \ldots, k_{s_i})$) for which $E \vdash_{1,3} \sigma_i(k_1, \ldots, k_{s_i}) = k$. A function σ_n is *recursive* if there is an E of this description[9]).

For the system of equations $(E_1 \ldots E_n)$ of Def. 2a can be proved to be a system E for Def. 2b thus: Clearly, $(E_1 \ldots E_n)$ is a system of equations in $\sigma_1, \ldots, \sigma_n$, and for each i and set of numerals $k_1, \ldots, k_{s_i}$, $(E_1 \ldots E_n) \vdash_{1,3} \sigma_i(k_1, \ldots, k_{s_i}) = k$ where k is the value of $\sigma_i(k_1, \ldots, k_{s_i})$ under Def. 2a. It remains to be shown that $(E_1 \ldots E_n) \vdash_{1,3} \sigma_i(k_1, \ldots, k_{s_i}) = l$ for l a numeral only when $l = k$. Now each equation of $(E_1 \ldots E_n)$ is verifiable (for each replacement of its numerical variables by numerals) by use of the values under Def. 2a, since, on examination, the supposition of the contrary is found to conflict with the hypothesis that for given i and numerals $k_1, \ldots, k_{s_i}$ there is only one numeral k such that $E_1^*, \ldots, E_{i-1}^*, E_i \vdash_{1,2} \sigma_i(k_1, \ldots, k_{s_i}) = k$[9a]). Moreover, R_1 and R_3 applied to verifiable equations yield verifiable equations. Hence, if $(E_1 \ldots E_n) \vdash_{1,3} \sigma_i(k_1, \ldots, k_{s_i}) = l$ where $k_1, \ldots, k_{s_i}$, l are numerals, the values of $\sigma_i(k_1, \ldots, k_{s_i})$ and l must be the same, i. e. l must be the value k of $\sigma_i(k_1, \ldots, k_{s_i})$ under Def. 2a.

The set of operations R_1, R_3 may be replaced in Def. 2b by a set R_i' ($i = 0, 1, 2, \ldots$) of single-valued binary operations, defined over all pairs of equations as follows:

R_{3i}' : *to pass from A and B to* $\mathrm{S}^{w_i}_{S(w_i)} A \Big|$.

R_{3i+1}': *to pass from A and B to* $\mathrm{S}^{w_i}_{0} A \Big|$.

R_{3i+2}': *to pass from A and $B = C$ to the result of replacing the occurrence of B in A beginning with the $i + 1^{st}$ symbol by C, if there is such an occurrence; otherwise, to A itself.*

For, under the conditions of Def. 2b, $E \vdash'_{0,1,2,\ldots} \sigma_i(k_1, \ldots, k_{s_i}) = l$ (l a numeral) when l is the value of $\sigma_i(k_1, \ldots, k_{s_i})$ under Def. 2b and only then (as is easily shown).

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9) A more general definition would not be obtained by allowing under R_3 also the substitution of B for C, since E may be chosen to include $b = a$ whenever $a = b$ is included.

9a) Similarly, the equations of the system E of Def. 2b are verifiable by use of the values under Def. 2b, if they are of the form $\sigma(a_1, \ldots, a_s) = b$.

We now assign numbers to symbols, expressions, finite sequences of expressions, etc., by the Gödel method [*loc. cit.*[4]) S. 179—182], letting numbers correspond to symbols thus:

$$\text{“}0\text{”} \ldots 1,\ \text{“}S\text{”} \ldots 3,\ \text{“}=\text{”} \ldots 5,\ \text{“},\text{”} \ldots 7,\ \text{“}(\text{”} \ldots 11,\ \text{“})\text{”} \ldots 13,$$

$$\text{“}w_i\text{”} \ldots p_{i+7},\ \text{“}\varrho_i\text{”} \ldots p^2_{i+7},$$

where p_i denotes the i^{th} prime number. Then if the numbers corresponding to $N_1, \ldots, N_k$ are $n_1, \ldots, n_k$, resp., the number corresponding to the sequence $N_1, \ldots, N_k$ is $p_1^{n_1} \ldots p_k^{n_k}$. Employing Gödel's notations (including the use of italics to indicate the correspondent for numbers of a given notion relating to expressions) and his methods of exhibiting the primitive recursiveness of functions and relations[9b]), we adopt 1—10 of his list, modifying 6, and define further primitive recursive functions and relations, as follows:

6. $n\,Gl\,x = \varepsilon y\,[y \leqq x \;\&\; x \,|\, (Pr(n))^y \;\&\; \overline{x \,|\, (Pr(n))^{y+1}}]$.

The finite sequence $n_1, \ldots, n_k$ of positive integers is represented by $p_1^{n_1} \ldots p_k^{n_k}$. Also, we may use the positive integer x to represent any sequence $n_1, \ldots, n_k$ of natural numbers such that $x = p_1^{n_1} \ldots p_k^{n_k}$. The modification in the definition of $n\,Gl\,x$ secures that $n\,Gl\,x$ always be the n^{th} member. The significances ascribed to $l(x)$, $x * y$, etc., refer only to the case $n_1, \ldots, n_k > 0$[10]).

11. $x \dot{-} y = \varepsilon z\,[z \leqq x \;\&\; x = y + z]$.

If $x \geqq y$, $x \dot{-} y = x - y$; if $x \leqq y$, $x \dot{-} y = 0$.

12. $\left[\frac{x}{y}\right] = \varepsilon z\,[z \leqq x \;\&\; (z+1)\,y > x]$.

13. $\mathrm{Rem}\,(x, y) = x \dot{-} \left(\left[\frac{x}{y}\right]\right) y$.

14. $\mathrm{Dy}\,(0) = 1$,

$$\begin{aligned}\mathrm{Dy}\,(k+1) = \varepsilon z\Big[z \leqq 3^k \;\&\; \{&[1\,Gl\,\mathrm{Dy}\,(k) < 2\,Gl\,\mathrm{Dy}\,(k) \;\&\; z = 2^{[1\,Gl\,\mathrm{Dy}\,(k)]+1}\,3^{2\,Gl\,\mathrm{Dy}\,(k)}]\\ &\vee [1\,Gl\,\mathrm{Dy}\,(k) \geqq 2\,Gl\,\mathrm{Dy}\,(k) > 0 \;\&\; z = 2^{1\,Gl\,\mathrm{Dy}\,(k)}\,3^{[2\,Gl\,\mathrm{Dy}\,(k)] \dot{-} 1}]\\ &\vee [2\,Gl\,\mathrm{Dy}\,(k) = 0 \;\&\; z = 3^{[1\,Gl\,\mathrm{Dy}\,(k)]+1}]\}\Big].\end{aligned}$$

[9b]) Also see Th. Skolem, Begründung der elementaren Arithmetik durch die rekurrierende Denkweise ohne Anwendung scheinbarer Veränderlichen mit unendlichem Ausdehnungsbereich, Videnskapsselskapets Skrifter 1923. I. Mat.-naturv. Kl., Nr. 6, S. 1—38.

[10]) Note that $l(1) = 0$ and $x * 1 = 1 * x = x\,[l(x) > 0]$.

Dy (k) represents the $k+1^{\text{st}}$ pair of numbers in the following order:

$$0\ 0;\ 0\ 1,\ 1\ 1,\ 1\ 0;\ 0\ 2,\ 1\ 2,\ 2\ 2,\ 2\ 1,\ 2\ 0;\ \ldots.$$

15. $v \operatorname{Occ} x \equiv (En)\,[0 < n \leqq l(x) \,\&\, v = n\,Gl\,x]$.

The *symbol* v occurs in the *expression* x.

16. $$Su\,x\binom{n}{y} = \varepsilon z\,\{z \leqq [Pr(l(x)+l(y))]^{x+y} \,\&\, (Eu, v)\,[u, v \leqq x \,\&\, x = u * R(n\,Gl\,x) * v \,\&\, z = u * y * v \,\&\, n = l(u)+1]\}.$$

„$Su\,x\binom{n}{y}$ entsteht aus x, wenn man an Stelle des n-ten Gliedes von x y einsetzt (vorausgesetzt, daß $0 < n \leqq l(x)$)" (Gödel, S. 184, Nr. 27).

17. $$\operatorname{Sb}(0, x, v, y) = x,$$

$$\operatorname{Sb}(k+1, x, v, y) = \varepsilon z\Big[z \leqq \operatorname{Sb}(k, x, v, y) + Su \operatorname{Sb}(k, x, v, y)\binom{k+1}{y} \,\&\, \Big\{[k+1\,Gl\,x \neq v \,\&\, z = \operatorname{Sb}(k, x, v, y)] \vee \Big[k+1\,Gl\,x = v \,\&\, z = Su \operatorname{Sb}(k, x, v, y)\binom{k+1}{y}\Big]\Big\}\Big].$$

$\operatorname{Sb}(k, x, v, y)$ is the result of substituting the *expression* y for the *symbol* v throughout the first k *symbols* of the *expression* x (if $k \leqq l(x)$).

18. $S(x, v, y) = \operatorname{Sb}(l(x), x, v, y)$.

$S(x, v, y)$ corresponds to the operation $\operatorname{S}\overset{v}{\underset{y}{x}}\Big|$ (if v is *a symbol* and x and y are *expressions*).

19. $$\operatorname{St}(x, n, a, y) = \varepsilon z\Big[z \leqq [Pr(l(x)+l(y))]^{x+y} \,\&\, \Big\{(Ep, q)\,[p, q \leqq x \,\&\, x = p * a * q \,\&\, l(p) = n \,\&\, z = p * y * q] \vee \Big[(p, q)\,[p, q \leqq x \,\&\, x = p * a * q \rightarrow l(p) \neq n] \,\&\, z = x\Big]\Big\}\Big].$$

$\operatorname{St}(x, n, a, y)$ is the result of substituting the *expression* y for the occurrence of the *expression* a in the *expression* x beginning with the $n+1^{\text{st}}$ *symbol*, if there is such an occurrence; otherwise, x itself.

20. $R_0''(i, x, y) = S\big(x, Pr(i+7), R(3) * E(Pr(i+7))\big)$

$R_1''(i, x, y) = S\big(x, Pr(i+7), R(1)\big)$.

$$R_2''(i, x, y) = St\big(x, i, \varepsilon p\,[p \leqq y \,\&\, (Eq)\,[q \leqq y \,\&\, y = p * R(5) * q]], \varepsilon q\,[q \leqq y \,\&\, (Ep)\,[p \leqq y \,\&\, y = p * R(5) * q]]\big).$$

$R_0''(i,x,y)$, $R_1''(i,x,y)$, $R_2''(i,x,y)$ correspond to the operations R'_{3i}, R'_{3i+1}, R'_{3i+2}, resp.

21. $$R'(n,x,y) = \varepsilon z\Big[z \leqq R_0''\Big(\Big[\frac{n}{3}\Big], x, y\Big) + R_1''\Big(\Big[\frac{n \dot{-} 1}{3}\Big], x, y\Big) + R_2''\Big(\Big[\frac{n \dot{-} 2}{3}\Big], x, y\Big) \,\&\, \Big\{\Big[n \,|\, 3 \,\&\, z = R_0''\Big(\Big[\frac{n}{3}\Big], x, y\Big)\Big] \vee \Big[n+2 \,|\, 3 \,\&\, z = R_1''\Big(\Big[\frac{n \dot{-} 1}{3}\Big], x, y\Big)\Big] \vee \Big[n+1 \,|\, 3 \,\&\, z = R_2''\Big(\Big[\frac{n \dot{-} 2}{3}\Big], x, y\Big)\Big]\Big\}\Big].$$

$R'(n,x,y)$ corresponds to the operation R'_n.

22. $Z(0) = R(1)$,
$Z(n+1) = R(3) * E(Z(n))$.

$Z(n)$ corresponds to the numeral $S(\ldots n \text{ times} \ldots S(0))$.

23. $$\mathrm{Eval}_p(n, y, x_1, \ldots, x_p) \equiv (Ex)\{x \leqq y \,\&\, y = R([Pr(n+7)]^2) * E(Z(x_1) * R(7) * \ldots * R(7) * Z(x_p)) * R(5) * Z(x)\}$$ (for a fixed number p).

y corresponds to an expression of the form $\varrho_n(x_1, \ldots, x_p) = x$, where x is a numeral.

24. $\mathrm{Val}(y) = \varepsilon x\{x \leqq y \,\&\, (Em)[m \leqq y \,\&\, y = m * Z(x)]\}$.

If y corresponds to an expression of the form $a = x$ where x is a numeral, then $\mathrm{Val}(y) = x$.

Supposing the function $\varphi(n,x,y)$ given, we define a series of functions as follows:

$$\psi(0, x, y) = x,$$
$$\psi(n+1, x, y) = \varphi(n, x, y).$$
$$\lambda(0, z) = l(z),$$
$$\lambda(k+1, z) = [k+1] \cdot \lambda(k, z)^2.$$
$$\tau(0, z) = z,$$
$$\tau(k+1, z) = \prod_{n=0}^{\lambda(k+1, z) \dot{-} 1} [Pr(n+1)] \exp\Big\{\psi\Big(\Big[\frac{n}{\lambda(k,z)^2}\Big], \Big[\big[1\, Gl\, \mathrm{Dy}(\mathrm{Rem}(n, \lambda(k,z)^2))\big] + 1\Big] Gl\, \tau(k,z), \Big[\big[2\, Gl\, \mathrm{Dy}(\mathrm{Rem}(n, \lambda(k,z)^2))\big] + 1\Big] Gl\, \tau(k,z)\Big)\Big\}.$$

$$\mu(n,z) = \varepsilon t\Big[t \leqq n \,\&\, n < \sum_{i=0}^{t} \lambda(i,z)\Big]$$
$$\nu(n,z) = \Big[\sum_{i=0}^{\mu(n,z)} \lambda(i,z)\Big] \dot{-} n.$$
$$\theta(z,m) = \nu(m,z)\, Gl\, \tau(\mu(m,z), z).$$

Then if z or $\tau(0,z)$ is the Gödel number for the sequence S_0 of the $\lambda(0,z)$ numbers $z_1, \ldots, z_l$ $(z_1, \ldots, z_l > 0)$, $\tau(k+1, z)$ is the Gödel number for the sequence S_{k+1} of the $\lambda(k+1, z)$ numbers $\psi(n, x, y)$, for $n = 0, \ldots, k$ and x and y ranging over S_k, in a certain order. Since $\psi(0, x, y) = x$, S_k includes all numbers in S_j for $0 \leqq j \leqq k$. When $l(z) > 0$, $\mu(n, z)$ and $\nu(n, z)$ as $n = 0, 1, 2, \ldots$ take successively the pairs of values $0\,\lambda(0, z), 0\,\lambda(0, z) - 1, \ldots, 0\,1;\ 1\,\lambda(1, z), 1\,\lambda(1, z) - 1, \ldots, 1\,1; \ldots$. Hence $\theta(z, m)$ for $m = 0, 1, 2 \ldots$ are the members of S_k $(k = 0, 1, 2, \ldots)$. But these are (with repetitions) the numbers obtainable from $z_1, \ldots, z_l$ by zero or more applications of the operations $\varphi(0, x, y), \varphi(1, x, y), \ldots$. Since $\theta(z, m)$ was defined in a manner which shows that it can be obtained from $\varphi(n, x, y)$ and known primitive recursive functions by substitutions and primitive recursions, we have proved:

I. *Given a function* $\varphi(n, x, y)$, *there is a function* $\theta(z, m)$, *primitive recursive in* $\varphi(n, x, y)$[11]), *such that, whenever* $z = p_1^{z_1} \ldots p_l^{z_l}$ $(z_1, \ldots, z_l > 0)$, *then* $\theta(z, 0), \theta(z, 1), \ldots$ *is an enumeration (with repetitions) of the least class* $C(x)$ *such that* $C(z_1), \ldots, C(z_l)$ *and* $(n, x, y)\left[C(x) \,\&\, C(y) \rightarrow C(\varphi(n, x, y))\right]$.

We note here the following two theorems for later use:

II. *Given a class* $A(x)$, *a relation* $x, y\,B\,z$, *and a number* k *which belongs to the least class* $C(x)$ *such that* $(x)[A(x) \rightarrow C(x)]$ *and* $(x, y, z)[C(x) \,\&\, C(y) \,\&\, x, y\,B\,z \rightarrow C(z)]$, *there is a function* $\eta(m)$, *primitive recursive in* $A(x)$ *and* $x, y\,B\,z$, *such that* $\eta(0), \eta(1), \ldots$ *is an enumeration (with repetitions) of* $C(x)$.

$\eta(m)$ is the function $\theta(R(k), m)$ when $\theta(z, m)$ is chosen as in I taking for $\varphi(n, x, y)$ the function

$$\varepsilon z \Big[z \leqq n + k \,\&\, \Big\{ \Big\{ n \,|\, 2 \,\&\, \Big[\Big(A\left[\left(\tfrac{n}{2}\right)\right] \,\&\, z = \left[\tfrac{n}{2}\right] \Big) \vee \Big(\overline{A\left(\left[\tfrac{n}{2}\right]\right)} \,\&\, z = k \Big) \Big] \Big\} \vee \Big\{ n + 1 \,|\, 2 \,\&\, \Big[\Big(x, y\,B\left[\tfrac{n \dot{-} 1}{2}\right] \,\&\, z = \left[\tfrac{n \dot{-} 1}{2}\right] \Big) \vee \Big(\overline{x, y\,B\left[\tfrac{n \dot{-} 1}{2}\right]} \,\&\, z = k \Big) \Big] \Big\} \Big\} \Big]$$ [12]).

If a member k of a class $R(x)$ is given, the class is enumerated (allowing repetitions) by the function $\varepsilon y\left[y \leqq m + k \,\&\, \{(R(m) \,\&\, y = m) \vee (\overline{R(m)} \,\&\, y = k)\}\right]$, which is primitive recursive in the class. Similarly:

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[11]) We call a function φ primitive recursive in other functions ψ_i, if φ becomes primitive recursive under the supposition that ψ_i are primitive recursive. $\prod_{n=0}^{\psi(\mathfrak{x}, \mathfrak{y})} \chi(\mathfrak{x}, \mathfrak{z}, n)$ and $\sum_{n=0}^{\psi(\mathfrak{x}, \mathfrak{y})} \chi(\mathfrak{x}, \mathfrak{z}, n)$ are primitive recursive in $\psi(\mathfrak{x}, \mathfrak{y})$ and $\chi(\mathfrak{x}, \mathfrak{z}, n)$. Here we use $\mathfrak{x}, \mathfrak{y}, \mathfrak{z}$ as abbreviations for $x_1, \ldots, x_n$, $y_1, \ldots, y_m$, $z_1, \ldots, z_l$, resp., and we shall continue to do so when convenient.

[12]) If $k = 0$, replace "$\lambda(0, z) = l(z)$" by "$\lambda(0, z) = 1$" in the definition of $\theta(z, m)$.

III. *Given a relation* $R(x, y)$ *and a number* k *such that* $(Ey)\,R(k, y)$, *there is a function* $\gamma(m)$, *primitive recursive in* $R(x, y)$, *such that* $\gamma(0), \gamma(1), \ldots$ *is an enumeration (allowing repetitions) of the class* $(Ey)\,R(x, y)$.

$$\gamma(m) = \varepsilon y\big[y \leqq [1\,Gl\,m] + k \;\&\; \{(R(1\,Gl\,m, 2\,Gl\,m) \;\&\; y = 1\,Gl\,m) \vee \overline{(R(1\,Gl\,m, 2\,Gl\,m)} \;\&\; y = k)\}\big].$$

By applying I, taking for $\varphi(n, x, y)$ the function $R'(n, x, y)$ (21), we obtain a primitive recursive function:

25. $H(z, m)$.

If z corresponds to a system of equations Z, $H(z, 0)$, $H(z, 1), \ldots$ is an enumeration (with repetitions) of the numbers corresponding to equations Y such that $Z \vdash'_{0, 1, 2, \ldots} Y$.

Now let $\varphi(\mathfrak{x})$ be a recursive function in the sense of Def. 2a or Def. 2b. Then there is a system E of equations defining φ recursively under Def. 2b; suppose that ϱ_a stands for φ in E. The system E has a Gödel number e. Using 23 and 25, if $R(\mathfrak{x}, y) \equiv \mathrm{Eval}_{r_a}(a, H(e, y), \mathfrak{x})$, then, by Def. 2b, $(\mathfrak{x})\,(Ey)\,R(\mathfrak{x}, y)$. Furthermore, using 24, if $\psi(y) = \mathrm{Val}(H(e, y))$, then $\varphi(\mathfrak{x}) = \psi(\varepsilon y[R(\mathfrak{x}, y)])$. We have now proved:

IV. *Every function recursive in the sense of Def.* 2a *(or Def.* 2b) *is expressible in the form* $\psi(\varepsilon y[R(\mathfrak{x}, y)])$, *where* $\psi(y)$ *is a primitive recursive function and* $R(\mathfrak{x}, y)$ *a primitive recursive relation and* $(\mathfrak{x})\,(Ey)\,R(\mathfrak{x}, y)$.

Thus the extension of general over primitive recursive functions consists only in that to substitutions and primitive recursions is added the operation of seeking indefinitely through the series of natural numbers for one satisfying a primitive recursive relation.

By Gödel S. 180 IV, $\varepsilon y[R(\mathfrak{x}, y)]$ is primitive recursive in $R(\mathfrak{x}, y)$ and any function $\chi(\mathfrak{x})$ which bounds y. Hence, in a certain sense, the length of the computation algorithm of a recursive function which is not also primitive recursive grows faster with the arguments than the value of any primitive recursive function[13]).

Given a relation $R(\mathfrak{x})$, the function $\varrho(\mathfrak{x})$ which is 0 or 1, according as $R(\mathfrak{x})$ holds or not, may be called the representing function of $R(\mathfrak{x})$. As with primitive recursions, we say that $R(\mathfrak{x})$ is recursive, if its representing function is recursive (under Def. 2a)[14]).

[13]) Besides the method, for demonstrating that a function is not primitive recursive (or not definable by given additional means, such as recursions with respect to n variables simultaneously), which consists in finding a lower bound for the values, we have the method, for demonstrating relationships of the opposite kind, which consists in finding an upper bound for the number of steps in the computation algorithm.

[14]) This is equivalent to saying that there is a recursive function $\varrho'(\mathfrak{x})$ such that $R(\mathfrak{x}) \sim [\varrho'(\mathfrak{x}) = 0]$, since then $\varrho(\mathfrak{x}) = 1 \dot{-} (1 \dot{-} \varrho'(\mathfrak{x}))$.

Let $R(\mathfrak{x}, y)$ be a recursive relation such that $(\mathfrak{x})(Ey)R(\mathfrak{x}, y)$. Then the function $\pi(\mathfrak{x}, y) = \prod_{i=0}^{y} \varrho(\mathfrak{x}, i)$, where $\varrho(\mathfrak{x}, y)$ is the representing function of $R(\mathfrak{x}, y)$, is recursive (under Def. 2a); and the function $\mu(\mathfrak{x}) = \varepsilon y[R(\mathfrak{x}, y)]$ satisfies the following relations in terms of $\pi(\mathfrak{x}, y)$:

$$
\begin{aligned}
&\sigma(0, \mathfrak{x}, y) = y, \\
(4) \qquad &\sigma(S(z), \mathfrak{x}, y) = \sigma(\pi(\mathfrak{x}, S(y)), \mathfrak{x}, S(y)), \\
&\mu(\mathfrak{x}) = \sigma(\pi(\mathfrak{x}, 0), \mathfrak{x}, 0).
\end{aligned}
$$

These equations (supplemented by the equations defining $\pi(\mathfrak{x}, y)$ recursively under Def. 2a) form a system E defining $\mu(\mathfrak{x})$ recursively under Def. 2a. Hence:

V. *If $R(\mathfrak{x}, y)$ is a recursive relation, and $(\mathfrak{x})(Ey)R(\mathfrak{x}, y)$, then $\varepsilon y[R(\mathfrak{x}, y)]$ is recursive (under Def. 2a).*

This shows that the converse of IV is true, and gives us as an operation of recursive definition the formation of $\varepsilon y[R(\mathfrak{x}, y)]$ from a recursive relation $R(\mathfrak{x}, y)$ such that $(\mathfrak{x})(Ey)R(\mathfrak{x}, y)$[15]. Also, the equivalence of Def. 2a and Def. 2b is now established:

VI. *The class of recursive functions under Def. 2b is identical with that under Def. 2a.*

For, as noted earlier, Def. 2b is not less general than Def. 2a, and now we have by IV and V that any function recursive under Def. 2b is expressible in the form $\psi(\mu(\mathfrak{x}))$ where $\psi(y)$ and $\mu(\mathfrak{x})$ are recursive under Def. 2a.

VII. *Let $R(\mathfrak{x}, y)$ be a relation such that for every $\mathfrak{x}$ $R(\mathfrak{x}, y)$ holds for infinitely many y's, and let $\nu(\mathfrak{x}, n)$ denote the n^{th} y such that $R(\mathfrak{x}, y)$ in order of magnitude. If $R(\mathfrak{x}, y)$ is recursive, then $\nu(\mathfrak{x}, n)$ is recursive.*

For $\nu(\mathfrak{x}, n)$ satisfies the relations $\nu(\mathfrak{x}, 0) = \varepsilon y[R(\mathfrak{x}, y)]$ and $\nu(\mathfrak{x}, S(n)) = \xi(\mathfrak{x}, \nu(\mathfrak{x}, n))$ where $\xi(\mathfrak{x}, z) = \varepsilon y[R(\mathfrak{x}, y) \,\&\, y > z]$, from which its recursiveness follows by use of V.

The converse of VII holds, since $R(\mathfrak{x}, y) \equiv (En)[n \leqq y \,\&\, \nu(\mathfrak{x}, n) = y]$, which is primitive recursive in $\nu(\mathfrak{x}, n)$.

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[15]) By IV, the use of this operation repeatedly and with $R(\mathfrak{x}, y)$ a general recursive relation gives no extension of the class of functions obtainable by a single application of it with $R(\mathfrak{x}, y)$ primitive recursive.

We had already as an operation of recursive definition the formation of $\varepsilon y[R(\mathfrak{x}, y)]$ from a recursive relation $R(\mathfrak{x}, y)$ such that there is a recursive function $\chi(\mathfrak{x})$ for which $R(\mathfrak{x}, y) \rightarrow y \leqq \chi(\mathfrak{x})$ (by Gödel, S. 180, IV). This and the present result correspond to different methods of expressing $\varepsilon y[R(\mathfrak{x}, y)]$ recursively in terms of $\varrho(\mathfrak{x}, y)$.

Thus, omitting the parameters $\mathfrak{x}$, an infinite class is recursively enumerable without repetitions in order of magnitude if and only if it is recursive.

VIII. *If the function $\zeta(x)$ is recursive and takes infinitely many values, and $\eta(n)$ denotes the n^{th} in order of first occurrence in $\zeta(0), \zeta(1), \ldots,$ then $\eta(n)$ is recursive.*

For $\eta(n) = \zeta(\nu(n))$ when $\nu(n)$ is chosen by VII for

$$R(y) \equiv (x)[x < y \rightarrow \zeta(x) \neq \zeta(y)].$$

Thus the recursive enumerability with repetitions of an infinite class implies its recursive enumerability without repetitions[16]).

§ 2.

The undecidability, in general, which systems of equations define recursive functions.

The definition of general recursive function offers no constructive process for determining when a recursive function is defined. This must be the case, if the definition is to be adequate, since otherwise still more general "recursive" functions could be obtained by the diagonal process.

In order to analyze the situation in detail, we utilize the correspondence of systems of equations E to numbers e, under which the problem, which systems E define functions recursively, becomes a number-theoretic one. We introduce for each particular value of n the following primitive recursive relation, where α_n denotes the least i for which $r_i = n$:

26. $$T_n(z, x_1, \ldots, x_n, y) \equiv \mathrm{Eval}_n(\alpha_n, H(z, y), x_1, \ldots, x_n).$$

The relation between the numbers and the recursive functions is simplified under the following definition:

Def. 2c. The number e *defines* (*recursively*) the function $\varphi(x_1, \ldots, x_n) = \mathrm{Val}\left(H(e, \varepsilon y[T_n(e, x_1, \ldots, x_n, y)])\right)$ if $(x_1, \ldots, x_n)(Ey)\, T_n(e, x_1, \ldots, x_n, y)$. A function $\varphi(x_1, \ldots, x_n)$ is *recursive* if there is an e of this description.

IX. *The class of recursive functions under Def.* 2c *is identical with that under Def.* 2a (2b).

For if $\varphi(x_1, \ldots, x_n)$ is recursive under Def. 2a (2b), the system of equations which defines φ recursively under Def. 2a (2b) has, after changing the notation if necessary so that φ is represented in it by ϱ_{α_n}, a Gödel number e which defines φ recursively under Def. 2c (cf. the proof of IV); and conversely, every function recursive under Def. 2c is recursive under Def. 2a (2b) by V (V and VI).

---

[16]) In XV below is given an example $(Ey)\, T_1(x, x, y)$ of a non-recursive class which by III is recursively enumerable.

If a number e defines a function $\varphi(x_1, \ldots, x_n)$ recursively under Def. 2c and is the Gödel number of a system E of equations, E is in general a system determining a multiple-valued function[17]), not necessarily a system defining a function recursively under Def. 2a or 2b.

What follows is stated for $n = 1$, and would hold similarly for any other fixed n[18]).

X. *If* $\theta(x)$ *is a recursive function, and* $(x)\,(Ey)\,T_1(\theta(x), x, y)$, *there is a number* f *such that* $(x)\,(Ey)\,T_1(f, x, y)$ *and* $\overline{(Eq)\,[\theta(q) = f]}$.

For then $\eta(x) = \mathrm{Val}\big(H\,(\theta(x), \varepsilon y\,[T_1(\theta(x), x, y)])\big) + 1$ is a recursive function (by V) such that, for every x for which $\theta(x)$ defines a function φ_x of one variable recursively, $\eta(x) = \varphi_x(x) + 1$ (by Def. 2c). Let f be a number defining $\eta(x)$ recursively. By Def. 2c, $(x)\,(Ey)\,T_1(f, x, y)$. Also, if there were a q such that $\theta(q)$ is f, we would have $\eta(x) = \varphi_q(x)$, which contradicts the preceding equality when x takes the value q.

In the case that for every x $\theta(x)$ is the Gödel number of a system E_x of equations defining a function φ_x of one variable recursively (φ_x being represented in E_x by ϱ_0), we have that the function $\varphi_x(x) + 1$ is recursive. Thus the diagonal procedure, applied to a sequence of recursive functions which are defined by systems of equations of which the Gödel numbers form a recursive sequence, does not lead outside the class of recursive functions.

XI. *The numbers which define functions* $\varphi(x)$ *recursively are not recursively enumerable, i. e. there is no recursive function* $\theta(m)$ *such that* $(m, x)\,(Ey)\,T_1(\theta(m), x, y)$ *and* $(z)\,\{(x)\,(Ey)\,T_1(z, x, y) \rightarrow (Em)\,[\theta(m) = z]\}$.

For, given any recursive function $\theta(m)$ such that $(m, x)\,(Ey)\,T_1(\theta(m), x, y)$, then a fortiori $(x)\,(Ey)\,T_1(\theta(x), x, y)$, and by X there is a number f such that $(x)\,(Ey)\,T_1(f, x, y)$ but $\overline{(Em)\,[\theta(m) = f]}$.

XII. *The class* $(x)\,(Ey)\,T_1(z, x, y)$ *of the numbers* z *which define functions* $\varphi(x)$ *recursively is not recursive.*

For if it were recursive, it would be enumerated by a recursive function, contradicting XI.

Indeed, given any recursive class $R(z)$ such that $(z)\,\{R(z) \rightarrow (x)\,(Ey)\,T_1(z, x, y)\}$, a number f such that $(x)\,(Ey)\,T_1(f, x, y)$ but $\overline{R(f)}$ is obtained by X, when $\theta(x) = \varepsilon y\,\big[(R(x)\;\&\; y = x) \vee (\overline{R(x)}\;\&\; y = k)\big]$, where k is any number such that $(x)\,(Ey)\,T_1(k, x, y)$.

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[17]) Then $\varphi(x_1, \ldots, x_n)$ is that one of the values x determined by E for which the Gödel number of $\varrho_{\alpha_n}(x_1, \ldots, x_n) = x$ occurs earliest in the list $H(e, 0), H(e, 1), \ldots$.

[18]) Since the means given for passing from definitions under Def. 2a (2b) to definitions under Def. 2c, and vice versa, are effective, the problem which we now study (which numbers define functions recursively) is equivalent to the one first proposed (which systems of equations define functions recursively).

The definability of a non-recursive class by use of quantifiers applied to a recursive relation gives the existence of undecidable number-theoretic propositions in certain formal logics from the consideration (somewhat different from that employed by Gödel) that otherwise the logics could be used to construct recursive definitions of the class.

XIII. *Given a formal logic S, suppose that the propositions* $(x)(Ey)T_1(z,x,y)$ $(z = 0, 1, 2, \ldots)$ *can be expressed in S by formulas* A_z, *and that numbers can be assigned to the formulas of S, in such a fashion that* (1) *to distinct formulas are assigned distinct numbers,* (2) *the class* $A(x)$ *of the numbers assigned to axioms, and the relation* $x, y\, B\, z$ *between numbers of being assigned to formulas in the relation of immediate consequence, are recursive,* (3) z *is a recursive function* $\beta(a_z)$ *of the number* a_z *of* A_z, (4) *the class* $C(n)$ *of the numbers* a_z *is recursive,* (5) *if* A_z *is provable, then* $(x)(Ey)T_1(z,x,y)$ *is true. Then there are* z*'s for which* A_z *is not provable although* $(x)(Ey)T_1(z,x,y)$ *is true*[19]).

For suppose that there is a number k such that A_k is provable. Then, by (2) and II, given a_k, there is a recursive function $H(m)$ which enumerates the numbers assigned to provable formulas of S; and, by (1), (3) and (4), the recursive function $\theta(y) = \beta\left(\varepsilon m \left[\{C(H(y)) \,\&\, m = H(y)\} \vee \{\overline{C(H(y))} \,\&\, m = a_k\}\right]\right)$ enumerates the z's for which A_z is provable. By (5), $(x)(Ey)T_1(\theta(x), x, y)$.

Hence, by X, there is a number f such that $\overline{(Eq)[\theta(q) = f]}$ (which implies that A_f is not provable in S) and $(x)(Ey)T_1(f, x, y)$[20]).

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19) The relation of "immediate consequence" we suppose to be a given relation between a formula and a pair of formulas, and the class of "provable formulas" to be the least class which contains the given class of "axioms" and has the property that Z is provable whenever X and Y are provable and Z is an immediate consequence of X and Y.

If more details of the structure of S were suitably specified, condition (5) could be given a more metamathematical appearance, such as the following (analogous to Gödel's condition of ω-Widerspruchsfreiheit, S. 187): for no relation $F(x,y)$ and natural number k are all of the formulas $F(k,0), F(k,1), \ldots, \overline{(Ex)(y)F(x,y)}$ provable. On the further assumption that for no relation $F(x,y)$ and sequence of natural numbers $k_0, k_1, \ldots$ are all of the formulas $F(0,k_0), F(1,k_1), \ldots,$ $\overline{(x)(Ey)F(x,y)}$ provable, the conclusion could be given the form, that there are z's for which A_z is formally undecidable, i. e. for which neither A_z nor $\bar{A}_z$ is provable. (The conditions need to be assumed merely for certain relations $F(x,y)$.)

20) The undecidable proposition A_f can be effectively constructed for a given logic, whenever the number a_k, recursive definitions of $A(x)$; $x, y\, B\, z$; $\beta(y)$ and $C(n)$, and effective means of constructing A_f from f, are given.

Whenever the supposition in this proof, that there is a k such that A_k is provable, is not realized, the theorem holds trivially.

XIV. *The function* $\varepsilon y\,[T_1(x,x,y)]$ *is non-recursive*[21]).

For, if $\varrho(x)$ is any recursive function, the function $\eta(x) = \mathrm{Val}\big(H(x,\varrho(x))\big)+1$ is recursive, and $\eta(x) = \mathrm{Val}\big(H(f,\varepsilon y[T_1(f,x,y)])\big)$ holds for any number f defining $\eta(x)$ recursively. Now if $\varrho(f) = \varepsilon y\,[T_1(f,f,y)]$, two different values are obtained for $\eta(f)$. Hence $\varrho(f) \neq \varepsilon y\,[T_1(f,f,y)]$. Thus $\varepsilon y\,[T_1(x,x,y)]$ differs from each recursive function for some value of x. Note also that $(E y)\,T_1(f,f,y)$.

XV. *The class* $(E y)\,T_1(x,x,y)$ *is non-recursive.*

Thus non-recursive functions can be defined by the schema

$$\tau(x) = \begin{cases} 0 \text{ if } (E y)\,R(x,y) \\ 1 \text{ if } \overline{(E y)\,R(x,y)} \end{cases},$$

where $R(x,y)$ is primitive recursive. This follows from XIV, since, if $(E y)\,R(k,y)$ and $\lambda(x) = [1 \dot{-} \tau(x)]\cdot x + \tau(x)\cdot k$, then $\varepsilon y\,[R(x,y)] = [1 \dot{-} \tau(x)]\cdot \varepsilon y\big[R(\lambda(x),y)\big]$, which is recursive if $\tau(x)$ is recursive.

To analyze the situation more fully, let $S(x)$ be any recursive class such that $(x)\,\{S(x) \rightarrow (E y)\,T_1(x,x,y)\}$, and $\sigma(x)$ the representing function of $S(x)$. If k is any number which defines a function recursively, then $(E y)\,T_1(k,k,y)$, and we set $\mu(x) = [1 \dot{-} \sigma(x)]\cdot x + \sigma(x)\cdot k$ and $\varrho(x) = [1 \dot{-} \sigma(x)]\cdot \varepsilon y\big[T_1(\mu(x),\mu(x),y)\big]$. $\varrho(x)$ is recursive, and as in the proof of XIV, there is an f such that $\varrho(f) \neq \varepsilon y\,[T_1(f,f,y)]$ and $(E y)\,T_1(f,f,y)$. If $S(f)$, then $\sigma(f) = 0$ and $\varrho(f) = \varepsilon y\,[T_1(f,f,y)]$. Hence $\overline{S(f)}$.

XVI. *The class* $\overline{(E y)\,T_1(x,x,y)}$ *is not recursively enumerable*[22]).

For by III, the complementary class $(E y)\,T_1(x,x,y)$ is enumerated by a recursive function $\gamma(m)$. Now if $\overline{(E y)\,T_1(x,x,y)}$ is enumerated by $\varkappa(m)$ and we set $\xi(m) = \varepsilon n\left\{\left[m \,\middle|\, 2 \,\&\, n = \gamma\left(\left[\frac{m}{2}\right]\right)\right] \vee \left[m+1 \,\middle|\, 2 \,\&\, n = \varkappa\left(\left[\frac{m \dot{-} 1}{2}\right]\right)\right]\right\}$, we have $(E y)\,T_1(x,x,y) \equiv \varepsilon m\,[\xi(m) = x]\,|\,2$, which would contradict XV if $\varkappa(m)$ were recursive.

XVII. *Given a recursive relation* $R(\mathfrak{x},\mathfrak{y})$, *there is a number* e *such that* $(\mathfrak{x})\,(E\,\mathfrak{y})\,R(\mathfrak{x},\mathfrak{y}) \equiv (x)\,(E y)\,T_1(e,x,y)$. *Given a recursive relation* $R(\mathfrak{y})$, *there is a number* e *such that* $(E\,\mathfrak{y})\,R(\mathfrak{y}) \equiv (E y)\,T_1(e,e,y)$[23]).

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[21]) We recall that $\varepsilon y\,[R(x,y)] = 0$ when $\overline{(E y)\,R(x,y)}$.

[22]) The proof given here is non-constructive. The writer has a constructive proof that for certain recursive relations $R(x,y)$ the class $\overline{(E y)\,R(x,y)}$ is not recursively enumerable. From that proof, the existence in certain formal logics of undecidable propositions involving only one quantifier (which can be concluded non-constructively from present results) is obtainable in the same manner as XIII.

[23]) From the great generality of the problems, which e's define recursively functions of one variable, and which e's "determine recursively" the e^{th} value of a function of one variable, as displayed by this theorem, the result, that they are not "effectively" soluble, could have been anticipated.

For, to every proposition of the form $(\mathfrak{x})(E\mathfrak{y})R(\mathfrak{x},\mathfrak{y})$, there is an equivalent proposition of the form $(x)(Ey)R(x,y)$ obtained by utilizing the recursive enumerability of n-tuples of natural numbers, or introducing fictive variables[24]); and the Gödel number e of the system E of equations which defines $\varepsilon y[R(x,y)]$ in the proof of V on the supposition that $(x)(Ey)R(x,y)$ satisfies the present theorem. Similarly, $(E\mathfrak{y})R(\mathfrak{y})$ has an equivalent $(Ey)R(y)$, and for e we may take the Gödel number of the equations defining $\varepsilon y[R(y) \,\&\, x = x]$ on the supposition that $(Ey)R(y)$[25]).

My thanks are due to Prof. Paul Bernays for the suggestion of improvements in the presentation.

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[24]) E. g. $(x_1, x_2, x_3)R(x_1, x_2, x_3) \equiv (x)(Ey)[R(1\,Gl\,x, 2\,Gl\,x, 3\,Gl\,x) \,\&\, y = y]$.

[25]) XV, XVI, and XVII are similar, respectively, to results obtained in a different connection by Prof. Alonzo Church (An unsolvable problem of elementary number theory, see Bull. Amer. Math. Soc. Abstract 41—5—205), Dr. J. B. Rosser (unpublished), and the present writer (A theory of positive integers in formal logic, Part II, Amer. Jour. Math. 57 No. 2, pp. 230 ff.).

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