1)

Total flips = 10

out of 10 flips we observe 4 heads

3 head in first 5 flips 1 head in last 5 flips

What is the new posterior for p?

$$pdf(p \mid 4H) = \frac{pr(4H \mid p) \cdot pdf(p)}{pr(4H)}$$

$pdf(p) = 1$ = prior = before any observation

$$pr(4H \mid p) = \binom{10}{4} p^4 (1-p)^6 = 210\, p^4 (1-p)^6$$

$$pr(4H) = \int_0^1 pr(4H \mid p) \cdot pdf(p)\, dp$$

$$= \int_0^1 210\, p^4 (1-p)^6 \, dp = \text{constant}$$

$$pdf(p \mid 4H) = \frac{210\, p^4 (1-p)^6}{\text{constant}} = c\, p^4 (1-p)^6$$

$$\text{argmax}\, pdf(p \mid 4H) = \frac{d}{dp}\left(p^4 (1-p)^6\right)$$

$$= 4p^3(1-p)^6 - 6p^4(1-p)^5 = 0$$

$$= 4(1-p) - 6p = 0$$

$$4 - 4p = 6p$$

$$4 = 10p$$

$$\boxed{p = 0.4}$$

This makes sense! We got 4/10 heads which would make $p=0.4$ the best guess for the probability of getting heads.

2) $\text{pdf}(p \mid 6H)$ when 10 flips are done

$$\text{pdf}(p \mid 6H) = \frac{\Pr(6H \mid p)\ \text{pdf}(p)}{\Pr(6H)}$$

$$\text{pdf}(p) = 1$$

$$\Pr(6H \mid p) = \binom{10}{6} p^6 (1-p)^4 = 210\, p^6 (1-p)^4$$

$$\Pr(6H) = \int_0^1 210\, p^6 (1-p)^4 \, dp = \text{constant}$$

$$\text{pdf}(p \mid 6H) = c\, p^6 (1-p)^4$$

$$\arg\max \text{pdf} = \frac{d}{dp}\left(p^6(1-p)^4\right)$$

$$= 6p^5(1-p)^4 - 4p^6(1-p)^3 = 0$$

$$6(1-p) - 4p = 0$$

$$6 - 6p = 4p$$

$$6 = 10p$$

$$\boxed{p = 0.6}$$

This makes sense! We got 6/10 heads which would make $p = 0.6$ the best guess for the probability of getting heads.

3) $E(x) = \int x\, f(x) = \iint x\, f(x,y)$

$$f(x,y) = 1$$

$$E(x) = \int_0^1 x\, dx \int_0^2 1\, dy = \left(\frac{x^2}{2}\Big|_0^1\right)\left(y\Big|_0^2\right)$$

$$= \left(\frac{1}{2}\right)(2) = \underbrace{1}$$

This is wrong though, $E(x)$ should be equal to 0.5 but I can't figure out what I am doing wrong in my integration.

$$\text{PDF of } X = \int f(x,y)\,dy = \int_0^2 1\,dy = \underbrace{2}_{\text{PDF of } X}$$

is constant therefore uniform distribution

Even though I got $E(X) = 1$, it should be $E(X) = 1/2$ if it is a uniform distribution between 0 and 1.

$$E(y) = \iint y\, f(x,y)\, dy\, dx = \int_0^1 \int_0^2 y\, dy\, dx$$

$$= (1)\cdot\left(\frac{y^2}{2}\Big|_0^2\right) = 1$$

$E(y) = 1$ } This result makes intuitive sense due to the symmetric form of y and how the majority of numbers are distributed around $y = 1$

$$\text{COV}(x,y) = E(xy) - E(x)\,E(y)$$

$$E(xy) = \int_0^1 \int_0^2 xy\, dy\, dx = \frac{1}{2} \cdot 2 = 1$$

$$\text{COV}(x,y) = 1 - \left(\tfrac{1}{2}\right)(1) = 1/2$$

Normalized version of covariance:

$$\rho_{x,y} = \frac{\text{cov}(x,y)}{\sqrt{\text{Var}(x)\ \text{Var}(y)}}$$

$$\text{Var}(x) = E(x^2) - E^2(x)$$

$$E(x^2) = \int_0^1 \int_0^2 x^2 \, dy\, dx = \left(\frac{x^3}{3}\Big|_0^1\right)(2) = 2/3$$

$$E^2(x) = 1/4$$

$$\text{Var}(x) = 2/3 - 1/4 = 5/12$$

$$\text{Var}(y) = E(y^2) - E^2(y)$$

$$E(y^2) = \int_0^1 \int_0^2 y^2 \, dy\, dx = \frac{y^3}{3}\Big|_0^2 = 8/3$$

$$E^2(y) = 1$$

$$\text{Var}(y) = 8/3 - 3/3 = 5/3$$

$$\rho_{x,y} = \frac{0.5}{\sqrt{5/12 \cdot 5/3}} = 0.6$$

Normalized version of covariance = 0.6