A Logic for Reasoning About Coherent Conditional Probability: A Modal Fuzzy Logic Approach

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Abstract. In this paper we define a logic to reason about coherent conditional probability, in the sense of de Finetti. Under this view, a conditional probability $\mu(\cdot \mid \cdot)$ is a primitive notion that applies over conditional events of the form "φ *given* ψ", where ψ is not the impossible event. Our approach exploits an idea already used by Hájek and colleagues to define a logic for (unconditional) probability in the frame of fuzzy logics. Namely, in our logic for each pair of classical propositions φ and ψ, we take the probability of the conditional event "φ *given* ψ", $\varphi|\psi$ for short, as the truth-value of the (fuzzy) modal proposition $P(\varphi \mid \psi)$, read as "$\varphi|\psi$ is probable". Based on this idea we define a fuzzy modal logic FCP(ŁΠ), built up over the many-valued logic Ł$\Pi\frac{1}{2}$ (a logic which combines the well-known Lukasiewicz and Product fuzzy logics), which is shown to be complete with respect to the class of probabilistic Kripke structures induced by coherent conditional probabilities. Finally, we show that checking coherence of a probability assessment to an arbitrary family of conditional events is tantamount to checking consistency of a suitable defined theory over the logic FCP(ŁΠ).

1 Introduction: Conditional Probability and Fuzzy Logic

Reasoning under uncertainty is a key issue in many areas of Artificial Intelligence. From a logical point of view, uncertainty basically concerns formulas that can be either true or false, but their truth-value is unknown due to incompleteness of the available information. Among the different models uncertainty, probability theory is no doubt the most relevant. One may find in the literature a number of logics to reason about probability, some of them rather early. We may cite [1,6,7,9,10,14,16,18,19,20,21,22,23,24] as some of the most relevant references. Besides, it is worth mentioning the recent book [15] by Halpern, where a deep investigation of uncertainty (not only probability) representations and uncertainty logics is presented.

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Nearly almost all the probability logics in the above references are based on classical two-valued logic (except for [10]). In this paper we develop a propositional *fuzzy* logic of (conditional) probability for which completeness results are provided. In [13] a new approach, further elaborated in [12] and in [11], was proposed to axiomatize logics of uncertainty in the framework of fuzzy logic. The basic idea consists in considering, for each classical (two-valued) proposition φ, a (fuzzy) modal proposition $P\varphi$ which reads "φ is probable" and taking as truth-degree of $P\varphi$ the probability of φ. Then one can define theories about the $P\varphi$'s over a particular fuzzy logic including, as axioms, formulas corresponding to the basic postulates of probability theory. The advantage of such an approach is that algebraic operations needed to compute with probabilities (or with any other uncertainty model) are embedded in the connectives of the many-valued logical framework, resulting in clear and elegant formalizations.

In reasoning with probability, a crucial issue concerns the notion of *conditional probability*. Traditionally, given a probability measure μ on an algebra of possible worlds W, if the agent observes that the actual world is in $A \subseteq W$, then the updated probability measure $\mu(\cdot \mid A)$, called conditional probability, is defined as $\mu(B \mid A) = \mu(B \cap A)/\mu(A)$, provided that $\mu(A) > 0$. If $\mu(A) = 0$ the conditional probability remains then undefined. This yields both philosophical and logical problems.

For instance, in [11] statements about conditional probability are handled by introducing formulas $P(\varphi \mid \psi)$ standing for $P\psi \to_\Pi P(\varphi \wedge \psi)$. Such a definition exploits the properties of Product logic implication $\to_\Pi$, whose truth function behaves like a truncated division:

$$e(\Phi \to_\Pi \Psi) = \begin{cases} 1, & \text{if } e(\Phi) \leq e(\Psi) \\ e(\Psi)/e(\Phi), & \text{otherwise.} \end{cases}$$

With such a logical modelling, whenever the probability of the conditioning event χ is 0, $P(\varphi \mid \chi)$ takes as truth-value 1. Therefore, this yields problems when dealing with zero probabilities.

To overcome such difficulties, an alternative approach (that goes back to the 30's with de Finetti, and later to the 60's with Rényi and Popper among others) proposes to consider conditional probability and conditional events as basic notions, not derived from the notion of unconditional probability. Coletti and Scozzafava's book [4] includes a rich elaboration of different issues of reasoning with *coherent* conditional probability, i.e. the conditional probability in the sense of de Finetti. We take from there the following definition (cf. [4]).

Definition 1. *Let $\mathcal{G}$ be a Boolean algebra and let $\mathcal{B} \subseteq \mathcal{G}$ be closed with respect to finite unions (additive set). Let $\mathcal{B}^0 = \mathcal{B} \setminus \{\emptyset\}$. A* conditional probability *on the set $\mathcal{G} \times \mathcal{B}^0$ of conditional events, denoted as $E|H$, is a function $\mu : \mathcal{G} \times \mathcal{B}^0 \to [0, 1]$ satisfying the following axioms:*

(i) $\mu(H \mid H) = 1$, *for all* $H \in \mathcal{B}^0$
(ii) $\mu(\cdot \mid H)$ *is a (finitely additive) probability on* $\mathcal{G}$ *for any given* $H \in \mathcal{B}^0$
(iii) $\mu(E \cap A \mid H) = \mu(E \mid H) \cdot \mu(A \mid E \cap H)$, *for all* $A \in \mathcal{G}$ *and* $E, H, E \cap H \in \mathcal{B}^0$.

In this paper we follow the above fuzzy logic approach to define a logic to reason about conditional probability in the sense of Definition 1 [1]. Thus, over the fuzzy logic $L\Pi\frac{1}{2}$ we directly introduce a modal operator P as primitive, and apply it to *conditional events* of the form $\varphi|\chi$. Unconditional probability, then, arises as non-primitive whenever the conditioning event is a (classical) tautology. The obvious reading of a statement like $P(\varphi \mid \chi)$ is "the conditional event "φ given χ" is probable". Similarly to the case mentioned above, the truth-value of $P(\varphi \mid \chi)$ will be given by a conditional probability $\mu(\varphi \mid \chi)$. It is worth mentioning a very related approach by Flaminio and Montagna [8] which deals with conditional probability in the frame of the fuzzy logic $L\Pi\frac{1}{2}$, but differs from ours in that they use non-standard probabilities.

The paper is structured as follows. After this introduction, in Section 2 we overview the basic facts about the fuzzy logic $L\Pi\frac{1}{2}$. In Section 3 we define our conditional probability logic FCP($L\Pi$) as a modal fuzzy logic over $L\Pi\frac{1}{2}$ and prove soundness and completeness results with respect to the intended probabilistic semantics. Then, in Section 4 we show how the problem of coherent conditional probability assessments can be cast as a problem of determining the logical consistency of a given theory in our logic. We end with some conclusions.

2 Logical Background: The $L\Pi\frac{1}{2}$ Logic

The language of the $L\Pi$ logic is built in the usual way from a countable set of propositional variables, three binary connectives $\rightarrow_L$ (Łukasiewicz implication), $\odot$ (Product conjunction) and $\rightarrow_\Pi$ (Product implication), and the truth constant $\bar{0}$. A truth-evaluation is a mapping e that assigns to every propositional variable a real number from the unit interval [0, 1] and extends to all formulas as follows:

$$\begin{aligned}
e(\bar{0}) &= 0, \\
e(\varphi \rightarrow_L \psi) &= \min(1 - e(\varphi) + e(\psi), 1), \\
e(\varphi \odot \psi) &= e(\varphi) \cdot e(\psi), \\
e(\varphi \rightarrow_\Pi \psi) &= \begin{cases} 1, & \text{if } e(\varphi) \le e(\psi) \\ e(\psi)/e(\varphi), & \text{otherwise} \end{cases}.
\end{aligned}$$

The truth constant $\bar{1}$ is defined as $\varphi \rightarrow_L \varphi$. In this way we have $e(\bar{1}) = 1$ for any truth-evaluation e. Moreover, many other connectives can be defined from those introduced above:

$$\begin{array}{rlrl}
\neg_L\varphi & \text{is } \varphi \rightarrow_L \bar{0}, & \neg_\Pi\varphi & \text{is } \varphi \rightarrow_\Pi \bar{0}, \\
\varphi \wedge \psi & \text{is } \varphi \& (\varphi \rightarrow_L \psi), & \varphi \vee \psi & \text{is } \neg_L(\neg_L\varphi \wedge \neg_L\psi), \\
\varphi \oplus \psi & \text{is } \neg_L\varphi \rightarrow_L \psi, & \varphi \& \psi & \text{is } \neg_L(\neg_L\varphi \oplus \neg_L\psi), \\
\varphi \ominus \psi & \text{is } \varphi \& \neg_L\psi, & \varphi \equiv \psi & \text{is } (\varphi \rightarrow_L \psi)\&(\psi \rightarrow_L \varphi), \\
\Delta\varphi & \text{is } \neg_\Pi\neg_L\varphi, & \nabla\varphi & \text{is } \neg_\Pi\neg_\Pi\varphi,
\end{array}$$

[1] Notice that somewhat similar definitions of conditional probability can be found in the literature. For instance, in [15] $\mathcal{B}^0$ is further required to be closed under supersets and $\mathcal{G} \times \mathcal{B}^0$ is called a Popper algebra. See also [4] for a discussion concerning weaker notions of conditional probability and their unpleasant consequences.

with the following interpretations:

$$e(\neg_L \varphi) = 1 - e(\varphi), \qquad e(\neg_\Pi \varphi) = \begin{cases} 1, \text{ if } e(\varphi) = 0 \\ 0, \text{ otherwise} \end{cases},$$
$$e(\varphi \wedge \psi) = \min(e(\varphi), e(\psi)), \qquad e(\varphi \vee \psi) = \max(e(\varphi), e(\psi)),$$
$$e(\varphi \oplus \psi) = \min(1, e(\varphi) + e(\psi)), \qquad e(\varphi \& \psi) = \max(0, e(\varphi) + e(\psi) - 1),$$
$$e(\varphi \ominus \psi) = \max(0, e(\varphi) - e(\psi)), \qquad e(\varphi \equiv \psi) = 1 - |e(\varphi) - e(\psi)|,$$
$$e(\Delta \varphi) = \begin{cases} 1, \text{ if } e(\varphi) = 1 \\ 0, \text{ otherwise} \end{cases}, \qquad e(\nabla \varphi) = \begin{cases} 1, \text{ if } e(\varphi) > 0 \\ 0, \text{ otherwise} \end{cases}.$$

The logic $L\Pi$ is defined Hilbert-style as the logical system whose axioms and rules are the following[2]:

(i) Axioms of Łukasiewicz Logic:
 (Ł1) $\varphi \rightarrow_L (\psi \rightarrow_L \varphi)$
 (Ł2) $(\varphi \rightarrow_L \psi) \rightarrow_L ((\psi \rightarrow_L \chi) \rightarrow_L (\varphi \rightarrow_L \chi))$
 (Ł3) $(\neg_L \varphi \rightarrow_L \neg_L \psi) \rightarrow_L (\psi \rightarrow_L \varphi)$
 (Ł4) $((\varphi \rightarrow_L \psi) \rightarrow_L \psi) \rightarrow_L ((\psi \rightarrow_L \varphi) \rightarrow_L \varphi)$

(ii) Axioms of Product Logic[3]:
 (A1) $(\varphi \rightarrow_\Pi \psi) \rightarrow_\Pi ((\psi \rightarrow_\Pi \chi) \rightarrow_\Pi (\varphi \rightarrow_\Pi \chi))$
 (A2) $(\varphi \odot \psi) \rightarrow_\Pi \varphi$
 (A3) $(\varphi \odot \psi) \rightarrow_\Pi (\psi \odot \varphi)$
 (A4) $(\varphi \odot (\varphi \rightarrow_\Pi \psi) \rightarrow_\Pi (\psi \odot (\psi \rightarrow_\Pi \varphi))$
 (A5a) $(\varphi \rightarrow_\Pi (\psi \rightarrow_\Pi \chi)) \rightarrow_\Pi ((\varphi \odot \psi) \rightarrow_\Pi \chi)$
 (A5b) $((\varphi \odot \psi) \rightarrow_\Pi \chi) \rightarrow_\Pi (\varphi \rightarrow_\Pi (\psi \rightarrow_\Pi \chi))$
 (A6) $((\varphi \rightarrow_\Pi \psi) \rightarrow_\Pi \chi) \rightarrow_\Pi (((\psi \rightarrow_\Pi \varphi) \rightarrow_\Pi \chi) \rightarrow_\Pi \chi)$
 (Π1) $\neg_\Pi \neg_\Pi \chi \rightarrow_\Pi (((\varphi \odot \chi) \rightarrow_\Pi (\psi \odot \chi)) \rightarrow_\Pi (\varphi \rightarrow_\Pi \psi))$
 (Π2) $\varphi \wedge \neg_\Pi \varphi \rightarrow_\Pi \bar{0}$

(iii) The following additional axioms relating Łukasiewicz and Product logic connectives:
 ($\neg$) $\neg_\Pi \varphi \rightarrow_L \neg_L \varphi$
 (Δ) $\Delta(\varphi \rightarrow_L \psi) \equiv \Delta(\varphi \rightarrow_\Pi \psi)$
 ($L\Pi$) $\varphi \odot (\psi \ominus \chi) \equiv (\varphi \odot \psi) \ominus (\varphi \odot \chi)$

(iv) Deduction rules of $L\Pi$ are modus ponens for $\rightarrow_L$ (modus ponens for $\rightarrow_\Pi$ is derivable), and necessitation for Δ: from φ derive $\Delta\varphi$.

The logic $L\Pi\frac{1}{2}$ is the logic obtained from $L\Pi$ by expanding the language with a propositional variable $\overline{\frac{1}{2}}$ and adding the axiom:

$$(L\Pi\tfrac{1}{2})\ \overline{\tfrac{1}{2}} \equiv \neg_L \overline{\tfrac{1}{2}}$$

Obviously, a truth-evaluation e for $L\Pi$ is easily extended to an evaluation for $L\Pi\frac{1}{2}$ by further requiring $e(\overline{\frac{1}{2}}) = \frac{1}{2}$.

[2] This definition, proposed in [3], is actually a simplified version of the original definition of $L\Pi$ given in [5].

[3] Actually Product logic axioms also include axiom A7 $[\bar{0} \rightarrow_\Pi \varphi]$ which is redundant in $L\Pi$.

From the above axiom systems, the notion of proof from a theory (a set of formulas) in both logics, denoted $\vdash_{L\Pi}$ and $\vdash_{L\Pi 1/2}$ respectively, is defined as usual. Strong completeness of both logics for finite theories with respect to the given semantics has been proved in [5]. In what follows we will restrict ourselves to the logic $L\Pi\frac{1}{2}$.

Theorem 1. *For any finite set of formulas T and any formula φ of $L\Pi\frac{1}{2}$, we have $T \vdash_{L\Pi 1/2} \varphi$ iff $e(\varphi) = 1$ for any truth-evaluation e which is a model*[4] *of T.*

As it is also shown in [5], for each rational $r \in [0, 1]$ a formula $\overline{r}$ is definable in $L\Pi\frac{1}{2}$ from the truth constant $\overline{\frac{1}{2}}$ and the connectives, so that $e(\overline{r}) = r$ for each evaluation e. Therefore, in the language of $L\Pi\frac{1}{2}$ we have a truth constant for each rational in [0, 1], and due to completeness of $L\Pi\frac{1}{2}$, the following book-keeping axioms for rational truth constants are provable:

$$\begin{array}{ll} (RL\Pi 1) & \neg_L \overline{r} \equiv \overline{1-r} \\ (RL\Pi 2) & \overline{r} \to_L \overline{s} \equiv \overline{\min(1, 1-r+s)} \\ (RL\Pi 3) & \overline{r} \odot \overline{s} \equiv \overline{r \cdot s} \\ (RL\Pi 4) & \overline{r} \to_\Pi \overline{s} \equiv \overline{r \Rightarrow_P s} \end{array}$$

where $r \Rightarrow_P s = 1$ if $r \leq s$, $r \Rightarrow_P s = s/r$ otherwise.

3 A Logic of Conditional Probability

In this section we define a fuzzy modal logic, built up over the many-valued logic $L\Pi\frac{1}{2}$, that we shall call FCP($L\Pi$) —FCP for Fuzzy Conditional Probability—, to reason about coherent conditional probability of crisp propositions.

The language of FCP($L\Pi$) is defined in two steps:

Non-modal formulas: they are built from a set V of propositional variables $\{p_1, p_2, \ldots p_n, \ldots\}$ using the classical binary connectives $\wedge$ and $\neg$. Other connectives like $\vee$, $\to$ and $\leftrightarrow$ are defined from $\wedge$ and $\neg$ in the usual way. Non-modal formulas (we will also refer to them as Boolean propositions) will be denoted by lower case Greek letters φ, ψ, etc. The set of non-modal formulas will be denoted by $\mathcal{L}$.

Modal formulas: they are built from elementary modal formulas of the form $P(\varphi \mid \chi)$, where φ and χ are non-modal formulas, using the connectives of $L\Pi$ ($\to_L$, $\odot$, $\to_\Pi$) and the truth constants $\overline{r}$, for each rational $r \in [0, 1]$. We shall denote them by upper case Greek letters Φ, Ψ, etc. Notice that we do not allow nested modalities.

Definition 2. *The axioms of the logic FCP($L\Pi$) are the following:*

(i) Axioms of Classical propositional Logic for non-modal formulas
(ii) Axioms of $L\Pi\frac{1}{2}$ for modal formulas

[4] We say that an evaluation e is a *model* of a theory T whenever $e(\psi) = 1$ for each $\psi \in T$.

(iii) Probabilistic modal axioms:

(FCP1) $P(\varphi \to \psi \mid \chi) \to_L (P(\varphi \mid \chi) \to_L P(\psi \mid \chi))$
(FCP2) $P(\neg\varphi \mid \chi) \equiv \neg_L P(\varphi \mid \chi)$
(FCP3) $P(\varphi \vee \psi \mid \chi) \equiv ((P(\varphi \mid \chi) \to_L P(\varphi \wedge \psi \mid \chi)) \to_L P(\psi \mid \chi)$
(FCP4) $P(\varphi \wedge \psi \mid \chi) \equiv P(\psi \mid \varphi \wedge \chi) \odot P(\varphi \mid \chi)$
(FCP5) $P(\chi \mid \chi)$

Deduction rules of FCP($L\Pi$) are those of $L\Pi$ (i.e. modus ponens *and* necessitation *for Δ), plus:*

(iv) necessitation *for P: from φ derive $P(\varphi \mid \chi)$*
(v) substitution of equivalents *for the conditioning event: from $\chi \leftrightarrow \chi'$, derive $P(\varphi \mid \chi) \equiv P(\varphi \mid \chi')$*

The notion of proof is defined as usual. We will denote that in FCP($L\Pi$) a formula Φ follows from a theory (set of formulas) T by $T \vdash_{FCP} \Phi$. The only remark is that the rule of necessitation for $P(\cdot \mid \chi)$ can only be applied to Boolean theorems.

The semantics for FCP($L\Pi$) is given by *conditional probability Kripke structures $K = \langle W, \mathcal{U}, e, \mu \rangle$*, where:

- W is a non-empty set of possible worlds.
- $e : V \times W \to \{0,1\}$ provides for each world a *Boolean* (two-valued) evaluation of the propositional variables, that is, $e(p,w) \in \{0,1\}$ for each propositional variable $p \in V$ and each world $w \in W$. A truth-evaluation $e(\cdot, w)$ is extended to Boolean propositions as usual. For a Boolean formula φ, we will write $[\varphi]_W = \{w \in W \mid e(\varphi, w) = 1\}$.
- $\mu : \mathcal{U} \times \mathcal{U}^0 \to [0,1]$ is a conditional probability over a Boolean algebra $\mathcal{U}$ of subsets of W[5] where $\mathcal{U}^0 = \mathcal{U} \setminus \{\emptyset\}$, and such that $([\varphi]_W, [\chi]_W)$ is μ-measurable for any non-modal φ and χ (with $[\chi]_W \neq \emptyset$).
- $e(\cdot, w)$ is extended to elementary modal formulas by defining

$$e(P(\varphi \mid \chi), w) = \mu([\varphi]_W \mid [\chi]_W)^6,$$

  and to arbitrary modal formulas according to $L\Pi\frac{1}{2}$ semantics, that is:

$$\begin{aligned} e(\overline{r}, w) &= r, \\ e(\Phi \to_L \Psi, w) &= \min(1 - e(\Phi, w) + e(\Psi, w), 1), \\ e(\Phi \odot \Psi, w) &= e(\Phi, w) \cdot e(\Psi, w), \\ e(\Phi \to_\Pi \Psi, w) &= \begin{cases} 1, & \text{if } e(\Phi, w) \leq e(\Psi, w) \\ e(\Psi, w)/e(\Phi, w), & \text{otherwise} \end{cases}. \end{aligned}$$

Notice that if Φ is a modal formula the truth-evaluations $e(\Phi, w)$ depend only on the conditional probability measure μ and not on the particular world w.

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[5] Notice that in our definition the factors of the Cartesian product are the same Boolean algebra. This is clearly a special case of what stated in Definition 1.

[6] When $[\chi]_W = \emptyset$, we define $e(P(\varphi \mid \chi), w) = 1$.

The truth-degree of a formula Φ in a conditional probability Kripke structure $K = \langle W, \mathcal{U}, e, \mu \rangle$, written $\|\Phi\|^K$, is defined as

$$\|\Phi\|^K = \inf_{w \in W} e(\Phi, w).$$

When $\|\Phi\|^K = 1$ we will say that Φ is valid in K or that K is a model for Φ, and it will be also written $K \models \Phi$. Let T be a set of formulas. Then we say that K is a model of T if $K \models \Phi$ for all $\Phi \in T$. Now let $\mathcal{M}$ be a class of conditional probability Kripke structures. Then we define the truth-degree $\|\Phi\|_T^{\mathcal{M}}$ of a formula in a theory T relative to the class $\mathcal{M}$ as

$$\|\Phi\|_T^{\mathcal{M}} = \inf\{\|\Phi\|^K \mid K \in \mathcal{M},\, K \text{ being a model of } T\} .$$

The notion of logical entailment relative to the class $\mathcal{M}$, written $\models_{\mathcal{M}}$, is then defined as follows:

$$T \models_{\mathcal{M}} \Phi \text{ iff } \|\Phi\|_T^{\mathcal{M}} = 1 .$$

That is, Φ logically follows from a set of formulas T if every structure of $\mathcal{M}$ which is a model of T also is a model of Φ. If $\mathcal{M}$ denotes the whole class of conditional probability Kripke structures we shall write $T \models_{FCP} \Phi$ and $\|\Phi\|_T^{FCP}$.

It is easy to check that axioms FCP1-FCP5 are valid formulas in the class of all conditional probability Kripke structures. Moreover, the inference rule of substitution of equivalents preserves truth in a model, while the necessitation rule for P preserves validity in a model. Therefore we have the following soundness result.

Lemma 1. (Soundness) *The logic FCP($L\Pi$) is sound with respect to the class of conditional probability Kripke structures.*

For any $\varphi, \psi \in \mathcal{L}$, define $\varphi \sim \psi$ iff $\vdash \varphi \leftrightarrow \psi$ in classical logic. The relation $\sim$ is an equivalence relation in the crisp language $\mathcal{L}$ and $[\varphi]$ will denote the equivalence class of φ, containing the propositions provably equivalent to φ. Obviously, the quotient set $\mathcal{L}/_{\sim}$ of classes of provably equivalent non-modal formulas in FCP($L\Pi$) forms a Boolean algebra which is isomorphic to a corresponding Boolean subalgebra $\mathbf{B}(\Omega)$ of the power set of the set Ω of Boolean interpretations of the crisp language $\mathcal{L}$[7]. For each $\varphi \in \mathcal{L}$, we shall identify the equivalence class $[\varphi]$ with the set $\{\omega \in \Omega \mid \omega(\varphi) = 1\} \in \mathbf{B}(\Omega)$ of interpretations that make φ true. We shall denote by $\mathcal{CP}(\mathcal{L})$ the set of conditional probabilities over $\mathcal{L}/_{\sim_{FCP}} \times (\mathcal{L}/_{\sim_{FCP}} \setminus [\perp])$ or equivalently on $\mathbf{B}(\Omega) \times \mathbf{B}(\Omega)^0$.

Notice that each conditional probability $\mu \in \mathcal{CP}(\mathcal{L})$ induces a conditional probability Kripke structure $\langle \Omega, \mathbf{B}(\Omega), e_\mu, \mu \rangle$ where $e_\mu(p, \omega) = \omega(p) \in \{0, 1\}$ for each $\omega \in \Omega$ and each propositional variable p. We shall denote by $\mathcal{CPS}$ the

[7] Actually, $\mathbf{B}(\Omega) = \{\{\omega \in \Omega \mid \omega(\varphi) = 1\} \mid \varphi \in \mathcal{L}\}$. Needless to say, if the language has only finitely many propositional variables then the algebra $\mathbf{B}(\Omega)$ is just the whole power set of Ω, otherwise it is a strict subalgebra.

class of Kripke structures induced by conditional probabilities $\mu \in \mathcal{CP}(\mathcal{L})$, i.e. $\mathcal{CPS} = \{\langle \Omega, \mathbf{B}(\Omega), e_\mu, \mu\rangle \mid \mu \in \mathcal{CP}(\mathcal{L})\}$. Abusing the language, we will say that a conditional probability $\mu \in \mathcal{CP}(\mathcal{L})$ is a *model* of a modal theory T whenever the induced Kripke structure $\Omega_\mu = \langle \Omega, \mathbf{B}(\Omega), e_\mu, \mu\rangle$ is a model of T. Besides, we shall often write $\mu(\varphi \mid \chi)$ actually meaning $\mu([\varphi] \mid [\chi])$.

Actually, for our purposes, we can restrict ourselves to the class of conditional probability Kripke structures $\mathcal{CPS}$. In fact, it is not difficult to prove the following lemma.

Lemma 2. *For each conditional probability Kripke structure $K = \langle W, \mathcal{U}, e, \mu\rangle$ there is a conditional probability $\mu^* : \mathbf{B}(\Omega) \times \mathbf{B}(\Omega)^0 \to [0,1]$ such that $\|P(\varphi \mid \chi)\|^K = \mu^*(\varphi \mid \chi)$ for all $\varphi, \chi \in \mathcal{L}$ such that $[\chi] \neq \emptyset$. Therefore, it also holds that $\|\Phi\|_T = \|\Phi\|_T^{\mathcal{CPS}}$ for any modal formula Φ and any modal theory T.*

As a consequence we have the following simple corollary.

Corollary 1. *For any modal theory T over FCP($L\Pi$) and non-modal formulas φ and χ (with $[\chi] \neq \emptyset$) the following conditions hold:*

(i) $T \models_{FCP} \bar{r} \to P(\varphi \mid \chi)$ iff $\mu(\varphi \mid \chi) \geq r$ for each $\mu \in \mathcal{CP}(\mathcal{L})$ model of T.
(ii) $T \models_{FCP} P(\varphi \mid \chi) \to \bar{r}$ iff $\mu(\varphi \mid \chi) \leq r$ for each $\mu \in \mathcal{CP}(\mathcal{L})$ model of T.

Now, we show that FCP($L\Pi$) is strongly complete for finite modal theories with respect to the intended probabilistic semantics.

Theorem 2. (Strong finite probabilistic completeness of FCP($L\Pi$)) *Let T be a finite modal theory over FCP($L\Pi$) and Φ a modal formula. Then $T \vdash_{FCP} \Phi$ iff $e_\mu(\Phi) = 1$ for each conditional probability model μ of T.*

Proof. The proof is an adaptation of the proof in [11], which in turn is based on [13,12] where the underlying logics considered were Łukasiewicz logic and Rational Pavelka logic rather than $L\Pi\frac{1}{2}$.

By soundness we have that $T \vdash_{FCP(L\Pi)} \Phi$ implies $T \models_{FCP(L\Pi)} \Phi$. We have to prove the converse. In order to do so, the basic idea consists in transforming modal theories over FCP($L\Pi$) into theories over $L\Pi\frac{1}{2}$.

Define a theory, called $\mathcal{F}$, as follows:

1. take as propositional variables of the theory variables of the form $f_{\varphi|\chi}$, where φ and χ are classical propositions from $\mathcal{L}$.
2. take as axioms of the theory the following ones, for each φ, ψ and χ:
   ($\mathcal{F}1$) $f_{\varphi|\chi}$, for φ being a classical tautology,
   ($\mathcal{F}2$) $f_{\varphi|\chi} \equiv f_{\varphi|\chi'}$, for any χ, χ' such that $\chi \leftrightarrow \chi'$ is a tautology,
   ($\mathcal{F}3$) $f_{\varphi\to\psi|\chi} \to_L (f_{\varphi|\chi} \to_L f_{\psi|\chi})$,
   ($\mathcal{F}4$) $f_{\neg\varphi|\chi} \equiv \neg_L f_{\varphi|\chi}$,
   ($\mathcal{F}5$) $f_{\varphi\vee\psi|\chi} \equiv [(f_{\varphi|\chi} \to_L f_{\varphi\wedge\psi|\chi}) \to_L f_{\psi|\chi}]$,
   ($\mathcal{F}6$) $f_{\varphi\wedge\psi|\chi} \equiv f_{\psi|\varphi\wedge\chi} \odot f_{\varphi|\chi}$,
   ($\mathcal{F}7$) $f_{\varphi|\varphi}$.

Then define a mapping $*$ from modal formulas to $L\Pi\frac{1}{2}$-formulas as follows:

1. $(P(\varphi \mid \chi))^* = f_{\varphi|\chi}$
2. $\overline{r}^* = \overline{r}$
3. $(\Phi \circ \Psi)^* = \Phi^* \circ \Psi^*$, for $\circ \in \{\rightarrow_L, \odot, \rightarrow_\Pi\}$

Let us denote by T^* the set of all formulas translated from T. First, by the construction of $\mathcal{F}$, one can easily check that for any Φ,

$$T \vdash_{FCP(L\Pi)} \Phi \text{ iff } T^* \cup \mathcal{F} \vdash_{L\Pi\frac{1}{2}} \Phi^*. \tag{1}$$

Notice that the use in a proof from $T^* \cup \mathcal{F}$ of instances of ($\mathcal{F}1$) and ($\mathcal{F}2$) corresponds to the use of the inference rules of necessitation for P and substitution of equivalents in $FCP(L\Pi)$, while instances of ($\mathcal{F}3$) – ($\mathcal{F}7$) obviously correspond to axioms (FCP1) - (FCP5) respectively.

Now, we prove that the semantical analogue of (1) also holds, that is,

$$T \models_{FCP(L\Pi)} \Phi \text{ iff } T^* \cup \mathcal{F} \models_{L\Pi\frac{1}{2}} \Phi^*. \tag{2}$$

First, we show that each $L\Pi\frac{1}{2}$-evaluation e which is model of $T^* \cup \mathcal{F}$ determines a conditional probabilistic Kripke model K_e of T such that $e(\Phi^*) = \|\Phi\|_e^K$ for any modal formula Φ. Actually, we can define the conditional probability μ_e on $\mathbf{B}(\Omega) \times \mathbf{B}(\Omega)^0$ as follows:

$$\mu_e([\varphi] \mid [\chi]) = e(f_{\varphi|\chi}).$$

So defined μ_e is indeed a conditional probability, but this is clear since by hypothesis e is a model of $\mathcal{F}$. Then, it is also clear that in the model $K_e = \Omega_{\mu_e}$ the truth-degree of modal formulas Φ coincides with the truth-evaluations $e(\Phi^*)$ since they only depend on the values of μ_e and e over the elementary modal formulas $P(\varphi \mid \chi)$ and the atoms $f_{\varphi|\chi}$ respectively.

Conversely, we have now to prove that each conditional probability Kripke structure $K = (W, \mathcal{U}, e, \mu)$ determines a $L\Pi\frac{1}{2}$-evaluation e_K model of $\mathcal{F}$ such that $e_K(\Phi^*) = \|\Phi\|^K$ for any modal formula Φ. Then, we only need to set

$$e_K(f_{\varphi|\chi}) = \begin{cases} \mu([\varphi]_W \mid [\chi]_W), & \text{if } [\chi]_W \neq \emptyset \\ 1, & \text{if } [\chi]_W = \emptyset \end{cases}.$$

It is easy to see then that e_K is a model of axioms $\mathcal{F}1 - \mathcal{F}7$, and moreover that for any modal formula Φ, we have $e_K(\Phi^*) = \|\Phi\|^K$. Hence we have proved the equivalence (2).

From (1) and (2), to prove the theorem it remains to show that

$$T^* \cup \mathcal{F} \vdash_{L\Pi\frac{1}{2}} \Phi^* \text{ iff } T^* \cup \mathcal{F} \models_{L\Pi\frac{1}{2}} \Phi^*.$$

Note that $L\Pi\frac{1}{2}$ is strongly complete but only for finite theories. We have that the initial modal theory T is finite, so is T^*. However $\mathcal{F}$ contains infinitely many instances of axioms $\mathcal{F}1 - \mathcal{F}7$. Nonetheless one can prove that such infinitely many instances can be replaced by only finitely many instances, by using propositional normal forms, again following the lines of [12, 8.4.12].

Take n propositional variables $p_1, \ldots, p_n$ containing at least all variables in T. For any formula φ built from these propositional variables, take the corresponding disjunctive normal form $(\varphi)_{dnf}$. Notice that there are 2^n different normal forms. Then, when translating a modal formula Φ into Φ^*, we replace each atom $f_{\varphi|\chi}$ by $f_{(\varphi)_{dnf}|(\chi)_{dnf}}$ to obtain its normal translation Φ^*_{dnf}. The theory T^*_{dnf} is the (finite) set of all Ψ^*_{dnf}, where $\Psi \in T$. The theory $\mathcal{F}_{dnf}$ is the *finite* set of instances of axioms $\mathcal{F}1 - \mathcal{F}7$ for disjunctive normal forms of Boolean formulas built from the propositional variables $p_1, \ldots, p_n$. We can now prove the following lemma.

Lemma 3. *(i)* $T^* \cup \mathcal{F} \vdash_{L\Pi\frac{1}{2}} \Phi^*$ *iff* $T^*_{dnf} \cup \mathcal{F}_{dnf} \vdash_{L\Pi\frac{1}{2}} \Phi^*_{dnf}$.
(ii) $T^* \cup \mathcal{F} \models_{L\Pi\frac{1}{2}} \Phi^*$ *iff* $T^*_{dnf} \cup \mathcal{F}_{dnf} \models_{L\Pi\frac{1}{2}} \Phi^*_{dnf}$.

The proof of is similar to [12, 8.4.13]. Finally, we obtain the following chain of equivalences:

$$\begin{array}{lll} T \vdash_{FCP} \Phi & \text{iff } T^* \cup \mathcal{F} \vdash_{L\Pi} \Phi^* & \text{by (i) above} \\ & \text{iff } T^*_{dnf} \cup \mathcal{F}_{dnf} \vdash_{L\Pi\frac{1}{2}} \Phi^*_{dnf} & \text{by (1) of Lemma 3} \\ & \text{iff } T^*_{dnf} \cup \mathcal{F}_{dnf} \models_{L\Pi\frac{1}{2}} \Phi^*_{dnf} & \text{by finite strong completeness of } L\Pi\frac{1}{2} \\ & \text{iff } T^* \cup \mathcal{F} \models_{L\Pi\frac{1}{2}} \Phi^* & \text{by (ii) of Lemma 3} \\ & \text{iff } T \models_{FCP} \Phi & \text{by (2) above} \end{array}$$

This completes the proof of theorem.

The following direct corollary exemplifies some kinds of deductions that are usually of interest.

Corollary 2. *Let T be a finite modal theory over FCP($L\Pi$) and let φ and χ be non-modal formulas, with $[\chi] \neq \emptyset$. Then:*

(i) $T \vdash_{FCP} \bar{r} \to P(\varphi \mid \chi)$ *iff* $\mu(\varphi \mid \chi) \geq r$, *for each conditional probability model μ of T.*
(ii) $T \vdash_{FCP} P(\varphi \mid \chi) \to \bar{r}$ *iff* $\mu(\varphi \mid \chi) \leq r$, *for each conditional probability model μ of T.*

It is worth pointing out that the logic FCP($L\Pi$) is actually very powerful from a knowledge representation point of view. Indeed, it allows to express several kinds of statements about conditional probability, such as purely comparative statements like "the conditional event $\varphi|\chi$ is at least as probable as the conditional event $\psi|\delta$" as

$$P(\psi \mid \delta) \to_L P(\varphi \mid \chi),$$

or numerical probability statements like

- "the probability of $\varphi|\chi$ is 0.8" as $P(\varphi \mid \chi) \equiv \overline{0.8}$,
- "the probability of $\varphi|\chi$ is at least 0.8" as $\overline{0.8} \to_L P(\varphi \mid \chi)$,
- "the probability of $\varphi|\chi$ is at most 0.8" as $P(\varphi \mid \chi) \to_L \overline{0.8}$,
- "$\varphi|\chi$ has positive probability" as $\neg_\Pi \neg_\Pi P(\varphi \mid \chi)$,

or even statements about *independence*, like "φ and ψ are independent given χ" as

$$P(\varphi \mid \chi \wedge \psi) \equiv P(\varphi \mid \chi).$$

4 Applications to the Coherence Problem

Another well-known solution to overcome the difficulties concerning conditional probability when dealing with zero probabilities consists in using non-standard probabilities. In this approach only the impossible event can take on probability 0, but non-impossible events can have an infinitesimal probability. Then the non-standard conditional probability $Pr^{\star}(\varphi \mid \psi)$ may be expressed as $Pr^{\star}(\varphi \wedge \psi)/Pr^{\star}(\psi)$, which can be taken then as the truth-value of the formula

$$P(\psi) \rightarrow_{\Pi} P(\varphi \wedge \psi),$$

where P is a (unary) modal operator standing for (unconditional) non-standard probability. This is the previously mentioned approach[8] followed by Flaminio and Montagna in [8], where the authors develop the logic $FP(S\text{Ł}\Pi)$ in which conditional probability can be treated along with both standard and non-standard probability. Standard probability Pr is recovered by taking the *standard part* of Pr^{*}. This is modelled in the logic by means of a unary connective S, so that the truth-value of $S(P\varphi)$ is the standard probability of φ. Furthermore, they show that the notion of coherence of a probabilistic assessment to a set of conditional events is tantamount to the consistency of a suitable defined theory over $FP(S\text{Ł}\Pi)$.

Definition 3 ([4]). *A probabilistic assessment* $\{Pr(\varphi_i \mid \chi_i) = \alpha_i\}_{i=1,n}$ *over a set of conditional events* $\varphi_i \mid \chi_i$ *(with* χ_i *not being a contradiction) is* coherent *if there is a conditional probability* μ*, in the sense of Definition 1, such that* $Pr(\varphi_i \mid \chi_i) = \mu(\varphi_i \mid \chi_i)$ *for all* $i = 1, \ldots, n$.

Remark that the above notion of *coherence* can be alternatively found in the literature in a different form, like in [2], in terms of a betting scheme.

Theorem 3 ([8]). *Let* $\kappa = \{Pr(\varphi_i \mid \chi_i) = \alpha_i : i = 1, \ldots, n\}$ *be a rational probabilistic assignment. Let* $\mathcal{B}$ *the Boolean algebra generated by* $\{\varphi_i, \chi_i \mid i = 1, \ldots, n\}$ *and let* Ω *and* $\emptyset$ *be its top element and its bottom element respectively. Then* κ *is coherent iff the theory* T^{*}_{κ} *consisting of the axioms of the form* $\neg_{\Pi}\neg_{\Pi} Pr(\psi)$ *for* $\psi \in \mathcal{B}\backslash\{\emptyset\}$*, plus the axioms* $S(P(\chi_i) \rightarrow_{\Pi} P(\varphi_i \wedge \chi_i)) \equiv \overline{\alpha_i}$ $(i = 1, \ldots, n)$ *is consistent in* $FP(SL\Pi)$*, i.e.* $T^{*}_{\kappa} \not\vdash_{FP(SL\Pi)} \overline{0}$.

The proof of this theorem is based on two characterizations of coherence, given in [4] and [17], using non-standard probabilities, and it is quite complicated. However in $FCP(L\Pi)$, contrary to $FP(S\text{Ł}\Pi)$, conditional probability is a primitive notion, then it can be easily shown that in the logic $FCP(L\Pi)$ an analogous theorem can be proved in a simpler way.

Theorem 4. *Let* $\kappa = \{Pr(\varphi_i \mid \chi_i) = \alpha_i : i = 1, \ldots, n\}$ *be a rational probabilistic assessment. Then* κ *is coherent iff the theory* $T_{\kappa} = \{P(\varphi_i \mid \chi_i) \equiv \overline{\alpha}_i : i = 1, \ldots, n\}$ *is consistent in* $FCP(L\Pi)$*, i.e.* $T_{\kappa} \not\vdash_{FCP(L\Pi)} \overline{0}$.

[8] A related approach due to Rašković et al. [21] deals with conditional probability by defining graded (two-valued) operators over the unit interval of a recursive non-archimedean field containing all rationals.

Proof. Remember that we are allowed to restrict ourselves to the subclass $\mathcal{CPS}$ of conditional probability structures. Now, suppose that T_κ is consistent. By strong completeness, there exists a model $\langle \Omega, \mathbf{B}(\Omega), e_\mu, \mu \rangle$ of T_κ, hence satisfying $\mu(\varphi_i \mid \chi_i) = \alpha_i$: therefore κ is coherent. Conversely, suppose κ is a coherent assessment. Then, there is a conditional probability μ which extends κ. Then the induced Kripke structure $\langle \Omega, \mathbf{B}(\Omega), e_\mu, \mu \rangle$ is a model of T_κ.

5 Conclusions

In this paper, we have been concerned with defining the modal logic $FCP(L\Pi)$ to reason about coherent conditional probability exploiting a previous fuzzy logic approach which deals with unconditional probabilities [11]. Conditional probability has been taken as a primitive notion, in order to overcome difficulties related to conditioning events with zero probabilities. $FCP(L\Pi)$ has been shown to be strongly complete with respect to the class of conditional probability Kripke structures when dealing with finite theories. Furthermore, we have proved that testing consistency of a suitably defined modal theory over $FCP(L\Pi)$ is tantamount to testing the coherence of an assessment to an arbitrary set of conditional events, as defined in [4].

To conclude, we would like to point out some possible directions of our future work. First, it will be interesting to study whether we could use a logic weaker than $L\Pi\frac{1}{2}$, since in fact we do not need in the probabilistic modal axioms to explicitly deal with the Product implication connective $\rightarrow_\Pi$. Thus, it seems it would be enough to use a logic including only the connectives $\rightarrow_L$ and $\odot$. Second, it will be worth studying theories also including non-modal formulas over the framework defined. Indeed, this would allow us to treat deduction for Boolean propositions as well as a logical representation of relationships between events, like, for instance when two events are incompatible or one follows from another. Clearly such an extension would enhance the expressive power of $FCP(L\Pi)$. Then, from a semantical point of view, we would be very close to the so-called *model-theoretic probabilistic logic* in the sense of Biazzo et al's approach [2] and the links established there to probabilistic reasoning under coherence and default reasoning (see also [20] for a another recent probability logic approach to model defaults). Actually, $FCP(L\Pi)$ can provide a (syntactical) deductive system for such a rich framework. Exploring all these connections will be an extremely interesting matter of research in the immediate future.

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