FOUNDATIONS OF INTUITIONISTIC LOGIC

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1. Introduction

Intuitionistic mathematics, as developed by Brouwer and Heyting, has two aspects. Its *negative* aspect, which is well known, rejects the basic notions of set-theoretic (or classical) mathematics. The *positive* aspect is this: the notions of 'construction (constructive function)' and 'constructive proof (of equality between two constructions)' are regarded as sufficiently clear for at least a part of mathematics to be built up systematically from some evident assertions about these notions. This aspect has been largely disregarded, probably because of current preoccupation with formal and first-order aspects of mathematics. In particular, when questions of interpretation are ignored, the first-order systems of Heyting can be set out as subsystems of the corresponding classical systems, and so any formal derivation in one of the former systems is also one of the latter. Thus, though Heyting intended his formal rules to be justified in terms of an intuitionistic interpretation they can also be justified on the basis of classical interpretations. Little attention has been paid to Heyting's higher-order systems [6] or, e.g., to Kleene's fragment of intuitionistic analysis [11], although it is only here that specifically intuitionistic notions begin to come into their own: e.g., the consistency of these higher-order systems is evident on their intended meaning, but more or less elaborate reinterpretations like Gödel's [5] or recursive realizability [11] are needed to establish their formal consistency classically. (Since consistency is expressible arithmetically, we have here an arithmetic statement which is proved more shortly by means of the basic intuitionistic notions than by classical ones.)

Our main purpose here is to enlarge the stock of formal rules of proof which follow directly from the meaning of the basic intuitionistic notions but not from the principles of classical mathematics so far formulated.[1] The specific problem which we have chosen to lead us to these rules is also of independent interest: *to set up a formal system, called 'abstract theory of constructions' for the basic notions mentioned above, in terms of which the formal rules of Heyting's predicate calculus can be interpreted.*

In other words, we give a formal semantic foundation for intuitionistic formal systems in terms of the abstract theory of constructions. This is analogous to the semantic foundation for classical systems [18] in terms of

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[1] In particular, the consistency of the new formal rules cannot be proved easily in the usual classical systems, and, perhaps, not at all.

abstract set theory.[2] It should be noted that in both cases the value of the *formal* semantic foundation is primarily technical. Thus, it seems to me, the informal indications of Heyting [7, p. 98] are quite sufficient to convey the meaning of the intuitionistic logical constants with the possible exception of quantifiers over ips, considered in Sec. 8. But, on the technical side, semantic foundations (i) are useful for independence proofs, since they set out minimal properties of the notions needed in the interpretation, and (ii) lead naturally 'beyond' the formal system considered and therefore seemed suited to our main purpose.

For the mathematician, their most useful application is, perhaps, this. Given a formal system, say in the notation of predicate logic, a classical interpretation assigns a set D_0 for the range of the variables and subsets D_i of $D_0^{n_i}$ to the n_i-placed predicate symbols occurring in the axioms of the given system. A 'non-standard' model uses then sets D different from the sets D^* of the intended interpretation; e.g., with (fragments of) arithmetic one uses a set D_0 of 'non-archimedean' integers instead of the set D_0 of natural numbers, but to date no really usable examples of such D_0 are known. In the intuitionistic case we have species[3] $\mathfrak{D}_0$, $\mathfrak{D}_i$ corresponding to D_0, D_i; but, in addition, we have a class of proofs $\mathfrak{P}$ as, roughly speaking, an extra parameter. In the intended interpretation, $\mathfrak{P}^*$ consists of *all* constructive proofs, while in a 'non-standard' interpretation $\mathfrak{P}$ may be any subclass of $\mathfrak{P}^*$ which satisfies the axioms laid down in formal semantics. Thus one gets non-standard models (with an easily described $\mathfrak{P}$, $\mathfrak{P} \neq \mathfrak{P}^*$), where $\mathfrak{D} = \mathfrak{D}^*$. Another application concerns *generalized inductive definitions* (cf., e.g., [17]). On the Dedekind-Frege interpretation of these definitions (as the least class satisfying the defining conditions) these are highly impredicative, since they involve quantification over all subclasses of the class of objects considered. On the intuitionistic interpretation they are much less so, and moreover much 'smaller' species will satisfy a given inductive definition, for now an object α is required to belong to the species defined only if α can be *proved by the restricted means of* $\mathfrak{P}$ to satisfy the premise in the inductive definition considered.

From a more logical point of view the abstract theory of constructions provides a systematic notation for various completeness results on intuitionistic logic (cf. end of Sec. 6). Of course, as pointed out in [13], these are perfectly intelligible from evident properties of proof and construction, and the mere possibility of finding some systematic notation could not have been doubted by anyone who really read Heyting's informal exposition. But the actual details have not been previously written down. Another logically

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[2]A semantic foundation of a formal system verifies that the formal theorems are valid on the intended interpretation of the symbolism, and a *formal* semantic foundation makes explicit just what properties of the notions which are used in the interpretation, are needed for this verification.

[3]The term is explained in Sec. 4.

interesting application of the work below is that it leads to a considerable simplification of [16].

It should be noted that all these applications are, scientifically speaking, independent of the negative aspect of intuitionistic mathematics. Psychologically it may well be that for many people intuitionistic mathematics becomes more interesting if they believe the alternatives to be ill-conceived.

2. Comparison with Set-Theoretic Foundations (of classical systems)

There is a striking parallelism between foundations based on set-theoretic notions on the one hand and intuitionistic ones on the other. We have a *negative* attitude of set-theoretically-minded logicians who regard the intuitionistic notions as intelligible only when formulated in some formal system.[4] Further, as shown by Gödel (again of course aside from interpretation), first-order classical systems can be formulated as subsystems of the corresponding intuitionistic systems.[5]

Note in passing that the parallelism has a disturbing side to it, since Frege's original formulation of abstract set theory, which corresponds to the abstract theory of constructions below, was inconsistent. Therefore I occasionally formulate below weakened versions of the 'natural' axioms such that a combinatorial consistency proof for the weak axioms can be given by means of hierarchies of formal systems.

3. Comparison with Finitist Foundations

It is clear that a semantic foundation cannot maximize the evidence for (or minimize the presuppositions of) given formal manipulations. The very fact that formal semantic foundations for a given system F need only *some* properties of the basic semantic notions shows that their *full* content is not needed to justify the formal rules of F (though it generally will be needed for the most obvious justification of F and for a well-motivated use of F). Further, the evidence of F is not increased by an explicit formulation of the restricted versions[6] of the semantic notions which satisfy the abstract theory, if, at the same time, one insists on defining these restrictions in terms of the basic semantic notions. Thus, if one is interested in evidence, one is forced to formulate a problem of foundations in terms of a group of notions introduced independently of and (if possible) in a more elementary manner

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[4]Which they proceed to reinterpret classically, thereby depriving themselves of whatever mathematical fecundity the basic intuitionistic notions may have (except of course their having suggested the reinterpretation).

[5]Naturally, this applies to (first-order) systems where under the intended interpretation the variables range over first-order objects, e.g., in arithmetic, but not in abstract set theory.

[6]I.e., restrictions of the notions of arbitrary construction and constructive proof in the intuitionistic case, and of the notion of set to a subclass of the collection of all sets in the classical case. 'Evidence (of)' is here used in the sense of the German 'Evidenz', and not of 'evidence (for)'.

than the basic semantic notions. This is evidently the line of reasoning which leads to *Hilbert's programme.*

In conception this does not conflict at all with a semantic foundation since, roughly speaking, Hilbert's programme begins where the semantic foundation leaves off. In practice, however, the remarkable success of Hilbert's programme (applied to first-order arithmetic) undermines an interest in intuitionism, and, all the more, its semantic foundations. For, granted that one is preoccupied with Heyting's first-order systems we have this: (i) as far as results and elegance of proof are concerned the classical rules are just as good and better, (ii) as far as evidence is concerned, finitist foundations make substantially fewer assumptions because they use only constructions applied to spatio-temporal objects, whereas in the intuitionistic case constructions are applied to abstract notions; this distinction was first formulated and stressed by Gödel [5]; and (iii) as far as a mathematically significant analysis of proofs [12] is concerned, Herbrand's theorem and other finitist interpretations have been generally more informative than translations into intuitionistic systems.[7]

The situation is changed when one tries to give a constructive analysis of second-order systems. Even if one's final aim is to avoid the problematic (highly abstract) general notions of construction and proof, as in the modified Hilbert programme [15], these notions are needed for a systematic approach. So to speak, intuitionism is to provide a *general theory of constructivity* which should allow a rational approach to specific problems of constructivity, just as a general theory of equations isolates the significant properties of equations which also help one with special cases. A very encouraging application of this kind is provided by Gödel [5], where, with the help of intuitionistic considerations, the consistency problem of classical arithmetic is broken down into several steps (cf. [15, para. 12]) and thus made much more transparent than, e.g., in [2].

4. The Intuitionistic Position (General Statement)

The *sense* of a mathematical assertion denoted by a linguistic object A is intuitionistically determined (or understood) if we have laid down what constructions constitute a *proof* of A, i.e., if we have a construction r_A such that, for any construction c, $r_A(c) = 0$ if c is a proof of A and $r_A(c) = 1$ if c is not a proof of A: the logical particles in this explanation are interpreted

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[7]Strictly speaking, if one uses intuitionistic concepts, exceptions can be formulated. Let $\mathfrak{A}$ be a prenex formula $(x_1)(Ey_1)\ldots(x_n)(Ey_n)A(x_1,\ldots,x_n,y_1,\ldots,y_n)$, e.g., of arithmetic, $\mathfrak{A}^-$ its negative version where $\vee$ and (E) are eliminated, $\mathfrak{A}^*$ be the (usual) prenex form of $\neg\mathfrak{A}$, namely, $(Ex_1)(y_1)\ldots(Ex_n)(y_n)\neg A(x_1,\ldots x_n,y_1,\ldots,y_n)$, $\mathfrak{A}^{**}$ the (second-order) prenex form of $\neg\mathfrak{A}^*$. Then intuitionistically, $\mathfrak{A}\rightarrow\neg\neg\mathfrak{A}\rightarrow\mathfrak{A}^-\rightarrow\neg\neg\mathfrak{A}^{**}\rightarrow\neg\mathfrak{A}^*$ but, in general, not conversely. For a classically provable $\mathfrak{A}$, the so-called no-counter-example interpretation yields $\mathfrak{A}^{**}$ (with explicit statement of the functionals involved), while Heyting's arithmetic yields $\mathfrak{A}^-$. But this improvement is not intelligible and certainly not significant unless the intuitionistic notions are adopted for other reasons.

truth functionally, since we are adopting the basic intuitionistic idealization that we can recognize a proof when we see one, and so r_A is decidable. (Note that this applies to *proof*, not *provability*). A *species* of n-tuples of constructions $a_1, \cdots, a_n$ is determined by a construction s where $s(c, a_1, \cdots, a_n) = 0$ if c is a proof that $\langle a_1, \cdots, a_n \rangle$ belong to the species, $s(c, a_1, \cdots, a_n) = 1$ otherwise. A construction c is to be thought of as the object constructed when c is of first order (function of zero arguments), and as a method of construction when c is of higher order. However, we do not make explicit type distinctions following, e.g., Heyting's practice in introducing the logical constants [7, p. 98].

Consider then a formula A of the predicate calculus whose relation symbols are $R_i(x_1, \cdots, x_{n_i})$, $1 \leqq i \leqq k$, where we use variables x in the formal system of Heyting and $a, b, c \cdots$ in the abstract theory of constructions. Suppose we are given 'interpretations' for the R_i, namely, species of n_i-tuples determined by $s_i(c, a_1, \cdots, a_{n_i})$. As Heyting indicated, one then obtains proof conditions for A in terms of s_i. In particular, if A is BoC, where o denotes a logical connective and r_B, r_C determine the senses of B and C, then we obtain r_A from r_B and r_C. In the case of quantifiers it is clear that the idea of an *infinite* range must be regarded as a *façon de parler* (about proof conditions), because intuitionistic mathematics does not recognize infinite extensions. Generally, one must be careful not to take over too mechanically the syntactic formation rules from classical calculi, since syntactic rules are motivated by considerations about the *sense* attached to linguistic objects, and these senses are different in classical and intuitionistic cases.

While the notion of 'proof of a formula A' is defined, the notion 'c is a proof of the extensional equality of the constructions a and b' is taken as primitive in our formulation below, and all assertions are reduced to this special form (for motivation, cf. the Remark on p. 206). In the formal system, $a = b$ is interpreted as essentially intensional[8] equality between a and b; $\pi(c, a, b) = 0$ as 'the construction c is a proof of the equality of a and b', where all three variables a, b, c are permitted to take arbitrary constructions as values.[9]

5. Abstract Theory of Constructions

We give a simultaneous description of *terms*, *formulae*, *axioms*, and *rules of inference*. We indicate the intended meaning of the terms and leave it to

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[8]The precise content is made explicit in the axioms below; e.g., if two definitions are convertible into one another in the very elementary manner considered below, the constructions so defined are considered equal, e.g., pair formation and inverses.

[9]This corresponds to the 'mixing' of mathematics and metamathematics stressed in the informal writings of intuitionists. It is to be distinguished from the interplay of protologic and formal rules in operative logic, because, in contrast to the latter, we consider proofs as constructions on constructions and not merely on symbolic expressions. Naturally, except in very elementary first-order contexts, different assertions hold for the intuitionistic and operative notions of construction.

the reader to verify that the axioms and rules hold for this meaning. When comparing the system with, e.g., abstract set theory, one finds that we have more *constant* terms because we do not have quantifiers. Note that the starred rules raise consistency questions.

TERMS. 0,1 (*two distinct first-order objects*); $f, g, h, \ldots$ (*distinct variables for constructions*): If a, b, c *are terms, then so are* $*a(b, c, \cdots)$ (*the result of applying the construction represented by* a *to* $b, c, \cdots$ *if this makes sense, i.e., if the number and types of the arguments fit, otherwise it is* a *itself*).

The corresponding unstarred rule permits such formation with variable a only in the context $\pi[c, a(b, c, \cdots), a_1(b_1, c_1, \cdots)]$.

$*\lambda_f\, a$ where f is a variable not bound in a. The corresponding unstarred rule requires that a actually contain f and adds the extra constants 0^1, 0^2, $0^3, \cdots$ with $0^1(a) = 0$, $0^2(a, b) = 0$, $0^3(a, b, c) = 0, \cdots$. Further, only *two-valued* a, introduced below, are permitted.

$d(a, b)$ (the pair of constructions a and b)

$d_1(a)$ (the first element of a if a is a pair, otherwise, e.g., a)

$d_2(a)$ (the second element of a)

$n(a)$ (generalization of truth functional negation, i.e., a construction on a, which, applied to $a = 0$ and $a = 1$, has the value $1-a$)

$k(a, b)$ (generalization of truth functional conjunction)

Abbreviation: $i(a, b)$ for $n\{k[a, n(b)]\}$ (hence generalization of truth functional implication);

$\pi(a, b, c)$ ($= 0$: a is a proof that b and c are extensionally equal);

$r_i(c)$, $s_i(c, a_1, \cdots, a_{n_i})$ (constructions determining particular assertions and particular species).

The following terms are not needed in the introduction of logical constants, but only in the proofs that they satisfy Heyting's formal rules.

$a * b$ (concatenation)

$a \circ b$ (particularization: if $\pi[a, \lambda_f c_1(f), \lambda_f c_2(f)] = 0$, then $\pi[a \circ b, c_1(b), c_2(b)] = 0$).

$\pi_0(a, b)$ ('verifiability' of π: if a proves $b = 0$, then $\pi_0(a, b)$ proves $\pi(a, b, 0) = 0$).

$\pi_1(a, b)$ ('contractibility' of π: if a proves $\pi(b, c, 0) = 0$, then $\pi_1(a, b)$ proves $c = 0$).

$r^i(a)$ ('verifiability' of r_i: if $r_i(a) = 0$, then $\pi[r^i(a), r_i(a), 0] = 0$).

$r^{(i)}(a, b)$ ('contractibility' of r_i: if a proves $r_i(b) = 0$, then $r_i[r^{(i)}(a,b)] = 0$).

For any sequence $\mathfrak{p}$ of equations between terms, $\pi_{\mathfrak{p}}$ is a term (if $\mathfrak{p}$ is a formal derivation in our system of $a(f_1, \cdots, f_k) = 0$, then $\pi_{\mathfrak{p}}$ represents an — intuitive — proof of the extensional equality of $\lambda_{f_1,\ldots,f_k} a(f_1, \cdots, f_k)$ and $\lambda_{f_1,\ldots,f_k} 0$).

TWO-VALUED TERMS. 0, 1; *for arbitrary* a, b, c, $r_i(a)$, $s_i(c, a_1, \ldots, a_{n_i})$, $0^1(a)$, $0^2(a, b)$, $\cdots$, $\pi(a, b, c)$; *for two-valued* a, b, $n(a)$, $k(a, b)$ (*two-valuedness expresses decidability*).

Notation: If a and b are two-valued, we write $\sim a$ for $n(a)$, a & b for $k(a, b)$, $a \supset b$ for $i(a, b)$ and $a \cup b$ for $n\{k[n(a), n(b)]\}$.

FORMULAE. *If* a *and* b *are terms,* $a = b$ *is a formula.*
($=$ denotes intensional equality at higher types, extensional equality at lowest type.)

AXIOMS. $[\lambda_f a(f)](b) = a(b)$,
$n(0) = 1, n(1) = 0, n[n(a)] = a$,
$k(0, a) = a, k(1, a) = 1, k(a, b) = k(b, a)$,
$k(\lambda_f a, \lambda_f b) = \lambda_f k(a, b), n(\lambda_f a) = \lambda_f n(a)$,
$d_1[d(a, b)] = a, d_2[d(a, b)] = b, d_1(\lambda_f a) = \lambda_f d_1(a), d_2(\lambda_f a) = \lambda_f d_2(a)$,
$d(\lambda_f a, \lambda_f b) = \lambda_f d(a, b)$.

If $I(A_1, \cdots, A_k)$ *is an identity of the classical propositional calculus (in the notation* $\sim$ *and* &) *and* $a_1, \cdots, a_k$ *are two-valued terms,*
$I(a_1, \cdots, a_k) = 0$ *is an axiom.*
$\{\pi(a, b, 0) \supset \pi[\pi_0(a, b), \pi(a, b, 0), 0]\} = 0$,
$\{\pi[a, \pi(b, c, 0), 0] \supset \pi[\pi_1(a, b), c, 0]\} = 0$.

(The corresponding starred rule is $*i[\pi(a, b, 0), b] = 0$,which is obvious on the intended interpretation, but troublesome for the consistency proof. The unstarred rule suffices for the semantic foundation of Heyting's predicate logic.)

$\{r_i(a) \supset \pi[r^i(a), r_i(a), 0]\} = 0$,
$\{\pi[a, r_i(b), 0] \supset r_i[r^{(i)}(a, b)]\} = 0$,
$[\{\pi(a, b, 0)\ \&\ \pi[a_1, i(b, c), d]\} \supset \pi(a * a_1, c, d)] = 0$,
$\{\pi[a, \lambda_f c(f), d] \supset \pi[a \circ b, c(b), d(b)]\} = 0$.

RULES OF INFERENCE. $a = a$,

$$\frac{a = b}{b = a}, \qquad \frac{a = b,\ b = c}{a = c}, \qquad \frac{a = b}{a(c) = b(c)}, \qquad \frac{a = b}{c(a) = c(b)}.$$

(The unstarred version of this rule is applied, when c is a π-term only to $\pi(a, a_1, b_1)$, not to $\pi(a_1, a, b_1)$ nor to $\pi(a_1, b_1, a)$. Note that the starred rule is strong; e.g., applied to $\pi(a_1, a, b_1)$, it means that even if the constructions a and b are only convertible into each other, the same construction a_1 proves the equality of a and b_1 and of b and b_1. This would of course not hold if *proof* were regarded in the syntactic sense).

Finally, if a does not contain f and if $\mathfrak{p}$ is a formal derivation of $[a \supset b(f_1, \cdots, f_k)] = 0$, then

$$\pi\{\pi_{\mathfrak{p}}, i[a, \lambda_{f_1, \ldots, f_k} b(f_1, \cdots, f_k)], \lambda_{f_1, \ldots, f_k} 0\} = 0$$

is an axiom.

REMARK. The rules above should be compared with Gödel's axioms and rules [3]. However, as noted already in [16], it is not particularly reasonable to apply truth functional connectives to provability statements. In fact, if $BA \rightarrow A$ and other properties of informal provability are assumed, even the consistency of the classical rules is not obvious, since it is not apparent (i) classically that the notion of provability has a well-defined extension, (ii) intuitionistically that it is decidable.

6. Introduction of Logical Constants

With every formula $A(x_1, \cdots, x_k)$ of predicate logic whose free variables are $x_1, \cdots, x_k$ and whose relation symbols are R_i we associate a term $r_A(c, a_1, \cdots, a_k)$ of Sec. 5 above with the following intended meaning: c proves that $a_1, \cdots, a_k$ satisfy $A(x_1, \cdots, x_k)$ if and only if $r_A(c, a_1, \cdots, a_k)=0$. To avoid excessive subscript we write for r_A

$$\Pi(c, a_1, \cdots, a_k; \ulcorner A(x_1, \cdots, x_k)\urcorner).$$

Also, we generally suppress free variables.

To distinguish (formal) objects of the predicate calculus from those of our system above, we use $\wedge, \vee, \twoheadrightarrow, \rightharpoondown$ for the non-truth functional connectives.

$$\Pi(c, a_1, \cdots, a_{n_i}; \ulcorner R_i(x_1, \cdots, x_{n_i})\urcorner) \quad \text{is} \quad s_i(c; a_1, \cdots, a_{n_i}).$$

Below we write c_1 for $d_1(c)$ and c_2 for $d_2(c)$.

$$\Pi(c; \ulcorner A \wedge B\urcorner) \quad \text{is} \quad \Pi(c_1; \ulcorner A\urcorner) \;\&\; \Pi(c_2; \ulcorner B\urcorner),$$

where here and below the use of truth functional connectives is permitted since Π is a two-valued term (by induction).

$$\Pi(c; \ulcorner A \vee B\urcorner) \quad \text{is} \quad \Pi(c; \ulcorner A\urcorner) \cup \Pi(c; \ulcorner B\urcorner),$$
$$\Pi(c; \ulcorner A \twoheadrightarrow B\urcorner) \quad \text{is} \quad \pi\{c_1, \lambda_f \Pi(f; \ulcorner A\urcorner) \supset \Pi[c_2(f); \ulcorner B\urcorner], \lambda_f 0\},$$
$$\Pi(c; \ulcorner \rightharpoondown A\urcorner) \quad \text{is} \quad \pi\{c_1, \lambda_f \Pi(f, \ulcorner A\urcorner) \supset \pi[c_2(f), 1, 0], \lambda_f 0\},$$
$$\Pi(c; \ulcorner (\mathrm{Ex})A(x)\urcorner) \quad \text{is} \quad \Pi(c_1, c_2; \ulcorner A(x)\urcorner),$$
$$\Pi(c; \ulcorner (x)A(x)\urcorner) \quad \text{is} \quad \pi\{c_1, \lambda_f \Pi[c_2(f), f; \ulcorner A(x)\urcorner], \lambda_f 0\}.$$

THEOREM. *For every formally derivable A of Heyting's predicate calculus HPC we obtain a term p_A such that $\Pi(p_A; \ulcorner A\urcorner) = 0$ is formally provable in our abstract theory of constructions above.*

COROLLARY. The theorem holds also if the quantifiers in A are relativized to given species of constructions (expressed by A_R), since if $\vdash_{HPC} A$, then also $\vdash_{HPC} A_R$.

For the proof, it is best to use Kleene's system $G3$ (and not e.g., Heyting's axiomatization in [7]). Let $\vdash$ mean provable in the abstract theory of constructions. One establishes, by induction with respect to the length of derivations in $G3$, that if $\vdash_{G3} \Gamma \rightarrow A$ and Γ consists of $C_1, C_2 \cdots C_r$, and $x_1, \cdots, x_k$ is a complete list of free variables occurring in Γ and A, then there is a term

$p(g; f_1, \cdots, f_k)$ such that

$$\{\Pi(g, f_1, \cdots, f_k; \ulcorner C_1 \wedge \cdots \wedge C_r \urcorner) \supset \Pi[p(g; f_1, \cdots, f_k), f_1, \cdots, f_k; \ulcorner A \urcorner]\} = 0$$

with the free variables $g, f_1, \cdots, f_k$.

One uses the following facts about our system above.

LEMMA 1. *For every formula A (of the predicate calculus) there is a term $X_A^{(c)}$ such that*

$$\{\Pi(c; \ulcorner A \urcorner) \supset \pi[\chi_A(c), \Pi(c; \ulcorner A \urcorner), 0]\} = 0.$$

(Verifiability of the Π-relation.)

LEMMA 2. *For every formula A there is a term $\psi_A(c, d)$ such that*

$$\{\pi[c, \Pi(d; \ulcorner A \urcorner), 0] \supset \Pi[\psi_A(c, d); \ulcorner A \urcorner]\} = 0.$$

(Contractibility of the Π-relation.)

LEMMA 3. *For every formula A there is a term $\tau_A(c)$ such that*

$$\{\pi(c, 1, 0) \supset \Pi[\tau_A(c); \ulcorner A \urcorner]\} = 0.$$

One uses the following derived rules of the abstract theory of constructions:

$$\frac{n(a) = 0}{a = 1}, \qquad \frac{n(a) = 1}{a = 0}, \qquad i(1, a) = 0, \qquad \frac{a = 0,\ i(a, b) = 0}{b = 0}.$$

REMARK. It is now clear why the relation $\pi(c, a, b) = 0$ is fundamental, and not e.g., $\Pi(c; \ulcorner A \urcorner) = 0$. For, starting with the latter for unanalyzed formulae A_i, $\Pi(c; \ulcorner A \urcorner)$ for composite A is defined from $\Pi(c; \ulcorner A_i \urcorner)$ by means of $\pi(c, a, b)$, but not conversely.

Relation to informal intuitionistic observations. (i) In our interpretation of the connective $\rightarrow$, $c_2(f)$ is totally defined. In contrast, Heyting [8, p. 334] insists on the need for partially defined functions. This need is greatly reduced by explicit use of parameters over proofs, as can be illustrated from the theory of recursively enumerable sets. Let $S = \{n: (\exists x)A(n, x)\}$, $T = \{n: (\exists x)B(n, x)\}$ A and B primitive recursive. For certain S and T, there are partially, but not potentially, recursive p such that $n \in S \rightarrow p(n) \in T$, illustrating a 'need' for partial functions. But if we regard a number m for which $A(n, m)$ holds as a 'proof' of $n \in S$, we eliminate this need in the following sense: there are recursive (and often primitive recursive) $p_1(n, m)$, $p_2(n, m)$ such that $A(n, m) \rightarrow p_1(n, m) \in T$ and, further, $A(n, m) \supset B[p_1(n, m), p_2(n, m)]$. Presumably, partial functions would be needed, if $\pi(c, a, b)$ is not assumed to be decidable, but only be 'verifiable' when it holds. In this case, the meaning of $\rightarrow$ could not be explained in terms of π and our generalizations of the truth functions, but only by having a (positive) logic in the abstract theory of constructions itself ('impredicativity of implication').

(ii) The Theorem above can be reformulated so as to express more directly that A is valid, i.e., that for all species R_i there is a 'proof' of A_R. (The extension to all species R_0 as domain of individuals [13] follows then from the corollary to the Theorem.)

We have to find a term $\chi(c; s_1, \cdots, s_k)$ with the following property: if c proves that s_i *define species*, then $\Pi[\chi(c), \ulcorner A \urcorner]$, where now the s_i are treated as arbitrary constructions. The condition of 'defining a species', i.e., two-valuedness, is not assumed as an axiom for s_i, but formulated in the premise: c_1 proves the equality of $\lambda_{g,f_1,\ldots,f_{n_i}} 0$ and $\lambda_{g,f_1,\ldots,f_{n_i}}\{\pi[c_2(g, f_1, \cdots, f_{n_i}), s_i(g, f_1, \cdots, f_{n_i}), 0] \cup \pi[c_2(g, f_1, \cdots, f_{n_i}), s_i(g, f_1, \cdots, f_{n_i}), 1]$. Thus we have here a (limited) possibility of *quantifying over all species.*

7. Consistency

A finitist consistency proof for the *unstarred* axioms and rules of our theory of constructions is obtained by an interpretation in quantifier-free formal systems. The main observation needed is the following

LEMMA. *Let F be a formal system which includes primitive recursive arithmetic and has a primitive recursive proof predicate (represented in F by a formula) $P(a, b)$ (for which the defining equations can be formally proved in F). Let $\ulcorner A(0^{(n)})\urcorner$ denote the term $t(n)$ of F which is the canonical representation in F of the Gödel number of the formula obtained by replacing x in $A(x)$ by the nth numeral. Then there is a primitive recursive $p_1(a, b)$ such that:*

$$\vdash_F P[a, \ulcorner P(0^{(b)}, 0^{(c)})\urcorner] \supset P[p_1(a, b), c]. \tag{1}$$

Observe first that, as in [9, pp. 312–324], there is a $q_1(b, c)$ such that

$$\vdash_F \sim P(b, c) \supset P[q_1(b, c), \ulcorner \sim P(0^{(b)}, 0^{(c)})\urcorner].$$

So, if $\sim P(b, c)$, the proofs in F with numbers a and $q_1(b, c)$ joined together yield a contradiction, and so there is a $q_2(a, b, c)$ which is a proof in F of the formula with number c. *Define* $p_1(a, b) = b$ if $P(b, c)$, and $= q_2(a, b, c)$ if $\sim P(b, c)$ and $P[a, \ulcorner P(0^{(b)}, 0^{(c)})\urcorner]$, c being determined by a. Hence the lemma.

The function p_1 in (1) provides what is needed to verify the axiom $\{\pi[a, \pi(b, c, 0), 0] \supset \pi[\pi_1(a, b), c, 0]\} = 0$ in the interpretation below. Since (1) is formally proved in F, one does not have to go outside F.[10] The interpretation of the unstarred theory is similar to the one sketched in [16].

[10] In particular, one does not need *hierarchies* of systems [16]. In contrast, the stronger axiom $(\pi[a, \pi(b, c, 0), 0] \supset \pi(b, c, 0)) = 0$ would require $P[a, \ulcorner P(0^{(b)}, 0^{(c)})\urcorner] \supset P(b,c)$, which is not provable in (a consistent) F. Find numerals b_0 and c_0 for which $\sim P(b_0, c_0)$, when we should have $\sim P[a, \ulcorner P(0^{(b_0)}, 0^{(c_0)})\urcorner]$, i.e., Con F. Similarly, as Gödel pointed out in [3] for his provability interpretation, there we should need:

$$(Ey)P[y, \ulcorner (Ez)P(z, 0^{(c)})\urcorner] \supset (Ex)P(x, c);$$

take $c = \ulcorner 0 = 1 \urcorner$, and so $(Ey)P(y, \ulcorner \neg \text{Con } F\urcorner) \supset \neg \text{Con } F$, i.e., Con $F \supset \text{Con}(F \cup \{\text{Con } F\})$, contrary to the second incompleteness theorem. Note, however, that hierarchies of systems are needed when induction is included.

The principal difference concerns equations $\pi(a, b, 0) = 0$ when b has the form $\lambda_f b_1(f)$ or $f(b_1)$. Suppose a', $b_1'(f)$, b_1' are the interpretations of a, $b_1(f)$, b_1. We replace *free* variables g in b_1' (but not in a') by $0^{(g)}$; if b is $\lambda_f b_1(f)$, we replace $\pi(a, b, 0) = 0$ by $P\{a', \ulcorner b_1'(f, 0^{(g)}) = 0\urcorner\}$; and if b is $f(b_1)$, by $P\{a', s[\ulcorner b_1'(0^{(g)})\urcorner, f]\}$, where $s(m, n)$ denotes the Gödel number of the term obtained by replacing the free variable in the nth term by the term with Gödel number m. In short, although the abstract theory does not distinguish between objects and their names, the interpretation reintroduces such a distinction; in fact, the starred rules were excluded just to make this possible.

Independence Proofs. The interpretation sketched above is an example of a 'non-standard' model of the kind mentioned in the introduction, since we have verified that the constructions and proofs *represented* in the formal system F satisfy the (starred part of the) abstract theory. Note that for *finitist* consistency or independence proofs it is necessary to formulate a translation in the syntactic style above, while from an intuitionistic point of view it is more natural to think in terms of informal proofs *represented* by formal derivations in F.

8. Extensions

Undoubtedly the most important open problem is to give a *foundation for the introduction of species by means of* (generalized) *inductive definitions for which the corresponding principles of proof by induction can be derived.* It may be remarked that on the basis of Brouwer's analysis [1] the introduction of linguistic expressions involving variables over ips is reduced to the use of generalized inductive definitions.[11] For according to Brouwer [1], an expression $A(\alpha)$ is well-determined for ips α if and only if the corresponding decision function r_A is in the species C of constructions determined by the following inductive definition:

For an arbitrary numerical constant c and variable f over constructions of N^N, $\lambda_f c \in C$; if, for each n, $r_n \in C$, then so does $\lambda_f r_{f(0)}[\lambda n f(n+1)]$.[12]

Granted this, combinations of such $A(\alpha)$ by means of first-order predicate logic are easily seen to have decision functions which are definable inductively as in C above (with essential use of the π-function of course). General quantification over ips is still obscure.

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[11]The misgivings about ips because they are 'incomplete' are out of place on the present approach because the sense of an assertion containing *any* variables, whether for ips or not, is determined by proof conditions and not by reference to a 'complete extension'. Note in passing that, for so-called *absolutely* free choice sequences, Theorem 4 of [14, p. 377] gives such conditions explicitly for $(\alpha)A(\alpha)$ when $A(\alpha)$ is of first order.

[12]Gödel has observed that on the *classical* interpretation of these two conditions, C consists of *all* those mappings from N^N to N which are continuous when N^N has the product topology. It was this fact that suggested the interpretation of [1] adopted in the text.

REMARK. Even the case of complete induction over the natural numbers requires further investigation along at least two lines: First, by analogy to the Frege-Dedekind method, one may define the *species* of natural numbers $[Z(c; n) : c$ is a proof that n is a natural number$]$ either by (i) c_1 proves (where $\equiv$ denotes extensional equality) if a_1 proves $b(0) = 0$, and a_2 proves $\lambda_f i[b(f), b(f * 1)] \equiv \lambda_f\ 0$, then $c_2(a, b)$ proves $b(n) = 0$; or by (ii) c_1 proves (for any constructions a, a', b, b') if a_1' proves that b is a species, a_1 proves $b(a_2', 0) = 0$, a_2 proves $\lambda_{f,g} i[b(g, f), b\{b'(g, f), f * 1\}] \equiv \lambda_{f,g} 0$, then $c_{21}(a, b, a', b')$ proves $b[c_{22}(a, b, a', b'), n] = 0$. Here (i) corresponds to taking the natural numbers as the intersection of all *decidable* inductive sets, (ii) of *all* inductive sets. These definitions can be straightforwardly framed in the present theory. The justification of the *rule* of induction depends then on *definition by induction*

$$\rho(0) = a, \quad \rho(n * 1) = b[n, \rho(n)],$$

for natural numbers n; but the existence of such a construction ρ is really plausible only if the species of natural numbers is *decidable* (or: a set, in Brouwer's terminology). The second approach would require a modification of the notation of Sec. 5; one would wish to express the idea that a natural number n is obtained by repeated application (iteration) of the operation $*1$, beginning with 0, and that this sequence of operations can be followed out step by step to get a proof of $A(n)$; i.e., a p_n such that $\Pi(p_n, n; \ulcorner A(x) \urcorner) = 0$ when $\Pi(p_0, 0; \ulcorner A(x) \urcorner) = 0$ and

$$\pi\{p_1', \lambda_{gf} \Pi(g, f; \ulcorner A(x) \urcorner) \supset \Pi[p_2'(g, f), f * 1; \ulcorner A(x) \urcorner], \lambda_{gf} 0\} = 0$$

are given.

A more isolated, but interesting, open problem is to formulate on the basis of the present work the question: Are $\wedge$, $\vee$, $\rightarrow$, $\neg$ the only intuitionistic propositional connectives? More generally, *what is an intuitionistic propositional connective*? As is well known, for the classical case the corresponding question has been satisfactorily settled by the identification of propositional connectives with truth functions.

REFERENCES

[1] BROUWER, L. E. J. Über Definitionsbereiche von Funktionen. *Mathematische Annalen*, Vol. 97 (1927), pp. 60–75.

[2] GENTZEN, G. Die Widerspruchsfreiheit der reinen Zahlentheorie. *Mathematische Annalen*, Vol. 112 (1936), pp. 493–565.

[3] GÖDEL, K. Eine Interpretation des intuitionistischen Aussagenkalküls. *Ergebnisse eines mathematischen Kolloquiums*, Vol. 4 (1932), pp. 39–40.

[4] GÖDEL, K. Zur intuitionistischen Arithmetik und Zahlentheorie. *Ergebnisse eines mathematischen Kolloquiums*, Vol. 4 (1932), pp. 34–38.

[5] GÖDEL, K. Über eine bisher noch nicht benützte Erweiterung des finiten Standpunktes. *Dialectica*, Vol. 12 (1958), pp. 280–287.

[6] HEYTING, A. Die formalen Regeln der intuitionistischen Logik. *Sitzungsberichte der Preussischen Akademie der Wissenschaften, Physikalisch-Mathematische Klasse* (1930), pp. 158–169.

[7] HEYTING, A. *Intuitionism. An Introduction.* Amsterdam, North-Holland Pub. Co., 1956, viii+133 pp.

[8] HEYTING, A. Blick von der intuitionistischen Warte. *Dialectica*, Vol. 12 (1958) pp. 332–345.

[9] HILBERT, D., and BERNAYS. *Grundlagen der Mathematik.* Vol. 2, Berlin, Springer, 1939, XII+498 pp.

[10] KLEENE, S. C. *Introduction to metamathematics.* Princeton, Van Nostrand Co, 1950, vi+550 pp.

[11] KLEENE, S. C. Realizability, in *Constructivity in mathematics*, A. Heyting, ed., Amsterdam, North-Holland Pub. Co. 1959, viii+297 pp.

[12] KREISEL, G. Mathematical significance of consistency proofs. *Journal of Symbolic Logic*, Vol. 23 (1958), pp. 26–50.

[13] KREISEL, G. Elementary completeness properties of intuitionistic logic with a note on negations of prenex formulae. *Journal of Symbolic Logic*, Vol. 23 (1958) pp. 317-330. Erratum, *ibid.*, p. vi.

[14] KREISEL G. A remark on free choice sequences and the topological completeness proofs. *Journal of Symbolic Logic*, Vol. 23 (1958), pp. 369–388.

[15] KREISEL, G. Hilbert's programme. *Dialectica*, Vol. 12 (1958), pp. 346–372.

[16] KREISEL, G. Ordinal logics and the characterization of informal concepts of proof Pp. 289–299 in *Proceedings of the International Congress of Mathematicians*, August 14–21, 1958, J. A. Todd,ed., Cambridge, University Press, 1960, lxiv+573 pp.

[17] LORENZEN, P. Logical reflection and formalism. *Journal of Symbolic Logic*, Vol. 23 (1958), pp. 241–249.

[18] TARSKI, A. Der Wahrheitsbegriff in den formalisierten Sprachen. *Studia philosophica*, Vol. 1 (1936), pp. 261–405.