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Full Models for Positive Modal Logic

Ramon Jansana[1)]

Departamento de Lògica, Història i Filosofia de la Ciència, Universitat de Barcelona, Baldiri i Reixach s/n, 08028 Barcelona, Spain[2)]

Abstract. The positive fragment of the local modal consequence relation defined by the class of all Kripke frames is studied in the context of Abstract Algebraic Logic. It is shown that this fragment is non-protoalgebraic and that its class of canonically associated algebras according to the criteria set up in [7] is the class of positive modal algebras. Moreover its full models are characterized as the models of the Gentzen calculus introduced in [3].

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1 Introduction

The study of Positive Modal Logic began in [6], and has been continued in [3] and [4]. It concerns the restriction of the modal local consequence relation defined by the class of all Kripke models to the propositional negation-free (or positive) modal language whose connectives are $\wedge$, $\vee$, $\Box$, $\Diamond$, $\top$, $\bot$. This restriction will be denoted $\mathcal{PML}$. The study of the expressive power of the positive modal language has been addressed in [10]. The present paper studies the algebraization of $\mathcal{PML}$ in the framework of Abstract Algebraic Logic (AAL from now on).

In [6] positive modal algebras are introduced and a representation theorem for them is proved. The equations that in that paper define positive modal algebras are obtained by translating into the algebraic language the postulates used there to axiomatize $\mathcal{PML}$. But the question about the exact connection between positive modal algebras and the logic $\mathcal{PML}$ is not addressed neither in [6], [3] nor in [4]. Do, for instance, positive modal algebras have a similar relation to $\mathcal{PML}$ as Boolean algebras have to classical logic? What kind of similarity do we have to look for?

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2) e-mail: jansana@mat.ub.es

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These questions can be precisely formulated and answered in the context of AAL. The question to be asked in this context is if the class of positive modal algebras is the class of algebras that corresponds to $\mathcal{PML}$ according to the general criteria developed in the theories about the algebraization of logic considered in AAL.

A key concept in AAL is algebraizable logic. Blok and Pigozzi introduced it in their seminal work [1]. A logical system is said to be algebraizable when its relation with its associated class of algebras is as strong as possible. In mathematical terms this means that there are two faithful interpretations between the consequence relation of the logical system and the equational consequence relation defined by the class of algebras, effected by two structural translations, one between formulas and sets of equations and the other between equations and sets of formulas, which are the inverse of each other. The paradigmatic example of algebraizable logic is classical logic (whose associated class of algebras is, of course, the class of Boolean algebras); among algebraizable logics also count intuitionistic logic, many-valued logics and the modal global consequence relations defined by classes of Kripke frames. Many logics are nevertheless non-algebraizable.

The class of algebras canonically associated with a given logic according to the theory of logical matrices (that is, the class of the algebraic reducts of the reduced matrices which are a model of the logic), is the right one for algebraizable logics and for the wider class of protoalgebraic logics. These logics, as it is frequently said, are the logics for which the semantics of logical matrices is well behaved from an algebraic point of view, meaning that many of the results of universal algebra have counterparts of specific logical interest in the theory of logical matrices for them. Outside the realm of protoalgebraic logics the situation is not so good. The class of algebras that associated with a non-protoalgebraic logic by the theory of logical matrices does not necessarily coincide with the class one would intuitively expect. An illustration of this phenomenon is found in the conjunction-disjunction fragment of classical logic. Here the expected class of algebras is the class of distributive lattices, but, as shown in [9], this is not the class of algebras associated with it by the theory of logical matrices. Another example is the logic $\mathcal{PML}$ studied in the present paper. As it turns out (see Theorem 9) $\mathcal{PML}$ is a non-protoalgebraic logic and the class of algebras associated with it by the theory of logical matrices is a proper subclass of the class of positive modal algebras (see Proposition 16).

In [7] a general theory of the algebraization of logic is developed using generalized matrices (there called abstract logics) as possible models for logical systems. A *generalized matrix* consists of an algebra $\boldsymbol{A}$ plus a set $\mathcal{C}$ of subsets of its domain A that are the closed sets of some finitary closure operator on A. Using generalized matrices, in [7] is proposed a canonical way to associate a class of algebras with each logical system $\boldsymbol{\mathcal{S}}$ that, as experience shows, supplies the expected results in the non-protoalgebraic case and gives in the protoalgebraic case exactly the class of algebras for the theory of logical matrices associated with $\boldsymbol{\mathcal{S}}$. Several results are proved in [7] that sustain the claim that this class of algebras is the natural class that corresponds to a given logical system. The present paper shows that the class of positive modal algebras is the class of algebras associated with $\mathcal{PML}$ according to these criteria laid down in [7] and studies the algebraization of $\mathcal{PML}$ along the lines established in that monograph.

Since algebraizable logics are protoalgebraic, outside the realm of protoalgebraic logics there is no chance of finding a logic with such strong ties with its associated class of algebras as classical logic has with the class of Boolean algebras. But in the early nineties it was observed that there is still a possibility of finding these links between a Gentzen calculus axiomatizing the logic and the class of algebras naturally associated with it. Part of the theory that studies this phenomena has started to be developed in [7] using the extension of the notion of algebraizable logic to Gentzen systems effected in [12] and [13] and the notion of full model of a logic. A *full model* of a logic $\boldsymbol{\mathcal{S}}$ is essentially a generalized matrix $\langle \boldsymbol{A}, \mathcal{C}\rangle$ that is a model of the system $\boldsymbol{\mathcal{S}}$ and is full in the sense that no set can be added to $\mathcal{C}$ without losing the property of being a model of $\boldsymbol{\mathcal{S}}$. The use of generalized matrices is crucial for this development due to their dual role in the theory of logical systems as models of logical systems and also as models of Gentzen-style calculi and the so-called Gentzen systems. This dual role allows us to relate the theory of the algebraization (here in a wide sense) of logical systems with the theory of the algebraization of Gentzen systems (in the strict sense of [12]), by establishing by means of its full models links between a logical system and a Gentzen style calculus axiomatizing it. This possibility opens the way to exploring a new sense, introduced by Pigozzi in [11] and explored to some extent in [8], in which a logical system can be said to be algebraizable. We can say that a logical system $\boldsymbol{\mathcal{S}}$ is *sequent-algebraizable* (or 2*nd-order algebraizable*) (Pigozzi says in [11] 'finitely algebraizable in the general sense') if it has an algebraizable Gentzen system, in the precise sense of [12], which is fully adequate, that is, whose models are exactly the full models of $\boldsymbol{\mathcal{S}}$. The notion, althought it is not an extension of the notion of algebraizable logic because there are many algebraizable logics which lack a fully adequate and algebraizable Gentzen calculus is coherent with it since if an algebraizable logic $\boldsymbol{\mathcal{S}}$ has a fully adequate Gentzen calculus, then this Gentzen calculus is algebraizable and its equivalent algebraic semantics is exactly the equivalent algebraic semantics of $\boldsymbol{\mathcal{S}}$. The work expounded in [7] and examples like $\boldsymbol{\mathcal{PML}}$ which, as proved in the present paper, is sequent-algebraizable show that 2nd-order-algebraizability is a notion of algebraization which deserves attention.

In the present paper, besides characterizing the class of algebras canonically associated with $\boldsymbol{\mathcal{PML}}$ as the class of positive modal algebras introduced by Dunn in [6], we characterize the class of full models of $\boldsymbol{\mathcal{PML}}$, which turns out to be the class of generalized matrices that are models of the Gentzen calculus G_m for $\boldsymbol{\mathcal{PML}}$ proposed in [6] and [3]. We finally prove that this Gentzen calculus is algebraizable in the sense of [12] and that its equivalent algebraic semantics is the variety of positive modal algebras. Therefore $\boldsymbol{\mathcal{PML}}$ is sequent-algebraizable. Since the variety of positive modal algebras does not have equationally definable principal congruences, this has the consequence that the Gentzen calculus G_m has no deduction theorem in the sense of [13].

2 Preliminaries

In this section we condense the essentials of the notions of Abstract Algebraic Logic (AAL) that will be used in the paper, and fix notation. For detailed expositions we refer to [1, 5, 7, 14].

2.1 Deductive systems, algebras and matrices

Let $\mathcal{L}$ be an algebraic similarity type (or set of connectives) that we fix throughout this section. All algebras considered, etc. will be of this type. Given an algebra $\boldsymbol{A}$, $\mathrm{Co}\,\boldsymbol{A}$ will denote the set of all congruences of $\boldsymbol{A}$, and if K is a class of algebras, $\mathrm{Co}_{\mathsf{K}}\boldsymbol{A}$ will denote the set of congruences of $\boldsymbol{A}$ whose quotient algebra belongs to K, that is, for every $\theta \in \mathrm{Co}\,\boldsymbol{A}$, $\theta \in \mathrm{Co}_{\mathsf{K}}\boldsymbol{A}$ iff $\boldsymbol{A}/\theta \in \mathsf{K}$. The elements of $\mathrm{Co}_{\mathsf{K}}\boldsymbol{A}$ are the so-called K-*congruences* of $\boldsymbol{A}$. The set of all homomorphisms from an algebra $\boldsymbol{A}$ to an algebra $\boldsymbol{B}$ is denoted by $\mathrm{Hom}(\boldsymbol{A}, \boldsymbol{B})$.

Let $\mathbf{Fm}$ be the absolutely free algebra of type $\mathcal{L}$ with a denumerable set of generators, called the *propositional variables* and whose set is denoted by Var. This algebra is the *formula algebra* of type $\mathcal{L}$ and the elements of its domain Fm are the *formulas* or *terms*. A (finitary) *deductive system* (or logic) of type $\mathcal{L}$ is a pair $\boldsymbol{\mathcal{S}} = \langle \mathbf{Fm}, \vdash_{\boldsymbol{\mathcal{S}}} \rangle$, where $\vdash_{\boldsymbol{\mathcal{S}}}$ is a relation, called the *entailment relation* of $\boldsymbol{\mathcal{S}}$, between sets of formulas and formulas with the following properties:

1. If $\varphi \in \Gamma$, then $\Gamma \vdash_{\boldsymbol{\mathcal{S}}} \varphi$.
2. If $\Gamma \vdash_{\boldsymbol{\mathcal{S}}} \varphi$ and for every $\psi \in \Gamma$, $\Delta \vdash_{\boldsymbol{\mathcal{S}}} \psi$, then $\Delta \vdash_{\boldsymbol{\mathcal{S}}} \varphi$.
3. If $\Gamma \vdash_{\boldsymbol{\mathcal{S}}} \varphi$, then for any substitution σ, $\sigma[\Gamma] \vdash_{\boldsymbol{\mathcal{S}}} \sigma(\varphi)$, where a *substitution* is an homomorphism from the formula algebra $\mathbf{Fm}$ into itself (substitution invariance).
4. If $\Gamma \vdash_{\boldsymbol{\mathcal{S}}} \varphi$, then there is a finite subset $\Gamma' \subseteq \Gamma$ such that $\Gamma' \vdash_{\boldsymbol{\mathcal{S}}} \varphi$ (finitarity).

From 1. and 2. it follows

5. If $\Gamma \vdash_{\boldsymbol{\mathcal{S}}} \varphi$, then for any ψ, $\Gamma \cup \{\psi\} \vdash_{\boldsymbol{\mathcal{S}}} \varphi$.

Deductive systems can be defined in many ways either syntactically or semantically. A *theory* of a deductive system $\boldsymbol{\mathcal{S}}$ ($\boldsymbol{\mathcal{S}}$-theory for short) is a set of formulas Γ that is closed under the entailment relation of $\boldsymbol{\mathcal{S}}$, that is, if $\Gamma \vdash_{\boldsymbol{\mathcal{S}}} \varphi$, then $\varphi \in \Gamma$. The set of $\boldsymbol{\mathcal{S}}$-theories will be denoted by $\mathrm{Th}\,\boldsymbol{\mathcal{S}}$.

Given a deductive system $\boldsymbol{\mathcal{S}}$ and an algebra $\boldsymbol{A}$ with universe A, a set $F \subseteq A$ is an $\boldsymbol{\mathcal{S}}$-*filter* if for any $h \in \mathrm{Hom}(\mathbf{Fm}, \boldsymbol{A})$, any set of formulas Γ and any formula φ, if $\Gamma \vdash_{\boldsymbol{\mathcal{S}}} \varphi$ and $h[\Gamma] \subseteq F$, then $h(\varphi) \in F$. The condition can be replaced by the corresponding condition with Γ finite. We denote the set of all $\boldsymbol{\mathcal{S}}$-filters of an algebra $\boldsymbol{A}$ by $\mathrm{Fi}_{\boldsymbol{\mathcal{S}}}\boldsymbol{A}$. The set of all $\boldsymbol{\mathcal{S}}$-filters of the formula algebra $\mathbf{Fm}$ is precisely the set $\mathrm{Th}\,\boldsymbol{\mathcal{S}}$ of all the theories of $\boldsymbol{\mathcal{S}}$. A *matrix* is a pair $\langle \boldsymbol{A}, F\rangle$, where $\boldsymbol{A}$ is an algebra and F is a subset of the universe of $\boldsymbol{A}$. A matrix $\boldsymbol{\mathcal{M}} = \langle \boldsymbol{A}, F\rangle$ is a (*matrix*) *model* of a deductive system $\boldsymbol{\mathcal{S}}$ if F is an $\boldsymbol{\mathcal{S}}$-filter of $\boldsymbol{A}$. Thus the matrix models of $\boldsymbol{\mathcal{S}}$ on the formula algebra are the matrices of the form $\langle \mathbf{Fm}, T\rangle$, where T is an $\boldsymbol{\mathcal{S}}$-theory.

Given an algebra $\boldsymbol{A}$ and a subset F of its universe, the *Leibniz congruence of* F *relative to* $\boldsymbol{A}$, denoted by $\Omega_{\boldsymbol{A}}(F)$, is the greatest congruence of $\boldsymbol{A}$ compatible with F, that is, that does not relate elements of F with elements outside F. The function $\Omega_{\boldsymbol{A}}$ is called the *Leibniz operator* on $\boldsymbol{A}$. A matrix $\langle \boldsymbol{A}, F\rangle$ is *reduced* if the Leibniz congruence $\Omega_{\boldsymbol{A}}(F)$ is the identity. The Leibniz congruence has the following useful characterization. Let $\boldsymbol{A}$ be an algebra and $F \subseteq A$, then $\langle a, b\rangle \in \Omega_{\boldsymbol{A}}(F)$ iff for every formula $\varphi(p, \overline{r})$ and every sequence $\overline{c}$ of elements of A, $\varphi^{\boldsymbol{A}}(a, \overline{c}) \in F$ iff $\varphi^{\boldsymbol{A}}(b, \overline{c}) \in F$.

The class of algebraic reducts of the reduced matrices that are models of a deductive system $\boldsymbol{\mathcal{S}}$ is usually denoted by $\mathrm{Alg}^* \boldsymbol{\mathcal{S}}$, that is,

$$\mathrm{Alg}^* \boldsymbol{\mathcal{S}} = \{\boldsymbol{A} : \text{ there is an } \boldsymbol{\mathcal{S}}\text{-filter } F \text{ of } \boldsymbol{A} \text{ such that } \langle \boldsymbol{A}, F\rangle \text{ is reduced}\}.$$

The class of algebras $\mathrm{Alg}^* \boldsymbol{\mathcal{S}}$ is the class of algebras canonically associated with the deductive system $\boldsymbol{\mathcal{S}}$ by the theory of matrices for deductive systems. In [7] it is argued at length that this class of algebras is not the right class in the case of non-protoalgebraic deductive systems. The right class, according to [7], is obtained by using the reduced generalized matrices we define below.

To every class of matrices M we can a associate a deductive system $\boldsymbol{\mathcal{S}}_{\mathsf{M}}$ defined by declaring that $\Gamma \vdash_{\boldsymbol{\mathcal{S}}_{\mathsf{M}}} \varphi$ iff there is a finite subset $\Gamma' \subseteq \Gamma$ such that for every $\boldsymbol{\mathcal{M}} = \langle \boldsymbol{A}, F\rangle \in \mathsf{M}$ and for every $h \in \mathrm{Hom}(\mathbf{Fm}, \boldsymbol{A})$ with $h[\Gamma'] \subseteq F$, $h(\varphi) \in F$.

A deductive system $\boldsymbol{\mathcal{S}}$ of type $\mathcal{L}$ is said to be *protoalgebraic* if for every algebra $\boldsymbol{A}$ the Leibniz operator is monotone on the set of all $\boldsymbol{\mathcal{S}}$-filters of $\boldsymbol{A}$, that is, if for any $\boldsymbol{\mathcal{S}}$-filters F and G of $\boldsymbol{A}$ with $F \subseteq G$ it holds that $\Omega_{\boldsymbol{A}}(F) \subseteq \Omega_{\boldsymbol{A}}(G)$. $\boldsymbol{\mathcal{S}}$ is said to be *weakly algebraizable* iff for every algebra $\boldsymbol{A}$ the Leibniz operator on $\boldsymbol{A}$ is an isomorphism between the lattice of $\boldsymbol{\mathcal{S}}$-filters of $\boldsymbol{A}$ and the lattice of congruences of $\boldsymbol{A}$ that in the quotient give an element of $\mathrm{Alg}^*(\boldsymbol{\mathcal{S}})$, that is, the lattice $\mathrm{Co}_{\mathrm{Alg}^*(\boldsymbol{\mathcal{S}})} \boldsymbol{A}$. If in addition the Leibniz operator commutes with inverse homomorphisms in the sense that if h is an homomorphism from $\boldsymbol{A}$ to $\boldsymbol{B}$, then for any $\boldsymbol{\mathcal{S}}$-filter G of $\boldsymbol{B}$, $h^{-1}[\Omega_{\boldsymbol{B}}(G)] = \Omega_{\boldsymbol{A}}(h^{-1}(G))$, the deductive system is said to be *algebraizable*, and $\mathrm{Alg}^*(\boldsymbol{\mathcal{S}})$ is said to be its *equivalent algebraic semantics*.

2.2 Generalized matrices and full models

A *finitary closed-set system* (fcss for short) on a set A is a family $\mathcal{C}$ of subsets of A that contains A and is closed under arbitrary intersections and under unions of upward directed subfamilies with respect to the inclusion relation. If $\mathcal{C}$ is a finitary closed-set system on a set A we define the closure operator $\mathrm{Clo}_{\mathcal{C}}$ on A associated with $\mathcal{C}$ by $\mathrm{Clo}_{\mathcal{C}}(X) = \bigcap\{F \in \mathcal{C} : X \subseteq F\}$, for each $X \subseteq A$. The closure operator $\mathrm{Clo}_{\mathcal{C}}$ is finitary in the following sense: if $a \in \mathrm{Clo}_{\mathcal{C}}(X)$, then there is a finite subset $Y \subseteq X$ such that $a \in \mathrm{Clo}_{\mathcal{C}}(Y)$. Moreover, given a finitary closure operator C on a set A, the family $\mathcal{C}_C$ of all subsets X of A such that $C(X) = X$ is a finitary closed-set system. It is well known that $\mathrm{Clo}_{\mathcal{C}_C} = C$, and that if $\mathcal{C}$ is a finitary closed-set system, then $\mathcal{C}_{\mathrm{Clo}_{\mathcal{C}_C}} = \mathcal{C}$.

A *generalized matrix* (g-matrix for short) is a pair $\boldsymbol{\mathcal{A}} = \langle \boldsymbol{A}, \mathcal{C}\rangle$, where $\boldsymbol{A}$ is an algebra and $\mathcal{C}$ is a finitary closed-set system on A. Usually we will denote the closure operator determined by $\mathcal{C}$ on A by $\mathrm{Clo}_{\boldsymbol{\mathcal{A}}}$. Sometimes we will identify the generalized matrix $\boldsymbol{\mathcal{A}} = \langle \boldsymbol{A}, \mathcal{C}\rangle$ with the pair $\langle \boldsymbol{A}, \mathrm{Clo}_{\boldsymbol{\mathcal{A}}}\rangle$. We will also refer to the closed-set system of a matrix $\boldsymbol{\mathcal{A}}$ by $\mathcal{C}_{\boldsymbol{\mathcal{A}}}$. Notice that for every deductive system $\boldsymbol{\mathcal{S}}$ the structure $\langle \mathbf{Fm}, \mathrm{Th}\,\boldsymbol{\mathcal{S}}\rangle$ is a generalized matrix. Its associated closure operator can be identified with the entailment relation $\vdash_{\boldsymbol{\mathcal{S}}}$. The finitarity of $\boldsymbol{\mathcal{S}}$ is essential for obtaining that $\mathrm{Th}\,\boldsymbol{\mathcal{S}}$ is closed under unions of upwards directed (by inclusion) subfamilies. Generalized matrices are called abstract logics in [7]; they are exactly the finitary abstract logics of that monograph.

A generalized matrix $\boldsymbol{\mathcal{A}} = \langle \boldsymbol{A}, \mathcal{C}\rangle$ is a *generalized model* (g-model for short) of a deductive system $\boldsymbol{\mathcal{S}}$ if every element of $\mathcal{C}$ is an $\boldsymbol{\mathcal{S}}$-filter, that is, if $\mathcal{C} \subseteq \mathrm{Fi}_{\boldsymbol{\mathcal{S}}}\boldsymbol{A}$. We denote the class of generalized models of a deductive system $\boldsymbol{\mathcal{S}}$ by GMod $\boldsymbol{\mathcal{S}}$. The g-matrix $\langle \mathbf{Fm}, \mathrm{Th}\,\boldsymbol{\mathcal{S}}\rangle$ is obviously a g-model of the deductive system $\boldsymbol{\mathcal{S}}$.

Given a generalized matrix $\boldsymbol{\mathcal{A}} = \langle \boldsymbol{A}, \mathcal{C}\rangle$, its *Tarski congruence*, which we denote by $\widetilde{\Omega}_{\boldsymbol{A}}(\mathcal{C})$ or sometimes by $\widetilde{\Omega}(\boldsymbol{\mathcal{A}})$, is defined as the greatest congruence of $\boldsymbol{A}$ compatible with every element of $\mathcal{C}$, that is, $\widetilde{\Omega}_{\boldsymbol{A}}(\mathcal{C}) = \bigcap_{F\in\mathcal{C}} \Omega_{\boldsymbol{A}}(F)$. The function $\widetilde{\Omega}_{\boldsymbol{A}}$ is called the *Tarski operator* on $\boldsymbol{A}$. A generalized matrix $\boldsymbol{\mathcal{A}} = \langle \boldsymbol{A}, \mathcal{C}\rangle$ is *reduced* if its Tarski congruence is the identity. The class of reduced g-models of a deductive system $\boldsymbol{\mathcal{S}}$ will be denoted by $\mathrm{GMod}^*\,\boldsymbol{\mathcal{S}}$ and the class of algebraic reducts of the elements of $\mathrm{GMod}^*\,\boldsymbol{\mathcal{S}}$ by Alg $\boldsymbol{\mathcal{S}}$, that is,

$$\mathrm{Alg}\,\boldsymbol{\mathcal{S}} = \{\boldsymbol{A} : \text{ there is an fcss on } \boldsymbol{A} \text{ such that } \langle \boldsymbol{A}, \mathcal{C}\rangle \in \mathrm{GMod}^*\,\boldsymbol{\mathcal{S}}\}.$$

This class of algebras is the class to be canonically associated with a deductive system $\boldsymbol{\mathcal{S}}$ according to the general algebraic semantics for deductive systems developed in [7]. For protoalgebraic deductive systems $\boldsymbol{\mathcal{S}}$, $\mathrm{Alg}^*\,\boldsymbol{\mathcal{S}} = \mathrm{Alg}\,\boldsymbol{\mathcal{S}}$.

A *strict homomorphism* from a g-matrix $\boldsymbol{\mathcal{A}} = \langle \boldsymbol{A}, \mathcal{C}\rangle$ to a g-matrix $\boldsymbol{\mathcal{B}} = \langle \boldsymbol{B}, \mathcal{D}\rangle$ is a homomorphism from $\boldsymbol{A}$ to $\boldsymbol{B}$ such that $\mathcal{C} = \{h^{-1}[F] : F \in \mathcal{D}\}$. Bijective strict homomorphisms are called *isomorphisms*, and surjective strict homomorphisms are called *bilogical morphisms* in [7]. If there is a strict surjective homomorphism from $\boldsymbol{\mathcal{A}} = \langle \boldsymbol{A}, \mathcal{C}\rangle$ onto $\boldsymbol{\mathcal{B}} = \langle \boldsymbol{B}, \mathcal{D}\rangle$ we write $\boldsymbol{\mathcal{A}} \succeq \boldsymbol{\mathcal{B}}$. In other words this means that $\boldsymbol{\mathcal{B}}$ is a strict homomorphic image of $\boldsymbol{\mathcal{A}}$. The most typical surjective strict homomorphism is reduction. Given a g-matrix $\boldsymbol{\mathcal{A}} = \langle \boldsymbol{A}, \mathcal{C}\rangle$, its *reduction* is the g-matrix $\boldsymbol{\mathcal{A}}^* = \langle \boldsymbol{A}/\widetilde{\Omega}(\boldsymbol{\mathcal{A}}), \mathcal{C}/\widetilde{\Omega}(\boldsymbol{\mathcal{A}})\rangle$, where $\boldsymbol{A}/\widetilde{\Omega}(\boldsymbol{\mathcal{A}})$ is the quotient algebra and $\mathcal{C}/\widetilde{\Omega}(\boldsymbol{\mathcal{A}}) = \{F/\widetilde{\Omega}(\boldsymbol{\mathcal{A}}) : F \in \mathcal{C}\}$. The projection homomorphism $\pi : \boldsymbol{A} \longrightarrow \boldsymbol{A}/\widetilde{\Omega}(\boldsymbol{\mathcal{A}})$ is a surjective strict homomorphism from $\boldsymbol{\mathcal{A}}$ onto $\boldsymbol{\mathcal{A}}^*$. It is known (see [7, Proposition 1.14]) that if $\boldsymbol{\mathcal{A}} \succeq \boldsymbol{\mathcal{B}}$, then $\boldsymbol{\mathcal{A}}^*$ is isomorphic to $\boldsymbol{\mathcal{B}}^*$.

A generalized matrix $\boldsymbol{\mathcal{A}} = \langle \boldsymbol{A}, \mathcal{C}\rangle$ is a *basic full g-model* of a deductive system $\boldsymbol{\mathcal{S}}$ if $\mathcal{C} = \mathrm{Fi}_{\boldsymbol{\mathcal{S}}}(\boldsymbol{A})$, and it is a *full g-model* of $\boldsymbol{\mathcal{S}}$ if there is a basic full g-model $\boldsymbol{\mathcal{B}}$ of $\boldsymbol{\mathcal{S}}$ such that $\boldsymbol{\mathcal{A}} \succeq \boldsymbol{\mathcal{B}}$, that is, if one of its strict homomorphic images is a basic full g-model of $\boldsymbol{\mathcal{S}}$. In [7] it is proved that if $\boldsymbol{\mathcal{A}} \succeq \boldsymbol{\mathcal{B}}$, then $\boldsymbol{\mathcal{A}}$ is a full g-model of $\boldsymbol{\mathcal{S}}$ iff $\boldsymbol{\mathcal{B}}$ is so. Moreover, the reduction of any full g-model is a basic full g-model. Clearly, in the perspective opened by the use of g-matrices, the basic full g-models are the most natural models to be considered, and since experience shows that the relevant logical notions are preserved under strict homomorphisms, the class of full g-models is the resulting class to deal with.

For a g-matrix $\boldsymbol{\mathcal{A}} = \langle \boldsymbol{A}, \mathcal{C}\rangle$, its *Frege relation* $\Lambda(\boldsymbol{\mathcal{A}})$ is defined by

$$\langle a, b\rangle \in \Lambda(\boldsymbol{\mathcal{A}}) \quad \text{iff} \quad \mathrm{Clo}_{\boldsymbol{\mathcal{A}}}(\{a\}) = \mathrm{Clo}_{\boldsymbol{\mathcal{A}}}(\{b\}).$$

It is easy to see that $\widetilde{\Omega}(\boldsymbol{\mathcal{A}})$ is the largest congruence of $\boldsymbol{A}$ included in $\Lambda(\boldsymbol{\mathcal{A}})$. We will also denote the Frege relation of a g-matrix $\langle \boldsymbol{A}, \mathcal{C}\rangle$ by $\Lambda_{\boldsymbol{A}}(\mathcal{C})$. For a deductive system $\boldsymbol{\mathcal{S}}$, the *interderivability relation* '$\varphi \vdash_{\boldsymbol{\mathcal{S}}} \psi$ and $\psi \vdash_{\boldsymbol{\mathcal{S}}} \varphi$' is the Frege relation of the g-matrix $\langle \mathbf{Fm}, \mathrm{Th}\,\boldsymbol{\mathcal{S}}\rangle$.

2.3 Gentzen systems

For the purposes of the present paper a *sequent* (of type $\mathcal{L}$) is a pair $\langle \Gamma, \varphi\rangle$, where Γ is a (possibly empty) finite set of formulas and φ is a formula. We will write $\Gamma \rhd \varphi$ instead of $\langle \Gamma, \varphi\rangle$ when working with sequents. A *Gentzen system* is a pair $\boldsymbol{\mathcal{G}} = \langle \mathbf{Fm}, \vdash_{\mathcal{G}}\rangle$, where $\mathbf{Fm}$ is the algebra of formulas and $\vdash_{\mathcal{G}}$ is a finitary closure operator on the set of sequents that is *substitution-invariant* in the sense that if

(1) $\{\Gamma_i \rhd \psi_i : i < n\} \vdash_{\mathcal{G}} \Gamma \rhd \varphi,$

then for every substitution $\sigma \in \mathrm{Hom}(\mathbf{Fm}, \mathbf{Fm})$

(2) $\{\sigma[\Gamma_i] \rhd \sigma(\psi_i) : i < n\} \vdash_{\mathcal{G}} \sigma[\Gamma] \rhd \sigma(\varphi).$

The expression (2) is called a *substitution instance of* (1). Notice that we use the notation $X \vdash_{\mathcal{G}} \Gamma \rhd \varphi$, for X a set of sequents, instead of the notation $\Gamma \rhd \varphi \in \vdash_{\mathcal{G}} (X)$ typical for closure operators. We will follow this practice throughout the paper.

A *Gentzen-style rule* is simply a pair $\langle X, \Gamma \rhd \varphi\rangle$, where X is a (possibly empty) finite set of sequents and $\Gamma \rhd \varphi$ is a sequent. A *substitution instance of a Gentzen-style rule* $\langle X, \Gamma \rhd \varphi\rangle$ is a Gentzen-style rule of the form $\langle \sigma[X], \sigma[\Gamma] \rhd \sigma(\varphi)\rangle$ for some substitution σ, where $\sigma[X] = \{\sigma[\Delta] \rhd \sigma(\psi) : \Delta \rhd \psi \in X\}$. A Gentzen-style rule $\langle X, \Gamma \rhd \varphi\rangle$ is *initial* if X is empty. As usual we will use the fraction notation for Gentzen-style rules

$$\frac{\Gamma_0 \rhd \varphi_0, \ldots, \Gamma_{n-1} \rhd \varphi_{n-1}}{\Gamma \rhd \varphi}.$$

We will say that a Gentzen system $\boldsymbol{\mathcal{G}} = \langle \mathbf{Fm}, \vdash_{\mathcal{G}}\rangle$ is *closed under a Gentzen-style rule* $\langle\{\Gamma_i \rhd \psi_i : i < n\}, \Gamma \rhd \varphi\rangle$, or also that this rule is a *sound rule* of $\boldsymbol{\mathcal{G}}$, if $\{\Gamma_i \rhd \psi_i : i < n\} \vdash_{\mathcal{G}} \Gamma \rhd \varphi$. Notice that this implies that for every substitution σ, $\{\sigma[\Gamma_i] \rhd \sigma(\psi_i) : i < n\} \vdash_{\mathcal{G}} \sigma[\Gamma] \rhd \sigma(\varphi)$.

A Gentzen system $\boldsymbol{\mathcal{G}} = \langle \mathbf{Fm}, \vdash_{\mathcal{G}}\rangle$ is said to be *structural* if it is closed under the structural rules of Weakening and Cut, and the structural rule $\langle \emptyset, p \rhd p\rangle$[3]. Every structural Gentzen system $\boldsymbol{\mathcal{G}}$ determines a deductive system $\boldsymbol{\mathcal{S}}_{\mathcal{G}}$ as follows

$$\Gamma \vdash_{\boldsymbol{\mathcal{S}}_{\mathcal{G}}} \varphi \quad \text{iff} \quad \text{there is a finite subset } \Delta \subseteq \Gamma \text{ such that } \emptyset \vdash_{\mathcal{G}} \Delta \rhd \varphi\,.$$

A *Gentzen calculus* G is a set of Gentzen-style rules. Every Gentzen calculus G defines a closure operator $\boldsymbol{\mathcal{G}}_{\mathrm{G}}$ on the set of sequents in the following sense. A sequent $\Gamma \rhd \varphi$ belongs to the closure of a set of sequents X if there is a finite sequence of sequents each one of whose elements is a substitution instance of an initial rule of the calculus or a sequent in X or is obtained applying a substitution instance of a rule of the calculus to previous elements in the sequence. Such a sequence is called a *proof in the calculus with premises in* X. If the Gentzen calculus has the structural rules, the Gentzen system it defines is structural. A sequent $\Gamma \rhd \varphi$ is *derivable* in a Gentzen calculus G if $\emptyset \vdash_{\mathcal{G}_{\mathrm{G}}} \Gamma \rhd \varphi$, that is, if there is a proof of $\Gamma \rhd \varphi$ without premises. A rule $\langle X, \Gamma \rhd \varphi\rangle$ is a *derived rule* of a Gentzen calculus G if $X \vdash_{\mathcal{G}_{\mathrm{G}}} \Gamma \rhd \varphi$. Notice that if a rule is a derived rule, so are all its substitution instances.

Generalized matrices can be used as models of Gentzen-style rules, Gentzen calculus and Gentzen systems; this shows their double nature as candidates to be models of both deductive systems and Gentzen systems and allows us to tie the algebraic theory of deductive systems with the algebraic theory of Gentzen systems.

A g-matrix $\boldsymbol{\mathcal{A}} = \langle \boldsymbol{A}, \mathcal{C}\rangle$ is a *model of a Gentzen-style rule* $\langle\{\Gamma_i \rhd \psi_i : i < n\}, \Gamma \rhd \varphi\rangle$ if for every homomorphism $h \in \mathrm{Hom}(\mathbf{Fm}, \boldsymbol{A})$, $h(\varphi) \in \mathrm{Clo}_{\boldsymbol{\mathcal{C}}}(h[\Gamma])$ whenever, for all

[3] We do not need to consider the other structural rules – Exchange and Contraction – because we consider sets as sets of premises in our sequents.

$i < n$, $h(\varphi_i) \in \mathrm{Clo}_{\mathcal{C}}(h[\Gamma_i])$. It is a *model of a Gentzen calculus* if it is a model of all its rules, and it is a *model of a Gentzen system* if it is a model of all its sound rules. The following fairly obvious facts should be mentioned: 1) if a g-matrix is a model of a Gentzen-style rule, it is also a model of all its substitution instances, 2) if a g-matrix is a model of a Gentzen calculus, then it is a model of the Gentzen system that it defines, and 3) if a g-matrix is a model of a Gentzen system $\mathcal{G}$, it is a g-model of the associated deductive system $\mathcal{S}_{\mathcal{G}}$.

A *structural translation t of sequents into equations* is a mapping that sends every sequent to a finite set of equations and satisfies the following property: for every sequent $\Gamma \rhd \varphi$ and every substitution σ, if $t(\Gamma \rhd \varphi) = \{\varepsilon_i \approx \delta_i : i < n\}$, then $t(\sigma[\Gamma] \rhd \sigma(\varphi)) = \{\sigma(\varepsilon_i) \approx \sigma(\delta_i) : i < n\}$. A *structural translation s from equations into sequents* is a mapping that sends every equation to a finite set of sequents and has the corresponding substitutions-invariace property, that is, for every equation $\varepsilon \approx \delta$ and every substitution σ, if $s(\varepsilon \approx \delta) = \{\Gamma_i \rhd \varphi_i : i < n\}$, then $s(\sigma(\varepsilon) \approx \sigma(\delta)) = \{\sigma[\Gamma_i] \rhd \sigma[\varphi_i] : i < n\}$. If t is a translation of sequents into equations and X is a set of sequents, the set of equations $t(X)$ is defined by $t(X) = \bigcup\{t(\Gamma \rhd \varphi) : \Gamma \rhd \varphi \in X\}$. If s is a translation from equations into sequents and E is a set of equations, the set of sequents $s(E)$ is defined by $s(E) = \bigcup\{s(\varphi \approx \psi) : \varphi \approx \psi \in E\}$.

A Gentzen system $\mathcal{G}$ is *algebraizable* if there is a class of algebras K and a substitution-invariant translation t from sequents into equations and a structural translation s from equations into sequents for which the following two conditions hold:

(i) $\{\Gamma_i \rhd \varphi_i : i \in I\} \vdash_{\mathcal{G}} \Gamma \rhd \varphi$ iff $t(\{\Gamma_i \rhd \varphi_i : i \in I\}) \vDash_{\mathsf{K}} t(\Gamma \rhd \varphi)$,

(ii) $\varphi \approx \psi \vDash_{\mathsf{K}} t(s(\varphi \approx \psi))$ and $t(s(\varphi \approx \psi)) \vDash_{\mathsf{K}} \varphi \approx \psi$,

where $\vDash_{\mathsf{K}}$ denotes the equational consequence defined by the class of algebras K as follows: $\{\varphi_i \approx \psi_i : i \in I\} \vDash_{\mathsf{K}} \varphi \approx \psi$ if for every algebra $\boldsymbol{A} \in \mathsf{K}$ and every homomorphism $h \in \mathrm{Hom}(\mathbf{Fm}, \boldsymbol{A})$, $h(\varphi) = h(\psi)$, whenever for all $i \in I$, $h(\varphi_i) = h(\psi_i)$. When to the right of "$\vDash_{\mathsf{K}}$" there is a set it means that every element of the set follows from the equations in the set to the left of "$\vDash_{\mathsf{K}}$". If two structural translations exist with the above properties the Gentzen system $\mathcal{G}$ and the equational consequence $\vDash_{\mathsf{K}}$ are also said to be *equivalent*.

The notion of algebraizable Gentzen system was first introduced in [12] and its theory developed in [13]. The theory of algebraizable Gentzen systems is an extension to this kind of system of the theory of algebraizable deductive systems developed by W. Blok and D. Pigozzi in [1]. It is also a particular case of the notion of equivalence between Gentzen systems introduced and studied in [13].

From [13] it follows that if a Gentzen system $\mathcal{G}$ is algebraizable, there is always a quasivariety K, which is unique, such that (i) and (ii) are satisfied for some substitution-invariant translations t and s.

3 The deductive system $\mathcal{PML}$ and positive modal algebras

3.1 The deductive system $\mathcal{PML}$

The minimum system of Positive Modal Logic, denoted by $\mathcal{PML}$, is the restriction of the minimum normal modal logic $\mathbf{K}$ with the local consequence relation to the positive, or negation-free, modal language with connectives $\wedge$, $\vee$, $\top$, $\bot$, $\Box$, $\Diamond$. That is, for any set of formulas Γ and any formula φ, $\Gamma \vdash_{\mathcal{PML}} \varphi$ iff for every Kripke model $\langle X, R, V\rangle$ and every point $w \in X$, if every $\psi \in \Gamma$ is true at w, then φ is also true at w. This is a (finitary) deductive system that together with some of its extensions is studied in [6] and [3]. It can be axiomatized by means of the Gentzen calculus G_m defined by the following rules, which we write in schematic form:

$$\frac{}{\varphi \rhd \varphi}\,, \qquad \frac{}{\rhd \top}\,, \qquad \frac{}{\Diamond\bot \rhd \bot}\,,$$

$$\frac{\Gamma \rhd \varphi}{\Gamma, \psi \rhd \varphi}\,, \qquad \frac{\Gamma \rhd \bot}{\Gamma \rhd \varphi}\,, \qquad \frac{\Gamma \rhd \varphi \quad \Gamma, \varphi \rhd \psi}{\Gamma \rhd \psi}\,,$$

$$\frac{\Gamma, \varphi, \psi \rhd \alpha}{\Gamma, \varphi \wedge \psi \rhd \alpha}\,, \qquad \frac{\Gamma \rhd \varphi \quad \Gamma \rhd \psi}{\Gamma \rhd \varphi \wedge \psi}\,,$$

$$\frac{\Gamma, \varphi \rhd \alpha \quad \Gamma, \psi \rhd \alpha}{\Gamma, \varphi \vee \psi \rhd \alpha}\,, \qquad \frac{\Gamma \rhd \varphi}{\Gamma \rhd \varphi \vee \psi}\,, \qquad \frac{\Gamma \rhd \psi}{\Gamma \rhd \varphi \vee \psi}\,,$$

$$[\Box\Diamond]\,\frac{\Gamma, \varphi \rhd \psi \vee \alpha}{\Box\Gamma, \Diamond\varphi \rhd \Diamond\psi \vee \Diamond\alpha}\,, \quad [\Diamond\Box]\,\frac{\Gamma \rhd \varphi \vee \psi}{\Box\Gamma \rhd \Box\varphi \vee \Diamond\psi}\,.$$

The Gentzen calculus G_m was introduced in [3]. The following completeness theorem can be proved using the proof of the completeness theorem in [3, Thm. 6.12] which is stablished for a slightly different Kripke-style semantics.

Theorem 1 (Completeness of G_m). *For every set of formulas Γ and every formula φ, $\Gamma \vdash_{\mathcal{PML}} \varphi$ iff there is a finite subset $\Delta \subseteq \Gamma$ such that the sequent $\Delta \rhd \varphi$ is a derivable sequent of* G_m.

The following sequents

1. $\Box(\varphi \wedge \psi) \rhd \Box\varphi \wedge \Box\psi\,,$
2. $\Box\varphi \wedge \Box\psi \rhd \Box(\varphi \wedge \psi)\,,$
3. $\Diamond(\varphi \vee \psi) \rhd \Diamond\varphi \vee \Diamond\psi\,,$
4. $\Diamond\varphi \vee \Diamond\psi \rhd \Diamond(\varphi \vee \psi)\,,$
5. $\Box(\varphi \vee \psi) \rhd \Box\varphi \vee \Diamond\psi\,,$
6. $\Box\varphi \wedge \Diamond\psi \rhd \Diamond(\varphi \wedge \psi)$
7. $\top \rhd \Box\top\,, \quad \Box\top \rhd \top\,,$
8. $\Diamond\bot \rhd \bot, \quad \bot \rhd \Diamond\bot$

are derivable sequents of G_m, and the rules

$$\frac{\Gamma \rhd \varphi}{\Box\Gamma \rhd \Box\varphi}\,, \qquad \frac{\psi \rhd \varphi}{\Diamond\psi \rhd \Diamond\varphi}$$

are derived rules; therefore the deductive system $\mathcal{PML}$ is closed under them in the sense that, for example in the case of the first rule, if $\Gamma \vdash_{\mathcal{PML}} \varphi$, then $\Box\Gamma \vdash_{\mathcal{PML}} \Box\varphi$.

3.2 Positive modal algebras

Positive modal algebras were introduced in [6] and their duality theory is developed in [4]. A *positive modal algebra* $\langle A, \wedge, \vee, \Box, \Diamond, 0, 1\rangle$ (pm-algebra for short) is an algebra $\boldsymbol{A}$ of the fixed similarity type $\{\wedge, \vee, \top, \bot, \Box, \Diamond\}$ such that $\langle A, \wedge, \vee, 0, 1\rangle$ is a bounded distributive lattice and for any a, $b \in A$,

1. $\Box(a \wedge b) = \Box a \wedge \Box b\,,$
2. $\Diamond(a \vee b) = \Diamond a \vee \Diamond b\,,$
3. $\Box a \wedge \Diamond b \leq \Diamond(a \wedge b)\,,$
4. $\Box(a \vee b) \leq \Box a \vee \Diamond b\,,$
5. $\Box 1 = 1\,,$
6. $\Diamond 0 = 0\,.$

By the definition it is clear that the class of pm-algebras forms a variety. We denote it by PMA. The following quasi-equational monotonicity conditions hold in every pm-algebra.

1. If $a \leq b$, then $\Box a \leq \Box b$, 2. If $a \leq b$, then $\Diamond a \leq \Diamond b$.

A *filter* of a pm-algebra $\boldsymbol{A}$ is simply a lattice filter of $\boldsymbol{A}$, and analogously for an ideal. A filter F of a pm-algebra $\boldsymbol{A}$ is a *prime filter* if it is proper ($\neq A$) and for every $a, b \in A$, if $a \vee b \in F$, then $a \in F$ or $b \in F$. The set of all non-empty filters of a pm-algebra $\boldsymbol{A}$ will be denoted by $\mathrm{Fi}\, \boldsymbol{A}$. In the paper we will use the Birkhoff-Stone Theorem for bounded distributive lattices that obviously holds for pm-algebras. We formulate it as a reminder.

Theorem 2. *If F is a filter of a pm-algebra $\boldsymbol{A}$ and I is an ideal of $\boldsymbol{A}$ disjoint from F, then there is a prime filter P such that $F \subseteq P$ and $I \cap P = \emptyset$.*

With the class of pm-algebras we can associate a deductive system defined by means of the order relation of the pm-algebras. This deductive system turns out to be (Theorem 6) the deductive system $\boldsymbol{\mathcal{PML}}$. Therefore the class of pm-algebras provides an algebraic semantics for the minimun system of Positive Modal Logic. We give the provisional name $\boldsymbol{\mathcal{PMA}}$ to the deductive system defined using the class of pm-algebras as follows:

$$\Gamma \vdash_{\boldsymbol{\mathcal{PMA}}} \varphi \quad \text{iff} \quad \begin{array}{l} \text{for all } \boldsymbol{A} \in \mathsf{PMA} \text{ and for all } v \in \mathrm{Hom}(\mathbf{Fm}, \boldsymbol{A}),\ v(\varphi) = 1, \\ \text{or there exist } \varphi_0, \ldots, \varphi_n \in \Gamma \text{ such that for all } \boldsymbol{A} \in \mathsf{PMA} \\ \text{and for all } v \in \mathrm{Hom}(\mathbf{Fm}, \boldsymbol{A}),\ v(\varphi_0) \wedge \cdots \wedge v(\varphi_n) \leq v(\varphi). \end{array}$$

It is not difficult to see that this deductive system is the deductive system defined by the class of all the matrices $\langle \boldsymbol{A}, F \rangle$, where $\boldsymbol{A}$ is a pm-algebra and F is a lattice filter of A.

Formulas φ and ψ are said to be *interderivable* in $\boldsymbol{\mathcal{PML}}$ if $\varphi \vdash_{\boldsymbol{\mathcal{PML}}} \psi$ and $\psi \vdash_{\boldsymbol{\mathcal{PML}}} \varphi$. The interderivability relation of $\boldsymbol{\mathcal{PML}}$ on the formula algebra is precisely the Frege relation of the g-matrix $\langle \mathbf{Fm}, \mathrm{Th}\, \boldsymbol{\mathcal{PML}} \rangle$, accordingly we denote it by $\Lambda(\boldsymbol{\mathcal{PML}})$. We state the following Lindenbaum-Tarski type theorem without the routine proof.

Theorem 3. *The interderivability relation of $\boldsymbol{\mathcal{PML}}$ is a congruence relation and the algebra $\mathbf{Fm}/\Lambda(\boldsymbol{\mathcal{PML}})$ is a positive modal algebra.*

It is useful to have the following lemma for further use. Let us denote the equivalence class of a formula φ modulo $\Lambda(\boldsymbol{\mathcal{PML}})$ by $[\varphi]$.

Lemma 4. *For all formulas φ and ψ, $[\varphi] \leq [\psi]$ in $\mathbf{Fm}/\Lambda(\boldsymbol{\mathcal{PML}})$ iff $\varphi \vdash_{\boldsymbol{\mathcal{PML}}} \psi$.*

Proof. The following chain of equivalences proves the lemma: $[\varphi] \leq [\psi]$ iff $[\varphi] \wedge [\psi] = [\varphi]$ iff $[(\varphi \wedge \psi)] = [\varphi]$ iff $(\varphi \wedge \psi)_{\boldsymbol{\mathcal{PML}}} \dashv\vdash_{\boldsymbol{\mathcal{PML}}} \varphi$ iff $\varphi \vdash_{\boldsymbol{\mathcal{PML}}} \psi$. □

Using this fact we can prove the algebraic soundness and completeness theorem of $\boldsymbol{\mathcal{PML}}$ relative to the semantics of pm-algebras. First we prove in the following lemma the finitary version of the result which has further applications.

Lemma 5. *For any formulas $\varphi, \varphi_0, \ldots, \varphi_n$,*

1. *$\varphi_0, \ldots, \varphi_n \vdash_{\boldsymbol{\mathcal{PML}}} \varphi$ iff for every pm-algebra $\boldsymbol{A}$ and every $v \in \mathrm{Hom}(\mathbf{Fm}, \boldsymbol{A})$, $v(\varphi_0) \wedge \cdots \wedge v(\varphi_n) \leq v(\varphi)$;*
2. *$\vdash_{\boldsymbol{\mathcal{PML}}} \varphi$ iff for every pm-algebra $\boldsymbol{A}$ and every $v \in \mathrm{Hom}(\mathbf{Fm}, \boldsymbol{A})$, $v(\varphi) = 1$.*

Proof. The proof of 2. is analogous to the proof of 1. The implication from left to right of 1. can be obtained using the completeness theorem for the calculus G_m by an induction on the length of a proof of a derivable sequent. To prove the other implication assume that for every pm-algebra $\boldsymbol{A}$ and every $v \in \mathrm{Hom}(\mathbf{Fm}, \boldsymbol{A})$, $v(\varphi_0) \wedge \cdots \wedge v(\varphi_n) \leq v(\varphi)$. Consider the algebra $\mathbf{Fm}/\Lambda(\mathcal{PML})$, which by Theorem 3 is a pm-algebra, and the homomorphism v defined by $v(p) = [p]$, for every $p \in \mathrm{Var}$. Then for every formula φ, $v(\varphi) = [\varphi]$. By assumption, $v(\varphi_0) \wedge \cdots \wedge v(\varphi_n) \leq v(\varphi)$. Hence, by Lemma 4, we obtain that $\varphi_0 \wedge \cdots \wedge \varphi_n \vdash_{\mathcal{PML}} \varphi$, and from this follows that $\varphi_0, \ldots, \varphi_n \vdash_{\mathcal{PML}} \varphi$. □

From Lemma 5, the definition of $\mathcal{PMA}$ and the finitarity of $\mathcal{PML}$ follows the algebraic completeness of $\mathcal{PML}$.

Theorem 6 (Algebraic completeness). *For any set of formulas Γ and any formula φ, $\Gamma \vdash_{\mathcal{PML}} \varphi$ iff $\Gamma \vdash_{\mathcal{PMA}} \varphi$.*

Two important consequences of Lemma 5 are the following Corollary 7 and Theorem 8.

Corollary 7. *In a pm-algebra $\boldsymbol{A}$ the $\mathcal{PML}$-filters are precisely the lattice filters.*

Proof. Assume that F is a $\mathcal{PML}$-filter. It is non-empty since $\vdash_{\mathcal{PML}} \top$. Moreover, since $p, q \vdash_{\mathcal{PML}} p \wedge q$, $p \wedge q \vdash_{\mathcal{PML}} p$ and $p \wedge q \vdash_{\mathcal{PML}} q$, it is easy to see using Lemma 5 that F is a lattice filter. Conversely, if F is a non-empty lattice filter, Lemma 5 and the finitarity of $\mathcal{PML}$ imply immediately that F is a $\mathcal{PML}$-filter. □

Theorem 8. *The algebra $\mathbf{Fm}/\Lambda(\mathcal{PML})$ is the free pm-algebra on a denumerable set of generators.*

Proof. It is clear that the set $\{[p] : p \in \mathrm{Var}\}$ is denumerable since two different propositional variables are not interderivable in $\mathcal{PML}$. Moreover, since Var generates the formula algebra, $\{[p] : p \in \mathrm{Var}\}$ generates the quotient algebra $\mathbf{Fm}/\Lambda(\mathcal{PML})$. Now we proceed to see that it is a free algebra in the variety of pm-algebras. Let $\boldsymbol{A}$ be a pm-algebra and h any function from $\{[p] : p \in \mathrm{Var}\}$ into $\boldsymbol{A}$. Consider the unique homomorphism $v : \mathbf{Fm} \longrightarrow \boldsymbol{A}$ such that $v(p) = h([p])$ for every $p \in \mathrm{Var}$, and define $h' : \mathbf{Fm} \longrightarrow \boldsymbol{A}$ by $h'([\varphi]) = v(\varphi)$. This function is well defined since by (the soundness part of) Lemma 5 if follows that if φ and ψ are interderivable in $\mathcal{PML}$, then $v(\varphi) = v(\psi)$. Now it is easy to check that h' it is an homomorphism that extends h. □

4 Classifying $\mathcal{PML}$

In this section we classify the deductive system $\mathcal{PML}$ according to the concepts that have become stable in AAL. First of all we see that $\mathcal{PML}$ is non-protoalgebraic and that the class of pm-algebras can not be the class of algebras canonically associated with any algebraizable deductive system, that is, it cannot be the equivalent algebraic semantics of any deductive system.

Theorem 9. *The minimum system of Positive Modal Logic $\mathcal{PML}$ is not protoalgebraic.*

P r o o f. Consider the four element chain distributive lattice with universe $A = \{0, b, a, 1\}$ ordered by $0 < b < a < 1$. Define the operations $\Box$ and $\Diamond$ by

$$\Box x = \begin{cases} x & \text{if } x \in \{0,1\}, \\ b & \text{if } x \in \{a,b\}; \end{cases} \qquad \text{and} \qquad \Diamond x = \begin{cases} x & \text{if } x \in \{0,1\}, \\ a & \text{if } x \in \{a,b\}. \end{cases}$$

It is not difficult to check that the sets $\{1\}$ and $\{1, a\}$ are $\boldsymbol{\mathcal{PML}}$-filters. The Leibniz congruence of $\{1\}$ is the relation

$$\Omega_{\boldsymbol{A}}(\{1\}) = \mathrm{Id}_A \cup \{\langle a, b\rangle, \langle b, a\rangle, \langle a, 0\rangle, \langle 0, a\rangle, \langle 0, b\rangle, \langle b, 0\rangle\}.$$

Moreover, $\langle b, 0\rangle \notin \Omega_{\boldsymbol{A}}(\{1, a\}$, because on the contrary $\langle \Diamond b, \Diamond 0\rangle = \langle a, 0\rangle \in \Omega_{\boldsymbol{A}}(\{1, a\})$, which is impossible since $0 \notin \{1, a\}$. Therefore, $\Omega_{\boldsymbol{A}}(\{1\}) \not\subseteq \Omega_{\boldsymbol{A}}(\{1, a\})$. Since $\{1\} \subseteq \{1, a\}$ we obtain that $\boldsymbol{\mathcal{PML}}$ is not protoalgebraic. $\Box$

T h e o r e m 10. *The variety* PMA *of positive modal algebras is not the class of algebras* $\mathrm{Alg}^* \boldsymbol{\mathcal{S}}$ *of any weakly algebraizable deductive system* $\boldsymbol{\mathcal{S}}$*; therefore it is not the equivalent algebraic semantics of any algebraizable deductive system.*

P r o o f. Let us consider the bounded distributive lattice with domain $\{0, a, b, 1\}$ and order $0 < a < b < 1$. Define the operators $\Box$ and $\Diamond$ as the identity function on $\{0, a, b, 1\}$. The resulting algebra is a pm-algebra $\boldsymbol{A}$. Assume that $\mathsf{PMA} = \mathrm{Alg}\, \boldsymbol{\mathcal{S}}$ for some algebraizable deductive system $\boldsymbol{\mathcal{S}}$. Then $\Omega_{\boldsymbol{A}}$ is an isomorphism between the lattice of all $\boldsymbol{\mathcal{S}}$-filters of $\boldsymbol{A}$ and the lattice of all congruences of $\boldsymbol{A}$ that in the quotient give an element of PMA. Since PMA is a variety, this last lattice is $\mathrm{Co}\, \boldsymbol{A}$. Now, since $\Box$ and $\Diamond$ are the identity function, $\mathrm{Co}\, \boldsymbol{A}$ is the set of all lattice congruences of the lattice reduct $\boldsymbol{A}^-$ of $\boldsymbol{A}$. Except for the inconsistent system, every algebraizable system has a non-empty set of theorems. Since $\mathrm{Co}\, \boldsymbol{A}$ has more than one element, $\boldsymbol{\mathcal{S}}$ cannot be inconsistent; on the contrary, there will be only one $\boldsymbol{\mathcal{S}}$-filter on $\boldsymbol{A}$. We see now that the Leibniz operator cannot be an isomorphism between any candidate to be the set of $\boldsymbol{\mathcal{S}}$-filters of some deductive system $\boldsymbol{\mathcal{S}}$ with theorems and the set $\mathrm{Co}\, \boldsymbol{A}^-$, both sets ordered by inclusion. The argument is the following one: $\mathrm{Co}\, \boldsymbol{A}^-$ has 8 elements, so we need 8 non-empty $\boldsymbol{\mathcal{S}}$-filters. To fulfill this requirement the least $\boldsymbol{\mathcal{S}}$-filter has to be a unitary set and has to be mapped to the least congruence, namely the identity. But $\Omega_{\boldsymbol{A}}(\{0\})$ identifies $1, a, b$, $\Omega_{\boldsymbol{A}}(\{1\})$ identifies $0, a, b$, $\Omega_{\boldsymbol{A}}(\{a\})$ identifies $b, 1$ and $\Omega_{\boldsymbol{A}}(\{b\})$ identifies $0, a$. $\Box$

The pm-algebra $\boldsymbol{A}$ of the above proof has the following property. Let φ^* be the formula that we obtain when deleting in the formula φ all the occurrences of modal operators. Since in $\boldsymbol{A}$ the modal operators are the identity, an equation $\varphi \approx \psi$ holds in $\boldsymbol{A}$ iff the equation $\varphi^* \approx \psi^*$ holds in $\boldsymbol{A}$. Thus $\boldsymbol{A}$ belongs to any subvariety of PMA axiomatized by a set of equations Eq such that the set $\mathrm{Eq}^* = \{\varphi^* \approx \psi^* : \varphi \approx \psi\}$ is a set of equations valid in any distributive lattice. Therefore we have the following

T h e o r e m 11. *Any subvariety of* PMA *axiomatized by a set of equations* Eq *such that the set* $\mathrm{Eq}^* = \{\varphi^* \approx \psi^* : \varphi \approx \psi\}$ *is a set of equations valid in any distributive lattice can not be the class of algebras* $\mathrm{Alg}^* \boldsymbol{\mathcal{S}}$ *of any weakly algebraizable deductive system* $\boldsymbol{\mathcal{S}}$ *and therefore it can not be the equivalent algebraic semantics of any algebraizable deductive system. The same is true for any subquasivariety of* PMA *with the analogous property.*

Examples of varieties to which the theorem applies are the classes of pm-algebras axiomatized by equations in the following list:

$$\Box x \wedge x \approx \Box x, \quad x \wedge \Diamond x \approx x, \quad \Box x \wedge \Box\Box x \approx \Box\Box x, \quad \Diamond\Diamond x \wedge \Diamond x \approx \Diamond x.$$

A deductive system $\mathcal{S}$ is said to be *selfextensional* if its interderivability relation, that is, the Frege relation of the g-matrix $\langle \mathbf{Fm}, \mathrm{Th}\,\mathcal{S}\rangle$, is a congruence. It is said to be *Fregean* if for any $\mathcal{S}$-theory T the Frege relation of the generalized matrix $\langle \mathbf{Fm}, [T)_{\mathrm{Th}\,\mathcal{S}}\rangle$ is a congruence, where $[T)_{\mathrm{Th}\,\mathcal{S}} = \{T' \in \mathrm{Th}\,\mathcal{S} : T \subseteq T'\}$.

Theorem 12. *The deductive system $\mathcal{PML}$ is selfextensional and is not Fregean.*

Proof. That $\mathcal{PML}$ is selfextensional follows from Theorem 3. That it is not Fregean follows from the following observation: we have

$$p, q \vdash_{\mathcal{PML}} p \wedge q, \quad p, p \wedge q \vdash_{\mathcal{PML}} q, \quad p, \Box q \nvdash_{\mathcal{PML}} \Box(p \wedge q),$$

as is easily seen using Kripke models. Let T be the $\mathcal{PML}$-theory generated by p. Then the observation implies that $\mathrm{Clo}_{[T)_{\mathrm{Th}\mathcal{PML}}}(p \wedge q) = \mathrm{Clo}_{[T)_{\mathrm{Th}\mathcal{PML}}}(q)$, but $\mathrm{Clo}_{[T)_{\mathrm{Th}\mathcal{PML}}}(\Box q) \neq \mathrm{Clo}_{[T)_{\mathrm{Th}\mathcal{PML}}}(\Box(p \wedge q))$. Thus the Frege relation of the g-matrix $\langle \mathbf{Fm}, [T)_{\mathrm{Th}\,\mathcal{PML}}\rangle$ is not a congruence. $\Box$

5 The class Alg $\mathcal{PML}$ and the full models of $\mathcal{PML}$

In this section we first characterize the class of algebras associated with the minimum system of Positive Modal Logic according to the criteria layed down in [7], that is, the class Alg $\mathcal{PML}$. It turns out to be, as expected, the class of positive modal algebras.

Lemma 13. *For every pm-algebra $\boldsymbol{A}$, the g-matrix $\langle \boldsymbol{A}, \mathrm{Fi}_{\mathcal{PML}}\,\boldsymbol{A}\rangle$ is reduced.*

Proof. The Frege relation of $\Lambda_{\boldsymbol{A}}(\mathrm{Fi}_{\mathcal{PML}}\,\boldsymbol{A})$ is the identity because if $a, b \in A$ and $a \neq b$, then there is a prime filter that separates them, which, by Corollary 7, implies that there is an element of $\mathrm{Fi}_{\mathcal{PML}}\,\boldsymbol{A}$ that separates them. Thus, since $\widetilde{\Omega}_{\boldsymbol{A}}(\mathrm{Fi}_{\mathcal{PML}}\,\boldsymbol{A}) \subseteq \Lambda_{\boldsymbol{A}}(\mathrm{Fi}_{\mathcal{PML}}\,\boldsymbol{A})$, we obtain that $\langle \boldsymbol{A}, \mathrm{Fi}_{\mathcal{PML}}\,\boldsymbol{A}\rangle$ is reduced. $\Box$

Proposition 14. *If $\boldsymbol{A}$ is an algebra and F is one of its $\mathcal{PML}$-filters, then $\boldsymbol{A}/\Omega_{\boldsymbol{A}}(F)$ is a pm-algebra.*

Proof. We prove only that $\boldsymbol{A}/\Omega_{\boldsymbol{A}}(F)$ satisfies the equation $\Box(x \wedge y) \approx \Box x \wedge \Box y$. That it satisfies the other equations is dealt with similarly. We have to see that for all a, $b \in A$, $\langle \Box(a \wedge b), \Box a \wedge \Box b\rangle \in \Omega_{\boldsymbol{A}}(F)$. To see this it is enough to prove that for every formula $\varphi(p, \overline{r})$ and every sequence $\overline{c}$ of elements of A,

$$(3) \qquad \varphi^{\boldsymbol{A}}(\Box(a \wedge b), \overline{c}) \in F \quad \text{iff} \quad \varphi^{\boldsymbol{A}}(\Box a \wedge \Box b, \overline{c}) \in F.$$

We know that $\Box(p \wedge q)$ and $\Box p \wedge \Box q$ are interderivable in $\mathcal{PML}$ and that the interderivability relation is a congruence relation, so $\varphi(\Box(p \wedge q), \overline{r})$ and $\varphi(\Box p \wedge \Box q, \overline{r})$ are interderivable, too. We can assume without loss of generality that q does not occur in the sequence $\overline{r}$. Then, since F is a $\mathcal{PML}$-filter, for any a, $b \in A$ and any sequence $\overline{c}$ of elements of A we obtain condition (3), as desired. $\Box$

Theorem 15. *The class* Alg$\mathcal{PML}$ *is the class of all positive modal algebras.*

Proof. Let $\boldsymbol{A}$ be a pm-algebra. Consider the g-matrix $\langle \boldsymbol{A}, \mathrm{Fi}_{\mathcal{PML}}\,\boldsymbol{A}\rangle$. By Lemma 13 it is a reduced g-matrix and by Lemma 7 it is a g-model of $\mathcal{S}$; thus, $\boldsymbol{A} \in \mathrm{Alg}\,\mathcal{PML}$. This shows that $\mathsf{PMA} \subseteq \mathrm{Alg}\,\mathcal{PML}$. Now we will prove that $\mathrm{Alg}\,\mathcal{PML} \subseteq \mathsf{PMA}$. Since PMA is a variety, it is closed under subdirect products.

From [7, Theorem 2.23] we know that $\mathrm{Alg}\,\boldsymbol{\mathcal{PML}}$ is the closure of $\mathrm{Alg}^*\boldsymbol{\mathcal{PML}}$ under subdirect products. Thus $\mathrm{Alg}^*\boldsymbol{\mathcal{PML}} \subseteq \mathsf{PMA}$ implies that $\mathrm{Alg}\,\boldsymbol{\mathcal{PML}} \subseteq \mathsf{PMA}$. We proceed to prove that $\mathrm{Alg}^*\boldsymbol{\mathcal{PML}} \subseteq \mathsf{PMA}$. Let $\boldsymbol{A} \in \mathrm{Alg}^*\boldsymbol{\mathcal{PML}}$. So there is a $\boldsymbol{\mathcal{PML}}$-filter F of $\boldsymbol{A}$ such that the matrix $\langle \boldsymbol{A}, F\rangle$ is reduced. From Proposition 14 it follows that $\boldsymbol{A} \in \mathsf{PMA}$. □

Now we will see that the class of algebras associated with $\boldsymbol{\mathcal{PML}}$ by the standard theory of logical matrices, that is, the class $\mathrm{Alg}^*\boldsymbol{\mathcal{PML}}$ is different from $\mathrm{Alg}\,\boldsymbol{\mathcal{PML}}$.

Proposition 16. *The class of algebras* $\mathrm{Alg}^*\boldsymbol{\mathcal{PML}}$ *is properly included in* $\mathrm{Alg}\,\boldsymbol{\mathcal{PML}}$.

Proof. Let us consider the algebra $\boldsymbol{A}$ which consists of the three element chain $0 < a < 1$ distributive lattice with the modal operators $\Box$ and $\Diamond$ defined as the identity function. This algebra is a pm-algebra and, therefore, its $\boldsymbol{\mathcal{PML}}$-filters are its lattice filters, that is, $\{1\}, \{a, 1\}$ and $\{0, a, 1\}$. It is easy to see for each one of them that the greatest congruence compatible with it is not the identity. Thus, $\boldsymbol{A}$ is not the algebraic reduct of any reduced matrix model of $\boldsymbol{\mathcal{PML}}$. Therefore, $\boldsymbol{A} \notin \mathrm{Alg}^*\boldsymbol{\mathcal{PML}}$; but being a pm-algebra, it belongs to $\mathrm{Alg}\,\boldsymbol{\mathcal{PML}}$. □

Having characterized the elements of $\mathrm{Alg}\,\boldsymbol{\mathcal{PML}}$ we can characterize the full g-models of $\boldsymbol{\mathcal{PML}}$. First we characterize the reduced basic full g-models. Although we could have used Theorem 21, we have opted for a more direct proof.

Proposition 17. *A g-matrix* $\langle \boldsymbol{A}, \mathcal{C}\rangle$ *is a reduced basic full g-model of* $\boldsymbol{\mathcal{PML}}$ *iff* $\boldsymbol{A}$ *is a pm-algebra and* $\mathcal{C} = \mathrm{Fi}_{\boldsymbol{\mathcal{PML}}}\,\boldsymbol{A}$.

Proof. We know from Lemma 13 that if $\boldsymbol{A}$ is a pm-algebra, then $\langle \boldsymbol{A}, \mathrm{Fi}_{\boldsymbol{\mathcal{PML}}}\,\boldsymbol{A}\rangle$ is reduced. By the definition of basic full g-model, this matrix is a reduced basic full g-model of $\boldsymbol{\mathcal{PML}}$. To prove the converse let $\langle \boldsymbol{A}, \mathcal{C}\rangle$ be a reduced basic full g-model of $\boldsymbol{\mathcal{PML}}$. Then, by the definition of basic full g-model, $\mathcal{C} = \mathrm{Fi}_{\boldsymbol{\mathcal{PML}}}\boldsymbol{A}$. We need to show that $\boldsymbol{A}$ is a pm-algebra. By Proposition 14 we know that for every $F \in \mathrm{Fi}_{\boldsymbol{\mathcal{PML}}}\,\boldsymbol{A}$, $\boldsymbol{A}/\Omega_{\boldsymbol{A}}(F)$ is a pm-algebra. Moreover, $\widetilde{\Omega}_{\boldsymbol{A}}(\mathrm{Fi}_{\boldsymbol{\mathcal{PML}}}\,\boldsymbol{A}) = \bigcap_{F\in \mathrm{Fi}_{\boldsymbol{\mathcal{PML}}}\boldsymbol{A}} \Omega_{\boldsymbol{A}}(F)$. Thus $\boldsymbol{A}/\widetilde{\Omega}_{\boldsymbol{A}}(\mathrm{Fi}_{\boldsymbol{\mathcal{PML}}}\,\boldsymbol{A})$ is a subdirect product of the family $\{\boldsymbol{A}/\Omega_{\boldsymbol{A}}(F) : F \in \mathrm{Fi}_{\boldsymbol{\mathcal{PML}}}\,\boldsymbol{A}\}$. Therefore, it is a subdirect product of a family of pm-algebras. Hence, as the class of pm-algebras forms a variety, it is a pm-algebra. Since $\langle \boldsymbol{A}, \mathrm{Fi}_{\boldsymbol{\mathcal{PML}}}\,\boldsymbol{A}\rangle$ is reduced by assumption, $\widetilde{\Omega}_{\boldsymbol{A}}(\mathrm{Fi}_{\boldsymbol{\mathcal{PML}}}\,\boldsymbol{A})$ is the identity on A. Hence, $\boldsymbol{A}$ is isomorphic to $\boldsymbol{A}/\widetilde{\Omega}_{\boldsymbol{A}}(\mathrm{Fi}_{\boldsymbol{\mathcal{PML}}}\,\boldsymbol{A})$, which shows that it is a pm-algebra. □

Now we will prove a pair of lemmas that will be needed in the proof of the theorem which characterizes the full g-models of $\boldsymbol{\mathcal{PML}}$.

Lemma 18. *Any reduced basic full g-model of* $\boldsymbol{\mathcal{PML}}$ *is a model of the rules in the Gentzen calculus* G_m *for conjunction and for disjunction.*

Proof. Let $\boldsymbol{\mathcal{A}}$ be a reduced basic full g-model of $\boldsymbol{\mathcal{PML}}$. By Proposition 17 we know that it is of the form $\langle \boldsymbol{A}, \mathrm{Fi}_{\boldsymbol{\mathcal{PML}}}\,\boldsymbol{A}\rangle$ for some pm-algebra $\boldsymbol{A}$. Moreover, in this situation we know (Corollary 7) that $\mathrm{Fi}_{\boldsymbol{\mathcal{PML}}}\,\boldsymbol{A}$ is the set of all lattice filters of $\boldsymbol{A}$. Thus for every set $X \subseteq A$, the closed set $\mathrm{Clo}_{\,\mathrm{Fi}_{\boldsymbol{\mathcal{PML}}}\boldsymbol{A}}(X)$ is the lattice filter generated by X. To simplify notation let us denote it by $F(X)$. Since we are in lattice structures and conjunction is interpreted as the infimum, to see that $\boldsymbol{\mathcal{A}}$ is a model of the rules for conjunction it is enough to see that for every $a, b, c \in A$, $a, b \in F(\{c, a \wedge b\})$ and $a \wedge b \in F(\{c, a, b\})$. This is obvious because the sets of the form $F(X)$ are lattice filters. Moreover, to see that $\boldsymbol{\mathcal{A}}$ is a model of the rules for disjunction it is enough to see first that for every $a, b, c, d \in A$, if $a \in F(\{c\})$ or $b \in F(\{c\})$, then $a \vee b \in F(\{c\})$, and that if $a \in F(\{c, b\})$ and $a \in F(\{c, d\})$, then $a \in F(\{c, b \vee d\})$. The first condition clearly holds. To prove the second one, assume the antecedent and that $a \notin F(\{c, b \vee d\})$. Then there is a prime filter P such that $a \notin P$ and $F(\{c, b \vee d\}) \subseteq P$. Hence, $b \in P$ or $d \in P$. Thus $F(\{c, b\}) \subseteq P$ or $F(\{c, d\}) \subseteq P$. In any case $a \in P$, which is absurd. $\Box$

Lemma 19. *Any reduced basic full g-model of $\boldsymbol{\mathcal{PML}}$ is a model of the rules in the Gentzen calculus* G_m *for the modal operators.*

Proof. Let $\boldsymbol{\mathcal{A}}$ be a reduced basic full g-model of $\boldsymbol{\mathcal{PML}}$. As in the previous proof we have that $\boldsymbol{\mathcal{A}}$ is of the form $\langle \boldsymbol{A}, \mathrm{Fi}_{\boldsymbol{\mathcal{PML}}} \boldsymbol{A}\rangle$ for some pm-algebra $\boldsymbol{A}$ and that $\mathrm{Fi}_{\boldsymbol{\mathcal{PML}}} \boldsymbol{A}$ is the set of all lattice filters of $\boldsymbol{A}$. As before we denote the closure operator associated with $\mathrm{Fi}_{\boldsymbol{\mathcal{PML}}} \boldsymbol{A}$ by F.

To see that $\boldsymbol{\mathcal{A}}$ is a model of the rule $[\Box\Diamond]$ it is enough to show that for every $a, b, c, d \in A$, if $a \vee b \in F(\{c, d\})$, then $\Diamond a \vee \Diamond b \in F(\{\Box c, \Diamond d\})$. Recall that any filter is the intersection of all prime filters that extend it. Assume that $a \vee b \in F(\{c, d\})$. Let us see that for every prime filter P with $\Box c$, $\Diamond d \in P$ it holds that $\Diamond a \vee \Diamond b \in P$. Assume that P is a prime filter with $\Box c$, $\Diamond d \in P$. Consider the filter $F(\{c, d\})$ and the ideal $A \setminus \Diamond^{-1}[P]$. We claim that $F(\{c, d\}) \cap (A \setminus \Diamond^{-1}[P]) = \emptyset$. Suppose the contrary. Then there is $e \notin \Diamond^{-1}[P]$ such that $c \wedge d \leq e$. Thus, $\Diamond(c \wedge d) \leq \Diamond e$. So $\Box c \wedge \Diamond d \leq \Diamond(c \wedge d) \leq \Diamond e$. Since $\Box c \in P$ and $\Diamond d \in P$, $\Diamond e \in P$, which is impossible. This proves the claim. Now, by the Birkhoff-Stone Theorem let Q be a prime filter that includes $F(\{c, d\})$ and is disjoint from $A \setminus \Diamond^{-1}[P]$. Then, as $F(\{c, d\}) \subseteq Q$, $a \vee b \in Q$. Thus, since $Q \subseteq \Diamond^{-1}[P]$, $\Diamond(a \vee b) \in P$. Therefore $\Diamond a \vee \Diamond b \in P$.

Now we consider the rule $[\Diamond\Box]$. To see that $\boldsymbol{\mathcal{A}}$ is one of its models it is enough to show that if $a \vee b \in F(\{c\})$, then $\Box a \vee \Diamond b \in F(\{\Box c\})$. To see this let us prove that if $a \vee b \in F(\{c\})$, then $\Box a \vee \Diamond b$ belongs to every prime filter containing $\Box c$. Assume that $a \vee b \in F(\{c\})$ and that P is a prime filter containing $\Box c$. It is easy to see that $\Box^{-1}[P]$ is a filter and that $c \in \Box^{-1}[P]$. Thus, since $a \vee b$ belongs to the filter generated by c, $a \vee b \in \Box^{-1}[P]$. Hence $\Box(a \vee b) \in P$. Since $\Box(a \vee b) \leq \Box a \vee \Diamond b$, we have $\Box a \vee \Diamond b \in P$.

Finally, it is clear that every reduced basic full g-model $\boldsymbol{\mathcal{A}} = \langle \boldsymbol{A}, \mathrm{Fi}_{\boldsymbol{\mathcal{PML}}} \boldsymbol{A}\rangle$ is a model of the only modal initial rule because in any pm-algebra $\Diamond 0 = 0$. $\Box$

Theorem 20. *The full g-models of $\boldsymbol{\mathcal{PML}}$ are exactly the g-matrices that are models of the Gentzen calculus* G_m.

Proof. We first prove that every full g-model is a model of G_m. In the present terminology [7, Proposition 4.5] says that if $\boldsymbol{A}$ and $\boldsymbol{B}$ are g-matrices with $\boldsymbol{A} \succeq \boldsymbol{B}$, then one of them is a model of a Gentzen calculus G iff the other one is also a model of G. Therefore, it is enough to show that every reduced basic full g-model of $\boldsymbol{\mathcal{PML}}$ is a model of G_m. To attain this goal it is sufficient to prove that they are models of the rules of G_m. From Lemmas 18 and 19 we know that every reduced basic full g-model $\langle \boldsymbol{A}, \mathrm{Fi}_{\boldsymbol{\mathcal{PML}}} \boldsymbol{A}\rangle$ of $\boldsymbol{\mathcal{PML}}$ is a model of the rules for conjunction, disjunction and for the modal operators. Moreover, it is a model of the structural rules because $\mathrm{Clo}_{\mathrm{Fi}_{\boldsymbol{\mathcal{PML}}} \boldsymbol{A}}$ is a closure operator, and a model of the initial rules and the inconsistency rule (rule 5. in our list).

To prove that every g-model of G_m is a full g-model of $\boldsymbol{\mathcal{PML}}$, let us assume that $\boldsymbol{\mathcal{A}} = \langle \boldsymbol{A}, \mathcal{C}\rangle$ is a g-model of G_m. Then it is a g-model of $\boldsymbol{\mathcal{PML}}$ because, by the completeness theorem for G_m, $\boldsymbol{\mathcal{PML}}$ is the deductive system defined by G_m. By [7, Proposition 4.5] we can assume without loss of generality that $\boldsymbol{\mathcal{A}}$ is reduced. Thus Theorem 15 and the definition of the class $\mathrm{Alg}\,\boldsymbol{\mathcal{PML}}$ imply that $\boldsymbol{A}$ is a pm-algebra. Now we proceed to show that $\mathcal{C} = \mathrm{Fi}_{\boldsymbol{\mathcal{PML}}}\,\boldsymbol{A}$. We will prove that for every $X \subseteq A$, $\mathrm{Clo}_{\mathcal{C}}(X)$ is the filter $F(X)$ generated by X. Since $\boldsymbol{A}$ is a g-model of $\boldsymbol{\mathcal{PML}}$ and $\boldsymbol{A}$ is a pm-algebra, $\mathcal{C} \subseteq \mathrm{Fi}_{\boldsymbol{\mathcal{PML}}}\,\boldsymbol{A}$. Thus, for every $X \subseteq A$, $\mathrm{Clo}_{\mathcal{C}}(X)$ is a filter that contains X, that is, $F(X) \subseteq \mathrm{Clo}_{\mathcal{C}}(X)$. To prove the other inclusion we can assume that X is non-empty since $\mathrm{Clo}_{\mathcal{C}}(\emptyset) = \mathrm{Clo}_{\mathcal{C}}(\{1\})$, because $\top$ is a theorem of $\boldsymbol{\mathcal{PML}}$. Let $a \in \mathrm{Clo}_{\mathcal{C}}(X)$. Since $\mathrm{Clo}_{\mathcal{C}}$ is a finitary closure operator, let $a_0, \ldots, a_{n-1} \in X$ be such that $a \in \mathrm{Clo}_{\mathcal{C}}(\{a_0, \ldots, a_{n-1}\})$. Then $\mathrm{Clo}_{\mathcal{C}}(\{a, a_0, \ldots, a_{n-1}\}) = \mathrm{Clo}_{\mathcal{C}}(\{a_0, \ldots, a_{n-1}\})$, and so, $\mathrm{Clo}_{\mathcal{C}}(\{a \wedge a_0 \wedge \cdots \wedge a_{n-1}\}) = \mathrm{Clo}_{\mathcal{C}}(\{a_0 \wedge \cdots \wedge a_{n-1}\})$. Since $\boldsymbol{\mathcal{A}}$ is a model of G_m, its Frege relation $\Lambda(\boldsymbol{\mathcal{A}})$ is a congruence relation, thus $\Lambda(\boldsymbol{\mathcal{A}}) = \widetilde{\Omega}(\boldsymbol{\mathcal{A}})$. Since by assumption $\boldsymbol{\mathcal{A}}$ is reduced, we obtain that $\Lambda(\boldsymbol{\mathcal{A}})$ is the identity. If follows then that $a \wedge a_0 \wedge \cdots \wedge a_{n-1} = a_0 \wedge \cdots \wedge a_{n-1}$. Hence, $a_0 \wedge \cdots \wedge a_{n-1} \leq a$. Therefore $a \in F(X)$. This concludes the proof of $F(X) = \mathrm{Clo}_{\mathcal{C}}(X)$, and the proof of the theorem. □

Using the results in [7] some interesting consequences can be obtained.

Theorem 21 (Bilogical Theorem). *For every g-matrix $\boldsymbol{\mathcal{A}} = \langle \boldsymbol{A}, \mathcal{C}\rangle$ the following three conditions are equivalent:*

1. *$\boldsymbol{\mathcal{A}}$ is a full g-model of $\boldsymbol{\mathcal{PML}}$.*
2. *$\boldsymbol{A}/\widetilde{\Omega}(\boldsymbol{\mathcal{A}})$ is a pm-algebra and $\mathcal{C}/\widetilde{\Omega}(\boldsymbol{\mathcal{A}})$ is the set of all filters of $\boldsymbol{A}/\widetilde{\Omega}(\boldsymbol{\mathcal{A}})$.*
3. *There is a strict surjective homomorphism between $\boldsymbol{\mathcal{A}}$ and a g-matrix $\boldsymbol{\mathcal{B}} = \langle \boldsymbol{B}, \mathcal{C}\rangle$, where $\boldsymbol{B}$ is a pm-algebra and $\mathcal{C} = \mathrm{Fi}(\boldsymbol{B})$.*

Proof. By Corollary 7, Theorem 15 and [7, Proposition 2.21]. □

Theorem 22 (Isomorphism Theorem). *For any algebra $\boldsymbol{A}$, the Tarski operator $\widetilde{\Omega}_{\boldsymbol{A}}$ is a dual order isomorphism between the set of all finite closed-set systems on A that determine a full g-model of $\boldsymbol{\mathcal{PML}}$, ordered by inclusion, and the set of all* PMA*-congruences of $\boldsymbol{A}$, also ordered by inclusion.*

Proof. This follows from Theorem 15 and [7, Theorem 2.30], taking into account that in the statement of [7, Theorem 2.30] the order of the full g-models over an algebra is defined in terms of the closure operators of the g-matrices instead of in terms of the finite closed-set systems. In the present paper we take the dual order, the inclusion between the finite closed-set systems. Thus we obtain a dual order isomorphism. □

6 Full adequacy and algebraizability of G_m

A Gentzen system $\boldsymbol{\mathcal{G}}$ is said to be *fully adequate* (strongly adequate in the terminology of [7]) for a deductive system $\boldsymbol{\mathcal{S}}$ with theorems if its models are exactly the full g-models of $\boldsymbol{\mathcal{S}}$. The notion of full adequacy for deductive systems without theorems is similar, with some slight technical modification. Since the deductive system $\boldsymbol{\mathcal{PML}}$ has theorems we do not enter into these subtleties. A fully adequate Gentzen system $\boldsymbol{\mathcal{G}}$ for a deductive system $\boldsymbol{\mathcal{S}}$ has $\boldsymbol{\mathcal{S}}$ as its associated deductive system $\boldsymbol{\mathcal{S}}_{\mathcal{G}}$

([7, Proposition 4.11]); essentially this happens because the g-matrices $\langle \mathbf{Fm}, \mathrm{Th}\,\boldsymbol{\mathcal{S}}\rangle$ and $\langle \mathbf{Fm}, \mathrm{Th}\,\boldsymbol{\mathcal{S}}_{\mathcal{G}}\rangle$ are both models of $\boldsymbol{\mathcal{G}}$. It is clear too that if a deductive system (with theorems) has a fully adequate Gentzen system, this system must be unique because two different Gentzen systems cannot have the same class of models ([7, Proposition 4.4]).

The notion of full adequacy is important in many respects and it has strong connections with well established concepts in the logical literature. For instance, in [8] it is proved that a weakly algebraizable deductive system $\boldsymbol{\mathcal{S}}$ has a fully adequate Gentzen system iff it has a form of the deduction theorem.

Theorem [20] can be paraphrased using the terminology just introduced as follows:

Theorem 23. *The Gentzen system defined by the Gentzen calculus* G_m *is fully adequate for* $\boldsymbol{\mathcal{PML}}$.

Now we will see that the Gentzen system defined by the Gentzen calculus G_m is algebraizable and that its equivalent algebraic semantics is the class of algebras $\boldsymbol{\mathcal{PML}}$. To this end we consider the translations $\mathrm{t}_\wedge$ from sequents into equations and the translation sq from equations into sequents defined by

$$\mathrm{t}_\wedge(\Gamma \rhd \varphi) = \{\bigwedge \Gamma \wedge \varphi \approx \bigwedge \Gamma\}, \quad \mathrm{sq}(\varphi \approx \psi) = \{\varphi \rhd \psi, \psi \rhd \varphi\}$$

(conjunctions are taken according to a fixed ordering of the formulas). This translations have been studied in [7] where the following theorem is proved. A deductive system $\boldsymbol{\mathcal{S}}$ is said to have the *Property of Conjunction* (PC for short) relative to a binary term $\wedge$ if (1) $p, q \vdash_{\boldsymbol{\mathcal{S}}} p \wedge q$, (2) $p \wedge q \vdash_{\boldsymbol{\mathcal{S}}} p$, and (3) $p \wedge q \vdash_{\boldsymbol{\mathcal{S}}} q$. Deductive systems with PC are frequently called *conjunctive* in the literature.

Theorem 24 ([7, Theorem 4.27]). *Any selfextensional deductive system* $\boldsymbol{\mathcal{S}}$ *with* PC *relative to a binary term* $\wedge$ *has a fully adequate Gentzen system which is equivalent to the equational consequence* $\vDash_{\mathrm{Alg}\,\boldsymbol{\mathcal{S}}}$ *(defined by the class of algebras canonically associated with* $\boldsymbol{\mathcal{S}}$*) by the translations* $\mathrm{t}_\wedge$ *and* sq*, and this Gentzen system can be axiomatized by the following rules:*

1. *The structural rules of Weakening and Cut.*
2. *For each sequent* $\Delta \rhd \varphi$ *with* $\Delta \vdash_{\boldsymbol{\mathcal{S}}} \varphi$*, the initial rule* $\dfrac{}{\Delta \rhd \varphi}$.
3. *For each n-ary connective* ϖ*, the congruence rules*
$$\frac{\{\varphi_i \rhd \psi_i, \psi_i \rhd \varphi_i : i < n\}}{\varpi(\varphi_0, \ldots, \varphi_{n-1}) \rhd \varpi(\varphi_0, \ldots, \varphi_{n-1})}\,.$$

From this result and the observations made above it follows that the Gentzen system defined by the Gentzen calculus G_m can also be axiomatized by the Gentzen calculus G_c whose rules are the rules obtained specializing to the deductive system $\boldsymbol{\mathcal{PML}}$ the rules in the theorem, that is, it can be axiomatized by all the sequents that hold in $\boldsymbol{\mathcal{PML}}$ taken as initial rules, the congruence rules and the structural rules. Therefore, the Gentzen calculus G_m is equivalent to the calculus G_c. We can say that the calculus G_c gives the *explicit content* of G_m in the sense that it shows which derived rules of G_m are in fact essentially added to the initial derived rules that simply mimic the behavior of $\boldsymbol{\mathcal{PML}}$.

Moreover, by the definition of algebraizable Gentzen system (given in Section 2) the theorem implies the following algebraizability result:

T h e o r e m 25. *The Gentzen calculus* G_m *is algebraizable and its equivalent algebraic semantics is the variety of positive modal algebras.*

As a corollary to Theorems 23 and 25 we obtain

T h e o r e m 26. *The logic* $\boldsymbol{\mathcal{PML}}$ *is sequent-algebraizable.*

To conclude the paper we show that the variety of pm-algebras does not have equationally definable principal congruences (EDPC). According to the last theorem and a result in [13] this implies that the Gentzen system defined by G_m does not have the deduction theorem in the sense of [13].

Recall that a variety of algebras K is said to have *first-order definable principal congruences* (DPC) if there is a first-order formula $\varphi(x, y, u, v)$ in the first-order language of the similarity type of K such that for every $\boldsymbol{A} \in \mathsf{K}$ and every $a, b, c, d \in A$,

$$(4) \qquad \langle a, b\rangle \in \theta(c, d) \quad \text{iff} \quad \boldsymbol{A} \vDash \varphi[a, b, c, d].$$

And that it is said to have *equationally definable principal congruences* (EDPC) if (4) holds for a formula $\varphi(x, y, u, v)$ which is a conjunction of equations.

A consequence of [2, Theorem 5.2], obtained specializing the theorem to varieties, is that if a variety has DPC, in particular EDPC, then the class of its simple algebras is closed under ultraproducts. In Examples 5.2.1 of [2] it is shown that the variety of modal algebras does not have DPC, and therefore does not have EDPC, by showing a family of simple modal algebras with a non-simple ultraproduct. Since every simple modal algebra has as $(\wedge, \vee, \top, \bot, \Box, \Diamond)$-reduct a pm-algebra and simplicity is conserved in passing to these reducts, we have a family of simple pm-algebras with a non-simple ultraproduct. Thus we have the following

T h e o r e m 27. *The variety* PMA *does not have equationally definable principal congruences.*

In [13] the notion of a deduction-detachment theorem for Gentzen calculi is introduced. Restricting it to the type of Gentzen calculus we consider in the present paper the definition runs as follows: A *possible deduction-detachment set* for a Gentzen system $\boldsymbol{\mathcal{G}}$ is a set of finite sets of sequents $E = \{E_{(n,m)} : n, m \in \omega\}$, where for $n, m \in \omega$, $E_{(n,m)}$ is a finite set of sequents in the different variables $p_0, \dots, p_{n-1}, p_n$ and $q_0, \dots, q_{m-1}, q_m$. A Gentzen system $\boldsymbol{\mathcal{G}}$ is said to have the *deduction-detachment theorem* if there is a possible deduction-detachment set E such that for every set of sequents X and every pair of sequents $\{\varphi_0, \dots, \varphi_{n-1}\} \rhd \varphi$, $\{\psi_0, \dots, \psi_{m-1}\} \rhd \psi$,

$$X, \{\varphi_0, \dots, \varphi_{n-1}\} \rhd \varphi_n \vdash_{\mathcal{G}} \{\psi_0, \dots, \psi_{m-1}\} \rhd \psi_m$$
iff
$$X \vdash_{\mathcal{G}} E_{(n,m)}(p_0/\varphi_0, \dots, p_{n-1}/\varphi_{n-1}, p_n/\varphi_n, q_0/\psi_0, \dots, q_{m-1}/\psi_{m-1}, q_m/\psi_m),$$

where $E_{(n,m)}(p_0/\varphi_0, \dots, p_{n-1}/\varphi_{n-1}, p_n/\varphi_n, q_0/\psi_0, \dots, q_{m-1}/\psi_{m-1}, q_m/\psi_m)$ is the result of substituting the formulas φ_i $(i < n)$ for the variables p_i, the formula φ_n for the variable p_n, the formulas ψ_j $(j < m)$ for the variables q_j, and the formula ψ_m for the variable q_m. In this case we say that E is a *deduction-detachment set* for $\boldsymbol{\mathcal{G}}$.

[13, Corollary 3.13] says that if $\boldsymbol{\mathcal{G}}$ is an algebraizable Gentzen system with equivalent algebraic semantics a variety K, then $\boldsymbol{\mathcal{G}}$ has DDT iff K has EDPC. Thus we conclude with the following

Theorem 28. *The Gentzen system defined by the Gentzen calculus* G_m *for positive modal logic does not have a deduction-detachment theorem.*

This fact is interesting for the following reason. There is no hope of finding a deduction-detachment theorem for the deductive system $\mathcal{PML}$ because it is not protoalgebraic and every deductive system with a deduction-detachment system is protoalgebraic. Nevertheless, it can in principle be the case that a deductive system without a deduction-detachment theorem has a fully adequate Gentzen system with it. This situation does not hold for $\mathcal{PML}$.

References

[1] Blok, W., and D. Pigozzi, Algebraizable Logics. Memoirs of the AMS, vol. 396. The American Mathematical Society, Providence 1986.

[2] Blok, W., and D. Pigozzi, Abstract algebraic logic and the deduction theorem. Bulletin of Symbolic Logic (to appear).

[3] Celani, S., and R. Jansana, A new semantics for positive modal logic. Notre Dame J. Formal Logic **38** (1997), 1 – 18.

[4] Celani, S., and R. Jansana, Priestley duality, a Sahlqvist theorem and a Goldblatt-Thomason theorem for positive modal logic. Logic Journal of the IGPL **7** (1999), 683 – 715.

[5] Czelakowski, J., Protoalgebraic Logics. Kluwer, Dordrecht 2001.

[6] Dunn, M., Positive modal logic. Studia Logica **55** (1995), 301 – 317.

[7] Font, J. M., and R. Jansana, A General Algebraic Semantics for Sentential Logics. Lecture Notes in Logic **7**, Springer-Verlag, Berlin-Heidelberg-New York 1996.

[8] Font, J. M., R. Jansana, and D. Pigozzi, Fully adequate Gentzen systems and closure properties of the class of full models. Preprint 1999.

[9] Font, J. M., and V. Verdú, Algebraic logic for classical conjunction and disjunction. Studia Logica, Special Issue on Algebraic Logic **50** (1991), 391 – 419.

[10] Kurtonina, N., and M. de Rijke, Simulating without Negation. J. Logic and Computation **7** (1997), 501 – 522.

[11] Pigozzi, D., Review of [7]. Studia Logica **64** (2000), 287 – 297.

[12] Rebagliato, J., and V. Verdú, On the algebraization of some Gentzen systems. Fundamenta Informaticae, Special Issue on Algebraic Logic and its Applications **18** (1993), 319 – 338.

[13] Rebagliato, J., and V. Verdú, Algebraizable Gentzen systems and the deduction theorem for Gentzen systems. Universitat de Barcelona, Mathematics Preprint Series 175 (1995).

[14] Wójcicki, R., Theory of Logical Calculi. Basic Theory of Consequence Operations. Synthese Library, vol. 199, Reidel, Dordrecht 1988.

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