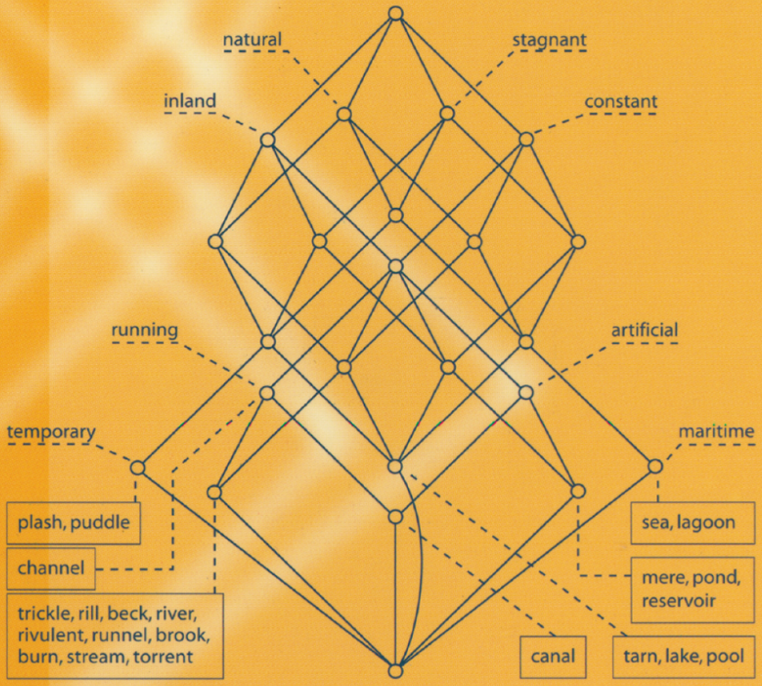
Bernhard Ganter
Rudolf Wille

Formal Concept Analysis

Mathematical Foundations



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Garrett Birkhoff
with his application-oriented view of lattice theory[1] and
Hartmut von Hentig
with his critical yet constructive understanding of science[2]
have had a decisive influence on the genesis of Formal Concept Analysis.

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[1] G. Birkhoff: *Lattice Theory.* Amer. Math. Soc., Providence. 1st edition 1940, 2nd (revised) edition 1948, 3rd (new) edition 1967.

[2] H. von Hentig: *Magier oder Magister? Über die Einheit der Wissenschaft im Verständigungsprozeß.* Klett, Stuttgart 1972.

Preface

Formal Concept Analysis is a field of applied mathematics based on the mathematization of *concept* and *conceptual hierarchy*. It thereby activates mathematical thinking for conceptual data analysis and knowledge processing.

The underlying notion of "concept" evolved early in the philosophical theory of concepts and still has effects today. For example, it has left its mark in the German standards DIN 2330 and DIN 2331. In mathematics it played a special role during the emergence of mathematical logic in the 19th century. Subsequently, however, it had virtually no impact on mathematical thinking. It was not until 1979 that the topic was revisited and treated more thoroughly. Since then, through a large number of contributions, Formal Concept Analysis has obtained such breadth that a systematic presentation is urgently needed, but can no longer be realized in one volume.

Therefore, the present book focuses on the mathematical foundations of Formal Concept Analysis, which can be regarded chiefly as a branch of applied lattice theory. A series of examples serves to demonstrate the utility of the mathematical definitions and results; in particular, to show how Formal Concept Analysis can be used for the conceptual unfolding of data contexts. These examples do not play the role of case studies in data analysis. A separate volume is intended for a comprehensive treatment of methods of conceptual data and knowledge processing. The general foundations of Formal Concept Analysis will also be treated separately.

It is perfectly possible to use Formal Concept Analysis when examining human conceptual thinking. However, this would be an application of the mathematical method and a matter for the experts in the respective science, for example psychology. The adjective "formal" in the name of the theory has a delimiting effect: we are dealing with a mathematical field of work, that derives its comprehensibility and meaning from its connection with well-established notions of "concept", but which does not strive to explain conceptual thinking in turn.

The mathematical foundations of Formal Concept Analysis are treated in seven chapters. By way of introduction, elements of mathematical order and lattice theory which will be used in the following chapters have been compiled in a *chapter "zero"*. However, all difficult notation and results from this chapter will be introduced anew later on. A reader who knows what is understood by a lattice in mathematics may skip this chapter.

The *first chapter* describes the basic step in the formalization: An elementary form of the representation of data (the "cross table") is defined mathematically ("formal context"). A formal concept of such a data context is then explained. The totality of all such concepts of a context in their hierarchy can be interpreted as a mathematical structure ("concept lattice"). It is also possible to allow more complex data types ("many-valued contexts"). These are then reduced to the basic type by a method of interpretation called "conceptual scaling".

The *second chapter* examines the question of how all concepts of a data context can be determined and represented in an easily readable diagram. In addition, implications and dependencies between attributes are dealt with. The *third chapter* supplies the basic notions of a structure theory for concept lattices, namely part- and factor structures as well as tolerance relations. In each case the extent to which these can be elaborated directly within the contexts is studied.

These mathematical tools are then used in the *fourth* and *fifth chapter*, in order to describe more complex concept lattices by means of decomposition and construction methods. Thus, the concept lattice can be split up into (possibly overlapping) parts, but it is also possible to use the direct product of lattices or of contexts as a decomposition principle. A further approach is that of substitution. In accordance with the same principles, it is possible to construct contexts and concept lattices. As an additional construction principle, we shall describe a method of doubling parts of a concept lattice.

The structural properties examined in mathematical lattice theory, for example the distributive law and its generalizations or notions of dimension, play a role in Formal Concept Analysis as well. This shall be treated in the *sixth chapter*. The *seventh chapter* finally deals with structure-comparing maps, examining various kinds of morphisms. Particular attention is given to the scale measures, occuring in the context of conceptual scaling.

We limit ourselves to a concise presentation of ideas for reasons of space. Therefore, we endeavour to give a complete reference to further results and the respective literature at the end of each chapter. However, we have only taken into account such contributions closely connected with the topic of the book, i.e., with the mathematical foundations of Formal Concept Analysis. The index contains all technical terms defined in this book, and in addition some particularly important keywords. The bibliography also serves as an author index.

The genesis of this book has been aided by the numerous lectures and activities of the "Forschungsgruppe Begriffsanalyse" (Research Group on Concept Analysis) at Darmstadt University of Technology. It is difficult to state in detail which kind of support was due to whom. Therefore, we can here only express our gratitude to all those who contributed to the work presented in this book.

Two years after the German edition, this English translation has been finished. In its content there are only a few minor changes. Although there is ongoing active work in the field, the mathematical foundations of Formal Concept Analysis have been stable over the last years.

The authors are extremely grateful to Cornelia Franzke for her precise and cooperative work when translating the book. They would also like to thank K.A. Baker, P. Eklund and R.J. Cole, M.F. Janowitz, and D. Petroff for their careful proofreading.

Contents

0. Order-theoretic Foundations

Formal Concept Analysis is based on mathematical order theory, in particular on the theory of complete lattices. The reader is not required to be familiar with these areas. The mathematical foundations are surveyed in this chapter. However, we limit ourselves to the most important facts, as there is no room for a comprehensive introduction to order theory. For this purpose, we refer to the bibliography listed at the end of this chapter. In general, the reader is supposed to have experience with mathematical texts: we use the technical language of mathematics, in particular of set theory, without further explanation.

In the first section we will introduce ordered sets, in the second complete lattices. These two sections constitute the basis for the following chapters. On the other hand, the third section, dealing with closure systems, and the fourth on Galois connections may be skipped at a first reading. Much of what they contain will be introduced again later under a different name. The second half of this chapter shows how the basic notions of Formal Concept Analysis have their roots in order and lattice theory. In this connection, we follow, in most aspects, the "classical" representation by **Garrett Birkhoff.**

0.1 Ordered Sets

Definition 1. A **binary relation** R between two sets M and N is a set of pairs (m, n) with $m \in M$ and $n \in N$, i.e., a subset of the set $M \times N$ of all such pairs. Instead of $(m, n) \in R$ we often write mRn. If $N = M$, we speak of a **binary relation on the set** M. R^{-1} denotes the **inverse relation** to R, that is the relation between N and M with $nR^{-1}m :\Leftrightarrow mRn$. $\diamond$

Definition 2. A binary relation R on a set M is called an **order relation** (or shortly an **order**), if it satisfies the following conditions for all elements $x, y, z \in M$:

1. xRx (reflexivity)
2. xRy and $x \neq y \Rightarrow$ not yRx (antisymmetry)
3. xRy and $yRz \Rightarrow xRz$ (transitivity)

For R we often use the symbol $\leq$ (for R^{-1} the symbol $\geq$), and we write $x < y$ for $x \leq y$ and $x \neq y$. As usual, we read $x \leq y$ as "x is less than or equal to y", etc. An **ordered set** is a pair $(M, \leq)$, with M being a set and $\leq$ an order relation on M.[1] ◇

Examples of ordered sets are: The real numbers $\mathbb{R}$ with the usual $\leq$-relation, but also the space $\mathbb{R}^n$ with

$$(x_1, x_2, \ldots, x_n) \leq (y_1, y_2, \ldots, y_n) :\Longleftrightarrow x_i \leq y_i \text{ for } i = 1, 2, \ldots n;$$

the natural numbers $\mathbb{N}$ with the divisibility relation $|$; the *power-set* $\mathfrak{P}(X)$ of all subsets of any set X with set inclusion. Even the equality relation $=$ is a (trivial) example of an order. Many further examples will be discussed in the following.

Definition 3. a is called a **lower neighbour** of b, if $a < b$ and there is no element of c fulfilling $a < c < b$. In this case, b is an **upper neighbour** of a, and we write $a \prec b$. ◇

Every finite ordered set $(M, \leq)$ can be represented by a **line diagram** (also called a *Hasse diagram* by many authors). The elements of M are depicted by small circles in the plane. If $x, y \in M$ with $x \prec y$, the circle corresponding to y is depicted above the circle corresponding to x (permitting sideways shifts), and the two circles are joined by a line segment. From such a diagram we can read off the order relation as follows: $x < y$ if and only if the circle representing y can be reached by an ascending path from the circle representing x. Figure 0.1 presents line diagrams for all ordered sets with up to four elements.

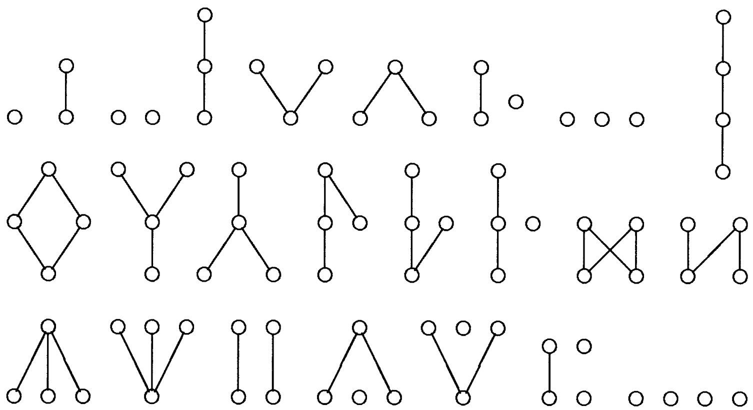
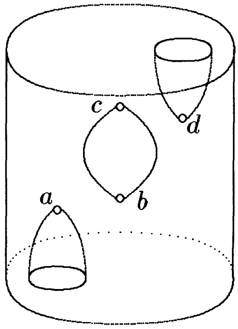


Figure 0.1 Line diagrams of all ordered sets with up to four elements.

[1] Instead of ordered sets, some authors equivalently speak of *partially ordered sets*.

Definition 4. Two elements x, y of an ordered set $(M, \leq)$ are called **comparable** if $x \leq y$ or $y \leq x$, otherwise they are **incomparable.** A subset of $(M, \leq)$ in which any two elements are comparable is called a **chain**; a subset in which any two elements are incomparable is called an **antichain.** The **width** of a finite ordered set $(M, \leq)$ is defined to be the maximal size of an antichain in $(M, \leq)$, for an arbitrary ordered set $(M, \leq)$ it is defined to be the supremum of the sizes of antichains in $(M, \leq)$. Similarly, the **length** is defined to be the supremum of the sizes of chains in $(M, \leq)$, minus one. $\diamond$

Definition 5. If $(M, \leq)$ is an ordered set and a, b, c, d are elements of M with $b \leq c$, we define the **interval**



$$[b, c] := \{x \in M \mid b \leq x \leq c\}.$$

The set

$$(a] := \{x \in M \mid x \leq a\}$$

is called a **principal ideal** and

$$[d) := \{x \in M \mid x \geq d\}$$

is called a **principal filter.**

Thus, $a \prec b$ is equivalent to $a < b$ and $[a, b] = \{a, b\}$. $\diamond$

Definition 6. A map $\varphi : M \longrightarrow N$ between two ordered sets $(M, \leq)$ and $(N, \leq)$ is called **order-preserving,** if

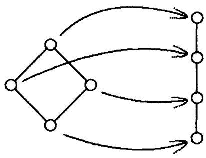
$$x \leq y \Rightarrow \varphi x \leq \varphi y$$

for all $x, y \in M$. If φ furthermore fulfills the converse implication

$$x \leq y \Leftarrow \varphi x \leq \varphi y,$$

φ is called an **order-embedding.** In this case, φ is necessarily injective. A bijective order-embedding is called **(order-) isomorphism.** $\diamond$

Not every bijective order-preserving map is an order-isomorphism, as the example shows. In order to prove that a certain order-preserving map φ is an isomorphism, it is usually shown that the inverse map φ^{-1} exists and is also order-preserving.



Bijective, order preserving, but not an isomorphism.

Definition 7. The (direct) **product** of two ordered sets $(M_1, \leq)$ and $(M_2, \leq)$ is defined to be the ordered set $(M_1 \times M_2, \leq)$ with

$$(x_1, x_2) \leq (y_1, y_2) :\iff x_1 \leq y_1 \text{ and } x_2 \leq y_2.$$

The definition of the product can be extended to any number of factors: If T is an index set and $(M_t, \leq)$, $t \in T$ are ordered sets, then

$$\bigtimes_{t\in T} (M_t, \leq) := (\bigtimes_{t\in T} M_t, \leq) \text{ with}$$

$$(x_t)_{t\in T} \leq (y_t)_{t\in T} :\iff x_t \leq y_t \text{ for all } t \in T.$$

$\diamond$

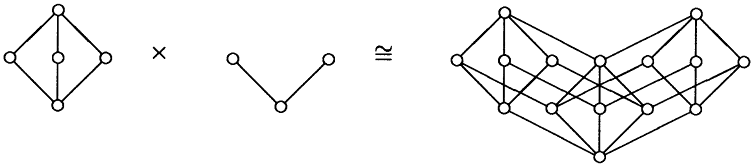


Figure 0.2 An example of a product of two ordered sets.

Definition 8. In order to be able to define the **cardinal sum** or **disjoint union** of two ordered sets, we first introduce the notation

$$\dot{M}_t := \{t\} \times M_t.$$

The sets $\dot{M}_1$ and $\dot{M}_2$ will then be disjoint copies of M_1 and M_2. We define

$$(M_1, \leq) + (M_2, \leq) := (\dot{M}_1 \cup \dot{M}_2, \leq),$$

the order relation being specified as follows:

$$(s, a) \leq (t, b) :\iff s = t \text{ and } a \leq b \text{ in } M_s.$$

This definition is also easily generalized in the case of any number of summands. $\diamond$

The Duality Principle for ordered sets. The inverse relation $\geq$ of an order relation $\leq$ is also an order relation. It is called the **dual** order of $\leq$. A line diagram of the dual ordered set $(M, \leq)^d := (M, \geq)$ can be obtained from the line diagram of $(M, \leq)$ by a horizontal reflection. If $(M, \leq) \cong (N, \leq)^d$, the two orders are called **dually isomorphic.**

We obtain the dual statement A^d of an order-theoretic statement A (which apart from purely logical components only contains the symbol $\leq$), if we replace in A the symbol $\leq$ by $\geq$. A holds for an ordered set, if and only if A^d holds for the dual ordered set. This Duality Principle is used to simplify definitions and proofs. If a theorem asserts two statements that are dual to each other, we usually prove only one of them, the other one follows "dually", i.e., with the same proof for the dual order.

Definition 9. Let $(M, \leq)$ be an ordered set and A a subset of M. A **lower bound** of A is an element s of M with $s \leq a$ for all $a \in A$. An **upper bound** of A is defined dually. If there is a largest element in the set of all lower bounds of A, it is called the **infimum** of A and is denoted by $\inf A$ or $\bigwedge A$; dually, a least upper bound is called **supremum** and denoted by $\sup A$ or $\bigvee A$. If $A = \{x, y\}$, we also write $x \wedge y$ for $\inf A$ and $x \vee y$ for $\sup A$. Infimum and supremum are frequently also called **meet** and **join**. $\diamond$

0.2 Complete Lattices

Definition 10. An ordered set $\mathbf{V} := (V, \leq)$ is a **lattice**, if for any two elements x and y in V the supremum $x \vee y$ and the infimum $x \wedge y$ always exist. $\mathbf{V}$ is called a **complete lattice**, if the supremum $\bigvee X$ and the infimum $\bigwedge X$ exist for any subset X of V. Every complete lattice $\mathbf{V}$ has a largest element, $\bigvee V$, called the **unit element** of the lattice, denoted by $\mathbf{1}_V$. Dually, the smallest element $\mathbf{0}_V$ is called the **zero element**. $\diamond$

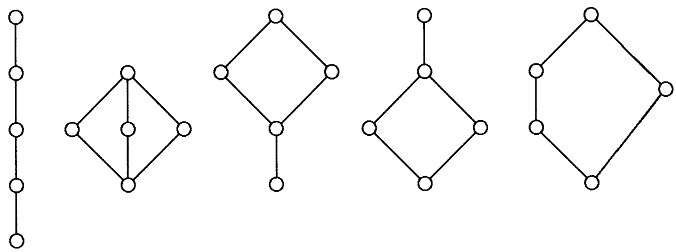


Figure 0.3 Line diagrams of the lattices with five elements.

The definition of a complete lattice presupposes that supremum and infimum exist for every subset X, in particular for $X = \emptyset$. We have $\bigwedge \emptyset = \mathbf{1}_V$ and $\bigvee \emptyset = \mathbf{0}_V$, from which it follows that $V \neq \emptyset$ for every complete lattice. Every non-empty finite lattice is a complete lattice.

We can reconstruct the order relation from the lattice operations infimum and supremum by

$$x \leq y \iff x = x \wedge y \iff x \vee y = y.$$

If T is an index set and $X := \{x_t \mid t \in T\}$ a subset of V, instead of $\bigvee X$ we also write $\bigvee_{t \in T} x_t$ and instead of $\bigwedge X$ we write $\bigwedge_{t \in T} x_t$. The operations of the supremum and infimum, respectively, are associative. The familiar particular case of the associative laws, i.e., $x \wedge (y \wedge z) = (x \wedge y) \wedge z$, respectively $x \vee (y \vee z) = (x \vee y) \vee z$, can be generalized as follows: If $\{X_t \mid t \in T\}$ is a set of subsets of V, then

$$\bigvee_{t\in T}\left(\bigvee X_t\right)=\bigvee\left(\bigcup_{t\in T}X_t\right) \text{ and dually } \bigwedge_{t\in T}\left(\bigwedge X_t\right)=\bigwedge\left(\bigcup_{t\in T}X_t\right).$$

The Duality Principle for lattices. The definitions of a lattice and a complete lattice, respectively, are self-dual: If $(V,\le)$ is a (complete) lattice, then so is $(V,\le)^d=(V,\ge)$. Therefore, the Duality Principle for ordered sets carries over to lattices: We obtain the dual statement of an order-theoretic statement, if we replace the symbols $\le$, $\vee$, $\wedge$, $\bigvee$, $\bigwedge$, 0_V, 1_V etc. by $\ge$, $\wedge$, $\vee$, $\bigwedge$, $\bigvee$, 1_V, 0_V etc.

Proposition 1. *An ordered set in which the infimum exists for every subset is a complete lattice.*

Proof. Let X be any subset of the ordered set. We have to prove that the supremum of X exists. The set S of all upper bounds of X has an infimum s (even if S is empty). Every element of X is a lower bound of S, i.e., $\le s$. Hence s itself is an upper bound of X and consequently the supremum. □

Examples of lattices. 1) For every set M the power-set $\mathfrak{P}(M)$, i.e., the set of all subsets of M, is ordered by set inclusion $\subseteq$ and $(\mathfrak{P}(M),\subseteq)$ is a complete lattice. In this case the lattice operations supremum and infimum are set union and intersection.

2) Every closed real interval $[a,b]$ in its natural order forms a complete lattice $([a,b],\le)$ with the usual infimum and supremum, respectively, as lattice operations. The ordered set $(\mathbb{R},\le)$, on the other hand, is a lattice, but it is not complete: It lacks a greatest and a least element.

We will give further examples of complete lattices from mathematics in section 0.3.

Definition 11. For an element v of a complete lattice $\boldsymbol{V}$ we define

$$\begin{aligned} v_* &:= \bigvee\{x\in V\mid x<v\} \\ \text{and}\quad v^* &:= \bigwedge\{x\in V\mid v<x\}. \end{aligned}$$

We call v **$\bigvee$-irreducible**[2], if $v\neq v_*$, i.e., if v cannot be represented as the supremum of strictly smaller elements. In this case, v_* is the unique lower neighbour of v. Dually, we call v **$\bigwedge$-irreducible**[3], if $v\neq v^*$. $J(\boldsymbol{V})$ denotes the set of all $\bigvee$-irreducible elements and $M(\boldsymbol{V})$ the set of all $\bigwedge$-irreducible elements. A set $X\subseteq V$ is called **supremum-dense** in V, if every element from V can be represented as the supremum of a subset of X and, dually, **infimum-dense**, if $v=\bigwedge\{x\in X\mid v\le x\}$ for all $v\in V$. ◇

[2] read: **supremum-irreducible**
[3] read: **infimum-irreducible**

Proposition 2. *An element v of a finite lattice is $\bigvee$-irreducible, if and only if it has exactly one lower neighbour, and $\bigwedge$-irreducible, if and only if it has exactly one upper neighbour. Every supremum-dense subset contains all $\bigvee$-irreducible elements and every infimum-dense subset contains all $\bigwedge$-irreducible elements. Conversely, in a finite lattice $\mathbf{V}$ the set $J(\mathbf{V})$ is supremum-dense and $M(\mathbf{V})$ is infimum-dense.*

Proof. v is $\bigvee$-irreducible, if and only if $v_* \neq v$. This, on the other hand, is equivalent to the fact that v_* is the largest element less than v, that is, it is in particular the only lower neighbour of v. For $\bigwedge$-irreducible elements we conclude dually. The second statement of the proposition is trivial, the third is proved inductively: Every element v which is not $\bigvee$-irreducible itself, is the supremum of strictly smaller elements. If those are suprema of $\bigvee$-irreducible elements, so is v. □

It is easy to state examples of complete lattices which contain neither $\bigvee$-irreducible nor $\bigwedge$-irreducible elements, as for instance the real interval $[0,1]$ in its natural order. The upper neighbours of the zero element are always $\bigvee$-irreducible (if they exist). They are called the **atoms** of the lattice. The **coatoms**, i.e., the lower neighbours of the unit element, are always $\bigwedge$-irreducible. A complete lattice in which every element is the supremum of atoms is called **atomistic.**

Definition 12. A subset U of a complete lattice $\mathbf{V}$ which is closed under suprema, i.e., for which holds

$$T \subseteq U \Rightarrow \bigvee T \in U,$$

is a **$\bigvee$-subsemilattice** of $\mathbf{V}$. Dually, a subset which is closed under infima is called a **$\bigwedge$-subsemilattice.** A subset which is closed under both suprema and infima is called a **complete sublattice.** ◇

Definition 13. A map $\varphi : V \longrightarrow W$ between two complete lattices $\mathbf{V}$ and $\mathbf{W}$ is called **supremum-preserving,** if[4]

$$\varphi \bigvee X = \bigvee \varphi(X)$$

holds for every subset X of V. Another name is **$\bigvee$-morphism,** and dually: **infimum-preserving map, $\bigwedge$-morphism.** If φ is both supremum-preserving and infimum-preserving, then φ is a **complete lattice homomorphism** or **complete homomorphism.** ◇

Every supremum-preserving map, in particular every complete homomorphism, is order-preserving. Conversely, every order-isomorphism between complete lattices is automatically a **lattice-isomorphism,** i.e., a bijective complete homomorphism.

---

[4] $\varphi(X)$ here stands for $\{\varphi x \mid x \in X\}$.

0.3 Closure Operators

Definition 14. A **closure system** on a set G is a set of subsets which contains G and is closed under intersections. Formally: $\mathfrak{A} \subseteq \mathfrak{P}(G)$ is a closure system if $G \in \mathfrak{A}$ and

$$\mathfrak{X} \subseteq \mathfrak{A} \Rightarrow \bigcap \mathfrak{X} \in \mathfrak{A}.$$

A **closure operator** φ on G is a map assigning a **closure** $\varphi X \subseteq G$ to each subset $X \subseteq G$ under the following conditions:

1. $X \subseteq Y \Rightarrow \varphi X \subseteq \varphi Y$ (monotony)
2. $X \subseteq \varphi X$ (extensity)
3. $\varphi\varphi X = \varphi X$ (idempotency)

◇

Closure system and closure operator are closely related, as shown by the following theorem:

Theorem 1. *If $\mathfrak{A}$ is a closure system on G then*

$$\varphi_{\mathfrak{A}} X := \bigcap \{A \in \mathfrak{A} \mid X \subseteq A\}$$

defines a closure operator on G. Conversely, the set

$$\mathfrak{A}_\varphi := \{\varphi X \mid X \subseteq G\}$$

of all closures of a closure operator φ is always a closure system, and

$$\varphi_{\mathfrak{A}_\varphi} = \varphi \quad \textit{as well as} \quad \mathfrak{A}_{\varphi_{\mathfrak{A}}} = \mathfrak{A}.$$

Proof.

– $\varphi_{\mathfrak{A}}$ is a closure operator: From $X \subseteq Y$ it follows that

$$\{A \in \mathfrak{A} \mid X \subseteq A\} \supseteq \{A \in \mathfrak{A} \mid Y \subseteq A\}.$$

Since set intersection is monotone, this implies

$$\varphi_{\mathfrak{A}} X = \bigcap \{A \in \mathfrak{A} \mid X \subseteq A\} \subseteq \bigcap \{A \in \mathfrak{A} \mid Y \subseteq A\} = \varphi_{\mathfrak{A}} Y.$$

Extensity is trivial. Idempotency: According to the definition of $\varphi_{\mathfrak{A}}$, each element of $\mathfrak{A}$ which contains X also contains $\varphi_{\mathfrak{A}} X$, and vice versa.

– $\mathfrak{A}_\varphi$ is a closure system: Let $\mathfrak{X} \subseteq \mathfrak{A}_\varphi$. On account of the extensity of φ we have $\bigcap \mathfrak{X} \subseteq \varphi(\bigcap \mathfrak{X})$. Because of monotony and idempotency, from $X \in \mathfrak{X}$ it always follows that $\varphi(\bigcap \mathfrak{X}) \subseteq \varphi X = X$, which implies $\varphi(\bigcap \mathfrak{X}) \subseteq \bigcap \mathfrak{X}$.

– $X \in \mathfrak{A} \Leftrightarrow X = \bigcap \{A \in \mathfrak{A} \mid X \subseteq A\} \Leftrightarrow \varphi_{\mathfrak{A}} X = X \Leftrightarrow X \in \mathfrak{A}_{\varphi_{\mathfrak{A}}}$.

– For $A \in \mathfrak{A}_\varphi$, $X \subseteq A$ is equivalent to $\varphi X \subseteq A$. Hence

$$\varphi_{\mathfrak{A}_\varphi} X = \bigcap\{A \in \mathfrak{A}_\varphi \mid X \subseteq A\} = \bigcap\{A \in \mathfrak{A}_\varphi \mid \varphi X \subseteq A\} = \varphi X,$$

since $\varphi X \in \mathfrak{A}_\varphi$.

□

Every closure system $\mathfrak{A}$ can be understood as the set of all closures of a closure operator. Thus, the elements of $\mathfrak{A}$ are called closures as well.

Proposition 3. *If $\mathfrak{A}$ is a closure system, then $(\mathfrak{A}, \subseteq)$ is a complete lattice with $\bigwedge \mathfrak{X} = \bigcap \mathfrak{X}$ and $\bigvee \mathfrak{X} = \varphi_{\mathfrak{A}} \bigcup \mathfrak{X}$ for all $\mathfrak{X} \subseteq \mathfrak{A}$. Conversely, every complete lattice is isomorphic to the lattice of all closures of a closure system.*

Proof. It is obvious that $\bigcap \mathfrak{X}$ is the infimum and thus (compare Proposition 1) that $\varphi_{\mathfrak{A}} \bigcup \mathfrak{X}$ is the supremum of $\mathfrak{X}$. If $(V, \leq)$ is a complete lattice, then the system $\{(x] \mid x \in V\}$ is a closure system, since $\bigcap_{y \in T}(y] = (\bigwedge T]$ holds for each subset $T \subseteq V$. □

However, a system of sets $\mathfrak{A} \subseteq \mathfrak{P}(G)$ for which $(\mathfrak{A}, \subseteq)$ is a complete lattice is not necessarily a closure system. Rather, such families of sets are precisely the image sets of monotonous, idempotent operators.

Examples. For many mathematical structures, the system of substructures is a closure system. The power-set evidently is a closure system. Other important examples are:

(1) **subspaces:** For any vector space V, the system $\mathfrak{U}(V)$ of all subspaces is a closure system. The complete lattice $(\mathfrak{U}(V), \subseteq)$ is called the **subspace lattice** of V; in this lattice $U_1 \vee U_2 = U_1 + U_2$ and more generally

$$\bigvee \mathfrak{X} = \{u_1 + u_2 + \cdots + u_n \mid \text{there are } U_1, \ldots, U_n \in \mathfrak{X} \text{ with } u_i \in U_i \text{ for } i \in \{1, \ldots, n\}\}.$$

(2) **subgroups:** For any group G, the set $\mathfrak{U}(G)$ of all subgroups of G is a closure system. The complete lattice $(\mathfrak{U}(G), \subseteq)$ is called **subgroup lattice** of G. Provided that G is commutative, $U_1 \vee U_2 = U_1 + U_2$, and more generally,

$$\bigvee \mathfrak{X} = \{u_1 + u_2 + \cdots + u_n \mid \text{there are } U_1, \ldots, U_n \in \mathfrak{X} \text{ with } u_i \in U_i \text{ for } i \in \{1, \ldots, n\}\}.$$

(3) **closed sets:** For a topological space T (for example, for $\mathbb{R}^n$), the set $\mathfrak{A}(T)$ of all closed sets of T is a closure system. In the complete lattice $(\mathfrak{A}(T), \subseteq)$ the supremum is equivalent to the topological closure of the union, i.e.,

$$\bigvee \mathfrak{X} = \overline{\bigcup \mathfrak{X}}.$$

(4) **convex sets:** For $\mathbb{R}^n$ the set $\mathfrak{C}(\mathbb{R}^n)$ of all convex subsets is a closure system, i.e., $(\mathfrak{C}(\mathbb{R}^n), \subseteq)$ is a complete lattice and in this lattice the supremum is the convex closure of the union.

(5) **the faces of polyhedra:** For a polyhedron P, the set $\mathfrak{S}(P)$ of all faces of P is a closure system. The complete lattice $(\mathfrak{S}(P), \subseteq)$ is called the **face lattice** of P; for those lattices there is no general "good" description of the suprema.

(6) **equivalence relations:** For a set M, the set $\mathfrak{E}(M)$ of all equivalence relations on M is a closure system on $M \times M$. The complete lattice $(\mathfrak{E}(M), \subseteq)$ is called the **lattice!of equivalence relations** of M; in this lattice

$$\bigvee \mathfrak{X} = \{(a,b) \in M \times M \mid \text{there are } R_1, \ldots, R_n \in \mathfrak{X} \text{ and } (x_i, x_{i+1}) \in R_i \text{ for } i \in \{1, \ldots, n\} \text{ with } a = x_1 \text{ and } b = x_{n+1}\}.$$

In lattice theory these lattices are examined for the structural laws they fulfill. In the following we will mention a few important properties, these and others will be discussed in Chapter 6.

Definition 15. A complete lattice $\boldsymbol{V}$ is called

– **distributive** if the following distributive laws hold for all $x, y, z \in V$:

$$\begin{array}{llcl} (\mathbf{D}_\wedge) & x \wedge (y \vee z) & = & (x \wedge y) \vee (x \wedge z) \\ (\mathbf{D}_\vee) & x \vee (y \wedge z) & = & (x \vee y) \wedge (x \vee z) \end{array}$$

– **completely distributive** if the following generalization of the two distributive laws for arbitrary infima and suprema holds for all index sets $S, T \neq \emptyset$:

$$(\mathbf{D}_{\vee\wedge}) \qquad \bigwedge \left\{ \bigvee \{x_{s,t} \mid t \in T\} \mid s \in S \right\} = \bigvee \left\{ \bigwedge \{x_{s,\varphi s} \mid s \in S\} \mid \varphi : S \to T \right\}.$$

– **modular** if the following law holds for all x, y and z:

$$x \leq z \Rightarrow x \vee (y \wedge z) = (x \vee y) \wedge z.$$

◇

Distributivity and modularity are **self-dual** properties: if they hold for a lattice $\boldsymbol{V}$, they also hold for $\boldsymbol{V}^d$. All above-mentioned properties transfer to complete sublattices. Power-set lattices are completely distributive, subspace lattices of vector spaces are modular.

For the special case $S := \{0, 1\}$, $x_{0,t} := x$ and $x_{1,t} := x_t$ for all $t \in T$, the property of complete distributivity yields the weaker law

$$(\mathbf{D}_{\wedge\bigvee}) \qquad x \wedge \bigvee_{t\in T} x_t = \bigvee_{t\in T} (x \wedge x_t).$$

The dual law $(\mathbf{D}_{\vee\bigwedge})$ holds in the lattice of the closed sets of any given topological space, and $(\mathbf{D}_{\wedge\bigvee})$ holds in the lattice of all open sets. Those lattices are not completely distributive in general.

0.4 Galois Connections

Definition 16. Let

$$\varphi : P \longrightarrow Q \quad \text{and} \quad \psi : Q \longrightarrow P$$

be maps between two ordered sets $(P, \leq)$ and $(Q, \leq)$. Such a pair of maps is called a **Galois connection** between the ordered sets if:

1. $p_1 \leq p_2 \Rightarrow \varphi p_1 \geq \varphi p_2$
2. $q_1 \leq q_2 \Rightarrow \psi q_1 \geq \psi q_2$
3. $p \leq \psi\varphi p$ and $q \leq \varphi\psi q$

The two maps then are called **dually adjoint** to each other. ◇

See Figure 0.4 for an example.

Proposition 4. *A pair (φ, ψ) of maps is a Galois connection if and only if:*

4. $p \leq \psi q \Leftrightarrow q \leq \varphi p$.

Proof. $p \leq \psi q$ by 1) yields $\varphi p \geq \varphi\psi q$ and by 3) $\varphi p \geq q$, i.e., one direction of 4). The other follows symmetrically. Conversely, from $\varphi p \leq \varphi p$ by 4) it follows that $p \leq \psi\varphi p$, i.e., 3). Thus, from $p_1 \leq p_2$ we can deduce that $p_1 \leq \psi\varphi p_2$, which by 4) yields $\varphi p_2 \leq \varphi p_1$. □

Proposition 5. *For every Galois connection (φ, ψ)*

$$\varphi = \varphi\psi\varphi \text{ and } \psi = \psi\varphi\psi.$$

Proof. With $q := \varphi p$ we obtain by condition 3) $\varphi p \leq \varphi\psi\varphi p$ and from $p \leq \psi\varphi p$ by 1) $\varphi p \geq \varphi\psi\varphi p$. □

The question, under which conditions a given map φ can be extended to a Galois connection, is answered by the following proposition.

Proposition 6. *A map*

$$\varphi : (M, \leq) \longrightarrow (N, \leq)$$

between two ordered sets has a dual adjoint, if the pre-image of each principal filter is a principal ideal. The dual adjoint is unique.

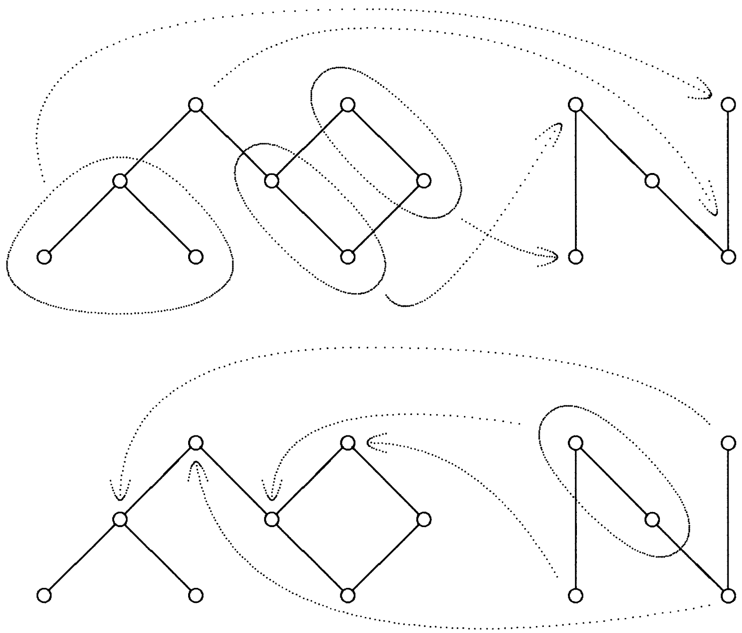


Figure 0.4 An example of a Galois connection.

Proof. For any $y \in N$ the pre-image of $[y)$ equals $\{x \in M \mid y \leq \varphi x\}$. If ψ is dually adjoint to φ, then by Proposition 4

$$\{x \in M \mid y \leq \varphi x\} = \{x \in M \mid x \leq \psi y\} = (\psi y].$$

This also proves the uniqueness of ψ. Conversely, we can define ψ by

$$(\psi y] := \varphi^{-1}([y)) = \{x \in M \mid y \leq \varphi x\}$$

and thereby obtain $x \leq \psi y \Leftrightarrow y \leq \varphi x$, which according to the proposition is characteristic of Galois connections. $\square$

In the case that the two ordered sets are complete lattices, the characterization can be further improved:

Proposition 7. *A map*

$$\varphi : (V, \leq) \longrightarrow (W, \leq)$$

between complete lattices has a dual adjoint if and only if

$$\varphi \bigvee_{t \in T} x_t = \bigwedge_{t \in T} \varphi x_t$$

holds for $x_t \in V$.

Proof. If ψ is dually adjoint to φ, then by Proposition 4

$$\begin{aligned} y \le \bigwedge_{t\in T} \varphi x_t &\Leftrightarrow y \le \varphi x_t \text{ for all } t \in T \\ &\Leftrightarrow x_t \le \psi y \text{ for all } t \in T \\ &\Leftrightarrow \bigvee_{t\in T} x_t \le \psi y \\ &\Leftrightarrow y \le \varphi \bigvee_{t\in T} x_t. \end{aligned}$$

If, conversely, $\varphi \bigvee_{t\in T} x_t = \bigwedge_{t\in T} \varphi x_t$, then φ definitely fulfills condition 1) in Definition 16. Defining $\psi y := \bigvee\{x \in V \mid y \le \varphi x\}$ for $y \in W$ we immediately obtain condition 2) as well as the first part of 3). For $y \in W$ it follows that $\varphi\psi y = \varphi \bigvee\{x \in V \mid y \le \varphi x\} = \bigwedge\{\varphi x \mid y \le \varphi x\} \ge y$, i.e., 3). Hence (φ, ψ) is a Galois connection. □

We are particularly interested in the special case of a Galois connection between two power-set lattices. If M and N are two sets and $\varphi : \mathfrak{P}(M) \to \mathfrak{P}(N)$ is a map (assigning a subset φA of N to each subset A of M) and ψ is a map from $\mathfrak{P}(N)$ to $\mathfrak{P}(M)$ such that conditions 1), 2) and 3) of Definition 16 are fulfilled (the order is set inclusion $\subseteq$), then this is briefly called a **Galois connection between** M **and** N. The connection with the closure operators is emphasized by the following proposition.

Proposition 8. *The map $A \mapsto \psi\varphi A$ is a closure operator on M and the map $B \mapsto \varphi\psi B$ is a closure operator on N. The maps φ and ψ, respectively, define dual isomorphisms between the corresponding closure systems.*

Proof. Monotony and extensity of the maps follow immediately from the definition of a Galois connection, and idempotency follows from Proposition 5. We can also see from this proposition that the closures in M are precisely the sets of the form $\psi B, B \subseteq N$, and those in N are precisely the sets of the form $\varphi A, A \subseteq M$. The maps $\psi B \mapsto \varphi\psi B$ and $\varphi A \mapsto \psi\varphi A$, respectively, are order-reversing and by Proposition 5 inverse to each other, i.e., bijective. □

Galois connections between power-set lattices and binary relations between their ground sets are closely interrelated. This is shown by the next theorem. In preparation, we introduce some new notation:

Definition 17. If $R \subseteq M \times N$ is a relation, we write

$$\begin{aligned} X^R &:= \{y \in N \mid xRy \text{ for all } x \in X\} \text{ for } X \subseteq M \\ \text{and} \quad Y^R &:= \{x \in M \mid xRy \text{ for all } y \in Y\} \text{ for } Y \subseteq N. \end{aligned}$$

◇

Since we have not presupposed that M and N are disjoint, this notation allows ambiguous formulations, which, however, can easily be avoided.

Theorem 2. *For every binary relation $R \subseteq M \times N$, a Galois connection (φ_R, ψ_R) between M and N is defined by*

$$\begin{array}{rrcll} & \varphi_R X & := & X^R & (= \{y \in N \mid xRy \text{ for all } x \in X\}) \\ \text{and} & \psi_R Y & := & Y^R & (= \{x \in M \mid xRy \text{ for all } y \in Y\}). \end{array}$$

If, conversely, (φ, ψ) is a Galois connection between M and N, then

$$\begin{array}{rcl} R_{(\varphi,\psi)} & := & \{(x,y) \in M \times N \mid x \in \psi\{y\}\} \\ & = & \{(x,y) \in M \times N \mid y \in \varphi\{x\}\} \end{array}$$

is a binary relation between M and N, $\varphi_{R_{(\varphi,\psi)}} = \varphi$, $\psi_{R_{(\varphi,\psi)}} = \psi$ and $R_{(\varphi_R,\psi_R)} = R$.

Proof. From Proposition 4 it can easily be seen that (φ_R, ψ_R) is a Galois connection and that the two sets used in order to define $R_{(\varphi,\psi)}$ are equal. According to this definition $(x,y) \in R_{(\varphi,\psi)} \Leftrightarrow y \in \varphi\{x\}$ and thus, by Proposition 7,

$$\begin{array}{rcl} \varphi X & = & \bigcap_{x \in X} \varphi\{x\} \\ & = & \bigcap_{x \in X} \varphi_{R_{(\varphi,\psi)}}\{x\} \\ & = & \varphi_{R_{(\varphi,\psi)}} X, \end{array}$$

i.e., $\varphi_{R_{(\varphi,\psi)}} = \varphi$ and correspondingly $\psi_{R_{(\varphi,\psi)}} = \psi$. The last statement $R_{(\varphi_R,\psi_R)} = R$ follows immediately from the equivalence $x \in \psi_R\{y\} \Leftrightarrow xRy$. $\square$

The use of the term "Galois connection" is not uniform. Some authors prefer to replace one of the ordered sets by its dual. We prefer to call such pairs of maps *residuated.* In the case of complete lattices we obtain:

Proposition 9. *To every $\bigwedge$-preserving map $\varphi : (V, \leq) \longrightarrow (W, \leq)$ between complete lattices there is a $\bigvee$-preserving map*

$$\psi : (W, \leq) \longrightarrow (V, \leq)$$

with

$$x \leq \varphi(y) \Leftrightarrow \psi(x) \leq y.$$

The maps φ and ψ uniquely determine each other: From φ we obtain ψ by

$$\psi(x) = \bigwedge\{y \mid x \leq \varphi(y)\},$$

and, conversely, φ results from ψ by

$$\varphi(y) = \bigvee\{x \mid \psi(x) \leq y\}.$$

In this case, ψ is called a **residuated map**, *φ is called the* **residual map**, *and the maps are* **adjoint** *to each other. If one of the maps is injective, the other one is surjective, and vice versa.*

Proof. This is an immediate consequence of Proposition 7 and the preceding propositions: If we replace $(W, \leq)$ by the dual lattice $(W, \geq)$, φ and ψ form a Galois connection. The relation between injectivity and surjectivity can be inferred from Proposition 5. □

0.5 Hints and References

The standard monograph on lattices and ordered sets remains Birkhoff's *Lattice Theory* [15]. Of the many other textbooks on these topics we particularly mention *Algebraic Theory of Lattices* by Crawley and Dilworth [29], also Davey and Priestley: *Introduction to Lattices and Order* [31], and Grätzer: *General Lattice Theory* [75]. Many facts about Galois connections and residuated maps can be found in the book *Residuation Theory* by Blyth and Janowitz [16].

1. Concept Lattices of Contexts

The basic notions of Formal Concept Analysis are those of a *formal context* and a *formal concept*. The adjective "formal" is meant to emphasize that we are dealing with mathematical notions, which only reflect some aspects of the meaning of *context* and *concept* in standard language. However, we will write out the adjective "formal" only in the definition and leave it out later for reasons of convenience, as we have in the title of the first section. Thus, it shall be understood that where we write *context* or *concept* we actually mean a *formal context* or a *formal concept*, respectively.

1.1 Context and Concept

Definition 18. A **formal context** $\mathbb{K} := (G, M, I)$ consists of two sets G and M and a relation I between G and M. The elements of G are called the **objects** and the elements of M are called the **attributes** of the context[1]. In order to express that an object g is in a relation I with an attribute m, we write gIm or $(g, m) \in I$ and read it as "the object g **has** the attribute m". $\diamond$

The relation I is also called the *incidence relation* of the context. Instead of $(g, m) \notin I$ we sometimes write $g\not{I}m$.

Example 1. The context in Figure 1.1 was used to plan a Hungarian educational film entitled "Living Beings and Water". Here the objects are the living beings mentioned in the film and the attributes are the properties which the film emphasizes.

A small context can be easily represented by a **cross table**, i.e., by a rectangular table the rows of which are headed by the object names and the columns headed by the attribute names. A cross in row g and column m means that the object g has the attribute m.

	m
	$\vdots$
g	$\cdots\cdots \times$

[1] Strictly speaking: "formal objects" and "formal attributes".

		a	b	c	d	e	f	g	h	i
1	Leech	×	×					×		
2	Bream	×	×					×	×	
3	Frog	×	×	×				×	×	
4	Dog	×		×				×	×	×
5	Spike – weed	×	×		×		×			
6	Reed	×	×	×	×		×			
7	Bean	×		×	×	×				
8	Maize	×		×	×		×			

Figure 1.1 Context of an educational film "Living Beings and Water". The attributes are: a: needs water to live, b: lives in water, c: lives on land, d: needs chlorophyll to produce food, e: two seed leaves, f: one seed leaf, g: can move around, h: has limbs, i: suckles its offspring.

Definition 19. For a set $A \subseteq G$ of objects we define

$$A' := \{m \in M \mid gIm \text{ for all } g \in A\}$$

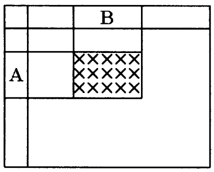
(the set of attributes common to the objects in A). Correspondingly, for a set B of attributes we define

$$B' := \{g \in G \mid gIm \text{ for all } m \in B\}$$

(the set of objects which have all attributes in B).[2] ◇

Definition 20. A **formal concept** of the context (G, M, I) is a pair (A, B) with $A \subseteq G, B \subseteq M$, $A' = B$ and $B' = A$. We call A the **extent** and B the **intent** of the concept (A, B). $\mathfrak{B}(G, M, I)$ denotes the set of all concepts of the context (G, M, I). ◇

We will give examples of concepts of the context in Figure 1.1 after Definition 21. The extent A and the intent B of a concept (A, B) are closely connected by the relation I. Each of the two parts determines the other and thereby the concept, since $B' = A$ and $A' = B$, respectively. The next proposition states further simple rules of this interaction:



Proposition 10. *If (G, M, I) is a context, $A, A_1, A_2 \subseteq G$ are sets of objects and B, B_1, B_2 are sets of attributes, then*

[2] The notation introduced here is convenient but sometimes insufficient. In order to improve comprehensibility it can be helpful to choose notations like $A^{\uparrow}$, $B^{\downarrow}$ to distinguish the derivation operators, or A^I, A^J to distinguish different relations.

1) $A_1 \subseteq A_2 \Rightarrow A_2' \subseteq A_1'$ *1')* $B_1 \subseteq B_2 \Rightarrow B_2' \subseteq B_1'$
2) $A \subseteq A''$ *2')* $B \subseteq B''$
3) $A' = A'''$ *3')* $B' = B'''$

$$\textit{4)}\ A \subseteq B' \iff B \subseteq A' \iff A \times B \subseteq I.$$

Proof. 1) If $m \in A_2'$, then gIm for all $g \in A_2$, i.e., in particular gIm for all $g \in A_1$, if $A_1 \subseteq A_2$ and thus $m \in A_1'$. 2) If $g \in A$, then gIm for all $m \in A'$, which implies $g \in A''$. 3) $A' \subseteq A'''$ follows immediately from 2'), and $A \subseteq A''$ together with 1) yields $A''' \subseteq A'$. 4) follows directly from the definition. □

The proposition shows that the two derivation operators form a Galois connection between the power-set lattices $\mathfrak{P}(G)$ and $\mathfrak{P}(M)$ (see Section 0.4). Hence we obtain (by Proposition 8) two closure systems on G and M, which are dually isomorphic to each other:

For every set $A \subseteq G$, A' is an intent of some concept, since (A'', A') is always a concept. A'' is the smallest extent containing A. Consequently, a set $A \subseteq G$ is an extent if and only if $A = A''$. The same applies to intents. The union of extents generally does not result in an extent. On the other hand, the intersection of any number of extents (respectively intents) is always an extent (intent), as is proved by the following proposition:

Proposition 11. *If T is an index set and, for every $t \in T$, $A_t \subseteq G$ is a set of objects, then*

$$\left(\bigcup_{t\in T} A_t\right)' = \bigcap_{t\in T} A_t'.$$

The same holds for sets of attributes.

Proof.

$$\begin{aligned} m \in \left(\bigcup_{t\in T} A_t\right)' &\iff gIm \text{ for all } g \in \bigcup_{t\in T} A_t \\ &\iff gIm \text{ for all } g \in A_t \text{ for all } t \in T \\ &\iff m \in A_t' \text{ for all } t \in T \\ &\iff m \in \bigcap_{t\in T} A_t'. \end{aligned}$$

□

The set of all extents of (G, M, I) is sometimes denoted by $\mathfrak{U}(G, M, I)$. For the set of all intents we write $\mathfrak{I}(G, M, I)$.

Definition 21. If (A_1, B_1) and (A_2, B_2) are concepts of a context, (A_1, B_1) is called a **subconcept** of (A_2, B_2), provided that $A_1 \subseteq A_2$ (which is equivalent to $B_2 \subseteq B_1$). In this case, (A_2, B_2) is a **superconcept** of (A_1, B_1), and we write $(A_1, B_1) \leq (A_2, B_2)$. The relation $\leq$ is called the **hierarchical order** (or simply **order**) of the concepts. The set of all concepts of (G, M, I) ordered in this way is denoted by $\underline{\mathfrak{B}}(G, M, I)$ and is called the **concept lattice** of the context (G, M, I). $\diamond$

Example 2. The context in Example 1 has 19 concepts. The line diagram in Figure 1.2 represents the concept lattice of this context.

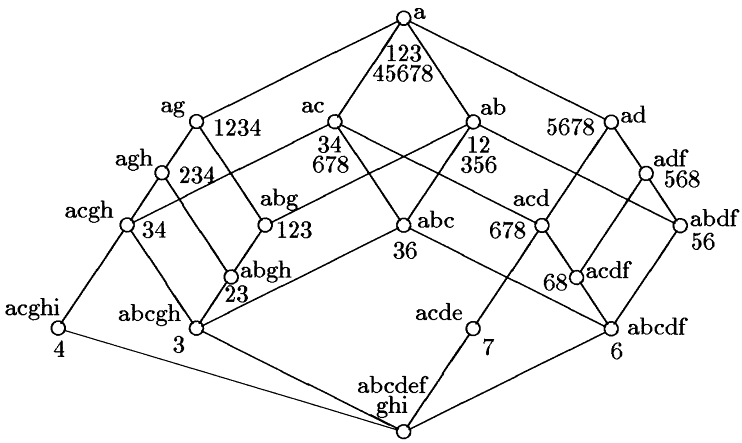


Figure 1.2 Concept lattice for the context of Figure 1.1

Theorem 3 (The Basic Theorem on Concept Lattices). *The concept lattice $\underline{\mathfrak{B}}(G, M, I)$ is a complete lattice in which infimum and supremum are given by:*

$$\bigwedge_{t \in T} (A_t, B_t) = \left(\bigcap_{t \in T} A_t, \left(\bigcup_{t \in T} B_t \right)'' \right)$$

$$\bigvee_{t \in T} (A_t, B_t) = \left(\left(\bigcup_{t \in T} A_t \right)'', \bigcap_{t \in T} B_t \right).$$

A complete lattice V is isomorphic to $\underline{\mathfrak{B}}(G, M, I)$ if and only if there are mappings $\tilde{\gamma} : G \to V$ and $\tilde{\mu} : M \to V$ such that $\tilde{\gamma}(G)$ is supremum-dense in V, $\tilde{\mu}(M)$ is infimum-dense in V and gIm is equivalent to $\tilde{\gamma}g \leq \tilde{\mu}m$ for all $g \in G$ and all $m \in M$. In particular, $V \cong \underline{\mathfrak{B}}(V, V, \leq)$.

Proof of the Basic Theorem. First, we will explain the formula for the infimum. Since $A_t = B_t'$ for each $t \in T$,

$$\left(\bigcap_{t \in T} A_t, \left(\bigcup_{t \in T} B_t \right)'' \right)$$

by Proposition 11 can be transformed into

$$\left(\left(\bigcup_{t\in T} B_t\right)', \left(\bigcup_{t\in T} B_t\right)''\right),$$

i.e., it has the form (X', X'') and is therefore certainly a concept. That this can only be the infimum, i.e., the largest common subconcept of the concepts (A_t, B_t), follows immediately from the fact that the extent of this concept is exactly the intersection of the extents of (A_t, B_t). The formula for the supremum is substantiated correspondingly. Thus, we have proven that $\underline{\mathfrak{B}}(G, M, I)$ is a complete lattice.

Now we prove, first for the special case $\boldsymbol{V} = \underline{\mathfrak{B}}(G, M, I)$, the existence of mappings $\tilde{\gamma}$ and $\tilde{\mu}$ with the required properties. We set

$$\tilde{\gamma}g := (\{g\}'', \{g\}') \text{ for } g \in G$$

$$\text{and} \quad \tilde{\mu}m := (\{m\}', \{m\}'') \text{ for } m \in M.$$

As claimed, we have $\tilde{\gamma}g \leq \tilde{\mu}m \iff \{g\}'' \subseteq \{m\}' \iff \{g\}' \supseteq \{m\} \iff m \in \{g\}' \iff gIm$. Furthermore, on account of the formulas proved above,

$$\bigvee_{g\in A} (\{g\}'', \{g\}') = (A, B) = \bigwedge_{m\in B} (\{m\}', \{m\}''),$$

holds for every concept (A, B), i.e., $\tilde{\gamma}(G)$ is supremum-dense and $\tilde{\mu}(M)$ is infimum-dense in $\underline{\mathfrak{B}}(G, M, I)$. More generally, if $\boldsymbol{V} \cong \underline{\mathfrak{B}}(G, M, I)$ and $\varphi : \underline{\mathfrak{B}}(G, M, I) \to \boldsymbol{V}$ is an isomorphism, we define $\tilde{\gamma}$ and $\tilde{\mu}$ by

$$\tilde{\gamma}g := \varphi(\{g\}'', \{g\}') \text{ for } g \in G$$

$$\text{and } \tilde{\mu}m := \varphi(\{m\}', \{m\}'') \text{ for } m \in M.$$

The properties claimed for these mappings are proved in a similar fashion.

If, conversely, $\boldsymbol{V}$ is a complete lattice and

$$\tilde{\gamma} : G \to V, \tilde{\mu} : M \to V$$

are mappings with the properties stated above, then we define

$$\varphi : \underline{\mathfrak{B}}(G, M, I) \to V$$

by

$$\varphi(A, B) := \bigvee \{\tilde{\gamma}(g) \mid g \in A\}.$$

Evidently, φ is order-preserving. In order to prove that φ is an isomorphism, we have to demonstrate that φ^{-1} exists and is also order-preserving. Therefore, we define

$$\psi x := (\{g \in G \mid \tilde{\gamma}g \leq x\}, \{m \in M \mid x \leq \tilde{\mu}m\}),$$

for $x \in V$ and demonstrate that ψx is a concept of (G, M, I):

$$\begin{aligned} h \in \{g \in G \mid \tilde{\gamma} g \leq x\} \quad &\Leftrightarrow \quad \tilde{\gamma} h \leq x \\ &\Leftrightarrow \quad \tilde{\gamma} h \leq \tilde{\mu} n \text{ for all } n \in \{m \in M \mid x \leq \tilde{\mu} m\} \\ &\Leftrightarrow \quad hIn \text{ for all } n \in \{m \in M \mid x \leq \tilde{\mu} m\} \\ &\Leftrightarrow \quad h \in \{m \in M \mid x \leq \tilde{\mu} m\}'. \end{aligned}$$

The second condition follows correspondingly. We have defined a map $\psi : V \to \underline{\mathfrak{B}}(G, M, I)$, and we can read off directly from the definition that ψ is order-preserving. Now we prove that $\varphi = \psi^{-1}$. We have

$$\varphi \psi x = \bigvee \{\tilde{\gamma} g \mid g \in G, \tilde{\gamma} g \leq x\} = x,$$

since $\tilde{\gamma}(G)$ is supremum-dense in $\boldsymbol{V}$. On the other hand, $\varphi(A, B) = \bigwedge\{\tilde{\mu} m \mid m \in B\}$, since $\tilde{\mu}(M)$ is infimum-dense in $\boldsymbol{V}$, and consequently

$$\begin{aligned} \psi\varphi(A, B) &= \psi \bigwedge \{\tilde{\mu} m \mid m \in B\} \\ &= (\{g \in G \mid \tilde{\gamma} g \leq \bigwedge \{\tilde{\mu} m \mid m \in B\}\}, \{\ldots\}') \\ &= (\{g \in G \mid \tilde{\gamma} g \leq \tilde{\mu} m \text{ for all } m \in B\}, \{\ldots\}') \\ &= (\{g \in G \mid gIm \text{ for all } m \in B\}, \{\ldots\}') \\ &= (B', B'') = (A, B). \end{aligned}$$

If we choose for a complete lattice $\boldsymbol{V}$ specifically $G := V$, $M := V$, $I := \leq$ and $\tilde{\gamma}$ as well as $\tilde{\mu}$ to be the identity of V, we obtain $\boldsymbol{V} \cong \underline{\mathfrak{B}}(G, M, I)$. □

The Duality Principle for Concept Lattices. Let (G, M, I) be a context. Then (M, G, I^{-1}) is also a context, in fact,

$$\underline{\mathfrak{B}}(M, G, I^{-1}) \cong \underline{\mathfrak{B}}(G, M, I)^d,$$

and

$$(B, A) \mapsto (A, B)$$

is an isomorphism.

In other words: if we exchange the roles of objects and attributes, we obtain the dual concept lattice. Thus, the Duality Principle extends to concept lattices.

The mappings $\tilde{\gamma}$ and $\tilde{\mu}$ which appear in the Basic Theorem indicate how the context can be identified in the concept lattice. This is elaborated by the following definition.

Definition 22. For an object $g \in G$ we write g' instead of $\{g\}'$ for the **object intent** $\{m \in M \mid gIm\}$ of the object g. Correspondingly, $m' := \{g \in G \mid gIm\}$ is the **attribute extent** of the attribute m. Retaining the symbols used in the Basic Theorem, we write γg for the **object concept** (g'', g') and μm for the **attribute concept** (m', m''). ◇

The line diagram in Figure 1.2 indicates the intent and the extent of every concept. The labelling can be simplified considerably by putting down each object and each attribute only once, namely at the circle for the respective object or attribute concept (see Figure 1.3). It is still possible to read off the context as well as all extents and intents from the line diagram: If one looks for the extent belonging to one of the little circles which represent the concepts, it consists of the objects located at this circle or the circles which can be reached by descending line paths from this circle. Correspondingly, the intent can be found by following all line paths going upwards from the circle and noting down the attributes assigned to these circles.

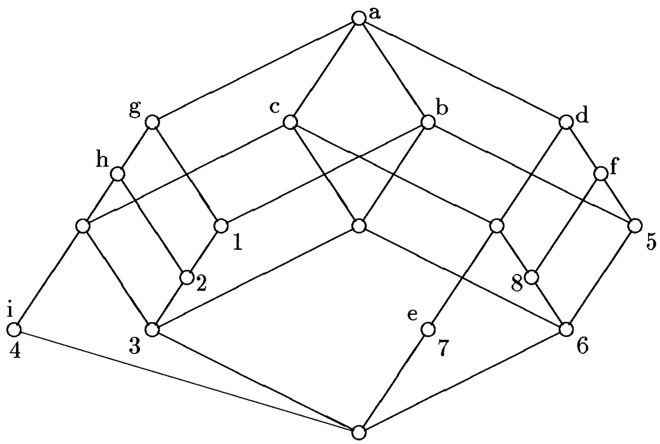


Figure 1.3 Line diagram with reduced labelling.

The sparing, reduced labelling enables us to enter the full names of the objects and attributes of the context in Figure 1.1 into the diagram. This improves the readability of the diagram, as can be seen in Figure 1.4.

1.2 Context and Concept Lattice

A context can be easily reconstructed from the system of all its concepts. G and M appear as the extent and the intent of the trivial *boundary concepts*: The set of all objects is the extent of the largest concept, $(\emptyset', \emptyset'') = (G, G')$. Dually, M is the intent of the least concept, $(\emptyset'', \emptyset') = (M', M)$. The incidence relation I is given by

$$I = \bigcup \{A \times B \mid (A, B) \in \mathfrak{B}(G, M, I)\}.$$

It is even easier to read off the context from the concept lattice, as the Basic Theorem shows. On the other hand, concept lattices of different contexts

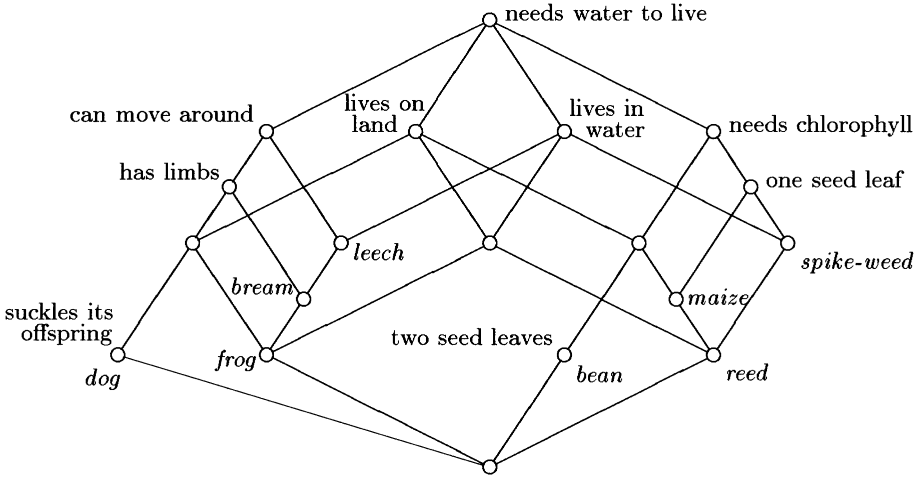


Figure 1.4 Concept lattice for the educational film "Living beings and water".

can well be isomorphic. The context manipulations which do not alter the structure of the concept lattice include the merging of objects with the same intents and attributes with the same extents, respectively:

Definition 23. A context (G, M, I) is called **clarified**, if for any objects $g, h \in G$ from $g' = h'$ it always follows that $g = h$ and, correspondingly, $m' = n'$ implies $m = n$ for all $m, n \in M$. $\diamond$

Example 3. Figure 1.5 shows a context which represents the service offers of an office supplies business. Below the clarified context.

Another feature which has no influence on the structure of the concept lattice are attributes which can be written as a combination of other attributes. More precisely: If $m \in M$ is an attribute and $X \subseteq M$ is a set of attributes with $m \notin X$ but $m' = X'$, then the attribute concept μm is the infimum of the attribute concepts $\mu x, x \in X$, i.e., the set $\mu(M \setminus \{m\})$ is also infimum-dense in $\underline{\mathfrak{B}}(G, M, I)$, and according to the Basic Theorem

$$\underline{\mathfrak{B}}(G, M, I) \cong \underline{\mathfrak{B}}(G, M \setminus \{m\}, I \cap (G \times (M \setminus \{m\}))).$$

The removal of **reducible attributes**, i.e., of attributes with $\bigwedge$-reducible attribute concepts and of **reducible objects**, i.e., of objects with $\bigvee$-reducible object concepts, is called **reducing** the context. **Full rows** and **full columns** are always reducible; thereby we mean objects g with $g' = M$ and attributes m with $m' = G$, respectively.

Definition 24. A clarified context (G, M, I) is called **row reduced**, if every object concept is $\bigvee$-irreducible, and **column reduced**, if every attribute

	Furniture	Computers	Copy-machines	Type-writers	Specialized machines
Consulting	×	×	×	×	×
Planning	×	×			
Assembly and installation	×	×	×	×	×
Instruction		×	×	×	×
Training, workshops		×			
Original spare parts and accessories	×	×	×	×	×
Repairs	×	×	×	×	×
Service contracts		×	×	×	

	Furniture	Computers	Copy machines and typewriters	Specialized machines
Consulting, assembly and installation, original spare parts and accessories, repairs	×	×	×	×
Planning	×	×		
Instruction		×	×	×
Training, workshops		×		
Service contracts		×	×	

Figure 1.5 Context and clarified context.

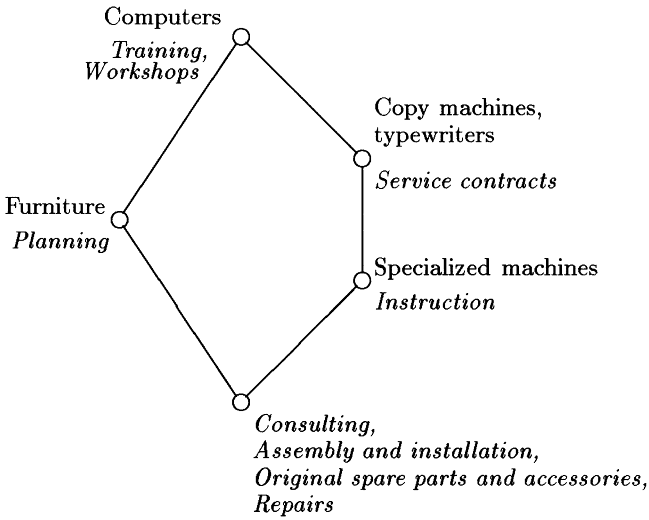


Figure 1.6 The concept lattice for the context of Figure 1.5.

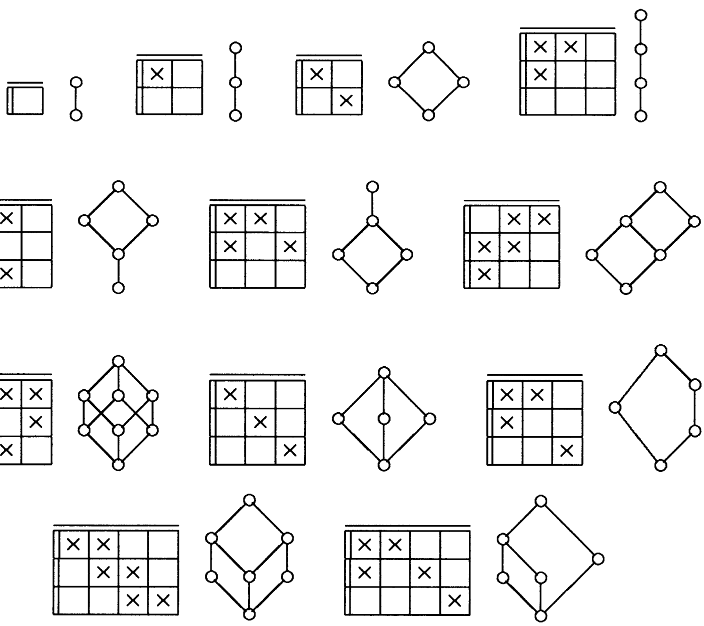


Figure 1.7 Reduced contexts with up to three objects, and their concept lattices. The context $(\emptyset, \emptyset, \emptyset)$ is omitted, as is its (one-element) concept lattice.

concept is $\bigwedge$-irreducible. A context, which is both row reduced and column reduced, is **reduced.** ◇

It is easy to find infinite contexts in which all attributes and all objects are reducible (see page 7). As a rule this means that we cannot simultaneously omit all reducible objects and attributes. This is no problem, however, in the case of **finite** concept lattices, since in a finite lattice each element is the join of $\bigvee$-irreducible and the meet of $\bigwedge$-irreducible elements (see Proposition 2, p. 7).

Proposition 12. *For every finite*[3] *lattice* $\boldsymbol{V}$ *there is -up to isomorphism*[4]*- a unique reduced context* $\mathbb{K}(\boldsymbol{V})$ *with* $\boldsymbol{V} \cong \underline{\mathfrak{B}}(\mathbb{K}(\boldsymbol{V}))$*, that is*

$$\mathbb{K}(\boldsymbol{V}) := (J(\boldsymbol{V}), M(\boldsymbol{V}), \leq).$$

□

This context is also called the **standard context** of the lattice $\boldsymbol{V}$. For practical work with contexts, the proposition has the following consequences: Every finite context can be brought into a reduced form without changing the structure of the concept lattice, and the latter is unique. We first clarify the context, i.e., we merge objects with the same intents and attributes with the same extents. Then we delete all objects, the intent of which can be represented as the intersection of other object intents, and correspondingly all attributes, the extent of which is the intersection of other attribute extents.

It is easy to reconstruct the concepts of the original context from those of the reduced context, if one has kept a record of the reduction process. If we denote, for a finite clarified context (G, M, I), the set of its irreducible objects by G_{irr} and the set of irreducible attributes by M_{irr}, the reduced context is $(G_{\text{irr}}, M_{\text{irr}}, I \cap (G_{\text{irr}} \times M_{\text{irr}}))$, and each concept (A, B) of (G, M, I) corresponds to the concept $(A \cap G_{\text{irr}}, B \cap M_{\text{irr}})$ of $(G_{\text{irr}}, M_{\text{irr}}, I \cap (G_{\text{irr}} \times M_{\text{irr}}))$. For every object $g \in G$ and every extent A of (G, M, I)

$$g \in A \iff g'' \cap G_{\text{irr}} \subseteq A \cap G_{\text{irr}},$$

holds dually for the attributes. If we note down the set $g'' \cap G_{\text{irr}}$ for every reducible object g and the set $m'' \cap M_{\text{irr}}$ for every reducible attribute m, it is easy to obtain the concepts of (G, M, I) from those of $(G_{\text{irr}}, M_{\text{irr}}, I \cap (G_{\text{irr}} \times M_{\text{irr}}))$.

There is another way to carry out the reduction of the clarified context, by means of the **arrow relations**, which will be defined next. These relations can conveniently be entered into the cross table, since they only apply to object-attribute-pairs which do not stand in the relation I.

---

[3] See also Proposition 14.c).

[4] Two contexts (G_1, M_1, I_1) and (G_2, M_2, I_2) are called **isomorphic**, if there are bijective mappings $\alpha : G_1 \to G_2, \beta : M_1 \to M_2$ with $gI_1m \Leftrightarrow (\alpha g)I_2(\beta m)$ for all $g \in G_1, m \in M_1$, see Definition 86 (p. 246).

	Group of 77	Non-aligned	LLDC	MSAC	OPEC	ACP
Afghanistan	×	×	×	×		
Algeria	×	×			×	
Angola	×	×				×
Antigua and Barbuda	×					×
Argentina	×					
Bahamas	×					×
Bahrain	×	×				
Bangladesh	×	×	×	×		
Barbados	×	×				×
Belize	×	×				×
Benin	×	×	×	×		×
Bhutan	×	×	×			
Bolivia	×	×				
Botswana	×	×	×			×
Brazil	×					
Brunei						
Burkina Faso	×	×	×	×		×
Burundi	×	×	×	×		×
Cambodia	×	×		×		
Cameroon	×	×		×		×
Cape Verde	×	×	×	×		×
Central African Rep.	×	×	×	×		×
Chad	×	×	×	×		×
Chile	×					
China						
Colombia	×	×				
Comoros	×	×	×			×
Congo	×	×				×
Costa Rica	×					
Cuba	×	×				
Djibouti	×	×	×			×
Dominica	×	×				×
Dominican Rep.	×					×

	Group of 77	Non-aligned	LLDC	MSAC	OPEC	ACP
Ecuador	×	×			×	
Egypt	×	×		×		
El Salvador	×			×		
Equatorial Guinea	×	×	×			×
Ethiopia	×	×	×	×		×
Fiji	×					×
Gabon	×	×			×	×
Gambia	×	×	×	×		×
Ghana	×	×	×	×		×
Grenada	×	×				×
Guatemala	×			×		
Guinea	×	×	×	×		×
Guinea-Bissau	×	×	×	×		×
Guyana	×	×		×		×
Haiti	×		×	×		×
Honduras	×			×		
Hong Kong						
India	×	×		×		
Indonesia	×	×			×	
Iran	×	×			×	
Iraq	×	×			×	
Ivory Coast	×	×		×		×
Jamaica	×	×				×
Jordan	×	×				
Kenya	×	×		×		×
Kiribati			×			×
Korea-North	×	×	×			
Korea-South	×					
Kuwait	×	×			×	
Laos	×	×	×	×		
Lebanon	×	×				
Lesotho	×	×	×	×		×
Liberia	×	×				×

The abbreviations stand for: LLDC := Least Developed Countries, MSAC := Most Seriously Affected Countries, OPEC := Organization of Petrol Exporting Countries, ACP := African, Caribbean and Pacific Countries.

Figure 1.8 Membership of developing countries in supranational groups. (Part 1).

	Group of 77	Non-aligned	LLDC	MSAC	OPEC	ACP
Libya	×	×			×	
Madagascar	×	×	×	×		×
Malawi	×	×	×			×
Malaysia	×	×				
Maledives	×	×	×			
Mali	×	×	×	×		×
Mauretania	×	×	×	×		×
Mauritius	×	×				×
Mexico	×					
Mongolia			×			
Morocco	×	×				
Mozambique	×	×		×		×
Myanmar	×		×	×		
Namibia	×					×
Nauru						
Nepal	×	×	×	×		
Nicaragua	×	×				
Niger	×	×	×	×		×
Nigeria	×	×			×	×
Oman	×	×				
Pakistan	×	×		×		
Panama	×	×				
Papua New Guinea	×					×
Paraguay	×					
Peru	×	×				
Philippines	×					
Qatar	×	×			×	
Réunion						
Rwanda	×	×	×	×		×
Samoa	×		×	×		×
São Tomé e Principe	×	×	×			×
Saudi Arabia	×	×			×	

	Group of 77	Non-aligned	LLDC	MSAC	OPEC	ACP
Senegal	×	×		×		×
Seychelles	×	×				×
Sierra Leone	×	×	×	×		×
Singapore	×	×				
Solomon Islands	×					×
Somalia	×	×	×	×		×
Sri Lanka	×	×		×		
St Kitts						
St Lucia	×	×				×
St Vincent& Grenad.	×					×
Sudan	×	×	×	×		×
Surinam	×	×				×
Swaziland	×	×				×
Syria	×	×				
Taiwan						
Tanzania	×	×	×	×		×
Thailand	×					
Togo	×	×	×			×
Tonga	×					×
Trinidad and Tobago	×	×				×
Tunisia	×	×				
Tuvalu			×			×
Uganda	×	×	×	×		×
United Arab Emirates	×	×			×	
Uruguay	×					
Vanuatu	×	×	×			×
Venezuela	×	×			×	
Vietnam	×	×	×			
Yemen	×	×	×	×		
Zaire	×	×	×			×
Zambia	×	×	×			×
Zimbabwe	×	×				×

Figure 1.8 Membership of developing countries in supranational groups. (Part 2). Source: *Lexikon Dritte Welt*, Rowohlt-Verlag, Reinbek 1993.

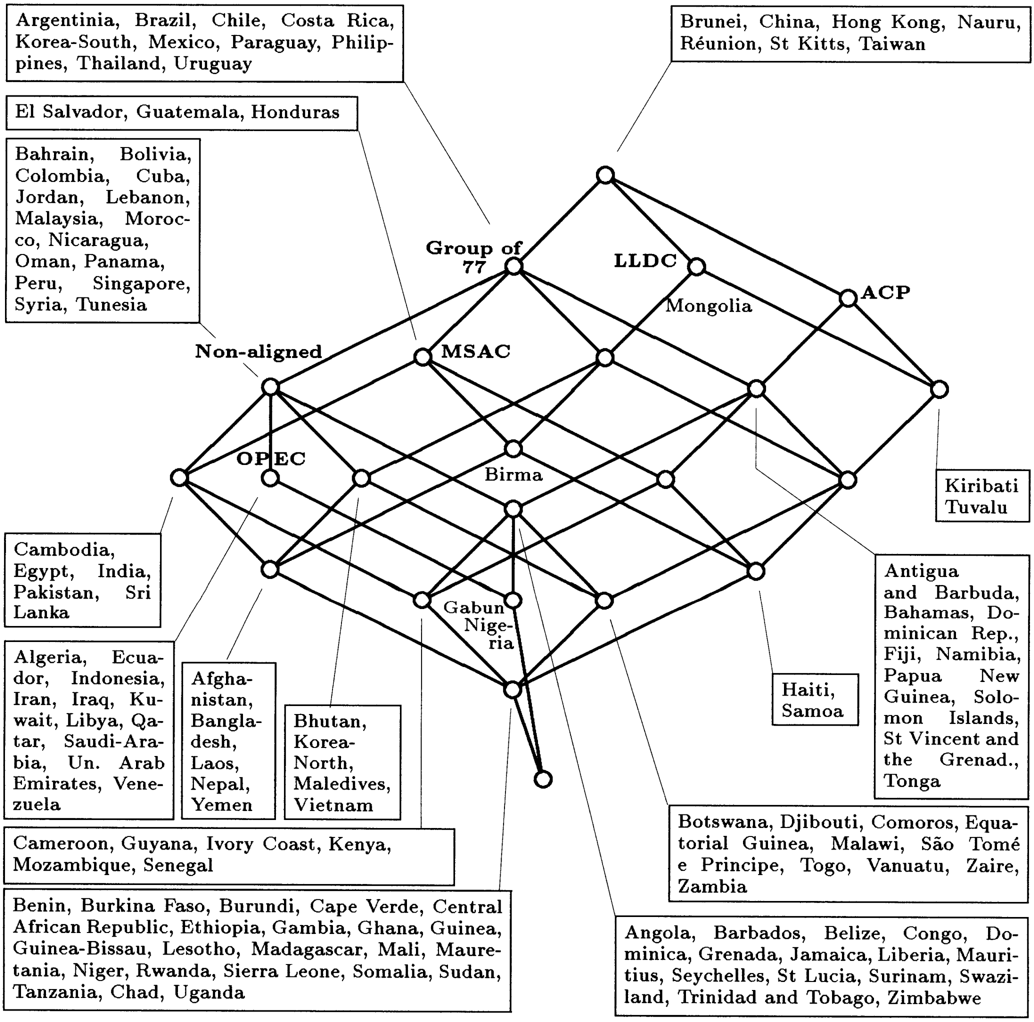


Figure 1.9 Concept lattice of the context of developing countries

Definition 25. If (G, M, I) is a context, $g \in G$ an object, and $m \in M$ an attribute, we write

$$
\begin{array}{lcl}
g \swarrow m & :\Longleftrightarrow & \begin{cases} g \not I m \text{ and} \\ \text{if } g' \subseteq h' \text{ and } g' \neq h', \text{ then } hIm, \end{cases} \\
g \nearrow m & :\Longleftrightarrow & \begin{cases} g \not I m \text{ and} \\ \text{if } m' \subseteq n' \text{ and } m' \neq n', \text{ then } gIn, \end{cases} \\
g \swarrow\!\!\!\!\nearrow m & :\Longleftrightarrow & g \swarrow m \text{ and } g \nearrow m.
\end{array}
$$

$\Diamond$

Thus, $g \swarrow m$ if and only if g' is maximal among all object intents which do not contain m. In other words: $g \swarrow m$ holds if and only if g does not have the attribute m, but m is contained in the intent of every proper subconcept of γg. If we now let

$$(\gamma g)_* := \bigvee \{\mathfrak{x} \in \underline{\mathfrak{B}}(G, M, I) \mid \mathfrak{x} < \gamma g\},$$

as in Definition 11 (p. 6) then $(\gamma g)_*$ is a subconcept of γg and γg is $\bigvee$-irreducible, if and only if $\gamma g \neq (\gamma g)_*$. This, on the other hand, is equivalent to the fact that there is an attribute m in the intent of $(\gamma g)_*$ which is not contained in the intent of γg, i.e., to $g \swarrow m$ for some $m \in M$. Therefore, we obtain

$$
\begin{array}{lcl}
g \swarrow m & \Longleftrightarrow & \gamma g \wedge \mu m = (\gamma g)_* \neq \gamma g \\
g \nearrow m & \Longleftrightarrow & \gamma g \vee \mu m = (\mu m)^* \neq \mu m.
\end{array}
$$

Example 4. Figure 1.10 shows the context from Figure 1.5 with the arrow relations; beside it the reduced context.

$\times$	$\times$	$\times$	$\times$
$\times$	$\times$	$\swarrow\!\!\!\!\nearrow$	$\swarrow$
$\swarrow\!\!\!\!\nearrow$	$\times$	$\times$	$\times$
$\nearrow$	$\times$	$\nearrow$	
$\nearrow$	$\times$	$\times$	$\swarrow\!\!\!\!\nearrow$

$\times$	$\swarrow\!\!\!\!\nearrow$	$\swarrow$
$\swarrow\!\!\!\!\nearrow$	$\times$	$\times$
$\nearrow$	$\times$	$\swarrow\!\!\!\!\nearrow$

Figure 1.10 Context with arrow relations, and the reduced context.

The significance of the arrow relations for the reduction of a context is shown by the next proposition:

Proposition 13. *The following statements hold for every context:*

a) γg *is* $\bigvee$*-irreducible* $\Longleftrightarrow$ *There is an* $m \in M$ *with* $g \swarrow m$.
b) μm *is* $\bigwedge$*-irreducible* $\Longleftrightarrow$ *There is a* $g \in G$ *with* $g \nearrow m$.

Furthermore, the following statements hold for every finite[5] *context:*

c) γg *is* $\bigvee$*-irreducible* $\iff$ *There is an* $m \in M$ *with* $g \swarrow\!\!\!\nearrow m$.
d) μm *is* $\bigwedge$*-irreducible* $\iff$ *There is a* $g \in G$ *with* $g \swarrow\!\!\!\nearrow m$.

Proof. This follows immediately from the above-mentioned observations together with Proposition 2. If we choose m' maximal with respect to $g \swarrow m$ (in a finite context this is certainly possible), then $g \nearrow m$, i.e., $g \swarrow\!\!\!\nearrow m$. □

In order to reduce a finite clarified context, we therefore first enter the arrow relations in the cross table and then delete all rows and columns not containing a double arrow. The condition of finiteness in Propositions 12 and 13 can be weakened:

Definition 26. A context (G, M, I) is called **doubly founded**, if, for every object $g \in G$ and every attribute $m \in M$ with $g \not I m$, there is an object $h \in G$ and an attribute $n \in M$ with

$$g \nearrow n \text{ and } m' \subseteq n' \quad \text{as well as} \quad h \swarrow m \text{ and } g' \subseteq h'.$$

A complete lattice $(V, \leq)$ is called **doubly founded**, if for any two elements $x < y$ of V there are elements $s, t \in V$ with:

s is minimal with respect to $s \leq y$, $s \not\leq x$, as well as
t is maximal with respect to $t \geq x$, $t \not\geq y$.

◇

By means of Proposition 13 we realize easily that the attribute n and the object h that appear in Definition 26 must be irreducible. The same applies to the lattice elements s and t in the second part of the definition: s must be $\bigvee$-irreducible and t must be $\bigwedge$-irreducible. This means that the property "doubly founded" implies the existence of "many" irreducible elements.

Proposition 14. *a) Every finite context is doubly founded.*

b) A context which does neither contain infinite chains $g_1, g_2, \ldots$ *of objects with* $g_1' \subset g_2' \subset \ldots$ *nor infinite chains* $m_1, m_2, \ldots$ *of attributes with* $m_1' \subset m_2' \subset \ldots$ *is doubly founded.*

c) Each concept of a doubly founded context is the supremum of $\bigvee$*-irreducible concepts and the infimum of* $\bigwedge$*- irreducible concepts. Hence Proposition 12 also applies to concept lattices of doubly founded contexts.*

d) If (G, M, I) *is doubly founded and* $g \in G$, $m \in M$*, the following hold true: if* $g \swarrow m$*, then there is an attribute* n *with* $g \swarrow\!\!\!\nearrow n$*, and if* $g \nearrow m$*, then there is an object* h *with* $h \swarrow\!\!\!\nearrow m$*. Hence parts c) and d) of Proposition 13 also apply to doubly founded contexts.*

[5] cf. also Proposition 14.d)

Proof. b) If $g \not\!I m_i$ holds and $g \nearrow m_i$ does not hold, according to the definition of the arrow relations there must be an attribute $m_{i+1} \neq m_i$ with $g \not\!I m_{i+1}$ and $m_i' \subset m_{i+1}'$. If in addition $g \swarrow m_i$, it follows that $g \swarrow m_{i+1}$. Starting from $m_1 := m$ we obtain by this argument a chain of attributes with increasing attribute extents. By assumption this chain must be finite and therefore end with an attribute $m_j =: n$ with $g \nearrow n$ and $g \swarrow n$, i.e., $g \swarrow\!\!\!\nearrow n$.
a) follows immediately from b).
c) Let (A, B) be a concept and $(C, D) := \bigvee\{\gamma x \mid x \in A, \gamma x \ \bigvee\text{-irreducible}\}$. We assume that $(C, D) < (A, B)$. Then there is $m \in D$, $g \in A$ with $g \not\!I m$. Hence there is an $h \in G$ with $h \swarrow m$ and $g' \subseteq h'$, i.e., $h \in A$. Because of $h \swarrow m$, γh is $\bigvee$-irreducible, i.e., $h \in C$ and thus $m \in h'$, which is a contradiction.
d) From $g \swarrow m$ it follows that $g \not\!I m$, hence there is an $n \in M$ with $m' \subseteq n'$ and $g \nearrow n$. All together we obtain $g \swarrow\!\!\!\nearrow n$. □

Proposition 15. *If $\mathfrak{B}(G, M, I)$ is doubly founded, so is (G, M, I). If a complete lattice $\mathbf{V}$ is not doubly founded, neither is the context $(V, V, \leq)$.*

Proof. Let $\mathfrak{B}(G, M, I)$ be doubly founded and $g \in G$, $m \in M$ with $g \not\!I m$, i.e., $\gamma g \not\leq \mu m$. For $\mathfrak{x} := \mu m$ and $\mathfrak{y} := \mu m \vee \gamma g$ there is a concept $\mathfrak{t}$ which is maximal with respect to $\mathfrak{t} \geq \mathfrak{x}$, $\mathfrak{t} \not\geq \mathfrak{y}$. On account of this property of maximality $\mathfrak{t}$ must be $\bigwedge$-irreducible. Hence there is an attribute n with $\mathfrak{t} = \mu n$. Thus we obtain $g \nearrow n$ and $m' \subseteq n'$. The second condition is obtained dually.

If $(V, V, \leq)$ is doubly founded, so must be $\mathbf{V}$: If $x < y$ in $\mathbf{V}$, then certainly $y \not\leq x$, i.e., $y \not\!I x$ in $(V, V, \leq)$. From the definition of the doubly foundedness of $(V, V, \leq)$ now follows the existence of an element $s \in V$ with $s \swarrow x$ and $y' \subseteq s'$, hence s is minimal with respect to $s \not\leq x$, $s \leq y$. The second condition can again be shown by means of the dual argument. □

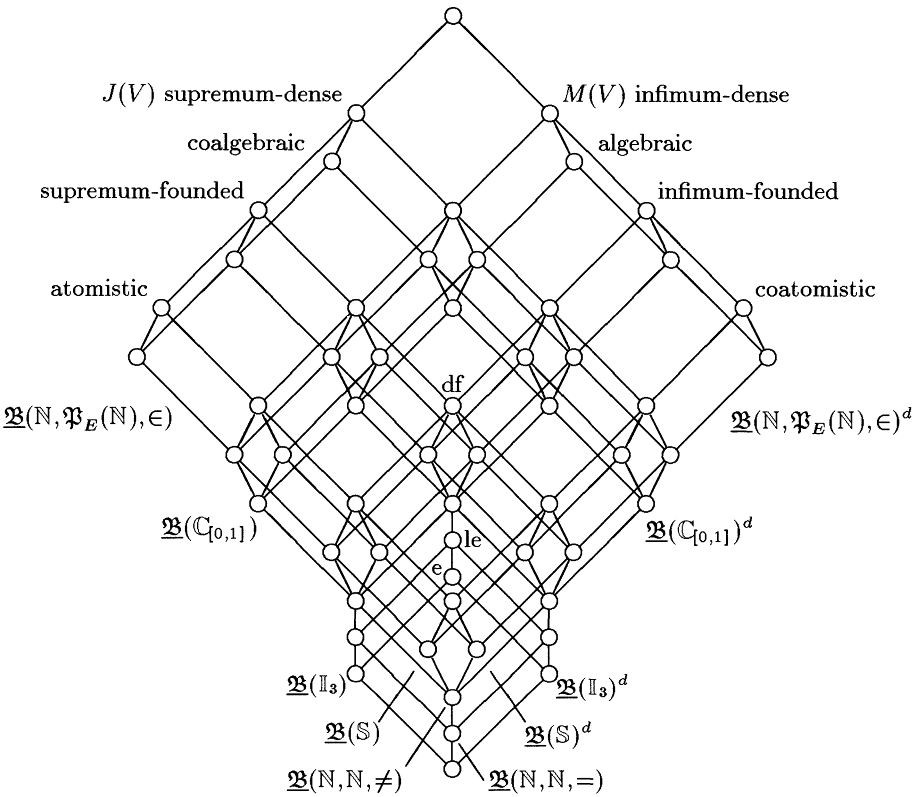
Thus, a complete lattice $\mathbf{V}$ is doubly founded if and only if *every* context (G, M, I) with $\mathbf{V} \cong \mathfrak{B}(G, M, I)$ is doubly founded. One should note, however, that the concept lattice of a doubly founded context does not necessarily have to be doubly founded, as shown by the example $(\mathbb{N}, \mathbb{N}, \leq)$.

It frequently occurs that a statement can be proved for all finite lattices but not for all complete lattices. We will (when possible) replace the condition of finiteness by "doubly foundedness". This is not in every case the strongest possible relaxation. The restriction to "doubly founded" is adopted for reasons of uniformity. Mathematical lattice theory uses numerous other conditions, some of which are represented by Figure 1.11 in their hierarchical order. We will only give a short explanation of the terminology used in this context: A complete lattice $(V, \leq)$ is **supremum-founded**, if for any two elements $x < y$ from V there is an element $s \in V$ which is minimal with respect to $s \leq y$, $s \not\leq x$. The dual property is "**infimum-founded**". A concept lattice $\mathfrak{B}(G, M, I)$ is **algebraic** (dually: **co-algebraic**), if for every subset $A \subseteq G$ from

$$E \subseteq A \Rightarrow E'' \subseteq A \text{ for every finite subset } E$$

it follows that $A = A''$. An ordered set is **chain finite**, if every chain contained by it is finite.

The lattice presented in Figure 1.11 is the result of an attribute exploration in accordance with Section 2.3, i.e., the represented implications between the properties are really provable. We will omit the proofs.



The abbreviations stand for: df := doubly founded, le := chain-finite, e := finite. $\mathfrak{P}_E(\mathbb{N})$ denotes the set of all finite subsets of the natural numbers. Furthermore, let $\mathbb{S}$ be the context arising as a subposition (see Section 1.4) of the contexts $(\mathbb{N},\mathfrak{P}_E(\mathbb{N}),\in)$ and $(\mathfrak{P}_E(\mathbb{N}),\mathfrak{P}_E(\mathbb{N}),=)$. $\mathbb{C}_{[0,1]}$ is the convex-ordinal scale for the real unit interval $([0,1],\leq)$, as defined in Section 1.4.

Figure 1.11 Foundedness compared with related conditions.

1.3 Many-valued Contexts

In standard language the word "attribute" is not only used for properties which an object may or may not have. Attributes such as "colour", "weight", "sex", "grade" have *values.* We call them *many-valued attributes,* in contrast to the *one-valued attributes* considered so far.

Definition 27. A **many-valued context** (G, M, W, I) consists of sets G, M and W and a ternary relation I between G, M and W (i.e., $I \subseteq G \times M \times W$) for which it holds that

$$(g, m, w) \in I \text{ and } (g, m, v) \in I \quad \text{always imply} \quad w = v.$$

The elements of G are called **objects**, those of M **(many-valued) attributes** and those of W **attribute values.**

$(g, m, w) \in I$ we read as "the attribute m has the value w" for the object g. (G, M, W, I) is called a n-**valued context**, if W has n elements. The many-valued attributes can be regarded as partial maps from G in W. Therefore, it seems reasonable to write $m(g) = w$ instead of $(g, m, w) \in I$. The **domain** of an attribute m is defined to be

$$\text{dom}(m) := \{g \in G \mid (g, m, w) \in I \text{ for some } w \in W\}.$$

The attribute m is called **complete**, if $\text{dom}(m) = G$. A many-valued context is **complete**, if all its attributes are complete. ◇

Like the **one-valued** contexts treated so far, many-valued contexts can be represented by tables, the rows of which are labelled by the objects and the columns labelled by the attributes. The entry in row g and column m then represents the attribute value $m(g)$. If the attribute m does not have a value for the object g, there will be no entry.[6]

Example 5. The many-valued context represented in the upper part of Figure 1.13 shows a comparison of the different possibilities of arranging the engine and the drive chain of a motorcar (cf. Figure 1.12).

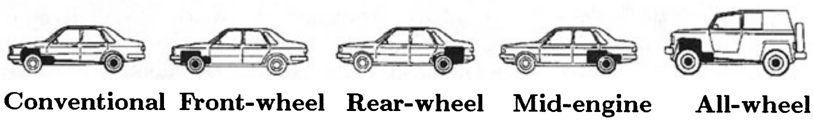


Figure 1.12 Drive concepts for motorcars.[7]

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[6] Further information on the role of the "empty cells" in a context will be given in the notes at the end of the chapter.

[7] Source: *Schlag nach! 100 000 Tatsachen aus allen Wissensgebieten.* BI-Verlag Mannheim, 1982.

How can we assign concepts to a many-valued context? We do this in the following way: The many-valued context is transformed into a one-valued one, in accordance with certain rules, which will be explained below. The concepts of this *derived* one-valued context are then interpreted as the concepts of the many-valued context. This interpretation process, however, called **conceptual scaling**, is not at all uniquely determined. The concept system of a many-valued context depends on the scaling. This may at first be confusing, but has proved to be an excellent instrument for a purposeful evaluation of data.

In the process of scaling, first of all each attribute of a many-valued context is interpreted by means of a context. This context is called *conceptual scale*.

Definition 28. A **scale** for the attribute m of a many-valued context is a (one-valued) context $\mathbb{S}_m := (G_m, M_m, I_m)$ with $m(G) \subseteq G_m$. The objects of a scale are called **scale values**, the attributes are called **scale attributes**. $\diamond$

Every context can be used as a scale. Formally there is no difference between a scale and a context. However, we will use the term "scale" only for contexts which have a clear conceptual structure and which bear meaning. Some particularly simple contexts are used as scales time and again. A summary (in tabular form) of the most important ones can be found at the end of the next section.

As already mentioned, the choice of the scale for the attribute m is not mathematically compelling, it is a matter of interpretation. The same is true for the second step in the process of **scaling**, the joining together of the scales to make a one-valued context. In the simplest case, this can be achieved by putting together the individual scales without connecting them. This is described below as *plain scaling.* Particularly when dealing with numerical scales this may well be unsatisfactory. In this case we need the scaling by means of a *composition operator.* For details we refer to the pointers at the end of the chapter.

In the case of **plain scaling** the derived one-valued context is obtained from the many-valued context (G, M, W, I) and the scale contexts $\mathbb{S}_m$, $m \in M$ as follows: The object set G remains unchanged, every many-valued attribute m is replaced by the scale attributes of the scale $\mathbb{S}_m$. If we imagine a many-valued context as represented by a table, we can visualize plain scaling as follows: Every attribute value $m(g)$ is replaced by the row of the scale context $\mathbb{S}_m$ which belongs to $m(g)$. A detailed description will be given in the following definition, for which we first introduce an abbreviation: The attribute set of the derived context is the disjoint union of the attribute sets of the scales involved. In order to make sure that the sets are disjoint, we replace the attribute set of the scale $\mathbb{S}_m$ by

$$\dot{M}_m := \{m\} \times M_m.$$

as in Definition 8 (p. 4).

Definition 29. If (G, M, W, I) is a many-valued context and $\mathbb{S}_m$, $m \in M$ are scale contexts, then the **derived context with respect to plain scaling** is the context (G, N, J) with

$$N := \dot{\bigcup_{m \in M}} M_m,$$

and

$$gJ(m, n) :\Longleftrightarrow m(g) = w \text{ and } wI_m n.$$

◇

Example 6. We obtain the one-valued context in Figure 1.13 as the derived context of the many-valued context presented above it, if we use the following scales:

$\mathbb{S}_{\mathrm{De}} := \mathbb{S}_{\mathrm{Dl}} :=$

	++	+	−
++	×	×	
+		×	
−			×

$\mathbb{S}_{\mathrm{R}} :=$

	++	+	−−
++	×	×	
+		×	
−−			×

$\mathbb{S}_{\mathrm{S}} :=$

	u	o	n	u/n
u	×			
o		×		
n			×	
u/n				×

$\mathbb{S}_{\mathrm{E}} := \mathbb{S}_{\mathrm{M}} :=$

	++	+	−	−−
++	×	×		
+		×		
−			×	
−−			×	×

$\mathbb{S}_{\mathrm{C}} :=$

	vl	l	m	h
vl	×	×		
l		×		
m			×	
h				×

If we had used the scale $\mathbb{S}_E$ for the attributes *De*, *Dl* and *R* as well, the derived context would have only turned out slightly different. The concept lattice is shown in Figure 1.14.

The formal definition of a context permits turning relations originating from any domain into contexts and examining their concept lattices, i.e., even contexts where an *interpretation* of the sets G and M as "objects" or "attributes" appears artificial. This is the case with many contexts from mathematics, and in this way we obtain concept lattices which often have structural properties occurring very rarely with empirical data sets. Nevertheless, these contexts are also of great importance for data analysis. They can be used for example as "ideal structures" or as scales for the scaling introduced above. The scales which are used by far most frequently, the *elementary scales* will be introduced now. Other scales will follow in the next section.

We will start with the definition of some operations which permit the construction of new contexts from given ones.

	De	Dl	R	S	E	C	M
Conventional	poor	good	good	understeering	good	medium	excellent
Front-wheel	good	poor	excellent	understeering	excellent	very low	good
Rear-wheel	excellent	excellent	very poor	oversteering	poor	low	very poor
Mid-engine	excellent	excellent	good	neutral	very poor	low	very poor
All-wheel	excellent	excellent	good	understeering/neutral	good	high	poor

	De			Dl			R			S				E				C				M			
	++	+	-	++	+	-	++	+	- -	u	o	n	u/n	++	+	-	- -	vl	l	m	h	++	+	-	- -
Conventional			×		×			×		×					×					×		×	×		
Front-wheel		×				×	×	×		×				×	×			×	×				×		
Rear-wheel	×	×		×	×				×		×					×			×				×		
Mid-engine	×	×		×	×			×				×				×	×		×					×	×
All-wheel	×	×		×	×			×					×		×						×			×	

De := drive efficiency empty; Dl := drive efficiency loaded; R := road holding/handling properties; S := self-steering effect; E := economy of space; C := cost of construction; M := maintainability
++ := excellent; + := good; - := poor; - - := very poor; u := understeering;
o := oversteering; n := neutral; vl := very low; l := low, m := medium; h := high.

Figure 1.13 A many-valued context: Drive concepts for motorcars. Below a derived one-valued context.

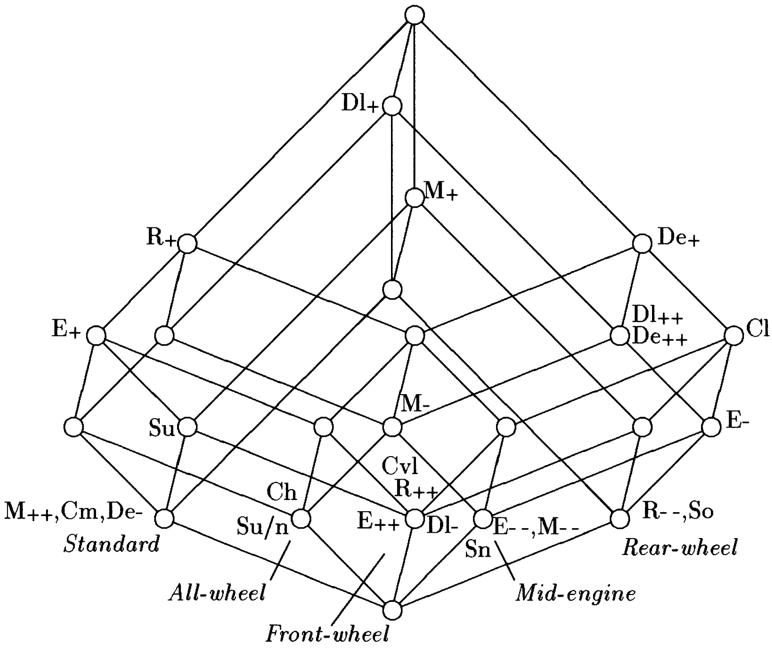


Figure 1.14 Concept lattice for the context of drive concepts.

Definition 30. Let $\mathbb{K} := (G, M, I)$, $\mathbb{K}_1 := (G_1, M_1, I_1)$ and $\mathbb{K}_2 := (G_2, M_2, I_2)$ be contexts. We will use the abbreviations $\dot{G}_j := \{j\} \times G_j$, $\dot{M}_j := \{j\} \times M_j$ and $\dot{I}_j := \{((j, g), (j, m)) \mid (g, m) \in I_j\}$ for $j \in \{1, 2\}$ in the following definition. It is:

$$\mathbb{K}^c := (G, M, (G \times M) \setminus I)$$

the **complementary context** to $\mathbb{K}$,

$$\mathbb{K}^d := (M, G, I^{-1})$$

the **dual context** to $\mathbb{K}$,

and, if $G = G_1 = G_2$,

$$\mathbb{K}_1 \mid \mathbb{K}_2 := (G, \dot{M}_1 \cup \dot{M}_2, \dot{I}_1 \cup \dot{I}_2)$$

the **apposition** of $\mathbb{K}_1$ and $\mathbb{K}_2$,

as well as dually, if $M = M_1 = M_2$,

$$\frac{\mathbb{K}_1}{\mathbb{K}_2} := (\dot{G}_1 \cup \dot{G}_2, M, \dot{I}_1 \cup \dot{I}_2)$$

the **subposition** of $\mathbb{K}_1$ and $\mathbb{K}_2$.

$$\mathbb{K}_1 \dot{\cup} \mathbb{K}_2 := (\dot{G}_1 \cup \dot{G}_2, \dot{M}_1 \cup \dot{M}_2, \dot{I}_1 \cup \dot{I}_2)$$

is the **disjoint union** of $\mathbb{K}_1$ and $\mathbb{K}_2$.

The context $\mathbb{K}^{cd}$ is called the **contrary context** to $\mathbb{K}$. $\Diamond$

By using $\dot{G}_i$ for $\{i\} \times G_i$ and $\dot{M}_i$, respectively, we intend to make sure that the sets are disjoint. However, strictly speaking, apposition and subposition under this definition become non-associative. We will overlook this fact and tacitly identify the contexts

$$(\ \mathbb{K}_1 \mid \mathbb{K}_2\ ) \mid \mathbb{K}_3 \qquad \text{and} \qquad \mathbb{K}_1 \mid (\ \mathbb{K}_2 \mid \mathbb{K}_3\ )\ .$$

The same applies to the subposition, even to hybrid forms of the two operations. We do not distinguish between

$$\frac{\mathbb{K}_1 \mid \mathbb{K}_2}{\mathbb{K}_3 \mid \mathbb{K}_4} \qquad \text{and} \qquad \frac{\mathbb{K}_1}{\mathbb{K}_3} \left| \frac{\mathbb{K}_2}{\mathbb{K}_4} \right. .$$

The two abbreviations

$$\begin{aligned} \times &:= (G, M, G \times M) \\ \emptyset &:= (G, M, \emptyset) \end{aligned}$$

are occasionally used without further describing the sets G and M, if they are evident from the context. For example

$$\begin{array}{c|c} \mathbb{K}_1 & \times \\ \hline \emptyset & \mathbb{K}_2 \end{array}$$

denotes the context $(\dot{G}_1 \cup \dot{G}_2, \dot{M}_1 \cup \dot{M}_2, \dot{I}_1 \cup \dot{I}_2 \cup (\dot{G}_1 \times \dot{M}_2))$, the concept lattice of which is isomorphic to the **vertical sum** of the concept lattices $\underline{\mathfrak{B}}(\mathbb{K}_1)$ and $\underline{\mathfrak{B}}(\mathbb{K}_2)$ (provided that $\mathbb{K}_1$ does not contain a full column and $\mathbb{K}_2$ does not contain a full row, cf. 4.3).

Each extent of $\mathbb{K}_1 \ \dot{\cup}\ \mathbb{K}_2$, apart from the extent $\dot{G}_1 \cup \dot{G}_2$, is entirely contained in one of the sets $\dot{G}_i$. The corresponding applies to the intents. Therefore, the concept lattice $\boldsymbol{V} := \underline{\mathfrak{B}}(\mathbb{K}_1 \ \dot{\cup}\ \mathbb{K}_2)$ is a **horizontal sum**, i.e., it is the union $\boldsymbol{V} = \boldsymbol{V}_1 \ \dot{\cup}\ \boldsymbol{V}_2$ of two sublattices which only overlap in the smallest and the largest element: $\boldsymbol{V}_1 \cap \boldsymbol{V}_2 = \{0_V, 1_V\}$. Provided that there are no full rows or columns in $\mathbb{K}_1$ and $\mathbb{K}_2$, we have $\boldsymbol{V}_i \cong \underline{\mathfrak{B}}(\mathbb{K}_i)$ or, more generally, $\boldsymbol{V}_i = \underline{\mathfrak{B}}(\dot{G}_1 \cup \dot{G}_2, \dot{M}_1 \cup \dot{M}_2, \dot{I}_i)$.

In Definition 28 we postulated that the values of the many-valued attribute had to be the objects of the scale. In the following *standardized scale* we frequently use $\mathbf{n} := \{1, 2, \ldots, n\}$ as the object set. In this case, in order to scale a many-valued attribute, we first have to rename the objects. The appropriate definitions for the isomorphy of scales will be introduced later, in Chapter 7.3 (p. 258 ff.).

Definition 31 (elementary scales, see also Figure 1.15**)**

Nominal scales. $\mathbb{N}_n := (\mathbf{n}, \mathbf{n}, =)$.

Nominal scales are used to scale attributes, the values of which mutually exclude each other. If an attribute for example has the values {*masculine, feminine, neuter*}, the use of a nominal scale suggests itself. We thereby obtain a *partition* of the objects into extents. In this case, the classes correspond to the values of the attribute.

	1	2	3	4
1	×			
2		×		
3			×	
4				×

The Nominal Scale $\mathbb{N}_4$.

(One-dimensional) ordinal scales. $\mathbb{O}_n := (\mathbf{n}, \mathbf{n}, \leq)$.

$\mathbb{O}_4 =$

	1	2	3	4
1	×	×	×	×
2		×	×	×
3			×	×
4				×

Ordinal scales scale many-valued attributes, the values of which are ordered and where each value implies the weaker ones. If an attribute has for instance the values {*loud, very loud, extremely loud*} ordinal scaling suggests itself. The attribute values then result in a chain of extents, interpreted as a hierarchy.

(One-dimensional) interordinal scales. $\mathbb{I}_n := \ (\mathbf{n}, \mathbf{n}, \leq) \mid (\mathbf{n}, \mathbf{n}, \geq)$.

$\mathbb{I}_4 =$

	≤ 1	≤ 2	≤ 3	≤ 4	≥ 1	≥ 2	≥ 3	≥ 4
1	×	×	×	×	×			
2		×	×	×	×	×		
3			×	×	×	×	×	
4				×	×	×	×	×

Questionnaires often offer opposite pairs as possible answers, as for example *active–passive*, *talkative–taciturn* etc., allowing a choice of intermediate values. In this case, we have a *bipolar* ordering of the values. This kind of attributes lend themselves to scaling by means of an interordinal scale. The extents of the interordinal scale are precisely the intervals of values, in this way, the betweenness relation is reflected conceptually. However, bipolar attributes often also lend themselves to *biordinal* scaling:

Biordinal scales. $\mathbb{M}_{n,m} := (\mathbf{n}, \mathbf{n}, \leq) \dot{\cup} (\mathbf{m}, \mathbf{m}, \geq)$.

$\mathbb{M}_{4,2} =$

	≤ 1	≤ 2	≤ 3	≤ 4	≥ 5	≥ 6
1	×	×	×	×		
2		×	×	×		
3			×	×		
4				×		
5					×	
6					×	×

In common usage we often use opposite pairs not in the sense of an interordinal scale, but simpler: each object is assigned one of the two poles, allowing graduations. The values {*very low, low, loud, very loud*} for example suggest this way of scaling: *loud* and *low* mutually exclude each other, *very loud* implies *loud*, *very low* implies *low*. We also find this kind of *partition with a hierarchy* in the names of the school marks: An excellent performance obviously is also very good, good, and satisfactory, but not unsatisfactory or a fail.

The **dichotomic scale.** $\mathbb{D} := (\{0,1\}, \{0,1\}, =)$
The dichotomic scale constitutes a special case, since it is isomorphic to the scales $\mathbb{N}_2$ and $\mathbb{M}_{1,1}$ and closely related to $\mathbb{I}_2$. It is frequently used to scale attributes with values of the kind {*yes, no*}.

	0	1
0	×	
1		×

A special case of plain scaling which frequently occurs is the case that all many-valued attributes can be interpreted with respect to the same scale or family of scales. Thus we speak of a **nominally scaled context**, if all scales $\mathbb{S}_m$ are nominal scales etc. We call a many-valued context **nominal**, if the nature of the data suggests nominal scaling; a many-valued context is called an **ordinal context** if for each attribute the set of values is ordered in a natural way. An example is presented in Figure 1.16, see also Figure 1.17.

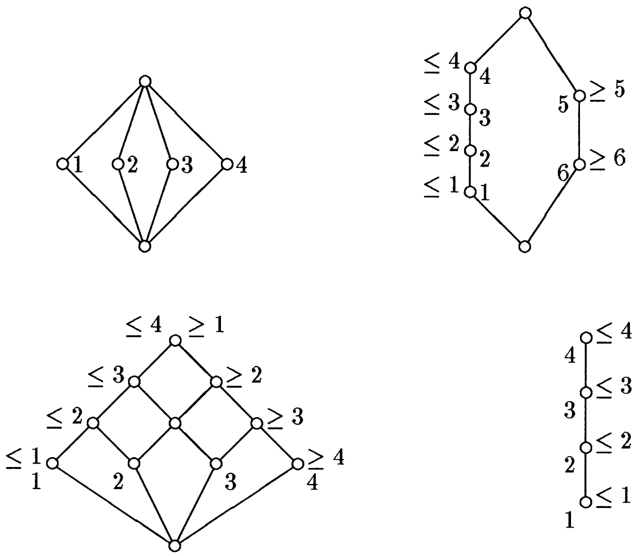


Figure 1.15 The concept lattices of the elementary scales are named after the scales. The figure shows a **nominal lattice**, $\underline{\mathfrak{B}}(\mathbb{N}_4)$, a **biordinal lattice**, $\underline{\mathfrak{B}}(\mathbb{M}_{4,2})$, an **interordinal lattice**, $\underline{\mathfrak{B}}(\mathbb{I}_4)$, and an **ordinal lattice**, $\underline{\mathfrak{B}}(\mathbb{O}_4)$. The ordinal lattice $\underline{\mathfrak{B}}(\mathbb{O}_n)$ is isomorphic to the n-element **chain** C_n.

Forum Romanum		B	GB	M	P
1	Arch of Septimus Severus	*	*	**	*
2	Arch of Titus	*	**	**	
3	Basilica Julia			*	
4	Basilica of Maxentius	*			
5	Phocas column		*	**	
6	Curia				*
7	House of the Vestals			*	
8	Portico of Twelve Gods		*	*	*
9	Tempel of Antonius and Fausta	*	*	***	*
10	Temple of Castor and Pollux	*	**	***	*
11	Temple of Romulus		*		
12	Temple of Saturn			**	*
13	Temple of Vespasian			**	
14	Temple of Vesta		**	**	*

Figure 1.16 Example of an ordinal context: Ratings of monuments on the Forum Romanum in different travel guides (B = Baedecker, GB = Les Guides Bleus, M = Michelin, P = Polyglott). The context becomes ordinal through the number of stars awarded. If no star has been awarded, this is rated zero.

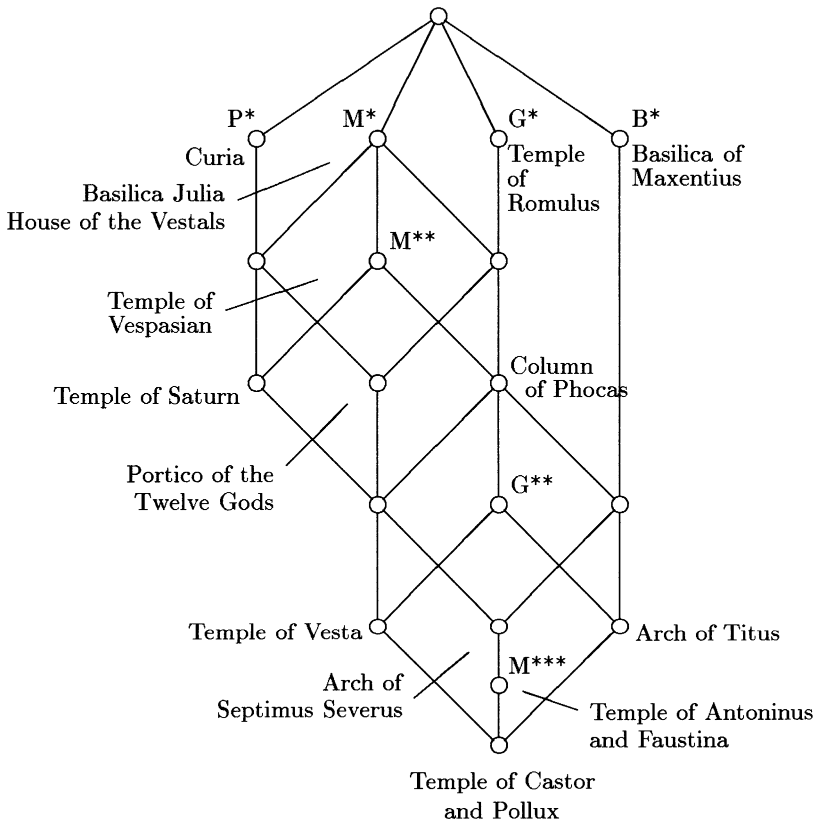


Figure 1.17 The concept lattice of the ordinal context from Figure 1.16.

1.4 Context Constructions and Standard Scales

We have formulated the following frequently used sum and product constructions for two contexts each, but the definitions can be easily generalized to any number of contexts. The additional statements on the concept lattices of the resulting contexts carry over.

Definition 32. The **direct sum** of two contexts is defined by[8]

$$\mathbb{K}_1 + \mathbb{K}_2 := (\dot{G}_1 \cup \dot{G}_2, \dot{M}_1 \cup \dot{M}_2, \dot{I}_1 \cup \dot{I}_2 \cup (\dot{G}_1 \times \dot{M}_2) \cup (\dot{G}_2 \times \dot{M}_1))$$

◇

The concept lattice of a sum of contexts is isomorphic to the product of its concept lattices. In the case of two contexts we therefore obtain

$$\underline{\mathfrak{B}}(\mathbb{K}_1 + \mathbb{K}_2) \cong \underline{\mathfrak{B}}(\mathbb{K}_1) \times \underline{\mathfrak{B}}(\mathbb{K}_2),$$

since (A, B) is a concept of $\mathbb{K}_1 + \mathbb{K}_2$ if and only if $(A \cap \dot{G}_i, B \cap \dot{M}_i)$ is a concept of $\mathbb{K}_i := (\dot{G}_i, \dot{M}_i, \dot{I}_i)$, for $i \in \{1, 2\}$. This means that the isomorphism is given by $(A, B) \mapsto ((A \cap \dot{G}_1, B \cap \dot{M}_1), (A \cap \dot{G}_2, B \cap \dot{M}_2))$.

Definition 33. The **semiproduct** is defined by

$$\mathbb{K}_1 \between \mathbb{K}_2 := (G_1 \times G_2, \dot{M}_1 \cup \dot{M}_2, \nabla)$$

with

$$(g_1, g_2) \nabla (j, m) :\Longleftrightarrow g_j I_j m \qquad \text{for } j \in \{1, 2\}.$$

◇

The extents of the semiproduct are precisely the sets of the form $A_1 \times A_2$, each set A_j being an extent of $\mathbb{K}_j$. This also yields the structure of the concept lattice $\underline{\mathfrak{B}}(\mathbb{K}_1 \between \mathbb{K}_2)$: Essentially, the concept lattice is the product of the concept lattices of the factor contexts, though there is a modification regarding the zero elements. Precisely, the instruction for the construction reads as follows: Provided that the extent of the corresponding concept is empty, we remove the zero element from each of the extents $\underline{\mathfrak{B}}(\mathbb{K}_j)$. Then we form the product of these ordered sets and, if we have previously removed an element, we add a new zero element to make a complete lattice. This lattice is then isomorphic to the concept lattice of the semiproduct.

Definition 34. The **direct product** is given by

$$\mathbb{K}_1 \times \mathbb{K}_2 := (G_1 \times G_2, M_1 \times M_2, \nabla)$$

$$\text{with } (g_1, g_2) \nabla (m_1, m_2) :\Longleftrightarrow g_1 I_1 m_1 \text{ or } g_2 I_2 m_2.$$

◇

[8] For the notation see Definition 8 (p. 4). A more general definition is given in Section 5.1.

The concept lattice of the direct product is called the *tensor product* of the concept lattices of the factor contexts. We will later discuss the tensor product in more detail (Sections 4.4, 5.4). We obtain the cross table of the direct product by replacing each empty cell in the table of $\mathbb{K}_1$ by a copy of $\mathbb{K}_2$ and each cross by a square full of crosses of the size of $\mathbb{K}_2$. For an example see Figures 4.19 (page 164) and 4.20.

Another context construction, the so-called **substitution sum**, where a context is inserted into an other context, will be described in section 4.3. The sum and the product of reduced contexts are reduced (cf. Corollary 74, p. 166). Reducible objects or attributes with empty intents or extents may occur in the case of the disjoint union. Semi products of reduced contexts are reduced if the factors (allowing for one exception at most) are **atomistic**, i.e., if they satisfy $g' \subseteq h' \Rightarrow g = h$.

It is easy to state numerous simple arithmetical rules for context constructions, which are useful for some proofs. In particular, the direct product is (up to isomorphism) commutative and associative; it is distributive over the direct sum, the apposition and the subposition. We note down one of these results for later:

Proposition 16.

$$(\mathbb{K}_1 + \mathbb{K}_2) \times \mathbb{K}_3 = (\mathbb{K}_1 \times \mathbb{K}_3) + (\mathbb{K}_2 \times \mathbb{K}_3).$$

Proof. We may assume that the three contexts $\mathbb{K}_i =: (G_i, M_i, I_i)$, $i \in \{1, 2, 3\}$, have disjoint object sets and disjoint attribute sets. By

$$(G_1 \cup G_2) \times G_3 = (G_1 \times G_3) \cup (G_2 \times G_3)$$

and

$$(M_1 \cup M_2) \times M_3 = (M_1 \times M_3) \cup (M_2 \times M_3),$$

the two contexts of the proposition have the same objects and attributes. For the incidence we find the same on both sides as well, namely

$$(g,h)I(m,n) \iff \begin{cases} g \in G_1 \text{ and } m \in M_2 & \text{or} \\ g \in G_2 \text{ and } m \in M_1 & \text{or} \\ hI_3n & \text{or} \\ g \in G_1, m \in M_1 \text{ and } gI_1m & \text{or} \\ g \in G_2, m \in M_2 \text{ and } gI_2m. & \end{cases} \qquad \square$$

We now state a list of interesting context families. Many of them have proved to be useful as scales. We provide a summary of these scales, including their basic meanings, in Figure 1.26 at the end of this section. Besides, these contexts serve as a reservoir of examples for mathematical reasoning.

(1) For every set S the **contranominal scale**

$$\mathbb{N}^c_S := (S, S, \neq)$$

is reduced. The concepts of this context are precisely the pairs $(A, S\setminus A)$ for $A \subseteq S$. The concept lattice is isomorphic to the power-set lattice of S, and thus has $2^{|S|}$ elements. If $S = \{1, 2, \ldots, n\}$ we write $\mathbb{N}^c_n$.

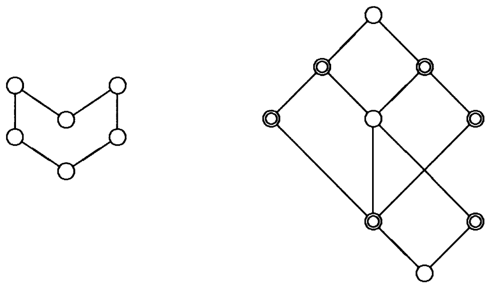


Figure 1.18 Example of an ordered set $(P, \leq)$ and its completion $\underline{\mathfrak{B}}(P, P, \leq)$.

(2) From an arbitrary ordered set $\boldsymbol{P} := (P, \leq)$ we obtain the **general ordinal scale**

$$\mathbb{O}_{\mathbf{P}} := (P, P, \leq).$$

Its concepts are precisely the pairs (X, Y) with $X, Y \subseteq P$ where X is the set of all lower bounds of Y and Y is the set of all upper bounds of X. This concept lattice is called the **Dedekind-MacNeille completion** of the ordered set $\boldsymbol{P}$. It is the smallest complete lattice in which $\boldsymbol{P}$ can be order-embedded, in the sense of the following theorem:

Theorem 4. (Dedekind's Completion Theorem) *For an ordered set* $(P, \leq)$

$$\iota x := ((x], [x)) \qquad \textit{for } x \in P$$

defines an embedding ι *of* $(P, \leq)$ *in* $\underline{\mathfrak{B}}(P, P, \leq)$*; moreover,* $\iota \bigvee X = \bigvee \iota X$ *or* $\iota \bigwedge X = \bigwedge \iota X$ *if the supremum or infimum of* X*, respectively, exists in* $(P, \leq)$*. If* κ *is an arbitrary embedding of* $(P, \leq)$ *in a complete lattice* V*, then there is always also an embedding* λ *of the ordered set* $\underline{\mathfrak{B}}(P, P, \leq)$ *in* V *with* $\kappa = \lambda \circ \iota$.

Proof. Evidently, the concepts of $(P, P, \leq)$ are precisely the pairs (A, B) with $A, B \subseteq P$ and

$$\begin{aligned} A = B^{\downarrow} &:= \{x \in P \mid x \leq y \text{ for all } y \in B\}, \\ B = A^{\uparrow} &:= \{y \in P \mid x \leq y \text{ for all } x \in A\}; \end{aligned}$$

in particular, all pairs $((x], [x))$ with $x \in P$ are concepts of $(P, P, \leq)$, which confirms ι as an embedding. If the supremum of X exists in $(P, \leq)$, then

$$\left[\bigvee X\right) = \bigcap_{x \in X} [x),$$

i.e., $\iota \bigvee X =$

$$= \left(\left(\bigvee X\right], \left[\bigvee X\right)\right) = \left(\left(\bigcap_{x \in X} [x)\right)^{\downarrow}, \bigcap_{x \in X} [x)\right) = \bigvee ((x], [x)) = \bigvee \iota X.$$

The equation for existing infima is shown dually.

With respect to the missing part of the proof we refer to Proposition 33 (p. 99). □

(3) From an arbitrary ordered set $\boldsymbol{P} := (P, \leq)$ we furthermore obtain the reduced context

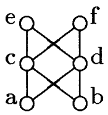
$$\mathbb{O}^{cd}_{\mathbf{P}} := (P, P, \not\geq),$$

which is called the **contraordinal scale**. In this case, the concepts are precisely the pairs (X, Y) with the following properties:
- $X \cup Y = P$ and $X \cap Y = \emptyset$,
- X is an **order ideal** in $\boldsymbol{P}$, i.e., $x \in X$ and $z \leq x$ always imply $z \in X$. Because of $X \cup Y = P$ and $X \cap Y = \emptyset$ this is equivalent to:
- Y is an **order filter** in $\boldsymbol{P}$, i.e., $y \in Y$ and $y \leq z$ always imply $z \in Y$.

The context $(P, P, \not\geq)$ is doubly founded, since

$$x \swarrow y \iff x \nearrow y \iff x = y$$

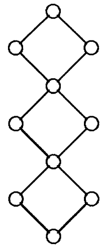
holds for $x, y \in P$. Hence if x is an object and y is an attribute with $x \not I y$ (i.e., $x \geq y$), then $x \nearrow x$ and $x' = P \setminus [x) \supset P \setminus [y) = y'$, hold for the *attribute* x, as required by Definition 26.
The concept lattice $\underline{\mathfrak{B}}(P, P, \not\geq)$ is isomorphic to the lattice of the order ideals of $\boldsymbol{P}$. A look at (1) shows that all concepts of the contraordinal scale are concepts of the contranominal scale $\mathbb{N}^c_P$ as well. We will prove later (Theorem 13, p. 112) that for this reason $\underline{\mathfrak{B}}(P, P, \not\geq)$ is a complete sublattice of $\underline{\mathfrak{B}}(P, P, \neq)$, which means that these lattices are completely distributive. Birkhoff's theorem (Theorem 39, p. 220) shows that the lattices constructed in this way, are precisely the doubly founded completely distributive lattices. In particular, every finite distributive lattice is isomorphic to the concept lattice of a contraordinal scale. The dual lattice, i.e., $\underline{\mathfrak{B}}(P, P, \not\leq)$, is often denoted by $2^{\mathbf{P}}$, because it is also isomorphic to the lattice of the order-preserving maps of $\boldsymbol{P}$ to the two-element lattice.



$(P, \leq)$

	a	b	c	d	e	f
a		×	×	×	×	×
b	×		×	×	×	×
c				×	×	×
d			×		×	×
e						×
f					×	

$(P, P, \not\geq)$



$\underline{\mathfrak{B}}(P, P, \not\geq)$

Figure 1.19 An ordered set $(P, \leq)$, the corresponding contraordinal scale and its concept lattice, i.e., the ideal lattice of $(P, \leq)$.

(4) We obtain an interesting special case of (3) by choosing the power-set of a set S as our ordered set $\boldsymbol{P}$, i.e., by considering the context

$$(\mathfrak{P}(S), \mathfrak{P}(S), \not\supseteq).$$

Because of $A \not\supseteq B \iff B \cap (S \setminus A) \neq \emptyset$, this context is isomorphic to

$$(\mathfrak{P}(S), \mathfrak{P}(S), \Delta) \quad \text{with} \quad X \Delta Y :\iff (X \cap Y) \neq \emptyset.$$

The concept lattice is called the **free completely distributive lattice** FCD(S). If for $S := \{1, 2, \ldots, n\}$ we denote the context $(\mathfrak{P}(S), \mathfrak{P}(S), \not\supseteq)$ by $\mathbb{A}_n$, we can state an easy recursion rule for the generation of these contexts:

$$\mathbb{A}_0 = \boxed{\emptyset} \quad \text{and} \quad \mathbb{A}_{n+1} = \begin{array}{c|c} \mathbb{A}_n & \times \\ \hline \mathbb{A}_n & \mathbb{A}_n \end{array}.$$

The construction can be generalized by taking an ordered set $(S, \leq)$ as the base set, the set $\mathcal{OI}(S, \leq)$ of the order ideals of $(S, \leq)$ as the object set and the set $\mathcal{OF}(S, \leq)$ of the order filters of $(S, \leq)$ as the attribute set. The concept lattice

$$\mathrm{FCD}(S, \leq) := (\mathcal{OI}(S, \leq), \mathcal{OF}(S, \leq), \Delta)$$

is called the **free completely distributive lattice over the ordered set** $(S, \leq)$.

(5) For an arbitrary ordered set $(P, \leq)$, we define a **filter** to be a subset of P which is an order filter and in which furthermore any two elements have a common lower bound. Hence $F \subseteq P$ is a filter if and only if the following two conditions are satisfied:
1. From $x \in F$ and $y \geq x$ it follows that $y \in F$,
2. for any two elements $x, y \in F$ there is an $u \in F$ with $u \leq x$ and $u \leq y$.

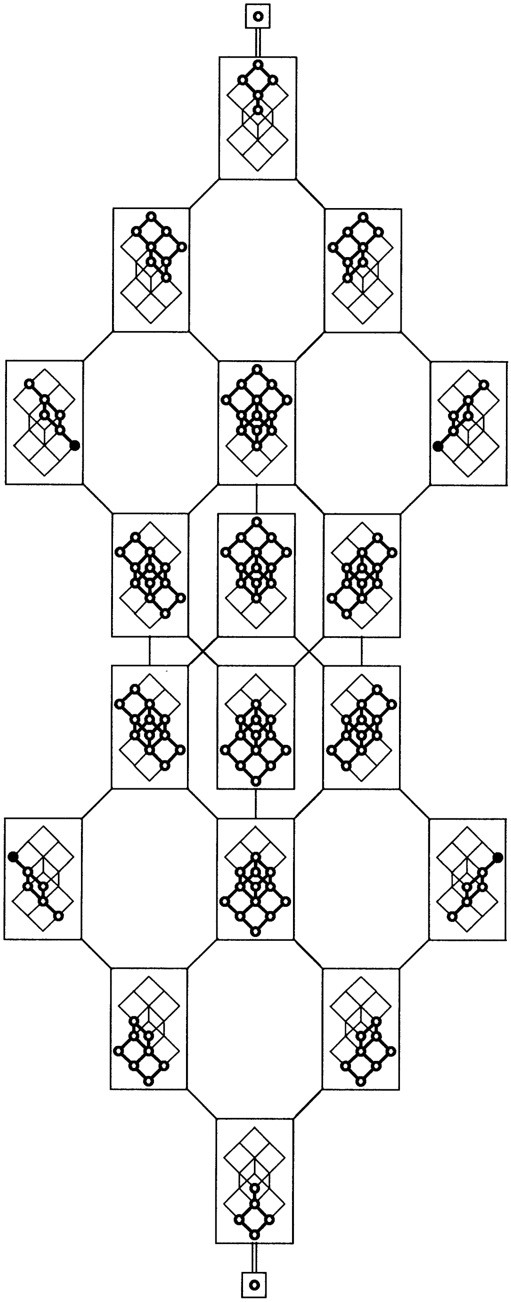


Figure 1.20 A nested line diagram of the free distributive lattice FCD(4). Such diagrams are introduced in 2.2. The one shown here is due to S. Thiele [175]. The method that led to it is explained in Example 14 (p. 215).

Dually, an **ideal** is defined to be a subset of P which is an order ideal and contains a common upper bound for any two elements contained in it. Filters in this sense are among other things the principal filters. Dually, each principal ideal is an ideal. The set of all filters is denoted by $\mathcal{F}(P,\leq)$, the set of all ideals by $\mathcal{I}(P,\leq)$. We obtain the doubly founded context

$$\mathbb{F}_{(P,\leq)} := (\mathcal{F}(P,\leq), \mathcal{I}(P,\leq), \Delta),$$

where again

$$F \Delta I :\iff F \cap I \neq \emptyset.$$

(6) Again from an ordered set $\boldsymbol{P} := (P, \leq)$ we obtain the **general interordinal scale**

$$\mathbb{I}_{\mathbf{P}} := (P, P, \leq) \mid (P, P, \geq),$$

the concept system of which we explain by means of the extents: the attribute extents are precisely the principal ideals and the principal filters of $\boldsymbol{P}$, the object extents are all intersections of those sets. These include all intervals[9]. In general, these are all sets which constitute *intersections of intervals.*

(7) By analogy with (6) we obtain the **convex-ordinal scale**

$$\mathbb{C}_{\mathbf{P}} := (P, P, \not\geq) \mid (P, P, \not\leq).$$

In this case, the extents are precisely the convex subsets of $\boldsymbol{P}$, i.e., those subsets which contain with any two elements a and b all elements c with $a \leq c \leq b$.

	1,a	1,b	1,c	1,d	1,e	1,f	2,a	2,b	2,c	2,d	2,e	2,f
a		×	×	×	×	×		×				
b	×		×	×	×	×	×					
c				×	×	×	×	×		×		
d			×		×	×	×	×	×			
e						×	×	×	×	×		×
f					×		×	×	×	×	×	

Figure 1.21 The convex-ordinal scale of the ordered set from Figure 1.19.

(8) Let S be a set and $s \in S$ an arbitrary element. If we now choose G to be the set of all two-element subsets of S and M to be the set of all subsets of $S \setminus \{s\}$, by the definition

$$\{x, y\} \diamond X :\Leftrightarrow |\{x, y\} \cap X| \neq 1$$

[9] in the sense of Definition 5 (p. 3), i.e., only the "closed" intervals.

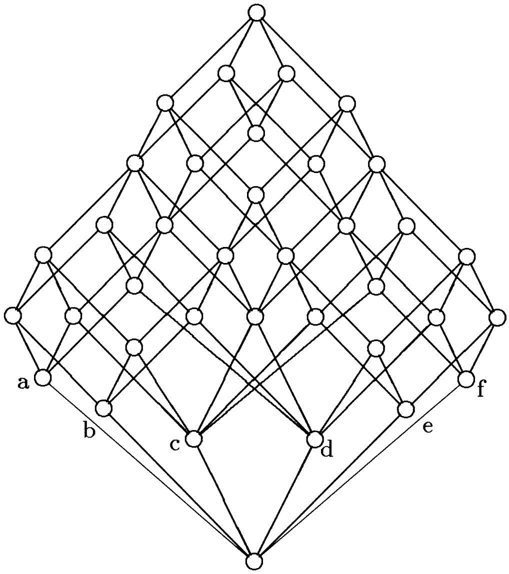


Figure 1.22 The concept lattice of the convex-ordinal scale from Figure 1.21.

we obtain a context $(G, M, \diamond)$ with $\binom{|S|}{2}$ objects and $2^{|S|-1}$ attributes, which is reduced except for one full column. Every extent of this context is a set of two-element subsets of S, i.e., it can be understood as a symmetric reflexive relation on S; actually, the relations occurring are precisely the equivalence relations on S. Hence the concept lattice $\underline{\mathfrak{B}}(G, M, \diamond)$ is isomorphic to the lattice $\mathfrak{E}(S)$ of equivalence relations. We can give a mnemonic rule for this context series as well. We get $\mathbb{P}_1 := (\emptyset, \{*\}, \emptyset)$ and obtain the $n+1$-st context of this series, $\mathbb{P}_{n+1}$, from the n-th as follows: We form the apposition of $\mathbb{P}_n$ with the cross table $\mathbb{P}_n^{\text{rev}}$, which is identical to $\mathbb{P}_n$, apart from the fact that the columns are written down in the reversed order.

$\mathbb{P}_n$	$\mathbb{P}_n^{\text{rev}}$
$2^n - 1 \quad \ldots \quad 2^{n-1}$	$2^{n-1} - 1 \quad \ldots \quad 0$

We add n further rows, which we fill with crosses such that the columns of this subcontext look like the binary representations of the numbers $2^n - 1, \ldots, 0$. An example is given in Figure 1.23.

(9) If R is a symmetric relation on S (easily visualized by the edges of an undirected graph) then with

10011001
32100123
76543210

$\longrightarrow$

×			×	×			×
×	×					×	×
×		×			×		×
×	×	×	×				
×	×			×	×		
×		×		×		×	

Figure 1.23 Context $\mathbb{P}_4$ for the lattice of equivalence relations on a 4-element set.

$$(S, S, R)$$

we obtain a context, the concepts of which are precisely the pairs (A, B), $A \subseteq S$, $B \subseteq S$, which are maximal with respect to the property that each element of A is in the relation R with each element of B (in the visualization these are maximal complete bipartite edge sets). Thus, together with (A, B), (B, A) is also a concept, and the map

$$(A, B) \mapsto (B, A)$$

is a **polarity**, i.e., an order-reversing bijection which is inverse to itself (another term for this is *involutory antiautomorphism*). Conversely, every complete **polarity lattice** (i.e., every complete lattice with a polarity) is isomorphic to the concept lattice of a context (S, S, R) with a symmetric relation R.
If the relation R is irreflexive, the extent and the intent of each concept must be disjoint and we have

$$(A, B) \wedge (B, A) = (\emptyset, \emptyset')$$

$$\text{and } (A, B) \vee (B, A) = (\emptyset', \emptyset),$$

i.e., (A, B) and (B, A) are **complementary** to each other: Their infimum is the smallest, their supremum the largest element of the concept lattice. A lattice with this kind of polarity is called an **ortholattice**; the complete ortholattices are (up to isomorphism) precisely the concept lattices of contexts with an irreflexive, symmetric relation.
There are many examples of such contexts in this book. They can be easily recognized if the cross table is represented symmetric to the main diagonal. The context $\mathbb{K}_{(2,3)}$ in Figure 1.24 is the context of a polarity lattice but not of an ortholattice. The same applies to the context in Figure 5.9 (p. 205), although this only becomes clear after an adroit reassembly of the cross table.

(10) If V is a finite dimensional vector space and V^* is the dual space of V, then

$$(V, V^*, \perp) \quad \text{with} \quad a \perp \varphi :\Longleftrightarrow \varphi a = 0$$

is a doubly founded context, the extents of which are precisely the subspaces of V.

For the special case of the vector spaces over $GF(2)$ there is again a simple recursion for the generation of these contexts: For

$$\mathbb{K}_{(d,2)} := (GF(2)^d, (GF(2)^d)^*, \perp)$$

it is easy to prove that

$$\mathbb{K}_{(d+1,2)} = \begin{array}{c|c} \mathbb{K}_{(d,2)} & \mathbb{K}_{(d,2)} \\ \hline \mathbb{K}_{(d,2)} & \mathbb{K}^c_{(d,2)} \end{array}.$$

An example is given in Figure 1.24.

×	×	×	×	×	×	×	×
×		×		×		×	
×	×			×	×		
×			×	×			×
×	×	×	×				
×		×			×		×
×	×					×	×
×			×		×	×	

Figure 1.24 $\mathbb{K}_{(3,2)}$, a context derived from the 3-dimensional vector space over the two-element field.

(11) If H is a Hilbert space and $\perp$ is the orthogonality relation, then the concept lattice of the context

$$(H, H, \perp)$$

is isomorphic to the (orthomodular) lattice of the closed subspaces of H; since $(U, U^\perp)$ is a concept for each such subspace U.

(12) The set of all permutations of the set $\{1, \ldots, n\}$ can be given a lattice order in a natural way. For this purpose we call a pair $(\varphi i, \varphi j)$ an **inversion** of the permutation φ if $i < j$ but $\varphi i > \varphi j$. If we order the permutations by

$$\sigma \leq \tau :\Longleftrightarrow \text{ every inversion of } \sigma \text{ is also an inversion of } \tau,$$

we obtain, as proved by Yanagimoto and Okamoto [217], a lattice Σ_n. There is a simple recursion rule for the description of the context: Putting

$$\mathbb{K}_0 := \mathbb{L}_0 := \boxed{\times} \qquad \text{and}$$

$$\mathbb{L}_{n+1} := \begin{array}{c|c} \emptyset & \mathbb{L}_n \\ \hline \mathbb{L}_n & \mathbb{L}_n \end{array}, \qquad \mathbb{K}_{n+1} := \begin{array}{c|c} \mathbb{K}_n & \mathbb{K}_n \\ \hline \mathbb{K}_n & \mathbb{L}_n \end{array},$$

then we obtain

$$\Sigma_n \cong \underline{\mathfrak{B}}(\mathbb{K}_n).$$

The contexts $\mathbb{K}_n$ are reduced except for the full rows and full columns. Σ_4 is presented in Figure 1.25.

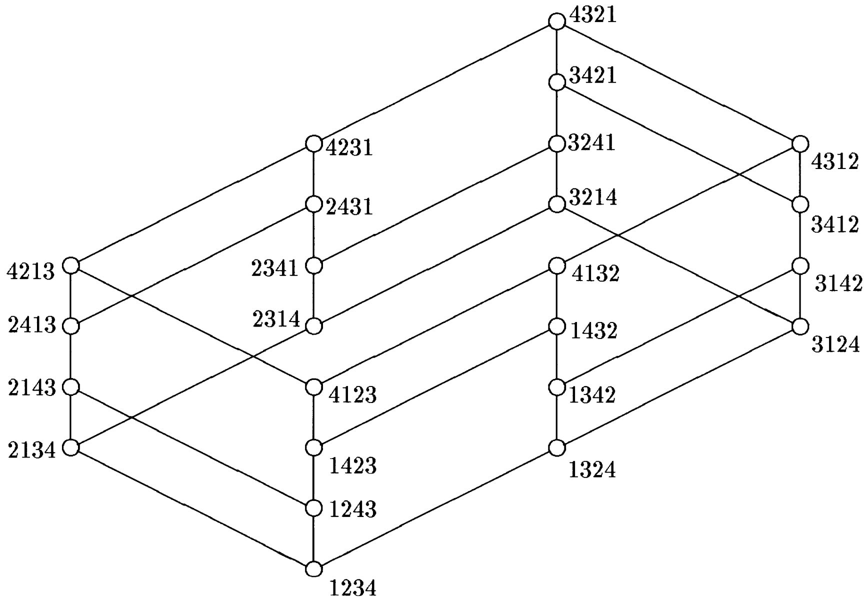


Figure 1.25 The lattice Σ_4 of the permutations of $\{1, 2, 3, 4\}$.

If the ordered sets occurring in the definitions for the standard scales are compounded, for example as a cardinal sum or as a direct product, it is to be expected that the respective scales can be split up. This is true, even if in different ways, as exemplified by the following rules:

Proposition 17.

$$\begin{aligned}
\mathbb{O}_{\mathbf{P}_1+\mathbf{P}_2} &= \mathbb{O}_{\mathbf{P}_1} \dot{\cup}\, \mathbb{O}_{\mathbf{P}_2} \\
\mathbb{I}_{\mathbf{P}_1+\mathbf{P}_2} &= \mathbb{I}_{\mathbf{P}_1} \dot{\cup}\, \mathbb{I}_{\mathbf{P}_2} \\
\mathbb{O}^{cd}_{\mathbf{P}_1+\mathbf{P}_2} &= \mathbb{O}^{cd}_{\mathbf{P}_1} + \mathbb{O}^{cd}_{\mathbf{P}_2} \\
\mathbb{C}_{\mathbf{P}_1+\mathbf{P}_2} &= \mathbb{C}_{\mathbf{P}_1} + \mathbb{C}_{\mathbf{P}_2} \\
\mathbb{O}^{cd}_{\mathbf{P}_1\times\mathbf{P}_2} &= \mathbb{O}^{cd}_{\mathbf{P}_1} \times \mathbb{O}^{cd}_{\mathbf{P}_2} \\
\mathbb{C}_{\mathbf{P}_1\times\mathbf{P}_2} &= \mathbb{O}^{cd}_{\mathbf{P}_1} \times \mathbb{O}^{cd}_{\mathbf{P}_2} \mid \mathbb{O}^{c}_{\mathbf{P}_1} \times \mathbb{O}^{c}_{\mathbf{P}_2}
\end{aligned}$$

Symbol	Definition	Name	Basic meaning
$\mathbb{O}_\mathbf{P}$	$(P, P, \leq)$	general ordinal scale	hierarchy
$\mathbb{O}_n$	$(\mathbf{n}, \mathbf{n}, \leq)$	one-dimensional ordinal scale	rank order
$\mathbb{N}_n$	$(\mathbf{n}, \mathbf{n}, =)$	nominal scale	partition
$\mathbb{M}_{n_1,\ldots,n_k}$	$\mathbb{O}_{\mathbf{n_1}+\cdots+\mathbf{n}_k}$	multiordinal scale	partition with rank orders
$\mathbb{M}_{m,n}$	$\mathbb{O}_{\mathbf{m}+\mathbf{n}}$	biordinal scale	two-class rank orders
$\mathbb{B}_n$	$(\mathfrak{P}(\mathbf{n}), \mathfrak{P}(\mathbf{n}), \subseteq)$	n-dimensional Boolean scale	dependency of attributes
$\mathbb{G}_{n_1,\ldots,n_k}$	$\mathbb{O}_{\mathbf{n_1}} \times \cdots \times \mathbb{O}_{\mathbf{n}_k}$	k-dimensional grid scale	multiple ordering
$\mathbb{O}_\mathbf{P}^{cd}$	$(P, P, \not\geq)$	contraordinal scale	hierarchy and independence
$\mathbb{N}_n^c$	$(\mathbf{n}, \mathbf{n}, \neq)$	contranominal scale	partition and independence
$\mathbb{D}$	$(\{0,1\}, \{0,1\}, =)$	dichotomic scale	dichotomy
$\mathbb{D}_k$	$\underbrace{\mathbb{D} \times \cdots \times \mathbb{D}}_{k-\text{times}}$	k-dimensional dichotomic scale	multiple dichotomy
$\mathbb{I}_\mathbf{P}$	$\mathbb{O}_\mathbf{P} \mid \mathbb{O}_\mathbf{P}^d$	general interordinal scale	betweenness relation
$\mathbb{I}_n$	$\mathbb{O}_n \mid \mathbb{O}_n^d$	one-dimensional interordinal scale	linear betweenness relation
$\mathbb{C}_\mathbf{P}$	$\mathbb{O}_\mathbf{P}^{cd} \mid \mathbb{O}_\mathbf{P}^c$	convex-ordinal scale	convex ordering

Figure 1.26 Standardized scales of ordinal type.

1.5 Hints and References

1.1 Formal Concept Analysis has been developed from the end of the seventies at the Faculty of Mathematics at Darmstadt University of Technology. The first programmatic publication on Formal Concept Analysis was

> Wille: *Restructuring lattice theory: An approach based on hierarchies of concepts* [191].

This publication contains already many of the ideas described in this book, including the proof of the Basic Theorem on Concept Lattices. There had been earlier proposals to make use of the mathematical possibilities offered by the results of Birkhoff (Theorem 2 on p. 14) for data analysis. The interpretation of the incidence relation as an object-attribute-relation was expressly mentioned in the first edition of Birkhoff's book on lattice theory. Remarkable approaches can be found in Barbut [8], see Barbut & Monjardet [9]. Some French authors therefore employ the term **treillis de Galois**, which was used in their works for "concept lattice" (German **Begriffsverband**).

The Darmstadt group was presumably the first, who systematically elaborated these possibilities into a method of data analysis and tested and further developed it in many applications. The decisive factor in the success of this work was among other things the formalization of "context" and the interpretation of "concept" as a *unity* of extension and intension.

The understanding of "concept" which is formalized here, has ramified and deep-reaching roots in philosophy, which are described in more detail elsewhere [210]. This tradition of thought finds expression even in the standards DIN 2330 and DIN 2331[10], which in turn were discussed by the Darmstadt group at the beginning of the development. Further information on the origin of Formal Concept Analysis and its intellectual background can be found in [209] and [42].

Quite certainly the mathematical substance of the Basic Theorem can be mainly attributed to Birkhoff [14], even if the second part has not been formulated there. This can be found - in an order-theoretic version - in J. Schmidt [151] and Banaschewski [5]. The general version presented in this book first appeared in [191]. It is not quite easy to attribute the intermediate steps to particular authors. The fact that a finite lattice is determined by its irreducibles was well known to lattice theorists. One source is Markowsky [122].

Generalizations of the model presented in this book have been discussed in several variants. The most important one in our view is the inclusion of many-valued contexts by means of conceptual scaling as introduced in 1.3 and 1.4. Lehmann and Wille [106] have outlined a **triadic concept analysis**, where the incidence relation is ternary and the concepts consist of three sets. The

[10] DIN stands for "Deutsche Industrienorm" and is characteristic for standards issued by the German National Bureau of Standards.

mathematical theory is at present still in its beginnings [211]. Umbreit [176] has examined in a comprehensive study how Formal Concept Analysis can be combined with the approaches of fuzzy logic. There are related elements in the work of Pawlak [134], Kent [94] and Burusco Juandeaburre & Fuentes-Gonzales [25]. Further approaches have been considered by Diday [39] and Marty [125]. A similar attempt at restructuring with respect to mathematical logic is made by [212].

Contexts with an additional structure, for example an additional operation, have also been examined. The respective concept lattices in this case carry additional structural properties as well. Examples are the polarity lattices and ortholattices introduced in 1.4. Generalizations can be found in Hoch [86]. Contexts with an algebraic structure have been examined by [178], [177], [179], [182] and by U. Wille [214], [215], contexts with a topological structure by Hartung [81], [82], [83] and contexts with a relational structure by Priß [137].

A pair (A,B) with $A \subseteq G$ and $B \subseteq M$ is called a **preconcept** of the context (G,M,I) if $A' \subseteq B$ and $B' \subseteq A$ (cf. [159]). If $A' = B$ *or* $B' = A$, this is called a **semiconcept** [116].

The example in Figure 1.1 has been taken from a pedagogical investigation, cf. Takács [172].

1.2 There is a simple way to assign a reduced context to every context $\mathbb{K} := (G,M,I)$:

$$\mathbb{K}^\circ := (G/\ker\gamma, M/\ker\mu, I^\circ),$$

the symbols having the following meanings: $\ker\gamma$ is the equivalence relation on G with

$$(g,h) \in \ker\gamma :\Longleftrightarrow \gamma g = \gamma h.$$

$\ker\mu$ is defined correspondingly. The equivalence classes of $\ker\gamma$ are the objects of $\mathbb{K}^\circ$, those of $\ker\mu$ the attributes. The incidence is defined by

$$([g]\ker\gamma, [m]\ker\mu) \in I^\circ :\Longleftrightarrow gIm.$$

The number of reduced contexts with four objects is 126, the number of reduced contexts with five objects is 13596. We do not know the further values. Even without the additional condition "reduced", it is not easy to determine the numbers. The following numbers have been calculated by the Bayreuth group around A. Kerber and R. Laue:

$\|G\|$ \ $\|M\|$	1	2	3	4	5
1	2	3	4	5	6
2	3	7	13	22	34
3	4	13	36	87	190
4	5	22	87	317	1053

Neither is it easy to determine the maximal possible number $f(n)$ of attributes in a reduced context with n objects. For small n we obtain

n	1	2	3	4	5
$f(n)$	1	2	4	7	13

.

Asymptotic results can be found in Kleitman [97].

The arrow relations have been introduced in [192] following the example of the *weak perspectivities* in congruence theory (cf. [75]). There were numerous forerunners. For example Day [33] already used the double arrow relation ("relation ρ") in order to characterize semidistributivity as well as a "relation C", which is closely related to the arrow relations, in order to describe the congruences of finite lattices. Doubly founded lattices have first been mentioned in [197]. Geyer [71] has examined possible configurations of the arrow relations.

1.3 Many valued contexts have been introduced already in [191]; of the numerous related models we would like to mention the *relational data bases* of Codd [26] but also the *information systems* of Pawlak [133] as well as the *Chu spaces* (cf. [136]). Their use in conceptual file and knowledge systems has been discussed in Vogt, Wachter & Wille [181], in Scheich, Skorsky, Vogt, Wachter & Wille [150] and in [207]. With respect to conceptual scaling of many-valued contexts see [65]. The term "scale" has been chosen in order to emphasize the connection with mathematical Theory of Measurement (cf. [103]), although the approaches differ considerably. Whereas in Measurement Theory a scale is usually understood to be a map to the real numbers, i.e., to a fixed structure, it has proved to be extraordinarily useful for conceptual scaling to be able to choose different scales for different many-valued attributes in accordance with their conceptual structure, even if the value set remains the same. Therefore, there are many ordinal scales in Concept Analysis, in contrast to Measurement Theory.

"Empty cells" of a one-valued context (i.e., pairs (g, m) with $(g, m) \notin I$) are considered to be *not concept forming.* If we want to use the negation for the formation of concepts, we have to **dichotomize** the respective attribute m, i.e., we have to introduce an additional attribute $\neg m$ with $(g, \neg m) \in I :\iff (g, m) \notin I$. "Empty cells" of a many-valued context (i.e., pairs (g, m) with $(g, m, w) \notin I$ for all $w \in W$) in the case of plain scaling usually result in empty cells in the derived context. If it is useful in terms of content, they can also be interpreted as values and be included into the scaling, see e.g. Figure 1.16 (p. 44).

Context constructions are treated in many papers, among other things in [58]. The complementation has been comprehensively examined in Deiters [37], [38]. Weinheimer [184] introduces the *product apposition* as a further construction.

The concept lattices of the powers (with respect to the semiproduct) of the dichotomic scale are precisely the "full concept lattices" in Lex [110].

Interpretations of scales have also been treated by Spangenberg and Wolff [158].

1.4 There is another definition of a product of contexts that suggests itself,

$$\mathbb{K}_1 \,\&\, \mathbb{K}_2 := (G_1 \times G_2, M_1 \times M_2, \&)$$

with

$$(g_1, g_2)\&(m_1, m_2) :\iff g_1 I_1 m_1 \ \underline{\text{and}}\ g_2 I_2 m_2.$$

This has been considered by various authors (Schaffert [149], Reuter [141], Erné [48]), but does not have the importance of the direct product in mathematical literature. The extents of $\mathbb{K}_1 \& \mathbb{K}_2$ are besides $G_1 \times G_2$ precisely the sets $U_1 \times U_2$ with

$$U_i \in \begin{cases} \mathfrak{U}(\mathbb{K}_i) & \text{if } G_i \text{ is an attribute intent of } \mathbb{K}_i, \\ \mathfrak{U}(\mathbb{K}_i) \setminus G_i & \text{if not.} \end{cases}$$

The concept lattice is therefore closely related to those of the context sum and the semiproduct.

The Dedekind Completion Theorem (which generalizes Dedekind's construction of the real numbers from the rational numbers) can be found in an order-theoretic version already in MacNeille [120] and J. Schmidt [151]. Compare also Banaschewski and Bruns [6].

The contraordinal scales are of central importance when treating distributive concept lattices [197]. Scaling by means of those scales has been carefully examined by Strahringer [165]. On this basis, Strahringer and Wille [166], [167] develop an *ordinal data analysis.*

Strahringer has also worked on convex-ordinal scaling [164]. Strahringer and Wille show in [168] that this kind of scaling lends itself to formulating a generalized *cluster analysis.* This has been further elaborated by Leonhard and Winterberg [109]. Formal Concept Analysis has also proved useful for the classification of ordinal cluster methods; compare Janowitz & Wille [89].

Further interesting contexts can be obtained from an ordered set $(P, \leq)$. For example, the concept lattice of the context $(P, P, \not\geq)$ can be interpreted as the lattice of the *maximal antichains* of $(P, \leq)$, see [196] and Reuter [143].

Free distributive lattices have been examined using the methods of Formal Concept Analysis in [205] and Bartenschlager [11], [10] and using closely related methods even earlier by Markowsky [123]. Compare also Luksch [112]. The extensive literature on this subject can be looked up in [11], [10]. The Δ-relation has been defined in [196].

Symmetric contexts have been treated by B. Schmidt [152] and Schaffert [149].

Flath ([54], [55]) has generalized the description of the irreducible elements of the lattice Σ_n of permutations by Bennett and Birkhoff [12] to multipermutations and used it among other things in order to determine the order dimension of those lattices with the methods of concept analysis.

2. Determination and Representation

Depending on the circumstances, the task of determining the concept lattice of a context can have different solutions. In the case of a small context, it is useful to start by drawing up a complete list of all concepts. This approach is treated in the first section of this chapter. In the second section, we discuss possibilities to generate line diagrams both automatically or by hand. A list of some dozens of concepts may already be quite difficult to survey, and it requires practice to draw good line diagrams of concept lattices with more than 20 elements. Nested line diagrams permit a satisfactory graphical representation of somewhat larger concept lattices. From some hundred elements at most, a complete graphical representation is no longer possible; in this case it is necessary to apply techniques for splitting up and representing lattices. These will be presented in later chapters.

Another determination problem arises if the context is not immediately available but must be inferred. We will discuss this case in the third section, which deals with the implications between attributes. This attribute logic can be extended to many-valued contexts, which shall be explained in section four of this chapter.

2.1 All Concepts of a Context

In principle, it is not difficult to find all the concepts of a formal context. The following proposition summarizes the naive possibilities of generating all concepts:

Proposition 18. *Each concept of a context (G, M, I) has the form (X'', X') for some subset $X \subseteq G$ and the form (Y', Y'') for some subset $Y \subseteq M$. Conversely, all such pairs are concepts.*

Every extent is the intersection of attribute extents and every intent is the intersection of object intents. □

However, the proposition does not immediately result in a method which is practicable. Only in the case of a very small context (G, M, I) it is reasonable to form the term (X'', X') for each subset X of G in order to generate all concepts. The second part of the proposition at least yields the possibility to calculate the concepts of a small context by hand. In order to do so, we draw up a list of concept extents. At the beginning, the list is empty. Then we proceed as follows:

First step. The extent G is entered into the list.

Then we carry out the following for each attribute $m \in M$ (the attributes are processed in an arbitrary order):

Step m. For each set A, entered into the list in an earlier step, we form the set

$$A \cap m'$$

and include it into the list, provided that it is not yet contained within it.

We can easily see that in the end the list contains precisely those sets which are the intersections of attribute extents. Those are, according to the proposition, precisely the concept extents. Then, by means of the context, we can find the concept intent for each such concept extent A. Thus we obtain a list of all concepts (A, A') of the context.

Step	Extent	Step	Extent	Step	Extent
1	$\{1,\ldots,8\}$		$\{6,7,8\}$		$\{1,2,3\}$
a			$\{6\}$		$\{3,4\}$
b	$\{1,2,3,5,6\}$	e	$\{7\}$		$\{3\}$
c	$\{3,4,6,7,8\}$		$\emptyset$	h	$\{2,3,4\}$
	$\{3,6\}$	f	$\{5,6,8\}$		$\{2,3\}$
d	$\{5,6,7,8\}$		$\{6,8\}$	i	$\{4\}$
	$\{5,6\}$	g	$\{1,2,3,4\}$		

Figure 2.1 List of concept extents for the context in Figure 1.1.

It is, in general, easier to determine concepts in this way, if we simultaneously draw a line diagram of the concept lattice. We will use the context in Figure 1.1 as an example of how this can be done. The intermediate steps are presented in Figure 2.2. First of all, we draw a small circle for the largest concept of the context. If there are attributes, the attribute extent of which contains all objects, we put their names down above the circle drawn. In our example, this would be "a". Then we determine the attributes the extents of which are maximal among the remaining attribute extents. In our example, we obtain b, c, d and g. The attribute concept of each of these attributes is represented by a small circle below the circle already drawn. These circles are then linked to the circle of the largest concept and the names of the new attributes are put down above the respective circles. Now we systematically form the infima of the concepts already represented and represent the newly generated concepts by small circles with their respective connecting lines. In

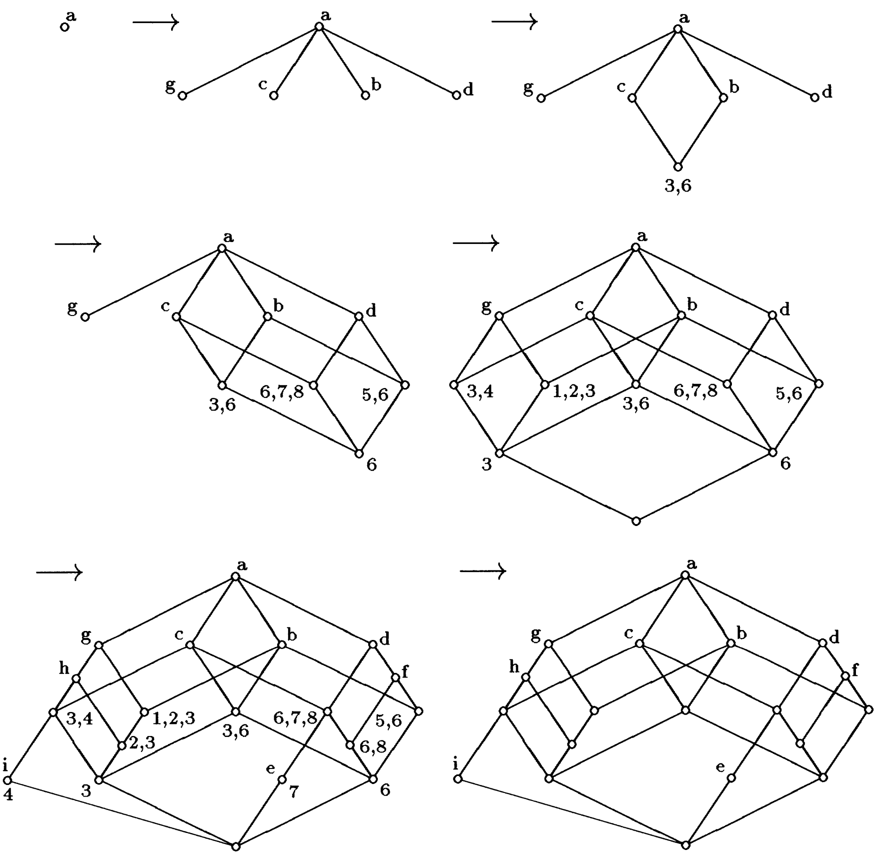


Figure 2.2 Intermediate steps in the drawing of Figure 1.3. As a last step we enter the object names. This also helps verifying whether the diagram is correct.

our example, this procedure is first applied to the concepts for b and c, then to those for b, c and d, and finally to those for b, c, d and g, making use of our knowledge about concepts already determined (if necessary one can note down the extent intersections temporarily at the respective circles in order to remember them for later). If we have drawn the line diagram for all concepts determined by this stage, we look for the attributes the extents of which are maximal among the attribute extents not already used. In our example, we obtain e, f and h. As above, we represent the attribute concepts and all new intersections of the concepts now available. In our example, we would have to go through this procedure one more time, that is for the attribute i. If we finally have worked our way through all the attributes, the resulting line diagram should be a correct representation of the concept lattice. In order to check this, we first delete the extents entered provisionally and then attach the object names (from below) to the concept circles, such that the equivalence $\gamma g \leq \mu m \iff gIm$ of the Basic Theorem is satisfied. If this is not possible for every concept, we have committed errors, and these may easily happen. According to our experience, it is easy to correct these errors. As a rule, it is useful to go over the line diagram again, in order to obtain a more readable diagram.

The algorithm for the determination of concepts described above becomes awkward for larger contexts, since it requires consulting the list again and again. For this reason, we next describe a faster algorithm for generating all extents, which has the additional advantage that it can easily be programmed. This algorithm only uses the closure operator $A \to A''$ of the context, i.e., it is an *algorithm for the generation of all closures of a given closure operator.*

First of all we consider the set of all subsets of G to be "in lexicographical order". For the sake of simplicity we assume that $G = \{1, 2, ..., n\}$. A subset $A \subseteq G$ is called **lectically smaller** than a subset $B \neq A$ if the smallest element which distinguishes A and B belongs to B. Formally:

$$A < B \quad :\Leftrightarrow \quad \exists_{i \in B \setminus A}\ A \cap \{1, 2, ..., i-1\} = B \cap \{1, 2, ..., i-1\}.$$

This defines a linear strict order on the power-set $\mathfrak{P}(G)$, i.e., for subsets $A \neq B$ always holds $A < B$ or $B < A$. The aim of the following is to find for an arbitrary given set $A \subseteq G$ the extent that is smallest after A with respect to this lectic order. If we have solved this, we can obviously generate all extents as follows: The lectically smallest concept extent is $\emptyset''$. The other extents are found incrementally by determining the one which is lectically closest to the last extent found. In the end, we obtain the lectically largest extent, namely G.

To make this precise, we define for $A, B \subseteq G, i \in G$,

$$A <_i B :\Leftrightarrow i \in B \setminus A \text{ and } A \cap \{1, 2, ..., i-1\} = B \cap \{1, 2, ..., i-1\}.$$
$$A \oplus i := ((A \cap \{1, 2, ..., i-1\}) \cup \{i\})''.$$

It is easy to verify the following statements:

(1) $A < B \Leftrightarrow A <_i B$ for one $i \in G$.
(2) $A <_i B$ and $A <_j C$ with $i < j \Rightarrow C <_i B$.
(3) $i \notin A \Rightarrow A < A \oplus i$.
(4) $A <_i B$ and B extent $\Rightarrow A \oplus i \subseteq B$, d.h. $A \oplus i \leq B$.
(5) $A <_i B$ and B extent $\Rightarrow A <_i A \oplus i$.

Theorem 5. *The smallest concept extent larger than a given set $A \subset G$ (with respect to the lectic order) is*

$$A \oplus i,$$

i being the largest element of G with $A <_i A \oplus i$.

Proof Let A^+ be the smallest extent after A with respect to the lectic order. On account of $A < A^+$, we get $A <_i A^+$ for some $i \in G$ by (1) and thus $A <_i A \oplus i$ by (5). By (4) it follows that $A \oplus i \leq A^+$, i.e., $A \oplus i = A^+$ because of $A < A \oplus i$. The fact that i is the largest element with $A <_i A \oplus i$ results from (2), since $A <_j A \oplus j$ with $j \neq i$ on account of $A \oplus i = A^+ < A \oplus j$ by (2) yields $j < i$. □

Theorem 5 shows how we can find the concept extent we are looking for. We summarize:

Algorithm for generating all extents of a given context (G, M, I): The lectically smallest extent is $\emptyset''$. For a given set $A \subset G$ we find the lectically next extent by checking all elements i of $G \setminus A$, starting from the largest one and continuing in a descending order until for the first time $A <_i A \oplus i$. $A \oplus i$ then is the "next" extent we have been looking for. □

No.	1	2	3	4	5	6	7	8	i	No.	1	2	3	4	5	6	7	8	i	No.	1	2	3	4	5	6	7	8	i
1					Ø				7	8					×	×	×	×	4	14		×	×						4
2							×		6	9				×					3	15		×	×	×					1
3						×			8	10			×						6	16	×	×	×						5
4						×		×	7	11			×			×			4	17	×	×	×		×	×			4
5						×	×	×	5	12			×	×					6	18	×	×	×	×					5
6					×	×			8	13			×	×		×	×	×	2	19	×	×	×	×	×	×	×	×	
7					×	×		×	7																				

Figure 2.3 List of the extents for the context in Figure 1.1 in a lectic order. Behind each extent A, the element i with $A^+ = A \oplus i$ is stated.

Because of the duality between objects and attributes, the algorithm can be transferred without changes to the intents; we only have to replace the set G by M. We can take advantage of the fact that the extents are being issued in a lectic order. If, for example,

$$C := \{1, 2, \ldots, c\}, \quad D := \{c + 1, c + 2, \ldots, d\} \subseteq G,$$

then as the lectic successors of C we first obtain those sets which contain C and are disjoint to D. A modification of the procedure (e.g. changes in the order of the elements of G) makes it possible to find for arbitrary subsets $C, D \subseteq G$ all concept extents A of (G, M, I) with $C \subseteq A, A \cap D = \emptyset$.

There are several implementations of this algorithm. The best-known is probably the program ConImp by Peter Burmeister, which is particularly common on DOS-computers. For the world of Unix there is a version named Concepts by Christian Lindig. Both programs are at present available for non-commercial purposes.[1]

The preconditions of the algorithm can be weakened in some respects. Therefore it permits several generalizations. Without substantial modifications of the proof we obtain the following theorem.

Theorem 6. *If $\mathcal{F}$ is a family of extents of the context (G, M, I) with the property*

$$A \in \mathcal{F} \textit{ and } i \in G \quad \Rightarrow \quad (A \cap \{1, \ldots, i-1\})'' \in \mathcal{F},$$

we obtain for an arbitrary subset $A \subseteq G$ the set A^+ which is the lectically next in $\mathcal{F}$ -if it exists- by

$$A^+ = A \oplus i,$$

i being the largest element of G for which $A <_i A \oplus i$ and simultaneously $A \oplus i \in \mathcal{F}$. □

We will give a simple example of possible applications of this theorem: If we want to find all partitions of a 7-element set not containing classes with more than three elements, we can use the context for the lattice of equivalence relations from 1.4.(8) (p. 52). The family $\mathcal{F}$ of partitions with the property specified is an order ideal and thus satisfies the condition in Theorem 6, i.e., it can be scanned with the modified algorithm.

2.2 Diagrams

The best and most versatile form of representation for a concept lattice is a well drawn line diagram. It is however tedious to draw such a diagram by hand and one would wish an automatic generation by means of a computer. We know quite a few algorithms to do this, but none which provides a general satisfactory solution. It is by no means clear which qualities make up a *good* diagram. It should be transparent, easily readable and should facilitate the interpretation of the data represented. How this can be achieved in each individual case depends however on the aim of the interpretation and on the

[1] e.g. free of charge via the Internet:
ftp.mathematik.th-darmstadt.de:/pub/department/software/conceptanalysis
or ftp.ips.cs.tu-bs.de:/pub/local/softech/misc.

structure of the lattice. Simple optimization criteria (minimization of the number of edge crossings, drawing in layers, etc.) often bring about results that are unsatisfactory. Nevertheless, automatically generated diagrams are a great help: they can serve as the starting point for drawing by hand. Therefore, we will describe simple methods of generating and manipulating line diagrams by means of a computer, later we suggest even better procedures with the aid of the structure theory for concept lattices.

		a	b	c	d	e	f	g
T_1	$(0,0),(6,0),(3,1)$		×	×	×		×	
T_2	$(0,0),(1,0),(0,1)$		×	×				×
T_3	$(0,0),(4,0),(1,2)$		×		×	×		
T_4	$(0,0),(2,0),(1,\sqrt{3})$	×		×	×	×		
T_5	$(0,0),(2,0),(5,1)$		×		×		×	
T_6	$(0,0),(2,0),(1,3)$		×	×	×	×		
T_7	$(0,0),(2,0),(0,1)$		×					×

a: equilateral, b: not equilateral, c: isosceles, d: oblique,
e: acute, f: obtuse, g: right.

Figure 2.4 A context for triangles.

As an illustration, we will use the context in Figure 2.4, in which triangles are classified according to properties such as *right-angled, equilateral,* etc. The choice of the triangles is not coincidental: the context is the result of an *attribute exploration*, a technique to be discussed in the next section. But for the moment we are only concerned with the question of how to obtain a line diagram for this context.

1 :	–				
2 :	1	–			
3 :	2	–			
4 :	1	–			
5 :	2	4	–		
6 :	3	5	–		
7 :	1	–			
8 :	7	–			
9 :	2	7	–		
10 :	9	–			
11 :	3	9	–		
12 :	4	7	–		
13 :	8	12	–		
14 :	5	9	12	–	
15 :	10	14	–		
16 :	6	11	14	–	
17 :	6	–			
18 :	13	15	16	17	–

We can use a computer program to obtain the concepts of the context and the edges of the line diagram. The *successor list*, displayed on the right, has been generated by means of the program CONIMP mentioned previously. We can read from it that the context has 18 concepts. These are denoted by the serial numbers 1,...,18. Behind the colon follow the upper neighbours of each concept. In the line diagram, an edge must be drawn to each of the upper neighbours, and those are all edges. Obviously concept no. 1 is the unit element of the concept lattice (since it has no upper neighbour) and no. 18 is the zero element (since 18 does not occur as an upper neighbour).

As a graph, the line diagram is already completely determined by this list. It can be used to sketch a diagram "from bottom to top": first of all we draw the smallest element (concept no. 18), above it the upper neighbours (13, 15, 16, 17), then their upper neighbours (6, 8, 10, 11, 12, 14), and so on. It is still open how points are arranged on paper. This can be done "intuitively" but will then require various iterations to develop a satisfactory diagram.

There is however an efficacious method to support the generation of a line diagram. This **geometrical method** is based on first understanding the lattice-theoretic structure through a geometrical representation of the concept lattice and then to find the best possible arrangement for the line diagram. This means that we draw as an intermediate step –by hand or with the aid of a computer– an auxiliary picture, which is then used to draw the actual line diagram. This auxiliary picture is called the **geometrical diagram**. Intuitively, we think of this diagram in the following way: we imagine that the lattice is realized by means of a three-dimensional line diagram and look down on the lattice from its highest point, i.e., from the unit element.

From the top, we first see the lower neighbours of the unit element. In the geometrical diagram they are represented by unconcealed circles into which we write the names of the respective elements. We continue to draw the geometrical diagram in accordance with the following rules:

1. An element with exactly one upper neighbour is represented by a circle which is partly covered by the upper neighbour.
2. An element with exactly two upper neighbours is represented by a connecting line segment between the two upper neighbours. The name of the element is written into a circle which is partly covered by this connecting line.
3. An element with exactly three upper neighbours is represented by a connecting triangle between the upper neighbours. The name of the element is written into the triangle.

Elements with $n > 3$ upper neighbours are represented analogously by an n-simplex connecting the upper neighbours. The largest and the smallest element of the lattice are omitted.

In this way we obtain the geometrical diagram in Figure 2.5. The individual steps are noted down in the following table. The necessary information has been taken from the above successor list.

2 lies immediately below 1: therefore a circle for 2.
3 immediately below 2: therefore a circle for 3, partly covered by the 2-circle.
4 immediately below 1: therefore a circle for 4.
5 immediately below 2 and 4: therefore a line segment for 5 between the 2-circle and the 4-circle.
6 immediately below 3 and 5: therefore a line segment for 6 between the 3-circle and the 5-line-segment.
7 immediately below 1: therefore a circle for 7.

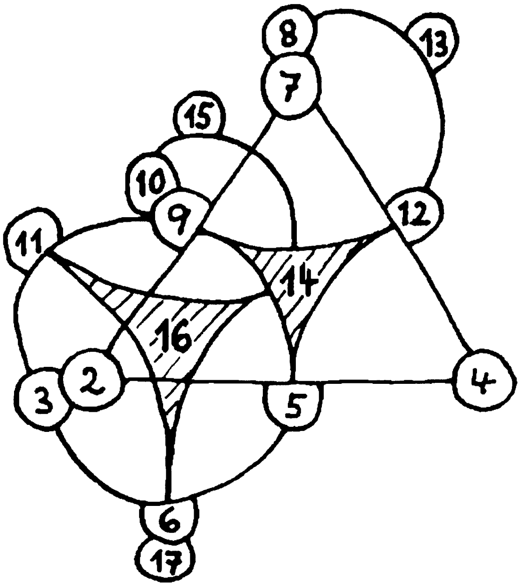


Figure 2.5 A geometrical diagram.

8 immediately below 7: therefore a circle for 8, partly covered by the 7-circle.
9 immediately below 2 and 7: therefore a line segment for 9 between the 2-circle and the 7-circle.
10 immediately below 9: therefore a circle for 10, partly covered by the 9-circle.
11 immediately below 3 and 9: therefore a line segment for 11 between the 3-circle and the 9-line-segment.
12 immediately below 4 and 7: therefore a line segment for 12 between the 4-circle and the 7-circle.
13 immediately below 8 and 12: therefore a line segment for 13 between the 8-circle and the 12-line-segment.
14 immediately below 5, 9 and 12: therefore a triangle for 14 between the 5-circle, the 9-circle and the 12-line-segment.
15 immediately below 10 and 14: therefore a line segment for 15 between the 10-circle and the 14-triangle.
16 immediately below 6, 11 and 14: therefore a triangle for 16 between the 6-line-segment, the 11-line-segment and the 14-triangle.
17 immediately below 6: therefore a circle for 17, partly covered by the 6-line-segment.

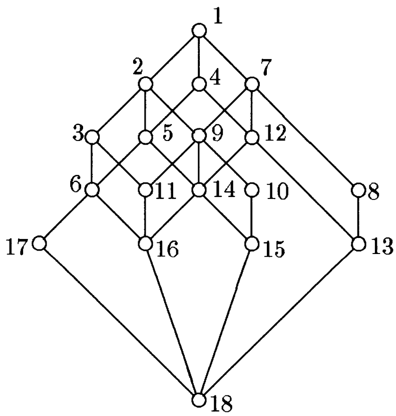


Figure 2.6 A line diagram for the lattice of triangle concepts.

It still remains to be said how a good line diagram can be obtained from the geometrical diagram. The derived line diagram for the concept lattice of the triangles is presented in Figure 2.6. If one already has some experience with the geometrical method, one can see from Figure 2.5, that the most striking substructure of the lattice consists of two Boolean cubes. But even without this experience, one can soon reach this conclusion by proceeding systematically. As a rule, one should start with the lower neighbours of the unit element being represented by unconcealed circles. In Figure 2.5 these are the 2-, 4- and 7-circle. These circles are connected pairwise by the line segments 5, 9 and 12, which in turn are connected by the 14-triangle. This shows that the concepts 1, 2, 4, 5, 7, 9, 12 and 14 form a Boolean sublattice. The question is, how these eight elements can best be arranged within the line diagram. After having drawn the unit element, it seems advisable to put the co-atoms 2 and 7 on the outside and 4 between them, since below both 2 and 7 there is another "point", which needs some space. The concepts 5, 7, 9 and 12 will be best placed in accordance with the *rule of parallelograms*, which says that one should (if possible) place an element in a way that it makes up a parallelogram together with three elements already represented and their connecting line segments. The resulting picture of the Boolean sublattice represents a cube standing on one of its corners. After the explanation given so far, it should not be difficult to recognize the second Boolean sublattice, consisting of the concepts 2, 3, 5, 6, 9, 11, 14 and 16. Since the cube representing it shares the elements 2, 5, 9 and 14 with the first cube, it is obvious how to continue the drawing. However, one further rule should be observed, the so-called *rule of lines*, according to which a line to a new "point" should be arranged in such a way that it continues some line segments already drawn. If we observe the rule of lines and the rule of parallelograms for the remaining elements 8, 10, 13, 15 and 17, we obtain from the geometrical diagram a satisfactory line diagram, to which we only have to add the zero element (no. 18) (cf. Figure 2.6). For the labelling with object and attribute names, additional information is required, which the program ConImp supplies by means of an *assignment list* (see Fig. 2.7).

Concept	:	Object	Concept	:	Attribute
8	:	T_7	2	:	oblique
10	:	T_5	3	:	acute
11	:	T_3	4	:	isosceles
13	:	T_2	7	:	not equilateral
15	:	T_1	8	:	right
16	:	T_6	10	:	obtuse
17	:	T_4	17	:	equilateral

Figure 2.7 The assignment to the concepts.

In general, it is advisable to draw the geometrical diagram as quickly as possible using the successor list. When doing so, one should not be afraid to draw segments of lines and surfaces rather boldly. Experience shows that this kind of diagrams can still be used as instructions for drawing good line diagrams. It is helpful to observe geometrical patterns and their respective realizations in the line diagrams. In some (relatively rare) cases, it is advisable to construct the line diagram from bottom to top; in this case one should use the so-called *predecessor list.*

Both of the procedures described above make use of the computer in order to obtain information necessary for a diagram. We will now explain a method where a computer generates a diagram and offers the possibility of improving it interactively. Programming details are irrelevant in this context. We will

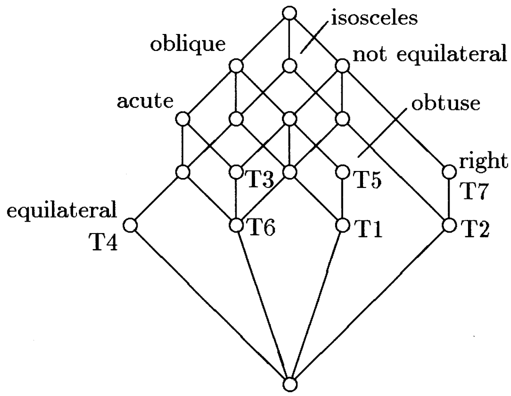


Figure 2.8 The labelled line diagram.

therefore only give a **positioning rule** which assigns points in the plane to the elements of a given ordered set $(P, \leq)$. If a and b are elements of P with $a < b$, the point assigned to a must be lower than the point assigned to b (i.e., it must have a smaller y-coordinate). This is guaranteed by our method. We will leave the computation of the edges and the checking for undesired coincidences to the programming. We do not even guarantee that our positioning is injective (which of course is necessary for a correct line diagram). This must also be checked if necessary.

Definition 35. A **set representation** of an ordered set $(P, \leq)$ is an order embedding of $(P, \leq)$ in the power-set of a set X, i.e., a map

$$\text{rep} : P \to \mathfrak{P}(X)$$

with the property

$$x \leq y \Leftrightarrow \text{rep}\, x \subseteq \text{rep}\, y.$$

$\diamond$

An example of a set representation for an arbitrary ordered set $(P, \leq)$ is the assignment

$$X := P, \quad a \mapsto (a].$$

In the case of a concept lattice

$$X := G, \quad (A, B) \mapsto A$$

$$\text{resp. } X := M, \quad (A, B) \mapsto M \setminus B$$

are representations which can be combined to

$$X := G \,\dot{\cup}\, M, \quad (A, B) \mapsto A \cup (M \setminus B).$$

It is sufficient to limit oneself to the irreducible objects and attributes[2].

For an **additive line diagram** of an ordered set $(P, \leq)$ we need a set representation $\text{rep} : P \to \mathfrak{P}(X)$ as well as a **grid projection**

$$\text{vec} : X \to \mathbb{R}^2,$$

assigning a real vector with a positive y-coordinate to each element of X. By

$$\text{pos}\, p := n + \sum_{x \in \text{rep}\, p} \text{vec}\, x$$

we obtain positioning of the elements of P in the plane. Here, n is a vector which can be chosen arbitrarily in order to shift the entire diagram. By only allowing positive y-coordinates for the grid projection we make sure that no element p is positioned below an element q with $q < p$.

Every finite line diagram can be interpreted as an additive diagram with respect to an appropriate set representation. For concept lattices we usually use the representation by means of the irreducible objects and/or attributes. The resulting diagrams are characterized by a great number of parallel edges, which improves their readability. Besides, it is particularly easy to manipulate these diagrams.

If we change –the set representation being fixed– the grid projection for an element $x \in X$, this means that all images of the order filter $\{p \in P \mid x \in \text{rep}\, p\}$ are shifted by the same distance and that all other images remain in the same position. In the case of the set representation by means of the irreducibles these order filters are precisely principal filters or complements of principal ideals, respectively. This means that we can manipulate the diagram by shifting principal filters or principal ideals, respectively, and leaving all other elements in position.

Experience shows that the set representation by means of the irreducible attributes is most likely to result in an easily interpretable diagram.

Occasionally, it can be convenient to represent a lattice as a part of a larger order. For this purpose, we draw a line diagram of the order but represent only those elements of the lattice by small circles which we actually mean. An example is shown in Figure 5.3 (p. 189).

Even carefully constructed line diagrams loose their readability from a certain size up, as a rule from around 50 elements up. One gets considerably further with the *nested line diagrams* which will be introduced next. However, these diagrams do not only serve to represent larger concept lattices. They offer the possibility to visualize how the concept lattice changes if we add further attributes.

The basic idea of the nested line diagram consists of delimiting parts of an ordinary diagram and replacing bundles of parallel lines between these parts by one line each. Thus, a nested line diagram consists of framed *boxes*, which

[2] For set representation see also Chapter 6.5

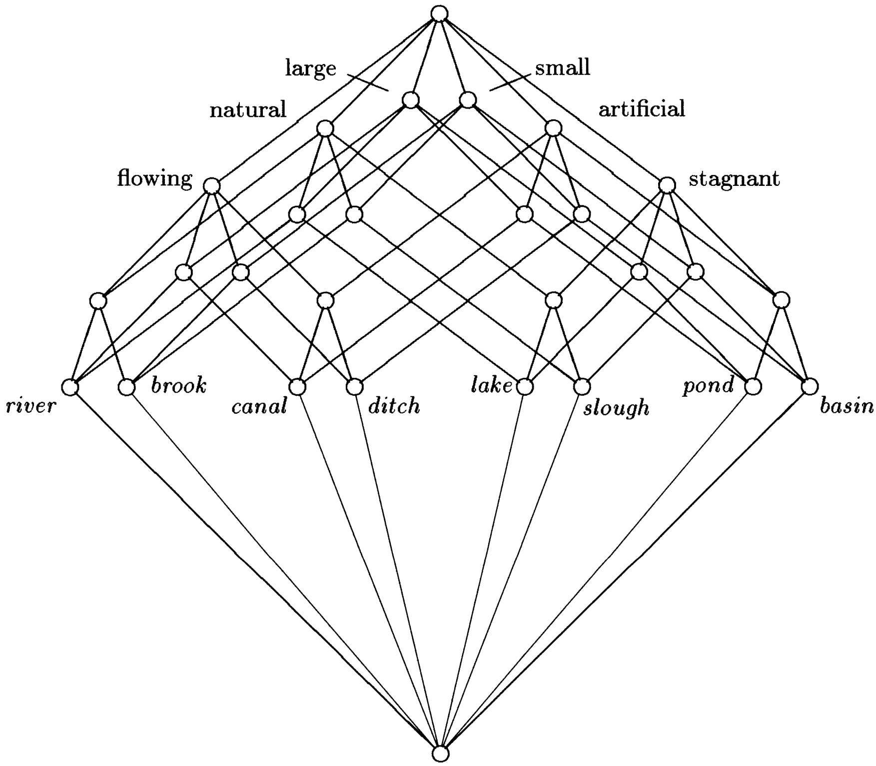


Figure 2.9 An additive line diagram of the concept lattice of a *lexical field "waters"*. The set representation is based on the irreducible attributes, i.e. the positioning of the attribute concepts determines that of all remaining concepts. If we interpret the line segments between the unit element and the attribute concepts as vectors, we obtain the position of an arbitrary concept by the sum of the vectors belonging to attributes of its concept intent starting from the unit element. Other diagrams for the same lattice can be found in Figure 2.10.

contain parts of the ordinary line diagram and which can be connected by lines. In the simplest case two boxes which are connected by a simple line are congruent. Here, the line indicates that circles which coincide if one box is put on top of the other are connected in the ordinary line diagram. A double line between two boxes means that each element of the upper box is larger than each element of the lower box. Figure 2.10 shows the concept lattice from the preceding section, once as an ordinary line diagram and once as a nested diagram. For reasons of comprehensibility we have left out the object and attribute names.

Furthermore, we allow that two boxes connected by a single line do not necessarily have to be congruent, but they may each contain a part of two congruent figures. In this case, the two congruent figures are drawn in the boxes as a "background structure", but the elements are only marked by circles if they are part of the respective substructures. The line connecting the two boxes then indicates that the respective pairs of elements of the background shall be connected with each other. Examples can be found in Figures 1.20 (p. 51) and 2.17 (p. 90).

Nested line diagrams originate from partitions of the set of attributes. The basis is the following Theorem:

Theorem 7. *Let (G, M, I) be a context and $M = M_1 \cup M_2$. The map*

$$(A, B) \mapsto (((B \cap M_1)', B \cap M_1), ((B \cap M_2)', B \cap M_2))$$

is a $\bigvee$-preserving order embedding of $\underline{\mathfrak{B}}(G, M, I)$ in the direct product of $\underline{\mathfrak{B}}(G, M_1, I \cap G \times M_1)$ and $\underline{\mathfrak{B}}(G, M_2, I \cap G \times M_2)$. The component maps

$$(A, B) \mapsto ((B \cap M_i)', B \cap M_i)$$

are surjective on $\underline{\mathfrak{B}}(G, M_i, I \cap G \times M_i)$.

Proof. If (A, B) is a concept of (G, M, I), then $B \cap M_i$ is the set of all attributes common to the objects of A in the context $(G, M_i, I \cap G \times M_i)$, i.e., it is an intent of this context. Hence the above-mentioned assignment is really a map into the product. The union of the intents $B \cap M_1$ and $B \cap M_2$ again yields B, i.e., the map is injective. The fact that it is furthermore $\bigvee$-preserving (and thus an order-embedding) can again be seen from the concept intents. It remains to be shown that the component maps are surjective. Let C be an intent of $(G, M_i, I \cap G \times M_i)$. Then $B := C^{II}$ is an intent of (G, M, I) with $B \cap M_i = C$, i.e., the image of the concept (B', B) of (G, M, I) under the ith component map is the concept with the intent C. $\square$

In order to sketch a nested line diagram, we proceed as follows: First of all we split up the attribute set: $M = M_1 \cup M_2$. This splitting up does not have to be disjoint. More important for interpretation purposes is the idea that the sets M_i bear meaning. Now, we draw line diagrams of the subcontexts $\mathbb{K}_i := (G, M_i, I \cap G \times M_i)$, $i \in \{1, 2\}$ and label them with the names of

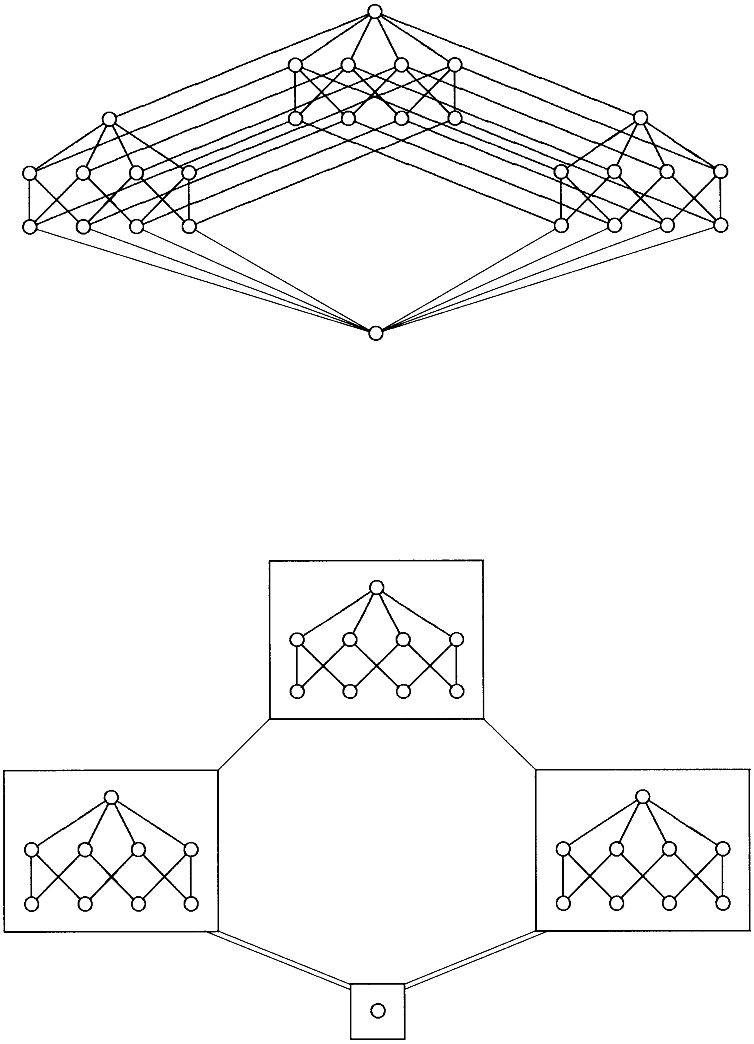


Figure 2.10 Line diagram and nested line diagram.

the objects and attributes, as usual. Then we sketch a nested diagram of the product of the concept lattices $\underline{\mathfrak{B}}(\mathbb{K}_i)$ as an auxiliary structure. For this purpose we draw a large copy of the diagram of $\underline{\mathfrak{B}}(\mathbb{K}_1)$, representing the lattice elements not by small circles but by congruent rectangular boxes, which contain each a diagram of $\underline{\mathfrak{B}}(\mathbb{K}_2)$.

By Theorem 7 the concept lattice $\underline{\mathfrak{B}}(G, M, I)$ is embedded in this product as a $\bigvee$-semilattice. If a list of the elements of $\underline{\mathfrak{B}}(G, M, I)$ is available, we can enter them into the product according to their intents. If not, we enter the object concepts the intents of which can be read off directly from the context, and form all suprema.

This at the same time provides us with a further, quite practicable method of determining a concept lattice by hand: split up the attribute set as appropriate, determine the (small) concept lattices of the subcontexts, draw their product in form of a nested line diagram, enter the object concepts and close it against suprema. This method is particularly advisable in order to arrive at a useful diagram quickly.

2.3 Implications between Attributes

An imaginary example shall serve as an introduction to the problem: imagine a manufacturer of computer hardware, whose different products can be combined in various ways but not arbitrarily. In order to obtain a conceptual structuring of the (reasonable) configurations, we would have to examine a context the objects of which are the combinations and the attributes of which are the components. If a list of these combinations is not available, we have to draw it up. This can be done on the basis of our knowledge about the existing possibilities of combining the elements.

In this case, the starting point of concept analysis is not an explicitly stated context. Rather, we infer the context and at the same time the concept system from the **attribute logic**, i.e., from the rules concerning the combination of attributes.

This method does not only suggest itself in the example discussed above. It often becomes necessary to classify a large number of objects with respect to a relatively small number of attributes, and it is frequently useless or impracticable to write down the whole context and to apply the procedures for the determination of the concept system which were described in the previous section. In such cases, the concept lattices can be inferred from the *implications between the attributes*, i.e., from statements of the following kind: "Every object with the attributes $a, b, c, \ldots$ also has the attributes $x, y, z, \ldots$". Formally, an **implication between attributes** (in M) is a pair of subsets of the attribute set M. It is denoted by $A \to B$. (When the sets are small, we shall omit the brackets (as we have done earlier), i.e., we shall write $A \to m$ instead of $A \to \{m\}$, etc.). In this section we examine the attribute implications which *hold* in a context. The amount of information contained by these implications is evidenced by the fact that we can reconstruct the structure of the concept lattice from them. Conversely, the implications between the attributes of a context can also be read off from the concept lattice. However, the systems of all implications between attributes which hold in a context tend to be very large and to contain many trivial implications. Therefore, we try to find subsystems which suffice to describe the concept lattice. First we give some simple definitions:

Definition 36. A subset $T \subseteq M$ **respects** an implication $A \to B$ if $A \not\subseteq T$ or $B \subseteq T$. T **respects a set** $\mathcal{L}$ of implications if T respects every single implication in $\mathcal{L}$. $A \to B$ **holds** in a set $\{T_1, T_2, \ldots\}$ of subsets if each of the subsets T_i respects the implication $A \to B$. $A \to B$ **holds in a context** (G, M, I) if it holds in the system of object intents. In this case, we also say that $A \to B$ is **an implication of the context** (G, M, I) or, equivalently, that within the context (G, M, I), A is **a premise of** B. $\diamond$

Proposition 19. *An implication $A \to B$ holds in (G, M, I) if and only if $B \subseteq A''$. It then automatically holds in the set of all concept intents as well.* $\square$

How can we read off an implication from the concept lattice? It is sufficient to describe this procedure for implications of the form $A \to m$, since $A \to B$ holds if and only if $A \to m$ holds for each $m \in B$. $A \to m$ holds if and only if $(m', m'') \geq (A', A'')$, i.e., if $\mu m \geq \bigwedge\{\mu n \mid n \in A\}$. This means that we have to check in the concept lattice whether the concept denoted by m is located above the infimum of all concepts denoted by an n from A.

It can occasionally be useful for the determination of the implications to replace the original context (G, M, I) by its complementary context $(G, M, (G \times M) \setminus I)$, in particular if the latter has considerably fewer concepts as (G, M, I). For $m \in M$ and $A \subseteq M$ the following equivalences hold: $m \in A'' \Leftrightarrow \{m\} \subseteq A'' \Leftrightarrow A' \subseteq m' \Leftrightarrow \bigcap\{n' \mid n \in A\} \subseteq m' \Leftrightarrow G \setminus m' \subseteq \bigcup\{G \setminus n' \mid n \in A\}$. Thus, $A \to m$ holds in the context (G, M, I) if and only if in the complementary context every object with the attribute m has at least one attribute n from A.

Proposition 20. *If $\mathcal{L}$ is a set of implications in M,*

$$\mathfrak{H}(\mathcal{L}) := \{X \subseteq M \mid X \textit{ respects } \mathcal{L}\}$$

is a closure system on M. If $\mathcal{L}$ is the set of all implications of a context, $\mathfrak{H}(\mathcal{L})$ is the system of all intents.

The *proof* is trivial. The respective closure operator can be described as follows: For a set $X \subseteq M$, let

$$X^{\mathcal{L}} := X \cup \bigcup\{B \mid A \to B \in \mathcal{L}, A \subseteq X\}.$$

We form the sets $X^{\mathcal{L}}$, $X^{\mathcal{L}\mathcal{L}}$, $X^{\mathcal{L}\mathcal{L}\mathcal{L}}, \ldots$ until we finally obtain a set $\mathcal{L}(X) := X^{\mathcal{L}\ldots\mathcal{L}}$ with $\mathcal{L}(X)^{\mathcal{L}} = \mathcal{L}(X)$ (in the case of infinite contexts it can be necessary to continue this process transfinitely. $\mathcal{L}(X)$ is then is the closure of X with respect to the closure system $\mathfrak{H}(\mathcal{L})$ which we have been looking for.

By means of the closure system $\mathfrak{H}(\mathcal{L})$ it is also possible to construct a context for every given set $\mathcal{L}$ of implications, the intents of which are precisely the sets respecting $\mathcal{L}$: $(\mathfrak{H}(\mathcal{L}), M, \ni)$ has this property. In addition to $\mathcal{L}$ in this context hold all implications which follow from $\mathcal{L}$ in the sense of the following definition:

Definition 37. An implication $A \to B$ **follows (semantically)** from a set $\mathcal{L}$ of implications between attributes if each subset of M respecting $\mathcal{L}$ also respects $A \to B$. A family of implications $\mathcal{L}$ is called **closed** if every implication following from $\mathcal{L}$ is already contained in $\mathcal{L}$.

A set $\mathcal{L}$ of implications of a context (G, M, I) is called **complete** if every implication of (G, M, I) follows from $\mathcal{L}$. ◇

In other words: An implication follows semantically from $\mathcal{L}$ if it holds in every system of sets in which $\mathcal{L}$ holds as well. This is the case if and only if $\mathfrak{H}(\mathcal{L}) = \mathfrak{H}(\mathcal{L} \cup \{A \to B\})$.

The closed sets of implications lend themselves to a syntactic characterization. This has been discussed comprehensively, for example in the book of Maier [121], from which we cite the following proposition (formulated by Armstrong [1]):

Proposition 21. *A set $\mathcal{L}$ of implications on M is closed if and only if the following conditions are satisfied for all $W, X, Y, Z \subseteq M$:*

1. $X \to X \in \mathcal{L}$,
2. *If* $X \to Y \in \mathcal{L}$, *then* $X \cup Z \to Y \in \mathcal{L}$,
3. *If* $X \to Y \in \mathcal{L}$ *and* $Y \cup Z \to W \in \mathcal{L}$, *then* $X \cup Z \to W \in \mathcal{L}$.

□

In order to demonstrate that a set $\mathcal{L}$ of implications of a context is complete, we have to show that every subset $T \subseteq M$ respecting $\mathcal{L}$ is an intent.

A first attempt to find a manageable complete set of implications consists in leaving out those implications which follow trivially from others or those which hold in any context. For instance, $A \to B$ holds whenever $B \subseteq A$, and from $A \to B$ and $C \subseteq B$ it always follows that $A \to C$. Correspondingly, from $A_j \to B_j$ for $j \in J$ it always follows that $\bigcup_{j\in J} A_j \to \bigcup_{j\in J} B_j$. If we eliminate the implications arising like that, there remain certain *implications with a proper premise*:

Definition 38. For an attribute set $A \subseteq M$ of a context (G, M, I) we denote by

$$A^\bullet := A'' \setminus (A \cup \bigcup_{n\in A} (A \setminus \{n\})'')$$

the set of those attributes contained in A'' but not in A or in the closure of any proper subset of A. We call A a **proper premise** if $A^\bullet \neq \emptyset$, i.e., if

$$A'' \neq A \cup \bigcup_{n \in A} (A \setminus \{n\})''.$$

In particular, $\emptyset$ is a proper premise if $\emptyset'' \neq \emptyset$. ◇

Proposition 22. *If T is a finite subset of M, then*

$$T'' = T \cup \bigcup \{A^\bullet \mid A \text{ is a proper premise with } A \subseteq T\}.$$

The set of all implications of the form

$$A \to A^\bullet, \quad A \text{ a proper premise,}$$

of a context with a finite attribute set is complete.

Proof. If $T = T''$ the assertion is trivial, thus let $m \in T'' \setminus T$. A subset A of T which is minimal with respect to the property $m \in A''$ has to be a proper premise, i.e., there is an implication $A \to A^\bullet$ with $m \in A^\bullet$. Since m had been chosen arbitrarily, the first assertion follows. If T respects all implications of the form $A \to A^\bullet$ and A is a proper premise, from what we have just proved it follows that $T'' = T$, i.e., that T is an intent. □

In certain respects, the set of proper premises is canonical with respect to the property described in Proposition 22. In order to state this more precisely, we first introduce a further term. A family of implications can be simplified by merging implications with the same premise. We call a family of implications **contracted** if there are no premises which occur more than once. If $\mathcal{L}$ is any contracted family of implications satisfying the condition of the proposition, i.e., with

$$T'' = T \cup \bigcup \{B \mid A \to B \in \mathcal{L}, A \subseteq T\} \quad \text{for all } T \subseteq M,$$

then $\mathcal{L}$ contains an implication $E \to F$ with $E^\bullet \subseteq F$ for every proper premise E, as can be seen easily, if we replace T by E in the condition.

In order to determine the proper premises of a doubly founded context (G, M, I), we can use the arrow relation $\swarrow$. Following Definition 36 we call an attribute set P a **proper premise of** an attribute m if P is a proper premise and $m \in P^\bullet$ holds.

Proposition 23. *P is a premise of m if and only if*

$$(M \setminus g') \cap P \neq \emptyset$$

holds for all $g \in G$ with $g \swarrow m$. P is a proper premise for m if and only if $m \notin P$ and P is minimal with respect to the property that $(M \setminus g') \cap P \neq \emptyset$ holds for all $g \in G$ with $g \swarrow m$.

Proof. For $g \in G$ and $P \subseteq M$ we have the equivalences

$$(M \setminus g') \cap P \neq \emptyset \iff P \not\subseteq g' \iff g \notin P'.$$

Since $P' \not\subseteq m'$ is equivalent to the fact that there is an object $g \in P'$ with $g \swarrow m$, the first assertion follows. The property of minimality of the proper premises yields the second assertion. □

According to the proposition, we obtain the proper premises by determining for every attribute m the minimal attribute sets P with $(M \setminus g') \cap P \neq \emptyset$ for all $g \swarrow m$.

Even the set of implications described in Proposition 22 is in general still redundant.

Definition 39. A set $\mathcal{L}$ of implications of a context is called **non-redundant** if none of the implications follows from the others. ◇

Guigues and Duquenne [74] have shown that there is a natural complete and non-redundant set of implications for every context with a finite attribute set M. For the following results, we make the *general assumption* that the attribute set M which occurs *is finite.* This permits a recursive definition of the basic notion of the pseudo-intent (which takes the place of the proper premise):

Definition 40. $P \subseteq M$ is called the **pseudo-intent** of (G, M, I) if and only if $P \neq P''$ and $Q'' \subseteq P$ holds for every pseudo-intent $Q \subseteq P$, $Q \neq P$. ◇

Theorem 8. *The set of implications*

$$\mathcal{L} := \{P \to P'' \mid P \textit{ pseudo-intent}\}$$

is non-redundant and complete.

Proof. Evidently, $\mathcal{L}$ holds in (G, M, I). In order to show that $\mathcal{L}$ is complete, we again have to show that every set $T \subseteq M$ respecting $\mathcal{L}$ is an intent. Each such set in particular respects all implications $Q \to Q''$ where Q is a pseudo-intent and $Q \subseteq T$. If we assume that $T \neq T''$, T itself satisfies the definition of a pseudo-intent and the implication $T \to T''$ is in $\mathcal{L}$ but is not respected by T, a contradiction.

In order to show that $\mathcal{L}$ is non-redundant, we consider an arbitrary pseudo-intent P and show that P respects the set $\mathcal{L} \setminus \{P \to P''\}$. In fact, if $Q \to Q''$ is an implication in $\mathcal{L} \setminus \{P \to P''\}$ with $Q \subseteq P$, then $Q'' \subseteq P$ must hold, since P is a pseudo-intent. □

In practice, the implications are not stated in the form $P \to P''$ but in the form $P \to (P'' \setminus P)$. We call this the **Duquenne-Guigues-Basis** or simply the **stem base** of the attribute implications. In the case of the developing countries (Figure 1.8) this basis consists of five implications (see Figure 2.11).

OPEC	$\to$	Group of 77, Non-aligned
MSAC	$\to$	Group of 77
Non-aligned	$\to$	Group of 77
Group of 77, Non-aligned, MSAC, OPEC	$\to$	LLDC, ACP
Group of 77, Non-aligned, LLDC, OPEC	$\to$	MSAC, ACP

Figure 2.11 Stem base for the context of developing countries.

Again it is possible to show that this family of implications is in a way canonical with respect to the properties stated. We will first note down a simple proposition:

Proposition 24. *If P and Q are concept or pseudo-intents with $P \not\subseteq Q$ and $Q \not\subseteq P$, then $P \cap Q$ is an intent.*

Proof. P as well as Q and thus also $P \cap Q$ respect all implications in $\mathcal{L}$ with the possible exception of $P \to P''$ and $Q \to Q''$. If $P \neq P \cap Q \neq Q$, then $P \cap Q$ also respects these implications, i.e., it is an intent. □

The following proposition shows among other things that there can be no complete set which contains fewer implications than there are pseudo-intents:

Proposition 25. *Every complete set Σ of implications contains an implication $A \to B$ with $A'' = P''$ for every pseudo-intent P.*

Proof. A pseudo-intent P is always not equal P''. Therefore, provided that Σ is complete, there must be at least one implication $A \to B$ in Σ which leads out of P, i.e., with $A \subseteq P$ and $B \not\subseteq P$. On account of $B \subseteq A''$, we get $A'' \not\subseteq P$, and thus $A'' \cap P$ cannot be a concept intent. By Proposition 24 this yields $P \subseteq A''$ and thus $P'' = A''$. □

The recursive definition of the pseudo-intents provides us with a first, although inefficient, algorithm for generating them. In the following we will develop a more practicable procedure. As an immediate consequence of Proposition 24 we obtain:

Proposition 26. *The set of all subsets of M which are intents or pseudo-intents of (G, M, I) is a closure system.* □

The closure operator for this closure system is obtained by a modification of the operator $\mathcal{L}$. Starting from a set X, we successively form

$$X^{\mathcal{L}^*} := X \cup \bigcup\{B \mid A \to B \in \mathcal{L}, A \subseteq X, A \neq X\}$$

$$X^{\mathcal{L}^*\mathcal{L}^*} := X^{\mathcal{L}^*} \cup \bigcup\{B \mid A \to B \in \mathcal{L}, A \subseteq X^{\mathcal{L}^*}, A \neq X^{\mathcal{L}^*}\}$$

and so on, until we finally obtain a set $\mathcal{L}^*(X)$ with $\mathcal{L}^*(X) = \mathcal{L}^*(X)^{\mathcal{L}^*}$. This set is the pseudo-intent or intent which we have been looking for. We should bear in mind that when using this method for generating a pseudo-intent P, we only need implications $A \to B$, the premise of which is a proper subset of P. This permits a recursive generation of the pseudo-intents by means of the algorithm described in 2.3 .

Algorithm for generating all intents and pseudo-intents of a context (G, M, I) in a lectic order. We assume that $M = \{1, 2, ..., n\}$. The symbol $<_i$ has been explained in 2.1.

The lectically smallest concept or pseudo-intent is $\emptyset$. For a given set B, we find the lectically next concept or pseudo-intent by checking all elements i of M, starting from the largest and continuing in a descending order, until we first obtain $B <_i \mathcal{L}^*((B \cap \{1, 2, \ldots, i-1\}) \cup \{i\})$. $\mathcal{L}^*((B \cap \{1, 2, \ldots, i-1\}) \cup \{i\})$ then is the concept or pseudo-intent we were looking for. □

We will now return to the opening question of this section: how can we determine the concept intents by means of the implications? We have seen that in order to do so, we do not need all the implications, but that a small subset of them is sufficient. So far we have only explained how these implications can be obtained from an available context. By means of the tools now on hand, however, we can also develop a method of generating sets of implications which are free of redundancies, even if the context is not or only partly available. This procedure, which is called **attribute exploration**, has proved successful in many applications. In practice, we use a computer which administers the sets of implications and is able to compute which information is still lacking. The implications are then determined *interactively*, i.e., in cooperation with the user.

The algorithm for the determination of the pseudo-intents permits a modification resulting in an interactive program: it is possible to modify the context by adding new objects, even while the generation of the list $\mathcal{L}$ of the implications is in progress. If the intents of these objects respect all implications determined so far, the computation for the new context can be continued with the results so far obtained. This is the content of the following proposition:

Proposition 27. *Let $\mathbb{K}$ be a context and let $P_1, P_2, \ldots, P_n$ be the first n pseudo-intents of $\mathbb{K}$ with respect to the lectic order. If $\mathbb{K}$ is extended by an object g the object intent g' of which respects the implications $P_i \to P_i''$, $i \in \{1, \ldots, n\}$, then $P_1, P_2, \ldots, P_n$ are also the lectically first n pseudo-intents of the extended context.*

This can be proved for example by induction on n. □

Therefore, if we have found a new pseudo-intent P, we can stop the algorithm and ask, whether the implication $P \to P''$ should be added to $\mathcal{L}$. The user can answer this question in the affirmative or add a counter-example, which must not contradict the implications he has confirmed so far. In the extreme case, the procedure can be started with a context the object set of which is empty. In this case, the user will have to enter all counter-examples, thereby creating a concept system with a given "attribute logic".

Instead of describing this program in detail, we shall demonstrate its functioning by means of an example. From a book on measurement theory [146] we take a list of properties of binary relations, which are used there in order to define different types of relations, see Figure 2.12.

	Property	Definition
r	reflexive	xRx for all $x \in S$
i	irreflexive	$\neg xRx$ for all $x \in S$
s	symmetric	$xRy \Rightarrow yRx$ for all $x, y \in S$
as	asymmetric	$xRy \Rightarrow \neg yRx$ for all $x, y \in S$
an	antisymmetric	xRy and $yRx \Rightarrow x = y$ for all $x, y \in S$
t	transitive	xRy and $yRz \Rightarrow xRz$ for all $x, y, z \in S$
nt	negatively transitive	$\neg xRy$ and $\neg yRz \Rightarrow \neg xRz$ for all $x, y, z \in S$
c	connex	xRy or yRx for all $x \neq y \in S$
sc	strictly connex	xRy or yRx for all $x, y \in S$

Figure 2.12 Properties of binary relations.

Which implications exist between those properties? For every one of these implications it is easy to determine whether it holds for all binary relations. Only a finite number of such implications is possible (since we are dealing with a finite number of attributes), but at any rate many more than we would care to list exist. Our algorithm should help us to discover "good" implications straight out. Implications which do not hold for all binary relations are refuted by stating counter-examples.

First of all we equip ourselves with a small supply of examples by considering all relations on the one or two-element set. Up to isomorphism there are twelve such relations (Figure 2.13).

Now we have a context to start with (generally, this context can even be empty). Of course, only implications which hold in this context can hold for all binary relations, but not vice versa. Please note that the four objects marked by a $\leftarrow$ are superfluous, since their intents are the intersections of other object intents and therefore respect all implications respected by the other objects. We will leave them out in the following. Now we use the algorithm in order to calculate the first pseudo-intent. The lectically smallest pseudo-intent in this context is $\{sc\}$, with $\{sc\}'' = \{r, t, nt, c, sc\}$. In other words, the implication

$$\{sc\} \to \{r, t, nt, c, sc\}$$

holds in all examples stated so far. Does it hold for binary relations in general? Of course not. A counter-example is for instance $S = \{0, 1, 2\}$, $R = S \times S \setminus \{(0,1), (1,2), (2,0)\}$. This relation is reflexive, antisymmetric,

S	R	r	i	s	as	an	t	nt	c	sc	
$\{0\}$	$\emptyset$		×	×	×	×	×	×	×		
$\{0\}$	$\{(0,0)\}$	×		×		×	×	×	×	×	
$\{0,1\}$	$\emptyset$		×	×	×	×	×	×			
$\{0,1\}$	$\{(0,0)\}$			×		×	×				$\leftarrow$
$\{0,1\}$	$\{(0,0),(1,1)\}$	×		×		×	×				
$\{0,1\}$	$\{(0,0),(0,1)\}$					×	×	×	×		$\leftarrow$
$\{0,1\}$	$\{(0,0),(1,0)\}$					×	×	×	×		$\leftarrow$
$\{0,1\}$	$S \times S \setminus \{(0,1)\}$	×				×	×	×	×	×	
$\{0,1\}$	$S \times S \setminus \{(0,0)\}$			×				×	×		$\leftarrow$
$\{0,1\}$	$\{(0,1)\}$		×		×	×	×	×	×		
$\{0,1\}$	$\{(0,1),(1,0)\}$		×	×				×	×		
$\{0,1\}$	$S \times S$	×		×			×	×	×	×	

Figure 2.13 Examples of binary relations.

No.	S	R
1	$\{0\}$	$\emptyset$
2	$\{0\}$	$\{(0,0)\}$
3	$\{0,1\}$	$\emptyset$
4	$\{0,1\}$	$\{(0,0),(1,1)\}$
5	$\{0,1\}$	$S \times S \setminus \{(0,1)\}$
6	$\{0,1\}$	$\{(0,1)\}$
7	$\{0,1\}$	$\{(0,1),(1,0)\}$
8	$\{0,1\}$	$S \times S$
9	$\{0,1,2\}$	$S \times S \setminus \{(0,1),(1,2),(2,0)\}$
10	$\{0,1,2\}$	$\{(0,1),(1,2),(2,0)\}$
11	$\{0,1,2\}$	$\{(0,1)\}$
12	$\{0,1,2\}$	$\{(0,1),(1,0)\}$
13	$\{0,1,2\}$	$S \times S \setminus \{(0,1)\}$
14	$\{0,1,2\}$	$S \times S \setminus \{(0,1),(1,0)\}$

Figure 2.14 A complete list of examples.

connex and strictly connex and has none of the other properties. We add this example to our context and again ask for the smallest pseudo-intent. It is still $\{sc\}$, but now $\{sc\}'' = \{r, c, sc\}$, and we have to check, whether the implication $\{sc\} \to \{r, c, sc\}$, which we abbreviate by

$$sc \to r, c,$$

holds for all binary relations. As a matter of fact, every strictly connex relation is reflexive and connex. Therefore, we can add this implication to the list of implications $\mathcal{L}$.

The next pseudo-intent is $\{t, c\}$ with $\{t, c\}'' = \{t, nt, c\}$. This suggests the implication

$$t, c \to nt,$$

which in fact holds for binary relations and therefore is added to the list, as well as the following one

$$an, nt \to t,$$

which results from the pseudo-intent $\{an, nt\}$ with

$$\{an, nt\}'' = \{an, t, nt\}.$$

After that, we obtain the pseudo-intent $\{as\}$, for which

$$\{as\}'' = \{i, as, an, t, nt\}$$

holds in the context of the examples. But the implication

$$as \to i, an, t, nt$$

does not hold generally, as the following example shows: $S := \{0, 1, 2\}$, $R := \{(0, 1), (1, 2), (2, 0)\}$. This relation has the attributes i, as, an, nt, and we add it to the context. Since it obviously respects all implications accepted so far, it has no consequences for the pseudo-intents found up to then (cf. Proposition 27).

In the following, we first confirm the implications $as \to i, an$ and $s, c \to nt$ as well as $s, an \to t$, then we state a counter-example for $i, t \to as, an, nt$ etc. The complete result is presented in Figures 2.14 to 2.17.

We point out that the premises of the implications in Figure 2.16 are precisely the pseudo-intents of the context in Figure 2.15.

The procedure does not guarantee that the resulting context is reduced (as in the example). Newly entered objects can make previously entered objects dispensable. It is possible to "row-reduce" the context during the process (i.e., to delete dispensable objects). This has no effect on the implications.

	r	i	s	as	an	t	nt	c	sc
1		×	×	×	×	×	×	×	
2	×		×		×	×	×	×	×
3		×	×	×	×	×	×		
4	×		×		×	×			
5	×				×	×	×	×	×
6		×		×	×	×	×	×	
7		×	×				×	×	
8	×		×			×	×	×	×
9	×				×			×	×
10		×		×	×			×	
11		×		×	×	×			
12		×	×						
13	×						×	×	×
14	×		×						

Figure 2.15 The context of the examples.

strictly connex → reflexive, connex
transitive, connex → negatively transitive
antisymmetric, negatively transitive → transitive
asymmetric → irreflexive, antisymmetric
symmetric, connex → negatively transitive
symmetric, antisymmetric → transitive
irreflexive, transitive → asymmetric, antisymmetric
irreflexive, antisymmetric → asymmetric
irreflexive, symmetric, asymmetric, antisymmetric, transitive → negatively transitive
reflexive, connex → strictly connex
reflexive, negatively transitive → connex, strictly connex
reflexive, symmetric, negatively transitive, connex, strictly connex → transitive
reflexive, irreflexive → all properties

Figure 2.16 A complete and non-redundant list of implications.

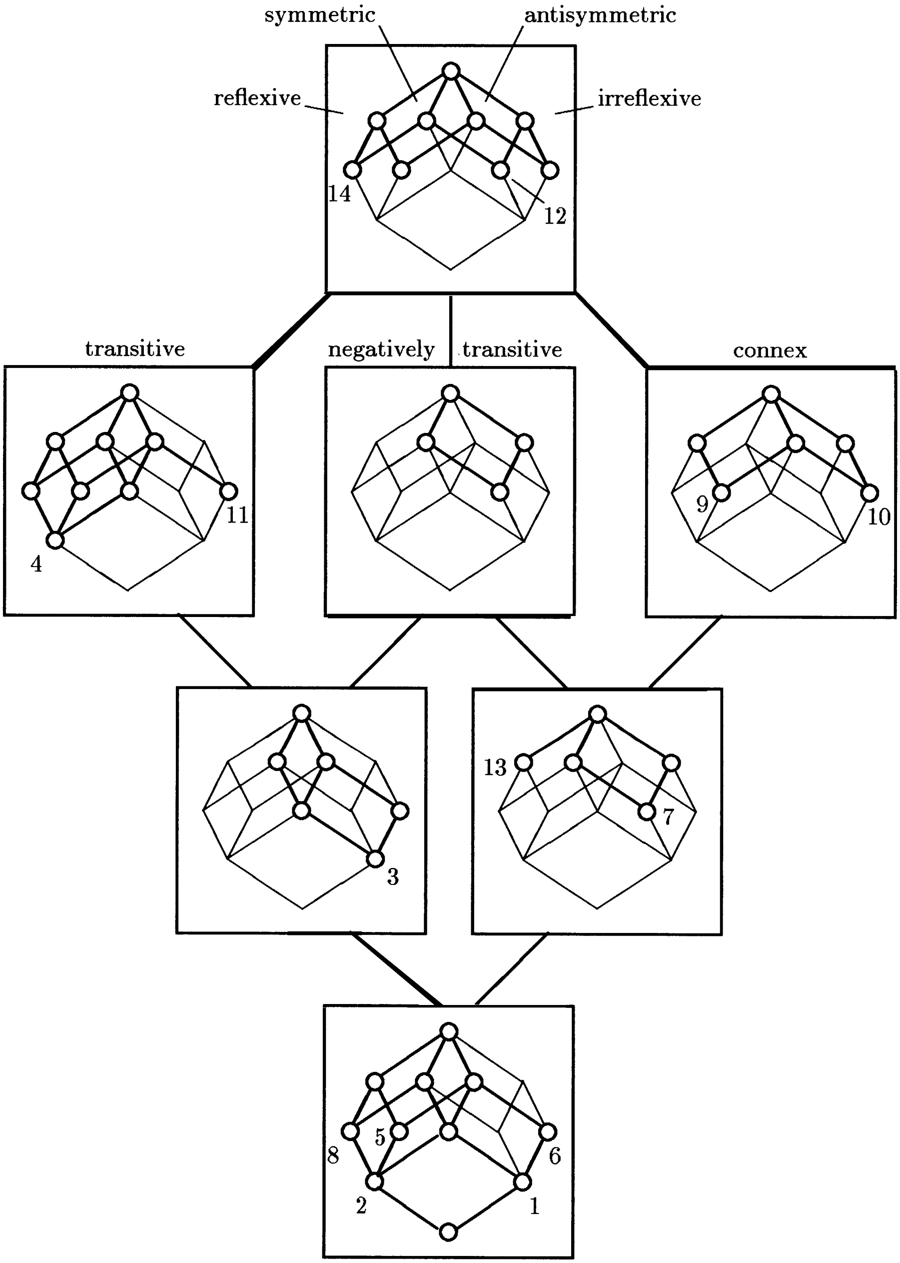


Figure 2.17 The concept lattice for the context of the binary relations.

2.4 Dependencies between Attributes

How can we apply the theory of implications in the case of many-valued contexts? The attribute implications in the derived context offer one approach, however elementary. Basically, it describes implications between the individual attribute values, at least as long as we keep to plain scaling.

In colloquial language we use the term **dependency** of many-valued attributes as exemplified by the following sentence

"The price of a real-estate depends on situation and size".

This is meant to express a simultaneous dependency of attribute values, perhaps even a gradual one, in the sense of "the larger the more expensive".

There are different notions of dependency for many-valued attributes, which correspond to the different possibilities of scaling. For an integration into a general theoretic framework, please refer to the corresponding literature.

We now describe the case of *functional dependency*, the (stronger) one of *ordinal dependency* and will indicate generalizations. For reasons of simplicity, we will first concentrate on complete many-valued contexts.

Definition 41. If $X \subseteq M$ and $Y \subseteq M$ are sets of attributes of a complete many-valued context (G, M, W, I), then we say that Y is **functionally dependent** on X if the following holds for every pair of objects $g, h \in G$:

$$(\forall_{m \in X}\ m(g) = m(h)) \Rightarrow (\forall_{n \in Y}\ n(g) = n(h)).$$

◇

That is to say, if two objects have the same values with respect to all attributes from X the same must be true for the attributes from Y. This notion of dependency is often used in the theory of relational databases. The term "functional" can be explained as follows: Y is functionally dependent on X if and only if there is a map $f : W^X \to W^Y$ with

$$f(m(g) \mid m \in X) = (n(g) \mid n \in Y) \qquad \text{for all } g \in G.$$

In the case of ordinal dependency, we consider an ordinal context, i.e., we have for each attribute $m \in M$ an order $\leq_m$ on the set $m(G)$ of the values of m. (We obtain the special case of functional dependency if we take the equality relation for each of those orders.)

Definition 42. Let (G, M, W, I) be a complete many-valued context and let $\leq_m$ be an order relation on the set $m(G)$ of the values of m for every attribute $m \in M$. If $X \subseteq M$ and $Y \subseteq M$ are sets of attributes, we call Y **ordinally dependent** on X if the following holds for each pair of objects $g, h \in G$:

$$(\forall_{m \in X}\ m(g) \leq_m m(h)) \Rightarrow (\forall_{n \in Y}\ n(g) \leq_n n(h)).$$

◇

Irrespective of which orders $\leq_m$ we have chosen, ordinal dependency always implies functional dependency, since from $m(g) = m(h)$ it follows that $m(g) \leq_m m(h)$ as well as $m(h) \leq_m m(g)$, and vice versa. Thus, one would expect that ordinal dependency is a kind of "order-preserving functional dependency". Intuitively, this is quite correct, but it is difficult to formulate, since the condition of being order-preserving is only required for the tuples $(m(g) \mid m \in X)$ that appear in the context. Not every map of this kind can be extended to form an order-preserving map of W^X to W^Y.

The ordinal dependencies (and as a special case within them the functional dependencies) of many-valued contexts can be expressed elegantly by implications of appropriate one-valued contexts. By means of the rule

$$(g,h)I_{\mathbb{O}}\, m \quad :\Longleftrightarrow m(g) \leq_m m(h),$$

we define a one-valued context

$$\mathbb{K}_{\mathbb{O}} := (G \times G, M, I_{\mathbb{O}})$$

for a complete many-valued context (G, M, W, I) with orders $\leq_m$ on the values. For the functional dependencies the context can be simplified further: It is possible to take advantage of the symmetry of the equality relation and to define

$$\mathbb{K}_{\mathbb{N}} := (\mathfrak{P}_2(G), M, I_{\mathbb{N}})$$

by

$$\{g,h\}I_{\mathbb{N}}\, m \quad :\Longleftrightarrow m(g) = m(h).$$

Then,

$$\mathfrak{P}_2(G) := \{\{g,h\} \mid g,h \in G, g \neq h\}.$$

The contexts defined in this way exactly fit the above-mentioned definitions of the dependencies and it is easy to prove the following proposition:

Proposition 28. *In (G, M, W, I) the attribute set Y is functionally dependent on X if and only if the implication $X \to Y$ holds in the context $\mathbb{K}_{\mathbb{N}}$. In (G, M, W, I) the attribute set Y is ordinally dependent on X if and only if the implication $X \to Y$ holds in the context $\mathbb{K}_{\mathbb{O}}$.* □

Hereby we have traced back the theory of functional and ordinal dependencies completely to the theory of implications. In particular, the algorithm mentioned in the previous section can also be used for the creation of a basis for the functional or ordinal dependencies, respectively.

The translation works even if the many-valued context (G, M, W, I) is not complete. In this connection, first of all we observe that Y is ordinally dependent on X if and only if this is true for every single attribute in Y, i.e., if $\{n\}$ is ordinally dependent on X for every $n \in Y$. This means that it is sufficient to state in which cases a single attribute is dependent on an attribute set. For the general case, this can be formulated as follows:

Definition 43. Let (G, M, W, I) be a many-valued context with an order relation $\leq_m$ on the set $m(G)$ of values for each attribute $m \in M$. If $X \subseteq M$ is a set of attributes and $n \in M$ is an attribute, we say that n is **ordinally dependent** on X if

$$\forall_{m \in X} \ \mathrm{dom}(n) \subseteq \mathrm{dom}(m)$$

and $n(g) \not\leq n(h)$ always implies that there exists an attribute $m \in X$ with $m(g) \not\leq m(h)$. $\diamond$

In order to adapt Proposition 28, we have to modify the definitions of the contexts $\mathbb{K}_{\mathbb{N}}$ and $\mathbb{K}_{\mathbb{O}}$. We introduce a copy $\hat{m}$ for every attribute $m \in M$ which is not complete. These new attributes have to be different from each other and must not belong to M. We add the set

$$\hat{M} := \{\hat{m} \mid \mathrm{dom}(m) \neq G\}$$

to the attribute set of the one-valued context. In the case of complete contexts, we have $\hat{M} = \emptyset$, in general

$$\mathbb{K}_{\mathbb{N}} := (\mathfrak{P}_2(G), M \mathbin{\dot{\cup}} \hat{M}, I_{\mathbb{N}}) \quad \text{and} \quad \mathbb{K}_{\mathbb{O}} := (G \times G, M \mathbin{\dot{\cup}} \hat{M}, I_{\mathbb{O}}),$$

with

$$\{g,h\} I_{\mathbb{N}}\, \hat{m} :\iff (g,h) I_{\mathbb{O}}\, \hat{m} :\iff g \in \mathrm{dom}(m) \text{ and } h \in \mathrm{dom}(m)$$

and, as above,

$$(g,h) I_{\mathbb{O}}\, m :\iff m(g) \leq_m m(h), \quad \{g,h\} I_{\mathbb{N}}\, m :\iff m(g) = m(h).$$

Proposition 28 can now be generalized as follows:

Proposition 29. *The attribute n is functionally (resp. ordinally) dependent on X if and only if the implications $\{\hat{n}\} \cup X \to n$ and $\hat{n} \to \hat{X}$ hold in the context $\mathbb{K}_{\mathbb{N}}$ (or in the context $\mathbb{K}_{\mathbb{O}}$, respectively).* $\square$

Do the approaches presented above extend to notions of dependency other than those of functional and ordinal dependency? For which cases is it possible to represent the dependencies of a many-valued context by means of the implications of an appropriate one-valued context?

Up to now there is no definite answer to these questions. An obvious generalization can be obtained if we consider (complete) many-valued contexts (G, M, W, I) with a given relation Θ_m on W for every attribute $m \in M$. We abbreviate the sequence of these relations by $\Theta := (\Theta_m \mid m \in M)$ and define an attribute set $Y \subseteq M$ to be **Θ-dependent** on a set $X \subseteq M$ if the following holds for each pair of objects $g, h \in G$:

$$(\forall_{m \in X} \ m(g) \Theta m(h)) \Rightarrow (\forall_{n \in Y} \ n(g) \Theta n(h)).$$

A possible interpretation of these kinds of dependency consists in viewing the Θ_m as tolerances or fuzziness. Then, a Θ-dependency describes a "fuzzy functional dependency".

Proposition 28 can be applied to this case without problems. The Θ-dependencies of (G, M, W, I) are precisely the implications of the context

$$\mathbb{K}_\Theta := (G \times G, M, I_\Theta) \qquad \text{with} \qquad (g,h) I_\Theta\, m \quad :\Longleftrightarrow \; m(g)\Theta_m m(h).$$

2.5 Hints and References

2.1 The algorithm in Theorem 5 has been taken from [56], see also [57]. Other algorithms were developed by Fay [53], Norris [131], Bordat [22]. For a comparison see Guenoche [79]. Further developments can be found in Ganter and Reuter [62], [59], Krolak–Schwerdt, Orlik and Ganter [104]. With respect to complexity see Skorsky [157]. A book by Vogt [180] describes a C++ class library for Formal Concept Analysis.

Schütt [154] gives an estimate of the number of concepts depending on $|I|$:

$$|\underline{\mathfrak{B}}(G, M, I)| \leq \frac{3}{2} \cdot 2^{\sqrt{|I|+1}} - 1 \qquad (\text{for } |I| > 2).$$

2.2 The example of the waters from Figure 2.9 has been taken from the paper [96] by Kipke and Wille. The automatic generation of diagrams has been discussed in detail in the works of Skorsky, Luksch and Wille, see [157], [113] and [204] but also Gepperth [69]. Besides, there are numerous implementations, the most widespread one is probably DIAGRAM for DOS by Frank Vogt. TOSCANA [101] is a commercially available program system which facilitates and improves the access to databases by means of elaborate nested line diagrams. See also [207], [208] as well as Kühn & Ries [105]. The geometrical method has been described in [201] and in [171] and has been supported by a program by Kark [93]. Skorsky [156] has examined the rule of parallelograms.

Other ways of representing contexts and concept lattices have been suggested, which we shall not discuss here. See [201], Bokowski and Kollewe [17], Kollewe [100], Lengnink [107], [108].

2.3 Implications and dependencies between attributes have already been examined in [191]. The implication base with the pseudo-intents was introduced into Formal Concept Analysis by Duquenne and Guigues [74], [45], Theorem 8 has also been taken from their book. Similar questions have also been of importance in the theory of relational databases. In this context see Maier [121], Ch. 5. Further investigations can be found in Wild [187], [186], [188].

Proper premises were introduced in [64], see also Rusch and Wille [147].

An implication $A \to B$ only holds in a context if *every* object having all the attributes from A also has all the attributes from B. Various authors have tried to weaken this condition. Burmeister [24] describes implications in the case of incomplete knowledge by means of a three-valued KLEENE-logic. This has also been implemented in his afore-mentioned program CONIMP. Luxenburger [117], [118], [119] examines **partial implications**, i.e., implications which only hold for part of the object set.

The results of Duquenne and Guigues permit a more effective algorithmic implementation of the attribute exploration process, which had already been suggested earlier by Wille. This has been described in [56], [57]. A remarkably early application of this technique was realized by Reeg and Weiß [139]. In the case of their investigation, the attribute set consisted of 50 common properties of finite lattices.

Stumme [170] allows exceptions and background implications.

The method of attribute exploration has been further developed in different ways (cf. [203]): On the one hand into **concept exploration** [198] (see also Klotz and Mann [98]), which instead of attributes uses concepts. A specialization of concept exploration to the distributive case which has practical applications to knowledge acquisition is presented by Stumme [169].

On the other hand, the attribute exploration can be developed into **rule exploration** where the implications are replaced by Horn clauses from predicate logic. This has been investigated by Zickwolff [219].

2.4 Most of the results of this section have been taken from [65]. A uniform theory of the dependency of many-valued attributes has been sketched in [200], compare also [64]. The Θ-dependencies can be looked up in Stöhr and Wille [163]. Umbreit [176] furthermore examines implications and dependencies between fuzzy attributes.

3. Parts and Factors

If one wishes to examine parts of a rather complex concept system, it seems reasonable to exclude some objects and/or attributes from the examination. We shall describe the effects of this procedure on the concept lattice. The concept lattice of a *subcontext* always has an order-embedding into that of the original context. Much more information can be obtained when dealing with *compatible* subcontexts, which will be introduced later in this section. It is easy to identify these particular subcontexts by means of the arrow relations. Thus we obtain a *factor lattice* of the original concept lattice. The interrelations between factor lattices, congruence relations and such subcontexts will be described in the second section.

The *complete sublattices* of a concept lattice can also be described through parts of the context, however not through subcontexts but through subrelations of the incidence relation I, with the object and attribute sets fixed. This kind of *closed relations* will be defined in the third section.

In the fourth section we shall introduce *tolerance relations*, i.e., generalized congruence relations which do not necessarily have to be transitive. It turns out that it is possible to introduce a factor lattice even for tolerance relations. Furthermore, a description within the context is possible: tolerances correspond to certain supersets of the incidence relation I, namely the *block relations.*

3.1 Subcontexts

Definition 44. If (G, M, I) is a context and if $H \subseteq G$ and $N \subseteq M$, then $(H, N, I \cap H \times N)$ is called a **subcontext** of (G, M, I).[1] $\diamond$

We open this section with the question of how the concept system of a subcontext is related to that of (G, M, I). If we merely leave out attributes, i.e., if for a set $N \subseteq M$ we consider the subcontext $(G, N, I \cap G \times N)$, the modification remains transparent. Every attribute extent of $(G, N, I \cap G \times N)$ is also an attribute extent of (G, M, I) and, since every concept extent is the intersection of attribute extents, we obtain:

[1] We write $I \cap H \times N$ for $I \cap (H \times N)$ and instead of $(H, N, I \cap H \times N)$ we sometimes simply use (H, N).

Proposition 30. *If $N \subseteq M$, then every extent of $(G, N, I \cap G \times N)$ is an extent of (G, M, I).* □

This means that the omission of attributes is equivalent to a coarsening of the closure system of the extents. The corresponding is true for the omission of objects. At the same time we obtain a natural embedding of the concept lattice of $(G, N, I \cap G \times N)$ into that of (G, M, I):

Proposition 31. *For $N \subseteq M$, the map*

$$\begin{aligned} \underline{\mathfrak{B}}(G, N, I \cap G \times N) &\to \underline{\mathfrak{B}}(G, M, I) \\ (A, B) &\mapsto (A, A') \end{aligned}$$

is a $\bigwedge$-preserving order-embedding. Dually, for $H \subseteq G$, the map

$$\begin{aligned} \underline{\mathfrak{B}}(H, M, I \cap H \times M) &\to \underline{\mathfrak{B}}(G, M, I) \\ (A, B) &\mapsto (B', B) \end{aligned}$$

is a $\bigvee$-preserving order-embedding. □

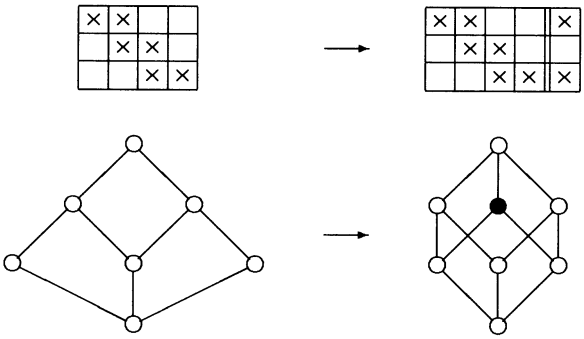


Figure 3.1 A $\bigwedge$-embedding of the concept lattice of a subcontext

An example is shown in Figure 3.1. If we combine the two parts of the proposition, we obtain:

Proposition 32. *If $H \subseteq G$ and $N \subseteq M$, the map*

$$\begin{aligned} \underline{\mathfrak{B}}(H, N, I \cap H \times N) &\to \underline{\mathfrak{B}}(G, M, I) \\ (A, B) &\mapsto (A'', A') \end{aligned}$$

is an order-embedding, and so is the map

$$(A, B) \mapsto (B', B'').$$ □

These order-embeddings are bijective if $(H, N, I \cap H \times N)$ is a **dense subcontext**, i.e., if γH is $\bigvee$-dense and dually μN is $\bigwedge$-dense in $\underline{\mathfrak{B}}(G, M, I)$.

If $\varphi : \underline{\mathfrak{B}}(G, M, I) \to V$ is an order-preserving mapping, then $\alpha := \varphi \circ \gamma$ and $\beta := \varphi \circ \mu$ are maps $\alpha : G \to V$, $\beta : M \to V$ with

$$gIm \Rightarrow \alpha g \leq \beta m.$$

If, conversely, (α, β) is a pair of maps satisfying this condition, then, for instance, the map

$$\varphi(A, B) := \bigvee_{g \in A} \alpha g$$

is order-preserving. A useful special case is considered in the next proposition:

Proposition 33. *An order-embedding of* $\underline{\mathfrak{B}}(G, M, I)$ *in a given complete lattice* V *exists if and only if there are maps* $\alpha : G \to V$, $\beta : M \to V$ *with*

$$gIm \iff \alpha g \leq \beta m.$$

Proof. If $\varphi : \underline{\mathfrak{B}}(G, M, I) \to V$ is an order-embedding, then $\alpha := \varphi \circ \gamma$ and $\beta := \varphi \circ \mu$ have the properties specified. If, conversely, (α, β) is a pair of maps with $gIm \iff \alpha g \leq \beta m$, then the map $\varphi(A, B) := \bigvee_{g \in A} \alpha g$ is order-preserving. We show that φ is, moreover, an order-embedding: If (A_1, B_1) and (A_2, B_2) are concepts and if $(A_1, B_1) \not\leq (A_2, B_2)$, then there exist an object $h \in A_1$ and an attribute $n \in B_2$ with $(h, n) \notin I$, i.e., $\alpha h \not\leq \beta n$. On the other hand, $\alpha g \leq \beta n$ holds for all $g \in A_2$, and we have $\alpha h \not\leq \bigvee\{\alpha g \mid g \in A_2\}$. Consequently, $\varphi(A_1, B_1)$ cannot be less than or equal to $\varphi(A_2, B_2)$. □

This means that the concept lattice of a subcontext is isomorphic to a suborder of the entire concept lattice (which is not necessarily a sublattice). The derivation operators with respect to a subcontext

$$(H, N, I \cap H \times N)$$

can be expressed in terms of those of (G, M, I): If $A \subseteq H$, then the set of common attributes with respect to $(H, N, I \cap H \times N)$ is equal to $A' \cap N$. Dually, the extent of $(H, N, I \cap H \times N)$ belonging to a set $B \subseteq N$ is equal to $B' \cap H$. However, the concepts of a subcontext cannot simply be derived from those of (G, M, I) by restricting their extent and intent to a subcontext. This can be done only for compatible subcontexts, which will be examined next.

Definition 45. A subcontext $(H, N, I \cap H \times N)$ is called **compatible** if the pair $(A \cap H, B \cap N)$ is a concept of the subcontext for every concept $(A, B) \in \underline{\mathfrak{B}}(G, M, I)$. ◇

Restricting the concepts to a compatible subcontext yields a map between the concept lattices, which necessarily has to be structure-preserving, as the following proposition shows:

			$\downarrow$	$\downarrow$	$\downarrow$
$\rightarrow$	×		×		
$\rightarrow$	×	×	×		×
$\rightarrow$		×			×
			×	×	×

Figure 3.2 Example of a compatible subcontext.

Proposition 34. *A subcontext $(H, N, I \cap H \times N)$ of (G, M, I) is compatible if and only if*

$$\Pi_{H,N}(A, B) := (A \cap H, B \cap N) \quad \text{for all } (A, B) \in \underline{\mathfrak{B}}(G, M, I)$$

defines a surjective complete homomorphism

$$\Pi_{H,N} : \underline{\mathfrak{B}}(G, M, I) \to \underline{\mathfrak{B}}(H, N, I \cap H \times N).$$

Proof. According to Definition 45, $(H, N, I \cap H \times N)$ is compatible if and only if $\Pi_{H,N}$ is a map. The fact that this map must necessarily be infimum-preserving can be recognized by examining the extents: The map $A \mapsto A \cap H$ is evidently $\bigcap$-preserving, and the infimum of concepts is defined in terms of the intersection of their extents. (cf. Basic Theorem). Dually, we infer that $\Pi_{H,N}$ is supremum-preserving. The surjectivity can be seen as follows: If $(C, C' \cap N)$ is a concept of $(H, N, I \cap H \times N)$, then $\Pi_{H,N}(C'', C') = (C'' \cap H, C' \cap N)$ is a concept with the same intent, i.e., the same concept. □

If there is a surjective complete homomorphism from a complete lattice $\boldsymbol{V}$ onto a complete lattice $\boldsymbol{W}$, then $\boldsymbol{W}$ is sometimes also called a (complete) **homomorphic image** of $\boldsymbol{V}$. Thus, the above proposition says that the concept lattice of a compatible subcontext of (G, M, I) is always a homomorphic image of $\underline{\mathfrak{B}}(G, M, I)$. For structure theory it is an important question whether the converse is true as well, i.e., whether every homomorphic image originates from a compatible subcontext. We shall defer this question until Section 3.2.

How can we recognize compatible subcontexts? We first give a technical condition, which is often used in proofs. For algorithms, however, the characterization by means of the arrow relations is more appropriate. We shall introduce it later on.

Proposition 35. *$(H, N, I \cap H \times N)$ is a compatible subcontext of (G, M, I), if and only if:*

a1) for every object $h \in H$ and every attribute $m \in M$ with $h \not I m$, there is some attribute $n \in N$ with $h \not I n$ and $m' \subseteq n'$,
a2) for every attribute $n \in N$ and every object $g \in G$ with $g \not I n$, there is some object $h \in H$ with $h \not I n$ and $g' \subseteq h'$.

Equivalent to these are the following conditions:

b1) $(A' \cap N)' \cap H \subseteq A''$ *for all* $A \subseteq G$,

b2) $(B' \cap H)' \cap N \subseteq B''$ *for all* $B \subseteq M$.

Proof. If $(H, N, I \cap H \times N)$ is compatible and $m \in M$, then $(m' \cap H, m'' \cap N)$ has to be a concept of the subcontext. If, therefore, $h \in H$ is an object with $h \not I m$, there must be an attribute $n \in m'' \cap N$ with $h \not I n$. This is precisely condition a1). a2) follows dually.

Now if a1) and a2) are satisfied, we show that b1) must hold: Assume that $A \subseteq G$, $h \notin A''$, $h \in H$. Then there exists some $m \in A'$ (i.e., $m' \supseteq A$) with $h \not I m$, i.e., by a1) some $n \in A' \cap N$ with $n' \supseteq A$ and $h \not I n$. Consequently, $h \notin (A' \cap N)'$ and thus $(A' \cap N)' \cap H \subseteq A''$. b2) follows correspondingly.

It remains to be shown that $(H, N, I \cap H \times N)$ is compatible if b1) and b2) are satisfied. Let (A, B) be a concept of (G, M, I). Then $(A \cap H)' \cap N \supseteq A' \cap N = B \cap N$ and, by applying b2), $(A \cap H)' \cap N = (B' \cap H)' \cap N \subseteq B'' \cap N = B \cap N$, i.e., $(A \cap H)' \cap N = B \cap N$ and dually $A \cap H = (B \cap N)' \cap H$. Therefore, $(A \cap H, B \cap N)$ is a concept of $(H, N, I \cap H \times N)$. □

In the case of doubly founded contexts, the compatible subcontexts can be easily identified by means of the arrow relations.

Definition 46. A subcontext $(H, N, I \cap H \times N)$ of a clarified context (G, M, I) is **arrow-closed** if the following holds: $h \nearrow m$ and $h \in H$ together imply $m \in N$, and $g \swarrow n$ and $n \in N$ together imply $g \in H$. ◇

Proposition 36. *Every compatible subcontext is arrow-closed.*

Every arrow-closed subcontext of a doubly founded context is compatible.

Proof. If $(H, N, I \cap H \times N)$ is compatible and $h \in H$, $m \in M$ are such that $h \nearrow m$, then by 35.a1) there is an attribute $n \in N$ with $h \not I n$ and $m' \subseteq n'$. Because of $h \nearrow m$, m' is maximal with respect to $h \not I m$, i.e., $m' = n'$, i.e., $m = n$, i.e., $m \in N$. Dually, $g \swarrow n$ and $n \in N$ yield $g \in H$.

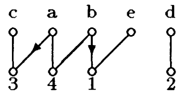
If, conversely, $(H, N, I \cap H \times N)$ is an arrow-closed subcontext of a doubly founded context, we can prove 35.a1): Let $h \in H$ be an object and let $m \in M$ be an attribute with $h \not I m$. By Definition 26 there exists an attribute n with $m' \subseteq n'$ and $h \nearrow n$, i.e., $n \in N$, which was to be proved. 35.a2) follows correspondingly. □

Thus, in the case of doubly founded contexts it is easy to determine the compatible subcontexts. We enter the arrow relations $\nearrow$ and $\swarrow$ into the context and examine the directed graph $(G \dot\cup M, \nearrow \cup \swarrow)$. The compatible subcontexts then correspond exactly to the arrow-closed components of the directed graph. If we furthermore assume that (G, M, I) is reduced, we can elegantly describe the arrow-closed subcontexts in terms of the concepts of a context. For this purpose we need the transitive closure of the arrow relations, as introduced in the following definition:

Definition 47. For $g \in G$ and $m \in M$ we write $g \swarrow\!\!\!\swarrow m$ if there are objects $g = g_1, g_2, \ldots, g_k \in G$ and attributes $m_1, m_2, \ldots, m_k = m \in M$ with $g_i \swarrow m_i$

	a	b	c	d	e
1	$\times$	$\swarrow$	$\times$		$\swarrow\!\!\!\!\nearrow$
2	$\times$	$\times$	$\times$	$\swarrow\!\!\!\!\nearrow$	$\times$
3	$\swarrow$	$\times$	$\swarrow\!\!\!\!\nearrow$		$\times$
4	$\swarrow\!\!\!\!\nearrow$	$\swarrow\!\!\!\!\nearrow$	$\times$	$\times$	$\times$

Context example



The graph of the arrows

	a	b	c	d	e
1	$\swarrow\!\!\!\swarrow$	$\swarrow\!\!\!\swarrow$			$\swarrow\!\!\!\swarrow$
2				$\swarrow\!\!\!\swarrow$	
3	$\swarrow\!\!\!\swarrow$	$\swarrow\!\!\!\swarrow$	$\swarrow\!\!\!\swarrow$		
4	$\swarrow\!\!\!\swarrow$	$\swarrow\!\!\!\swarrow$			

The relation $\swarrow\!\!\!\swarrow$

$\not\swarrow\!\!\!\swarrow$	a	b	c	d	e
1			$\times$	$\times$	
2	$\times$	$\times$	$\times$		$\times$
3				$\times$	$\times$
4			$\times$	$\times$	$\times$

$(G, M, \not\swarrow\!\!\!\swarrow)$

Figure 3.3 With reference to Proposition 37

for $i \in \{1, \ldots, k\}$ and $g_j \nearrow m_{j-1}$ for $j \in \{2, \ldots, k\}$. The complement (the negation) of this relation is denoted by $\not\swarrow\!\!\!\swarrow$, i.e., $g \not\swarrow\!\!\!\swarrow m \Leftrightarrow$ not $g \swarrow\!\!\!\swarrow m$. $\diamond$

Proposition 37. *Let (G, M, I) be a reduced doubly founded context. Then $(H, N, I \cap H \times N)$ is an arrow-closed subcontext if and only if $(G \setminus H, N)$ is a concept of the context $(G, M, \not\swarrow\!\!\!\swarrow)$.*

Proof. From the presuppositions *doubly founded* and *reduced* it follows by Proposition 13 (p. 31) that for every object g there is an attribute m with $g \swarrow\!\!\!\!\nearrow m$, and dually. First of all, let $(H, N, I \cap H \times N)$ be arrow-closed. If $g \in H$ and $g \swarrow\!\!\!\!\nearrow m$, then $m \in N$ must be true, i.e., $g \in H$ holds if and only if there is an $n \in N$ with $g \swarrow\!\!\!\swarrow n$. Consequently, $g \in G \setminus H$ if and only if $g \not\swarrow\!\!\!\swarrow n$ for all $n \in N$, i.e., $G \setminus H = N^{\not\swarrow\!\!\!\swarrow}$. Now assume that $m \in M \setminus N$ and $g \swarrow\!\!\!\!\nearrow m$. $g \in H$ is impossible because $(H, N, I \cap H \times N)$ is arrow-closed. Therefore, we get $g \in G \setminus H$, $g \swarrow\!\!\!\swarrow m$ and thus $m \notin (G \setminus H)^{\not\swarrow\!\!\!\swarrow}$. This shows $(G \setminus H)^{\not\swarrow\!\!\!\swarrow} \subseteq N$, which means that $(G \setminus H, N)$ is a concept of $(G, M, \not\swarrow\!\!\!\swarrow)$.

For the converse we assume that $(G \setminus H, N)$ is a concept of $(G, M, \not\swarrow\!\!\!\swarrow)$. From $g \swarrow n$ and $n \in N$ it immediately follows that $g \in H$; if we have $h \nearrow m$ and $h \in H$, it remains to be proved that $m \in N$. Assuming that $m \notin N$, there would be an object $g \in (G \setminus H)$ with $g \swarrow\!\!\!\swarrow m$, and because of $h \in H$ an attribute $n \in N$ with $h \swarrow\!\!\!\swarrow n$. Taken together, $g \swarrow\!\!\!\swarrow m$, $h \nearrow m$ and $h \swarrow\!\!\!\swarrow n$ yield $g \swarrow\!\!\!\swarrow n$, which is impossible. Thus, $(H, N, I \cap H \times N)$ is arrow-closed. $\square$

Proposition 38. *Every compatible subcontext of a clarified (resp. reduced, resp. doubly founded) context is clarified (resp. reduced, resp. doubly founded).*

The arrow relations are inherited by compatible subcontexts, i.e., $g \swarrow m$ holds in $(H, N, I \cap H \times N)$ if and only if $g \in H$, $m \in N$ and $g \swarrow m$ hold in (G, M, I), and the corresponding is true for $\nearrow$.

Proof. Several times in the proof we use the following argument: If $h_1, h_2 \in H$ are objects with $h_1' \cap N \subseteq h_2' \cap N$, then $h_1' \subseteq h_2'$. This follows from Proposition 35: If m were an attribute with $m \in h_1' \setminus h_2'$, we should obtain by a1) an attribute $n \in N \cap (h_1' \setminus h_2')$, which in the case of $h_1' \cap N \subseteq h_2' \cap N$ is impossible. (Of course, the corresponding applies to the attributes.) This immediately yields the first assertion: $h_1' \cap N = h_2' \cap N$ implies $h_1' = h_2'$, i.e., objects with the same object intents in the subcontext have the same object intents in general.

If $h \in H$ is irreducible in (G, M, I), then there exists an attribute m with $h \not I m$ and gIm for every $g \in G$ with $g' \supsetneq h'$. By 35.a1) we find an $n \in N$ with $h \not I n$ and $n' \supseteq m'$, i.e., with gIn for all $g \in G$ with $g' \supsetneq h'$ and particularly gIn for all $g \in H$ with $g' \cap N \supsetneq h' \cap N$. Together with the dual consideration this shows that a compatible subcontext of a reduced context is reduced as well.

Now concerning the arrows: First of all we assume that $h \in H$ and $n \in N$ and that $h \swarrow n$ in (G, M, I). Then h' is maximal with respect to $n \notin h'$. According to our previous considerations, in this case $h' \cap N$ is maximal with respect to $n \notin h' \cap N$. This means that $h \swarrow n$ also holds in $(H, N, I \cap H \times N)$. Thus, all arrows from (G, M, I) are being preserved in $(H, N, I \cap H \times N)$.

Is it possible, conversely, to infer $h \swarrow n$ in (G, M, I) from $h \swarrow n$ in $(H, N, I \cap H \times N)$? If not, there would have to be an object $g \in G$ with $g \not I n$ and $g' \supset h'$ and furthermore by Proposition 35.a2) an object $h_2 \in H$ with $h_2 \not I n$ and $h_2' \supseteq g'$, from which would follow $h_2' \supset h'$ and $h_2 \not I n$ (the fact that $h_2' \cap N = h' \cap N$ is impossible, again follows from our first consideration). Thus, h' would not be maximal among the extents of $(H, N, I \cap H \times N)$ which do not contain n, in contradiction to the precondition $h \swarrow n$.

We are still lacking the proof that the property of being doubly founded is inherited by compatible subcontexts. Thus, assume that $h \in H, n \in N$ and $h \not I n$. If (G, M, I) is doubly founded there is an attribute $m \in M$ with $h \nearrow m$ and $n' \subseteq m'$. We apply Proposition 35.a1) and obtain an attribute $n_2 \in N$ with $h \not I n_2$ and $m' \subseteq n_2'$. It follows that $m' = n_2'$ and thus that $h \nearrow n_2$, which according to what we have just proved transfers to $(H, N, I \cap H \times N)$. One of the conditions of doubly foundedness is proved thereby, the other one follows dually. □

We should also mention that dense subcontexts are always compatible:

Proposition 39. *For a subcontext $(H, N, I \cap H \times N)$ of (G, M, I) the following statements are equivalent:*

1. $(H, N, I \cap H \times N)$ *is dense.*
2. $(H, N, I \cap H \times N)$ *is compatible and the map $\Pi_{H,N}$ is injective.*
3. *for every concept (A, B) of (G, M, I),*

$$(A \cap H)'' = A \text{ and } (B \cap N)'' = B.$$

Proof. 1) $\Leftrightarrow$ 3): $(H, N, I \cap H \times N)$ is dense if and only if both, for every object $g \in G$ there is a subset $X \subseteq H$ with $\gamma g = \bigvee_{x \in X} \gamma x$, i.e., with $g' = X'$ and thus $g \in X''$, and the dual condition holds for the attributes. Because of $\gamma x \leq \gamma g$ for all $x \in X$ we have $X \subseteq g''$ and thus the condition from 3) for the case $A = g''$. The more general condition follows without difficulty.
3) $\Rightarrow$2): In order to show that $(H, N, I \cap H \times N)$ is compatible, we prove the conditions b) from Proposition 35: If $A \subseteq G$, then A' is an intent and on account of 3) it satisfies $(A' \cap N)'' = A'$, which yields $(A' \cap N)' = A''$

and thus b1). b2) is dual. The injectivity of $\Pi_{H,N}$ immediately follows from $(A \cap H)'' = A$.
2) $\Rightarrow$ 3): If $\Pi_{H,N}$ is injective and (A, B) is a concept of (G, M, I), then $(A \cap H)'' = A$ must hold, otherwise $(A \cap H)''$ and A would be different extents having the same intersection with H. □

3.2 Complete Congruences

In the preceding section we have seen that, for a compatible subcontext $(H, N, I \cap H \times N)$ of (G, M, I), the map $\Pi_{H,N}$ is a surjective complete homomorphism of $\mathfrak{B}(G, M, I)$ onto $\mathfrak{B}(H, N, I \cap H \times N)$, i.e., that the concept lattice of the subcontext is always a homomorphic image of $\mathfrak{B}(G, M, I)$. We shall now examine whether the converse is true as well, i.e., whether every surjective complete homomorphism, every homomorphic image can be described in terms of a subcontext. In the case of finite contexts this is true, in the case of infinite contexts not in general. In order to clarify the situation, we require a notion from lattice theory, namely that of the complete congruence relation.

Definition 48. A **complete congruence relation** of a complete lattice $\boldsymbol{V}$ is an equivalence relation Θ on $\boldsymbol{V}$ satisfying:

$$x_t \Theta y_t \text{ for } t \in T \Rightarrow (\bigwedge_{t \in T} x_t)\Theta(\bigwedge_{t \in T} y_t) \text{ and } (\bigvee_{t \in T} x_t)\Theta(\bigvee_{t \in T} y_t).$$

We define

$$[x]\Theta := \{y \in V \mid x\Theta y\},$$

which is the equivalence class of Θ containing x. The **factor lattice**

$$\boldsymbol{V}/\Theta := \{[x]\Theta \mid x \in V\}$$

has the order

$$[x]\Theta \leq [y]\Theta :\Leftrightarrow x\Theta(x \wedge y) \qquad (\Leftrightarrow (x \vee y)\Theta y).$$

In order to demonstrate that this is really an order relation we can, for instance, argue as follows: If we define

$$x_\Theta := \bigwedge\{y \in \boldsymbol{V} \mid y\Theta x\} = \bigwedge[x]\Theta$$

$$\text{and } x^\Theta := \bigvee\{y \in \boldsymbol{V} \mid y\Theta x\} = \bigvee[x]\Theta$$

for $x \in \boldsymbol{V}$, this immediately yields $[x]\Theta = \left[x_\Theta, x^\Theta\right]$ and $[x]\Theta \leq [y]\Theta \Leftrightarrow x_\Theta \leq y_\Theta \Leftrightarrow x^\Theta \leq y^\Theta$. Thus, the **congruence classes**, i.e., the classes of a congruence relation, are intervals. They are ordered according to their smallest, or, which is the same, according to their largest elements. We infer that

$$\bigwedge_{t\in T}[x_t]\Theta = \left[\bigwedge_{t\in T} x_t\right]\Theta \text{ and } \bigvee_{t\in T}[x_t]\Theta = \left[\bigvee_{t\in T} x_t\right]\Theta.$$

◇

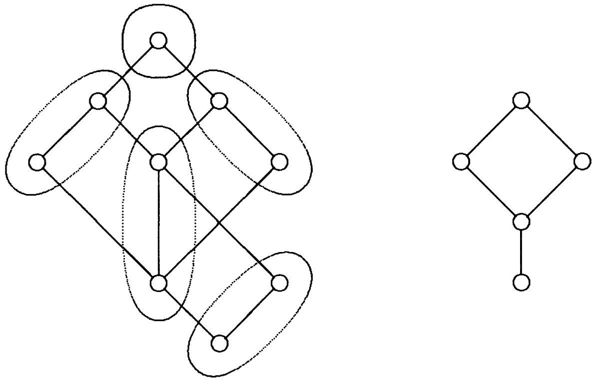


Figure 3.4 Congruence and factor lattice

If in the following we speak of congruence relations or **congruences**, we mean complete congruence relations. The significance of the congruences for the problem we are concerned with is revealed by the following *Homomorphism Theorem.* It asserts among other things that every homomorphic image of a complete lattice can be found within the lattice itself, namely as a factor lattice.

Theorem 9. (Homomorphism Theorem) *If Θ is a complete congruence relation of a complete lattice $\mathbf{V}$, then $x \mapsto [x]\Theta$ is a complete homomorphism of $\mathbf{V}$ onto $\mathbf{V}/\Theta$. If, conversely, $\varphi : \mathbf{V}_1 \longrightarrow \mathbf{V}_2$ is a surjective complete homomorphism between complete lattices, then*

$$\ker\varphi := \{(x, y) \in V_1 \times V_1 \mid \varphi(x) = \varphi(y)\}$$

is a complete congruence relation of $\mathbf{V}_1$; besides,

$$[x]\ker\varphi \mapsto \varphi(x)$$

describes an isomorphism of $\mathbf{V}_1/\ker\varphi$ onto $\mathbf{V}_2$.

Proof. The homomorphism properties of the map $x \mapsto [x]\Theta$ have been proved above. $\ker\varphi$ is evidently an equivalence relation. Moreover,

$$\begin{aligned}(x_t, y_t) \in \ker\varphi \text{ for all } t \in T \quad &\Leftrightarrow \quad \varphi(x_t) = \varphi(y_t) \text{ for all } t \in T\\ &\Rightarrow \quad \varphi\left(\bigwedge_{t\in T} x_t\right) = \bigwedge_{t\in T}\varphi(x_t)\end{aligned}$$

$$= \bigwedge_{t\in T} \varphi(y_t) = \varphi\left(\bigwedge_{t\in T} y_t\right)$$
$$\Rightarrow \left(\bigwedge_{t\in T} x_t\right) \ker\varphi \left(\bigwedge_{t\in T} y_t\right);$$

the $\bigvee$-compatibility follows by analogy. $([x]\ker\varphi) \mapsto \varphi(x)$ describes a bijection which evidently satisfies the conditions for homomorphisms. □

The map $x \mapsto [x]\Theta$ is occasionally denoted by π_Θ and is called **canonical projection** onto the factor lattice.

We shall supply one further result from lattice theory, which we shall need later on. It describes which equivalence relations are congruences in a given lattice.

Theorem 10 (Characterization of Complete Congruence Relations). *An equivalence relation Θ on a complete lattice $\mathbf{V}$ is a complete congruence relation of $\mathbf{V}$ if and only if every equivalence class of Θ is an interval of $\mathbf{V}$ (we then assume that $[x]\Theta =: \left[x_\Theta, x^\Theta\right]$ for all $x \in V$), the lower bounds of these intervals being closed under suprema (i.e., $\bigvee_{t\in T} x_\Theta^t = y_\Theta$) and the upper bounds of these intervals being closed under infima (i.e., $\bigwedge_{t\in T} x_t^\Theta = z^\Theta$).*

Proof. Let Θ be a complete congruence relation of $\mathbf{V}$. Then for $x \in \mathbf{V}$, $x_\Theta := \bigwedge[x]\Theta$ and $x^\Theta := \bigvee[x]\Theta$ are elements of $[x]\Theta$. If $x_\Theta \le y \le x^\Theta$, from $y\Theta y$ and $x_\Theta \Theta x^\Theta$ it follows that

$$y = (x_\Theta \vee y)\Theta(x^\Theta \vee y) = x^\Theta$$

and thus $y \in [x]\Theta$. This means that $[x]\Theta = \left[x_\Theta, x^\Theta\right]$. The maps $x \mapsto x_\Theta$ and $x \mapsto x^\Theta$ form a Galois-connection between $\mathbf{V}$ and $\mathbf{V}^d$, since $x \le y \Rightarrow x_\Theta \le y_\Theta$, $x \ge y \Rightarrow x^\Theta \ge y^\Theta$, $x \le (x_\Theta)^\Theta$, $x \ge (x^\Theta)_\Theta$. By Proposition 7 it follows that $x \mapsto x_\Theta$ is a $\bigvee$-homomorphism and $x \mapsto x^\Theta$ is a $\bigwedge$-homomorphism of $\mathbf{V}$ in itself. In particular $\bigvee_{t\in T} x_\Theta^t = \left(\bigvee_{t\in T} x^t\right)_\Theta$ and $\bigwedge_{t\in T} x_t^\Theta = \left(\bigwedge_{t\in T} x_t\right)^\Theta$, which was to be proved. Conversely, we assume that Θ is an equivalence relation on $\mathbf{V}$ the equivalence classes of which are intervals with supremum-dense lower bounds and infimum-dense upper bounds. Assume that $x_t \Theta y_t$ for $t \in T$. Then $(x_t)_\Theta = (y_t)_\Theta$ and $(x_t)^\Theta = (y_t)^\Theta$ for $t \in T$. Consequently,

$$\bigvee_{t\in T} (x_t)_\Theta \le \bigvee_{t\in T} x_t, \quad \bigvee_{t\in T} y_t \le \bigvee_{t\in T} (x_t)^\Theta.$$

Since, in general, $a \le b$ implies $a^\Theta \le b^\Theta$ (because $a \le b$ yields $a \le a^\Theta \wedge b^\Theta \le a^\Theta$ and thus, since $a^\Theta \wedge b^\Theta$ is an upper interval bound, $a^\Theta \wedge b^\Theta = a^\Theta$), from $(x_s)_\Theta \le \bigvee_{t\in T}(x_t)_\Theta$ for every $s \in T$ we furthermore obtain $(x_s)^\Theta = ((x_s)_\Theta)^\Theta \le (\bigvee_{t\in T}(x_t)_\Theta)^\Theta$ and thus $\bigvee_{t\in T}(x_t)^\Theta \le \left(\bigvee_{t\in T}(x_t)_\Theta\right)^\Theta$. Therefore:

$$\bigvee_{t\in T} x_t, \bigvee_{t\in T} y_t \in \left[\bigvee_{t\in T}(x_t)_\Theta, (\bigvee_{t\in T}(x_t)_\Theta)^\Theta\right],$$

i.e., $(\bigvee_{t\in T} x_t)\,\Theta\,(\bigvee_{t\in T} y_t)$. Dually, we can show the $\bigwedge$-compatibility of Θ, whereby we have proved that Θ is a complete congruence relation.

□

In the light of the Homomorphism Theorem our question "What is the connection between the compatible subcontexts and the congruences?" comes up again. From the Homomorphism Theorem it immediately follows that the concept lattice of a compatible subcontext $(H, N, I \cap H \times N)$ of (G, M, I) is always isomorphic to a factor lattice of $\underline{\mathfrak{B}}(G, M, I)$: $(H, N, I \cap H \times N)$ induces a complete congruence $\Theta_{H,N}$ on $\underline{\mathfrak{B}}(G, M, I)$, namely the kernel of the complete homomorphism $\Pi_{H,N}$, and we get

$$\underline{\mathfrak{B}}(H, N, I \cap H \times N) \cong \underline{\mathfrak{B}}(G, M, I)/\Theta_{H,N},$$

with

$$(A_1, B_1)\Theta_{H,N}(A_2, B_2) \Leftrightarrow A_1 \cap H = A_2 \cap H \Leftrightarrow B_1 \cap N = B_2 \cap N.$$

It is easy to identify the smallest and the largest elements of the congruence classes. If (A, B) is a concept, the smallest element of the congruence class $[(A, B)]\Theta_{H,N}$ is the concept $((A\cap H)'', (A\cap H)')$ and the largest is the concept $((B \cap N)', (B \cap N)'')$.

We say that a complete congruence Θ is **induced by a subcontext** if there is a compatible subcontext $(H, N, I \cap H \times N)$ with $\Theta = \Theta_{H,N}$. Using the connection between compatible subcontexts and congruences, we shall prove the following: in the case of a doubly founded concept lattice every congruence is induced by a subcontext. Provided that the context is reduced, this subcontext is uniquely determined by the congruence. These restricting preconditions are not superfluous, i.e., the general theory is somewhat more complicated.

First, we shall examine the problem of uniqueness. A congruence can be induced by various subcontexts. These, however, only differ in their reducible objects and attributes. Among all the possible subcontexts there is always a largest one.

Proposition 40. *If a complete congruence Θ is induced by a subcontext $(H, N, I \cap H \times N)$, then*

$$H \subseteq G_\Theta := \{g \in G \mid \gamma g \text{ is the smallest element of a } \Theta\text{-class}\} \text{ and}$$

$$N \subseteq M_\Theta := \{m \in M \mid \mu m \text{ is the largest element of a } \Theta\text{-class}\}.$$

In this case, Θ is also induced by the compatible subcontext

$$(G_\Theta, M_\Theta, I \cap G_\Theta \times M_\Theta).$$

Proof. If Θ is the congruence induced by $(H, N, I \cap H \times N)$ (i.e., $\Theta = \Theta_{H,N}$), then according to the above-mentioned description of the smallest elements of the $\Theta_{H,N}$-classes we have:

$$g \in G_\Theta \Leftrightarrow \text{ there is } X \subseteq H \text{ with } X' = g'.$$

This immediately yields $H \subseteq G_\Theta$ and dually $N \subseteq M_\Theta$.

Why is $(G_\Theta, M_\Theta, I \cap G_\Theta \times M_\Theta)$ compatible? We use Proposition 35 and prove condition a1): If $g \in G_\Theta$ and $m \in M$ with $g \not I m$, then there is a set $X \subseteq H$ with $g' = X'$, i.e., in particular some $h \in H$ with $h \not I m$ and $g' \subseteq h'$. Since $(H, N, I \cap H \times N)$ is compatible, there is an attribute $n \in N$ with $h \not I n$ and $m' \subseteq n'$. This means that $g \not I n$, $n \in N \subseteq M_\Theta$ and $m' \subseteq n'$ hold as well. Thus, condition a1) is satisfied, and a2) can be shown dually.

Finally, it remains to be proved that $(G_\Theta, M_\Theta, I \cap G_\Theta \times M_\Theta)$ induces the same congruence as $(H, N, I \cap H \times N)$. In order to do so, it suffices to show that from $(A_1, B_1)\Theta(A_2, B_2)$ it always follows that $A_1 \cap G_\Theta = A_2 \cap G_\Theta$; the converse implication immediately follows from $H \subseteq G_\Theta$ and $N \subseteq M_\Theta$. We assume that $g \in A_1 \cap G_\Theta$. Then there is some $X \subseteq H$ with $X' = g'$, consequently $X \supseteq B_1$ and therefore $X'' \subseteq A_1$, from which it follows that $X = X \cap H \subseteq A_1 \cap H = A_2 \cap H$ and thus $g \in A_2$. □

Hence, it is possible to "saturate" a compatible subcontext by adding reducible objects and attributes, without changing the corresponding congruence.

Definition 49. A subcontext $(H, N, I \cap H \times N)$ of (G, M, I) is called **saturated** if:

from $g \in G, X \subseteq H$ and $X' = g'$ it follows that $g \in H$ and

from $m \in M, Y \subseteq N$ and $Y' = m'$ it follows that $m \in N$. ◇

The preceding proposition together with this definition immediately yields:

Proposition 41. *If a congruence Θ is induced by some subcontext, then it is also induced by a saturated subcontext, which is then equal to*

$$(G_\Theta, M_\Theta, I \cap G_\Theta \times M_\Theta).$$

In a reduced context every subcontext is saturated. □

Now we turn to the second part of the question: Which congruences are induced by subcontexts? On account of the propositions we know that $H = G_\Theta$ and $N = M_\Theta$ can be chosen, if Θ is at all induced by a subcontext. It is easy to state congruences which do not have this form, these examples are however infinite. The following propositions provide an exact clarification.

Proposition 42. *A complete congruence relation Θ is induced by a subcontext if and only if $\{[\gamma h]\Theta \mid h \in G_\Theta\}$ is supremum-dense and $\{[\mu n]\Theta \mid n \in M_\Theta\}$ is infimum-dense in $\underline{\mathfrak{B}}(G, M, I)/\Theta$.*

Proof. If Θ is induced by a subcontext, then

$$\underline{\mathfrak{B}}(H, N, I \cap H \times N) \cong \underline{\mathfrak{B}}(G, M, I)/\Theta,$$

and the isomorphism $(A, B) \mapsto [(A, B)]\Theta$ maps the supremum-dense set $\{\gamma h \mid h \in H\}$ onto $\{[\gamma h]\Theta \mid h \in H\}$, i.e., this set is supremum-dense in $\underline{\mathfrak{B}}(G, M, I)/\Theta$, and dually $\{[\mu n]\Theta \mid n \in N\}$ is infimum-dense. Because of $H \subseteq G_\Theta$ and $N \subseteq M_\Theta$ thus the direction "$\Rightarrow$" of the assertion follows.

We begin the proof of the other direction by showing that $(G_\Theta, M_\Theta, I \cap G_\Theta \times M_\Theta)$ is compatible under the conditions specified. Assume that $h \in G_\Theta$ and that $m \in M$ with $h \not I m$. Then $[\gamma h]\Theta \not\leq [\mu m]\Theta$ and, since $\{[\mu n]\Theta \mid n \in M_\Theta\}$ is infimum-dense in $\underline{\mathfrak{B}}(G, M, I)/\Theta$, there is some $n \in M_\Theta \setminus h'$ with $\mu n \geq \mu m$, i.e., $n' \supseteq m'$. This yields 35.a1) and dually 35.a2).
In order to show that Θ is induced by $(G_\Theta, M_\Theta, I \cap G_\Theta \times M_\Theta)$, we have to prove that

$$(A, B)\Theta(C, D) \Leftrightarrow A \cap G_\Theta = C \cap G_\Theta$$

for $(A, B), (C, D) \in \underline{\mathfrak{B}}(G, M, I)$. Let $(\underline{A}, \underline{B})$ be the smallest concept in the Θ-class containing (A, B). For $h \in G_\Theta$ we have $\gamma h \leq (A, B) \Leftrightarrow \gamma h \leq (\underline{A}, \underline{B})$, since γh also is the smallest element of a Θ-class, and $(\underline{A}, \underline{B}) = \bigvee\{\gamma h \mid h \in A \cap G_\Theta\}$ because $\{[\gamma h]\Theta \mid h \in G_\Theta\}$ is supremum-dense. (A, B) and (C, D) are congruent if and only if the classes $[(A, B)]\Theta$ and $[(C, D)]\Theta$ have the same smallest element, i.e., if $A \cap G_\Theta = C \cap G_\Theta$. □

If $[v]\Theta$ is $\bigvee$-irreducible in $\underline{\mathfrak{B}}(G, M, I)/\Theta$, then the smallest element of the congruence class $[v]\Theta$ must also be $\bigvee$-irreducible and thus must be an object concept γg with $g \in G_\Theta$. Hence, the set $\{[\gamma h]\Theta \mid h \in G_\Theta\}$ contains all $\bigvee$-irreducible elements, and likewise $\{[\mu n]\Theta \mid n \in M_\Theta\}$ contains all $\bigwedge$-irreducible elements of $\underline{\mathfrak{B}}(G, M, I)/\Theta$. Thus, from Proposition 42 we can infer:

Proposition 43. *If $\underline{\mathfrak{B}}(G, M, I)/\Theta$ is doubly founded, then Θ is induced by a subcontext.* □

If we only regard the concept lattice up to isomorphism, every congruence is induced by a subcontext of an appropriate context: Every complete lattice $\boldsymbol{V}$ can be represented as a concept lattice $\underline{\mathfrak{B}}(V, V, \leq)$; by Proposition 42, in this representation every complete congruence is induced by a subcontext. In the case of doubly founded contexts we can go even further. For this purpose we first of all transfer Proposition 38:

Proposition 44. *Every factor lattice of a doubly founded complete lattice is doubly founded.*

Proof. Let $[x]\Theta < [y]\Theta$ be two elements of $\underline{\mathfrak{B}}(G, M, I)/\Theta$ and assume w.l.o.g. that x is the largest and y the smallest element of its class, i.e., that $x = \bigvee[x]\Theta$ and $y = \bigwedge[y]\Theta$. Hence, in the doubly founded concept lattice $\underline{\mathfrak{B}}(G, M, I)$ we find an element s that is minimal with respect to $s \leq y$, $s \not\leq x \wedge y$. We claim that $[s]\Theta$ has the corresponding property of minimality with respect to $[x]\Theta < [y]\Theta$. It is certain that $[s]\Theta \leq [y]\Theta$. On the other hand $[s]\Theta \leq [x]\Theta$ is impossible, since it would yield $s \vee x \in [x]\Theta$ and thus $s \vee x \leq x$. Consequently, we have $[s]\Theta \leq [y]\Theta$, $[s]\Theta \not\leq [x]\Theta$ and shall prove the minimality: If $[r]\Theta < [s]\Theta$, $r = \bigwedge[r]\Theta$ being the smallest element of its class, then $r < s$ and, because of the property of minimality of s, $r \leq x$ and hence $[r]\Theta \leq [x]\Theta$. The second condition is proved dually. □

If we combine Propositions 43 and 44, we obtain:

Theorem 11. *If $\underline{\mathfrak{B}}(G, M, I)$ is doubly founded, then every complete congruence relation is induced by a subcontext.* □

At the end of this section, we shall use the above results in order to analyze the system of all complete congruence relations of a concept lattice $\boldsymbol{V}$. This set of congruences is ordered by set inclusion $\subseteq$; it even forms a closure system on $V \times V$ and thus a complete lattice, the **lattice $\mathfrak{C}(\boldsymbol{V})$ of the complete congruence relations of $\boldsymbol{V}$.**

If we suppose that $\boldsymbol{V}$ is doubly founded, then we may assume that $\boldsymbol{V}$ is the concept lattice of a reduced, doubly founded context (G, M, I). This yields the following simplifications: Every compatible subcontext of a reduced context is saturated (cf. Definition 41), the compatible subcontexts are precisely the arrow-closed subcontexts (Proposition 36), and every complete congruence is induced by a subcontext (Theorem 11). Thus, in this case the arrow-closed subcontexts correspond bijectively to the complete congruences.

The order of the congruence relations is also reflected by the subcontexts: If Θ and Ψ are two congruences of $\boldsymbol{V}$, then

$$\begin{aligned}
\Theta \subseteq \Psi \Leftrightarrow \quad & (A, B)\Theta(C, D) \Rightarrow (A, B)\Psi(C, D) \\
& \text{for all } (A, B), (C, D) \in \boldsymbol{V} \\
\Leftrightarrow \quad & A \cap G_\Theta = C \cap G_\Theta \Rightarrow A \cap G_\Psi = C \cap G_\Psi \text{ and} \\
& B \cap M_\Theta = D \cap M_\Theta \Rightarrow B \cap M_\Psi = D \cap M_\Psi \\
& \text{for all } (A, B), (C, D) \in \boldsymbol{V} \\
\Leftrightarrow \quad & G_\Psi \subseteq G_\Theta \text{ and } M_\Psi \subseteq M_\Theta.
\end{aligned}$$

Hence, if we order the subcontexts by

$$(H_1, N_1, I \cap H_1 \times N_1) \leq (H_2, N_2, I \cap H_2 \times N_2)$$

$$:\Leftrightarrow H_1 \subseteq H_2 \text{ and } N_1 \subseteq N_2,$$

under the preconditions specified, the ordered set of the arrow-closed subcontext is dually isomorphic to the lattice of complete congruences. Now, however, the union as well as the intersection of arrow-closed subcontexts are arrow-closed too. Therefore, the lattice of arrow-closed subcontexts is completely distributive. Thus, Proposition 37 makes it easy to state a context for the congruence lattice as well.

Theorem 12. *The congruence lattice of a doubly founded concept lattice* $\underline{\mathfrak{B}}(G, M, I)$ *is isomorphic to the completely distributive lattice* $\underline{\mathfrak{B}}(G, M, \not\swarrow\!\!\!\nearrow)$.

Proof. If (G, M, I) is reduced, then every congruence is induced by exactly one subcontext $(H, N, I \cap H \times N)$. Furthermore we know by Proposition 37 that those subcontexts correspond to the concepts of $(G, M, \not\swarrow\!\!\!\nearrow)$: $(H, N, I \cap H \times N)$ induces a congruence if and only if $(G \setminus H, N)$ is such a concept.

The order of those subcontexts is dual to that of the concepts of $(G, M, \not\swarrow\!\!\!\nearrow)$ as well as to that of the congruences. This means that the latter two must be isomorphic to each other.

For the structure of $\underline{\mathfrak{B}}(G, M, \not\swarrow\!\!\!\nearrow)$, however, it is irrelevant whether (G, M, I) is reduced, provided that $\underline{\mathfrak{B}}(G, M, I)$ is doubly founded. In this case, we can switch to the reduced context $(G_{\text{irr}}, M_{\text{irr}}, I \cap G_{\text{irr}} \times M_{\text{irr}})$ with G_{irr} and M_{irr} being the set of irreducible objects and attributes, respectively. The $\swarrow\!\!\!\nearrow$-relation is inherited by this subcontext, since in Definition 47, apart from g and m, there only appear irreducible objects and attributes. Therefore, $\underline{\mathfrak{B}}(G, M, \not\swarrow\!\!\!\nearrow)$ and $\underline{\mathfrak{B}}(G_{\text{irr}}, M_{\text{irr}}, \not\swarrow\!\!\!\nearrow)$ are isomorphic, every concept of $(G, M, \not\swarrow\!\!\!\nearrow)$ is of the form $\big((G \setminus G_{\text{irr}}) \cup \underline{A}, \underline{B} \cup (\underline{A}^{\not\swarrow\!\!\!\nearrow} \cap (M \setminus M_{\text{irr}}))\big)$, $(\underline{A}, \underline{B})$ being a concept of $(G_{\text{irr}}, M_{\text{irr}}, \not\swarrow\!\!\!\nearrow)$. □

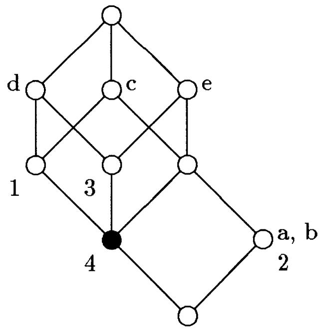


Figure 3.5 The congruence lattice of the lattice from Figure 3.4 is at the same time the lattice of arrow-closed subcontexts of Figure 3.3. The marked element corresponds to the congruence from Figure 3.4 and the compatible subcontext in Definition 45.

3.3 Closed Subrelations

Definition 50. A relation $J \subseteq I$ is called a **closed relation** of the context (G, M, I) if every concept of the context (G, M, J) is also a concept of (G, M, I). $\diamond$

Theorem 13. *If J is a closed relation of (G, M, I), then $\underline{\mathfrak{B}}(G, M, J)$ is a complete sublattice of $\underline{\mathfrak{B}}(G, M, I)$ with $J = \bigcup\{A \times B \mid (A, B) \in \underline{\mathfrak{B}}(G, M, J)\}$. Conversely, for every complete sublattice U of $\underline{\mathfrak{B}}(G, M, I)$ the relation*

$$J := \bigcup\{A \times B \mid (A, B) \in U\}$$

is closed and $\underline{\mathfrak{B}}(G, M, J) = U$.

Proof. Let J be a closed relation of (G, M, I). According to the definition, $\underline{\mathfrak{B}}(G, M, J)$ is a subset of $\underline{\mathfrak{B}}(G, M, I)$ containing $(M^I, M) = (M^J, M)$ as well as $(G, G^I) = (G, G^J)$[2]. The characterization of the suprema and infima in the Basic Theorem shows that $\underline{\mathfrak{B}}(G, M, J)$ is a complete sublattice of $\underline{\mathfrak{B}}(G, M, I)$. The relation $J = \bigcup\{A \times B \mid (A, B) \in \underline{\mathfrak{B}}(G, M, J)\}$ holds for every context (G, M, J).

Now, conversely, let U be a complete sublattice and

$$J := \bigcup\{A \times B \mid (A, B) \in U\}.$$

We have to show that J is a closed relation with $U = \underline{\mathfrak{B}}(G, M, J)$. It is evident that $U \subseteq \underline{\mathfrak{B}}(G, M, J)$. Thus, it remains to be shown that every concept of (G, M, J) belongs to U. We first prove this for the object concepts: Assume that $g \in G$ and $D := \bigcap\{A \mid (A, B) \in U, g \in A\}$. D is an extent of (G, M, J), and consequently $g^{JJ} \subseteq D$. For every attribute $m \in g^J$ there exists a concept $(A, B) \in U$ with $(g, m) \in A \times B$ and because of $D \subseteq A$ it follows that $m \in D^J$. Therefore, $g^J = D^J$ and $g^{JJ} = D$. This shows that for every $g \in G$ the concept (g^{JJ}, g^J) belongs to U. Every concept of $\underline{\mathfrak{B}}(G, M, J)$ is however the supremum of such object concepts, thus $U \supseteq \underline{\mathfrak{B}}(G, M, J)$, which remained to be proved. $\square$

This means that the closed relations are in a one-to-one correspondence to the complete sublattices. The map

$$\mathbf{C}(U) := \bigcup\{A \times B \mid (A, B) \in U\}$$

maps the set of complete sublattices bijectively onto the map of closed relations of $\underline{\mathfrak{B}}(G, M, I)$. It is furthermore order-preserving, $U_1 \subseteq U_2 \Leftrightarrow J_1 \subseteq J_2$. However, $\mathbf{C}$ is neither $\bigcup$– nor $\bigcap$–preserving. The intersection of closed relations does not necessarily have to be closed, the closed relations in general

[2] In conformity with Definition 17 (p. 13) we write X^I or X^J instead of X', in order to make clear when we are referring to the context (G, M, I) or to (G, M, J), respectively.

do not form a closure system. This is surprising in so far as the family of complete sublattices does form a closure system: for every subset T of a complete lattice, the intersection of all lattices containing T is also a sublattice (namely the complete **sublattice generated** by T).

If $\mathfrak{J}$ is a family of closed relations and $D := \bigcap \mathfrak{J}$, then there is, nonetheless, always a largest closed relation in D, namely

$$J := \bigcup \{A \times B \mid (A, B) \text{ concept}, A \times B \subseteq D\}.$$

This is easily seen if we consider the following: If (A, B) is a concept of (G, M, I) and J is a closed relation with $A \times B \subseteq J$, then (A, B) is also a concept of (G, M, J). Thus, the concepts (A, B) with $A \times B \subseteq D$ are precisely those which are contained in each of the sublattices $\underline{\mathfrak{B}}(G, M, L)$, $L \in \mathfrak{J}$. This means that they form precisely the intersection of those sublattices, i.e., they themselves are a complete sublattice. These considerations yield the following proposition:

Proposition 45. *For every set $T \subseteq \underline{\mathfrak{B}}(G, M, I)$ of concepts, there is a smallest closed relation J of (G, M, I) containing all sets $A \times B$ with $(A, B) \in T$. $\underline{\mathfrak{B}}(G, M, J)$ is the complete sublattice of $\underline{\mathfrak{B}}(G, M, I)$ generated by T.* □

	a	b	c	d	e	f	g	h
1			×					×
2	×	×	×					
3	×	×	⊗		⊗			
4			⊗	⊗	⊗			
5			⊗	⊗	⊗	⊗	⊗	
6		⊗	⊗		⊗		⊗	
7			×			×	×	
8				×	×	×	×	
9		⊗	⊗	⊗	⊗	⊗	⊗	
10		×	×				×	

Figure 3.6 Example of a closed relation in a context, from [60].

How can we recognize whether a relation is closed? A first clue is provided by the next proposition.

Proposition 46. *A subrelation $J \subseteq I$ is closed if and only if*

$$X^{JJ} \supseteq X^{JI}$$

holds for each subset $X \subseteq G$ and for each subset $X \subseteq M$.

Proof. (X^{JJ}, X^J) is a concept of (G, M, I), if and only if $X^{JJ} = X^{JI}$ and $X^{JJI} = X^J$. If we set $Y := X^J$ the second condition can be rewritten as $Y^{JI} = Y^{JJ}$ because of $X^J = X^{JJJ}$. However, the inclusion $Y^{JJ} \subseteq Y^{JI}$ holds for every subrelation. □

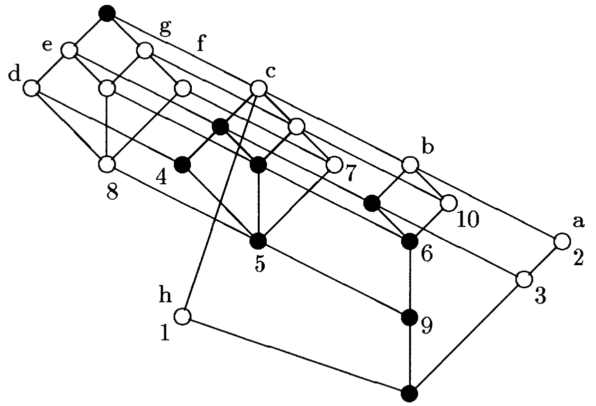


Figure 3.7 Diagram of the concept lattice for the context from Figure 3.6. The sublattice consisting of the blackened elements belongs to the above-mentioned closed relation.

The following characterization is somewhat more ambitious:

Proposition 47. *The closed relations of a context (G, M, I) are precisely those subrelations $J \subseteq I$ which satisfy the following condition:*

(C) *$(g, m) \in I \setminus J$ implies $(h, m) \notin I$ for some $h \in G$ with $g^J \subseteq h^J$ as well as $(g, n) \notin I$ for some $n \in M$ with $m^J \subseteq n^J$.*

Proof. Let J be a closed relation of (G, M, I) and assume that $(g, m) \in I \setminus J$. (g^{JJ}, g^J) is a concept of (G, M, I), i.e., $g^J = g^{JJI}$. Since $m \notin g^J$, there is some $h \in g^{JJ}$ with $m \notin h^I$, i.e., with $(h, m) \notin I$ and $g^J \subseteq h^J$. The second part of (**C**) follows dually.

Conversely, let $J \subseteq I$ be a relation satisfying (**C**) and let (A, B) be a concept of (G, M, J). We have to show that (A, B) is a concept of (G, M, I), i.e., that $A = B^I$ and $B = A^I$. $B \subseteq A^I$ is trivial, we show $B \supseteq A^I$. If we assume that there is an attribute $m \in A^I$ which is not an element of $B = A^J$, then there should be an object $g \in A$ with $(g, m) \notin J$ but $(g, m) \in I$. By means of condition (**C**) we should find some $h \in G$ with $m \notin h^I$ and $h^J \supseteq g^J \supseteq B$. Because of $h^J \supseteq B$, however, in this case we should obtain $h \in A$ which would contradict $m \in A^I$. $A = B^I$ is proved dually, i.e., J is closed. □

Proposition 48. *If J is a closed relation and*

$$(H, N, I \cap H \times N)$$

is a compatible subcontext of (G, M, I), then $J \cap H \times N$ is a closed relation of $(H, N, I \cap H \times N)$ and $(H, N, J \cap H \times N)$ is a compatible subcontext of (G, M, J).

Proof. If (A, B) is a concept of (G, M, I), then $(A \cap H, B \cap N)$ is a concept of $(H, N, I \cap H \times N)$. This holds in particular for the concepts of (G, M, J), in which case $(A \cap H, B \cap N)$ is even a concept of $(H, N, J \cap H \times N)$. Each concept of $(H, N, J \cap H \times N)$ originates in this way, i.e., $J \cap H \times N$ is closed and $(H, N, J \cap H \times N)$ is compatible. $\square$

The proposition has the following background: A homomorphism maps sublattices onto sublattices. If $(H, N, I \cap H \times N)$ is a compatible subcontext and J is a closed relation, then $\Pi_{H,N}$ maps the sublattice $\underline{\mathfrak{B}}(G, M, J)$ onto the sublattice $\underline{\mathfrak{B}}(H, N, J \cap H \times N)$.

Proposition 48 contains as a special case the statement that a closed relation remains closed if we omit reducible objects and attributes. By Proposition 38, a dense subcontext is always compatible. In the following proposition we establish a connection between closed relations and the arrow relations. First, we shall explain an abbreviation used in this connection:

$$\swarrow \cup \nearrow := \{(g, m) \mid g \swarrow m \text{ or } g \nearrow m\}.$$

In the following proposition this refers to the arrow relations in the context (G, M, J):

Proposition 49. *Let (G, M, J) be a doubly founded clarified context. Then the following statement holds: J is a closed relation of (G, M, I) if and only if*

$$J \subseteq I \subseteq G \times M \setminus (\swarrow \cup \nearrow).$$

Proof. If J is a closed relation of (G, M, I) and $(g, m) \in I \setminus J$, then by Proposition 47 there exists some h with $(h, m) \notin J$ and $g^J \subseteq h^J$, i.e., $g^J \subset h^J$ and thus $(g, m) \notin \swarrow$.

If, on the other hand, $J \subseteq I \subseteq G \times M \setminus (\swarrow \cup \nearrow)$, then, according to Proposition 46, it suffices to show for given $X \subseteq G$ (and dually for $X \subseteq M$) that $X^{JJ} \supseteq X^{JI}$. Hence, assume that $X \subseteq G$, $B := X^J$ and $g \in B^I$. If we had $g \notin B^J$, then there would be an attribute $m \in B$ with $(g, m) \notin J$ and furthermore, on account of the doubly-foundedness, an attribute n with $g \nearrow n$ and $n^J \supseteq m^J$, i.e., in particular $n \in B$ and consequently $(g, n) \in I$, in contradiction to $g \nearrow n$. $\square$

Full rows and full columns of a context belong to every closed relation and it is sometimes awkward to have to carry them along. For simplification purposes, we therefore occasionally use the notation

$$\Box := M' \times M \cup G \times G'.$$

The relation $\Box$ consists precisely of the trivial incidences in I. In the following proposition we simply assume that $\Box = \emptyset$ and give some simple examples of closed relations.

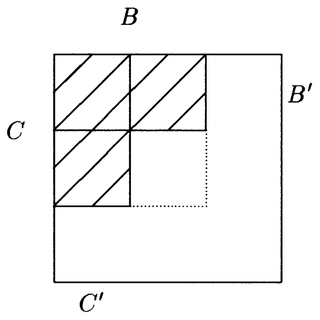


Figure 3.8 With reference to Proposition 51: Between (B', B) and (C, C') lie precisely the concepts of the context $(C, B, I \cap C \times B)$.

Proposition 50. *If* (A, B) *and* (C, D) *are concepts of a context* (G, M, I) *with* $G' = \emptyset = M'$, *then*

$$I \cap A \times M, \quad I \cap G \times D, \quad I \cap (A \times M \cup G \times D)$$

and, if $(A, B) \leq (C, D)$, *even* $I \cap C \times B$ *are closed relations with*

$$\mathfrak{B}(G, M, I \cap A \times M) = \{(G, \emptyset)\} \cup ((A, B)]$$
$$\mathfrak{B}(G, M, I \cap G \times D) = \{(\emptyset, M)\} \cup [(C, D))$$
$$\mathfrak{B}(G, M, I \cap (A \times M \cup G \times D)) = ((A, B)] \cup [(C, D))$$
$$\mathfrak{B}(G, M, I \cap C \times B) = \{(\emptyset, M), (G, \emptyset)\} \cup [(A, B), (C, D)].$$

Proof. It suffices to undertake the proof for $J := I \cap C \times B$. It is clear that $(G, \emptyset)$ and $(\emptyset, M)$ are concepts of $\mathfrak{B}(G, M, J)$. Furthermore every concept $(X, Y) \in \mathfrak{B}(G, M, I)$ with $(A, B) \leq (X, Y) \leq (C, D)$ is also a concept of (G, M, J), since $X \times Y \subseteq I \cap C \times B \subseteq J$. Hence, assume that $(X, Y) \in \mathfrak{B}(G, M, J)$. We may assume that $X \subseteq C$ and $Y \subseteq B$. Because of $A \times B \subseteq J$, $Y \subseteq B$ immediately yields $A = B^J \subseteq Y^J = X$, i.e., $A \subseteq X$ and thus $X^I \subseteq A^I = B$. With $X^J = X^I \cap B$ it follows that $X^I = X^J$. Dually, we recognize that $Y^I = Y^J$, i.e., that $(X, Y) \in \mathfrak{B}(G, M, I)$. □

As an immediate consequence we obtain:

Proposition 51. *If* (A, B) *and* (C, D) *are concepts of* (G, M, I) *with* $(A, B) \leq (C, D)$, *then*

$$[(A, B), (C, D)] = \mathfrak{B}(C, B, I \cap C \times B). \quad \square$$

Example 7. We demonstrate this proposition by means of the concepts $\gamma 5$ and μc in the context from Figure 3.6. We have

$$\gamma 5 = (5'', 5') = (\{5, 9\}, \{c, d, e, f, g\}),$$

$$\mu c = (c', c'') = (G \setminus \{8\}, \{c\}).$$

According to the proposition, $[\gamma 5, \mu c] = \mathfrak{B}(C, B, I \cap C \times B)$ with

$$\begin{aligned} C &:= \{1, 2, 3, 4, 5, 6, 7, 9, 10\} \\ \text{and } B &:= \{c, d, e, f, g\}. \end{aligned}$$

It turns out that in this subcontext the objects 1, 2, 3, 5, 9 and 10 as well as the attribute c are reducible. The reduced context is presented in Figure 3.9 together with its concept lattice. It is an interval in the center of Figure 3.7.

	d	e	f	g
4	×	×		
6		×		×
7			×	×

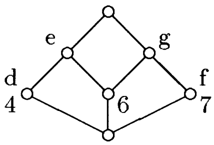


Figure 3.9 The concept lattice of this subcontext is isomorphic to an interval.

We give two further examples of closed relations:

If Γ is a group of automorphisms of the context (G, M, I), i.e., of pairs of maps (α, β) with

$$\alpha : G \to G, \quad \beta : M \to M, \quad gIm \iff \alpha(g)I\beta(m),$$

then we obtain a closed relation I_Γ by means of the definition

$$\begin{aligned} (g, m) \in I_\Gamma \quad &:\iff \quad gI\beta(m) \text{ for all } (\alpha, \beta) \in \Gamma \\ &(\iff \quad \alpha(g)Im \text{ for all } (\alpha, \beta) \in \Gamma), \end{aligned}$$

as can be easily proved. The corresponding sublattice $\underline{\mathfrak{B}}(G, M, I_\Gamma)$ consists precisely of those concepts (A, B) of (G, M, I) which are invariant under Γ, i.e., for which holds

$$(\alpha A, \beta B) = (A, B) \quad \text{for all } (\alpha, \beta) \in \Gamma.$$

It may happen that a closed relation differs very little from the incidence relation I, in the extreme case only by one "cross". This case corresponds to the *dismantling* of doubly irreducible elements. Therefore, we shall give a short description, only sketching the order-theoretic results and referring to the corresponding literature for the proofs.

An element a of an ordered set shall be called **doubly irreducible** if a has exactly one lower neighbour a_* and exactly one upper neighbour a^* and furthermore the conditions

$$x < a \Rightarrow x \leq a_*, \quad x > a \Rightarrow x \geq a^*$$

(which are dispensable in the finite) are satisfied. According to Proposition 2 (p. 7), in a complete lattice the doubly irreducible elements are precisely those which are $\bigvee$-irreducible as well as $\bigwedge$-irreducible.

We talk about the **dismantling** of a doubly irreducible element a in an ordered set $(P, \leq)$ if we mean the transition from $(P, \leq)$ to the ordered set

$$(P \setminus \{a\}, (P \setminus \{a\})^2 \cap \leq).$$

In the following we shall write $(P \setminus \{a\}, \leq)$ for this ordered set .

If we dismantle a doubly irreducible element a of a complete lattice $\boldsymbol{V}$, then we obtain a complete sublattice $\boldsymbol{V} \setminus \{a\}$. Obviously, the property of being doubly irreducible is also necessary for this purpose. We get a further converse:

Proposition 52. *If a is a doubly irreducible element of $(P, \leq)$, then $(P \setminus \{a\}, \leq)$ is a complete lattice if and only if $(P, \leq)$ is a complete lattice.* □

We omit the (easy) proof and point to an application instead: If we want to determine whether a given ordered set $(P, \leq)$ is a complete lattice, we can first remove doubly irreducible elements and then examine the remaining structure. If $(P, \leq)$ is finite we can gradually dismantle *all* doubly irreducible elements until there finally remains a **DI-kernel** without doubly irreducible elements.

It can be shown by means of a simple argument that the DI-kernel is unique, i.e., that it does not depend on the order in which the doubly irreducible elements are being dismantled.

Dismantling an element corresponds to cancelling a cross in the context:

Proposition 53. *If $\mathfrak{a} = \gamma g = \mu m$ is a doubly irreducible concept of a clarified context (G, M, I), then*

$$\mathfrak{B}(G, M, I) \setminus \{\mathfrak{a}\} = \mathfrak{B}(G, M, I \setminus \{(g, m)\}).$$

Proof. We have already noted that $\mathfrak{B}(G, M, I) \setminus \{\mathfrak{a}\}$ is a complete sublattice. By Theorem 13 the corresponding closed relation is given by

$$J := \bigcup \{A \times B \mid (A, B) \neq \mathfrak{a}\}.$$

Now, if $(h, n) \in I$ is an arbitrary incident object-attribute pair, then $(h, n) \in h'' \times h'$ and $(h, n) \in n' \times n''$. Hence, from $(h, n) \notin J$ it follows that $(h'', h') = \mathfrak{a} = (n', n'')$, i.e., (since (G, M, I) is clarified) $h = g$ and $n = m$. □

3.4 Block Relations and Tolerances

Definition 51. Let V be a complete lattice. A **complete tolerance relation** on V is a relation $\Theta \subseteq V \times V$ which is reflexive, symmetric and compatible with suprema and infima, i.e., for which holds

$$x_t \Theta y_t \text{ for } t \in T \Rightarrow (\bigwedge_{t \in T} x_t) \Theta (\bigwedge_{t \in T} y_t) \text{ and } (\bigvee_{t \in T} x_t) \Theta (\bigvee_{t \in T} y_t).$$

Hence, a complete tolerance relation is a congruence relation if it is transitive. $\diamond$

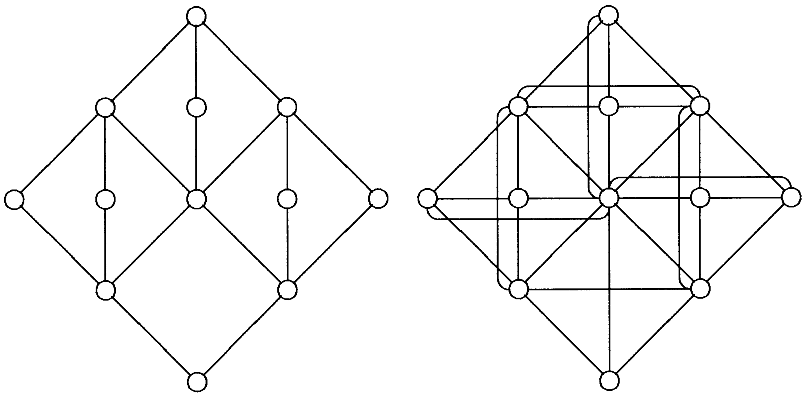


Figure 3.10 The pairs which are linked together in the figure on the right (including the pairs of neighbouring elements) form part of a tolerance relation of the lattice on the left.

Proposition 54. *If Θ is a complete tolerance relation on V, then it follows alone from $a\Theta b$ and $x, y \in [a \wedge b, a \vee b]$ that $x\Theta y$.*

Proof. From $a\Theta b$ and $a\Theta a$ it follows that $a\Theta a \wedge b$ and correspondingly $b\Theta a \wedge b$. This yields $a \wedge b \; \Theta \; a \vee b$. It follows that $x \vee (a \wedge b) \; \Theta \; x \vee (a \vee b)$, i.e., $x\Theta(a \vee b)$ and correspondingly $(a \vee b)\Theta y$. Because of $x, y \leq a \vee b$ we obtain $x\Theta y$. $\square$

Definition 52. If Θ is a complete tolerance relation and $a \in V$, we define

$$a_\Theta := \bigwedge\{x \in V \mid a\Theta x\} \text{ and } a^\Theta := \bigvee\{x \in V \mid a\Theta x\}.$$

The intervals $[a]_\Theta := [a_\Theta, (a_\Theta)^\Theta]$, $(a \in V)$ are called the **blocks** of Θ. $\diamond$

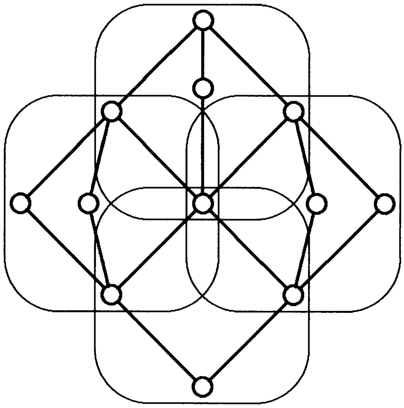


Figure 3.11 The blocks of the tolerance relation from Figure 3.10.

a_Θ is the smallest (and dually a^Θ is the largest) element related with a under Θ . The dual definition $[a]^\Theta := [(a^\Theta)_\Theta, a^\Theta]$ also yields the blocks of Θ: Because of $((a_\Theta)^\Theta)_\Theta = a_\Theta$ and $((a^\Theta)_\Theta)^\Theta = a^\Theta$ we obtain $[a]_\Theta = [a_\Theta]^\Theta$ as well as $[a]^\Theta = [a^\Theta]_\Theta$. From $a\Theta b$ and $a \le b$ it follows that $b_\Theta \le a \le b$, i.e., $a \in [b]_\Theta$. Correspondingly, from $a\Theta b$ and $a \ge b$ it always follows that $a \in [b]^\Theta$. The blocks of a tolerance relation do not have to be disjoint, unless we are dealing with a congruence relation. We have

$$[a]_\Theta \cap [b]_\Theta \neq \emptyset \iff a_\Theta \le (b_\Theta)^\Theta \text{ and } b_\Theta \le (a_\Theta)^\Theta.$$

Proposition 55. *The blocks of Θ are precisely the maximal subsets X of $\boldsymbol{V}$ with $x\Theta y$ for all $x, y \in X$.*

Proof. Because of Proposition 54 we have $x\Theta y$ for all $x, y \in [a]_\Theta$. Now if z is an arbitrary element with $z\Theta a$ and $z\Theta a_\Theta$, we obtain $z \ge a_\Theta$ and $z \le (a_\Theta)^\Theta$, i.e., $z \in [a]_\Theta$. Hence, every block is maximal with regard to the property specified. If X is an arbitrary maximal set of elements of $\boldsymbol{V}$ which are pairwise related under Θ, then from the compatibility it follows that $\bigwedge X$ and $\bigvee X$ are elements of X. Hence, because of the maximality, $X = [\bigwedge X, \bigvee X]$ and $(\bigwedge X)^\Theta = \bigvee X$, $(\bigvee X)_\Theta = \bigwedge X$, i.e., X is a block of Θ. □

Proposition 56. *The map $x \overset{\varphi}{\mapsto} x_\Theta$ is a $\bigvee$-morphism and the map $x \overset{\psi}{\mapsto} x^\Theta$ is a $\bigwedge$-morphism. The two maps are adjoint to each other.*

Proof. We show that (φ, ψ) is a Galois connection between $\boldsymbol{V}$ and $\boldsymbol{V}^d$. We have $x \le y \Rightarrow x_\Theta \le y_\Theta \Rightarrow x_\Theta \ge^d y_\Theta$, $x \le^d y \Rightarrow x \ge y \Rightarrow x^\Theta \ge y^\Theta$, $x \le (x_\Theta)^\Theta$ and $x \ge (x^\Theta)_\Theta \Rightarrow x \le^d (x^\Theta)_\Theta$. Hence,

$$\varphi(\bigvee x_t) = \bigwedge^d \varphi(x_t) = \bigvee \varphi(x_t)$$

and correspondingly

$$\psi(\bigwedge x_t) = \psi(\bigvee^d x_t) = \bigwedge \psi(x_t). \qquad \square$$

Definition 53. The set of all blocks of a complete tolerance relation of $\boldsymbol{V}$ is denoted by $\boldsymbol{V}/\Theta$ and ordered by

$$B_1 \leq B_2 :\iff \bigwedge B_1 \leq \bigwedge B_2 \qquad (\iff \bigvee B_1 \leq \bigvee B_2).$$

$\diamond$

The definition says that the smallest elements of the blocks are ordered in the same way as their largest elements. This is correct because of $x_\Theta \leq y_\Theta \iff (x_\Theta)^\Theta \leq (y_\Theta)^\Theta$. In fact, even more is true: The set of the upper bounds of the blocks is closed under infima, that of the lower bounds is closed under suprema, in analogy to the case of the complete congruences (cf. Theorem 10, p. 106). This is described by the following theorem:

Theorem 14. *With the order described above, $\boldsymbol{V}/\Theta$ is a complete lattice (the* **factor lattice** *of $\boldsymbol{V}$ by Θ). The following equations hold for blocks B_t and for elements x_t, $t \in T$, of $\boldsymbol{V}$ respectively:*

$$\bigvee_{t\in T} B_t = [\bigvee_{t\in T} \bigwedge B_t]^\Theta \quad \textit{resp.} \quad \bigwedge_{t\in T} B_t = [\bigwedge_{t\in T} \bigvee B_t]_\Theta$$
$$\bigvee [x_t]_\Theta = [\bigvee x_t]_\Theta \quad \textit{resp.} \quad \bigwedge [x_t]^\Theta = [\bigwedge x_t]^\Theta$$

Proof. The proofs of the equations follow easily from Proposition 56. $\square$

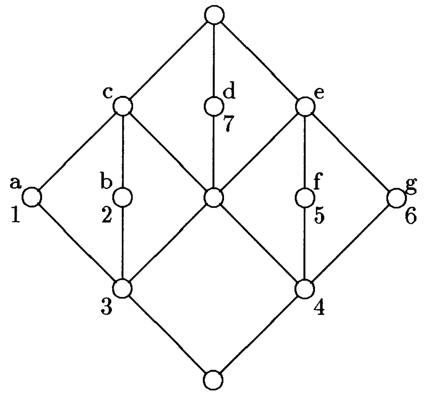
How can we describe complete tolerance relations of concept lattices in terms of the contexts?

Definition 54. By a **block relation** of a context (G, M, I) we mean a relation $J \subseteq G \times M$ which satisfies the following conditions:

1. $I \subseteq J$,
2. for every object $g \in G$, g^J is an intent of (G, M, I),
3. for every attribute $m \in M$, m^J is an extent of (G, M, I).

$\diamond$

We can use this definition as a starting point for some observations: If J is a block relation of (G, M, I), then every extent of (G, M, J) is an extent of (G, M, I) and every intent of (G, M, J) is an intent of (G, M, I). The intersection of any number of block relations of (G, M, I) is again a block relation, since $g^{\bigcap J_t} = \bigcap g^{J_t}$, and the intersection of intents is always an intent, and dually. Hence, the block relations of (G, M, I) form a closure system and thus a complete lattice.



	a	b	c	d	e	f	g
1	×	•	×	•	•		
2	•	×	×	•	•		
3	×	×	×	×	×	•	•
4	•	•	×	×	×	×	×
5			•	•	×	×	•
6			•	•	×	•	×
7			•	×	•		

Figure 3.12 The block relation J belonging to the tolerance from Figure 3.10 additionally contains the pairs marked by dots.

Theorem 15. *The lattice of all block relations of (G, M, I) is isomorphic to the lattice of all complete tolerance relations of $\underline{\mathfrak{B}}(G, M, I)$. The map β assigning to any complete tolerance relation Θ the block relation defined by*

$$g\beta(\Theta)m :\Longleftrightarrow \gamma g\Theta(\gamma g \wedge \mu m) \quad (\Longleftrightarrow (\gamma g \vee \mu m)\Theta\mu m)$$

is an isomorphism. Conversely,

$$(A, B)\beta^{-1}(J)(C, D) \Longleftrightarrow A \times D \cup C \times B \subseteq J$$

yields the tolerance corresponding to a block relation J.

Proof. First, we show that $J := \beta(\Theta)$ is a block relation. Since Θ is reflexive, $I \subseteq J$. According to the definition,

$$g^J = \{m \in M \mid \gamma g\Theta(\gamma g \wedge \mu m)\}.$$

We claim that this is an intent of (G, M, I). For this purpose we consider the concept

$$\bigwedge\{\mu m \mid \gamma g\Theta(\gamma g \wedge \mu m)\} = (g^{JI}, g^{JII}).$$

If n is an attribute of this concept, we get

$$\mu n \geq \bigwedge\{\mu m \mid \gamma g\Theta(\gamma g \wedge \mu m)\},$$

and hence also

$$\gamma g \wedge \mu n \geq \bigwedge\{\gamma g \wedge \mu m \mid \gamma g\Theta(\gamma g \wedge \mu m)\}.$$

If we are aware that this infimum is in a Θ-relation with γg, we recognize that $(\gamma g \wedge \mu n)\Theta\gamma g$, i.e., that $n \in g^J$. Hence, $g^J = g^{JII}$ is an intent. Dually we prove that every set of the form m^J is an extent of (G, M, I).

Now we start from a block relation J and define a relation $\tau(J)$ on $\underline{\mathfrak{B}}(G, M, I)$ by

$$(A, B)\tau(J)(C, D) :\iff A \times D \cup B \times C \subseteq J.$$

Evidently $\tau(J)$ is reflexive and symmetric, and, if T is an index set and there are concepts with $(A_t, B_t)\Theta(C_t, D_t)$ for $t \in T$, we argue as follows: For $g \in A_t$ we have $g^J \supseteq D_t = C_t^I$ and consequently $g^{JI} \subseteq C_t^{II} = C_t$. Hence, for $g \in \bigcap_{t\in T} A_t$ we obtain $g^{JI} \subseteq \bigcap_{t\in T} C_t$, i.e.,

$$g^J = g^{JII} \supseteq \left(\bigcap_{t\in T} C_t\right)^I,$$

which proves

$$\bigcap_{t\in T} A_t \times \left(\bigcap_{t\in T} C_t\right)^I \subseteq J.$$

Analogously we show that

$$\bigcap_{t\in T} C_t \times \left(\bigcap_{t\in T} A_t\right)^I \subseteq J,$$

and altogether we have proved

$$\bigwedge_{t\in T} (A_t, B_t)\tau(J) \bigwedge_{t\in T} (C_t, D_t).$$

The dual argument proves, moreover, that $\tau(J)$ is compatible with suprema, i.e., that it is a complete tolerance relation.

Both maps β and τ are evidently order-preserving. In order to prove the theorem we furthermore have to show that they are inverse to each other. Let Θ be a complete tolerance relation of $\underline{\mathfrak{B}}(G, M, I)$. We want to show that

$$(A, B)\Theta(C, D) \iff (A, B)\tau(\beta(\Theta))(C, D).$$

According to Proposition 54, we may limit ourselves to the special case $(A, B) > (C, D)$. We have

$$\begin{aligned}
(A, B)\Theta(C, D) &\iff \gamma g \vee (C, D)\Theta(C, D) \text{ for all } g \in A \\
&\iff \gamma g\Theta\gamma g \wedge (C, D) \text{ for all } g \in A \\
&\iff \gamma g\Theta\gamma g \wedge \mu m \text{ for all } g \in A \text{ and } m \in D \\
&\iff A \times D \subseteq \beta(\Theta) \\
&\iff (A, B)\tau(\beta(\Theta))(C, D).
\end{aligned}$$

For the last part of the proof let J be a block relation of (G, M, I). Then

$$\begin{array}{rcl}
(g,m)\in J & \iff & g\in m^J \\
 & \iff & g^{II}\subseteq g^{JJ}\cap m^J \text{ and} \\
(g^{II}\cap m^I)^I=(g^I\cup m^{II})^{II} & \subseteq & (g^J\cup m^{JJ})^{JJ}=(g^{JJ}\cap m^J)^J \\
 & \iff & g^{II}\times(g^{II}\cap m^I)^I\subseteq J \\
 & \iff & (\gamma g)\tau(J)(\gamma g\wedge\mu m) \\
 & \iff & (g,m)\in\beta(\tau(J)).
\end{array}$$

This proves that $J=\beta(\tau(J))$. □

Corollary 57. *If Θ is a tolerance relation on $\underline{\mathfrak{B}}(G,M,I)$ and $J:=\beta(\Theta)$ is the corresponding block relation, then*

$$\underline{\mathfrak{B}}(G,M,I)/\Theta\cong\underline{\mathfrak{B}}(G,M,J).$$

More precisely, we have:

1. *$\left[(B^I,B),(C,C^I)\right]$ is a block of Θ if and only if (C,B) is a concept of (G,M,J).*
2. *The map*
$$\left[(B^I,B),(C,C^I)\right]\mapsto(C,B)$$
is an isomorphism of the lattice of the blocks of Θ onto the concept lattice of (G,M,J).
3. *If (C,B) is a concept of (G,M,J), then*
$$\left[(B^I,B),(C,C^I)\right]=\mathfrak{B}(C,B,I\cap C\times B)$$
for the corresponding block of Θ

Proof. According to Theorem 15, two concepts $(A,B)\leq(C,D)$ of $\underline{\mathfrak{B}}(G,M,I)$ stand in the relation Θ to each other if and only if $C\times B\subseteq J$, i.e., if $B\subseteq C^J$ and $C\subseteq B^J$. For an arbitrary concept (X,Y) of (G,M,I) therefore

$$\begin{array}{rcl}
(X,Y)_\Theta & = & (X^{JI},X^J)\quad\text{and} \\
(X,Y)^\Theta & = & (Y^J,Y^{JI})\quad\text{and consequently} \\
((X,Y)_\Theta)^\Theta & = & (X^{JJ},X^{JJI}).
\end{array}$$

If we assume that $B:=X^J$ and $C:=X^{JJ}$, then (C,B) is a concept of (G,M,J) and the block $[(X,Y)]_\Theta$ proves to be of the form we claimed:

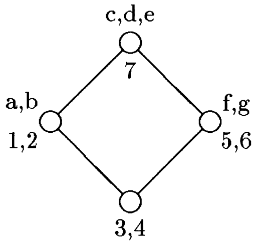
$$[(X,Y)]_\Theta=\left[(X,Y)_\Theta,((X,Y)_\Theta)^\Theta\right]=\left[(B^I,B),(C,C^I)\right].$$

Therefore, the map

$$[(X,Y)]_\Theta\mapsto(X^{JJ},X^J)$$

is an order isomorphism mapping the blocks of Θ onto the concepts of (G,M,J).

The third part of the assertion follows from Proposition 51. □



	c	d	e	f	g
3	×	×	×		
4	×	×	×	×	×
5			×	×	
6			×		×

Figure 3.13 The concept lattice $\mathfrak{B}(G, M, J)$ of the block relation J is isomorphic to the factor lattice by the tolerance relation. As an example, we state the subcontext belonging to the concept $(\{3, 4, 5, 6\}, \{c, d, e, f, g\})$ of J. Its concept lattice is isomorphic to the corresponding block of the tolerance.

We close this section with two observations. The first deals with the question which families of intervals form the systems of the blocks of a tolerance. This is answered by the following theorem:

Theorem 16. *Let $\boldsymbol{V}$ be a complete lattice, T an index set and*

$$\mathcal{F} := \{[\underline{x}_t, \overline{x}_t] \mid t \in T\}$$

a family of intervals from $\boldsymbol{V}$ which are assumed to be pairwise distinct, i.e., $s \neq t \Rightarrow [\underline{x}_s, \overline{x}_s] \neq [\underline{x}_t, \overline{x}_t]$. Then the following conditions are equivalent:

1. *$\mathcal{F}$ is the family of the blocks of a complete tolerance relation on a complete sublattice of $\boldsymbol{V}$.*
2. *a) The set $\{\overline{x}_t \mid t \in T\}$ of the upper bounds of the intervals is $\bigwedge$-closed.*
   b) The set $\{\underline{x}_t \mid t \in T\}$ of the lower bounds of the intervals is $\bigvee$-closed.
   c) The upper and the lower bounds are ordered in the same way, i.e., $\underline{x}_s \leq \underline{x}_t \iff \overline{x}_s \leq \overline{x}_t$.
3. *There is an order $\leq$ on T with respect to which $(T, \leq)$ is a complete lattice and there are maps*

$$\underline{\alpha} : T \to \boldsymbol{V}, \qquad \text{injective and } \bigvee\text{-preserving,}$$

$$\overline{\alpha} : T \to \boldsymbol{V}, \qquad \text{injective and } \bigwedge\text{-preserving,}$$

with $\underline{\alpha}(s) \leq \underline{\alpha}(t) \iff \overline{\alpha}(s) \leq \overline{\alpha}(t)$, and $\mathcal{F} = \{[\underline{\alpha}(t), \overline{\alpha}(t)] \mid t \in T\}$.

Proof. $1 \Rightarrow 2$: Every block of a complete tolerance relation Θ is of the form $[x]^\Theta = [(x^\Theta)_\Theta, x^\Theta]$, i.e., the upper bounds of the blocks are precisely the elements of the form x^Θ. From Proposition 56 it follows that

$$\bigwedge\{x_t^\Theta \mid t \in T\} = (\bigwedge\{x_t \mid t \in T\})^\Theta,$$

i.e., the result is again an upper bound of a block. This proves a) and dually we infer b). c) again follows from Proposition 56: $x^\Theta \leq y^\Theta \iff (x^\Theta)_\Theta \leq (y^\Theta)_\Theta$.

2 $\Rightarrow$ 3: We order T by $s \leq t :\Longleftrightarrow \underline{x}_s \leq \underline{x}_t$. Because of c) this is equivalent to $\overline{x}_s \leq \overline{x}_t$. Hence, the map defined by $\underline{\alpha}(t) := \underline{x}_t$ is an order-isomorphism of $(T, \leq)$ on $\{\underline{x}_t \mid t \in T\}$, correspondingly $\overline{\alpha}(t) := \overline{x}_t$ defines an order-isomorphism of $(T, \leq)$ on $\{\overline{x}_t \mid t \in T\}$. Therefore, according to a) and b), $(T, \leq)$ is a complete lattice.

3 $\Rightarrow$ 1: We first show that under the conditions mentioned under 3, the set

$$U := \bigcup\{[\underline{\alpha}(t), \overline{\alpha}(t)] \mid t \in T\}$$

is a complete sublattice of $\boldsymbol{V}$. Hence, let $x_s, s \in S$ be a sequence of elements from U. For every $s \in S$ there is a $t_s \in T$ with $x_s \in [\underline{\alpha}(t_s), \overline{\alpha}(t_s)]$. We claim that $\bigvee_{s \in S} x_s$ also lies in such an interval, namely in $[\underline{\alpha}(t), \overline{\alpha}(t)]$ with $t := \bigvee_{s \in S} t_s$. Since $\underline{\alpha}$ is $\bigvee$-preserving, $\underline{\alpha}(t) \leq \bigvee_{s \in S} x_s$ is obvious, and since $\overline{\alpha}$ is order-preserving, for every $s \in S$ we have $x_s \leq \overline{\alpha}(t_s) \leq \overline{\alpha}(\bigvee_{s \in S} t_s)$ and thus $\bigvee_{s \in S} x_s \leq \overline{\alpha}(t)$. Hence, U is closed with regard to suprema as well as with respect to infima, as the dual argument shows.

The relation

$$\Theta := \{(x, y) \mid \exists_{t \in T}\ x, y \in [\underline{\alpha}(t), \overline{\alpha}(t)]\}$$

evidently is reflexive and symmetric. Moreover, from $(x_s, y_s) \in \Theta, s \in S$ it follows that $x_s, y_s \in [\underline{\alpha}(t_s), \overline{\alpha}(t_s)]$ holds for suitable $t_s \in T$, i.e.,

$$\bigvee\{x_s \mid s \in S\}, \bigvee\{y_s \mid s \in S\} \in [\underline{\alpha}(\bigvee t_s), \overline{\alpha}(\bigvee t_s)],$$

i.e.,

$$(\bigvee\{x_s \mid s \in S\}, \bigvee\{y_s \mid s \in S\}) \in \Theta.$$

Hence, Θ is $\bigvee$-compatible (and dually, of course, $\bigwedge$-compatible as well), i.e., it is a complete tolerance.

Finally we have to show that the intervals $[\underline{\alpha}(t), \overline{\alpha}(t)]$ are in fact the blocks of Θ, i.e., the maximal sets of elements which are pairwise related under the relation Θ. If $[u, v]$ is a block, then $(u, v) \in \Theta$, i.e., $u, v \in [\underline{\alpha}(t), \overline{\alpha}(t)]$ for some t, hence every block of Θ is of this form. On the other hand, every $[\underline{\alpha}(t), \overline{\alpha}(t)]$ is maximal too; from $[\underline{\alpha}(s), \overline{\alpha}(s)] \subseteq [\underline{\alpha}(t), \overline{\alpha}(t)]$ we can infer $\overline{\alpha}(s) \leq \overline{\alpha}(t)$, that is $s \leq t$, as well as $\underline{\alpha}(t) \leq \underline{\alpha}(s)$, that is $t \leq s$. Together this yields $s = t$. □

The second observation establishes a link between compatible subcontexts and block relations. Transitive tolerance relations are congruence relations, hence we can describe a complete congruence relation (under suitable conditions, cf. Theorem 11) in two ways: in terms of a compatible subcontext and in terms of a block relation.

Proposition 58. *Let Θ be a complete congruence relation of a doubly founded concept lattice $\mathfrak{B}(G, M, I)$, $(G_\Theta, M_\Theta, I \cap G_\Theta \times M_\Theta)$ the corresponding saturated compatible subcontext and $J = \beta(\Theta)$ the block relation for Θ. Then:*

$$\begin{array}{rcl}(g,m)\in J & \iff & g''\cap G_\Theta \subseteq m' \\ & \iff & m''\cap M_\Theta \subseteq g' \\ G_\Theta & = & \{g\in G \mid g^I = g^J\} \\ M_\Theta & = & \{m\in M \mid m^I = m^J\}.\end{array}$$

Proof. According to Theorem 15, $(g,m)\in J \iff (\gamma g, \gamma g \wedge \mu m)\in\Theta$. Two concepts are congruent if and only if their extents have the same intersection with G_Θ. In the present case, this means $g''\cap G_\Theta = g''\cap m'\cap G_\Theta$ which corresponds to the first line of the assertion. The second line can be inferred dually.

According to the definition, g is in G_Θ if and only if γg is the smallest element of a Θ-class. This is equivalent to the fact that for every attribute m from $(\gamma g, \gamma g\wedge\mu m)\in\Theta$ it already follows that $\mu m \geq \gamma g$, i.e., that $(g,m)\in J$ always implies $(g,m)\in I$. □

The connection between congruence relations and block relations, according to the theorem, can also be explained as follows: If φ is an homomorphism, $\Theta = \ker\varphi$ and J is the block relation belonging to Θ, then

$$(g,m)\in J \iff \varphi\gamma g \leq \varphi\mu m.$$

3.5 Hints and References

3.1 Compatible subcontexts as well as their characterization by means of the arrow relations were introduced in [192]. Proposition 37 has been taken from a paper by Knecht and Wille [99].

3.2 Congruence relations belong to the standard subjects of textbooks on abstract algebra. Theorem 9 is "classical". Books on lattice theory, however, usually examine lattice congruence (where the requirement is compatibility with suprema and infima of *finite sets* only). These congruences differ considerably from the complete congruence relations. Of course, every complete congruence is a lattice congruence in the weaker sense. But whereas the lattice congruences always form a distributive sublattice of the lattice of equivalence relations, the situation in the case of the complete congruences is more complicated: The supremum of two complete congruences does not have to coincide with the supremum as equivalence relations, and *every* complete lattice is isomorphic to the lattice of the complete congruence relations of a suitable concept lattice (Teo [174], Grätzer [76]). Therefore Theorem 10 cannot simply be derived from the corresponding theorems for algebras, but follows [190]. With regard to the congruence theory for concept lattices see also Reuter and Wille [144].

3.3 Closed relations were introduced in [198], in order to simplify the description of subdirect products, which had been tackled in [193]. The concept lattice from Figure 3.7 originally resulted from an analysis of biological data, see [60]. The dismantling of doubly irreducible elements was examined by Duffus and Rival [44]. They also proved the uniqueness of the DI-kernel. A very simple proof was given by Farley [52]. Distributive lattices that are generated by their doubly irreducible elements were examined by Monjardet & Wille [128].

It is possible to describe in terms of structure what happens if we add "a cross" to the incidence relation I. Here, we shall limit ourselves to cardinality: The concept lattice can become larger but it can also become smaller. A simple estimate by Skorsky shows

$$\frac{1}{2} \leq \frac{|\underline{\mathfrak{B}}(G, M, I \cup \{(g,m)\})|}{\underline{\mathfrak{B}}(G, M, I)} \leq \frac{3}{2}.$$

The concept lattice can only become smaller if $g \swarrow\!\!\!\nearrow m$. Assume w.l.o.g. that (G, M, I) is clarified. If neither $g \nearrow m$ nor $g \swarrow m$, then, by Proposition 49, $\underline{\mathfrak{B}}(G, M, I)$ is a complete sublattice of $\underline{\mathfrak{B}}(G, M, I \cup \{(g,m)\})$.

3.4 Czedli [30] had discovered that tolerance relations also yield a factor lattice. Bandelt [7] examined this connection in more detail. The interrelation between complete tolerance relations and block relations was first described in [195]. Wille has also suggested the use of tolerance relations in order to obtain counting formulas by means of the Möbius function. In this context see also [140] and Vogt [178].

4. Decompositions of Concept Lattices

A complex concept lattice can possibly be split up into simpler parts. Here the mathematical model must prove its worth by providing efficacious and versatile methods for the decomposition. Every such decomposition principle can be reversed to make a construction method. Therefore, some of the following subjects will be taken up again in the next chapter with this second focus.

If a lattice can be represented as a sublattice of a direct product, this is called a *subdirect decomposition*. The theory described in the preceding chapter permits an elegant description of these decompositions by means of the context. This is the subject of the first section.

The tolerance relations introduced in 3.4 result in coverings of the concept lattice by overlapping intervals. This fact will be used as a principle of decomposition in the second section.

A surprisingly versatile context operation consists in inserting one context into another one. We shall explain this in more detail in the third section and describe the corresponding lattice construction, the *substitution product*. Then we shall use some effort to prove a decomposition theorem for this product (Theorem 25).

In the fourth section we shall finally introduce the *tensor product* of complete lattices by means of the direct product of the contexts. Similarly, as in the case of the direct product of lattices (which corresponds to the context sum), tensorial decomposability is rare. Therefore, we transfer the idea of the subdirect product, which we explain in 4.1, to contexts and obtain the notion of the *subtensorial decomposition* of concept lattices.

4.1 Subdirect Decompositions

The **direct product** of ordered sets has already been introduced in Definition 7. We shall repeat it here for the special case of complete lattices:

Definition 55. Let T be an arbitrary index set. For a family $(\mathbf{V}_t)_{t\in T}$ of complete lattices, the **product** is defined to be

$$\bigtimes_{t\in T} \mathbf{V}_t := \left(\bigtimes_{t\in T} V_t, \leq\right)$$

with

$$(x_t)_{t\in T} \leq (y_t)_{t\in T} :\Leftrightarrow x_t \leq y_t \text{ for all } t \in T.$$

The lattices $\mathbf{V}_t$, $t \in T$ are the **factors** of the product, and the maps

$$\pi_s : \bigtimes_{t\in T} V_t \longrightarrow V_s$$

with

$$\pi_s((x_t)_{t\in T}) := x_s$$

defined for $s \in T$ are the **canonical projections**. ◇

Without difficulty we prove:

Proposition 59. *Every product of complete lattices is a complete lattice. The infimum and the supremum can be formed component-wise. The canonical projections are surjective complete homomorphisms.* □

The direct product of concept lattices corresponds to the direct sum of the contexts, cf. Definition 34.

Definition 56. A **(complete)**[1] **subdirect product** of complete lattices is a complete sublattice of the direct product for which the canonical projection maps onto the factors are all surjective.

A **subdirect decomposition** of a complete lattice $\mathbf{V}$ is a family Θ_t, $t \in T$, of complete congruence relations of $\mathbf{V}$ with

$$\bigcap_{t\in T} \Theta_t = \Delta,$$

where Δ denotes the trivial congruence $\Delta := \{(x,x) \mid x \in V\}$. The lattices $\mathbf{V}/\Theta_t$, $t \in T$, are called the **factors** of the subdirect decomposition. ◇

Theorem 17. *If $\mathbf{V}$ is a complete subdirect product of the lattices $\mathbf{V}_t, t \in T$, then the kernels of the canonical projections*

$$\{\ker \pi_t \mid t \in T\}$$

form a subdirect decomposition of $\mathbf{V}$. Conversely, for every subdirect decomposition Θ_t, $t \in T$ of $\mathbf{V}$, by

$$\iota(v) := ([v]\Theta_t)_{t\in T}$$

we obtain an isomorphism of $\mathbf{V}$ onto a subdirect product of the factor lattices $\mathbf{V}/\Theta_t$, $t \in T$.

[1] For stylistic reasons, we will frequently leave out the adjective "complete", i.e., in the following "subdirect" should be replaced by "completely subdirect" where necessary.

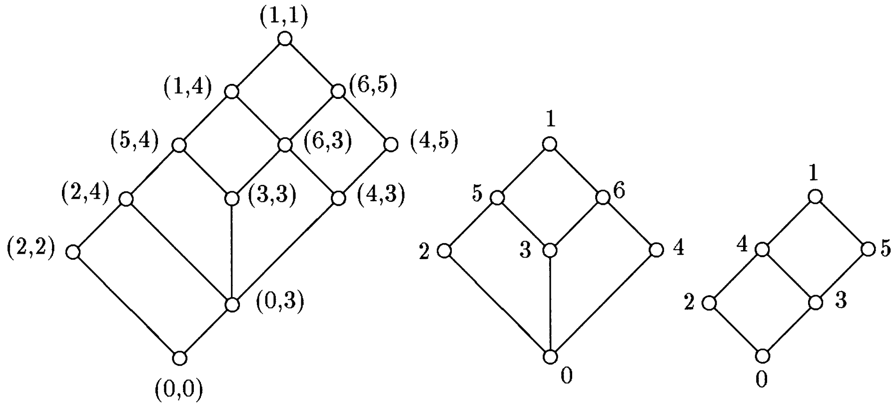


Figure 4.1 The lattice on the left is a subdirect product of the two lattices on the right.

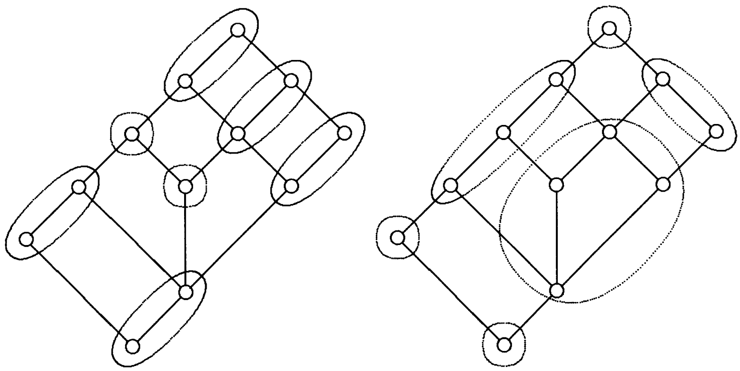


Figure 4.2 The two congruences represented here form a subdirect decomposition of the lattice in Figure 4.1. The factor lattices by these congruences are precisely the factors of the subdirect product represented in the figure above.

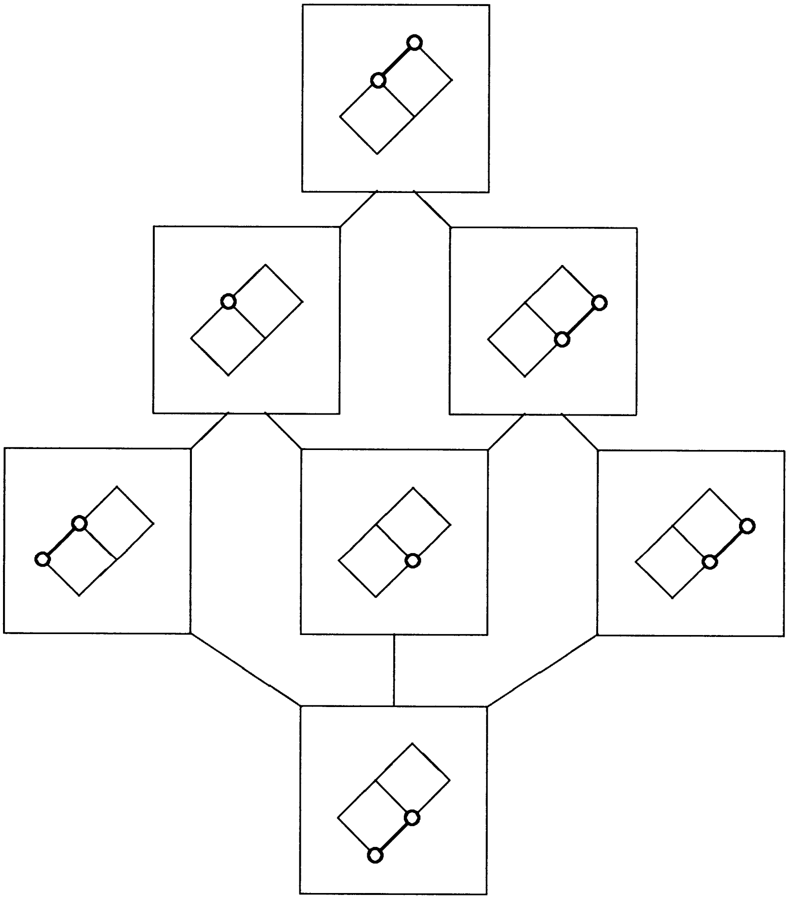


Figure 4.3 The nested line diagram helps us to follow the definition of the subdirect product.

Proof. The kernels of the canonical projections are congruences. Hence, in order to prove the first part, we only have to show that their intersection is the trivial congruence Δ. Two elements $(v_t)_{t\in T}$ and $(w_t)_{t\in T}$ of the direct product are different if there is some $s \in T$ with $v_s \neq w_s$, i.e., with $\pi_s(v) \neq \pi_s(w)$, which is equivalent to $(v,w) \notin \ker \pi_s$. Hence, from $v \neq w$ it follows that $(v,w) \notin \bigcap_{t\in T} \ker \pi_t$, which was to be proved.

Now we show that the map ι defined in the second part has the properties claimed. ι is a complete homomorphism, since for an arbitrary family $(v_j)_{j\in J}$ of elements of $\boldsymbol{V}$ we have

$$\begin{aligned}
\iota\left(\bigwedge_{j\in J} v_j\right) &= \left(\left[\bigwedge_{j\in J} v_j\right]\Theta_t\right)_{t\in T} \\
&= \left(\bigwedge_{j\in J} [v_j]\,\Theta_t\right)_{t\in T} \\
&= \bigwedge_{j\in J} ([v_j]\,\Theta_t)_{t\in T} \\
&= \bigwedge_{j\in J} \iota\,(v_j)\,.
\end{aligned}$$

Likewise, we show that ι preserves suprema. ι is injective, since from $v, w \in \boldsymbol{V}$, $v \neq w$ follows the existence of some $t \in T$ with $(v,w) \notin \Theta_t$. This means, however, that $[v]\Theta_t \neq [w]\Theta_t$, i.e., that $\iota(v) \neq \iota(w)$.

Hence, ι is an isomorphism of $\boldsymbol{V}$ onto the complete sublattice $\iota(V)$ of the product $\times_{t\in T} (\boldsymbol{V}/\Theta_t)$. It remains to be shown that the canonical projections

$$\pi_s : \iota(\boldsymbol{V}) \longrightarrow V_s, \qquad s \in T,$$

are surjective. This follows from

$$\pi_s(\iota(V)) = \{[v]\Theta_s \mid v \in V\} = \boldsymbol{V}/\Theta_s. \qquad \square$$

In the definition of the subdirect decomposition, we have not excluded the trivial case that one of the congruences is the trivial congruence Δ. The lattices which only allow such decompositions are described by the following definition:

Definition 57. A complete lattice $\boldsymbol{V}$ is called **(completely) subdirectly irreducible** if every subdirect decomposition of $\boldsymbol{V}$ contains the trivial congruence Δ. ◇

This property can also be formulated as follows: If $\boldsymbol{V}$ is isomorphic to a subdirect product of lattices $\boldsymbol{V}_t$, $t \in T$, then $\boldsymbol{V}$ is canonically isomorphic to one of the factors V_t. (*Canonically isomorphic* here means that the canonical projection π_t is bijective and is therefore an isomorphism from $\boldsymbol{V}$ to $\boldsymbol{V}_t$).

It is particularly easy to read off this property from the lattice of congruence relations of $\boldsymbol{V}$, since $\boldsymbol{V}$ is subdirectly irreducible if and only if this lattice has exactly one atom:

Proposition 60. *A complete lattice $\boldsymbol{V}$ is subdirectly irreducible if and only if $\boldsymbol{V}$ has a smallest non-trivial*[2] *congruence, i.e., a complete congruence relation Θ with $\Theta \neq \Delta$ and $\Theta \leq \Psi$ for all complete congruences $\Psi \neq \Delta$.*

Proof. If Θ is such a congruence and if ι is an isomorphism of $\boldsymbol{V}$ onto a subdirect product $\iota(V)$ of lattices $\boldsymbol{V}_t, t \in T$, then, because of

$$\bigcap_{t \in T} \ker(\pi_t \circ \iota) = \Delta,$$

it is not possible that

$$\Theta \leq \ker(\pi_t \circ \iota) \text{ for all } t \in T.$$

Hence, $\ker(\pi_t \circ \iota) = \Delta$ for at least one $t \in T$.

If, on the other hand, there is no such minimal congruence, then

$$\bigcap\{\Theta \mid \Theta \in \mathfrak{C}(\boldsymbol{V}), \Theta \neq \Delta\} = \Delta$$

for the family $\mathfrak{C}(\boldsymbol{V})$ of all congruences of $\boldsymbol{V}$. Hence, these congruences form a proper subdirect decomposition of $\boldsymbol{V}$. □

The examination of subdirect decompositions can be carried out directly on the context if we use the interplay between congruence relations, compatible and arrow-closed subcontexts which we have developed in the preceding sections. In order to do so, we must presuppose that the lattice $\boldsymbol{V}$ we examine is doubly founded and thus isomorphic to the concept lattice of a reduced context $\mathbb{K}$, since in this case the congruences are in one-to-one correspondence to the arrow-closed subcontexts of $\mathbb{K}$.

Proposition 61. *If (G, M, I) is a reduced context of a doubly founded concept lattice, then the subdirect decompositions of $\underline{\mathfrak{B}}(G, M, I)$ correspond bijectively to the families of arrow-closed subcontexts $(G_t, M_t, I \cap G_t \times M_t)$ with $\bigcup_{t \in T} G_t = G$ and $\bigcup_{t \in T} M_t = M$.*

Proof. According to the observations preceding Theorem 12, $\bigcap_{t \in T} \Theta_t = \Delta$ holds for a family Θ_t, $t \in T$ of congruences if and only if $\bigcup_{t \in T} G_t = G$ and $\bigcup_{t \in T} M_t = M$ holds for the corresponding arrow-closed subcontexts $(G_t, M_t, I \cap G_t \times M_t)$. □

It is particularly easy to recognize the subcontexts belonging to subdirectly irreducible factors of $\underline{\mathfrak{B}}(G, M, I)$. We must be aware that for every object g there is always a smallest arrow-closed subcontext containing g. We

[2] We allow the total congruence $V \times V$.

shall call such a subcontext a **1-generated** arrow-closed subcontext. (The corresponding is true for the attributes, but since in a reduced context every object is connected to an attribute by a double arrow and vice versa, it suffices to concentrate on one of the sets G or M, respectively).

Proposition 62. *A doubly founded reduced context (G, M, I) is 1-generated if and only if $\underline{\mathfrak{B}}(G, M, I)$ is subdirectly irreducible.*

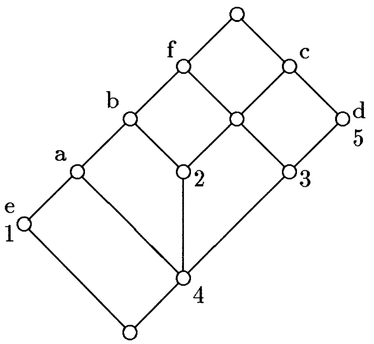
Proof. If (G, M, I) is 1-generated, then for every family $(G_t, M_t, I \cap G_t \times M_t)$, $t \in T$, of arrow-closed subcontexts there is some $t \in T$ with $G_t = G$ and $M_t = M$. By means of Proposition 61 we recognize that this is equivalent to subdirect irreducibility. □

Theorem 18. *Every doubly founded complete lattice has a subdirect decomposition into subdirectly irreducible factors.*

Proof. W.l.o.g. we may assume that $\boldsymbol{V}$ is the concept lattice of a reduced context (G, M, I). We may then assume that $\boldsymbol{V} = \underline{\mathfrak{B}}(G, M, I)$. For $g \in G$ let $(G_g, M_g, I \cap G_g \times M_g)$ denote the smallest arrow-closed subcontext of (G, M, I) containing g. Proposition 36 (p. 101) shows that this subcontext is reduced as well. According to Proposition 62 the corresponding concept lattice is subdirectly irreducible. Hence, together with Proposition 61,

$$(G_g, M_g, I \cap G_g \times M_g), \quad g \in G,$$

provides a subdirect decomposition of $\boldsymbol{V}$ into subdirectly irreducible factors. □



	a	b	c	d	e	f
1	×	×	⤢	$\swarrow$	×	×
2	⤢	×	×	⤢		×
3	$\swarrow$	⤢	×	×		×
4	×	×	×	×	⤢	×
5			×	×		⤢

Figure 4.4 Using the arrow relations in the context, we can examine which subdirect decompositions are possible for the concept lattice.

To close this section, we apply the theory we have developed to the example given in the beginning (Figure 4.1). A representation of the lattice

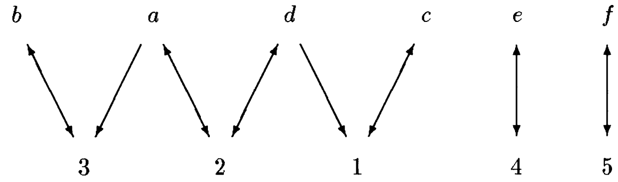


Figure 4.5 We recognize five 1-generated subcontexts: one for each object.

as a concept lattice is presented in Figure 4.4. The arrow relations of the context are shown in Figure 4.5 as a graph. We can read off that there are exactly five 1-generated subcontexts. In the main, there is only one subdirect decomposition of this lattice into subdirectly irreducible factors. The respective subcontexts are generated by the objects 2, 4 and 5. We can add the subcontexts generated by the objects 1 or 3, they are, however, contained in the subcontext generated by 2 and therefore dispensable.

The subdirect decomposition shown in Figure 4.2 corresponds to the two subcontexts $(\{1,2,3\},\{a,b,c,d\})$ and $(\{1,4,5\},\{c,e,f\})$. We recognize that the second factor can be chosen smaller: object 1 is superfluous, the second subcontext can be replaced by $(\{4,5\},\{e,f\})$. Figure 4.6 shows the corresponding congruence. It can replace the second congruence in Figure 4.2. The factor lattice obtained from this congruence has three elements.

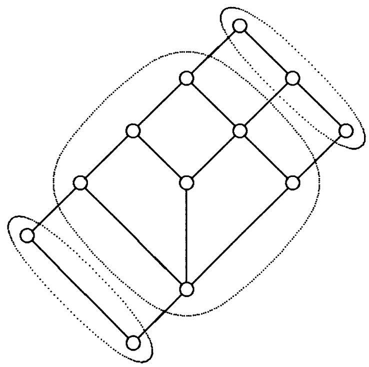


Figure 4.6 A coarser congruence may be chosen for the second congruence in the subdirect decomposition from Figure 4.2.

4.2 Atlas-decompositions

A map which is meant to represent a larger area on a larger scale necessarily becomes unwieldy. Usually, we make shift by splitting it up into an atlas: a collection of manageable part maps covering the desired area, together with a general map as well as additional information, which show how the individual maps are related.

An analogous procedure for large unwieldy lattices can be introduced with the help of the tolerance relations. According to Theorem 14 (p. 121), a tolerance relation on a complete lattice $\mathbf{V}$ provides a decomposition of $\mathbf{V}$ into intervals which are themselves elements of a complete lattice. Thus, $\mathbf{V}$ can be understood as being constructed from a family of complete lattices, whose indices in turn form a complete lattice. This means that the blocks of the tolerance appear in the role of the part maps in the atlas and the factor lattice acts as the "general map".

It is, however, in the case of the tolerances not quite as easy as in the case of topographic maps to explain the interconnection between the part maps. In the general case, this is done through a family of adjoint pairs of mappings. It is easier in the case of glued tolerances: here those maps ensue automatically.

Definition 58. Let Q be a complete lattice and, furthermore, let $\mathbf{V}_q$ be a complete lattice for every element $q \in Q$. Let

$$\varphi_q^r : \mathbf{V}_q \to \mathbf{V}_r \quad \text{and} \quad \psi_q^r : \mathbf{V}_r \to \mathbf{V}_q$$

be maps for every pair $q \leq r$ in Q. Such a family

$$(\mathbf{V}_q \mid q \in Q)$$

is called a **Q-atlas** if the following conditions are satisfied:

0. $\mathbf{V}_q \subseteq \mathbf{V}_r \Rightarrow q = r$.
1. $\mathbf{V}_q \cap \mathbf{V}_r$ is an order filter of V_q and an order ideal of $\mathbf{V}_r$ if $q \leq r$.
2. $\{q \in Q \mid x \in \mathbf{V}_q\}$ is an interval $[x_{\min}, x_{\max}]$ in Q for every $x \in \bigcup_{q \in Q} \mathbf{V}_q$.
3. $\varphi_q^q x = x = \psi_q^q x$ for all $x \in \mathbf{V}_q$.
4. $\varphi_q^r x \leq y$ holds in $\mathbf{V}_r$ if and only if $x \leq \psi_q^r y$ holds in $\mathbf{V}_q$.
5. $\varphi_r^s \varphi_q^r = \varphi_q^r$ and $\psi_q^r \psi_r^s = \psi_q^r$.
6. $\varphi_q^r x = \varphi_{q \vee s}^{r \vee s} x$ for all $x \in \mathbf{V}_q \cap \mathbf{V}_{q \vee s}$ and $\psi_s^t = \psi_{s \wedge r}^{t \wedge r}$ for all $y \in \mathbf{V}_t \cap \mathbf{V}_{t \wedge r}$.

The **sum** of the Q-atlas is defined as the pair

$$\left(\bigcup_{q \in Q} \mathbf{V}_q, \sqsubseteq \right)$$

with

$$x \sqsubseteq y : \Longleftrightarrow x_{\min} \leq y_{\min} \text{ and } \varphi_{x_{\min}}^{y_{\min}} x \leq y$$

for all $x, y \in \bigcup_{q \in Q} \mathbf{V}_q$. ◇

Condition (4) says that for $q \leq r$ the two maps φ_q^r and ψ_q^r are mutually *adjoint* in the sense of Proposition 9 (p. 14), as can easily be seen from Propositions 4 and 6. Hence, φ_q^r is $\bigvee$-preserving and ψ_q^r is $\bigwedge$-preserving. If we denote the boundary elements of $\boldsymbol{V}_t$ by 0_t and 1_t, respectively, we obtain in particular $\varphi_q^r 0_q = 0_r$ and $\psi_q^r 1_r = 1_q$.

Proposition 63.

$$x_{\min} \leq y_{\min} \quad \textit{and} \quad \varphi_{x_{\min}}^{y_{\min}} x \leq y$$

is equivalent to

$$x_{\max} \leq y_{\max} \quad \textit{and} \quad x \leq \psi_{x_{\max}}^{y_{\max}} y.$$

Proof. Let $x_{\min} \leq y_{\min}$ and $\varphi_{x_{\min}}^{y_{\min}} x \leq y$. In the following we shall use the abbreviations

$$q := x_{\min}, \quad r := y_{\min}, \quad s := x_{\max}, \quad t := y_{\max}.$$

By (6) we obtain

$$y \geq \varphi_q^r x = \varphi_s^{r \vee s} x.$$

Hence, $\varphi_q^r x$ is an element of $\boldsymbol{V}_r \cap \boldsymbol{V}_{r \vee s}$, and by (1) it follows that $y \in \boldsymbol{V}_{r \vee s}$, i.e., $s \leq t$. Since $r \geq x_{\min}$, we have $s \wedge r \in [x_{\min}, x_{\max}]$ and $x \in \boldsymbol{V}_{s \wedge r}$ follows from (2). Together with (6) and (3) we obtain

$$\varphi_q^{s \wedge r} x = x.$$

Thus

$$\varphi_{s \wedge r}^r x = \varphi_{s \wedge r}^r \varphi_q^{s \wedge r} x = \varphi_q^r x \leq y$$

because of (5). With (4) and (6) we obtain

$$x \leq \psi_{s \wedge r}^r y = \psi_s^t y.$$

The opposite direction of the equivalence follows dually. □

Theorem 19. *The sum of a Q-atlas is a complete lattice $\boldsymbol{V}$, in which infima and suprema can be described as follows:*

$$\bigsqcup_{t \in T} x_t = \bigvee_{t \in T} \varphi_{q_t}^r x_t \quad \textit{and} \quad \bigsqcap_{t \in T} x_t = \bigwedge_{t \in T} \psi_s^{q_t} x_t$$

with $x_t \in \boldsymbol{V}_{q_t}$, $r := \bigvee_{t \in T} q_t$ and $s := \bigwedge_{t \in T} q_t$. The complete lattices $\boldsymbol{V}_q$, $q \in Q$ are precisely the blocks of a complete tolerance relation Θ of $\boldsymbol{V}$, and $q \mapsto \boldsymbol{V}_q$ describes an isomorphism of Q onto $\boldsymbol{V}/Q$; furthermore, we have

$$\varphi_q^r = x \sqcup 0_r \quad \textit{for all } x \in \boldsymbol{V}_q$$

and

$$\psi_q^r y = y \sqcap 1_q \quad \textit{for all } y \in \boldsymbol{V}_r.$$

In this way we obtain a bijective assignment between the complete tolerance relations on a complete lattice and the representations of this lattice as the sum of a Q-atlas.

Proof. Without difficulty we prove that $\sqsubseteq$ is an order which corresponds, moreover, on each of the $\boldsymbol{V}_q$ to the order given there. Furthermore, we prove that $x \sqsubseteq y$ always follows from $\varphi_q^r x = y$: By (6) we first obtain $\varphi_{x_{\max}}^{r \vee x_{\max}} x = y$ and thus $x_{\max} \leq y_{\max}$. With

$$\varphi_{r \vee x_{\max}}^{y_{\max}} y = \varphi_{y_{\max}}^{y_{\max}} y = y$$

we get $\varphi_{x_{\max}}^{y_{\max}} x = y$ and, because of (4), $x \leq \psi_{x_{\max}}^{y_{\max}} y$. Consequently, $x \sqsubseteq y$ holds according to Proposition 63. Dually we infer that $x = \psi_q^r y$ always implies $x \sqsubseteq y$.

We now show that the supremum has the form specified in the theorem: Assume that $x_t \in \boldsymbol{V}_{q_t}$ and $x_t \sqsubseteq y$ for $t \in T$. By Proposition 63 and (2) we obtain $y \in \boldsymbol{V}_{y_{\min} \vee q_t}$ for all $t \in T$ and thus $y \in \boldsymbol{V}_{y_{\min} \vee r}$ for $r := \bigvee_{t \in T} q_t$. Therefore, from $\varphi_{x_{\min}}^{y_{\min}} x_t \leq y$ it follows that $\varphi_{q_t}^{y_{\min} \vee q_t} x_t \leq y$ because of (6) and

$$\varphi_{q_t}^r \sqsubseteq \varphi_{q_t}^{y_{\min} \vee r} x_t = \varphi_{y_{\min} \vee q_t}^{y_{\min} \vee r} \varphi_{q_t}^{y_{\min} \vee q_t} x_t \sqsubseteq \varphi_{y_{\min} \vee q_t}^{y_{\min} \vee r} y = y$$

because of (5), (6) and (3). Thereby we have proved $\bigvee_{t \in T} \varphi_{q_t}^r x_t \sqsubseteq y$. Since $\bigvee_{t \in T} \varphi_{q_t}^r x_t$ is an upper bound of each x_t, $t \in T$, $\bigvee_{t \in T} \varphi_{q_t}^r x_t$ is the supremum of the x_t, $t \in T$, in $(V, \sqsubseteq)$. Because of Proposition 63, the dual proof yields the equation for the infimum. Thus, we have proved that the sum of a Q-atlas is always a complete lattice.

For $x, y \in \boldsymbol{V}$ now assume that

$$x \Theta y :\Longleftrightarrow x, y \in \boldsymbol{V}_q \text{ for at least one } q \in Q.$$

The proved description of the suprema and infima immediately yields that Θ is a complete tolerance relation of $\boldsymbol{V}$. We can use Proposition 55 (p. 120) to show that the $\boldsymbol{V}_q$ are precisely the blocks of Θ. This requires however to prove the maximality of the $\boldsymbol{V}_q$. Assume therefore that $q \in Q$ and that y is an element with

$$x \Theta y \quad \text{for all } x \in \boldsymbol{V}_q.$$

In particular, we have $y \Theta 0_q$, which yields $\{0_q, y\} \subseteq \boldsymbol{V}_r$ for some $r \in Q$ and because $0_r \leq 0_q$ it follows that $r \leq q$. Likewise, we obtain from $y \Theta 1_q$ some $s \in Q$ with $y \in \boldsymbol{V}_s$ and $s \geq q$. In all, we have $q \in [y_{\min}, y_{\max}]$ and thus by (2) $y \in \boldsymbol{V}_q$, which together with (0) yields the assertion.

Evidently, $q \mapsto \boldsymbol{V}_q$ describes an isomorphism of Q on $\boldsymbol{V}/\Theta$. For $q \leq r$, $x \in \boldsymbol{V}_q$ and $y \in \boldsymbol{V}_r$ we have

$$x \sqcup 0_r = \varphi_q^r x \vee \varphi_r^r 0_r = \varphi_q^r x$$

and dually $y \sqcap 1_q = \psi_q^r y$. If we define for an arbitrary complete tolerance relation Ξ morphisms between its blocks through these equations, we obtain a $\boldsymbol{V}/\Xi$-atlas whose sum again is $\boldsymbol{V}$. This shows the bijective assignment we claimed. $\square$

The conditions become simpler in the case of tolerances with overlapping neighbourhoods.

Definition 59. A complete tolerance relation Θ of a lattice $\boldsymbol{V}$ has **overlapping neighbourhoods** if

$$B_1 \prec B_2 \text{ in } \boldsymbol{V}/\Theta \quad \text{implies} \quad B_1 \cap B_2 \neq \emptyset.$$

Let $\Sigma(\boldsymbol{V})$ denote the smallest tolerance relation comprising all pairs (x, y) with $x \prec y$ in $\boldsymbol{V}$.

In the case of doubly founded lattices, a tolerance with overlapping neighbourhoods is called **glued**, and $\Sigma(\boldsymbol{V})$ is called the **skeleton tolerance.** The factor lattice $\boldsymbol{V}/\Sigma(\boldsymbol{V})$ is then called the **skeleton** of $\boldsymbol{V}$. $\diamond$

The intersection of any number of tolerance relations is again a tolerance relation. Therefore $\Sigma(\boldsymbol{V})$ is well-defined.

Theorem 20. *$\Sigma(\boldsymbol{V})$ is the smallest tolerance relation of $\boldsymbol{V}$ with overlapping neighbourhoods. In particular, the skeleton tolerance is the smallest glued tolerance.*

Proof. First we show that $\Sigma(\boldsymbol{V})$ has overlapping neighbourhoods. For this purpose, let $B_1 =: [a, b]$ and $B_2 =: [c, d]$ be blocks of $\Sigma(\boldsymbol{V})$ with $B_1 \prec B_2$. We show that the assumption $B_1 \cap B_2 = \emptyset$ leads to a contradiction. It would imply that $b < b \vee c$ and from $b < x \leq d$ it would follow that generally

$$B_1 = [b]_{\Sigma(V)} < [x]_{\Sigma(V)} \leq [d]_{\Sigma(V)} = B_2,$$

i.e., because of $B_1 \prec B_2$, that $[x]_{\Sigma(V)} = B_2$ and thus $x \geq c$. For this reason we have that $b \vee c = \bigwedge\{x \mid b < x \leq d\}$, from which we can infer that $b \prec b \vee c$ and consequently $(b, b \vee c) \in \Sigma(V)$. Because of $(b \vee c)_{\Sigma(V)} = c$ this yields $c \leq b \leq d$, i.e., the desired contradiction $b \in B_2$.

It remains to be shown that $\Sigma(\boldsymbol{V})$ is smallest among the tolerance relations with overlapping neighbourhoods. Hence, let Θ be any such tolerance relation and $x \prec y$ a pair of neighbouring elements of $\boldsymbol{V}$. We must show that there is a block of Θ containing x and y. This is certainly the case if $x \in [y]_\Theta$, i.e., we may assume $x \notin [y]_\Theta$ and in particular $[x]_\Theta < [y]_\Theta$. Now we consider an arbitrary block $[u, v]$ with $[x]_\Theta \leq [u, v] < [y]_\Theta$. Because of $u < y_\Theta \leq y$, $y \in [u, v]$ would immediately follow from $y \leq v$. It would imply $y \Theta u$ and thus a contradiction $u < y_\Theta$. Hence, $y \not\leq v$ and, because of $x = (v \wedge y)\ \Theta\ (v \wedge y^\Theta) = v$, we obtain $x \Theta v$ and thus $x \in [u, v]$. Hence, every block between $[x]_\Theta$ and $[y]_\Theta$ contains x. Since the lower bounds of the blocks are closed under suprema, this also holds for

$$B_x := \bigvee\{B \in \boldsymbol{V}/\Theta \mid [x]_\Theta \leq B < [y]_\Theta\}.$$

Hence, this block must be a lower neighbour of $[y]_\Theta$. Since Θ has overlapping neighbourhoods, it follows that $B_x \cap [y]_\Theta \neq \emptyset$, i.e., $\bigvee B_x \in [y]_\Theta$. If $y \notin B_x$, then $y \wedge \bigvee B_x = x$ and thus $x \in [y]_\Theta$, contradicting $[x]_\Theta < [y]_\Theta$! $\square$

The corresponding block relation $\beta(\Sigma(\boldsymbol{V}))$ can be described easily.

Theorem 21. *Let (G, M, I) be a doubly founded context and let*

$$\Sigma := \Sigma(\underline{\mathfrak{B}}(G, M, I))$$

be the skeleton tolerance. Then the following statements hold for the corresponding block relation $J := \beta(\Sigma)$:

a) J is the smallest block relation of (G, M, I) containing all pairs (g, m) with $g \swarrow\!\!\!\nearrow m$.
b) J contains all pairs (g, m) with $g \swarrow m$ or $g \nearrow m$.

Proof. b) From $g \swarrow m$ it follows that $\gamma g \wedge \mu m = (\gamma g)_* \prec \gamma g$, i.e., $(g, m) \in J$ according to the definition of β. Dually we show that J furthermore contains all pairs (g, m) with $g \nearrow m$.

a) Let K be a block relation comprising all pairs (g, m) with $g \swarrow\!\!\!\nearrow m$ and let furthermore (A, B) and (C, D) be concepts with $(A, B) \prec (C, D)$. We want to show that (A, B) and (C, D) are related under the tolerance relation $\beta^{-1}(K)$ belonging to K. For this purpose, we consider an object $g \in C$ and an attribute $m \in B$ with $g \not I m$. Since the context is doubly founded, we find an object h with $h' \supseteq g'$ and $h \swarrow m$, i.e., in particular $h \not I m$. We again make use of the fact that the context is doubly founded to get an attribute n with $n' \supseteq m'$ and $h \nearrow n$ and thus $h \swarrow\!\!\!\nearrow n$ (since $\mu n \geq \mu m \geq (\gamma h)_*$). Consequently, $(h, n) \in K$, which is equivalent to $(\gamma h, \gamma h \wedge \mu n) \in \beta^{-1}(K)$, or shorter $(\gamma h, (\gamma h)_*) \in \beta^{-1}(K)$. From that we infer $((A, B) \vee \gamma h, (A, B) \vee (\gamma h)_*) \in \beta^{-1}(K)$, i.e., $((C, D), (A, B)) \in \beta^{-1}(K)$, as claimed. □

Now we concentrate on the case of glued tolerances; in particular we presuppose that the lattices are doubly founded. We describe the system of the blocks of a glued tolerance in abstract terms:

Definition 60. Let $\boldsymbol{V}_q$, $q \in Q$ be a family of doubly founded complete lattices. Let the index set Q be a lattice of finite length. We call $(\boldsymbol{V}_q \mid q \in Q)$ a **Q-atlas with overlapping neighbour maps**, if for each two elements $q, r \in Q$ the following conditions are satisfied:

0. $\boldsymbol{V}_q \subseteq \boldsymbol{V}_r \Rightarrow q = r$.
1. If $q \leq r$, then $\boldsymbol{V}_q \cap \boldsymbol{V}_r$ is an order filter in $\boldsymbol{V}_q$ and an order ideal in $\boldsymbol{V}_r$.
2. If q is a lower neighbour of r, then $\boldsymbol{V}_q \cap \boldsymbol{V}_r \neq \emptyset$.
3. The orders of $\boldsymbol{V}_q$ and $\boldsymbol{V}_r$ coincide on the intersection $\boldsymbol{V}_q \cap \boldsymbol{V}_r$.
4. $\boldsymbol{V}_q \cap \boldsymbol{V}_r \subseteq \boldsymbol{V}_{q \wedge r} \cap \boldsymbol{V}_{q \vee r}$.
5. $q \leq r \leq s \Rightarrow \boldsymbol{V}_q \cap \boldsymbol{V}_s \subseteq \boldsymbol{V}_r$

◇

This is compatible with Definition 58. The maps postulated there ensue canonically in the glued case, as the following proposition shows.

Proposition 64. *A Q-atlas with overlapping neighbour maps is a Q-atlas in the sense of 58 if we define the maps φ_q^r and ψ_q^r as follows:*

1. $\varphi_q^q\, x := \psi_q^q\, x := x$ *for all* $x \in \boldsymbol{V}_q$*;*
2. *let*
$$\varphi_q^r\, x := x \vee 0_r \quad (\text{in } \boldsymbol{V}_q),$$
$$\psi_q^r\, y := y \wedge 1_q \quad (\text{in } \boldsymbol{V}_r)$$
for $q \prec r$*;*
3. *let*
$$\varphi_q^r := \varphi_{q_{m-1}}^{q_m} \circ \cdots \circ \varphi_{q_0}^{q_1},$$
$$\psi_q^r := \psi_{q_0}^{q_1} \circ \cdots \circ \psi_{q_{m-1}}^{q_m}$$
for $q = q_0 \prec q_1 \prec \cdots \prec q_m = r$*.*

Proof. Conditions 0) and 1) are the same as in Definition 58. Condition 2) of Definition 58 follows like this: The set

$$\{q \in Q \mid x \in \boldsymbol{V}_q\}$$

is by 5) convex and by 4) closed against $\vee$ and $\wedge$, i.e., an interval in the lattice Q (which is of finite length).

The remaining conditions refer to the maps φ_q^r and ψ_q^r. First, we have to show that the maps in the manner specified are well-defined for all $q \leq r$. In the case that $q \prec r$, then $\boldsymbol{V}_q \cap \boldsymbol{V}_r$ by 2) is not empty and, with 1), we find $0_r \in \boldsymbol{V}_q$. Hence, the supremum $x \vee 0_r$ can be formed within $\boldsymbol{V}_q$ for all $x \in \boldsymbol{V}_q$ and by 1) lies in $\boldsymbol{V}_r$.

Since Q is of finite length, for any two elements $q < r$ in Q there exists at least one, but possibly several chains of neighbour elements between q and r. We have to show that the definition of φ_q^r is independent of the choice of such a chain (the proof of ψ_q^r then works analogously). Hence, let $q_0 \prec q_1 \prec \cdots \prec q_m$ and $r_0 \prec r_1 \prec \cdots \prec r_n$ be chains with $q_0 = r_0$ and $q_m = r_n$. The proof works through induction on the length of the interval $[q_0, q_m]$.

1[st] *case:* $q_1 \vee r_1 = q_m$.
Since the smallest elements of $\boldsymbol{V}_{q_1}$ and $\boldsymbol{V}_{r_1}$ belong to $\boldsymbol{V}_{q_0}$, the supremum $0_{q_1} \vee 0_{r_1}$ can be formed in $\boldsymbol{V}_{q_0}$. This element belongs to $\boldsymbol{V}_{q_1}$ as well as to $\boldsymbol{V}_{r_1}$; i.e., the two lattices are not disjoint. Because of 4) we have

$$\boldsymbol{V}_{q_1} \cap \boldsymbol{V}_{r_1} \subseteq \boldsymbol{V}_{q_0} \cap \boldsymbol{V}_{q_m},$$

i.e., $\boldsymbol{V}_{q_0}$ and $\boldsymbol{V}_{q_m}$ are not disjoint either, and 0_{q_m} has to be an element of $\boldsymbol{V}_{q_0}$. Because of 5), 0_{q_m} therefore belongs to all lattices $\boldsymbol{V}_{q_i}$, $i \in \{0, \ldots, m\}$

and to all lattices $\boldsymbol{V}_{r_j}$, $j \in \{0, \ldots, n\}$. The sets $\boldsymbol{V}_{q_0} \cap \boldsymbol{V}_{q_i}$ and $\boldsymbol{V}_{q_0} \cap \boldsymbol{V}_{r_j}$ are therefore all nonempty. Since by 1) they are order ideals, the elements 0_{q_i} and 0_{r_j}, respectively, must all belong to $\boldsymbol{V}_{q_0}$.

Hence, for $x \in \boldsymbol{V}_{q_0}$ it follows that

$$(\varphi^{q_m}_{q_{m-1}} \circ \cdots \circ \varphi^{q_1}_{q_0})x = (\ldots((x \vee 0_{q_1}) \vee 0_{q_2}) \vee \ldots) \vee 0_{q_m},$$

and all those suprema are being formed in $\boldsymbol{V}_{q_0}$. Therefore,

$$\varphi^{q_m}_{q_0}\, x = x \vee 0_{q_1} \vee \cdots \vee 0_{q_m} = x \vee 0_{q_m}$$

and correspondingly

$$\varphi^{r_n}_{r_0}\, x = x \vee 0_{r_1} \vee \cdots \vee 0_{r_n} = x \vee 0_{r_n},$$

which on account of $q_m = r_n$ yields the desired result.
2nd *case:* $q_1 \vee r_1 < q_m$. Assume that

$$\begin{aligned} q_1 = s_1 \prec s_2 \prec \cdots \prec s_j &= q_1 \vee r_1 \\ r_1 = t_1 \prec t_2 \prec \cdots \prec t_k &= q_1 \vee r_1 \end{aligned}$$

and $s_j \prec s_{j+1} \prec \cdots \prec s_l = q_m$ as well as $t_{k+i} = s_{j+i}$ for $i \in \{1, \ldots, l-j\}$. By the induction hypothesis

$$\varphi^{s_j}_{s_{j-1}} \circ \cdots \circ \varphi^{s_2}_{s_1} \circ \varphi^{q_1}_{q_0} = \varphi^{t_k}_{t_{k-1}} \circ \cdots \circ \varphi^{t_2}_{t_1} \circ \varphi^{r_1}_{r_0},$$

$$\varphi^{q_m}_{q_{m-1}} \circ \cdots \circ \varphi^{q_2}_{q_1} = \varphi^{s_l}_{s_{l-1}} \circ \cdots \circ \varphi^{s_2}_{s_1}$$

and

$$\varphi^{r_n}_{r_{n-1}} \circ \cdots \circ \varphi^{r_2}_{r_1} = \varphi^{t_{l-j}}_{t_{l-j-1}} \circ \cdots \circ \varphi^{t_2}_{t_1}.$$

Now, by insertion we obtain

$$\varphi^{q_m}_{q_{m-1}} \circ \cdots \circ \varphi^{q_1}_{q_0} = \varphi^{r_n}_{r_{n-1}} \circ \cdots \circ \varphi^{r_1}_{r_0}.$$

Thereby we have shown that the definition of φ^r_q is independent of the choice of the chain.

Furthermore, we have to prove (4), i.e., that the pairs of maps φ^r_q, ψ^r_q, $q \leq r$ are adjoint pairs.. If $q \prec r$, then this follows immediately from the definition, since for $x \in \boldsymbol{V}_q$, $y \in \boldsymbol{V}_r$ we have

$$\varphi^r_q\, x \leq y \iff x \vee 0_r \leq y \iff x \leq y$$

and dually

$$x \leq \psi^r_q y \iff x \leq y \wedge 1_q \iff x \leq y.$$

Thus, we get

$$\begin{array}{ll} & \varphi^{q_m}_{q_{m-1}} \circ \cdots \circ \varphi^{q_1}_{q_0}\, x \leq y \\ \iff & \varphi^{q_{m-1}}_{q_{m-2}} \circ \cdots \circ \varphi^{q_1}_{q_0}\, x \leq \psi^{q_m}_{q_{m-1}}\, y \\ \iff & \vdots \\ \iff & x \leq \psi^{q_1}_{q_0} \circ \cdots \circ \psi^{q_m}_{q_{m-1}}\, y. \end{array}$$

Condition (5) follows immediately from parts (1) and (3) of the definition. What remains to be shown is condition 6) of Definition 58. In order to do so, we first consider the case $q \prec r$, for which from $x \in \boldsymbol{V}_q \cap \boldsymbol{V}_{q \vee s}$ it follows that:

$$\varphi^r_q x = x \vee 0_r = x \vee 0_r \vee 0_{q \vee s} = x \vee 0_{r \vee s} = \varphi^{r \vee s}_{q \vee s} x$$

(if $q \vee s \prec r \vee s$ does not hold, then the last of these equalities is inferred as above for $\varphi^{q_m}_{q_0}$). The general case is obtained by concatenation along a chain of neighbours. □

The **sum** of a Q-atlas with overlapping neighbour maps given by $(\boldsymbol{V}_q \mid q \in Q)$ is described by

$$\left(\bigcup_{q \in Q} \boldsymbol{V}_q, \leq \right),$$

$\leq$ being the transitive closure of the union of the orders on the summands.

Theorem 22. *The sum of a Q-atlas with overlapping neighbour maps is a complete lattice $\boldsymbol{V}$ where the summands $\boldsymbol{V}_q$, $q \in Q$ are precisely the blocks of a complete tolerance relation Θ and where $q \mapsto \boldsymbol{V}_q$ describes an isomorphism of Q onto $\boldsymbol{V}/Q$.*

Conversely, in a complete lattice $\boldsymbol{V}$ the blocks of a tolerance Θ with overlapping neighbourhoods, for which $Q := \boldsymbol{V}/\Theta$ is of finite length, always form a Q-atlas with overlapping neighbour maps whose sum is $\boldsymbol{V}$.

Proof. First of all, we shall prove that the order $\sqsubseteq$ of the Q-atlas, which is described by Proposition 64, is equal to the transitive closure $\leq$ of the union of the orders on the summands. According to the definition, $\sqsubseteq$ on the summands $\boldsymbol{V}_q$ coincides with their respective orders, which is the reason why from $x \leq y$ always follows $x \sqsubseteq y$. If, conversely, $x \sqsubseteq y$, i.e., $x_{\min} \leq y_{\min}$ and $\varphi^{y_{\min}}_{x_{\min}}\, x \leq y$, then for

$$x_{\min} = q_0 \prec q_1 \prec \cdots \prec q_m = y_{\min}$$

as in Proposition 64 it follows that

$$\varphi^{y_{\min}}_{x_{\min}}\, x = (\ldots((x \vee 0_{q_1}) \vee 0_{q_2}) \vee \ldots) \vee 0_{q_m},$$

which yields $x \leq y$. Thus, we have proved that

$$\left(\bigcup_{q\in Q} V_q, \leq\right) = \left(\bigcup_{q\in Q} V_q, \sqsubseteq\right).$$

Therefore, Proposition 64 yields the assertions of the first part of Theorem 19. Furthermore, we get that in a complete lattice $\boldsymbol{V}$ the blocks of a glued tolerance Θ form a Q-atlas. It only remains to be shown that this Q-atlas is in fact a Q-atlas with overlapping neighbour maps. Conditions 0), 1), 2) and 3) of Definition 60 are obviously satisfied. From $0_q \leq x \leq 1_q$ and $0_r \leq x \leq 1_r$ it follows that

$$0_{q\vee r} = 0_q \vee 0_r \leq x \leq 1_q \wedge 1_r = 1_{q\wedge r},$$

which proves 4). 5) can be seen from the fact that $q \leq r \leq s$ and $0_s \leq x \leq 1_q$, because of $0_r \leq 0_s$ and $1_q \leq 1_r$, immediately yield $0_r \leq x \leq 1_r$. $\square$

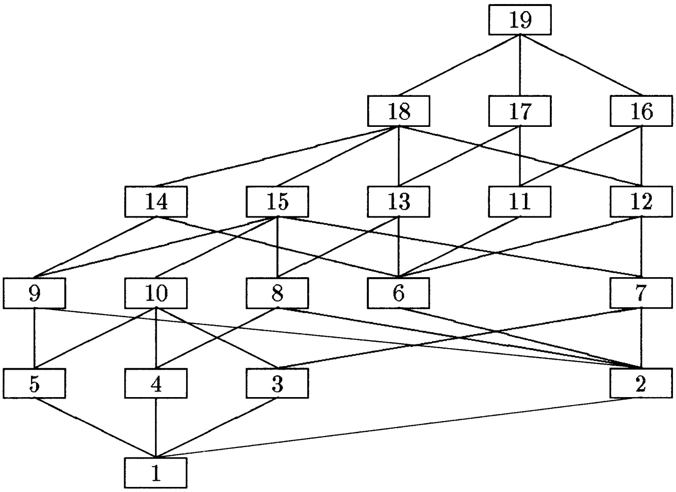


Figure 4.7 Computer-generated lattice diagram

	10	14	15	16	17	18
2	$\swarrow\!\!\nearrow$	$\times$	$\times$	$\times$	$\times$	$\times$
3	$\times$	$\swarrow\!\!\nearrow$	$\times$	$\times$	$\swarrow\!\!\nearrow$	$\times$
4	$\times$	$\swarrow\!\!\nearrow$	$\times$	$\swarrow\!\!\nearrow$	$\times$	$\times$
5	$\times$	$\times$	$\times$	$\swarrow\!\!\nearrow$	$\swarrow\!\!\nearrow$	$\times$
6		$\times$	$\swarrow\!\!\nearrow$	$\times$	$\times$	$\times$
11		$\swarrow$		$\times$	$\times$	$\swarrow\!\!\nearrow$

Figure 4.8 The standard context for the lattice from Figure 4.7

Theorem 22 can be applied quite practically for the representation in diagrams, provided that the lattice which is to be represented has a tolerance with overlapping neighbourhoods. This can be checked by entering the arrow relations into the context and enriching the relation

$$J := I \cup \swarrow \cup \nearrow$$

in accordance with the conditions in Definition 54 (p. 121), until a block relation is obtained (namely $\beta(\Sigma)$). According to Corollary 57 (p. 124), we obtain the blocks ("maps") as concept lattices of subcontexts. Diagrams are created from these. The overlaps can be read off, even in the case of a reduced labelling of the individual maps, since every concept is stated with its correct extent and intent. According to Theorem 22, the lattice is uniquely described by this set of diagrams. The correctness of the atlas can be verified by means of the conditions in Definition 60.

We shall demonstrate this using the example of a lattice of subgroups. Figure 4.7 shows a computer-generated diagram, which was taken from a book on orthomodular lattices [91]. The standard context for this lattice (cf. page 27), including the arrow relations, is presented in Figure 4.8. A short examination shows that $J := I \cup \swarrow \cup \nearrow$ is already a block relation, i.e., that it is equal to $\beta(\Sigma)$.

The concept lattice of (G, M, J) is a three-element chain. The subcontexts belonging to the blocks of the relation are represented together with their concept lattices in Figure 4.9. In the case of the lattice presented in the middle of this figure, it can be easily seen how the blocks overlap. The smallest element of this block is the concept with the extent $\{2\}$. We discover it in the lower lattice on the right side. The largest element of the lower lattice has the intent $\{15, 18\}$. It can easily be found in the middle lattice.

Hence, the lower and the middle lattice have the five elements of the interval

$$[(\{2\}, \{14, 15, 16, 17, 18\}), (\{2, 3, 4, 5\}, \{15, 18\})]$$

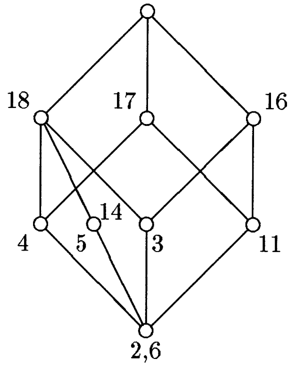
in common. Analogous are the middle and the upper lattice, which overlap in the interval

$$[\gamma 6, \mu 18],$$

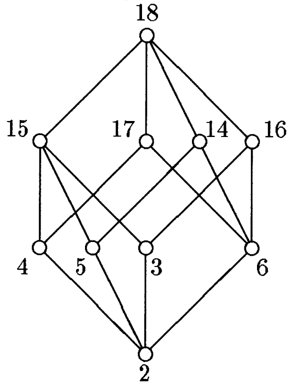
also having five elements.

In the present case, it turns out to be particularly convenient that we have chosen congruent diagrams for the overlap areas. Thereby, it becomes possible to superimpose the individual part-diagrams, Figure 4.10), and, thus, to obtain a new diagram for the lattice (Figure 4.11), which, due to its construction method, reflects the structure particularly well.

	14	16	17	18
2	×	×	×	×
3		×		×
4			×	×
5	×			×
6	×	×	×	×
11		×	×	



	14	15	16	17	18
2	×	×	×	×	×
3		×	×		×
4		×		×	×
5	×	×			×
6	×		×	×	×



	10	14	15	16	17	18
2		×	×	×	×	×
3	×		×	×		×
4	×		×		×	×
5	×	×	×			×

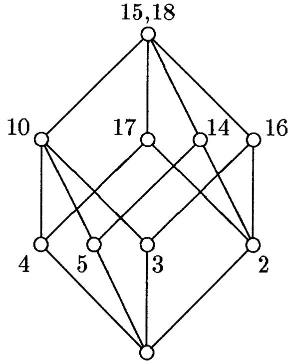


Figure 4.9 The blocks of the skeleton tolerance

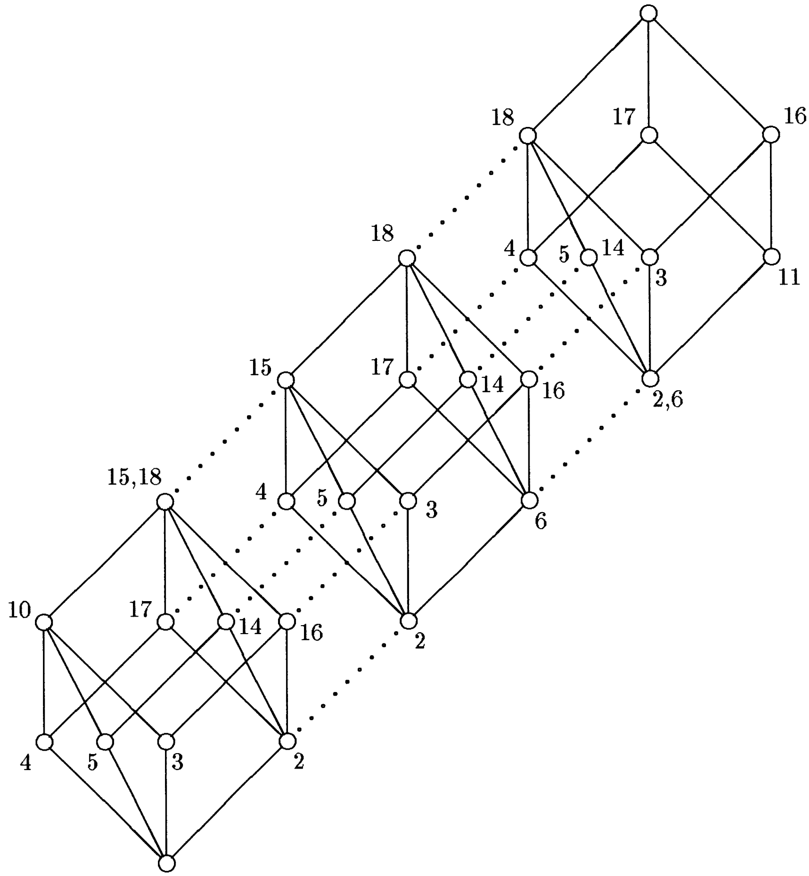


Figure 4.10 Atlas of the part diagrams. The dotted lines link equal concepts in the different part diagrams. If the diagram is contracted along those lines, we obtain the diagram in Figure 4.11.

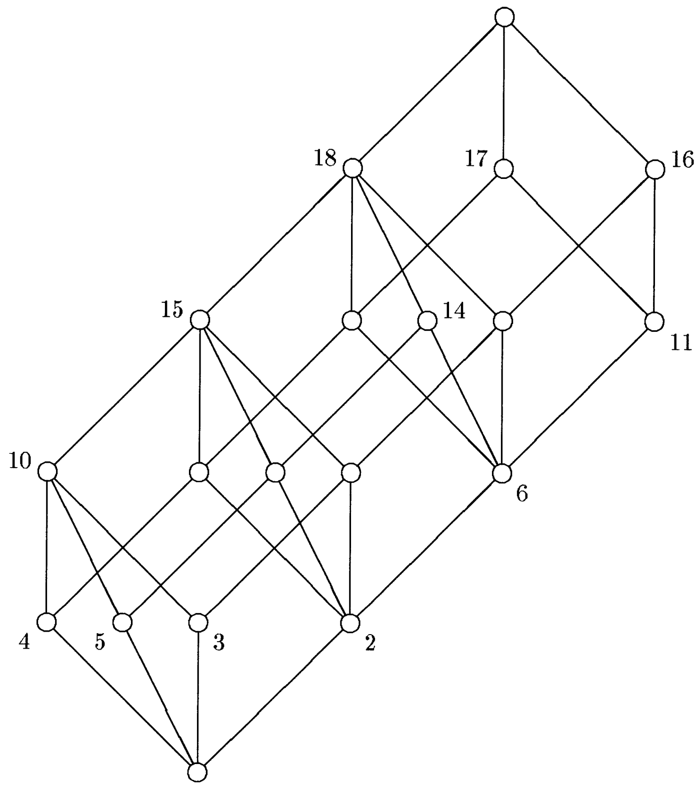


Figure 4.11 The same lattice as in Figure 4.7, but with a diagram which better reflects the structure.

4.3 Substitution

In the case of the *substitution sum* a context is inserted into another one at the place of an "empty cell", i.e., a non-incident object-attribute pair. We can visualize the construction by imagining the respective row and column suitably multiplied, so that there is room for the context which is to be inserted. For reasons of convenience we presuppose that the two contexts are disjoint and non-empty, and thus we obtain the following definition.

Definition 61. Let $\mathbb{K}_1 = (G_1, M_1, I_1)$ and $\mathbb{K}_2 = (G_2, M_2, I_2)$ be contexts and $(g, m) \notin I_1$ a non-incident object-attribute pair in $\mathbb{K}_1$. We presuppose that $G_2 \neq \emptyset \neq M_2$ and $(G_1 \setminus \{g\}) \cap G_2 = \emptyset = (M_1 \setminus \{m\}) \cap M_2$. We define **substitution sum** of $\mathbb{K}_1$ with $\mathbb{K}_2$ on (g, m) to be the context

$$\mathbb{K}_1 (g, m)\mathbb{K}_2 := (G, M, I)$$

with $G := (G_1 \setminus \{g\}) \dot\cup G_2$, $M := (M_1 \setminus \{m\}) \dot\cup M_2$ and

$$I := \{(h, n) \in I_1 \mid h \neq g, n \neq m\} \cup G_2 \times g^{I_1} \cup m^{I_1} \times M_2 \cup I_2.$$

We speak of a **proper substitution sum** if $G_2^{I_2} = \emptyset = M_2^{I_2}$ holds[3]. Then g^{I_1} is an intent and m^{I_1} is an extent of (G, M, I). The corresponding concepts shall be denoted by a and b. $\diamond$

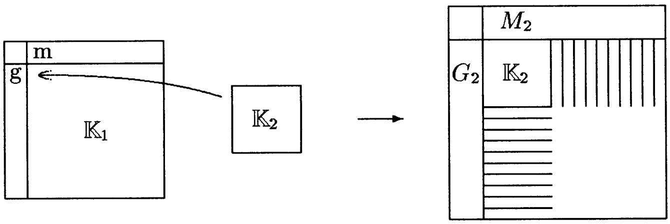


Figure 4.12 To form the substitution sum $\mathbb{K}_1 (g, m)\mathbb{K}_2$, the context $\mathbb{K}_2$ is inserted into "the empty cell" (g, m) of $\mathbb{K}_1$. The hatchings in the resulting context are meant to indicate that every object $\notin G_2$ is incident either with all or with no element from M_2 and, dually, that all objects from G_2 have the same intents with regard to the attributes $\notin M_2$.

From the definition it immediately follows that the substitution sum is restrictedly associative:

[3] The cases $\mathbb{K}_2 \cong (\{g\}, \{m\}, \emptyset)$, i.e., $\mathbb{K}_1 (g, m)\mathbb{K}_2 = \mathbb{K}_1$, as well as $\mathbb{K}_1 = (\{g\}, \{m\}, \emptyset)$ are admitted. In the following we shall consider proper substitution sums only.

Proposition 65.

$$\mathbb{K}_1(g,m)(\mathbb{K}_2(h,n)\mathbb{K}_3) = (\mathbb{K}_1(g,m)\mathbb{K}_2)(h,n)\mathbb{K}_3,$$

if $g \in G_1$, $m \in M_1$, $h \in G_2$, $n \in M_2$ *and* $(g,m) \notin I_1$, $(h,n) \notin I_2$. $\square$

The substitution sum generalizes several of the context operations we have already introduced. The context sum, the disjoint union and the construction from page 41 leading up to the vertical sum of the concept lattices can be obtained as special cases of the form

$$(\mathbb{K}_0(g,m)\mathbb{K}_1)(h,n)\mathbb{K}_2$$

if we choose the contexts

$$\begin{array}{|c|cc|} \hline & m & n \\ \hline g & & \times \\ h & \times & \\ \hline \end{array}, \qquad \begin{array}{|c|cc|} \hline & m & n \\ \hline g & & \\ h & & \\ \hline \end{array}, \qquad \text{resp.} \qquad \begin{array}{|c|cc|} \hline & m & n \\ \hline g & & \times \\ h & & \\ \hline \end{array}$$

for $\mathbb{K}_0$, since as results we obtain

$$\begin{array}{c|c} \mathbb{K}_1 & \times \\ \hline \times & \mathbb{K}_2 \end{array}, \qquad \begin{array}{c|c} \mathbb{K}_1 & \emptyset \\ \hline \emptyset & \mathbb{K}_2 \end{array} \qquad \text{and} \qquad \begin{array}{c|c} \mathbb{K}_1 & \times \\ \hline \emptyset & \mathbb{K}_2 \end{array}$$

We shall examine how the concept lattice of the substitution sum is related to those of the summands. It turns out that the concept lattice of the second summand is "hung up" several times in the concept lattice of the first summand, similarly to the sails of a ship in the rigging. (see Figure 4.13).

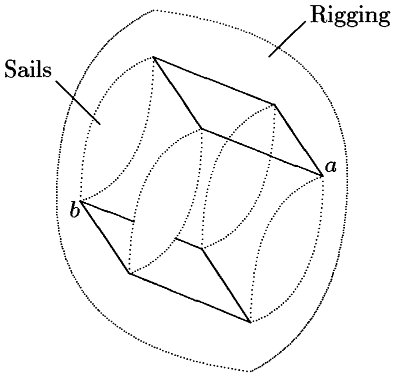


Figure 4.13 Sails and rigging

The first proposition shows that we can rediscover $\underline{\mathfrak{B}}(\mathbb{K}_1)$ as a sublattice:

Proposition 66. *The* **rigging** *of a proper substitution sum*

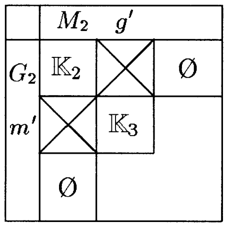
$$(G, M, I) = \mathbb{K}_1(g, m)\mathbb{K}_2,$$

defined as

$$U := \underline{\mathfrak{B}}(G, M, I \setminus I_2),$$

is a complete sublattice of $\underline{\mathfrak{B}}(\mathbb{K}_1(g, m)\mathbb{K}_2)$ *that is isomorphic to* $\underline{\mathfrak{B}}(\mathbb{K}_1)$. *$U$ contains in particular all concepts which are $\geq a$ or $\leq b$.*

Proof. Proposition 47 (p. 114) shows that $I \setminus I_2$ is closed. The context $(G, M, I \setminus I_2)$ is up to clarification identical to $\mathbb{K}_1$, i.e., the isomorphy follows from Theorem 13 (p. 112). A concept $(A, B) \leq b$ satisfies $A \subseteq m^{I_1}$ and therefore $A \times B \cap I_2 = \emptyset$, which implies $(A, B) \in U$. □



The remaining concepts of $\mathbb{K} := (G, M, I)$, i.e., those which do not belong to $\underline{\mathfrak{B}}(G, M, I \setminus I_2)$, all contain "one cross from I_2" and thus are entirely contained in the subcontext $(G_2 \cup m^{I_1}, M_2 \cup g^{I_1})$. This subcontext, according to the definition of $\mathbb{K}$, is the sum of $\mathbb{K}_2$ and $\mathbb{K}_3 := (m^{I_1}, g^{I_1}, I_3 := I \cap (m^{I_1} \times g^{I_1}))$, i.e., the concept lattice of this subcontext is isomorphic to $\underline{\mathfrak{B}}(\mathbb{K}_2) \times \underline{\mathfrak{B}}(\mathbb{K}_3)$. We do not claim that we thereby obtain a sublattice, but by Proposition 32 (p. 98) we know that we find an order-embedding of this concept lattice into $\underline{\mathfrak{B}}(\mathbb{K})$ by assigning the concept (A^{II}, A^I) of $\mathbb{K}$ to every concept (A, B) of $\mathbb{K}_2 + \mathbb{K}_3$. The proposition just mentioned suggests a further order-embedding, namely $(A, B) \mapsto (B^I, B^{II})$. In the present case, however, this yields the same map, since $A^I = B$ or $B^I = A$ for every concept (A, B) of $\mathbb{K}_2 + \mathbb{K}_3$.

We denote the image of this mapping by P and summarize:

Proposition 67. *The map*

$$\begin{aligned} \varphi : \underline{\mathfrak{B}}(\mathbb{K}_2 + \mathbb{K}_3) &\to \underline{\mathfrak{B}}(\mathbb{K}_1(g, m)\mathbb{K}_2), \\ (A, B) &\mapsto (A^{II}, A^I) \quad (= (B^I, B^{II})), \end{aligned}$$

is an order-embedding, mapping the concept with the extent G_3 $(= m^{I_1})$ onto b and the concept with the intent M_3 $(= g^{I_1})$ onto a. The range P covers all concepts which do not belong to U. □

We call P the **sails** of the substitution sum. The following theorem shows that the two parts, rigging and sails, determine the structure of the concept lattice of a substitution sum. However, we must indicate how U and P are joined together. Before doing so, we shall introduce the corresponding lattice construction.

Definition 62. For complete lattices U and W, $|W| > 1$ and elements $a, b \in U$ with $a \not\leq b$, we define the **substitution product** $U(a,b)W$ of U and W on (a,b) to be the concept lattice of the (proper) substitution sum

$$U(a,b)W := \underline{\mathfrak{B}}\left((U,U,\leq)(a,b)(W_0,W_1,\leq)\right)$$

with $W_0 := W \setminus \{0_W\}$ and $W_1 := W \setminus \{1_W\}$. $\diamond$

Hence, according to this definition, the substitution product of U and W is the concept lattice of the context (G,M,I) with

$$\begin{aligned} G &= (U \setminus \{a\}) \cup (W \setminus \{0_W\}), \\ M &= (U \setminus \{b\}) \cup (W \setminus \{1_W\}), \end{aligned}$$

and

$$gIm \iff \begin{cases} g \leq m \text{ in } U, & \text{if } g \in U, m \in U \\ g \leq b \text{ in } U, & \text{if } g \in U, m \in W \\ a \leq m \text{ in } U, & \text{if } g \in W, m \in U \\ g \leq m \text{ in } W, & \text{if } g \in W, m \in W \end{cases}$$

The rigging of this substitution sum is naturally isomorphic to U; therefore, we also denote it by U. We also take over the names a and b for the corresponding elements of the substitution product.

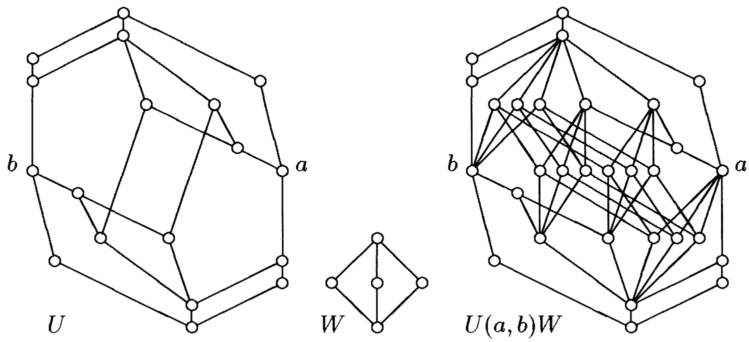


Figure 4.14 A substitution product.

Theorem 23 (Properties of the substitution product). *The concept lattice V of the substitution sum*

$$V = \underline{\mathfrak{B}}(\mathbb{K}_1(g,m)\mathbb{K}_2)$$

with rigging U and sails P has the properties (Subst 1) – (Subst 4), specified below. a, b are as in the propositions and $W := \underline{\mathfrak{B}}(\mathbb{K}_2)$. Conversely, every *complete lattice V satisfying (Subst 1) – (Subst 4) is isomorphic to $U(a,b)W$. In particular,*

$$\underline{\mathfrak{B}}(\mathbb{K}_1(g,m)\mathbb{K}_2) \cong \underline{\mathfrak{B}}(\mathbb{K}_1)(\gamma g, \mu m)\underline{\mathfrak{B}}(\mathbb{K}_2).$$

(Subst 1) $V = U \cup P$, $P \cap U \subseteq (b] \cup [a)$, $a, b \in U$, $a \not\leq b$.

(Subst 2) U *is a complete sublattice of* V,

(Subst 3) W *is order-isomorphic to* $P \cap (a]$, *and*

$$p = (p \wedge a) \vee (p \wedge b) \quad \textit{as well as} \quad p = (p \vee a) \wedge (p \vee b)$$

for all $p \in P$,

(Subst 4) If $u \in U$ *and* $p \in P \setminus U$, *then* $u \leq p \Rightarrow u \leq b$ *and* $u \geq p \Rightarrow u \geq a$.

Proof. In Propositions 66 and 67 we have already shown that $\underline{\mathfrak{B}}(\mathbb{K}_1(g,m)\mathbb{K}_2)$ has the properties (Subst 1) and (Subst 2), and furthermore that P is order-isomorphic to the direct product of $\underline{\mathfrak{B}}(\mathbb{K}_2)$ and $\underline{\mathfrak{B}}(\mathbb{K}_3)$. This yields the first part of (Subst 3). A concept $p = (X, Y)$ of $\mathbb{K}_1(g,m)\mathbb{K}_2$ which does not belong to U satisfies $X \subseteq G_2 \cup m^{I_1}$ and $Y \cap M_2 \neq \emptyset$. Every subconcept of it which belongs to U must therefore have an extent which is entirely contained in m^{I_1}, i.e., it is $\leq b$. This proves (Subst 4). Now we can infer the second part of (Subst 3), since an upper bound of $(p \wedge a) \vee (p \wedge b)$ which is less than or equal to p must be contained in P and must therefore be equal to p.

We now assume, conversely, that V has the properties (Subst 1) – (Subst 4). First we derive some information from the structure of P. From (Subst 3) we infer that $P \subseteq [a \wedge b, a \vee b]$. The maps $p \mapsto p \vee b$ and $q \mapsto q \wedge a$ are isomorphisms between $P \cap (a]$ and $P \cap [b)$ which are inverse to each other, since it holds for $p \leq a$ that

$$(p \vee b) \wedge a = (p \vee b) \wedge (p \vee a) = p,$$

and dually. In this way we do not only obtain the isomorphism $\sigma : W \to P \cap (a]$ postulated in (Subst 3) but a further isomorphism $\tau : W \to P \cap [b)$ by virtue of $\tau(x) := \sigma(x) \vee b$. From (Subst 1) we can see that the elements of $P \cap (a]$, with the exception of the boundary elements a and $a \wedge b$, do not belong to U. The corresponding is true for $P \cap [b)$.

In order to prove the isomorphy of V with $U(a,b)W$, we use the Basic Theorem on Concept Lattices. The context defining $U(a,b)W$ has the object set $G = (U \backslash \{a\}) \cup (W \backslash \{0_W\})$ and the attribute set $M = (U \backslash \{b\}) \cup (W \backslash \{1_W\})$ (cf. the explanation following Definition 62). The maps

$$\tilde{\gamma} : G \to V, \quad \tilde{\mu} : M \to V,$$

which are defined by

$$\tilde{\gamma}(x) := \begin{cases} x & \text{if } x \in U \\ \sigma(x) & \text{if } x \in W \end{cases} \qquad \text{and} \qquad \tilde{\mu}(x) := \begin{cases} x & \text{if } x \in U \\ \tau(x) & \text{if } x \in W, \end{cases}$$

satisfy the conditions mentioned in the Basic Theorem, as we shall show. For this purpose we have to prove that $\tilde{\gamma}(G)$ is $\bigvee$-dense and $\tilde{\mu}(M)$ is $\bigwedge$-dense in V and that gIm is equivalent to $\tilde{\gamma}g \leq \tilde{\mu}m$.

We have $\tilde{\gamma}(G) = U \cup (P \cap (a])$, since $\sigma(0_W) = a \wedge b$ according to (Subst 2) is an element of U. Likewise, $\tilde{\mu}(M) = U \cup (P \cap [b))$. Because of $p = (p \wedge a) \vee (p \wedge b)$, every element $p \notin U$ is the supremum of some element of $P \cap (a]$ and some element of $(b]$, i.e., in any case of elements of $\tilde{\gamma}(G)$. Dually, we show that $\tilde{\mu}(M)$ is infimum-dense.

In order to prove $gIm \iff \tilde{\gamma}g \leq \tilde{\mu}m$, we distinguish four different cases, depending on whether $g \in U$ or $g \in W$ and whether $m \in U$ or $m \in W$. If both are contained in U, the assertion is obviously right. If $g \in U$, $m \in W$, then by Definition 62 gIm is equivalent to $g \leq b$ in U. On the other hand, $\tilde{\mu}(m) = \tau(m) \notin U$ or $\tilde{\mu}(m) = b$, i.e., by (Subst 4) $\tilde{\gamma}g \leq \tilde{\mu}m \iff \tilde{\gamma}g \leq b$, and, because of $\tilde{\gamma}g = g$ the conditions are equivalent. The case $g \in W, m \in U$ is treated dually. Finally, we have to deal with the case $g, m \in W$: In this case we have $gIm \iff g \leq m \in W \iff \sigma(g) \leq \sigma(m) \iff \sigma(g) \leq \sigma(m) \vee b$ (since from $\sigma(g) \leq \sigma(m) \vee b$ it follows that $\sigma(g) = \sigma(g) \wedge a \leq (\sigma(m) \vee b) \wedge b = \sigma(m)$), and because of $\tilde{\gamma}(g) = \sigma(g)$ as well as $\tilde{\mu}(m) = \tau(m) = \sigma(m) \vee b$ everything has been proved. □

It is easy to derive further information on the set P. In the following proposition we compile some information (without proof).

Proposition 68 (Further Properties). *The concept lattice*

$$\underline{\mathfrak{B}}(\mathbb{K}_1\,(g,m)\mathbb{K}_2\,)$$

of a proper substitution sum has (using the notations in Theorem 23) the following properties:

(Subst 5) The sails P are isomorphic to a direct product

$$P \cong (P \cap (a]) \times (P \cap (b]).$$

The elements of $P \cap (b]$ are precisely those of the form $x = (x \vee a) \wedge b$.

(Subst 6) It holds for $x \in P \cap (a]$, $y \in P \cap [b)$ that

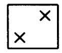
$$x \leq y \quad \iff \quad x \leq y \wedge a \quad \iff \quad x \vee b \leq y.$$

(Subst 7) Each element of $P \cap (a]$ is the supremum of object concepts in $P \cap (a]$, and each element of $P \cap [b)$ is the infimum of attribute concepts in $P \cap [b)$.

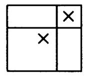
(Subst 8) If 1_W is $\bigvee$-irreducible, then a is an object concept. If 0_W is $\bigwedge$-irreducible, then b is an attribute concept.

□

The substitution product of a lattice $\boldsymbol{V}$ with the two-element lattice is always isomorphic to $\boldsymbol{V}$. Therefore, we call $\boldsymbol{V}$ **substitutionally indecomposable** if $\boldsymbol{V}$ has more than two elements and if from $\boldsymbol{V} \cong \boldsymbol{V}_1(a,b)\boldsymbol{V}_2$ it always follows that $|\boldsymbol{V}_1| = 2$ or $|\boldsymbol{V}_2| = 2$. We conclude our investigation by settling the question: under which circumstances does substitutional decomposability of a concept lattice $\underline{\mathfrak{B}}(\mathbb{K})$ imply decomposability of the context $\mathbb{K}$? This implication does not apply generally, since due to the addition (in special cases also the removal) of reducible objects and attributes a context can loose its property of being a substitution sum.



substitutionally indecomposable



substitutionally decomposable

In the following theorem (which we shall prepare by means of a proposition,) we therefore switch over to a dense subcontext.

Proposition 69. *A context $\mathbb{K}$ is isomorphic to a proper substitution sum of contexts with concept lattices isomorphic to U and W, if and only if there is a lattice isomorphism ψ of $\underline{\mathfrak{B}}(\mathbb{K})$ onto a substitution product $U(a,b)W$ with*

$$\psi\gamma(G) \subseteq U \cup (a] \quad \textit{and} \quad \psi\mu(M) \subseteq U \cup [b),$$

for which additionally the following special condition is satisfied:

- *if 0_W is $\bigwedge$-irreducible, then $b \in \psi\mu(M)$,*
- *if 1_W is $\bigvee$-irreducible, then $a \in \psi\gamma(G)$.*

Proof. If $\mathbb{K} = \mathbb{K}_1(g,m)\mathbb{K}_2$, then, by the preceding theorem,

$$\underline{\mathfrak{B}}(\mathbb{K}) \cong \underline{\mathfrak{B}}(\mathbb{K}_1)(\gamma g, \mu m)\underline{\mathfrak{B}}(\mathbb{K}_2).$$

If $h \in G$ is an object, then the object concept belongs either to the rigging U or it is contained in the object set G_2; then, however, $\gamma h \leq \gamma g = a$. If 1_W is $\bigvee$-irreducible, then 1_W is an object concept in $\mathbb{K}_2$, this object then also belongs to $\mathbb{K}$ and is mapped under $\psi\gamma$ on a. We argue dually for the attributes.

Now, conversely, let ψ be an isomorphism with the properties stated in the proposition. Let P again be the sails of $U(a,b)W$, and furthermore assume that $g_a \notin G$ and $m_b \notin M$. We define a context $\mathbb{K}_1$ by

$$G_U := \{g \in G \mid \psi\gamma g \in U \setminus \{a\}\}, \quad G_1 := G_U \cup \{g_a\}$$

$$M_U := \{m \in M \mid \psi\mu m \in U \setminus \{b\}\}, \quad M_1 := M_U \cup \{m_b\}$$

$$I_1 := I \cap (G_U \times M_U) \cup \{(g_a, m) \mid \psi\mu m \geq a\} \cup \{(g, m_b) \mid \psi\gamma g \leq b\}.$$

The concept lattice of this context $\mathbb{K}_1 := (G_1, M_1, I_1)$ is isomorphic to U, since $\psi\gamma(G)$ is $\bigvee$-dense in $U(a,b)W$, and by (Subst 4) $(\psi\gamma(G) \cap U) \cup \{a\}$

then is $\bigvee$-dense in U. Dually, $(\psi\mu(M) \cap U) \cup \{b\}$ is infimum-dense in U. Therefore, the maps $\gamma_1 : G_1 \to U$ and $\mu_1 : M_1 \to U$ being defined by

$$\gamma_1(g) := \begin{cases} \psi\gamma g, & \text{if } g \in G_U, \\ a, & \text{if } g = g_a, \end{cases} \quad \text{and} \quad \mu_1(m) := \begin{cases} \psi\mu m, & \text{falls } m \in M_U, \\ b, & \text{if } m = m_b, \end{cases}$$

by the Basic Theorem are sufficient for the isomorphy of $\underline{\mathfrak{B}}(\mathbb{K}_1)$ with U, since evidently $gIm \iff \gamma_1 g \leq \mu_1 m$.

The context $\mathbb{K}_2 := (G_2, M_2, I_2)$ is explained through

$$G_2 := G \setminus G_U, \qquad M_2 := M \setminus M_U, \qquad I_2 := I \cap G_2 \times M_2.$$

We claim that $\underline{\mathfrak{B}}(\mathbb{K}_2) \cong W(\cong P \cap (a])$ and argue again with the help of the Basic Theorem, using the maps $\gamma_2 : G_2 \to P \cap (a]$ and $\mu_2 : M_2 \to P \cap (a]$, which are defined as follows:

$$\gamma_2 g := \psi\gamma g, \qquad \mu_2 m := (\psi\mu m) \wedge a.$$

With (Subst 6) we obtain

$$\psi\gamma g \leq (\psi\mu m) \wedge a \iff \psi\gamma g \leq \psi\mu m,$$

i.e., $\gamma_2 g \leq \mu_2 m \iff (g, m) \in I_2$. (Subst 7) says that the object concepts are $\bigvee$-dense in $P \cap (a]$. By (Subst 1), however, $P \cap (a]$ does not contain elements of U with the exception of a and $a \wedge b$. Therefore, $\gamma_2 G_2$ is $\bigvee$-dense in the lattice $P \cap (a]$, since by (Subst 8) the largest element a is also a supremum of elements of $\gamma_2 G_2$. If we take into account that by (Subst 3) $x \mapsto x \wedge a$ is an isomorphism of $P \cap [b)$ on $P \cap (a]$, we obtain that, dually, $\mu_2 M_2$ is $\bigwedge$-dense in $P \cap (a]$.

Without difficulty we verify that

$$\mathbb{K} = \mathbb{K}_1 (g_a, m_b) \mathbb{K}_2. \qquad \square$$

Theorem 24. *If $\underline{\mathfrak{B}}(\mathbb{K}) \cong U(a,b)W$, then there is a dense subcontext $\mathbb{K}_0$ of $\mathbb{K}$ which is a proper substitution sum of contexts with concept lattices isomorphic to U and W, provided that the isomorphism $\psi : U(a,b)W \to \underline{\mathfrak{B}}(\mathbb{K})$ can be chosen such that the following (necessary) extra condition is satisfied:*

- *If 1_W is $\bigvee$-irreducible, then $\psi(a)$ is an object concept γg_a of $\mathbb{K}$.*
- *If 0_W is $\bigwedge$-irreducible, then $\psi(b)$ is an attribute concept μm_b of $\mathbb{K}$.*

Proof. If

$$\psi : \underline{\mathfrak{B}}(\mathbb{K}) \to U(a,b)W$$

is an isomorphism, so that ψ^{-1} satisfies the extra condition, then we define subcontext $\mathbb{K}_0 := (G_0, M_0, I \cap G_0 \times M_0)$ through

$$\begin{aligned} G_0 &:= \{g \mid \psi\gamma g \in U \cup (a]\} \\ M_0 &:= \{m \mid \psi\mu m \in U \cup [b)\}. \end{aligned}$$

We only have to prove that $\mathbb{K}_0$ is a dense subcontext, since in this case Proposition 69 yields the rest of the assertion.

For this purpose we show that γG_0 is $\bigvee$-dense. The corresponding assertion M_0 is proved dually. Since ψ is an isomorphism, we can prove instead that $\psi\gamma G_0$ is $\bigvee$-dense in $U(a,b)W$. $\psi\gamma G$ is certainly $\bigvee$-dense, i.e., every element $s \in U(a,b)W$ can be represented as a supremum

$$s = \bigvee X, \quad X := (s] \cap \psi\gamma G.$$

We distinguish four different cases:

$s \in U, s \not\geq a$ By (Subst 4), $X \subseteq U$.

$s \leq a$ Then, trivially, $X \subseteq (a]$.

$s \in P$ By (Subst 3), $s = (s \wedge a) \vee (s \wedge b)$ and $s \wedge b \in U$, $s \wedge a \leq a$. Therefore, we have $s = \bigvee(X \cap (a]) \vee \bigvee(X \cap U)$.

$s \geq a$ By (Subst 3) for every $p \in P$, $a \vee p = a \vee (p \wedge b)$ with $p \wedge b \in U$. Hence, each element of X which does not belong to U, can be replaced by elements of $(a] \cup (b]$, and those in turn are suprema of elements of $\psi\gamma G \cap ((a] \cup (b])$.

This means that in the end those elements of $\psi\gamma G$ are sufficient which are contained in $U \cup (a]$, i.e., the images of G_0, as claimed. □

We shall use these results to prove a theorem on "unique prime factor decomposition" for the substitution product of finite lattices. However, this does not work quite smoothly, the extra condition makes itself felt and prevents a result without exceptions. The decisive technical aid is a refinement result, which shows that two substitution products can only be isomorphic if they are made up of the same factors.

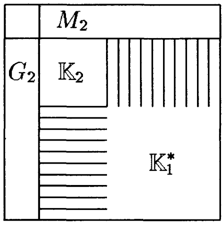
Proposition 70. *Let V_1, V_2, V_3, V_4 be doubly founded complete lattices. Assume that V_3 and V_4 each have a $\bigvee$-reducible unit element and a $\bigwedge$-reducible zero element. If*

$$V_1(a_1, b_1)V_3 \cong V_2(a_2, b_2)V_4$$

holds for suitable elements a_1, a_2, b_1, b_2, then there are lattices W_1, W_2, W_3, W_4 and elements $c_1, \ldots, c_4$ as well as $d_1, \ldots, d_4$ with

$$\begin{aligned} V_1 &\cong W_1(c_1, d_1)W_2, & V_2 &\cong W_1(c_2, d_2)W_3, \\ V_3 &\cong W_3(c_3, d_3)W_4, & V_4 &\cong W_2(c_4, d_4)W_4. \end{aligned}$$

The proof of the proposition can be illustrated by Figure 4.15. It represents a substitution sum $\mathbb{K} := \mathbb{K}_1(g, m)\mathbb{K}_2$. The subcontext $\mathbb{K}_2$ is drawn in, $\mathbb{K}_1^* =: (G_1^*, M_1^*, I_1^*)$ denotes the context resulting from $\mathbb{K}_1$ by omission of the object g and of the attribute m. We obtain a context which is isomorphic to $\mathbb{K}_1$ by adding an arbitrary non-incident object-attribute pair from $\mathbb{K}_2$.



$$\mathbb{K} = \mathbb{K}_1(\ ,\ )\mathbb{K}_2.$$

Figure 4.15 With reference to the proof of Proposition 70.

Hence, except for the names of g and m, $\mathbb{K}_1$ is also given, and those names are irrelevant for the substitution sum. Therefore, in this situation we write $\mathbb{K} = \mathbb{K}_1(\ ,\ )\mathbb{K}_2$ as an abbreviation for the fact that $\mathbb{K} = \mathbb{K}_1(g, m)\mathbb{K}_2$ holds for suitable g, m.

Proof. Assume that

$$V := V_1(c_1, d_1)V_3 \cong V_2(c_2, d_2)V_4$$

and that $\mathbb{K}$ is the (reduced) standard context for V. We can apply Theorem 24, since the extra condition is irrelevant because of the additional preconditions. Hence, $\mathbb{K}$ is in two ways a substitution sum:

$$\mathbb{K} \cong \mathbb{K}_1(g_1, m_1)\mathbb{K}_3 \cong \mathbb{K}_2(g_2, m_2)\mathbb{K}_4,$$

with

$$\mathfrak{B}(\mathbb{K}_i) \cong V_i \qquad \text{and} \qquad \mathbb{K}_i =: (G_i, M_i, I_i)$$

for $i \in \{1, \ldots, 4\}$. The presuppositions of the proposition guarantee that $\mathbb{K}_3$ and $\mathbb{K}_4$ have neither full nor empty rows or columns.

Hence, the isomorphy of the substitution products corresponds to the isomorphy of two substitution sums. In the following, we shall make use of this circumstance. There are, however, some complications, since the result for substitution sums which corresponds to Proposition 70 does not hold in general. If, however, we manage to find contexts $\mathbb{L}_1$, $\mathbb{L}_2$, $\mathbb{L}_3$, $\mathbb{L}_4$ with

$$\begin{tikzcd}[column sep=small] & W_1 \arrow[dl, "V_1"'] \arrow[dr, "V_2"] & \\ W_2 \arrow[dr, "V_4"'] & & W_3 \arrow[dl, "V_3"] \\ & W_4 & \end{tikzcd}$$

$$\mathbb{K}_1 = \mathbb{L}_1(\ ,\ )\mathbb{L}_2, \qquad \mathbb{K}_2 = \mathbb{L}_1(\ ,\ )\mathbb{L}_3,$$
$$\mathbb{K}_3 = \mathbb{L}_3(\ ,\ )\mathbb{L}_4, \qquad \mathbb{K}_4 = \mathbb{L}_2(\ ,\ )\mathbb{L}_4,$$

$$\begin{tikzcd}[column sep=small] & \mathbb{L}_1 \arrow[dl, "\mathbb{K}_1"'] \arrow[dr, "\mathbb{K}_2"] & \\ \mathbb{L}_2 \arrow[dr, "\mathbb{K}_4"'] & & \mathbb{L}_3 \arrow[dl, "\mathbb{K}_3"] \\ & \mathbb{L}_4 & \end{tikzcd}$$

then the statement of the proposition results from Theorem 23. This is true for some special cases. These will be dealt with first:

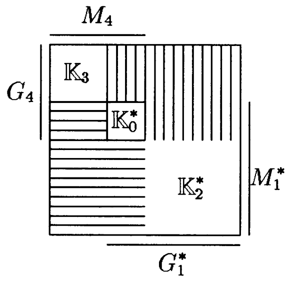


Figure 4.16 With reference to the proof of Proposition 70.

1st case: $G_3 \subset G_4$ and $M_3 \subset M_4$ (see Figure 4.16).
We define a subcontext

$$\mathbb{K}_0^* := (G_0^*, M_0^*, I_0^*)$$

through

$$G_0^* := G_4 \setminus G_3$$

and

$$M_0^* := M_4 \setminus M_3.$$

As above we expand this context into a context $\mathbb{K}_0$ by adding a non-incident pair from $\mathbb{K}_3$. The latter cannot contain any full rows or full columns, because $\mathbb{K}_4$ does not contain any either. Therefore, we have

$$\mathbb{K}_1^* = \mathbb{K}_2(\;,\;)\mathbb{K}_0^*, \quad \mathbb{K}_1 = \mathbb{K}_2(\;,\;)\mathbb{K}_0, \quad \mathbb{K}_4 = \mathbb{K}_0(\;,\;)\mathbb{K}_3$$

If, as a fourth context, we add the trivial context $\square := (\{g\}, \{m\}, \emptyset)$, we obtain the desired refinement with

$$\begin{array}{ll} \mathbb{K}_1 = \mathbb{K}_2(\;,\;)\mathbb{K}_0 & \qquad \mathbb{K}_2 = \mathbb{K}_2(\;,\;)\square \\ \mathbb{K}_3 = \square(\;,\;)\mathbb{K}_3 & \qquad \mathbb{K}_4 = \mathbb{K}_0(\;,\;)\mathbb{K}_3 \end{array}$$

(cf. the left diagram of Figure 4.17). Of course, thereby we have also dealt with the converse case $G_4 \subset G_3$, $M_4 \subset M_3$.
2nd case: $G_3 \cap G_4 = \emptyset = M_3 \cap M_4$.
We define (middle figure)

$$\begin{aligned} G_0^{**} &:= G \setminus (G_3 \cup G_4), \\ M_0^{**} &:= M \setminus (M_3 \cup M_4). \end{aligned}$$

Let $\mathbb{K}_0$ be the subcontext with the object set $G_0 := G_0^{**} \cup \{g, h\}$ and the attribute set $M_0 := M_0^{**} \cup \{m, n\}$, with (g, m) and (h, n) being non-incident object-attribute pairs of $\mathbb{K}_3$ or $\mathbb{K}_4$, respectively.

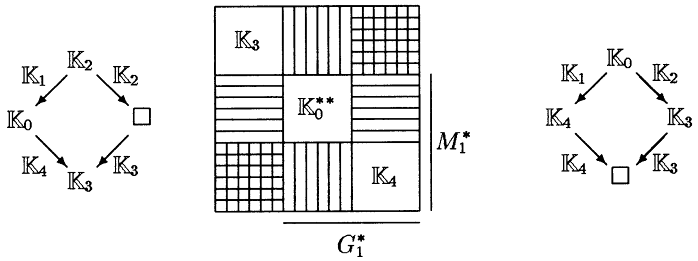


Figure 4.17 With reference to the proof of Proposition 70.

We recognize that $\mathbb{K}_0(g,m)\mathbb{K}_4$ is equal to $\mathbb{K}_1$. Furthermore, $\mathbb{K}_2 = \mathbb{K}_0(h,n)\mathbb{K}_3$. Once again by using the trivial context $\square$ we obtain the refinement represented in the diagram on the right, which yields the statement of the proposition.

The cases $G_3 \subseteq G_4$, $M_4 \subseteq M_3$ (resp. dually) and $G_3 \cap G_4 \neq \emptyset$, $M_3 \cap M_4 = \emptyset$ turn out to be trivial: If, for instance, $G_3 \subseteq G_4$ and $m \in M_3 \setminus M_4$, then $m' \cap G_4 = \emptyset$ or $m' \cap G_4 = G_4$ and consequently $m' \cap G_3 = \emptyset$ or $m' \cap G_3 = G_3$, contrary to our presuppositions. If $M_3 \cap M_4 = \emptyset$, then $G_3 \subseteq G_4$ is obviously impossible, since otherwise $\mathbb{K}_3$ would have constant columns. So, if $g \in G_3 \cap G_4$, $h \in G_3 \setminus G_4$ and $m \in M_4$, then from gIm it immediately follows that hIm and thus hIn for all $n \in M_4$, which in turn necessitates gIn for all $n \in M_4$. Similarly, from $g\not{I}m$ it follows that $g' \cap M_4 = \emptyset$. This means that $\mathbb{K}_4$ would contain a full row or an empty row, which is contrary to the presuppositions.

What remains is the case that the two subcontexts $\mathbb{K}_3$ and $\mathbb{K}_4$ intersect non-trivially. We can proceed similarly as we have done so far and introduce contexts $\mathbb{L}_1, \ldots, \mathbb{L}_4$, as presented in Figure 4.18:

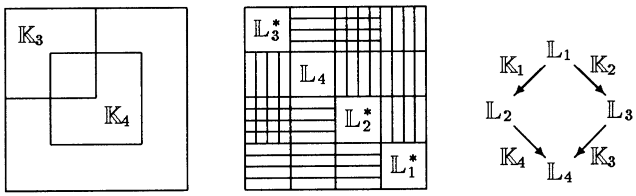


Figure 4.18 With reference to the proof of Proposition 70.

With $H_1^* := G \setminus (G_3 \cup G_4)$, $N_1^* := M \setminus (M_3 \cup M_4)$, $H_2^* := G_4 \setminus G_3$, $N_2^* := M_4 \setminus M_3$, $H_3^* := G_3 \setminus G_4$, $N_3^* := M_3 \setminus M_4$ and $H_4 := G_3 \cap G_4$, $N_4 := M_3 \cap M_4$ we define subcontexts $\mathbb{L}_1 \ldots, \mathbb{L}_4$, which indeed yield the refinement indicated in the diagram, provided that the substitution sums appearing in the process are proper substitution sums. However, it may happen that $\mathbb{L}_4$ contains full rows or full columns (for $\mathbb{L}_2$ and $\mathbb{L}_3$ this shall be excluded, as in the first case).

A simple trick helps us along: We extend the context $\mathbb{K}$ by an object g_∞ and an attribute m_∞ with $g'_\infty := g'_4 \cap M_1^*$ and $m'_\infty := m'_4 \cap G_1^*$, $g_4 \in G_4$ and $m_4 \in M_4$ being chosen arbitrarily. The new object and the new attribute are reducible in $\mathbb{K}$ as well as in the subcontexts $\mathbb{K}_1, \ldots, \mathbb{K}_4$, i.e., the respective concept lattices are isomorphic. If by $\mathbb{L}_4^+$ we denote the context resulting from $\mathbb{L}_4$, we obtain

$$\begin{array}{ll} \mathbb{K}_1 = \mathbb{L}_1(\,,\,)\mathbb{L}_2 & \mathbb{K}_2 = \mathbb{L}_1(\,,\,)\mathbb{L}_3 \\ \mathbb{K}_3 = \mathbb{L}_3(\,,\,)\mathbb{L}_4^+ & \mathbb{K}_4 = \mathbb{L}_2(\,,\,)\mathbb{L}_4^+, \end{array}$$

and thus the assertion. □

By means of this proposition we can finally prove the result which we asserted above. In this context, the three-element lattice C_3 plays a special role, because it is the only substitutionally indecomposable lattice which does not satisfy the additional condition from Proposition 70.

Theorem 25. *If no substitutional decomposition of the finite lattice $\boldsymbol{V}$ contains a factor isomorphic to C_3, then any two substitutional decompositions of $\boldsymbol{V}$ into indecomposable lattices have the same length and pairwise isomorphic factors.*

Proof. According to Proposition 65 each substitution decomposition can be brought into a left-bracketed form. Hence, let

$$V \cong ((\ldots(M_1(\,,\,)M_2)\ldots)(\,,\,)M_{m-1})(\,,\,)M_m$$

$$V \cong ((\ldots(N_1(\,,\,)N_2)\ldots)(\,,\,)N_{n-1})(\,,\,)N_n$$

be two decompositions of $\boldsymbol{V}$ into indecomposable factors $M_1, \ldots, M_m$ or N_1, …, N_n, respectively (the names of the elements in brackets are irrelevant for the proof). Assume that n is the largest possible length which this kind of decomposition of $\boldsymbol{V}$ can have. We proceed by induction on n.

According to Proposition 70 there are lattices $W_1, \ldots, W_4$ with

$$\begin{array}{rcl} ((\ldots(M_1(\,,\,)M_2)(\,,\,)M_3\ldots)(\,,\,)M_{m-1} & \cong & W_1(\,,\,)W_2, \\ ((\ldots(N_1(\,,\,)N_2)(\,,\,)N_3\ldots)(\,,\,)N_{n-1} & \cong & W_1(\,,\,)W_3, \end{array}$$

$$M_m \cong W_3(\,,\,)W_4 \quad \text{and} \quad N_n \cong W_2(\,,\,)W_4.$$

Since M_m and N_n are substitutionally indecomposable, $|W_2| = |W_3| = 2$ or $|W_4| = 2$. In the first case $M_m \cong W_4 \cong N_n$ and

$$(\ldots(M_1(\,,\,)M_2)\ldots)M_{m-1} \cong W_1 \cong (\ldots(N_1(\,,\,)N_2)\ldots)N_{n-1},$$

and the assertion follows by means of induction. If $|W_4| = 2$, then $M_m \cong W_3$ and $N_n \cong W_2$, and we have

$$(W_1(\ ,\ )N_n)(\ ,\ )M_m \cong \mathbf{V} \cong (W_1(\ ,\ )M_m)(\ ,\ )N_n$$

Since N_1 is indecomposable, for $n = 2$ we obtain $|W_1| = 2$, and thus $m = 2$, $M_1 \cong N_2$ and $N_1 \cong M_2$. If $n > 2$ we can infer that every substitutional decomposition of W_1 has at most $n - 2$ factors, because otherwise we would have a decomposition of $\mathbf{V}$ with more than n factors. By induction we can infer that all decompositions of W_1 into indecomposable factors have the same number k of factors. This number, however, must equal $n - 2$, since

$$W_1(\ ,\ )M_m \cong (\ldots((N_1(\ ,\ )N_2)(\ ,\ )N_3)\ldots)(\ ,\ )N_{n-1}.$$

Also by the induction hypothesis, every decomposition of this lattice has precisely $n - 1$ factors. □

4.4 Tensorial Decompositions

Definition 63. Let T be an index set. The **direct product** of contexts $\mathbb{K}_t := (G_t, M_t, I_t), t \in T$ is defined to be the context

$$\bigtimes_{t \in T} \mathbb{K}_t := (\bigtimes_{t \in T} G_t, \bigtimes_{t \in T} M_t, \nabla),$$

with

$$g \nabla m :\Longleftrightarrow \exists_{t \in T} \quad g_t I_t m_t$$

for $g := (g_t)_{t \in T}$ and $m := (m_t)_{t \in T}$. ◇

We had introduced this definition already in Section 1.4 for the special case of two factors:

$$\mathbb{K}_1 \times \mathbb{K}_2 := (G_1 \times G_2, M_1 \times M_2, \nabla),$$

$$(g_1, g_2)\nabla(m_1, m_2) :\Longleftrightarrow g_1 I_1 m_1 \text{ or } g_2 I_2 m_2.$$

For reasons of simplicity we will use the following abbreviations throughout this section:

$$G := \bigtimes_{t \in T} G_t, \quad M := \bigtimes_{t \in T} M_t, \quad g := (g_t)_{t \in T} \quad \text{and} \quad m := (m_t)_{t \in T}.$$

A tiresome complication in the notation stems from the trivial case of the "full rows" and "full columns". We use the notation introduced in Section 3.3

$$\blacksquare := M^{\nabla} \times M \cup G \times G^{\nabla}.$$

	a	b
1	×	
2		

$\mathbb{K}_1$

×

	c	d	e
3	×		
4		×	
5			×

$\mathbb{K}_2$

=

		a	a	a	b	b	b
		c	d	e	c	d	e
1	3	×	×	×	×		
1	4	×	×	×		×	
1	5	×	×	×			×
2	3	×			×		
2	4		×			×	
2	5			×			×

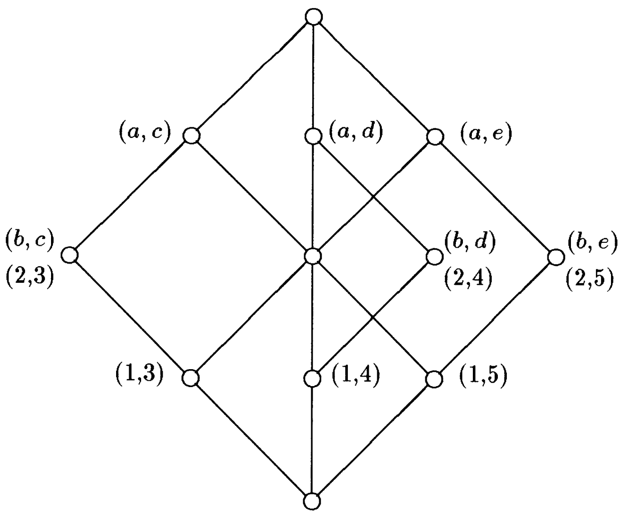
$\mathbb{K}_1 \times \mathbb{K}_2$

=

		a	b	a	b	a	b
		c	c	d	d	e	e
1	3	×	×	×		×	
2	3	×	×				
1	4	×		×	×	×	
2	4			×	×		
1	5	×		×		×	×
2	5					×	×

$\mathbb{K}_1 \times \mathbb{K}_2$

Figure 4.19 The direct product of two small contexts.



$\underline{\mathfrak{B}}(\mathbb{K}_1 \times \mathbb{K}_2)$

Figure 4.20 The concept lattices of the direct product of the contexts from Figure 4.19.

However, the reader can assume without great loss of generality that full rows and columns do not occur, and can set $G^{\nabla} = M^{\nabla} = \square = \emptyset$ everywhere.

Every trivial context having only one concept, i.e., every context of the form $(G, M, G \times M)$, acts like a zero element for the direct product: If one of the factors is trivial, so is the product. Occasionally, we have to exclude this case.

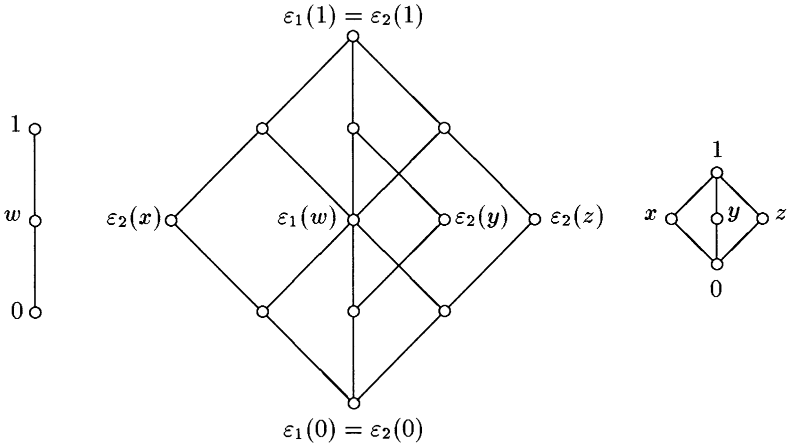


Figure 4.21 The maps $\varepsilon_i : \underline{\mathfrak{B}}(\mathbb{K}_i) \to \underline{\mathfrak{B}}(\mathbb{K}_1 \times \mathbb{K}_2)$.

Proposition 71. *For every $t \in T$, the relation*

$$\nabla_t := \{(g, m) \in G \times M \mid g_t I_t m_t\} \cup \square$$

is a closed subrelation. If $\nabla \neq G \times M$, then the corresponding sublattice $\underline{\mathfrak{B}}(G, M, \nabla_t)$ is isomorphic to $\underline{\mathfrak{B}}(\mathbb{K}_t)$, and the map

$$\varepsilon_t : \underline{\mathfrak{B}}(\mathbb{K}_t) \to \underline{\mathfrak{B}}(G, M, \nabla)$$

with

$$\varepsilon_t(A, B) := (\{g \in G \mid g_t \in A\} \cup M^{\nabla}, \{m \in M \mid m_t \in B\} \cup G^{\nabla})$$

is a canonical lattice-embedding.

Proof. The fact that ∇_t is closed can be proved easily, for instance by means of Proposition 47 (p. 114): If $(g, m) \in \nabla \setminus \nabla_t$, then in particular $m^{\nabla} \neq G$, i.e., we can choose an object $\tilde{g} \notin m^{\nabla}$. Using this object, we define an object h to be

$$h_s := \begin{cases} \tilde{g}_s & \text{if } s \neq t \\ g_s & \text{if } s = t \end{cases}.$$

We get $(h,m) \notin \nabla$ and $h^\nabla = h^{\nabla_t}$.

It is a matter of routine to prove that ε_t has the properties claimed. □

Proposition 72. *If g, h are objects of the direct product $\times_{t\in T}\mathbb{K}_t$, then*

$$g^\nabla \subseteq h^\nabla \iff \begin{cases} h \in M^\nabla & \textit{or} \\ g_t^{I_t} \subseteq h_t^{I_t} & \textit{for all } t \in T \end{cases}.$$

Proof. "$\Rightarrow$": We presuppose the negation of the right side. Assume that $m \notin h^\nabla$ and $n_t \in g_t^{I_t} \setminus h_t^{I_t}$ for some $t \in T$. Consider the attribute $\tilde{m}$, defined by

$$\tilde{m}_s := \begin{cases} m_s & \text{if } s \neq t \\ n_t & \text{if } s = t \end{cases}.$$

Then $\tilde{m} \in g^\nabla \setminus h^\nabla$, i.e., $g^\nabla \nsubseteq h^\nabla$. The direction "$\Leftarrow$" is trivial. □

Proposition 73.

$$g \swarrow m \iff \forall_{t\in T} \quad g_t \swarrow m_t,$$

$$g \nearrow m \iff \forall_{t\in T} \quad g_t \nearrow m_t.$$

Proof. We prove only the first statement. First of all we notice that, because of $(g,m) \notin \nabla \iff \forall_t (g_t, m_t) \notin I_t$, we can limit ourselves to non-incident pairs. If, for some t, $g_t \swarrow m_t$ does not hold, there must be an object $h_t \in G_t$ with $g_t' \subseteq h_t'$, $g_t' \neq h_t'$ and $(h_t, m_t) \notin I_t$. The object $\tilde{g}$, defined by

$$\tilde{g}_s := \begin{cases} g_s & \text{if } s \neq t \\ h_t & \text{if } s = t \end{cases}$$

satisfies

$$g^\nabla \subseteq \tilde{g}^\nabla, \quad g^\nabla \neq \tilde{g}^\nabla \quad \text{and} \quad (\tilde{g}, m) \notin \nabla,$$

which yields $\neg(g \swarrow m)$.

Analogously, we infer the converse direction: If $g_t \swarrow m_t$ holds for all $t \in T$, then we certainly have $(g,m) \notin \nabla$ and we only have to consider an object h with $g^\nabla \subseteq h^\nabla$, $g^\nabla \neq h^\nabla$. By Proposition 72 we obtain $g_s' \subset h_s'$, $g_s' \neq h_s'$ for some $s \in T$, from which, because of $g_s \swarrow m_s$, it immediately follows that $h_s I_s m$ and thus $h \nabla m$. □

Together with Proposition 13 (p. 31) this yields

Corollary 74. *An object g of a direct product is irreducible if and only if all g_t are irreducible. The corresponding is true for attributes.*

The direct product of reduced contexts is reduced, the direct product of doubly founded contexts is doubly founded. □

Our interest lies in the concept lattice of the direct product. We shall call this lattice the *tensor product* of the factor lattices $\underline{\mathfrak{B}}(\mathbb{K}_t)$. Thereby we obtain a new lattice construction and thus a new decomposition principle. However, in order to do so we have to show that the tensor product is independent (up to isomorphism) of the choice of the underlying contexts $\mathbb{K}_t$. This is the result of the theorem which follows the next definition.

Definition 64. The **tensor product** of complete lattices $\mathbf{V}_t$, $t \in T$ is defined as

$$\bigotimes_{t\in T} \mathbf{V}_t := \underline{\mathfrak{B}}(\bigtimes_{t\in T} (V_t, V_t, \leq)),$$

i.e., for the special case of two factors as

$$\mathbf{V}_1 \otimes \mathbf{V}_2 := \underline{\mathfrak{B}}(V_1 \times V_2, V_1 \times V_2, \nabla)$$

with

$$(g_1, g_2)\nabla(m_1, m_2) :\Longleftrightarrow g_1 \leq m_1 \text{ or } g_2 \leq m_2. \qquad \diamond$$

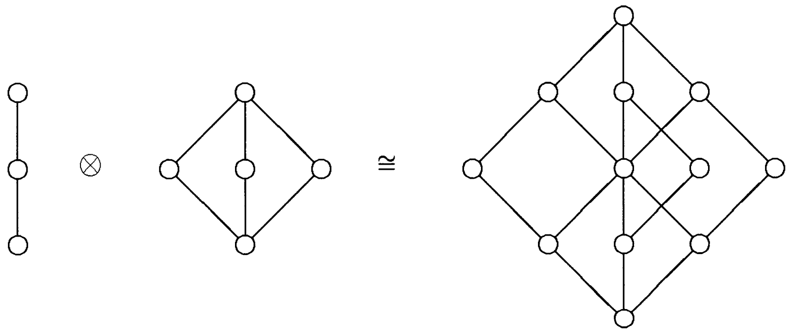


Figure 4.22 $\underline{\mathfrak{B}}(\mathbb{K}_1) \otimes \underline{\mathfrak{B}}(\mathbb{K}_2) \cong \underline{\mathfrak{B}}(\mathbb{K}_1 \times \mathbb{K}_2)$

Theorem 26. *The concept lattice of a direct product of contexts is isomorphic to the tensor product of the concept lattices of the factor contexts:*

$$\underline{\mathfrak{B}}(\bigtimes_{t\in T} \mathbb{K}_t) \cong \bigotimes_{t\in T} \underline{\mathfrak{B}}(\mathbb{K}_t).$$

For the *proof* we use the Basic Theorem on Concept Lattices. According to Definition 64, the tensor product $\bigotimes_{t\in T} \underline{\mathfrak{B}}(\mathbb{K}_t)$ is the concept lattice of the context (G, M, ∇) with

$$G = M = \bigtimes_{t\in T} \underline{\mathfrak{B}}(\mathbb{K}_t)$$

and

$$(A_t, B_t)_{t\in T} \nabla (C_t, D_t)_{t\in T} \iff \exists_{t\in T} A_t \subseteq C_t.$$

In order to prove the isomorphy we claimed, we have to give maps

$$\tilde{\gamma} : G \to \underline{\mathfrak{B}}(\bigtimes_{t\in T} \mathbb{K}_t) \qquad \text{and} \qquad \tilde{\mu} : M \to \underline{\mathfrak{B}}(\bigtimes_{t\in T} \mathbb{K}_t)$$

which have the properties postulated by the Basic Theorem. We choose

$$\tilde{\gamma}((A_t, A_t')_{t\in T}) := (\{g \mid \forall_t\, g_t \in A_t\} \cup M^\nabla, \{m \mid \exists_t\, m_t \in A_t'\})$$

$$\tilde{\mu}((B_t', B_t)_{t\in T}) := (\{g \mid \exists_t\, g_t \in B_t'\}, G^\nabla \cup \{m \mid \forall_t\, m_t \in B_t\}).$$

First, we have to show that these are concepts, i.e., that

$$\begin{aligned} \{g \mid \forall_t\, g_t \in A_t\}^\nabla &= \{m \mid \exists_t\, m_t \in A_t'\} \quad \text{and} \\ M^\nabla \cup \{g \mid \forall_t\, g_t \in A_t\} &= \{m \mid \exists_t\, m_t \in A_t'\}^\nabla. \end{aligned}$$

The inclusions $\subseteq$ are trivial. Therefore, let m be an attribute with $m_t \notin A_t'$ for all $t \in T$. Then for every $t \in T$ there exists an object $g_t \in A_t$ with $(g_t, m_t) \notin I_t$; and thus $g := (g_t)_{t\in T}$ satisfies $(g, m) \notin \nabla$. This proves the inclusion $\supseteq$ in the first case, the second case as well as the dual proof for $\tilde{\mu}$ are analogous.

Next we show that $\tilde{\gamma}G$ is supremum-dense by proving that $\tilde{\gamma}G$ contains all object concepts of $\underline{\mathfrak{B}}(\bigtimes_{t\in T} \mathbb{K}_t)$. We have (with $g := (g_t)_{t\in T}$)

$$\tilde{\gamma}((g_t'', g_t')_{t\in T}) = (\ldots, \{m \mid \exists_{t\in T}\, m_t \in g_t\}) = (\ldots, g^\nabla).$$

Finally, we have

$$\begin{aligned} & \tilde{\gamma}((A_t, B_t)_{t\in T}) \leq \tilde{\mu}((C_t, D_t)_{t\in T}) \\ \iff & \{g \in G \mid \forall_{t\in T}\, g_t \in A_t\} \cup M^\nabla \subseteq \{g \in G \mid \exists_{t\in T}\, g_t \in C_t\} \\ \iff & \exists_{t\in T}\, A_t \subseteq C_t, \end{aligned}$$

since $\forall_t A_t \not\subseteq C_t \iff \exists_{g\in G} \forall_{t\in T}\, g_t \in A_t \setminus C_t$. However, the condition $\exists_{t\in T}\, A_t \subseteq C_t$ is equivalent to $(A_t, B_t)_{t\in T} \nabla (C_t, D_t)_{t\in T}$, which remained to be proved. □

Theorem 27. *The congruence lattice of a tensor product of finitely many doubly founded lattices is isomorphic to the tensor product of the congruence lattices:*

$$\mathfrak{C}(\bigotimes_{t\in T} V_t) \cong \bigotimes_{t\in T} \mathfrak{C}(V_t)$$

Proof. According to Theorem 12 (p. 111), $\mathfrak{C}(V_t) \cong \underline{\mathfrak{B}}(G_t, M_t, \not\swarrow\!\!\nearrow)$, where $g_t \swarrow\!\!\swarrow m_t$ in V_t if and only if there are objects $g_1, \ldots, g_n$ and attributes $m_1, \ldots, m_n$ with

$$g_t = g_1 \swarrow m_1 \nwarrow g_2 \swarrow \cdots \nwarrow g_n \swarrow m_n = m_t.$$

If such a sequence of elements of length n exists, then it also exists for every number which is larger than n, since in this case there is an attribute $k \in V_t$ with $g_t \swarrow\!\!\!\nearrow k$ and consequently $g_t \swarrow k \nwarrow g_t$, whereby the sequence can be extended arbitrarily. We can apply Proposition 73 and, since T is finite, we obtain

$$g \swarrow\!\!\!\swarrow m \text{ in } \bigtimes_{t\in T} \mathbb{K}_t \quad \Longleftrightarrow \quad g_t \swarrow\!\!\!\swarrow m_t \text{ in } \mathbb{K}_t \text{ for all } t \in T,$$

and consequently

$$g \not\swarrow\!\!\!\swarrow m \iff g_t \not\swarrow\!\!\!\swarrow m_t \text{ for some } t \in T.$$

Therefore,

$$(G, M, \not\swarrow\!\!\!\swarrow) = (\bigtimes_{t\in T} (G_t, M_t, \not\swarrow\!\!\!\swarrow)),$$

which together with Theorem 12 proves the assertion. □

The tensor product has been defined as the concept lattice of the context

$$(G, M, \nabla) := \bigtimes_{t\in T} (V_t, V_t, \leq),$$

with $G^\nabla = \{m \mid \exists_t\ m_t = 0\}$ and $M^\nabla = \{g \mid \exists_t\ g_t = 0\}$. The concept lattices of the factor contexts are naturally isomorphic to the lattices $\boldsymbol{V}_t$. Therefore, it seems reasonable to denote the embedding of $\boldsymbol{V}_s$ into $\bigotimes_{t\in T} \boldsymbol{V}_t$ with the same letter as the corresponding embedding in Proposition 71. Hence, we define $\varepsilon_s : \boldsymbol{V}_s \to \bigotimes_{t\in T} \boldsymbol{V}_t$ through

$$\varepsilon_s(x_s) := (\{g \in G \mid g_s \leq x_s\} \cup M^\nabla, \{m \in M \mid x_s \leq m_s\} \cup G^\nabla),$$

with $(G, M, \nabla) := \bigtimes_{t\in T}(V_t, V_t, \leq)$.

Proposition 75. *For each object concept $y := \tilde{\gamma}(g)$ and for each attribute concept $z := \tilde{\mu}(m)$ of the tensor product as well as for every subset $S \subseteq T$, we have*

$$y \leq \bigvee_{s\in S} \varepsilon_s(x_s) \iff \exists_{s\in S}\ y \leq \varepsilon_s(x_s)$$

$$z \geq \bigwedge_{s\in S} \varepsilon_s(x_s) \iff \exists_{s\in S}\ z \geq \varepsilon_s(x_s).$$

Proof. In other words, the proposition claims that the extent of $\bigvee_{s\in S} \varepsilon_s(x_s)$ is exactly the union of the extents of the $\varepsilon_s(x_s), s \in S$, and dually. This immediately results from the explicit descriptions of these sets which were given above. □

Hence, the sublattices $\varepsilon_t(V_t)$ are *mutually distributive*: The supremum resp. infimum of elements from different $\varepsilon_t(V_t)$ can be obtained by forming the union of the extents or intents, respectively. This implies a calculation rule which will be formulated in the following definition:

Definition 65. We call two subsets X and Y of a complete lattice **mutually distributive** if the following inequalities hold for every index set S and for every pair of sequences $(x_s)_{s\in S}$, $(y_s)_{s\in S}$ of elements $x_s \in X$, $y_s \in Y$:

$$\bigvee_{s\in S}(x_s \wedge y_s) \geq \bigwedge_{R\subseteq S}(\bigvee_{r\in R} x_r \vee \bigvee_{s\in S\setminus R} y_s),$$

$$\bigwedge_{s\in S}(x_s \vee y_s) \leq \bigvee_{R\subseteq S}(\bigwedge_{r\in R} x_r \wedge \bigwedge_{s\in S\setminus R} y_s). \qquad \diamond$$

The inequalities can be replaced by equations without changing the statement, since the respective other directions hold in every lattice.

Proposition 76. *If $\mathbb{V}_i$ and $\mathbb{V}_j$, $(i \neq j)$ are factors of a tensor product*

$$\bigotimes_{t\in T} \mathbb{V}_t,$$

then the sublattices $\varepsilon_i(\mathbb{V}_i)$ and $\varepsilon_j(\mathbb{V}_j)$ are mutually distributive.

Proof. We only prove the first inequality

$$\bigvee_{s\in S}(\varepsilon_i(x_s) \wedge \varepsilon_j(y_s)) \geq \bigwedge_{R\subseteq S}(\bigvee_{r\in R} \varepsilon_i(x_r) \vee \bigvee_{s\in S\setminus R} \varepsilon_j(y_s)).$$

For this purpose, it suffices to prove that every attribute concept z which is $\geq$ the left side is also $\geq$ the right side of the inequality. Hence, let z be an attribute concept and assume that

$$R_z := \{r \in S \mid \varepsilon_i(x_r) \leq z\}.$$

Then we obviously have $\bigvee_{r\in R_z} \varepsilon_i(x_r) \leq z$ and can follow the following chain of inferences:

$$\begin{array}{lll}
 & & z \geq \bigvee_{s\in S}(\varepsilon_i(x_s) \wedge \varepsilon_j(y_s)) \\
\iff & \forall_{s\in S} & z \geq \varepsilon_i(x_s) \wedge \varepsilon_j(y_s) \\
\iff & \forall_{s\in S} & z \geq \varepsilon_i(x_s) \text{ or } z \geq \varepsilon_j(y_s) \\
\iff & \forall_{s\in S\setminus R_z} & z \geq \varepsilon_j(y_s) \\
\iff & & z \geq \bigvee_{r\in R_z} \varepsilon_i(x_r) \vee \bigvee_{s\in S\setminus R_z} \varepsilon_j(y_s) \\
\Longrightarrow & & z \geq \bigwedge_{R\subseteq S}(\bigvee_{r\in R} \varepsilon_i(x_r) \vee \bigvee_{s\in S\setminus R} \varepsilon_j(y_s))
\end{array}$$

In the case of the second equivalence we have used Proposition 75. $\square$

The formulation “mutually distributive” suggests the following result:

Theorem 28. *The tensor product of completely distributive lattices is completely distributive.*

Proof. We leap ahead to the characterization of complete distributivity through a context condition in Theorem 40 (p. 221) and show that this condition can be transferred from the factors to a direct product of contexts. In order to improve the readability we replace the expression $h \in k''$ by the (equivalent) statement $k' \subseteq h'$.

Assume that $\mathbb{K}_t := (G_t, M_t, I_t)$, $t \in T$ are contexts and that $(G, M, I) := \bigtimes_{t \in T} \mathbb{K}_t$. Assume further that $g \in G$ and $m \in M$ are elements with $(g, m) \notin I$. Then, we have

$$(g_t, m_t) \notin I_t$$

for all $t \in T$. Moreover, if the $\mathbb{K}_t$ satisfy the condition from Theorem 40, there exist elements $h_t \in G_t$ as well as $n_t \in M_t$ with

$$(h_t, m_t) \notin I_t, (g_t, n_t) \notin I_t \text{ and } k_t \subseteq h_t \text{ for all } k \in G_t \setminus n_t'$$

for every $t \in T$. We set $h := (h_t)_{t \in T}$ and $n := (n_t)_{t \in T}$ and find $(h, m) \notin I$ and $(g, n) \notin I$. If now $k \in G \setminus n'$, i.e., $k_t \in G_t \setminus n_t'$ for all $t \in T$, then for every $t \in T$ it holds that

$$k_t' \subseteq h_t'$$

and, according to Proposition 72, consequently

$$k' \subseteq h',$$

which was to be proved. □

We had generalized the direct product of lattices to the subdirect product in order to obtain a more versatile decomposition principle. We can proceed similarly in the case of the tensor product. Two possibilities suggest themselves: On the one hand, we can form a *subtensorial product* of complete lattices in analogy to the subdirect product. On the other hand, we can introduce a *subdirect product* of contexts. If we do this correctly, both constructions are equivalent.

Definition 66. A **subtensorial product** of complete lattices $\mathbf{V}_t$, $t \in T$ is a factor lattice

$$\bigotimes_{t \in T} \mathbf{V}_t \Big/ \Theta$$

of the tensor product for which the restrictions of the projection mapping

$$\pi_\Theta : \bigotimes_{t \in T} \mathbf{V}_t \to \bigotimes_{t \in T} \mathbf{V}_t \Big/ \Theta, \qquad x \mapsto [x]\Theta$$

onto the sublattices $\varepsilon_t(\boldsymbol{V}_t)$ are all injective.

A **subtensorial decomposition** of a complete lattice $\boldsymbol{V}$ then is a sequence $(\boldsymbol{V}_t \mid t \in T)$ of complete sublattices of $\boldsymbol{V}$ for which there is an isomorphism ψ of $\boldsymbol{V}$ onto a subtensorial product $\bigotimes_{t\in T} \boldsymbol{V}_t / \Theta$ with

$$\pi_\Theta(\varepsilon_t(\boldsymbol{V}_t)) = \psi(\boldsymbol{V}_t), \quad t \in T. \qquad \diamond$$

Subtensorial decompositions can be characterized internally. For reasons of simplicity, we limit ourselves to the case $T = \{1, 2\}$.

Proposition 77. *A pair $(\boldsymbol{V}_1, \boldsymbol{V}_2)$ of complete sublattices is a subtensorial decomposition of $\boldsymbol{V}$ if and only if $\boldsymbol{V}_1$ and $\boldsymbol{V}_2$ are mutually distributive and their union generates $\boldsymbol{V}$.*

Proof. If $\boldsymbol{V}_1$ and $\boldsymbol{V}_2$ are mutually distributive sublattices of $\boldsymbol{V}$, then by Theorem 37 (p. 205), $\otimes 3$, there is a complete homomorphism

$$\varphi : \boldsymbol{V}_1 \otimes \boldsymbol{V}_2 \to \boldsymbol{V},$$

for which it holds that $\varphi \circ \varepsilon_t = id_{V_t}$ for $t = 1$ and $t = 2$. If $V_1 \cup V_2$ generates $\boldsymbol{V}$, then this morphism must be surjective, i.e., $\boldsymbol{V}$ then is a factor lattice of $\boldsymbol{V}_1 \otimes \boldsymbol{V}_2$.

If, conversely, $(\boldsymbol{V}_1, \boldsymbol{V}_2)$ is a subtensorial decomposition, then $\boldsymbol{V}_1$ and $\boldsymbol{V}_2$ are mutually distributive, since this property is inherited by factor lattices. Their union generates $\boldsymbol{V}$, because $\boldsymbol{V}_1 \otimes \boldsymbol{V}_2$ is generated by $\varepsilon(\boldsymbol{V}_1) \cup \varepsilon(\boldsymbol{V}_2)$. $\square$

In the case of doubly founded concept lattices, the notation for subtensorial products can be further simplified. By Theorem 26, $\bigotimes_{t\in T} \underline{\mathfrak{B}}(\mathbb{K}_t)$ is isomorphic to the concept lattice of the direct product

$$(G, M, \nabla) := \bigtimes_{t\in T} \mathbb{K}_t$$

of the respective contexts. The closed relation ∇_t always corresponds to the sublattice $\varepsilon_t(\underline{\mathfrak{B}}(\mathbb{K}_t))$. In the doubly founded case, we can be sure that a subtensorial product is always induced by a compatible subcontext $(H, N, \nabla \cap H \times N)$ of (G, M, ∇). Such subcontexts are described by the following definition:

Definition 67. A **subdirect product** of contexts

$$\mathbb{K}_t := (G_t, M_t, I_t), \quad t \in T,$$

is a compatible subcontext

$$(H, N, \nabla \cap H \times N)$$

of the direct product $\bigtimes_{t\in T} \mathbb{K}_t$ having the property that for each $t \in T$ the subcontext

$$(H_t, N_t, I_t \cap H_t \times N_t)$$

with

$$H_t := \{h_t \mid h \in H\} \quad \text{and} \quad N_t := \{n_t \mid n \in N\}$$

is dense in $\mathbb{K}_t$. ◇

Proposition 78. *The subdirect products of contexts are precisely the compatible subcontexts of subtensorial products.*

Proof. According to Proposition 38 (p. 102), the condition (here restricted to $\varepsilon_t(\underline{\mathfrak{B}}(\mathbb{K}_t)))$ that the map $\Pi_{H,N}$ is injective, is equivalent to the fact that the subcontext $(H, N, \nabla_t \cap H \times N)$ is dense in (G, M, ∇_t). This in turn is, according to the same proposition, equivalent to the fact that

$$(A \cap H)^{\nabla_t \nabla_t} = A \quad \text{and} \quad (B \cap N)^{\nabla_t \nabla_t} = B$$

holds for each concept (A, B) of (G, M, ∇_t). If we set $A_t := \{g_t \mid g \in A\}$, we recognize by means of the description of the concepts of (G, M, ∇_t) in Proposition 71 that

$$\{g_t \mid g \in A \cap H\} = A_t \cap H_t$$

and therefore

$$(A \cap H)^{\nabla_t \nabla_t} = \{g \in G \mid g_t \in (A_t \cap H_t)^{I_t I_t}\}.$$

Consequently,

$$(A \cap H)^{\nabla_t \nabla_t} = A \iff (A_t \cap H_t)^{I_t I_t} = A_t.$$

This is again the condition from Proposition 38. Hence, $(H, N, \nabla_t \cap N \times N)$ is dense in (G, M, ∇_t), if and only if $(H_t, N_t, I_t \cap H_t \times N_t)$ is dense in $\mathbb{K}_t$. □

The restriction of the closed relations ∇_t to such a subcontext then yields the subrelations $J_t := \nabla_t \cap H \times N$ with

$$\underline{\mathfrak{B}}(H, N, J_t) \cong \underline{\mathfrak{B}}(\mathbb{K}_t).$$

We now want to find out under which conditions a context is *isomorphic* to a subdirect product. For this purpose we define

Definition 68. A **subdirect decomposition** of a context

$$\mathbb{K} := (G, M, I)$$

is a family $(I_t)_{t \in T}$ of subrelations of I with the following properties:

1. $I = \bigcup_{t \in T} I_t$

2. There are surjective maps

$$\alpha : G \to H, \qquad \beta : M \to N$$

onto a subdirect product $(H, N, \nabla \cap H \times N)$ with

$$gI_t m \iff \alpha g \nabla_t \beta m.$$

$\diamond$

If $(H, N, \nabla \cap H \times N)$ is a subdirect product, then $(\nabla_t \cap H \times N)_{t \in T}$ is a subdirect decomposition, as can be easily recognized by choosing the identical map for α and β, respectively.

The maps α and β do not have to be injective. Nevertheless, from $\alpha g = \alpha h$ it always follows that $g^{I_t} = h^{I_t}$ for all $t \in T$ and in particular $g' = h'$. Hence, (α, β) is "up to clarification" an isomorphism of (G, M, I) onto $(H, N, \nabla \cap H \times N)$ and even of (G, M, I_t) onto $(H, N, \nabla_t \cap H \times N)$ for all $t \in T$. In particular, the contexts $\mathbb{K}_t := (G, M, I_t)$ and $(H, N, \nabla_t \cap H \times N)$ have isomorphic concept lattices. It suggests itself to use these contexts as the factors of the direct product (note that such factors are not further specified in the definition). We show that this is possible, but before that, we clarify the contexts. For this purpose we define for each of the contexts

$$\mathbb{K}_t := (G, M, I_t)$$

equivalence relations Θ_t on G and Ψ_t on M through

$$\begin{aligned} (g, h) \in \Theta_t \quad &:\iff \quad g^{I_t} = h^{I_t} \\ (m, n) \in \Psi_t \quad &:\iff \quad m^{I_t} = n^{I_t}. \end{aligned}$$

Hence, $\Theta_t = \ker \gamma_t$ and $\Psi_t = \ker \mu_t$. The context

$$\mathbb{K}_t^\circ := (G/\Theta_t, M/\Psi_t, \bar{I}_t)$$

with

$$([g]\Theta_t, [m]\Psi_t) \in \bar{I}_t :\iff (g, m) \in I_t$$

then is the corresponding clarified context.

Then, we can naturally assign a subcontext of the direct product

$$\bigtimes_{t \in T} \mathbb{K}_t^\circ = (\bigtimes_{t \in T} G/\Theta_t, \bigtimes_{t \in T} M/\Psi_t, \overline{\nabla})$$

of the clarified contexts $\mathbb{K}_t^\circ$ to the context $\mathbb{K} := (G, M, I)$.

The symbol $\overline{\nabla}$ is only used for a better distinction. It denotes the incidence of the direct product, i.e.,

$$(g_t)_{t \in T} \overline{\nabla}_s (m_t)_{t \in T} \iff g_s I_s m_s.$$

The role of the maps α and β from Definition 68 is taken over by the maps $\underline{\iota}$ and $\bar{\iota}$, which are defined as follows:

$$\underline{\iota} : G \to \bigtimes_{t \in T} G/\Theta_t, \qquad g \mapsto ([g]\Theta_t)_{t \in T}$$

$$\bar{\iota} : M \to \bigtimes_{t \in T} M/\Psi_t, \qquad m \mapsto ([m]\Psi_t)_{t \in T}.$$

The image context

$$(\underline{\iota}G, \bar{\iota}M, \overline{\nabla} \cap \underline{\iota}G \times \bar{\iota}M)$$

obviously has the property that the projection maps

$$([g]\Theta_t)_{t \in T} \mapsto [g]\Theta_s, \qquad ([m]\Psi_t)_{t \in T} \mapsto [m]\Psi_s$$

are surjective on the factor contexts whereby their images are certainly dense. Furthermore, we have

$$gI_t m \quad (\text{in } \mathbb{K}) \qquad \Longleftrightarrow \qquad \alpha g \overline{\nabla}_t \beta m \quad (\text{in } (\underline{\iota}G, \bar{\iota}M, \overline{\nabla} \cap \underline{\iota}G \times \bar{\iota}M)).$$

Hence, if it is compatible, this subcontext is certainly a subdirect product.

Proposition 79. *$(I_t)_{t \in T}$ is a subdirect decomposition, if and only if the subcontext*

$$(\underline{\iota}G, \bar{\iota}M, \overline{\nabla} \cap \underline{\iota}G \times \bar{\iota}M)$$

of $\bigtimes_{t \in T} \mathbb{K}_t^\circ$ is compatible.

Proof. If this subcontext is compatible, it evidently satisfies all conditions of a subdirect decomposition. The other direction is more laborious. Hence, let $(H, N, \nabla \cap H \times N)$ be a compatible subcontext of an arbitrary direct product $\bigtimes_{t \in T}(\tilde{G}_t, \tilde{M}_t, J_t)$ and let $\alpha : G \to H$, $\beta : M \to N$ be mappings satisfying $gI_t m \iff \alpha g \nabla_t \beta_m$ as in Definition 68. We have to show that under these conditions the subcontext mentioned in the proposition is also compatible. For this purpose, we use the characterization from Proposition 35 (p. 100). Hence, let $\underline{h} \in \underline{\iota}G$ be an object of the subcontext, i.e., $\underline{h} = ([g]\Theta_t)_{t \in T}$ for an object $g \in G$. Furthermore, let $m := ([m_t]\Psi_t)_{t \in T}$ be an arbitrary attribute of the direct product of the $\mathbb{K}_t^\circ$ with $(\underline{h}, m) \notin \overline{\nabla}$. Then, we have to show that there is an attribute $\bar{n} \in \bar{\iota}M$ with

$$(\underline{h}, \bar{n}) \notin \overline{\nabla} \quad \text{and} \quad m^{\overline{\nabla}} \subseteq \bar{n}^{\overline{\nabla}}.$$

From the preconditions we obtain

$$([g]\Theta_t, [m_t]\Psi_t) \notin \bar{I}_t$$

for all $t \in T$, i.e.,

$$(g, m_t) \notin I_t \quad \text{for all } t \in T.$$

Then, we also have

$$(\alpha g, \beta m_t) \notin \nabla_t \quad \text{for all } t \in T.$$

If

$$\alpha g =: (g_t)_{t\in T}$$

and

$$\beta m_s =: (m_t^s)_{t\in T} \qquad \text{for every } s \in T,$$

then we have

$$(g_t, m_t^t) \notin J_t \quad \text{for all } t \in T.$$

If we set

$$l := (m_t^t)_{t\in T},$$

then l is an attribute of the direct product with $(\alpha g, l) \notin \nabla$. Since $(H, N, \nabla \cap H \times N)$ is compatible, there must be an attribute $\beta n \in N$ with

$$(\alpha g, \beta n) \notin \nabla \quad \text{and} \quad l^{\nabla} \subseteq (\beta n)^{\nabla}.$$

If we set $\beta n =: (n_t)_{t\in T}$, then with Proposition 72

$$(m_t^t)^{\nabla_t} \subseteq (n_t)^{\nabla_t} \qquad \text{for all } t \in T,$$

i.e.,

$$(\beta m_t)^{\nabla_t} \subseteq (\beta n)^{\nabla_t} \qquad \text{for all } t \in T,$$

which yields

$$m_t^{I_t} \subseteq n_t^{I_t} \qquad \text{for all } t \in T.$$

Hence, we obtain

$$([m_t]\Psi_t)^{\bar{I}_t} \subseteq ([n_t]\Psi_t)^{\bar{I}_t} \quad \text{for all } t \in T$$

and with $\overline{n} := \overline{\iota}n$

$$m^{\overline{\nabla}} \subseteq \overline{n}^{\overline{\nabla}}.$$

Because of $(\alpha g, \beta n) \notin \nabla$, we have $(g, n) \notin I$ and therefore $(g, n) \notin I_t$ for all $t \in T$, which yields the statement

$$(\underline{h}, \overline{n}) \notin \overline{\nabla},$$

which we were still lacking. □

Proposition 79 contains a structural description of subdirect products. It is particularly easy to make use of this fact when we are dealing with a doubly founded context. We shall explain this in the following theorem. The notations used for this purpose are to be understood as follows: A family $(I_t)_{t\in T}$ of subrelations of (G, M, I) is called doubly founded if each of the contexts (G, M, I_t) is doubly founded. The arrow relations $\swarrow_t$ and $\nearrow_t$ also refer to these contexts.

Theorem 29. *A doubly founded family $(I_t)_{t\in T}$ of subrelations is a subdirect decomposition of (G, M, I) if and only if:*

1. $I = \bigcup_{t\in T} I_t$,
2. *if $g_t \swarrow_t m$ for all $t \in T$, then there exists an object $h \in G$ with $g_t^{I_t} = h^{I_t}$ for all $t \in T$,*
3. *if $g \nearrow_t m_t$ for all $t \in T$, then there exists an attribute $n \in M$ with $m_t^{I_t} = n^{I_t}$ for all $t \in T$.*

Proof. From Proposition 79 we infer that $(I_t)_{t\in T}$ is a subdirect decomposition if and only if the subcontext

$$(\underline{\iota}G, \bar{\iota}M, \overline{\nabla} \cap \underline{\iota}G \times \bar{\iota}M)$$

is closed. Because of the condition of foundedness, this is equivalent to its being arrow closed. This is, however, precisely what is postulated in conditions 2) and 3) of the above theorem. Condition 2), for instance, is a rephrasing of the condition

$$([g_t]\Theta_t)_{t\in T} \swarrow \bar{\iota}m \quad \text{implies} \quad ([g_t]\Theta_t) \in \underline{\iota}G,$$

since $([g_t]\Theta_t)$ forms part of $\underline{\iota}G$, if there is an object $h \in G$ with $[g_t]\Theta_t = [h]\Theta_t$ for all $t \in T$. □

Together with Proposition 77 this can be extended into a practicable condition. For this purpose, we call a pair (x, y) of a doubly founded lattice $\boldsymbol{V}$ **weakly distributive** if

$$g \leq x \vee y \iff g \leq x \text{ or } g \leq y$$

holds for every $\bigvee$-irreducible element $g \in J(\boldsymbol{V})$ and dually

$$m \geq x \wedge y \iff m \geq x \text{ or } m \geq y$$

holds for every $\bigwedge$-irreducible element $m \in M(\boldsymbol{V})$. Hence, two concepts (A_1, B_1) and (A_2, B_2) certainly form a weakly distributive pair if $A_1 \cup A_2$ is an extent and $B_1 \cup B_2$ is an intent. We have seen above that this is always the case for pairs (x, y) of elements of a tensor product with $x \in \varepsilon_i(\boldsymbol{V}_i)$, $y \in \varepsilon_j(\boldsymbol{V}_j)$ and $i \neq j$. This implied that those sublattices were mutually distributive. In fact, the following statement can be shown by means of the same proof as as was given for Proposition 76:

Proposition 80. *If all pairs (x_t, y_t), $t \in T$ are weakly distributive, then*

$$\bigvee_{t\in T}(x_t \wedge y_t) = \bigwedge_{S\subseteq T}\Big(\bigvee_{s\in S} x_s \vee \bigvee_{t\in T\setminus S} y_t\Big)$$

holds as well as the dual equation. □

Theorem 30. *Let $\mathbf{V}_1$ and $\mathbf{V}_2$ be complete sublattices of $\mathbf{V}$ and let $\mathbf{V}$, $\mathbf{V}_1$ and $\mathbf{V}_2$ be doubly founded. Then $(\mathbf{V}_1, \mathbf{V}_2)$ is a subtensorial decomposition of $\mathbf{V}$ if and only if the union $\mathbf{V}_1 \cup \mathbf{V}_2$ generates the lattice $\mathbf{V}$ and every pair (x_1, x_2) with $x_1 \in \mathbf{V}_1$ and $x_2 \in \mathbf{V}_2$ is weakly distributive.*

Proof. From the Propositions 77 and 80 it immediately follows that the conditions specified are sufficient for a subtensorial decomposition. It remains to be shown that weak distributivity is necessary as well. For this purpose we use Theorem 29 for the standard context

$$\mathbb{K}(\mathbf{V}) := (J(\mathbf{V}), M(\mathbf{V}), \leq)$$

of $\mathbf{V}$. Consider an element $x_1 \in \mathbf{V}_1$ and an element $x_2 \in \mathbf{V}_2$ as well as a $\bigvee$-irreducible element $g \in J(\mathbf{V})$ with $g \not\leq x_1$ and $g \not\leq x_2$. Then there are elements $m_1, m_2 \in M(\mathbf{V})$ with $x_1 \leq m_1$, $x_2 \leq m_2$ and $g \nearrow_1 m_1$, $g \nearrow_2 m_2$, where $\nearrow_t$ is the arrow relation with respect to the closed subrelation I_t, which belongs to $\mathbf{V}_t$. According to Theorem 29, there is an attribute $m \in M(\mathbf{V})$ with $m^{I_1} = m_1^{I_1}$ and $m^{I_2} = m_2^{I_2}$. Condition 1 of Theorem 29 forces $m^I = m^{I_1} \cup m^{I_2}$, from which it follows that $g \not\leq m$. Since $m = m_1 \vee m_2 \geq x_1 \vee x_2$, this implies $g \not\leq x_1 \vee x_2$, which is one of the conditions of weak distributivity. The other follows dually. □

Corollary 81. *Two doubly founded complete sublattices $\mathbf{V}_1$ and $\mathbf{V}_2$ whose union generates a sublattice that is also doubly founded are mutually distributive if and only if every pair (x_1, x_2) with $x_1 \in \mathbf{V}_1$ and $x_2 \in \mathbf{V}_2$ is weakly distributive.* □

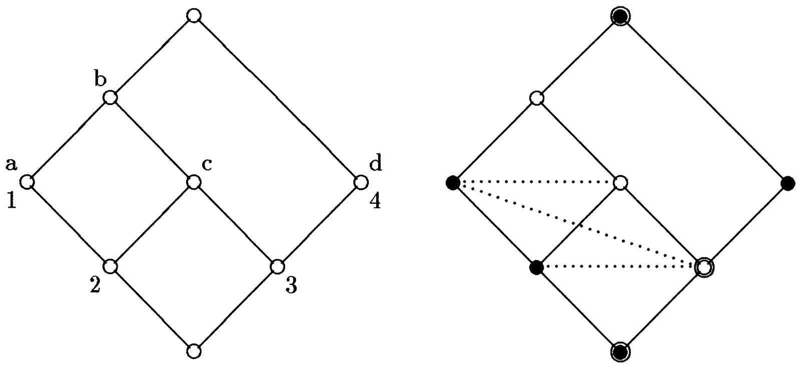


Figure 4.23 The dotted lines in the diagram on the right link the weakly distributive pairs of incomparable elements.

Example 8. We examine the lattice represented in Figure 4.23 to see if it has a subtensorial decomposition. Apart from pairs of comparable elements (which are automatically weakly distributive), the lattice only contains three weakly distributive pairs. They are represented in the diagram on the right side by the dotted lines. We recognize that there is only one non-trivial decomposition into two sublattices that satisfy the conditions in Theorem 30. Those two sublattices are marked in the right diagram by the filled or doubled circles, respectively.

I	a	b	c	d
1	×	×		
2	×	×	×	
3		×	×	×
4				×

I_1	a	b	c	d
1	×	×	$\swarrow\!\!\nearrow$	$\nearrow$
2	×	×	×	$\swarrow\!\!\nearrow$
3	$\swarrow\!\!\nearrow$	$\swarrow\!\!\nearrow$	$\swarrow$	×
4	$\swarrow\!\!\nearrow$	$\swarrow\!\!\nearrow$	$\swarrow$	×

I_2	a	b	c	d
1		$\swarrow\!\!\nearrow$	$\swarrow\!\!\nearrow$	$\swarrow\!\!\nearrow$
2		$\swarrow\!\!\nearrow$	$\swarrow\!\!\nearrow$	$\swarrow\!\!\nearrow$
3	$\swarrow\!\!\nearrow$	×	×	×
4		$\swarrow\!\!\nearrow$	$\swarrow\!\!\nearrow$	$\swarrow\!\!\nearrow$

$\overline{I_1}$	a,b	c	d
1	×		
2	×	×	
3,4			×

$\overline{I_2}$	a	b,c,d
1,2,4		
3		×

Figure 4.24 Context for the lattice from Figure 4.23, together with the closed relations for the sublattices. Below, the clarified contexts.

$\overline{\nabla}$		a,b		c		d		
		a	b,c,d	a	b,c,d	a	b,c,d	
1	1,2,4	×	×		$\swarrow\!\!\nearrow$		$\nearrow$	←
	3	×	×	$\swarrow\!\!\nearrow$	×	$\nearrow$	×	
2	1,2,4	×	×	×	×		$\swarrow\!\!\nearrow$	←
	3	×	×	×	×	$\swarrow\!\!\nearrow$	×	
3,4	1,2,4		$\swarrow\!\!\nearrow$		$\swarrow$	×	×	←
	3	$\swarrow\!\!\nearrow$	×	$\swarrow$	×	×	×	←
		↑	↑		↑		↑	

Figure 4.25 The direct product of the two clarified contexts from Figure 4.24. The context from Proposition 79 is marked by the arrows on the margin.

Figure 4.24 represents the context $\mathbb{K}(\boldsymbol{V})$ and the closed relations belonging to the sublattices, including their arrow relations. For those small contexts, it is easy to verify that the conditions in Theorem 29 are indeed satisfied. Therefore, we can switch over to the clarified contexts to find a concrete representation as a subtensorial product. Proposition 79 explains how we have to proceed. The subcontext

$$(\underline{\iota}G, \bar{\iota}M, \overline{\nabla} \cap \underline{\iota}G \times \bar{\iota}M)$$

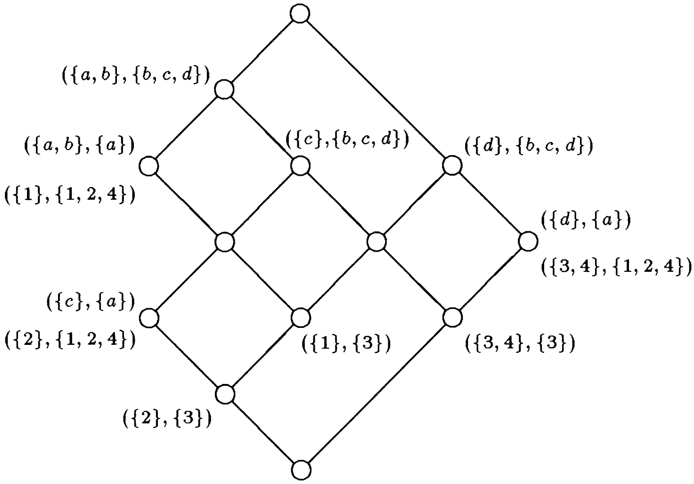


Figure 4.26 The concept lattice for the context from Figure 4.25 is the tensor product of the sublattices from Figure 4.23.

of $\bigtimes_{t\in T}\mathbb{K}_t^\circ$, which is mentioned there, is marked in Figure 4.25 through arrows on the margin. From the arrow relations we can see that it is arrow closed and thus compatible. Finally, Figure 4.26 shows the concept lattice of the context from Figure 4.25, i.e., the tensor product of the two sublattices forming the subtensorial decomposition. The lattice we started with can be recognized as an interval below the largest element; its elements are indeed separated by the projection map.

4.5 Hints and References

4.1 The introductory lines of this section, approximately up to Proposition 60 are analogous to known results of General Algebra. The complete lattices, however, do not form part of the structure classes treated in General Algebra and therefore require separate proofs. Some of them can be found in Pierce [135]. The concept-analytic results are based mainly on [192].

4.2 This section follows [195]. The decomposing and gluing technique described in this section was developed and successfully employed by Herrmann [85], an updated version can be found in Day and Herrmann [34]. Vogt [178] has employed the technique of atlas-decomposition when investigating the structure of subgroup lattices of finite Abelian groups.

4.3 The substitution sum and the substitution product were used by Luksch and Wille [115] for the concept-analytic evaluation of pair comparison tests. They were formally introduced in [114] and thoroughly examined by Stephan

[161], [160]. From the dissertation of Stephan we have taken in particular Theorem 25 and the preparatory Proposition 70. However, we have introduced a little change in the notation compared with the literature we quote and now write $U(a,b)V$, where in earlier publications appeared $U(b,a)V$.

4.4 Tensor products of complete lattices have been introduced in many articles. Our presentation (Sections 4.4, 5.4) does not claim to be complete, but is meant to supply the basic knowledge. The definition of the tensor product discussed in this section has been taken from [197] and the generalization in [206]. Precursors can be found among other things in Waterman [183], Mowat [129] and Shmuely [155]. A description of the extents and intents of direct products of contexts (as G_κ-ideals) can be found in [206]. The significance of this product for category theory was discussed by Erné [49]. Other sources are Bandelt [7], Raney [138] and Kalmbach [92].

Subtensorial products are treated in [67].

5. Constructions of Concept Lattices

A construction method by means of which we obtain from two contexts $\mathbb{K}_1$ and $\mathbb{K}_2$ a new context, let us say $\mathbb{K}$, can only be a useful construction principle for concept lattices, if it is *invariant under reduction*. This means that, if the same construction is applied to contexts whose concept lattices are isomorphic to those of $\mathbb{K}_1$ and $\mathbb{K}_2$, then the concept lattice of the result should be isomorphic to that of $\mathbb{K}$.

We have already presented some such methods in the first chapter. Now we shall describe four constructions in detail.

In the case of the *subdirect product* we consider sublattices of direct products. In 4.1 we have already examined how we can recognize the corresponding closed relations of the context sum. Now we are going to show how such relations can be constructed as a *fusion* of contexts.

Although the subdirect products are of central significance for General Algebra, they are rarely regarded as means of construction. One reason is their ambiguity. A subdirect product is not uniquely determined by stating its factors. This can however be easily remedied by choosing fixed generating systems in the factors. Thereby we obtain the P-product of algebraic structures and the P-fusion of contexts. A possible application of this construction consists in jointly unfolding different data sets which relate to the same situation.

The atlas-gluings introduced in 4.2 will be supplemented by a method in which the lattices are glued "sideways". This can be depicted particularly easily if the the overlap area of the lattices involved is the union of an ideal and a filter.

The third section deals with the technique of *doubling* convex subsets of a concept lattice. This construction has been used successfully in mathematical lattice theory, among other things for the examination of free lattices.

Finally we shall return to the tensor product of complete lattices. We shall give a lattice-theoretic characterization of this product and introduce the *tensorial operations*, by means of which we can trace back calculation within a tensor product to calculation within its factors.

5.1 Subdirect Product Constructions

We introduced the *(direct) sum* of contexts in Definition 34 (p. 46), however, we formulated it only for the case of two contexts. More generally, for a family of contexts $\mathbb{K}_t := (G_t, M_t, I_t)$, $t \in T$ we define the sum by

$$\sum_{t\in T}(G_t, M_t, I_t) := \left(\dot{\bigcup} G_t, \dot{\bigcup} M_t, \dot{\bigcup} I_t \cup \bigcup_{s\neq t} G_s \times M_t\right),$$

presupposing that the sets G_t, $t \in T$ as well as the sets $M_t, t \in T$ are pairwise disjoint. If necessary, this can be enforced by previously replacing every context $\mathbb{K}_t := (G_t, M_t, I_t)$ by the isomorphic context $\dot{\mathbb{K}}_t := (\dot{G}_t, \dot{M}_t, \dot{I}_t)$ with $\dot{G}_t := \{t\} \times G_t$ and $\dot{M}_t := \{t\} \times M_t$, as in Definition 34 (p. 46). Then we obtain

Theorem 31. *The concept lattice of a sum of contexts is isomorphic to the product of its concept lattices:*

$$\underline{\mathfrak{B}}(\sum_{t\in T}\mathbb{K}_t) \cong \bigtimes_{t\in T} \underline{\mathfrak{B}}(\mathbb{K}_t).$$

The map

$$(A, B) \mapsto ((A \cap G_t, B \cap M_t) \mid t \in T)$$

is a natural isomorphism.

The projection map on $\underline{\mathfrak{B}}(\mathbb{K}_t)$ combined with this isomorphism is the map

$$(A, B) \mapsto (A \cap G_t, B \cap M_t).$$

The corresponding compatible subcontext is $\mathbb{K}_t =: (G_t, M_t, I_t)$.

Proof. We only have to show that the concepts of the sum context are precisely the pairs (A, B) with $A \subseteq \dot{\bigcup} G_t$, $B \subseteq \bigcup M_t$ which have the property that for every $t \in T$ the restriction $(A \cap G_t, B \cap M_t)$ is a concept of $\mathbb{K}_t$. This is easy: By means of the definition, we realize that for a set $A_t \subseteq G_t$ the derivation in the sum context can be determined as follows:

$$A_t' = A^{I_t} \cup \bigcup_{s\neq t} M_t.$$

For an arbitrary subset $A \subseteq G$ we therefore get (with $A_t := A \cap G_t$)

$$A' = (\bigcup_{t\in T} A_t)' = \bigcap_{t\in T} A_t' = \bigcup_{t\in T} A_t^{I_t}.$$

Dually, the extents of the sum context are precisely the unions of extents of the summands, which yields the isomorphy we claimed. The statement on the compatible subcontexts can be verified by means of Proposition 34 (p. 100). □

In this section we want to characterize complete subdirect products, i.e., complete sublattices of a direct product for which the projection maps are surjective, in the language of contexts. We have already taken first steps in this direction, since to every complete sublattice corresponds a closed relation J in the sum context and, by Proposition 48 (p. 114), the surjectivity of the projection maps is equivalent to the condition $J \cap G_t \times M_t = I_t$. This is summarized in the following proposition:

Proposition 82. *The subdirect products of concept lattices $\underline{\mathfrak{B}}(G_t, M_t, I_t)$ are in one-to-one correspondence to the closed relations J of the sum context $\sum_{t \in T}(G_t, M_t, I_t)$ with $J \cap G_t \times M_t = I_t$ for all $t \in T$.* □

We want to describe such relations J more precisely. For this purpose we need the notion of a *bond* between contexts, which will be treated in more detail in section 7.2. To simplify the formulation we introduce an abbreviated notation: If $\mathbb{K}_s := (G_s, M_s, I_s)$ and $\mathbb{K}_t := (G_t, M_t, I_t)$ are contexts and if $J_{st} \subseteq G_s \times M_t$ is a relation, then for $X \subseteq G_s$, $Y \subseteq M_t$ we write

$$X^t \quad \text{instead of} \quad X^{J_{st}} \qquad \text{and} \qquad Y^s \quad \text{instead of} \quad Y^{J_{st}}.$$

Definition 69. A **bond** from a context $\mathbb{K}_s := (G_s, M_s, I_s)$ to a context $K_t := (G_t, M_t, I_t)$ is a relation $J_{st} \subseteq G_s \times M_t$ for which the following is true:

- g^t is an intent of $\mathbb{K}_t$ for every object $g \in G_s$
- m^s is an extent of $\mathbb{K}_s$ for every attribute $m \in M_t$. ◇

A bond can be well illustrated in the imagery of the cross tables by writing the two contexts diagonally below each other and entering the bond in the right upper quadrant (a bond from $\mathbb{K}_t$ to $\mathbb{K}_s$ can be entered into the left lower quadrant). Each row of J_{st} has to give an intent of $\mathbb{K}_t$ and each column of J_{st} has to give an extent of $\mathbb{K}_s$.

$\mathbb{K}_s$	J_{st}
	$\mathbb{K}_t$

Proposition 83. *If J_{rs} is a bond from $\mathbb{K}_r$ to $\mathbb{K}_s$ and if J_{st} is a bond from $\mathbb{K}_s$ to $\mathbb{K}_t$, then for $g \in G_r$, $m \in M_t$ the following holds:*

$$m \in g^{sst} \iff m^s \supseteq g^{ss} \iff g^s \supseteq m^{ss} \iff g \in m^{ssr}.$$

Proof. (cf. Figure 5.1.) The equivalence in the middle

$$m^s \supseteq g^{ss} \iff g^s \supseteq m^{ss}$$

follows immediately from Proposition 10 (p. 18), because g^s is an intent and m^s is an extent of $\mathbb{K}_s$. The other two equivalences are dual to each other and result easily from the definitions. We have for example

$$\begin{aligned} m \in g^{sst} &\iff m \in g^{ss\prime} && (\text{since } m \in M_t) \\ &\iff m' \supseteq g^{ss} \\ &\iff m^s \supseteq g^{ss} && (\text{since } g^{ss} \subseteq G_s \text{ and } m^s = m' \cap G_s). \end{aligned}$$

□

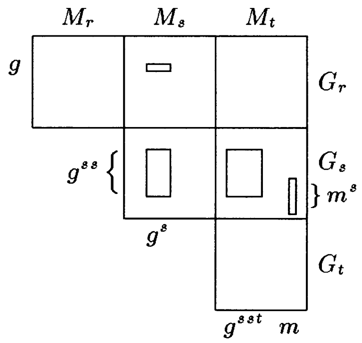


Figure 5.1 With reference to the proof of Proposition 83.

Proposition 84. *If J_{rs} is a bond from $\mathbb{K}_r$ to $\mathbb{K}_s$ and J_{st} is a bond from $\mathbb{K}_s$ to $\mathbb{K}_t$, then*

$$J_{rs} \circ J_{st} := \{(g,m) \in G_r \times M_t \mid g^{ss} \subseteq m^s\}$$

is a bond from $\mathbb{K}_r$ to $\mathbb{K}_t$. For an arbitrary bond J_{rt} from $\mathbb{K}_r$ to $\mathbb{K}_t$, $J_{rt} \subseteq J_{rs} \circ J_{st}$ if and only if

$$g^t \subseteq g^{sst} \text{ for all } g \in G_r$$

or, equivalently, if $m^r \subseteq m^{ssr}$ for all $m \in M_t$.

Proof. According to Proposition 83

$$\{m \in M_t \mid (g,m) \in J_{rs} \circ J_{st}\} = \{m \in M_t \mid g^{ss} \subseteq m^s\} = g^{sst},$$

for fixed $g \in G_r$, i.e., this set is an intent of $\mathbb{K}_t$. Dually we show that

$$\{g \in G_r \mid (g,m) \in J_{rs} \circ J_{st} = m^{ssr}\}$$

holds for every $m \in M_t$. Thus, $J_{rs} \circ J_{st}$ is a bond. The second assertion of the proposition follows from the same argument. □

We can visualize the definition of $J_{rs} \circ J_{st}$ without falling back on the contexts $\mathbb{K}_r$ and $\mathbb{K}_t$, since we have

$$J_{rs} \circ J_{st} = \bigcup_{(A,B) \in \underline{\mathfrak{B}}(\mathbb{K}_s)} B^r \times A^t,$$

which can be easily interpreted with the help of Figure 5.2:

We add (g,m) to $J_{rs} \circ J_{st}$ whenever there is a concept (A,B) of $\mathbb{K}_s$ satisfying $B \subseteq g^s$ and $A \subseteq m^s$. Therefore, $\mathbb{K}_s$ is a dense subcontext of the context in Figure 5.2.

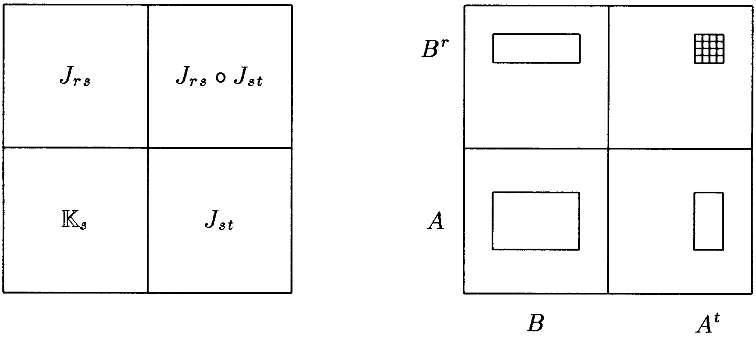


Figure 5.2 With reference to the definition of $J_{rs} \circ J_{st}$.

Proposition 85. *If $J \subseteq I$ is a subrelation in the sum context $(G, M, I) := \sum_{t \in T} \mathbb{K}_t$ with the property that*

$$I_t = J_{tt} := J \cap G_t \times M_t$$

holds for all $t \in T$, then the following statements are equivalent

1. *J is a closed relation.*
2. *The $J_{st} := J \cap (G_s \times M_t)$ are bonds and*

$$J_{rt} \subseteq J_{rs} \circ J_{st}$$

holds for all $r, s, t \in T$.

Proof. If J is closed, then, for every object $g \in G$, (g^J, g^{JJ}) is a concept of $\sum \mathbb{K}_t$. We learn from Theorem 31 that the subcontexts $\mathbb{K}_s$ are all compatible, and draw the two following conclusions:

First, $g^s = g^J \cap M_s$ must be an intent of $\mathbb{K}_s$. This, together with the dual argument for attributes, shows that all J_{st} must be bonds.

Second, $(g^{JJ} \cap G_s, g^J \cap M_s)$ must be a concept of $\mathbb{K}_s$. Hence, $g^{JJ} \cap G_s = g^{ss}$ and in particular $g^{ss} \subseteq g^{JJ}$, from which we can infer

$$g^{sst} \supseteq g^{JJt} = g^t$$

for arbitrary $t \in T$. This holds for every object g and, according to Proposition 84, implies $J_{rt} \subseteq J_{rs} \circ J_{st}$. Hence, we have proved (2).

If, on the other hand, we presuppose (2), then we can use Proposition 47 (p. 114) to prove that J is a closed relation. Hence, let $(g, m) \in I \setminus J$, which because of $I_t = J_{tt}$ implies that $g \in G_r$ and $m \in M_s$ hold for suitable $r \neq s$. Since J_{rs} is a bond from $\mathbb{K}_r$ to $\mathbb{K}_s$, g^s is an intent of $\mathbb{K}_s$. Therefore, there must be an object $h \in g^{ss}$ with $(h, m) \notin I$. According to Proposition 84 we have

$$g^t \subseteq g^{sst} \subseteq h^t$$

for every $t \in T$, and consequently $g^J \subseteq h^J$. This, together with the dual argument, shows that the condition of Proposition 47 is satisfied: J is closed. □

The Propositions 82 and 85 can be summarized as follows:

Theorem 32. *For a subrelation $J \subseteq I$ in the sum context $(G, M, I) := \sum_{t \in T} \mathbb{K}_t$, the following statements are equivalent:*

1. *J is a closed relation and corresponds to a subdirect product of the $\underline{\mathfrak{B}}(\mathbb{K}_t)$, $t \in T$.*
2. *J is a closed relation and $I_t = J_{tt} \quad (:= J \cap G_t \times M_t)$ holds for all $t \in T$.*
3. *The $J_{st} := J \cap G_s \times M_t$ are bonds from $\mathbb{K}_s$ to $\mathbb{K}_t$ with $J_{tt} = I_t$ and $J_{rt} \subseteq J_{rs} \circ J_{st}$ for all $r, s, t \in T$.*

□

Note that the last point of the theorem does not contain the requirement that J is closed. The latter follows (as in Proposition 85) as a consequence if J is made up of bonds, as specified above. If the contexts $\mathbb{K}_t$ are all reduced, then (G, M, J) is reduced as well. This follows from Proposition 48 (p. 114).

In order to be able to use the subdirect product as a construction method, we introduce the following notion:

Definition 70. If P is a set, $\boldsymbol{V}$ is a complete lattice and $\alpha : P \to \boldsymbol{V}$ is a map, then we call $(\boldsymbol{V}, \alpha)$ a (complete) ***P*-lattice** if $\boldsymbol{V}$ is generated by $\{\alpha p \mid p \in P\}$.

When $P := \{1, 2, \ldots, n\}$ we also speak of a (complete) n-lattice. If $(P, \leq)$ is an ordered set, then we call $(\boldsymbol{V}, \alpha)$ a (complete) **$(P, \leq)$-lattice** if α is furthermore order-preserving.

To a family $(\boldsymbol{V}_t, \alpha_t)$, $t \in T$, of complete P-lattices we can naturally assign a complete sublattice of the direct product $\times_{t \in T} \boldsymbol{V}_t$, namely the sublattice which is generated by the elements

$$\alpha p := (\alpha_t p \mid t \in T), \qquad p \in P.$$

We call this lattice the ***P*-product** of the lattices $(\boldsymbol{V}_t, \alpha_t)$. As a symbol for P-products of two lattices we use $\overset{P}{\times}$ or $\overset{n}{\times}$, respectively. ◇

Example 9. In Figure 5.3 a 4-product of three small chains is represented as a sublattice of the direct product. The elements of the set $P = \{1, 2, 3, 4\}$ are written below those lattice elements on which they are mapped by α. In the diagram on the right, only the elements represented by small circles belong to the 4-product; the additional lines have been drawn to indicate the situation of the lattice within the direct product.

If the lattices involved are concept lattices, we use the following obvious terms: $(\mathbb{K}, \alpha)$ is called ***P*-context**, if $(\underline{\mathfrak{B}}(\mathbb{K}), \alpha)$ is a P-lattice. In this case, α

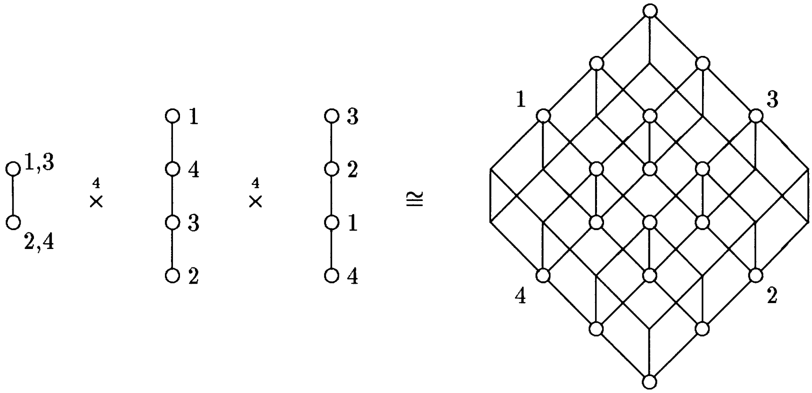


Figure 5.3 A simple example of a 4-product.

maps the elements of P onto concepts of $\mathbb{K}$; for those images we most often write $(A^p, B^p) := \alpha p$. Then we call $(\underline{\mathfrak{B}}(\mathbb{K}), \alpha)$ "the concept lattice of the P-context $(\mathbb{K}, \alpha)$", for short.

Evidently, the P-product is a complete subdirect product, since the canonical projection π_t maps the generating system $\{\alpha p \mid p \in P\}$ of the P-product onto $\{\alpha_t p \mid p \in P\}$, i.e., onto a generating system of $\boldsymbol{V}_t$. Therefore, π_t must be surjective. Hence, according to Theorem 32, to the P-product of P-concept lattices there corresponds in a natural way a closed relation J in the sum context. This is described more precisely in the following theorem, in which we shall again use the abbreviation $J_{st} := J \cap G_s \times M_t$:

Theorem 33. *The closed relation J of the sum context $\sum_{t \in T} \mathbb{K}_t$ which belongs to a P-product of P-concept lattices $(\underline{\mathfrak{B}}(\mathbb{K}_t), \alpha_t)$ is characterized by the following properties:*

1. *for all $t \in T$, $J_{tt} = I_t$,*
2. *for all $s, t \in T, s \neq t$, J_{st} is the smallest bond from $\mathbb{K}_s$ to $\mathbb{K}_t$ which contains the sets*
$$A_s^p \times B_t^p, \qquad p \in P,$$
A_s^p and B_t^p being defined to be $\alpha_s p =: (A_s^p, B_s^p)$ and $\alpha_t p =: (A_t^p, B_t^p)$.

Proof. From Definition 69 it immediately follows that the intersection of bonds is a bond. Therefore, there is always a smallest bond from $\mathbb{K}_s$ to $\mathbb{K}_t$, $s \neq t$ which entirely contains the sets

$$A_s^p \times B_t^p, \qquad p \in P.$$

If we denote this bond by J_{st} and set $J_{tt} := I_t$, then obviously

$$J := \bigcup_{s,t \in T} J_{st}$$

is precisely the relation which is characterized by the two conditions of the theorem. We shall show first that J satisfies condition 3) of Theorem 32.

For this purpose we consider, for fixed $r, s, t \in T$, $p \in P$, an attribute $m \in B_t^p$. Because of $A_s^p \times B_t^p \subseteq J_{st}$ we certainly have $m^s \supseteq A_s^p$. Likewise, because of $A_r^p \times B_s^p \subseteq J_{rs}$, we can infer for each object that $g \in A_r^p$ $g^s \supseteq B_s^p$ and, since (A_s^p, B_s^p) is a concept of $\mathbb{K}_s$, even $g^{ss} \subseteq A_s^p$. Hence, we have $g^{ss} \subseteq A_s^p \subseteq m^s$ and obtain, together with Proposition 84,

$$\begin{aligned}(g, m) \in A_r^p \times B_t^p \quad &\Rightarrow \quad g^{ss} \subseteq m^s \\ &\Rightarrow \quad (g, m) \in J_{rs} \circ J_{st}.\end{aligned}$$

Since this is correct for all $g \in A_r^p$ and all $m \in B_t^p$, we have

$$A_s^p \times B_t^p \subseteq J_{rs} \circ J_{st}.$$

Proposition 84 furthermore states that $J_{rs} \circ J_{st}$ is a bond. Since we have assumed that J_{rt} is the smallest bond containing all those sets, it follows that

$$J_{rt} \subseteq J_{rs} \circ J_{st}.$$

Thus, the third condition of Theorem 32 is satisfied and consequently J is the closed relation of a subdirect product of the $\underline{\mathfrak{B}}(\mathbb{K}_t)$.

It remains to be shown that J is the right closed relation, i.e., that it really corresponds to the P-product specified. This is generated by the elements $\{\alpha p \mid p \in P\}$; the corresponding concepts in the sum context are

$$(A^p, B^p), \qquad A^p := \bigcup A_t^p, \quad B^p := \bigcup B_t^p, \quad t \in T.$$

J contains all sets $A^p \times B^p$, and every closed relation containing those sets belongs to a subdirect product and therefore has to satisfy the third condition of Theorem 32. J is the smallest closed relation for which this is true, and is therefore, according to Proposition 45 (p. 113), the closed relation belonging to the sublattice generated by (A^p, B^p). □

The context construction described in Theorem 33, which corresponds to the P-product, is here given a name:

Definition 71. The ***P*-fusion** of a family $(\mathbb{K}_t, \alpha_t)$, $t \in T$ of P-contexts is the P-context

$$((G, M, J), \alpha),$$

in the case that $J \subseteq I$ is the subrelation in the sum context $(G, M, I) := \sum_{t \in T} \mathbb{K}_t$ which is characterized by the conditions of Theorem 33 and in the case that α is the map defined as follows: If for any $t \in T$ $\alpha_t p =: (A_t^p, B_t^p)$, then

$$\alpha p := (\dot{\bigcup_{t \in T}} A_t^p, \dot{\bigcup_{t \in T}} B_t^p).$$

In the case of two P-context we use

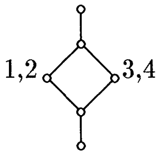
$$(\mathbb{K}_1, \alpha_1) \overset{P}{+} (\mathbb{K}_2, \alpha_2)$$

as the symbol for the P-fusion. ◇

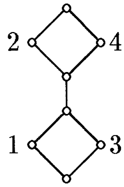
Then, of course we have:

Corollary 86. *The concept lattice of a P-fusion of contexts is isomorphic to the P-product of its concept lattices.* □

Example 10. We calculate the 4-product of the two 4-lattices

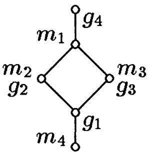


and



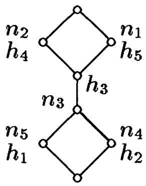
.

In this case, the method used in the preceding example would lead to complicated intermediate steps. Therefore, we determine the corresponding 4-standard contexts and obtain:



	m_1	m_2	m_3	m_4
g_1	×	×	×	
g_2	×	×		
g_3	×		×	
g_4				

$\alpha_1(1) = \alpha_1(2) = (\{g_1, g_2\}, \{m_1, m_2\})$
$\alpha_1(3) = \alpha_1(4) = (\{g_1, g_3\}, \{m_1, m_3\})$



	n_1	n_2	n_3	n_4	n_5
h_1	×	×	×		×
h_2	×	×	×	×	
h_3	×	×			
h_4		×			
h_5	×				

$\alpha_2(1) = (\{h_1\}, \{n_1, n_2, n_3, n_5\})$
$\alpha_2(2) = (\{h_1, h_2, h_3, h_4\}, \{n_2\})$
$\alpha_2(3) = (\{h_2\}, \{n_1, n_2, n_3, n_4\})$
$\alpha_2(4) = (\{h_1, h_2, h_3, h_5\}, \{n_1\})$

Now we form the 4-fusion of these two contexts, as described in Theorem 33, i.e., we form the disjoint union of the two contexts and add the sets $\{g_1, g_2\} \times \{n_1, n_2, n_3, n_5\}$, $\{g_1, g_2\} \times \{n_2\}$, $\{g_1, g_3\} \times \{n_1, n_2, n_3, n_4\}$, $\{g_1, g_3\} \times \{n_1\}$ for $J_{1,2}$ as well as the sets $\{h_1\} \times \{m_1, m_2\}$, $\{h_1, h_2, h_3, h_4\} \times \{m_1, m_2\}$, $\{h_2\} \times \{m_1, m_3\}$, $\{h_1, h_2, h_3, h_5\} \times \{m_1, m_3\}$ for $J_{2,1}$ to the incidence. In the present case this has already resulted in bonds. In general, the incidence must be extended until bonds are obtained. As a result we obtain the following 4-context:

	m_1	m_2	m_3	m_4	n_1	n_2	n_3	n_4	n_5
g_1	×	×	×		×	×	×	×	×
g_2	×	×			×	×	×		×
g_3	×		×		×	×	×	×	
g_4									
h_1	×	×	×		×	×	×		×
h_2	×	×	×		×	×	×	×	
h_3	×	×	×		×	×			
h_4	×	×				×			
h_5	×		×		×				

$$\alpha(1) = (\{g_1, g_2, h_1\}, \{m_1, m_2, n_1, n_2, n_3, n_5\})$$
$$\alpha(2) = (\{g_1, g_2, h_1, h_2, h_3, h_4\}, \{m_1, m_2, n_2\})$$
$$\alpha(3) = (\{g_1, g_3, h_2\}, \{m_1, m_3, n_1, n_2, n_3, n_4\})$$
$$\alpha(4) = (\{g_1, g_3, h_1, h_2, h_3, h_5\}, \{m_1, m_3, n_1\})$$

The corresponding concept lattice is isomorphic to the 4-product we have been looking for. It is presented in Figure 5.4.

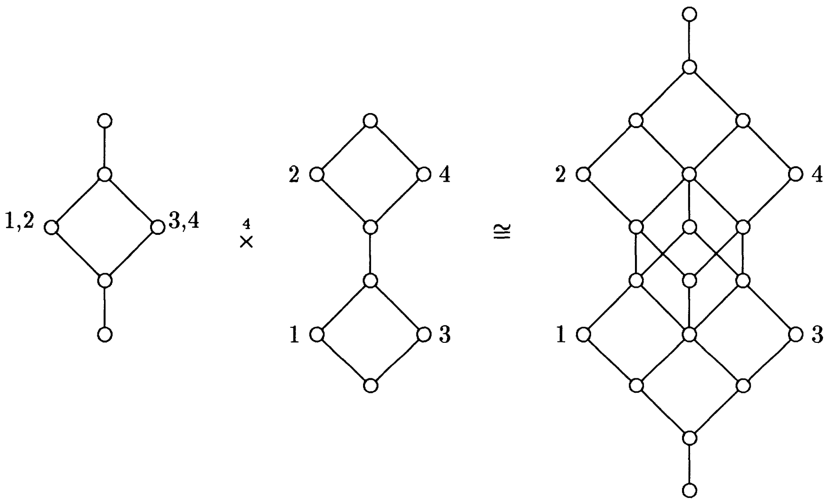


Figure 5.4 The 4-lattice on the right is the concept lattice of the 4-fusion calculated in Example 10 and is consequently isomorphic to the 4-product of the factors on the left.

For the special case that the contexts $\mathbb{K}_t$ have the same objects and attributes, there is a natural choice for the set P. In the following definition we presuppose merely for reasons of convenience that G and M are disjoint.

Definition 72. Contexts $\mathbb{K}_t := (G, M, I_t)$ with a fixed object set G and a fixed attribute set M can be interpreted as P-contexts $(\mathbb{K}_t, \alpha_t)$ with $P := G \dot\cup M$ and

$$\begin{array}{rcll} \alpha_t g & := & (g^{tt}, g^t) \in \underline{\mathfrak{B}}(\mathbb{K}_t) & \text{for } g \in G, \\ \alpha_t m & := & (m^t, m^{tt}) \in \underline{\mathfrak{B}}(\mathbb{K}_t) & \text{for } m \in M. \end{array}$$

If $((G, M, I), \alpha)$ is the P-fusion of such a family of P-contexts, then (G, M, I) is called the **fusion** of the contexts $\mathbb{K}_t$. ◇

5.2 Gluings

The geometric nature of the lattice diagrams suggests a simple construction, namely that of putting together lattices to form larger lattices by gluing them together along common substructures. Such a possibility has already been introduced in Section 4.2, but as a decomposition principle.

Such methods do indeed play a role in the construction, but they turn out to be complicated in the details and are not always easy to manage. The same is true for the corresponding context operation, the *union*. Under suitable additional conditions, however, we obtain a smooth and practicable theory.

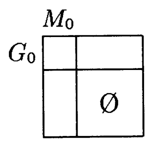
Proposition 87. *Let (G_0, G_0') and (M_0', M_0) be concepts of (G, M, I). Then*

$$(M_0', M_0) \le (X, Y) \text{ or } (X, Y) \le (G_0, G_0'),$$

holds for every (X, Y) of (G, M, I) if and only if

$$I \subseteq G \times M_0 \cup G_0 \times M.$$

Proof. "$\Rightarrow$": If $(g, m) \in I$, then $(M_0', M_0) \le (g'', g')$ (i.e., $m \in M_0$) or $(g'', g') \le (G_0, G_0')$ (i.e., $g \in G_0$). "$\Leftarrow$": If $I \subseteq G \times M_0 \cup G_0 \times M$ and (X, Y) is a concept with $X \not\subseteq G_0$, then $X' \subseteq M_0$ and therefore $(M_0', M_0) \le (X, Y)$. □



If we are confronted with the situation described in the proposition, the concept lattice is made up in a simple way of two lattices, namely of the ideal $((G_0, G_0')]$ and the filter $[(M_0', M_0))$, which overlap in the (possibly empty) interval $[(M_0', M_0), (G_0, G_0')]$. We speak of the **Hall-Dilworth gluing** (cf. Definition 60 on page 141), in the special case $(G_0, G_0') = (M_0', M_0)$ of the **vertical sum**, which has already been mentioned on page 41. There we also introduced the **horizontal sum**, where two lattices are "glued together sideways" by identifying the two largest and the two smallest elements. In Section 4.3 we showed that the substitution product generalizes both constructions.

Whereas the Hall-Dilworth gluing is the simplest case of an atlas-construction in the sense of Section 4.2, another way of generalizing the horizontal sum suggests itself, namely the *horizontal gluing*, where we allow the lattices involved to overlap in more than the two border elements. The general situation, namely the situation that a concept lattice is the union of sublattices, is treated in the next (rather trivial) proposition:

Proposition 88. *For relations $J_t \subseteq G \times M$, $t \in T$, the following statements are equivalent:*

1. $\mathfrak{B}(G, M, \bigcup_{t\in T} J_t) = \bigcup_{t\in T} \mathfrak{B}(G, M, J_t)$.
2. *The J_t are closed relations of $(G, M, \bigcup_{t\in T} J_t)$ and*

$$A \times B \subseteq \bigcup_{t\in T} J_t \Rightarrow \exists_{s\in T}\ A \times B \subseteq J_s.$$

□

The proof is simple, but the result is not very rewarding. In order to obtain a condition which is easier to manage, we limit ourselves to two lattices and assume that the overlapping is the union of an ideal and a filter, i.e., that it has the form described in Proposition 87.

Definition 73. A complete lattice $\boldsymbol{V}$ is an **ideal-filter gluing** of two sublattices U_1 and U_2 if:

1. $\boldsymbol{V} = U_1 \cup U_2$
2. $x \leq y$ in $\boldsymbol{V}$ implies $\{x, y\} \subseteq U_1$ or $\{x, y\} \subseteq U_2$.
3. $U_1 \cap U_2 = (a] \cup [b)$ for suitable elements $a, b \in \boldsymbol{V}$.

◇

An ideal-filter gluing can be recognized by the context:

Theorem 34. *The following conditions are equivalent:*

1. $\underline{\mathfrak{B}}(G, M, I)$ *is an ideal-filter gluing of complete sublattices U_1 and U_2. $J_1 := C(U_1)$ and $J_2 := C(U_2)$ are the corresponding closed relations.*
2. *J_1 and J_2 are closed relations of (G, M, I) and*
   a) *$g^I = g^{J_1}$ or $g^I = g^{J_2}$ holds for every object g and, dually, $m^I = m^{J_1}$ or $m^I = m^{J_2}$ holds for every attribute m,*
   b) *there is an extent G_0 and an intent M_0 of (G, M, I) with $J_1 \cap J_2 = (G_0 \times M \cup G \times M_0) \cap I$.*

Proof. 1) $\Rightarrow$ 2a): Every object concept γg belongs to one of the sublattices, but $\gamma g \in U_i$ is equivalent to $g^I = g^{J_i}$.
1) $\Rightarrow$ 2b): By assumption $U_1 \cap U_2 = (\mathfrak{a}] \cup [\mathfrak{b})$ for suitable concepts $\mathfrak{a} =: (G_0, G_0')$ and $\mathfrak{b} =: (M_0', M_0)$. It is evident that $J_1 \cap J_2 \supseteq (G_0 \times M \cup G \times M_0) \cap I$, what remains to be shown is the other inclusion. Assume that $(g, m) \in J_1 \cap J_2$ and $m \notin M_0$. Then there are concepts $(X_1, Y_1) \in U_1$ and $(X_2, Y_2) \in U_2$ with $(g, m) \in X_i \times Y_i$. The infimum of these concepts is comparable with both of them, i.e., one of the three concepts (X_1, Y_1), (X_2, Y_2) and $(X_1, Y_1) \wedge (X_2, Y_2)$ is not in $U_1 \cap U_2$. The intent of this concept contains m, i.e., it cannot be a subset of M_0. Hence, the concept is contained in the ideal $((G_0, G_0')]$, and we obtain $X_1 \cap X_2 \subseteq G_0$, which implies $g \in G_0$.

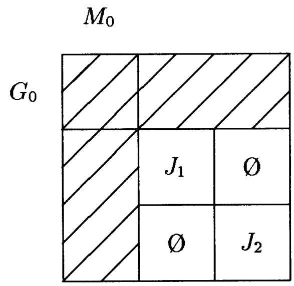


Figure 5.5 Context of an ideal-filter-gluing. In the hatched area J_1 and J_2 coincide with I; this is at the same time the closed relation belonging to the overlapping of the lattices.

2) $\Rightarrow$ 1): We have to show that all concepts and all comparabilities between concepts of (G, M, I) originate from one of the sublattices U_i. Let us assume that there is a concept $(X, Y) \in \underline{\mathfrak{B}}(G, M, I)$ that belongs neither to U_1 nor to U_2, so that neither $X \times Y \subseteq J_1$ nor $X \times Y \subseteq J_2$. Then there must be pairs $(g, m), (h, n) \in X \times Y$ with $(g, m) \in J_1 \setminus J_2$ and $(h, n) \in J_2 \setminus J_1$, and from the presuppositions it follows that $g, h \notin G_0$ and $m, n \notin M_0$. Hence, (g, n) cannot belong to $J_1 \cap J_2$, but certainly $(g, n) \in I$, since $(g, n) \in X \times Y$. From $(g, m) \in J_1 \setminus J_2$ we infer $g^I = g^{J_1}$, i.e., $(g, n) \in J_1$, and from $(h, n) \in J_2 \setminus J_1$ we dually infer $n^I = n^{J_2}$, i.e., $(g, n) \in J_2$, which is a contradiction.
If $(X, Y) \in \underline{\mathfrak{B}}(G, M, J_1)$ and $(U, V) \in \underline{\mathfrak{B}}(G, M, J_2)$ and $(U, V) \leq (X, Y)$, then $U \subseteq X$ and $V \subseteq Y$, i.e., $U \times Y \subseteq J_1 \cap J_2$. This implies $U \subseteq G_0$ (i.e., $(U, V) \in \underline{\mathfrak{B}}(G, M, J_1)$) or $Y \subseteq M_0$ (i.e., $(X, Y) \in \underline{\mathfrak{B}}(G, M, J_2)$), in any case $\{(U, V), (X, Y)\} \subseteq U_i$ for $i = 1$ or $i = 2$. □

The characterization in Theorem 34 leads the way to the corresponding context construction. We define

Definition 74. The **union** of two contexts $\mathbb{K}_1 := (G_1, M_1, I_1)$ and $\mathbb{K}_2 := (G_2, M_2, I_2)$ is the context

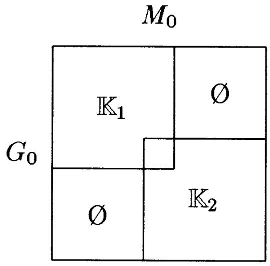
$$\mathbb{K}_1 \cup \mathbb{K}_2 := (G_1 \cup G_2, M_1 \cup M_2, I_1 \cup I_2).$$

We call $\mathbb{K}_1 \cup \mathbb{K}_2$ a **gluing** of the contexts $\mathbb{K}_1$ and $\mathbb{K}_2$ if the following conditions are satisfied:

1. $G_0 := G_1 \cap G_2$ is an extent of $\mathbb{K}_1 \cup \mathbb{K}_2$.
2. $M_0 := M_1 \cap M_2$ is an intent of $\mathbb{K}_1 \cup \mathbb{K}_2$.
3. $I_0 := I_1 \cap I_2 = I_1 \cap G_0 \times M_0 = I_2 \cap G_0 \times M_0$.

◇

The context gluing is not the exact counterpart of the lattice gluing. The preconditions are weaker. The condition that $\mathbb{K}_1$ and $\mathbb{K}_2$ coincide on $G_0 \times M_0$, does not at all enforce that the extents contained in G_0 and the intents contained in M_0 are also the same in both contexts. This is however necessarily true in the case of an ideal-filter gluing. Therefore, it is rather surprising that the following theorem holds true. There is a snag in the theorem, which we have to point out. It says that the concept lattice of the gluing of two contexts $\mathbb{K}_1$ and $\mathbb{K}_2$ is the ideal-filter gluing of two sublattices, but it does *not* say that those sublattices are isomorphic to $\mathfrak{B}(\mathbb{K}_1)$ and $\mathfrak{B}(\mathbb{K}_2)$. In fact, this is generally not the case.



Theorem 35. *$\mathfrak{B}(\mathbb{K})$ is the ideal-filter gluing of two complete sublattices U_1 and U_2 with*

$$U_1 \cap U_2 = ((G_0, G_0')] \cup [(M_0', M_0))$$

if and only if $\mathbb{K}$ is the gluing of two subcontexts $\mathbb{K}_1 := (G_1, M_1, I_1)$ and $\mathbb{K}_2 := (G_2, M_2, I_2)$ with

$$G_0 = G_1 \cap G_2 \quad \textit{and} \quad M_0 = M_1 \cap M_2.$$

Proof. One direction immediately follows from Theorem 34: If $\mathbb{K} := (G, M, I)$ satisfies condition 2 of Theorem 34, then with

$$\begin{array}{rrcl} & G_1 & := & \{g \in G \mid g^I = g^{J_1}\}, \\ & M_1 & := & \{m \in M \mid m^I = m^{J_1}\}, \\ & G_2 & := & G_0 \cup (G \setminus G_1), \\ & M_2 & := & M_0 \cup (M \setminus M_1) \\ \text{and} & I_1 & := & I \cap G_1 \times M_1, \\ & I_2 & := & I \cap G_2 \times M_2 \end{array}$$

we evidently obtain contexts $\mathbb{K}_1$ and $\mathbb{K}_2$ wit $\mathbb{K} = \mathbb{K}_1 \cup \mathbb{K}_2$ and $G_0 = G_1 \cap G_2$, $M_0 = M_1 \cap M_2$ and $I_0 = I_1 \cap I_2 = I_1 \cap G_0 \times M_0 = I_2 \cap G_0 \times M_0$. Since G_0 is an extent and M_0 is an intent of $\mathbb{K}$, $\mathbb{K}$ is a gluing of $\mathbb{K}_1$ and $\mathbb{K}_2$.

For the opposite direction we have to show that $\mathbb{K}$ satisfies the second condition of Theorem 34. For this purpose we set

$$\begin{array}{rcl} J_1 & := & I_1 \cup I \cap (G_0 \times M \cup G \times M_0) \\ J_2 & := & I_2 \cup I \cap (G_0 \times M \cup G \times M_0). \end{array}$$

Then, we have to prove that J_1 and J_2 are closed relations. We show this for $J := J_1$, with the help of Proposition 46 (p. 113): Let $X \subseteq G$ be arbitrary. If $X \not\subseteq G_0$, then $X^J \subseteq M_1$ and because of

$$J \cap G \times M_1 = I \cap G \times M_1$$

it follows that $X^{JJ} = X^{II}$. If $X \subseteq G_0$, then $X^J = X^I$ and therefore $X^{JJ} \subseteq X^{JI} = X^{II} \subseteq G_0$, since G_0 is an extent of $\mathbb{K}$. Because of $J \cap G_0 \times M = I_0 \cap G_0 \times M$ and $X^{II} \subseteq G_0$ we have $X^{IJ} = X^{II}$, i.e., $X^{JJ} = X^{II} = X^{JI}$. □

The closed relations corresponding to the sublattices have been specified in the proof. They coincide with I_1 resp. I_2 up to a modification "below G_0" and "above M_0". This modification does not apply if I_1 is dense in J_1 and I_2 is dense in J_2. This is the case if G_0 is an extent of $\mathbb{K}_1$ as well as of $\mathbb{K}_2$ and if furthermore every subset $T \subseteq G_0$ satisfies

$$T \text{ is extent of } \mathbb{K}_1 \iff T \text{ is extent of } \mathbb{K}_2$$

and the corresponding is true for M_0. Those conditions have the effect that the ideals generated by the concept with the extent G_0 are isomorphic in $\underline{\mathfrak{B}}(\mathbb{K}_1)$ and $\underline{\mathfrak{B}}(\mathbb{K}_1)$ and that the same is true for the filters of the concepts with the intents $\subseteq M_0$. The fact that $\mathbb{K}_1$ and $\mathbb{K}_2$ coincide in I_0 implies that those isomorphisms can be generated by a single map.

In particular, we have: An ideal-filter gluing of two concept lattices is isomorphic to the concept lattice of the gluing of the contexts involved.

In practice, the task that usually crops up is the slightly generalized one of having to glue two lattices together which do not have elements in common, but in which an isomorphism of the ideal-filter pair of one lattice onto a corresponding pair of the other lattice is given. In order to implement this construction for concept lattices, one first modifies the respective contexts $\mathbb{K}_1$ and $\mathbb{K}_2$ in such a way that both the object concepts in the two ideals and the attribute concepts in the two filters coincide. This can be achieved through mutual enrichment, and in the case of doubly founded contexts even through reduction. The objects and attributes of these concepts are given the same name, if they are mapped onto each other by the isomorphism. For the remainder one makes the two contexts disjoint. The concept lattice of the gluing of the contexts modified in this way is then the ideal-filter gluing of suitable isomorphic copies of their concept lattices, as desired.

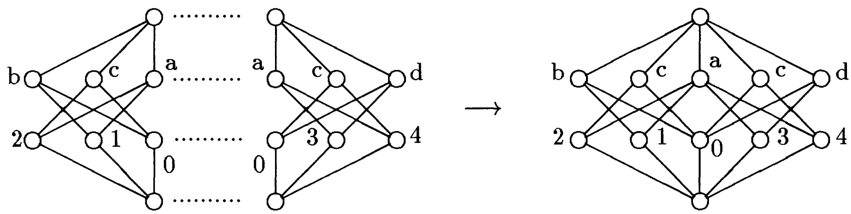


Figure 5.6 Ideal-filter-gluing of two cubes

	a	b	c
0		×	×
1	×	×	
2	×		×

$\mathbb{K}_1$

$\cup$

	a	d	e
0		×	×
3	×	×	
4	×		×

$\mathbb{K}_2$

$=$

	a	b	c	d	e
0		×	×	×	×
1	×	×			
2	×		×		
3	×			×	
4	×				×

$\mathbb{K}_1 \cup \mathbb{K}_2$

Figure 5.7 The context gluing belonging to Figure 5.6.

As an example, Figure 5.6 shows an ideal-filter gluing of two Boolean lattices. The corresponding context gluing is presented in Figure 5.7. We recognize that $\mathbb{K}_1 \cup \mathbb{K}_2$ is furthermore a context sum and that consequently the lattice generated by the gluing is a direct product. In general, for context gluings we have:

$$(\mathbb{K}_0 + \mathbb{K}_1) \cup (\mathbb{K}_0 + \mathbb{K}_2) = \mathbb{K}_0 + (\mathbb{K}_1 \cup \mathbb{K}_2).$$

5.3 Local Doubling

A further construction principle consists in suitably doubling a part of a lattice, for example an interval. We first describe the context construction and then derive the corresponding lattice construction.

A context manipulation which has no influence at all on the structure of the concept lattice is the "inverse reduction", i.e., the addition of reducible attributes or objects: To a context (G, M, I) we add for example a set N of new attributes and supplement the relation I in such a manner that for every $n \in N$, the attribute extent n' is a extent of (G, M, I). In this case, (G, M, I) is a dense subcontext of the new context and, consequently, the concept lattices are isomorphic. The same is true, if instead we add a set H of new objects to (G, M, I) and make sure that every such object $h \in H$ is reducible with respect to G, i.e., that h' is an intent of (G, M, I).

However, if we carry out both extensions simultaneously, the concept lattice changes considerably. A first clue in this connection is contained in the next few propositions.

Proposition 89. *Let $(G \cup H, M \cup N, J)$ be a context with $G \cap H = M \cap N = \emptyset$ and $J \cap H \times N = \emptyset$. The subcontext (G, M, I) with $I := J \cap G \times H$ satisfies the following conditions:*

1. *for every object $h \in H$, h' is an intent of (G, M, I),*
2. *for every attribute $n \in N$, n' is an extent of (G, M, I)*

if and only if (G, M, I) is compatible.

This can be *proved* without effort by means of conditions a1) and a2) of Proposition 35 (p. 100). □

Hence, under the conditions of the proposition, the map $\Pi_{G,M}$ with

$$(A,B) \mapsto (A \cap G, B \cap M)$$

is a surjective complete homomorphism (Proposition 34, p. 100), which, as the next proposition shows, has small pre-image sets:

Proposition 90. *Let (C,D) be a concept of (G,M,I). Then there is at least one and at most two concepts (A,B) of $(G \cup H, M \cup N, J)$ with $(C,D) = (A \cap G, B \cap M)$, namely*

$$(C, C^J) \quad or \quad (D^J, D).$$

(C, C^J) is a concept of $(G \cup H, M \cup N, J)$ if and only if

there is an attribute $n \in N$ with $C \subseteq n^J$
or there is no object $h \in H$ with $D \subseteq h^J$.

Proof. Because of $J \cap H \times N = \emptyset$, one of the possibilities $A \subseteq G$ or $B \subseteq M$ holds for every concept (A,B) of $(G \cup H, M \cup N, J)$, which, under the condition that $(C,D) = (A \cap G, B \cap M)$, implies

$$A = C \text{ (and thus } B = C^J) \quad \text{or} \quad B = D \text{ (and thus } A = D^J).$$

(C, C^J) is not a concept, if $C^J \subseteq M$ (i.e., $C^J = C^I = D$), but $D^J \not\subseteq G$ (i.e., $D^J \neq C$). For D we argue correspondingly. □

The proposition states the possible pre-images for a concept $(C,D) \in \mathfrak{B}(G,M,I)$ in a somewhat tricky formulation. Therefore, we repeat the description of the different cases in the form of a table:

$C \subseteq n^J$ for some $n \in N$	$D \subseteq h^J$ for some $h \in H$	(C, C^J) is a concept	(D^J, D) is a concept	equal ?
yes	yes	yes	yes	$(C, C^J) < (D^J, D)$
yes	no	yes	no	
no	yes	no	yes	
no	no	yes	yes	$(C, C^J) = (D^J, D)$

Thus it is only in the first case that (C,D) has two different pre-images with respect to $\Pi_{G,M}$. Additionally, we note down:

Proposition 91. *If $(G \cup H, M \cup N, J)$ is a context with the properties specified in the preceding proposition, then:*
(A,B) is a concept of $(G \cup H, M \cup N, J)$ if and only if $(A \cap G, B \cap M)$ is a concept of (G,M,I) and we are dealing with one of the following three cases

1. $A \subseteq G, B \subseteq M$, $A = B^J$, $B = A^J$

2. $A \subseteq G, B = A^J \not\subseteq M$
3. $B \subseteq M, A = B^J \not\subseteq G$.

Proof. According to Proposition 89, for every concept (A, B) of $(G \cup H, M \cup N, J)$, the restriction $(A \cap G, B \cap M)$ is a concept of (G, M, I), and, since we have presupposed that $J \cap H \times N = \emptyset$, it follows that $A \times B \cap H \times N = \emptyset$, i.e., $A \subseteq G$ or $B \subseteq N$, and thus one of the cases 1)–3) must hold.

If, conversely, A is an extent of (G, M, I) and $A^J \not\subseteq M$, then (A, A^J) by the preceding proposition is a concept of $(G \cup H, M \cup N, J)$. If $A^J = A^I$, then $B = A^J$ is an intent of (G, M, I), and, under the condition that $B^J \neq B^I$, we can argue dually. What remains is the trivial case $A^I = A^J$ and $B^I = B^J$. □

In the case of a doubly founded context, it is particularly easy to check whether the conditions of Proposition 89 are satisfied. We can apply Proposition 36 (p. 101) and obtain a condition which is easy to manage algorithmically.

	M	N
G		$\not\nearrow$
H	$\not\swarrow$	$\emptyset$

Proposition 92. *A doubly founded context* $(G \cup H, M \cup N, J)$ *with* $G \cap H = \emptyset$ *and* $M \cap N = \emptyset$ *has the properties specified in Proposition 89 if and only if the following conditions are satisfied:*

1. $J \cap H \times N = \emptyset$,

2. $h \swarrow m$, $h \in H$ *together imply* $m \in N$,

3. $g \nearrow n$, $n \in N$ *together imply* $g \in H$.

□

We have made no restrictions concerning the choice of the sets H and N. However, it turns out that we can make a very special choice without loss of generality.

Definition 75. Assume that $\mathfrak{C} \subseteq \mathfrak{B}(G, M, I)$ is a convex set of concepts and w.l.o.g. that $\mathfrak{C} \cap (G \cup M) = \emptyset$. Then

$$\mathbb{K}[\mathfrak{C}] := (G \cup \mathfrak{C}, M \cup \mathfrak{C}, I_{\mathfrak{C}}),$$

$I_{\mathfrak{C}}$ being defined as follows:

$$I_{\mathfrak{C}} \cap G \times M := I, \quad I_{\mathfrak{C}} \cap \mathfrak{C} \times \mathfrak{C} := \emptyset$$

and, for $(C, D) \in \mathfrak{C}$, $g \in G$ and $m \in M$,

$$\begin{aligned} g\, I_{\mathfrak{C}}\, (C, D) \quad &:\Longleftrightarrow \quad g \in C, \\ (C, D)\, I_{\mathfrak{C}}\, m \quad &:\Longleftrightarrow \quad m \in D. \end{aligned}$$

◇

	M	N
G	I	
H		$\emptyset$

$(G \cup H, M \cup N, J)$

	M	$\mathfrak{C}$
G	I	$\in$
$\mathfrak{C}$	$\ni$	$\emptyset$

$\mathbb{K}[\mathfrak{C}]$

Figure 5.8 In the case of the doubling construction H and N can be replaced by the same convex set $\mathfrak{C}$.

Evidently, the context defined in this way satisfies the conditions from Proposition 89, $\mathfrak{C}$ assuming the role of H as well as of N.

Proposition 93. *If we define $\mathfrak{C} \subseteq \mathfrak{B}(G, M, I)$ by specifying for $(C, D) \in \mathfrak{B}(G, M, I)$ that*

$$(C, D) \in \mathfrak{C} \quad :\Longleftrightarrow \quad \exists_{h \in H} \exists_{n \in N} \; C \subseteq n^J \text{ and } D \subseteq h^J,$$

then $\mathfrak{C}$ is convex and

$$\underline{\mathfrak{B}}(G \cup H, M \cup N, J) \cong \underline{\mathfrak{B}}(\mathbb{K}[\mathfrak{C}]).$$

Proof. Let $H_0 := \{h \in H \mid h^{JIJ} \cap N \neq \emptyset\}$. N_0 is defined dually. If $h \in H \setminus H_0$, then $h^J = (h^{JI})^J$, i.e., h is reducible. $(G \cup H_0, M \cup N_0, J)$ is dense in $(G \cup H, M \cup N, J)$ and therefore has the same concept lattice up to an isomorphism. Therefore, we may presuppose $H = H_0$, $N = N_0$. This simplifies the argumentation, since we have for every $h \in H$ some $n \in N$ with $h^{JI} \subseteq n^J$, from which we may infer that the concept $\gamma_I(h) := (h^{JI}, h^J)$ of (G, M, I) belongs to $\mathfrak{C}$. Likewise, we may assign the concept $\mu_I(n) := (n^J, n^{JI}) \in \mathfrak{C}$ to every attribute $n \in N$. In the context $\mathbb{K}[\mathfrak{C}]$ those concepts are objects and attributes. For the object $\gamma_I(h)$ we have

$$\gamma_I(h)^{I_{\mathfrak{C}}} = h^J,$$

and for the attribute $\mu_I(n)$ we have

$$\mu_I(n)^{I_{\mathfrak{C}}} = n^J.$$

If (A, B) is a concept of (G, M, I) and $h \in H$, then we have

$$B \subseteq h^J \iff B \subseteq \gamma_I(h)^{I_{\mathfrak{C}}}$$

and dually. Since furthermore for every concept $(C, D) \in \mathfrak{C}$ by the definition of $\mathfrak{C}$ there exists some $h \in H$ and some $n \in N$ with $\mu_I(n) \leq (C, D) \leq \gamma_I(h)$, we have

$$\exists_{h \in H} \; B \subseteq h^J \iff \exists_{\mathfrak{c} \in \mathfrak{C}} \; B \subseteq \mathfrak{c}^{I_{\mathfrak{C}}},$$

and the dual statement, which implies

$$A^J \cap H \neq \emptyset \iff A^{I_{\mathfrak{C}}} \cap \mathfrak{C} \neq \emptyset \quad \text{and} \quad B^J \cap N \neq \emptyset \iff B^{I_{\mathfrak{C}}} \cap \mathfrak{C} \neq \emptyset.$$

Now we define for $(A, B) \in \mathfrak{B}(G \cup H, M \cup N, J)$

$$\varphi(A, B) := \begin{cases} (A, A^{I_{\mathfrak{C}}}) & \text{if } A \subseteq G \\ (B^{I_{\mathfrak{C}}}, B) & \text{if } B \subseteq M \end{cases},$$

and claim that we have thereby defined an isomorphism

$$\varphi : \underline{\mathfrak{B}}(G \cup H, M \cup N, J) \to \underline{\mathfrak{B}}(\mathbb{K}[\mathfrak{C}]).$$

First of all, we note that $\varphi(A, B)$ by Proposition 91 is defined for every concept $(A, B) \in \underline{\mathfrak{B}}(G \cup H, M \cup N, J)$. By means of the equivalence proved above and again by means of Proposition 91 we conclude that $\varphi(A, B)$ is in fact always a concept of $\underline{\mathfrak{B}}(\mathbb{K}[\mathfrak{C}])$ and even that every such concept occurs. Hence, φ is a bijection. The fact that φ is also an order isomorphism is elementary because of the simple shape of the concepts involved. □

Thanks to Proposition 92 we may concentrate on the context construction $\mathbb{K} \mapsto \mathbb{K}[\mathfrak{C}]$, because it covers the general case. The content of Proposition 90, specialized to the context $\mathbb{K}[\mathfrak{C}]$, reads as follows:

Proposition 94. *For every concept (C, D) of $\mathbb{K}$ there is at least one and at most two concepts (A, B) of $\mathbb{K}[\mathfrak{C}]$ with $(A \cap G, B \cap M) = (C, D)$, namely*

$$(C, C^{I_{\mathfrak{C}}}) \quad or \quad (D^{I_{\mathfrak{C}}}, D).$$

$(C, C^{I_{\mathfrak{C}}})$ and $(D^{I_{\mathfrak{C}}}, D)$ are both concepts of $\mathbb{K}[\mathfrak{C}]$. They are distinct if and only if $(C, D) \in \mathfrak{C}$.

Proof. What remains to be proved is only the last sentence. By Proposition 90 there are two concepts (A, B) with $(A \cap G, B \cap M) = (C, D)$ if and only if there are elements $\mathfrak{h}, \mathfrak{n} \in \mathfrak{C}$ with $C \subseteq \mathfrak{h}^{I_{\mathfrak{C}}}$, $D \subseteq \mathfrak{n}^{I_{\mathfrak{C}}}$, i.e., $\mathfrak{h} \leq (C, D) \leq \mathfrak{n}$. Since $\mathfrak{C}$ is convex, this is equivalent to $(C, D) \in \mathfrak{C}$. □

Definition 76. For a convex subset C of a complete lattice $\boldsymbol{V} := (V, \leq)$ we define the complete lattice $\boldsymbol{V}[C] := (V[C], \leq)$ to be

$$V[C] := (V \setminus C) \cup (C \times \{0, 1\})$$

and

$$x \leq y :\iff \begin{cases} x, y \in V \setminus C \text{ and } x \leq y \text{ in } \boldsymbol{V} \\ \text{or } x \in V \setminus C, y = (y_0, i), y_0 \in C, x \leq y_0 \text{ in } \boldsymbol{V} \\ \text{or } y \in V \setminus C, x = (x_0, i), x_0 \in C, x_0 \leq y \text{ in } \boldsymbol{V} \\ \text{or } x = (x_0, i), y = (y_0, j) \in C \times \{0, 1\}, i \leq j \text{ and } x_0 \leq y_0. \end{cases}$$

◇

The assertion that a complete lattice is defined in this way requires a proof. It follows from the next theorem.

Theorem 36. *If* $\mathfrak{C} \subseteq \underline{\mathfrak{B}}(\mathbb{K})$ *is convex, then*

$$\underline{\mathfrak{B}}(\mathbb{K})[\mathfrak{C}] \cong \underline{\mathfrak{B}}(\mathbb{K}[\mathfrak{C}]).$$

Proof. We show that the rule

$$\varphi(A,B) := \begin{cases} (A\cap G, B\cap M), & \text{if } (A\cap G, B\cap M) \notin \mathfrak{C}, \\ ((A\cap G, B\cap M), 0), & \text{if } (A, B\cap M) \in \mathfrak{C}, \\ ((A\cap G, B\cap M), 1), & \text{if } (A\cap G, B) \in \mathfrak{C}, \end{cases}$$

defines an isomorphism

$$\varphi : \underline{\mathfrak{B}}(\mathbb{K}[\mathfrak{C}]) \to \underline{\mathfrak{B}}(\mathbb{K})[\mathfrak{C}].$$

By Proposition 91, for every concept (A,B) of $\mathbb{K}[\mathfrak{C}]$, at least one of $(A\cap G, B)$ and $(A, B\cap M)$ is always a concept of $\mathbb{K}$ and, by Proposition 94, a concept (C,D) of $\mathbb{K}$ has two pre-images under $\Pi_{G,M}$ if and only if $(C,D) \in \mathfrak{C}$.

Therefore, φ is a bijective map. It remains to be shown that φ is also an order isomorphism. We have

$$\begin{aligned}
(A_1,B_1) < (A_2,B_2) &\iff A_1 \subset A_2 \\
&\iff A_1\cap G \subset A_2 \cap G \text{ or} \\
&\qquad A_1 \cap G = A_2 \cap G \text{ and } A_1 \subset A_2 \\
&\iff A_1\cap G \subset A_2\cap G \text{ or } (A_1\cap G, B_1 \cap M) \in \mathfrak{C} \\
&\iff \varphi(A_1,B_1) < \varphi(A_2,B_2).
\end{aligned}$$

□

In practice, we would if possible reduce the context $\mathbb{K}[\mathfrak{C}]$. A look at the arrow relations shows us how: If $\mathfrak{c} \in \mathfrak{C}$ is an object of $\mathbb{K}[\mathfrak{C}]$, then by Proposition 92 $\mathfrak{c} \swarrow \mathfrak{d}$ can only hold for one attribute $\mathfrak{d} \in \mathfrak{C}$. We discover quickly that precisely the minimal resp. maximal elements of $\mathfrak{C}$ are irreducible. If we define

$$\mathfrak{C}_{min} := \{\mathfrak{c} \in \mathfrak{C} \mid \mathfrak{c} \text{ is minimal in } \mathfrak{C}\}$$

$$\text{and } \mathfrak{C}_{max} := \{\mathfrak{c} \in \mathfrak{C} \mid \mathfrak{c} \text{ is maximal in } \mathfrak{C}\},$$

then for $\mathfrak{c}, \mathfrak{d} \in \mathfrak{C}$ we have

$$\mathfrak{c} \swarrow \mathfrak{d} \iff \mathfrak{c} \in \mathfrak{C}_{min} \text{ and } \mathfrak{c} \leq \mathfrak{d},$$

$$\mathfrak{c} \nearrow \mathfrak{d} \iff \mathfrak{d} \in \mathfrak{C}_{max} \text{ and } \mathfrak{c} \leq \mathfrak{d}.$$

If, therefore, we assume that $\mathfrak{C}$ has enough minimal and maximal elements, i.e., that

$$\mathfrak{C} = \bigcup\{[\mathfrak{c}, \mathfrak{d}] \mid \mathfrak{c} \in \mathfrak{C}_{min}, \mathfrak{d} \in \mathfrak{C}_{max}, \mathfrak{c} \leq \mathfrak{d}\},$$

then $\mathbb{K}[\mathfrak{C}]$ is doubly founded (provided that $\mathbb{K}$ is doubly founded), and the context $\mathbb{K}[\mathfrak{C}]$ has (up to isomorphism) the same concept lattice as

$$\mathbb{K}[\mathfrak{C}]_r := (G \cup \mathfrak{C}_{min}, M \cup \mathfrak{C}_{max}, I_{\mathfrak{C}} \cap (G \cup \mathfrak{C}_{min}) \times (M \cup \mathfrak{C}_{max})).$$

Particularly simple is $\mathbb{K}[\mathfrak{C}]_r$ in the case of the **interval doubling**, that is in the case that $\mathfrak{C}$ is an interval $\mathfrak{C} = [(B', B), (C, C')]$ of $\underline{\mathfrak{B}}(\mathbb{K})$, because in this case $\mathfrak{C}_{min}$ and $\mathfrak{C}_{max}$ are both one-element sets and we have

$$\mathbb{K}[\mathfrak{C}]_r = (G \cup \{(B', B)\}, M \cup \{(C, C')\}, J)$$

with

$$J \cap G \times M = I, \quad (B', B)^J := B \quad \text{and} \quad (C, C')^J := C.$$

We note this down as a proposition:

Proposition 95. *A doubly founded context is of the form $\mathbb{K}[\mathfrak{C}]$ for an interval $\mathfrak{C} \subseteq \underline{\mathfrak{B}}(\mathbb{K})$, if and only if there is an object h and an attribute n with*

$$h \swarrow\!\!\!\nearrow n, \quad g \swarrow n \Rightarrow g = h, \quad h \nearrow m \Rightarrow m = n.$$

Because in this case with $H := \{h\}$ and $N := \{n\}$ the conditions of Proposition 92 are evidently satisfied. □

Example 11. We consider the possible bracketings of a product $x_0 x_1 \cdots x_n$ of $n+1$ variables $x_0, \ldots, x_n$. Since the names of the variables are of no consequence, we replace them by dots. Thus, (..)((..).) stands for $(x_0 x_1)((x_2 x_3) x_4)$, etc. We can order these bracketings by agreeing that a term becomes larger if subterms are replaced according to the rule

$$A(BC) \longrightarrow (AB)C.$$

Tamari [173] observed that this induces an order which turns the set of all bracketings of $n + 1$ symbols into a lattice; this lattice is therefore called the **Tamari lattice** $\mathbb{T}_n$. Bennett and Birkhoff [13] have determined the irreducibles of these lattices. This makes it possible to state a (reduced) context for the Tamari lattice $\mathbb{T}_n$. With $S := \{1, 2, \ldots, n\}$ and $\mathfrak{P}_2(S) := \{\{i, j\} \mid i, j \in S, i \neq j\}$ this is the context

$$(\mathfrak{P}_2(S), \mathfrak{P}_2(S), I),$$

the incidence I for $i < j$ and $p < q$ being defined by

$$\{i, j\} I \{p, q\} :\Longleftrightarrow \text{ not } (p \leq i < q \leq j).$$

Geyer [73] has stated a recursion rule for these contexts, which can be recognized by means of the example $n = 5$ in Figure 5.9.

	$\begin{smallmatrix}1\\2\end{smallmatrix}$	$\begin{smallmatrix}2\\3\end{smallmatrix}$	$\begin{smallmatrix}1\\3\end{smallmatrix}$	$\begin{smallmatrix}3\\4\end{smallmatrix}$	$\begin{smallmatrix}2\\4\end{smallmatrix}$	$\begin{smallmatrix}1\\4\end{smallmatrix}$	$\begin{smallmatrix}4\\5\end{smallmatrix}$	$\begin{smallmatrix}3\\5\end{smallmatrix}$	$\begin{smallmatrix}2\\5\end{smallmatrix}$	$\begin{smallmatrix}1\\5\end{smallmatrix}$
1 2	$\swarrow\!\!\!\!\nearrow$	×	×	×	×	×	×	×	×	×
2 3	×	$\swarrow\!\!\!\!\nearrow$	$\swarrow$	×	×	×	×	×	×	×
1 3	$\nearrow$	×	$\swarrow\!\!\!\!\nearrow$	×	×	×	×	×	×	×
3 4	×	×	×	$\swarrow\!\!\!\!\nearrow$	$\swarrow$	$\swarrow$	×	×	×	×
2 4	×	$\nearrow$		×	$\swarrow\!\!\!\!\nearrow$	$\swarrow$	×	×	×	×
1 4	$\nearrow$	×	$\nearrow$	×	×	$\swarrow\!\!\!\!\nearrow$	×	×	×	×
4 5	×	×	×	×	×	×	$\swarrow\!\!\!\!\nearrow$	$\swarrow$	$\swarrow$	$\swarrow$
3 5	×	×	×	$\nearrow$			×	$\swarrow\!\!\!\!\nearrow$	$\swarrow$	$\swarrow$
2 5	×	$\nearrow$		×	$\nearrow$		×	×	$\swarrow\!\!\!\!\nearrow$	$\swarrow$
1 5	$\nearrow$	×	$\nearrow$	×	×	$\nearrow$	×	×	×	$\swarrow\!\!\!\!\nearrow$

Figure 5.9 The reduced context belonging to the Tamari lattice $\mathbb{T}_5$.

The context shows a salient structure of the arrow relations: The (square) cross table can be arranged in such a way that all double arrows are on the main diagonal, all upward arrows are above and all downward arrows are below the main diagonal. In this particular case the "lowest" object h and the corresponding attribute with regard to $\swarrow\!\!\!\!\nearrow$ n evidently satisfy the conditions 95. This means that the context is generated by interval doubling from the subcontext obtained by omitting h and n.

However, this subcontext has again the same structure of the arrow relations. Hence, the procedure can be repeated until there remains nothing. This means that the Tamari lattice can be generated by iterated interval doubling from the one-element lattice. Figure 5.10 shows the Tamari lattice $\mathbb{T}_4$ including its "genesis": At the edges, we have noted at which stage of the iterated interval doubling they have been generated. In descending order, congruences arise, which gradually factorize the lattice until a one-element lattice is reached.

5.4 Tensorial Constructions

By means of the distributive law introduced in 4.4 it is possible to formulate a lattice-theoretic characterization of the tensor product which does not fall back upon the notion of a context. We shall only treat the case of tensor products with two factors, the general case is not essentially different but one needs time to grow accustomed to it.

Theorem 37. *The tensor product $\mathbf{V}_1 \otimes \mathbf{V}_2$ has the following properties:*

$\otimes 1$*) $\mathbf{V}_1 \otimes \mathbf{V}_2$ is a complete lattice, and*

$$\varepsilon_1 : \mathbf{V}_1 \hookrightarrow \mathbf{V}_1 \otimes \mathbf{V}_2, \qquad \varepsilon_2 : \mathbf{V}_2 \hookrightarrow \mathbf{V}_1 \otimes \mathbf{V}_2$$

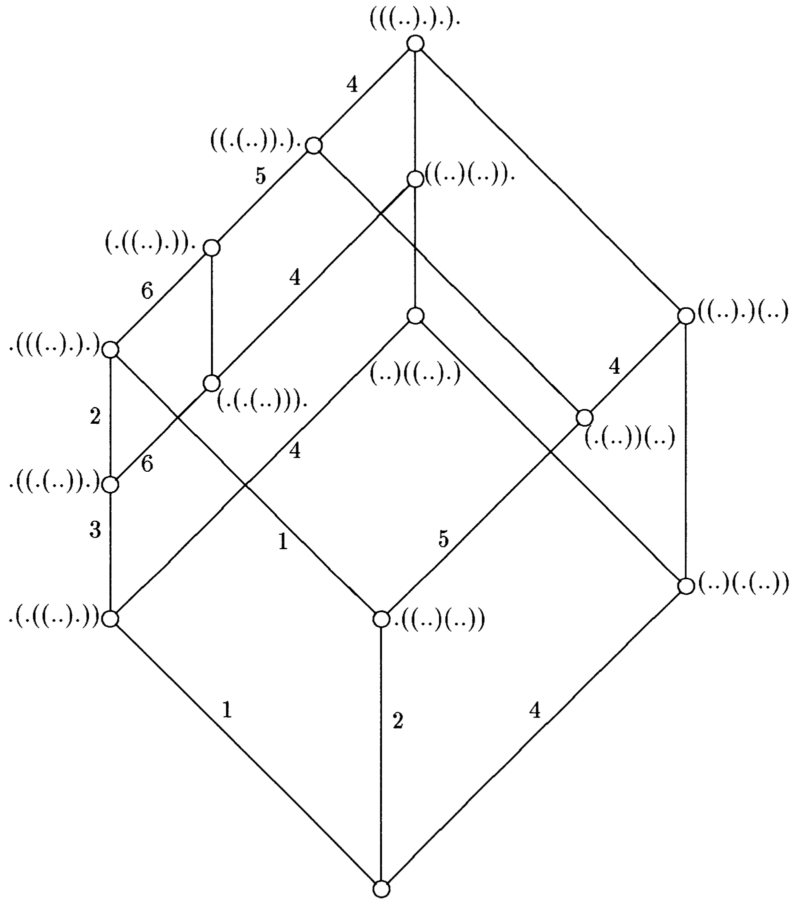


Figure 5.10 The Tamari lattice $\mathbb{T}_4$. The numbers at the edges indicate the recursive structure of the lattice, which was generated by interval doubling.

are complete lattice embeddings.

$\otimes 2$) *The complete sublattices $\varepsilon_1(\boldsymbol{V}_1)$ and $\varepsilon_2(\boldsymbol{V}_2)$ are mutually distributive.*

$\otimes 3$) *If $\boldsymbol{V}$ is a complete lattice satisfying $\otimes 1$) and $\otimes 2$), i.e., if there are embeddings*

$$\alpha_1 : \boldsymbol{V}_1 \hookrightarrow \boldsymbol{V} \qquad \text{and} \qquad \alpha_2 : \boldsymbol{V}_2 \hookrightarrow \boldsymbol{V}$$

with the property that the complete sublattices $\alpha_1(\boldsymbol{V}_1)$ and $\alpha_2(\boldsymbol{V}_2)$ are mutually distributive, then there is a complete homomorphism

$$\varphi : \boldsymbol{V}_1 \otimes \boldsymbol{V}_2 \to \boldsymbol{V}$$

with

$$\alpha_1 = \varphi \circ \varepsilon_1, \qquad \text{and} \qquad \alpha_2 = \varphi \circ \varepsilon_2.$$

$\otimes 4$) *The union $\varepsilon_1(\boldsymbol{V}_1) \cup \varepsilon_2(\boldsymbol{V}_2)$ of the two sublattices generates $\boldsymbol{V}_1 \otimes \boldsymbol{V}_2$.*

By these properties the tensor product is characterized up to isomorphism.

Proof. The properties $\otimes 1$), $\otimes 2$) and $\otimes 4$) have already been proved. We can easily see that the properties $\otimes 1$)–$\otimes 4$) are characteristic, since for every lattice with these properties, from $\otimes 3$) we immediately obtain an isomorphism to the tensor product.

What remains to be shown is $\otimes 3$). Hence, let $\boldsymbol{V}$ be a lattice with the properties specified in $\otimes 3$). First, we work out the following sub-claim: *For every subset $X \subseteq \boldsymbol{V}_1 \times \boldsymbol{V}_2$ we have*

$$\bigvee_{(x_1,x_2)\in X} (\alpha_1(x_1) \wedge \alpha_2(x_2)) = \bigwedge_{(y_1,y_2)\in X^\nabla} (\alpha_1(y_1) \vee \alpha_2(y_2)).$$

For this purpose, we make use of the condition that the two image sets are mutually distributive and obtain:

$$\begin{aligned} \bigvee_{(x_1,x_2)\in X} (\alpha_1(x_1) \wedge \alpha_2(x_2)) &= \bigwedge_{R\subseteq X} \Big(\bigvee_{x_1\in R} \alpha_1(x_1) \vee \bigvee_{x_2\in X\setminus R} \alpha_2(x_2)\Big) \\ &= \bigwedge_{R\subseteq X} \Big(\alpha_1\Big(\bigvee_{x_1\in R} x_1\Big) \vee \alpha_2\Big(\bigvee_{x_2\in X\setminus R} x_2\Big)\Big). \end{aligned}$$

By means of the notations $y_1^R := \bigvee_{x_1\in R} x_1$ and $y_2^R := \bigvee_{x_2\in X\setminus R} x_2$ we simplify this to

$$\bigvee_{(x_1,x_2)\in X} (\alpha_1(x_1) \wedge \alpha_2(x_2)) = \bigwedge_{R\subseteq X} (\alpha_1(y_1^R) \vee \alpha_2(y_2^R)).$$

Every element of X belongs to R or $X \setminus R$, therefore, either its first component must be $\leq y_1^R$ or its second component $\leq y_2^R$, in any case we have $(x_1, x_2)\nabla(y_1^R, y_2^R)$ for all $(x_1, x_2) \in X$, and consequently $(y_1^R, y_2^R) \in X^\nabla$, independent of R. This proves

$$\bigwedge_{R\subseteq X} (\alpha_1(y_1^R) \vee \alpha_2(y_2^R)) \geq \bigwedge_{(y_1,y_2)\in X^\nabla} (\alpha_1(y_1) \vee \alpha_2(y_2)).$$

If, on the other hand, for $(y_1, y_2) \in X^\nabla$ we specifically choose $R := \{(x_1, x_2) \in X \mid x_1 \leq y_1\}$, then $X \setminus R \subseteq \{(x_1, x_2) \in X \mid x_2 \leq y_2\}$ and therefore $\alpha_1(y_1^R) \vee \alpha_2(y_2^R) \leq \alpha_1(y_1) \vee \alpha_2(y_2)$, from which it follows that

$$\bigwedge_{R\subseteq X} (\alpha_1(y_1^R) \vee \alpha_2(y_2^R)) \leq \bigwedge_{(y_1,y_2)\in X^\nabla} (\alpha_1(y_1) \vee \alpha_2(y_2))$$

and thus the sub-claim. Now we define a map $\varphi : \boldsymbol{V}_1 \otimes \boldsymbol{V}_2 \to \boldsymbol{V}$ by

$$\varphi(A, B) := \bigvee_{(x_1,x_2)\in A} \alpha_1(x_1) \wedge \alpha_2(x_2) = \bigwedge_{(y_1,y_2)\in B} \alpha_1(y_1) \vee \alpha_2(y_2).$$

We have to show that φ is a complete homomorphism. Because of the symmetry of the definition it suffices to prove the property "$\bigvee$-preserving". For this purpose we use the sub-claim and obtain for an arbitrary subset $\{(A_t, B_t) \mid t \in T\} \subseteq \boldsymbol{V}_1 \otimes \boldsymbol{V}_2$

$$\begin{aligned}
\bigvee_{t\in T} \varphi(A_t, B_t) &= \bigvee_{t\in T} \bigvee_{(x_1,x_2)\in A_t} (\alpha_1(x_1) \wedge \alpha_2(x_2)) \\
&= \bigvee_{(x_1,x_2)\in\bigcup A_t} (\alpha_1(x_1) \wedge \alpha_2(x_2)) \\
&= \bigwedge_{(y_1,y_2)\in(\bigcup A_t)^\nabla} (\alpha_1(x_1) \vee \alpha_2(x_2)) \\
&= \bigwedge_{(y_1,y_2)\in\bigcap B_t} (\alpha_1(x_1) \vee \alpha_2(x_2)) \\
&= \varphi(\bigvee_{t\in T} (A_t, B_t)),
\end{aligned}$$

as desired.

Finally, we have to examine the connection between the maps α_i and ε_i. For this purpose we recall the definition of the ε_i (in particular of ε_1), from which it follows that, for an arbitrary $x \in \boldsymbol{V}_1$, the extent of $\varepsilon_1(x)$ is given by $\{(x_1, x_2) \in V_1 \times V_2 \mid x_1 \leq x \text{ or } x_2 = 0\}$. Thereby we obtain

$$\begin{aligned}
\varphi(\varepsilon_1(x)) &= \bigvee_{\substack{x_1\leq x\\ x_2\leq 1}} (\alpha_1(x_1) \wedge \alpha_2(x_2)) \vee \bigvee_{\substack{x_1\leq 1\\ x_2\leq 0}} (\alpha_1(x_1) \wedge \alpha_2(x_2)) \\
&= \alpha_1(x) \wedge \alpha_2(1) = \alpha_1(x).
\end{aligned}$$

□

The maps γ and μ mapping onto the object and attribute concepts, respectively, are related to the ε_t. We have $\gamma(x_1, x_2) =$

$(\{(g_1, g_2) \mid g_1 \leq x_1 \text{ and } g_2 \leq x_2\} \cup M^\nabla, \{(m_1, m_2) \mid x_1 \leq m_1 \text{ or } x_2 \leq m_2\})$,

and $\mu(x_1, x_2) =$

$(\{(g_1, g_2) \mid g_1 \leq x_1 \text{ or } g_2 \leq x_2\}, G^\nabla \cup \{(m_1, m_2) \mid x_1 \leq m_1 \text{ and } x_2 \leq m_2\})$,

and therefore

$$\gamma(x_1, x_2) = \varepsilon_1(x_1) \wedge \varepsilon_2(x_2) \quad \text{and} \quad \mu(x_1, x_2) = \varepsilon_1(x_1) \vee \varepsilon_2(x_2).$$

It has proved worthwhile to introduce special symbols for these maps.

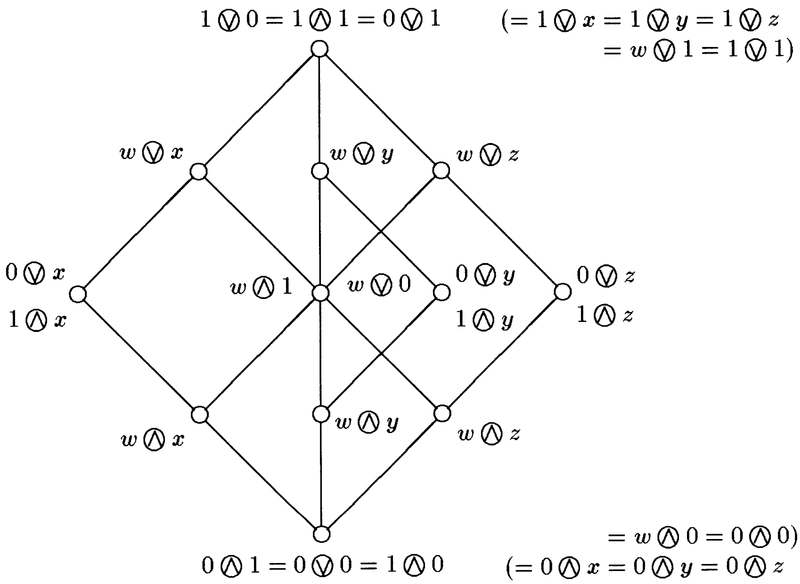
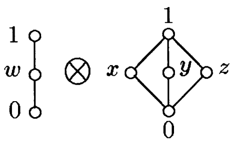


Figure 5.11 

with the tensorial operations.

Definition 77. If V_1 and V_2 are complete lattices, then the **tensorial operations**

$$\textcircled{\scriptsize$\wedge$} : V_1 \times V_2 \to V_1 \otimes V_2, \quad \text{and} \quad \textcircled{\scriptsize$\vee$} : V_1 \times V_2 \to V_1 \otimes V_2$$

are defined by

$$x_1 \owedge x_2 := \gamma(x_1, x_2) = \varepsilon_1(x_1) \wedge \varepsilon_2(x_2),$$
$$x_1 \ovee x_2 := \mu(x_1, x_2) = \varepsilon_1(x_1) \vee \varepsilon_2(x_2). \qquad \diamond$$

Proposition 96. *The tensorial operations satisfy the following arithmetic rules:*

$$x_1 \owedge 1 = \varepsilon_1(x_1) = x_1 \ovee 0, \qquad 1 \owedge x_2 = \varepsilon_2(x_2) = 0 \ovee x_2,$$

$$x_1 \owedge x_2 = x_1 \ovee 0 \wedge 0 \ovee x_2, \qquad x_1 \ovee x_2 = x_1 \owedge 1 \vee 1 \owedge x_2,$$

$$\bigwedge_{s\in S} x_1^s \owedge x_2^s = (\bigwedge_{s\in S} x_1^s) \owedge (\bigwedge_{s\in S} x_2^s), \qquad \bigvee_{s\in S} x_1^s \ovee x_2^s = (\bigvee_{s\in S} x_1^s) \ovee (\bigvee_{s\in S} x_2^s),$$

$$\bigvee_{s\in S} x_1 \owedge x_2^s = x_1 \owedge (\bigvee_{s\in S} x_2^s), \qquad \bigwedge_{s\in S} x_1 \ovee x_2^s = x_1 \ovee (\bigwedge_{s\in S} x_2^s),$$

$$\bigvee_{s\in S} x_1^s \owedge x_2 = (\bigvee_{s\in S} x_1^s) \owedge x_2, \qquad \bigwedge_{s\in S} x_1^s \ovee x_2 = (\bigwedge_{s\in S} x_1^s) \ovee x_2,$$

$$\bigwedge_{s\in S} (x_1^s \ovee x_2^s) = \bigvee_{R\subseteq S} ((\bigwedge_{r\in R} x_1^r \owedge 1) \wedge (\bigwedge_{s\in S\setminus R} 1 \owedge x_2^s)),$$

$$\bigvee_{s\in S} (x_1^s \owedge x_2^s) = \bigwedge_{R\subseteq S} ((\bigvee_{r\in R} x_1^r \ovee 0) \vee (\bigvee_{s\in S\setminus R} 0 \ovee x_2^s)).$$

Proof. All these rules result immediately from the definitions, apart from the last two, for which we have to consult Proposition 76: Because of $x_1 \owedge x_2 = \varepsilon_1(x_1) \wedge \varepsilon_2(x_2)$, $x_1 \ovee x_2 = \varepsilon_1(x_1) \vee \varepsilon_2(x_2)$ and the rules mentioned in the first line, the equations are precisely the translation of the circumstance that the sublattices $\varepsilon_1(\mathbf{V}_1)$ and $\varepsilon_2(\mathbf{V}_2)$ are mutually distributive. $\square$

Thiele [175] has impressively demonstrated how to obtain readable diagrams of tensor products of small lattices. First, the idea of the P-product developed in Section 5.1 is transferred to the tensor product and it is agreed that:

Definition 78. For a P-lattice $(\mathbf{V}_1, \alpha_1)$ and a Q-lattice $(\mathbf{V}_2, \alpha_2)$ with $P \cap Q = \emptyset$,

$$(\mathbf{V}_1, \alpha_1) \otimes (\mathbf{V}_2, \alpha_2) := (\mathbf{V}, \alpha)$$

is the $P \dot\cup Q$-sublattice of $\mathbf{V}_1 \otimes \mathbf{V}_2$ for which the map

$$\alpha : P \dot\cup Q \to \mathbf{V}_1 \otimes \mathbf{V}_2$$

is defined as follows:

$$\alpha(r) := \begin{cases} \varepsilon_1 \alpha_1(r) & \text{if } r \in P, \\ \varepsilon_2 \alpha_2(r) & \text{if } r \in Q. \end{cases} \qquad \diamond$$

By means of Theorem 37 we quickly convince ourselves of the fact that this indeed defines a $P \dot\cup Q$-lattice.

If $\boldsymbol{V}_1$ and $\boldsymbol{V}_2$ are concept lattices, we can introduce the corresponding context operation:

Definition 79. For a P-context $(\mathbb{K}_1, \alpha_1)$ and a Q-context $(\mathbb{K}_2, \alpha_2)$ with $P \cap Q = \emptyset$ we define

$$(\mathbb{K}_1, \alpha_1) \times (\mathbb{K}_2, \alpha_2) := (\mathbb{K}_1 \times \mathbb{K}_2, \alpha)$$

to be the $P \dot\cup Q$-context for which the map

$$\alpha : P \dot\cup Q \to \underline{\mathfrak{B}}(\mathbb{K}_1 \times \mathbb{K}_2)$$

is explained as in Definition 78. ◇

From these two definitions it immediately follows that

$$\underline{\mathfrak{B}}((\mathbb{K}_1, \alpha_1) \times (\mathbb{K}_2, \alpha_2)) \cong (\underline{\mathfrak{B}}(\mathbb{K}_1), \alpha_1) \otimes (\underline{\mathfrak{B}}(\mathbb{K}_2), \alpha_2).$$

Thiele has shown that the product defined in this way is distributive over the P-fusion, i.e., that it is possible to transfer Proposition 16 (p. 47) to this case. This is the content of the following theorem.

Theorem 38. *If* $(\mathbb{K}_1, \alpha_1)$ *and* $(\mathbb{K}_2, \alpha_2)$ *are both* P*-contexts and if* $(\mathbb{K}_3, \alpha_3)$ *is a* Q*-context with* $P \cap Q = \emptyset$*, then*

$$((\mathbb{K}_1, \alpha_1) \overset{P}{+} (\mathbb{K}_2, \alpha_2)) \times (\mathbb{K}_3, \alpha_3)$$

$$= ((\mathbb{K}_1, \alpha_1) \times (\mathbb{K}_3, \alpha_3)) \overset{P \dot\cup Q}{+} ((\mathbb{K}_2, \alpha_2) \times (\mathbb{K}_3, \alpha_3)).$$

Proof. Both sides of the equation claimed describe closed relations of

$$\mathbb{K} := (\mathbb{K}_1 + \mathbb{K}_2) \times \mathbb{K}_3 = \mathbb{K}_1 \times \mathbb{K}_3 + \mathbb{K}_2 \times \mathbb{K}_3$$

(cf. Proposition 16). If we are able to show that the map α is also the same in both cases, nothing remains to be proved, since in this case the sublattices generated by

$$\{\alpha x \mid x \in P \dot\cup Q\}$$

and thus the corresponding closed relations must also be the same. This can be checked easily; the main problem is that of a transparent notation. We again use the abbreviations

$$(A_t^x, B_t^x) := \alpha_t x \quad \text{for } t \in \{1, 2, 3\} \text{ and } x \in P \dot\cup Q.$$

Furthermore, we write $(\mathbb{K}_{12}, \alpha_{12})$ for the P-context $(\mathbb{K}_1, \alpha_1) \overset{P}{+} (\mathbb{K}_2, \alpha_2)$ and agree on the abbreviation

$$(A_1, B_1) + (A_2, B_2) := (A_1 \cup A_2, B_1 \cup B_2).$$

Then we have

$$\alpha_{12} p = \alpha_1 p + \alpha_2 p.$$

For the embedding maps we use the symbols ε_1, ε_2, ε_{12} and ε_3 in the obvious way, in the case of ε_3, however, we have to differentiate: We write ε_3^i, if we are working in the product $\mathbb{K}_i \times \mathbb{K}_3$. For reasons of readability we presuppose that none of the contexts contains any full columns or full rows. Thus, the trivials terms M^∇ and G^∇ disappear when we evaluate the maps ε_i by means of the formula stated in Proposition 71.

For the left-hand side we obtain, if $p \in P$,

$$\begin{aligned} \alpha p &= \varepsilon_{12} \alpha_{12} p \\ &= \varepsilon_{12}(A_1^p \cup A_2^p, B_1^p \cup B_2^p) \\ &= ((A_1^p \cup A_2^p) \times G_3, (B_1^p \cup B_2^p) \times M_3) \end{aligned}$$

and for $q \in Q$

$$\begin{aligned} \alpha q &= \varepsilon_3^{12} \alpha_3 q \\ &= \varepsilon_3^{12}(A_3^q, B_3^q) \\ &= ((G_1 \cup G_2) \times A_3^q, (M_1 \cup M_2) \times B_3^q). \end{aligned}$$

On the right-hand side we calculate for $p \in P$

$$\begin{aligned} \alpha p &= \varepsilon_1 \alpha_1 p + \varepsilon_2 \alpha_2 p \\ &= (A_1^p \times G_3, B_1^p \times M_3) + (A_2^p \times G_3, B_2^p \times M_3) \\ &= ((A_1^p \cup A_2^p) \times G_3, (B_1^p \cup B_2^p) \times M_3) \end{aligned}$$

and for $q \in Q$

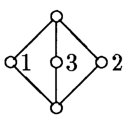
$$\begin{aligned} \alpha q &= \varepsilon_3^1 \alpha_3 q + \varepsilon_3^2 \alpha_3 q \\ &= (G_1 \times A_3^q, M_1 \times B_3^q) + (G_2 \times A_3^q, M_2 \times B_3^q) \\ &= ((G_1 \cup G_2) \times A_3^q, (M_1 \cup M_2) \times B_3^q). \end{aligned}$$

□

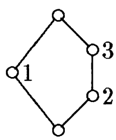
Because of the nice applications offered by this theorem, we substantiate it by several examples. The first one simply demonstrates the situation reflected by the theorem.

Example 12. Consider the two 3-contexts $(\mathbb{K}_1, \alpha_1)$ and $(\mathbb{K}_2, \alpha_2)$, which are given as follows

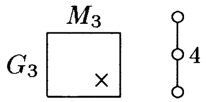
	M_1		
	×		
G_1		×	
			×



	M_2		
	×	×	
G_2	×		
			×



as well as the $\{4\}$-context $(\mathbb{K}_3, \alpha_3)$ with the following illustration:



With $P := \{1,2,3\}$ and $Q := \{4\}$ we recognize in Figure 5.12 on the left the context

$$((\mathbb{K}_1, \alpha_1) \overset{P}{+} (\mathbb{K}_2, \alpha_2)) \times (\mathbb{K}_3, \alpha_3)$$

and on the right

$$((\mathbb{K}_1, \alpha_1) \times (\mathbb{K}_3, \alpha_3)) \overset{P\dot{\cup}Q}{+} ((\mathbb{K}_2, \alpha_2) \times (\mathbb{K}_3, \alpha_3)).$$

We also recognize that both contexts are equal.

		M_1			M_2			M_1			M_2	
	×			×	×	×	×			×	×	×
G_1		×		×	×			×		×	×	
			×	×	×				×	×	×	
	×	×	×	×	×		×	×	×	×	×	
G_2	×	×	×	×			×	×	×	×		
	×					×	×					×
	×			×	×	×	×	×	×	×	×	×
G_1		×		×	×	×	×	×	×	×	×	×
			×	×	×		×	×	×	×	×	×
	×	×	×	×	×		×	×	×	×	×	×
G_2	×	×	×	×			×	×	×	×	×	×
	×					×	×	×	×	×	×	×

=

		M_1			M_1			M_2			M_2	
	×			×			×	×	×	×	×	×
G_1		×			×		×	×		×	×	
			×			×	×	×		×	×	
	×			×	×	×	×	×	×	×	×	×
G_1		×		×	×	×	×	×		×	×	×
			×	×	×	×	×	×		×	×	×
	×	×	×	×	×	×	×	×		×	×	
G_2	×	×	×	×	×	×	×			×		
	×			×					×			×
	×	×	×	×	×	×	×	×		×	×	×
G_2	×	×	×	×	×	×	×			×	×	×
	×			×	×	×			×	×	×	×

Figure 5.12 According to Theorem 38 both contexts are equal.

Corollary 97. *For two P-lattices $(\boldsymbol{V}_1, \alpha_1)$, $(\boldsymbol{V}_2, \alpha_2)$ and a Q-lattice $(\boldsymbol{V}_3, \alpha_3)$ (with $P \cap Q = \emptyset$) we have*

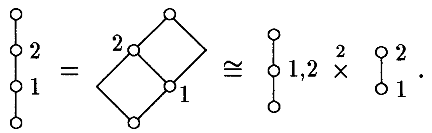
$$((\boldsymbol{V}_1, \alpha_1) \overset{P}{\times} (\boldsymbol{V}_2, \alpha_2)) \otimes (\boldsymbol{V}_3, \alpha_3)$$

$$\cong ((\boldsymbol{V}_1, \alpha_1) \otimes (\boldsymbol{V}_3, \alpha_3)) \overset{P\cup Q}{\times} (\boldsymbol{V}_2, \alpha_2) \otimes (\boldsymbol{V}_3, \alpha_3)).$$

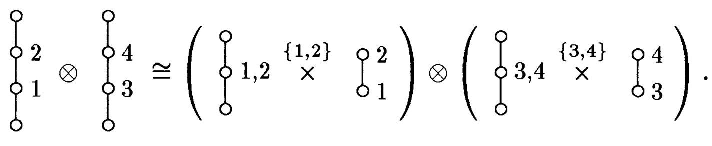
□

This corollary also goes back to Thiele. He has used it skillfully in order to draw diagrams of tensor products. As a first application we calculate the tensor product of two four-element chains.

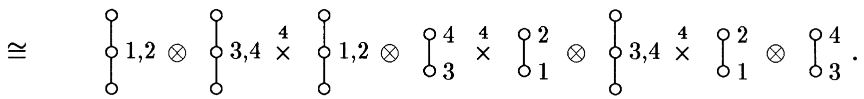
Example 13. In order to calculate the tensor product of two four-element chains, we make use of the fact that such a chain can be written as a 2-product of a two-element and a three-element chain:



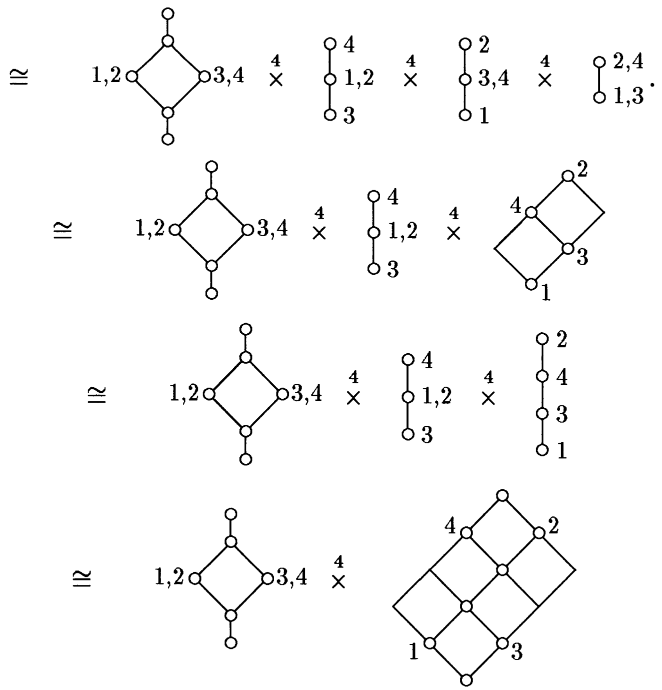
If we decompose both factors in this way, we obtain

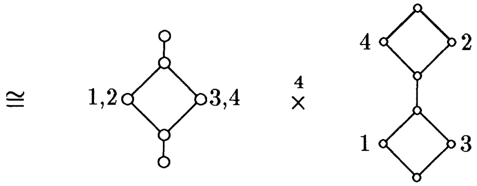


Using Corollary 97 this can be multiplied out. The convention "tensor product first, then the P-product" saves brackets. The above expression yields



As can be easily seen by means of the contexts, the tensor product of two three-element chains has six elements. With respect to the tensor product, two-element chains behave like neutral elements. Using these observations, we can convert the expression into a mere 4-product:



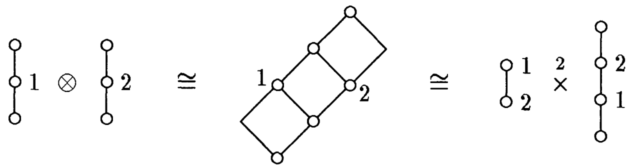


This product has already been calculated in Example 10 (p. 191). The result is presented in Figure 5.4.

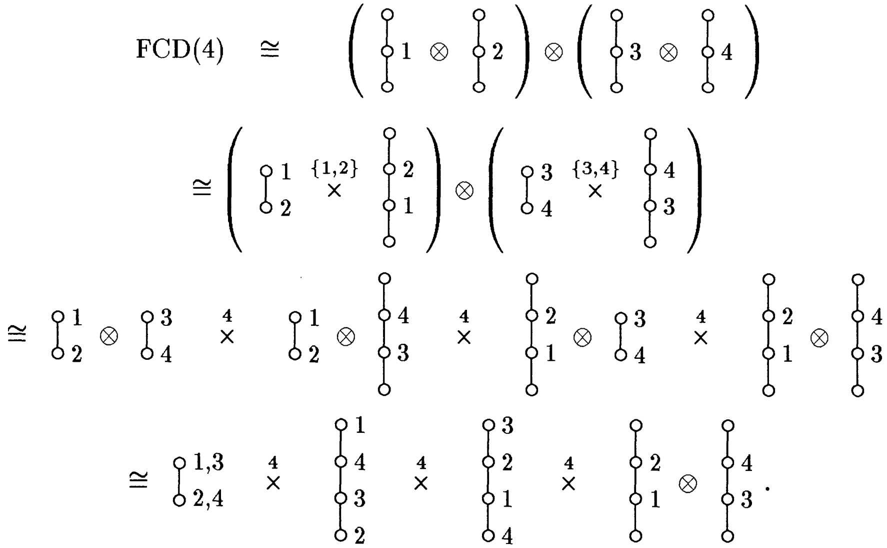
Example 14. A particularly nice application of this method is Thiele's representation of a free distributive lattice with four generators as a subdirect product. The nested line diagram obtained thereby is presented in Figure 1.20 (p. 51).

In general, it is true that FCD(n) is isomorphic to the n-th tensor power of the three-element lattice [196]. This means that FCD(4) can be obtained as the tensor product of four three-element chains.

We make use of the fact that a tensor product of two three-element chains can be rewritten as a 2-product:



Thereby we obtain



The tensor product of two four-element chains has already been calculated in Example 13; the 4-product of the remaining factors was treated in Example

9, see Figure 5.3. Hence the 4-product presented in Figure 5.13 is isomorphic to the tensor product of four three-element chains, and thus also to the free completely distributive lattice with four generators. This is how the diagram in Figure 1.20 (p. 51) has been obtained.

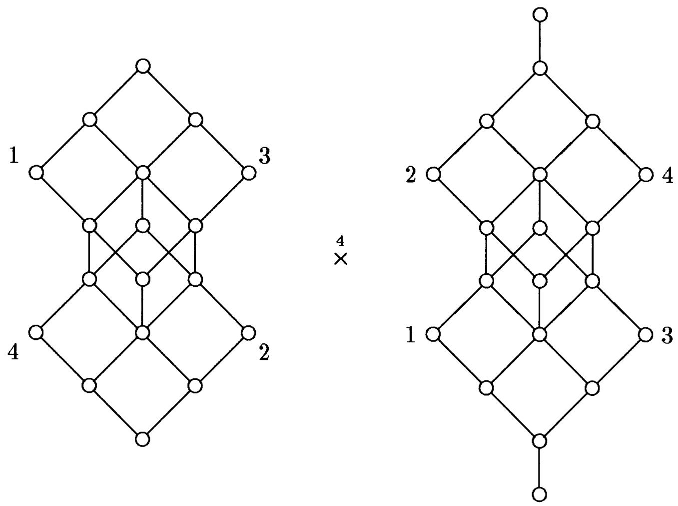


Figure 5.13 The free distributive lattice FCD(4) as a 4-product.

5.5 Hints and References

5.1 Section 5.1 follows [199] and the predecessor of this article, [194].

With regard to the role of the bond product $J_{r,s} \circ J_{s,t}$ compare also Proposition 113 (p. 256).

P-products of lattices are a long-standing subject of one of the authors of this book, see [189] and [190]. Bartenschlager [10] and Thiele [175] in their work make ample use of the P-product as a mathematical construction tool.

5.2 The results of this section are based on the doctoral thesis of S. Gürgens [80], we have, however, changed the notations. It also contains further-reaching results. Gürgens states an algorithm which determines whether $\mathbb{K}$ is the gluing of two contexts. Furthermore, she studies the simultaneous gluing of several lattices or contexts, respectively. Her model were the gluings of Boolean lattices in the theory of orthomodular lattices, cf. Greechie [77].

5.3 Local doubling was introduced by Day [32], first for intervals and then more generally. It played an important role in the framework of the examination of free lattices, see also Day [33], Nation [130] and Day, Nation &

Tschantz [35]. Day even stated a concept-analytic version of the construction of interval doubling. Our representation mainly follows Geyer [72].

The fact that the bracketings form a lattice was first published by Tamari [173]. Later, Huang and Tamari [87] gave a simpler proof. The concept-analytic investigation goes back to Geyer [73].

5.4 The direct product of contexts in particular has proved to be a natural product for conceptual scales. Products of the elementary scales have been examined and illustrated by many diagrams in Thiele [175]. This article, which has already been cited several times, also contains the result that the direct product of two closed relations is again closed. Hence, tensor products of sublattices lead to sublattices of the tensor product.

Strahringer [164] describes products of convex-ordinal scales. [200] uses the direct product for the general modeling of dependencies between many-valued attributes. Stumme [169] uses it for distributive concept exploration.

The set of all order-preserving maps from an ordered set $\mathbf{P}$ into a complete lattice $\boldsymbol{V}$ also forms a complete lattice, when ordered point-wise. This lattice is denoted by $\boldsymbol{V}^{\mathbf{P}}$. Occasionally, a formula is used which establishes a connection between this lattice and the lattice $\underline{2}^{\mathbf{P}}$ of all order-preserving maps of $\mathbf{P}$ into the two-element lattice $\underline{2}$:

$$\boldsymbol{V}^{\mathbf{P}} \cong \underline{2}^{\mathbf{P}} \otimes \boldsymbol{V}.$$

6. Properties of Concept Lattices

Mathematical lattice theory classifies lattices according to their structural properties. The most important such property, namely *distributivity*, has already been mentioned in Section 0.3 and has been used several times since then. Now we shall examine it a little more closely. For this purpose, we concentrate on doubly founded lattices, a choice that simplifies many things. Furthermore, we shall examine other interesting properties, for example *modularity* and *semimodularity*, which play a particularly important role in geometry. We shall show how *semidistributivity* and *local distributivity* can be described by means of the arrow relations and what the consequences of these properties are for the associated closure operators. The last section deals with different notions of *dimension* of lattices, in particular with that of order dimension.

6.1 Distributivity

Already in Definition 15 (p. 10) we have introduced variants of the distributive law: A complete lattice $\boldsymbol{V}$ is called **distributive** if the following two (mutually equivalent) laws

$$\begin{array}{llll} (\mathbf{D}_\wedge) & x \wedge (y \vee z) & = & (x \wedge y) \vee (x \wedge z) \\ (\mathbf{D}_\vee) & x \vee (y \wedge z) & = & (x \vee y) \wedge (x \vee z) \end{array}$$

hold, and it is called **completely distributive** if the following generalization to arbitrary infima and suprema is satisfied for all index sets $S, T \neq \emptyset$:

$$(\mathbf{D}_{\vee\wedge}) \qquad \bigwedge_{s\in S} \bigvee_{t\in T} x_{s,t} = \bigvee_{\varphi:S\to T} \bigwedge_{s\in S} x_{s,\varphi(s)}.$$

This law is also equivalent to its dual $(\mathbf{D}_{\wedge\vee})$. One direction of the law $(\mathbf{D}_{\vee\wedge})$, namely the inequality

$$\bigwedge_{s\in S} \bigvee_{t\in T} x_{s,t} \geq \bigvee_{\varphi:S\to T} \bigwedge_{s\in S} x_{s,\varphi(s)},$$

holds in every complete lattice, since, for fixed φ, $\bigwedge_{s\in S} x_{s,\varphi(s)}$ is always less than or equal to the left-hand side.

We frequently use a stricter version of the law of complete distributivity, which can however be derived from the one mentioned above. We allow the set T to vary with $s \in S$; for this purpose we replace T by a family of sets

$$\{T_s \mid s \in S\}.$$

The place of the maps $\varphi : S \to T$ is taken by the elements

$$\varphi \in \bigtimes_{s \in S} T_s$$

of the direct product of these sets (which we abbreviate as $\bigtimes T_s$). This version of the law $(\mathbf{D}_{\wedge\vee})$ then reads

$$\bigwedge_{s \in S} \bigvee_{t \in T_s} x_{s,t} = \bigvee_{\varphi \in \bigtimes T_s} \bigwedge_{s \in S} x_{s,\varphi(s)}.$$

The inequality "$\geq$" again holds in every complete lattice.

Proofs for the equivalences we have claimed can be found in the books cited on lattice theory, in particular Balbes & Dwinger [3]. The following useful characterization of distributive lattices has been taken from Birkhoff's "Lattice Theory":

Proposition 98. *A lattice is distributive if and only if $a \wedge x = a \wedge y$ and $a \vee x = a \vee y$ always imply $x = y$.* □

Examples of completely distributive complete lattices are the power-set lattices, and more generally lattices of the order ideals of ordered sets, as stated by the following well known theorem:

Theorem 39. (Theorem of Birkhoff) *If $\mathbf{D}$ is a completely distributive complete lattice in which the set $J(\mathbf{D})$ of $\bigvee$-irreducible elements is supremum-dense, then*

$$x \mapsto (x] \cap J(\mathbf{D})$$

describes an isomorphism of $\mathbf{D}$ onto the closure system of all order ideals of $(J(\mathbf{D}), \leq)$. Conversely, for every ordered set $(P, \leq)$ the closure system of all order ideals is a completely distributive lattice $\mathbf{D}$, in which

$$J(\mathbf{D}) = \{(x] \mid x \in P\}$$

is supremum-dense.

Proof. For $x \in D$, $(x] \cap J(\mathbf{D})$ is obviously an order ideal of $(J(\mathbf{D}), \leq)$. If A is an order ideal of $(J(\mathbf{D}), \leq)$ and $a := \bigvee A$, then $A \subseteq (a] \cap J(\mathbf{D})$ and even $A = (a] \cap J(\mathbf{D})$, as the following consideration shows: For $x \in D$ we have

$$x \in (a] \iff x \leq a \iff x \leq \bigvee A \iff x = x \wedge \bigvee A.$$

By means of the distributive law we obtain

$$x \in (a] \iff x = x \wedge \bigvee A = \bigvee \{x \wedge y \mid y \in A\}.$$

If additionally x is $\bigvee$-irreducible, then $x = \bigvee\{x \wedge y \mid y \in A\}$ can only occur if $x = x \wedge y$ holds for some $y \in A$, i.e., if $x \leq y$ holds for some $y \in A$. Since A is an order ideal in $J(\boldsymbol{D})$, this yields $x \in A$.

Hence if $J(\boldsymbol{D})$ is supremum-dense in $\boldsymbol{D}$,

$$x \mapsto (x] \cap J(\boldsymbol{D})$$

describes a bijection, which, because of

$$x \leq y \iff (x] \cap J(\boldsymbol{D}) \subseteq (y] \cap J(\boldsymbol{D}),$$

is even a lattice isomorphism.

The intersection and the union of an arbitrary number of order ideals of an ordered set $(P, \leq)$ are again order ideals. Therefore, the lattice of all order ideals is a complete sublattice of the power-set lattice of P and thus is completely distributive. Every order ideal A is the union and hence the supremum of principal ideals:

$$A = \bigcup_{a \in A} (a],$$

and, because $(a]_* = (a] \setminus \{a\}$, every principal ideal is $\bigvee$-irreducible. □

Theorem 40. *A concept lattice $\underline{\mathfrak{B}}(G, M, I)$ is completely distributive if and only if for every non-incident object-attribute pair*

$$(g, m) \notin I$$

there exist an object $h \in G$ and an attribute $n \in M$ with $(g, n) \notin I$, $(h, m) \notin I$ and $h \in k''$ for all $k \in G \setminus \{n\}'$.

Proof. The following statement holds for every concept lattice:

$$\bigwedge_{s \in S} \bigvee_{t \in T_s} (A_{s,t}, B_{s,t}) \geq \bigvee_{\varphi \in \times T_s} \bigwedge_{s \in S} (A_{s,\varphi(s)}, B_{s,\varphi(s)}).$$

Assume that the left-hand side is strictly greater than the right-hand side. Then there exist

$$g \in \bigcap_{s \in S} (\bigcap_{t \in T} B_{s,t})' \quad \text{and} \quad m \in \bigcap_{\varphi \in \times T_s} (\bigcap_{s \in S} A_{s,\varphi(s)})'$$

with $(g, m) \notin I$. Now if there are

$h \in G$ and $n \in M$ with $(g,n) \notin I, (h,m) \notin I$ and $h \in k''$ for all $k \in G \setminus n'$,

then $n \in \bigcap_{t \in T} B_{s,t}$ cannot hold for $s \in S$, because of $g \in (\bigcap_{t \in T} B_{s,t})'$. Therefore, there is a $\hat{\varphi} \in \times_{s \in S} T_s$ with $n \notin B_{s,\hat{\varphi}(s)}$ for all $s \in S$. From $h \in k''$ and $k \in G \setminus n'$ it follows that $h \in A_{s,\hat{\varphi}(s)}$ for all $s \in S$. Since, however, $m \in (\bigcap_{s \in S} A_{s,\hat{\varphi}(s)})'$, this results in a contradiction with $(h,m) \notin I$. Hence, the equation follows from the conditions specified. In order to be able to use complete distributivity to prove the inverse direction, we first argue that

$$(g'', g') = \bigwedge_{\varphi} \bigvee_{n \in M \setminus g'} (\varphi(n)'', \varphi(n)')$$

holds for every object $g \in G$, provided the maps φ under the $\bigwedge$-operator are chosen as follows:

$$\varphi \in \bigtimes_{n \in M \setminus g'} (G \setminus n').$$

Thus φ runs over all maps that assign to each attribute n which is not incident with g an object $\varphi(n)$ which is not incident with n. Hence, a possible choice is $\varphi(n) := g$ for all n, whereby we obtain the direction "$\geq$" of the statement. For the other direction we note that $n \notin \varphi(n)'$. Hence, n can still less be contained in the intent of the supremum $\bigvee_{n \in M \setminus g'} (\varphi(n)'', \varphi(n)')$, i.e., this intent is a subset of g'.

We have to show that –provided that the concept lattice is completely distributive– it is possible, for arbitrary $g \in G$ and $m \in M$ with $(g,m) \notin I$, to find some $h \in G$ and some $n \in M$ which satisfy the conditions specified in the theorem.

With the help of the preliminary considerations and by applying the distributive law we obtain

$$(g'', g') = \bigwedge_{\varphi} \bigvee_{n \in M \setminus g'} (\varphi(n)'', \varphi(n)') = \bigvee_{n \in M \setminus g'} \bigwedge_{k \in G \setminus n'} (k'', k').$$

Thus, there is some $n \in M \setminus g'$ with $m \notin (\bigcap_{k \in G \setminus n'} k'')'$ and consequently some $h \in \bigcap_{k \in G \setminus n'} k''$ with $(h,m) \notin I$. The elements elements h and n we obtain satisfy the conditions specified. □

It is easy to give complete lattices which are distributive but not completely distributive. However, these examples cannot be doubly founded, as the following theorem shows:

Theorem 41. *For a doubly founded concept lattice* $\mathbf{V} := \underline{\mathfrak{B}}(G,M,I)$, *the following conditions are equivalent:*

1. $\mathbf{V}$ *is distributive.*
2. $\mathbf{V}$ *is completely distributive.*
3. *From* $g \swarrow\!\!\!\nearrow m$ *and* $g \nearrow n$ *it follows that* $\mu m = \mu n$.
4. *From* $g \swarrow\!\!\!\nearrow m$ *and* $h \swarrow m$ *it follows that* $\gamma g = \gamma h$.

5. *$\boldsymbol{V}$ is isomorphic to a complete subdirect product of two-element lattices.*
6. *$\boldsymbol{V}$ is isomorphic to a complete sublattice of a power-set lattice.*
7. *$\boldsymbol{V}$ is isomorphic to the complete lattice of all order filters of an ordered set.*

If (G, M, I) is reduced, the following conditions are equivalent to those stated so far:

8. *Every proper premise is a singleton set.*
9. *$g \nearrow m$ implies $g \swarrow\!\!\!\!\!\nearrow m$, $g \swarrow m$ implies $g \swarrow\!\!\!\!\!\nearrow m$,
$g \swarrow\!\!\!\!\!\nearrow m$ and $g \swarrow\!\!\!\!\!\nearrow n$ imply $m = n$ and $g \swarrow\!\!\!\!\!\nearrow m$ and $h \swarrow\!\!\!\!\!\nearrow m$ imply $g = h$.*

Proof. We show that $1 \Rightarrow 3 \Leftrightarrow 4 \Rightarrow 9 \Rightarrow 5 \Rightarrow 6 \Rightarrow 2 \Rightarrow 8 \Rightarrow 7 \Rightarrow 1$.
$1 \Rightarrow 3$: From $g \swarrow\!\!\!\!\!\nearrow m$ and $g \nearrow n$ we infer that γg is $\bigvee$-irreducible and that $\gamma g \wedge \mu m \leq \gamma g_*$ and $\gamma g \wedge \mu n \leq \gamma g_*$. Hence, $\mu m \vee \mu n \not\geq \gamma g$, because by the distributive law we obtain $\gamma g \wedge (\mu m \vee \mu n) = (\gamma g \wedge \mu m) \vee (\gamma g \wedge \mu n) \leq \gamma g_* \vee \gamma g_*$ $= \gamma g_* < \gamma g$. Since we had presupposed μm and μn to be maximal $\not\geq \gamma g$, we obtain $\mu m = \mu m \vee \mu n = \mu n$, q.e.d.
$3 \Rightarrow 4$: Assume that $g \swarrow\!\!\!\!\!\nearrow m$, $h \swarrow m$ and $\gamma g \neq \gamma h$. Then $h' \not\subseteq g'$, i.e., there exists an attribute n with hIn, $g \not{I} n$. Then there is an attribute $\tilde{n}$ with $g \nearrow \tilde{n}$ and $n' \subseteq \tilde{n}'$, however, because of 3, $\gamma m = \gamma \tilde{n}$, from which we infer that $n' \subseteq m'$. This is however contradictory to $h \in n'$, $h \notin m'$.

Correspondingly we show $4 \Rightarrow 3$.
$3, 4 \Rightarrow 9$: If $g \nearrow n$ and g is irreducible, then there exists an m with $g \swarrow\!\!\!\!\!\nearrow m$, i.e., by 3, $m = n$ and thus $g \swarrow\!\!\!\!\!\nearrow n$. This shows, together with the dual argument, that a reduced context satisfying 3 and 4 allows only double arrows. The further assertion, namely that the double arrow relation represents a bijection between G and M, now follows immediately from 3 and 4, respectively.
$9 \Rightarrow 5$: According to Proposition 62 (p. 135), the subdirectly irreducible factors of $\boldsymbol{V}$ exactly correspond to the concept lattices of the one-generated subcontexts of (G, M, I) (we may presuppose (G, M, I) to be reduced). By 9, however, all one-generated subcontexts are of the same, trivial form: they are composed of an object and an attribute which are not related to each other by I. Hence, every subdirect factor of $\underline{\mathfrak{B}}(G, M, I)$ is a two-element lattice.
$5 \Rightarrow 6$: The mapping $(x_t)_{t \in T} \mapsto \{t \in T \mid x_t = \mathbf{1}\}$ is an isomorphism of the T-fold power of the two-element lattice onto the complete lattice of all subsets of T.
$6 \Rightarrow 2$: The fact that the power-set lattices are completely distributive is known from elementary set theory (and is moreover easy to prove); hence, we have 2.
$2 \Rightarrow 8$: If (G, M, I) is a reduced context with a completely distributive concept lattice, then

$$B'' = \bigcup_{b \in B} b''$$

holds for every set $B \subseteq M$, since

$$\begin{aligned} m \in B'' &\Leftrightarrow \mu m \geq \bigwedge\{\mu n | n \in B\} \\ &\Leftrightarrow \mu m = \mu m \vee \bigwedge\{\mu n | n \in B\} \\ &\Leftrightarrow \mu m = \bigwedge\{\mu m \vee \mu n | n \in B\} \\ &\Leftrightarrow \mu m = \mu m \vee \mu n \text{ for some } n \in B \\ &\qquad \text{(since } \mu m \text{ is } \bigwedge\text{-irreducible)} \\ &\Leftrightarrow m \in n'' \text{ for some } n \in B. \end{aligned}$$

This is obviously equivalent to condition 8.
$8 \Rightarrow 7$: The set M of (irreducible) attributes is ordered by $m \leq n :\Leftrightarrow n' \subseteq m'$. From 8 it follows that the intents are precisely the order filters of M with respect to this order.
$7 \Rightarrow 1$: Union and intersection of order filters again yield order filters, i.e., the distributivity follows from that of the set operations. □

Every subdirect product[1] of (completely) distributive lattices is (completely) distributive.

6.2 Semimodularity and Modularity

The lattice of the subspaces of a vector space, or more generally the lattice of the submodules of a module, has a particular structural property: it satisfies the *modular law.* The lattices of normal subgroups of groups are also modular. A weaker property, *semimodularity* can be defined in different ways. Some of these definitions make use of the neighbourhood relation $x \prec y$ (compare Definition 3) and therefore refer meaningfully only to lattices with certain finiteness requirements.

Definition 80. A lattice $\boldsymbol{V}$ is called **semimodular** if

$$x \wedge y \prec y \Rightarrow x \prec x \vee y$$

holds for each two elements x, y. It is **modular** if it satisfies the following law for all x, y and z:

$$x \leq z \Rightarrow x \vee (y \wedge z) = (x \vee y) \wedge z,$$

and **graded** if there is a **rank function** $r(x)$ assigning a natural number to each element of $\boldsymbol{V}$ with

$$r(0) = 0 \text{ and } x \prec y \quad \Rightarrow \quad r(y) = r(x) + 1.$$

We say that $\boldsymbol{V}$ satisfies the **weak condition of semimodularity** if

$$x \wedge y \prec x, y \quad \Rightarrow \quad x, y \prec x \vee y,$$

and the **strong condition of semimodularity** if the following is true:

[1] compare the footnote on page 130

If x, y and z are elements with $x < z$, $y \vee x = y \vee z$ and $y \wedge x = y \wedge z$, then there exists an element $d \leq y$ with $y \wedge x < d$ and $(x \vee d) \wedge z = x$.

◇

There is a characterization of modular lattices which is quite analogous to that of the distributive lattices in Proposition 98. The proof can again be found in Birkhoff's book.

Proposition 99. *A lattice is modular if and only if $x \leq y$, $a \wedge x = a \wedge y$ and $a \vee x = a \vee y$ always imply $x = y$.*

A modular lattice which is not distributive contains elements x, y, z with $x \vee y = x \vee z = y \vee z$ and $x \wedge y = x \wedge z = y \wedge z$, but $x \neq y$. □

The hierarchy of these lattice properties has been thoroughly examined and is described in detail elsewhere. Here we note only the simplest statements:

Proposition 100. *Every distributive lattice is modular. The modular law implies the strong condition of semimodularity (and its dual), and from this follows semimodularity.*

Proof. The first statement results from a comparison of Propositions 98 and 99. The strong condition of semimodularity holds in every modular lattice already because its premise is never satisfied: If x, y, z are elements with $x < y$ and $y \vee x = y \vee z$ as well as $y \wedge x = y \wedge z$, the modular law yields

$$x \vee (y \wedge z) = (x \vee y) \wedge z.$$

However, the left-hand side of this equation equals x, the right-hand side equals z, which is contradictory to $x < z$.

The fact that the strong condition of semimodularity implies semimodularity can be seen as follows: Let y, x be elements with $y \wedge x \prec y$, but $x \not\prec y \vee x$. Then there must be an element z with $x < z < y \vee x$, and $y \wedge x \prec y$ forces $y \wedge z = y \wedge x$. Condition 2 now yields an element d with $y \wedge x < d \leq y$, from which, because of $y \wedge x \prec y$, we can immediately infer $d = y$. This implies, however, that $x \vee d = y \vee x$, i.e., d cannot satisfy the required condition $(x \vee d) \wedge z = x$. □

In a reduced context $g \swarrow m$ is equivalent to $(\gamma g)_* = \gamma g \wedge \mu m$. If the concept lattice is semimodular, this implies $\gamma g \vee \mu m = (\mu m)^*$, i.e., $g \nearrow m$. Hence, such a context cannot contain "proper downward arrows". In the modular case all arrows must even be double arrows. However, these conditions are by no means sufficient for semimodularity, to say nothing of modularity. The following theorem provides a characterization of these properties in the language of contexts. Preparatory to it, we need an abbreviation: If g is an object in a context (G, M, I), let

$$g_\bullet := \{x \in G \mid \gamma x < \gamma g\}.$$

If γg is $\bigvee$-irreducible, then $g_\bullet$ is precisely the extent of $(\gamma g)_*$.

Theorem 42. *For a doubly founded concept lattice* $\mathbf{V} := \underline{\mathfrak{B}}(G, M, I)$*, the following conditions are equivalent:*

1. $\mathbf{V}$ *is semimodular.*
2. $\mathbf{V}$ *satisfies the strong condition of semimodularity.*
3. *The following* exchange condition *holds in* (G, M, I)*:*

$$g_\bullet \subseteq A, h \in (A \cup \{g\})'' \text{ and } h \notin A'' \Rightarrow g \in (A \cup \{h\})''.$$

4. *From* $g \swarrow m$*,* $g \swarrow n$*,* hIm *and* $h\not{I}n$ *it follows that there is an attribute* p *with* $h\not{I}p$*,* gIp *and* $m' \cap n' \subseteq p'$*.*

If $\mathbf{V}$ *is finite, the following conditions are equivalent to those stated so far:*

5. $\mathbf{V}$ *satisfies the weak condition of semimodularity.*
6. $\mathbf{V}$ *is graded and has a rank function with*

$$r(x) + r(y) \geq r(x \wedge y) + r(x \vee y).$$

Note: Contrary to the formulation of the theorem, the proof only uses one of the conditions of foundedness. If we add the other one, we can slightly improve the result. For instance, in condition (4) $h\not{I}p$ can in this case be replaced by $h \nearrow p$. This makes it possible to show that factor lattices of doubly founded semimodular lattices are again semimodular.

Proof. $1 \Rightarrow 2$: Let y, x and z be elements with $x < z$ and $y \vee x = y \vee z$ and $y \wedge x = y \wedge z$. First we show that there is an element d with $y \wedge x \prec d \leq y$. Because of the foundedness there exists an element s minimal with respect to $s \leq y$, $s \not\leq y \wedge x$. s is necessarily $\bigvee$-irreducible, and $s_* \leq y \wedge x$, i.e., $s \wedge (y \wedge x) = s_* \prec s$. If now we set $d := s \vee (y \wedge x)$, we obtain $y \wedge x \prec d$ from the first condition of the proposition. A repeated application of the condition yields $x \prec x \vee d$. $x \vee d \leq z$ would imply $z \geq x \vee d \geq d \geq s$ and, because of $y \geq s$, also $y \wedge z = y \wedge x \geq s$, which would be contradictory to the definition of s. Hence $z \wedge (x \vee d) = x$. $2 \Rightarrow 1$ has already been proved in Proposition 100.

$3 \Rightarrow 1$: Let x, y be concepts with $x \wedge y \prec y$ and $x = (A, B)$, and let furthermore z be a concept between x and $x \vee y$: $x < z < x \vee y$. The foundedness yields an element s which is minimal with respect to $s \leq y$, $s \not\leq x$, s then is $\bigvee$-irreducible and thus an object concept, i.e., $s = \gamma h$ for some $h \in G$. Because of $s_* \leq x$ we have $h_\bullet \subseteq A$, furthermore we have $x \vee y = x \vee s$. If we now choose an object g which is contained in the extent of the concept z but which does not form part of A, we obtain

$$g \notin (A \cup h_\bullet)'', \quad g \in (A \cup \{h\})'',$$

from which by (3) it follows that $h \in (A \cup \{g\})''$. This extent is however contained in that of z, which results in a contradiction with $\gamma h \not\leq z$.

$1 \Rightarrow 3$: Trivially, condition (3) is satisfied if g is reducible. Therefore, we can restrict ourselves to the case $(\gamma g)_* \prec \gamma g$. The preconditions of (3) describe three concepts, namely (A'', A'), $((A \cup \{h\})'', (A \cup \{h\})')$ and $((A \cup \{g\})'', (A \cup \{g\})')$ with $A'' \subset (A \cup \{h\})'' \subseteq (A \cup \{g\})''$. Since $g_\bullet \subseteq A$ is true, $\gamma g \wedge (A'', A') = \gamma g_*$, and thus, because of semimodularity,

$$((A \cup \{g\})'', (A \cup \{g\})')$$

is an upper neighbour of (A'', A'), which forces

$$((A \cup \{g\})'', (A \cup \{g\})') = ((A \cup \{h\})'', (A \cup \{h\})').$$

$1 \Rightarrow 4$: Let g, h, m, n be as specified. If we choose x minimal with respect to $x \leq \gamma h$, $x \not\leq \mu n$, then x is an object concept, satisfying the same preconditions as γh. Furthermore, for every attribute p with $x \not\leq \mu p$ we have $h \not I p$. Hence, we may assume without loss of generality that $x = \gamma h$, i.e., in particular that γh is $\bigvee$-irreducible and that $\gamma h_* \leq \mu m \wedge \mu n$. Because of (1), $\gamma h \vee (\mu m \wedge \mu n)$ is an upper neighbour of $\mu m \wedge \mu n$ which is $\leq \mu m$. Also because of (1) $\mathfrak{y} := \gamma g \vee (\mu m \wedge \mu n)$ is an upper neighbour of $\mu m \vee \mu n$, but because of $\gamma g \not\leq \mu m$ it is different from $\gamma h \vee (\mu m \wedge \mu n)$. Hence, there must be a distinguishing attribute p which is contained in the intent of the concept $\mathfrak{y}$ but which h does not have. This attribute satisfies the conditions specified.
$4 \Rightarrow 1$: First, we convince ourselves that it suffices to prove condition (1) for object concepts y: If y is arbitrary and $x \wedge y \prec y$ is a lower neighbour of y, we find an object concept $\gamma g \leq y$ which is minimal with respect to $\gamma g \not\leq x \wedge y$, which implies that $\gamma g_* = \gamma g \wedge (x \wedge y) = \gamma g \wedge x$ is a lower neighbour of γg. If we now may apply (1), we obtain $x \prec x \vee \gamma g = x \vee \gamma g \vee (x \wedge y) = x \vee y$, since $x \wedge y < \gamma g \vee (x \wedge y) \leq y$, i.e., $\gamma g \vee (x \wedge y) = y$.

Now we assume that there are an object g and an x such that $x \wedge \gamma g = \gamma g_*$ is a lower neighbour of γg but $x \vee \gamma g$ is not an upper neighbour of x, i.e., that $x < z < x \vee \gamma g$ for some z. Then there exist an object h in the extent of z which is not contained in the extent of x, an attribute n which is contained in the intent of x but not in the intent of z and an attribute m which is contained in the intent of z but does not apply to g. Thereby the prerequisites of (4) are all satisfied and we may conclude that there has to be an attribute p which has $m' \cap n'$ in its extent, i.e., which satisfies $x \leq \mu p$, with gIp and $h \not I p$. However, from gIp it follows that $\mu p \geq x \vee \gamma g \geq z$ and from $h \not I p$ it follows that $\mu p \not\geq z$, a contradiction!
$1 \Rightarrow 5$ is trivial.
$5 \Rightarrow 6$: Let n be the length of a maximal chain in $\mathbf{V}$. We define a function r on $\mathbf{V}$ by $r(1) := n$ and, for $x \neq 1$,

$$r(x) := \max\{r(y) \mid x \prec y\} - 1.$$

If r were not a rank function, then there would have to be elements $x \prec y$ with $r(x) + 1 \neq r(y)$, and, of all such examples, we could choose one with maximal y, furthermore, by the definition of r, there would be a further upper neighbour z of x with $r(z) = r(x)+1$. By the condition of semimodularity $y \vee z$

would be an upper neighbour of y and of z, by the condition of maximality; then we would have $r(y) = r(y \vee z) - 1 = r(z)$, a contradiction!

In order to prove the rank inequality claimed, we again derive a contradiction from the assumption of a counter-example: Let a, b, x, y be elements with $a = x \wedge y$ and $b = x \vee y$ which do not satisfy the inequality. Let these elements be chosen such that a is maximal among all counter-examples and that b is minimal among all counter-examples with the smallest element a. If now y is an upper neighbour of $x \wedge y$, then we choose an upper neighbour $\underline{x}$ of $x \wedge y$ lying below x. Since y cannot lie below x (otherwise the inequality would be satisfied), $\underline{x} \neq y$, and $\overline{y} := y \vee \underline{x}$ is an upper neighbour of y with $x \wedge \overline{y} = \underline{x}$ and $x \vee \overline{y} = x \vee y$. For the elements x and $\overline{y}$ the rank inequality holds by assumption and thus also for x and y. If y is not an upper neighbour of x, we can find an upper neighbour $\underline{y}$ of $x \wedge y$ which lies below y. The application of the rank inequality to the elements x and $\underline{y}$ and to y and $x \vee \underline{y}$ yields the statement.
$6 \Rightarrow 1$ is again trivial. □

In the atomistic case we have $g_{\bullet} = \emptyset$ for all $g \in G$. Thus, the exchange condition simplifies to yield the known form

$$h \in (A \cup \{g\})'', \quad h \notin A'' \quad \Rightarrow \quad g \in (A \cup \{h\})''.$$

6.3 Semidistributivity and Local Distributivity

Definition 81. A complete lattice $\boldsymbol{V}$ is called

– **semidistributive**, if the following laws hold for all $x, y, z \in \boldsymbol{V}$:

$$(\mathbf{SD}_\vee) \qquad x \vee y = x \vee z \Rightarrow x \vee y = x \vee (y \wedge z) = x \vee z$$
$$(\mathbf{SD}_\wedge) \qquad x \wedge y = x \wedge z \Rightarrow x \wedge y = x \wedge (y \vee z) = x \wedge z.$$

If $\boldsymbol{V}$ satisfies $(\mathrm{SD}_\vee)$, $\boldsymbol{V}$ is called **join-semidistributive**, dually, a complete lattice satisfying $(\mathrm{SD}_\wedge)$ is called **meet-semidistributive.**

– **locally distributive** or **join-distributive**, if $\boldsymbol{V}$ is semimodular and every modular sublattice is distributive. Lattices satisfying the dual condition are called **meet-distributive.**

A representation of a lattice element a as a supremum $a = \bigvee X$ is called **irredundant** if $a \neq \bigvee(X \setminus \{x\})$ holds for every $x \in X$. Obviously, the elements of X must be pairwise incomparable. Here, we will mainly deal with irredundant $\bigvee$-representations in chain-finite lattices. This constitutes a simplification in many respects, because in this kind of lattice from each $\bigvee$-representation we can choose a finite and then also an irredundant $\bigvee$-representation.

Special attention is given to the $\bigvee$-representations through $\bigvee$-irreducible elements. If a has exactly one such representation, we say that a has a **unique**

irredundant $\bigvee$-representation. The addition "through irreducibles" is often omitted; it is however implied.

An irredundant $\bigvee$-representation $a = \bigvee X$ is called **canonical** if, for every irredundant representation $a = \bigvee Y$ and for every $x \in X$, there exists some $y \in Y$ with $x \leq y$. If an element of a finite lattice has a canonical representation, the latter necessarily consists of $\bigvee$-irreducible elements. Note, however, that the refinement property is required with respect to all irredundant representations. As is well known, in a doubly founded lattice every element a is the supremum of $\bigvee$-irreducible elements: $a = \bigvee\{x \in J(\mathbf{V}) \mid x \leq a\}$. $e \in J(\mathbf{V})$ is an **extremal point** of a if e is indispensable, i.e., if

$$a \neq \bigvee\{x \in J(\mathbf{V}) \setminus \{e\} \mid x \leq a\}.$$

The extremal points are part of *every* $\bigvee$-representation of a through irreducibles. A **base point** of a is a $\bigvee$-irreducible element $b \leq a$ with

$$b \not\leq \bigvee\{x \in J(\mathbf{V}) \mid x \leq a, b \not\leq x\}.$$

In a concept lattice the extremal points of a concept (A, A') are precisely the object concepts γg with $g \in A$ but

$$g \notin (A \setminus \{h \mid g' = h'\})''.$$

Therefore, such an object is called an **extremal point of the extent** A. Correspondingly, g is a **base point of the extent** A if $g \in A$ but

$$g \notin (A \setminus \{h \mid g' \supseteq h'\})''. \qquad \diamond$$

Theorem 43. *For a doubly founded concept lattice $\mathbf{V} = \underline{\mathfrak{B}}(G, M, I)$, the following statements are equivalent:*

1. *$\mathbf{V}$ satisfies $(SD_\vee)$.*
2. *For all $g, h \in G$ and all $m \in M$ we have:*

$$g \swarrow\!\!\nearrow m \text{ and } h \swarrow\!\!\nearrow m \text{ imply } g' = h'.$$

If $\mathbf{V}$ is finite, or, more generally, if $\mathbf{V}$ satisfies the additional condition

(*) if $\mathfrak{x} < \mathfrak{a}$, then there exists a lower neighbour $\mathfrak{u}$ of $\mathfrak{a}$ with $\mathfrak{x} \leq \mathfrak{u} \prec \mathfrak{a}$,

then the following conditions are equivalent to the above-stated ones:

3. *Every element of $\mathbf{V}$ has a canonical $\bigvee$-representation.*
4. *Every extent is the closure of its base points.*

The condition $(SD_\wedge)$ can be characterized dually.

Proof. $1 \Leftrightarrow 2$: We assume without loss of generality that (G, M, I) is the context of a concept lattice satisfying $(\mathrm{SD}_\vee)$. From $g \swarrow\!\!\!\nearrow m$ and $h \swarrow\!\!\!\nearrow m$ we infer that

$$\mu m \vee \gamma g = \mu m^* = \mu m \vee \gamma h$$

and then with the help of semidistributivity

$$\mu m^* = \mu m \vee (\gamma g \wedge \gamma h).$$

Now γg_* as well as γh_* is less than or equal to μm. The above equation can only be true if neither $\gamma g \wedge \gamma h \leq \gamma g_*$ nor $\leq \gamma h_*$. This forces $\gamma g = \gamma h$.

Now we conversely assume that (G, M, I) satisfies the above-stated condition for the arrow relations and show that $\underline{\mathfrak{B}}(G, M, I)$ is join-semidistributive. Let furthermore x, y and z be elements of $\underline{\mathfrak{B}}(G, M, I)$ with

$$x \vee y = x \vee z > x \vee (y \wedge z).$$

Then there exists an element t maximal with respect to

$$t \geq x \vee (y \wedge z), \quad t \not\geq x \vee y.$$

t is $\bigwedge$-irreducible, i.e., $t = \mu m$ for some $m \in M$, and $\mu m^* \geq x \vee y$. Now, $y \not\leq \mu m$, since

$$y \vee \mu m \geq y \vee x \vee (y \wedge z) = x \vee y,$$

and we obtain

$$y \vee \mu m = \mu m^*.$$

Hence there is a concept s which is minimal with respect to $s \leq y, s \not\leq \mu m$. s is $\bigvee$-irreducible; consequently it is an object concept $s = \gamma g$ for some $g \in G$ and $\gamma g_* \leq \mu m$, i.e., $g \swarrow\!\!\!\nearrow m$.

Likewise, we can find some $h \in G$ with $h \swarrow\!\!\!\nearrow m$ and $\gamma h \leq z$. The presupposition forces $\gamma g = \gamma h$, from which in turn it follows that $\gamma g \leq z$, i.e., $\gamma g \leq y \wedge z \leq x \vee (y \wedge z) \leq \mu m$. Contradiction!

$1 \Rightarrow 4$ for lattices satisfying the additional condition: Let $\mathfrak{a}$ be a concept. For every lower neighbour $\mathfrak{u}$ of $\mathfrak{a}$ there exists, because of the foundedness, an element $\bar{\mathfrak{u}} \leq \mathfrak{a}$ which is minimal with respect to $\bar{\mathfrak{u}} \not\leq \mathfrak{u}$. This element is uniquely determined by $\mathfrak{u}$, since if $\mathfrak{x}$ is an arbitrary element satisfying $\mathfrak{x} \leq \mathfrak{a}$, $\mathfrak{x} \not\leq \mathfrak{u}$, we have

$$\mathfrak{u} \vee \bar{\mathfrak{u}} = \mathfrak{a} = \mathfrak{u} \vee \mathfrak{x}$$

and by application of $(\mathrm{SD}_\vee)$ we obtain

$$\mathfrak{u} \vee (\bar{\mathfrak{u}} \wedge \mathfrak{x}) = \mathfrak{a},$$

which, because of the minimality of $\bar{\mathfrak{u}}$, immediately yields $\bar{\mathfrak{u}} \leq \mathfrak{x}$. Hence for $\mathfrak{x} \leq \mathfrak{a}$ we have

$$\mathfrak{x} \not\leq \mathfrak{u} \iff \bar{\mathfrak{u}} \leq \mathfrak{x}.$$

$\bar{\mathfrak{u}}$ is $\bigvee$-irreducible, and consequently there exists an object g with $\bar{\mathfrak{u}} = \gamma g$. We show that g is a base point of $\mathfrak{a}$: If A is the extent of $\mathfrak{a}$ and $h \in A$ is arbitrary then we have

$$h' \subseteq g' \iff \gamma h \geq \gamma g \iff \gamma h \geq \bar{\mathfrak{u}} \iff \gamma h \not\leq \mathfrak{u}.$$

Hence, in this case, $A \setminus \{h \mid g' \supseteq h'\}$ is entirely contained in the extent of $\mathfrak{u}$, from which follows the desired base point property:

$$g \notin (A \setminus \{h \mid g' \supseteq h'\})''.$$

Now assume that

$$\mathfrak{x} := \bigvee\{\bar{\mathfrak{u}} \mid \mathfrak{u} \prec \mathfrak{a}\}.$$

We claim that $\mathfrak{x} = \mathfrak{a}$. If this were not the case, because of the additional condition there would exist a lower neighbour $\mathfrak{u}$ of $\mathfrak{a}$ with $\mathfrak{x} \leq \mathfrak{u} \prec \mathfrak{a}$, and $\bar{\mathfrak{u}} \not\leq \mathfrak{x}$ would yield a contradiction.

$4 \Rightarrow 3$: If every extent is the closure of its base points, i.e., for every lattice element $\mathfrak{a}$ we have that

$$\mathfrak{a} = \bigvee\{\mathfrak{b} \mid \mathfrak{b} \text{ is base point of } \mathfrak{a}\},$$

then this representation is canonical. It is furthermore irredundant and, if $\mathfrak{a} = \bigvee Y$ is an arbitrary irredundant representation and $\mathfrak{b}$ is a base point of $\mathfrak{a}$, then from $\mathfrak{b} \leq \bigvee Y$ and the fact that $J(\boldsymbol{V})$ is $\bigvee$-dense it follows immediately that there has to be some $\mathfrak{y} \in Y$ with $\mathfrak{b} \leq \mathfrak{y}$.

$3 \Rightarrow 1$: We consider an element $\mathfrak{v}$ and a canonical $\bigvee$-representation $\mathfrak{v} = \bigvee X$. From

$$\mathfrak{v} = \mathfrak{v}_\circ \vee \mathfrak{y} = \mathfrak{v}_\circ \vee \mathfrak{z}$$

it follows that $\mathfrak{y} \geq \mathfrak{x}$ for all $\mathfrak{x} \in X$ with $\mathfrak{x} \not\leq \mathfrak{v}_\circ$ and $\mathfrak{z} \geq \mathfrak{x}$ for all $\mathfrak{x} \in X$ with $\mathfrak{x} \not\leq \mathfrak{v}_\circ$. Together this forces

$$\mathfrak{v} = \mathfrak{v}_\circ \vee (\mathfrak{y} \wedge \mathfrak{z}). \qquad \square$$

Examples of lattices satisfying the conditions of Theorem 39 can easily be obtained by means of the technique of local doubling (5.3). If $\boldsymbol{V}$ is a doubly founded semidistributive lattice and $\mathfrak{C} \subseteq \boldsymbol{V}$ is a convex subset with a smallest element, then $\boldsymbol{V}[\mathfrak{C}]$ is also semidistributive, as can be seen by the arrow relations. The Tamari lattice (cf. Figures 5.9, 5.10, p. 206) satisfies ($\mathrm{SD}_\vee$) and ($\mathrm{SD}_\wedge$).

Theorem 44. *For a doubly founded concept lattice $\boldsymbol{V} := \underline{\mathfrak{B}}(G, M, I)$ the following statements are equivalent:*

1. *If g and h are irreducible objects then $g \nearrow m$ and $h \nearrow m$ imply $g' = h'$.*
2. *$\boldsymbol{V}$ has a neighbourhood-preserving $\bigwedge$-embedding into a power-set lattice.*
3. *Every extent is the closure of its extremal points.*

4. For every concept extent A and for all $g, h \in G \setminus A$ with $g' \neq h'$ we have:

$$g \in (A \cup \{h\})'' \text{ implies } h \notin (A \cup \{g\})'' \quad \textbf{(Anti-exchange Axiom).}$$

If V is chain-finite, then the following condition is equivalent to those we have mentioned so far:

5. V is meet-distributive.
6. Every element has a unique irredundant $\bigvee$-representation.

Proof. $1 \Rightarrow 4$: If A is an extent and $g \notin A$, we find an attribute m with $A \subseteq m'$ and $g \nearrow m$. From hIm it would follow that $g \notin (A \cup \{h\})''$; hence $h \not I m$ would hold and we would find an attribute n with $h \nearrow n$ and $m' \subseteq n'$. $m' = n'$ would imply $g' = h'$ already because of (1), hence $g \in n'$; this would mean, however, that $A \cup \{g\} \subseteq n'$ and consequently $h \notin (A \cup \{g\})''$.

$4 \Rightarrow 2$: We pass on to the concept lattice of the clarified context and show that the closure system of the extents is embedded in the power-set of G in such a way that it preserves neighbourhoods. Let A_1, A_2 be extents with $A_1 \subset A_2$, and let (A_1, A_1') be a lower neighbour of (A_2, A_2'). If now $g, h \in A_2 \setminus A_1$, then, because of the neighbourhood $(A_1 \cup \{g\})'' = A_2 = (A_1 \cup \{h\})''$, which, by the Anti-exchange Axiom, yields $g = h$.

$2 \Rightarrow 3$: We begin this part of the proof with three preliminary considerations:

Every lattice having a neighbourhood-preserving $\wedge$-embedding

$$\varphi : V \to \mathfrak{P}(X)$$

into a power-set lattice is dually semimodular, since from $a \prec a \vee b$ it follows that $\varphi(a)$ and $\varphi(a \vee b)$ only differ in one element. This transfers to $\varphi(a \wedge b)$ and $\varphi(b)$, which means that they also must be neighbours, which implies $a \wedge b \prec b$.

Secondly we show that a doubly founded lattice satisfying (2) also satisfies the additional condition (*) in Theorem 43. If $a < b$, then because V is doubly founded, there exists some $\bigwedge$-irreducible element $t \in V$ with

$$a \leq t, \quad b \not\leq t, \quad b \leq t^*,$$

and with $t \prec t^*$ because of the dual semimodularity it follows that

$$a \leq b \wedge t \prec b.$$

Hence every element of V is either $\bigvee$-irreducible or is the supremum of its lower neighbours.

For the third preliminary consideration we take an arbitrary element $x \in X$. Among all elements of V whose image contains x, there is a smallest one, namely

$$v_x := \bigwedge \{v \in V \mid x \in \varphi(v)\}.$$

v_x is $\bigvee$-irreducible (or the smallest element of $\boldsymbol{V}$), since there can only be one lower neighbour u of v because the image

$$\varphi(u) = \varphi(v_x) \setminus \{x\}$$

of the latter is uniquely determined. If w is an arbitrary $\bigvee$-irreducible element with $x \in \varphi(w)$, $x \notin \varphi(w_*)$, then $w = w_* \vee v_x$, i.e., $w = v_x$.

Now we consider an arbitrary element a of $\boldsymbol{V}$. If the extent a were not the closure of its extremal points, then there would be a lower neighbour $u \prec a$, the extent of which would contain all extremal points of a. $\varphi(a)$ and $\varphi(u)$ only differ by one element, let us say x of X. If w is a $\bigvee$-irreducible element with $w \vee u = a$, then $u \wedge w$ is a lower neighbour of w, i.e., w_*, and we have $x = \varphi(w) \setminus \varphi(w_*)$, from which it follows that $w = v_x$. Hence v_x is an extremal point of a that does not lie below u. Contradiction!

$3 \Rightarrow 1$: If m is an irreducible attribute and A is the extent of μm^*, then by (3) there is an extremal point h of A with $h \notin m'$. $A^- := A \setminus \{x \mid h' = x'\}$ then is an extent containing m'. Now consider an arbitrary object $g \in A \setminus m'$. We have $(m' \cup \{g\})'' = A$, and therefore A^- cannot contain $m' \cup \{g\}$. Consequently, $g \in \{x \mid h' = x'\}$, i.e., $g' = h'$. Hence μm and μm^* differ only by one single object concept.

$3 \Rightarrow 6$: If an extent is the closure of its extremal points, this is its only irredundant $\bigvee$-representation.

$3 \Leftarrow 6$ for chain-finite lattices: In a chain-finite lattice every $\bigvee$-representation contains an irredundant one, i.e., the uniqueness implies that every extent A has a smallest subset E with $E'' = A$. This must however consist of extremal points, since otherwise there would be some $e \in E$ with $e \in (A \setminus \{e\})''$, and $A \setminus \{e\}$ would be a generating set of A not containing E.

$2 \Rightarrow 5$: We have already shown above that from (2) follow dual semimodularity and the additional condition from Theorem 43. It remains to be shown that every modular sublattice is distributive. If this were not the case, by Proposition 99 there would be elements x, y, z with $x \vee y = x \vee y = y \vee z$, $x \wedge y = x \wedge z = y \wedge z$, but $x \neq y$. Choose a lower neighbour u of $x \vee y$ with $z \leq u$. $\varphi(x \vee y) \setminus \varphi(u)$ consists of exactly one element, let us say g. Since $\varphi(x) \not\subseteq \varphi(u)$ and $\varphi(y) \not\subseteq \varphi(u)$, $g \in \varphi(x) \cap \varphi(y) = \varphi(x \wedge y)$, from which, because of $x \wedge y \leq z$ follows the contradiction $g \in \varphi(z) \subseteq \varphi(u)$.

$5 \Rightarrow 1$ for chain-finite lattices: Because of dual semimodularity $g \nearrow m$ implies $g \swarrow m$. Hence we can presuppose $g \swarrow\!\!\!\nearrow m$, $h \swarrow\!\!\!\nearrow m$ and we have to show $\gamma g = \gamma h$. The element $\gamma g \vee \gamma h$ is $\leq \mu m^*$, but not $\leq \mu m$. Therefore if $\gamma g \neq \gamma h$ we can find three different lower neighbours a_0, b_0, c_0 of the element $\gamma g \vee \gamma h$ with $a_0 \geq \gamma g$, $b_0 \geq \gamma h$, $b_0 \not\geq \gamma g$ and $c_0 \leq \mu m$, and furthermore a descending chain $a_0 \succ a_1 \succ \ldots \succ a_n = \gamma g$ of neighbouring elements. Because of meet-distributivity the meets $b_1 := a_0 \wedge b_0$ and $c_1 := a_0 \wedge c_0$ are distinct from each other. Furthermore, they both cannot be equal to a_1, since otherwise it would follow that $\gamma g \leq \mu m$ or $\gamma g \leq b_0$. If we continue this argument, we get that in each case the elements a_i, b_i and c_i are different lower neighbours of a_{i-1} for $i \in \{1, \ldots, n\}$, and similarly that $a_n \wedge b_n \neq a_n \wedge c_n$. Therefore $a_n = (a_n \wedge b_n) \vee (a_n \wedge c_n)$ cannot be $\bigvee$-irreducible; this however is a contradiction because $a_n = \gamma g$. □

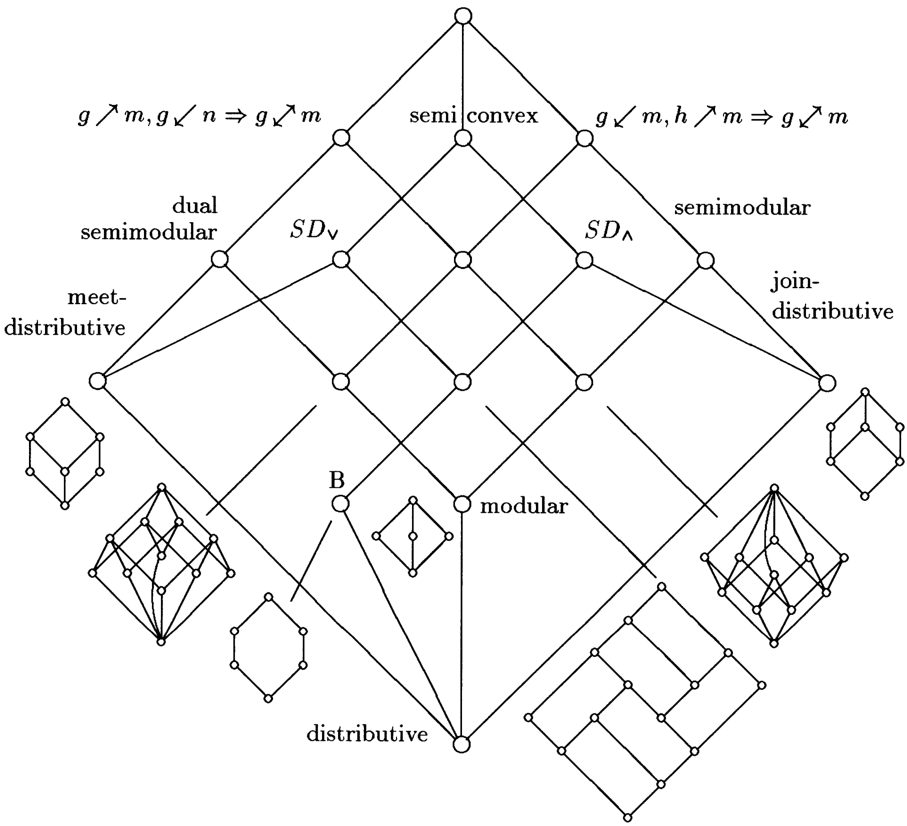


Figure 6.1 Generalizations of the Distributive Law.

The first condition of Theorem 44 obviously implies the second condition of Theorem 43. Hence finite meet-distributive lattices also satisfy ($\mathrm{SD}_\vee$).

In a context in which there are no "proper upward arrows" (which means that $g \nearrow m$ always implies $g \mathrel{\swarrow\!\!\!\nearrow} m$), the converse is also true. This holds in particular for atomistic contexts, in which from $(g, m) \notin I$ it already follows that $g \swarrow m$.

Figure 6.1 shows the implications between the above mentioned lattice properties (for doubly founded complete lattices). Two of the attributes still lack an explanation: According to A. Day, a lattice is **semiconvex** if it satisfies the following condition:

$$x \wedge y = x \wedge z, \quad x \vee y = x \vee z \quad \Rightarrow \quad x \leq z.$$

B stands for the property of being a bounded homomorphic image of a free lattice. This property has a simple characterization in the language of contexts: it is equivalent to the fact that the objects and attributes of the (reduced) context can be ordered in such a way that every $\swarrow\!\!\!\!\nearrow$ is on the diagonal, every $\nearrow$ below the diagonal and every $\swarrow$ above the diagonal.

The convex-ordinal scales $\mathbb{C}_{(P,\leq)}$ (see Section 1.4), at least if $(P,\leq)$ is finite, are doubly founded. The extents are the convex subsets of $(P,\leq)$, the extremal points are the maximal and minimal elements of such subsets. Hence Condition (3) of Theorem 44 is satisfied and therefore finite convex-ordinal scales are meet-distributive.

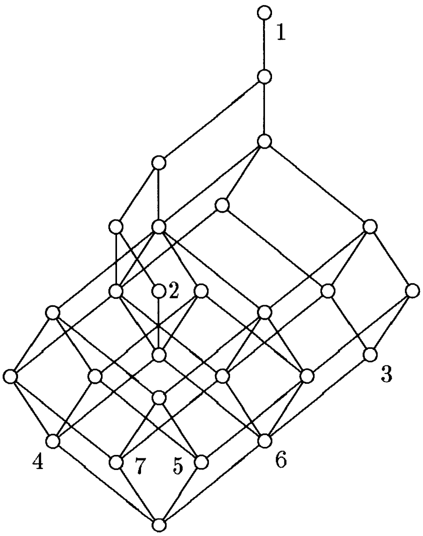


Figure 6.2 The additively saturated subsets of $\{1,\ldots,7\}$.

The Anti-exchange Axiom holds in the closure system of the convex sets of an arbitrary metric space. But it also holds in other connections, for example, if we can assign a *weight* $wt(g)$ to every object g, such that $wt(g) = wt(h)$ only if $g' = h'$ and such that

$$g \notin A'', g \in (A \cup \{h\})'' \Rightarrow wt(g) \geq wt(h).$$

The last condition can be interpreted as saying that the weight of g must be at least as big as that of h if g is generated by means of h. A simple example in this connection is presented in Figure 6.2. If $G \subseteq \mathbb{N}$ is an arbitrary set, then a subset $T \subseteq G$ will be called **additively saturated** if from $a, b \in T, a+b \in G$ it already follows that $a+b \in T$. The additively saturated subsets of G form a closure system which with $wt(g) := g$ obviously satisfies the above-mentioned condition. Hence the corresponding lattice is meet-distributive. Figure 6.2 shows an example with $G := \{1, 2, \ldots, 7\}$.

6.4 Dimension

Definition 82. An ordered set $(P, \leq)$ has **order dimension**

$$\dim(P, \leq) = n$$

if and only if it can be embedded in a direct product of n chains[2] and n is the smallest number for which this is possible. ◇

We will show that the order dimension can also be well described in the language of Formal Concept Analysis. For this purpose we start with some simple observations, on which we will base our statements and which at the same time are formulated in such a way that they can also be used for possible variations of the notion of order dimension. An example of such a variation is that of the k-dimension: The **k-dimension** $\dim_k(P, \leq)$ of an ordered set $(P, \leq)$ is the smallest number of chains of cardinality k in whose product it can be order-embedded. What is usually examined are embeddings of arbitrary ordered sets. We shall concentrate on embeddings of concept lattices. This is not a serious restriction, however: The Dedekind Completion Theorem (Theorem 4, p. 48) shows that $(P, \leq)$ can be embedded in a complete lattice if and only if $\underline{\mathfrak{B}}(P, P, \leq)$ can also be embedded in this lattice. Therefore, we have the following theorem:

Theorem 45.

$$\begin{aligned}\dim(P, \leq) &= \dim \underline{\mathfrak{B}}(P, P, \leq)\\ \dim_k(P, \leq) &= \dim_k \underline{\mathfrak{B}}(P, P, \leq)\end{aligned}$$

□

As a direct corollary of Proposition 33 (p. 99) we obtain:

Proposition 101. *There is an order embedding of* $\underline{\mathfrak{B}}(G, M, I)$ *in a product* $\times_{t \in T} \mathbf{V}_t$*, if and only if there are pairs of maps* $(\alpha_t, \beta_t), t \in T$ *with the following properties:*

1. $\alpha_t : G \to \mathbf{V}_t, \quad \beta_t : M \to \mathbf{V}_t,$
2. $(g, m) \in I \Rightarrow \alpha_t g \leq \beta_t m$ *for all* $t \in T$,
3. $(g, m) \notin I \Rightarrow \alpha_t g \not\leq \beta_t m$ *for some* $t \in T$.

□

[2] Meaning an order embedding according to Definition 6 (p. 3).

We can express this fact differently by replacing the maps α_t and β_t by relations J_t with $(g,m) \in J_t :\Longleftrightarrow \alpha_t g \leq \beta_t m$:

Proposition 102. *There is an order-embedding of $\underline{\mathfrak{B}}(G,M,I)$ in a product $\times_{t\in T} \mathbf{V}_t$ if and only if there exist contexts (G,M,J_t) and order-embeddings of $\underline{\mathfrak{B}}(G,M,J_t)$ in $\mathbf{V}_t$ for $t \in T$ such that*

$$I = \bigcap_{t\in T} J_t \qquad \square$$

In the case of a doubly founded context, Proposition 102 can be further improved by weakening the condition $I = \bigcap_{t\in T} J_t$: The statement remains correct if instead we only presuppose that

$$I \subseteq \bigcap_{t\in T} J_t \text{ and } \{(g,m) \mid g \swarrow m \text{ or } g \nearrow m\} \cap \bigcap_{t\in T} J_t = \emptyset.$$

This follows from Proposition 49 (p. 115).

Definition 83. A **Ferrers relation** is a relation $F \subseteq G \times M$ with

$$(g,m) \in F, \quad (h,n) \in F, \quad (g,n) \notin F \quad \Rightarrow \quad (h,m) \in F.$$

The **Ferrers dimension** $\operatorname{fdim}(G,M,I)$ of a context (G,M,I) is the smallest number of Ferrers relations $F_t \subseteq G \times M, t \in T$ with $I = \bigcap_{t\in T} F_t$. $\diamond$

If we imagine (G,M,F) as a cross table, it is easy to visualize the Ferrers condition: The definition excludes the subcontext $\begin{array}{|c|c|}\hline \times & \\ \hline & \times \\ \hline\end{array}$, which does not occur if and only if the table can be brought into a stair-shaped form by rearranging the rows and columns. This is also the basis of the following proposition:

Proposition 103. *$F \subseteq G\times M$ is a Ferrers relation if and only if $\underline{\mathfrak{B}}(G,M,F)$ is a chain.*

Proof. Let F be a Ferrers relation and let $(A_1,B_1), (A_2,B_2)$ be two concepts of $\underline{\mathfrak{B}}(G,M,F)$. If $(A_1,B_1) \not\leq (A_2,B_2)$, then there are an object $g \in A_1$ and an attribute $n \in B_2$ with $(g,n) \notin F$. For every $m \in B_1$ we have $(g,m) \in F$ and for every $h \in A_2$ we have $(h,n) \in F$. Hence from the Ferrers condition it follows that $(h,m) \in F$ for all $h \in A_2, m \in B_1$ and thus $(A_2,B_2) \leq (A_1,B_1)$. This means that any two concepts of $\underline{\mathfrak{B}}(G,M,F)$ are comparable and $\underline{\mathfrak{B}}(G,M,F)$ thus is a chain. The reverse direction is even easier. $\square$

The determination of the Ferrers dimension is a task which is generally difficult to solve (since it is $\mathcal{NP}$-complete). Nevertheless, in the case of small contexts it can be carried out by hand. In this connection it is convenient to make use of the fact that the complement of a Ferrers relation is again a Ferrers relation. Hence the Ferrers dimension of (G,M,I) is also equal to the smallest number of Ferrers relations F_t covering the empty cells of the cross table, i.e., with $G \times M \setminus I = \bigcup_{t\in T} F_t$. It is, however, not always possible to choose this covering to be *disjoint*.

Theorem 46. *The Ferrers dimension of (G, M, I) is equal to the order dimension of the concept lattice $\underline{\mathfrak{B}}(G, M, I)$:*

$$\dim \underline{\mathfrak{B}}(G, M, I) = \mathrm{fdim}(G, M, I).$$

The order dimension of an ordered set $(P, \leq)$ is equal to the Ferrers dimension of $(P, P, \leq)$:

$$\dim(P, \leq) = \mathrm{fdim}(P, P, \leq).$$

Proof. This follows immediately from the Propositions 102 and 103, because, obviously, a complete lattice can be embedded in a chain if and only if it is a chain itself. □

×	×	1	×	1	1	1
2	×	×	4	×	1	1
2	3	×	×	2	×	1
2	3	3	×	×	3	×
×	3	3	4	×	×	1
2	×	2	2	2	×	×
×	3	×	4	4	4	×

$(3,0,0,0)$	$(3,0,3,3)$
$(2,1,0,1)$	$(3,3,0,3)$
$(1,2,1,0)$	$(2,2,1,3)$
$(0,2,3,0)$	$(3,2,3,0)$
$(1,0,2,1)$	$(2,1,3,2)$
$(0,3,0,0)$	$(1,3,2,2)$
$(0,0,1,3)$	$(0,3,3,3)$

Figure 6.3 Point-line context of the projective plane PG(2,2), with Ferrers relations. The Ferrers dimension, and thus the order dimension of the plane, is 4; a covering of the 28 empty boxes with less than four Ferrers relations is impossible, since each Ferrers relation in this example can have eight elements at most. On the right is an embedding of PG(2,2) into a product of four chains; the first column gives the images of the points, the second those of the lines.

Theorem 46 can be strengthened in several respects. We can include the lengths of the chains involved and we can examine whether it would not suffice to cover the arrow relations. Both are possible and the two possibilities can even be combined. For this purpose, we first define the *length* of a Ferrers relation:

Definition 84. The **length** of a Ferrers relation $F \subseteq G \times M$ is the length of the concept lattice $\underline{\mathfrak{B}}(G, M, F)$. By $\mathrm{fdim}_k(G, M, I)$ let us denote the smallest number of Ferrers relations $F_t \subseteq G \times M$, $t \in T$, of length $\leq k$ with $I = \bigcap_{t \in T} F_t$. A Ferrers relation is **k-step**, if $k = |\{g^F \mid g \in G\}|$. ◇

If we imagine a k-step Ferrers relation represented as a table and arranged in the form of stairs, then these stairs really have k steps. The number of steps is equal to the length if full rows and full columns are counted separately. The following trivial observation is quite helpful:

Proposition 104. *The complement $G \times M \setminus F$ of a k-step Ferrers relation $F \subseteq G \times M$ is a Ferrers relation of length k.*

$\mathrm{fdim}_k(G, M, I)$ *is equal to the smallest number of at most $(k-1)$-step Ferrers relations whose union is $G \times M \setminus I$.* □

Now Theorem 46 can be transferred without a new proof:

Theorem 47.

$$\dim_k \underline{\mathfrak{B}}(G,M,I) = \operatorname{fdim}_k(G,M,I)$$

$$\dim_k(P,\leq) = \operatorname{fdim}_k(P,P,\leq). \qquad \square$$

×	×	1	×	1	1	1
2	×	×	2	×	1	1
3	3	×	×	3	×	3
3	3	4	×	×	5	×
×	4	4	4	×	×	4
2	×	2	2	2	×	×
×	5	×	5	5	5	×

Figure 6.4 The 3-dimension of PG(2,2) is 5. In arithmetic terms a covering with four Ferrers relations of the length 3 would be conceivable, however, such a covering does not exist. The 2-dimension is 7.

Of particular interest is the 2-dimension, i.e., the smallest number of squares by which the complement of I can be covered. The direct products of two-element chains are precisely the power-set lattices. Hence the 2-dimension of a complete lattice is also the smallest possible size of a set representation in the sense of Definition 35 (p. 74). We show a connection with another form of representation:

Definition 85. A **set representation** of a context (G,M,I) on a set T is a pair of maps $\alpha : G \to \mathfrak{P}(T)$, $\beta : M \to \mathfrak{P}(T)$ assigning a subset of T to each object and to each attribute, in such a way that:

$$gIm \iff \alpha g \cap \beta m \neq \emptyset.$$

We speak of a **complementary set representation** if the condition

$$gIm \iff \alpha g \cap \beta m = \emptyset$$

is satisfied. $\diamond$

In Section 1.4 we had defined the contexts for free distributive lattices by means of set representations.

Proposition 105. *The following statements are equivalent:*

1. *(G,M,I) has a complementary set representation in T.*
2. *$\underline{\mathfrak{B}}(G,M,I)$ has a set representation in T.*
3. $\dim_2 \underline{\mathfrak{B}}(G,M,I) \leq |T|$.

Proof.

$1 \Rightarrow 3$: If (α, β) is a complementary set representation in T, then the relations

$$\begin{aligned} F_t &:= \{g \mid t \in \alpha g\} \times \{m \mid t \in \beta m\} \\ &= \{(g,m) \mid t \in \alpha g \cap \beta m\} \\ &\subseteq G \times M \setminus I \end{aligned}$$

are 1-step Ferrers relations (or empty) with $\bigcup_{t \in T} F_t = G \times M \setminus I$, and by Theorem 47 it follows that $\dim_2 \underline{\mathfrak{B}}(G, M, I) \leq |T|$.

$3 \Rightarrow 2$: The products of 2-element chains are precisely the power-set lattices and, therefore, $\dim_2 \underline{\mathfrak{B}}(G, M, I) \leq n$ is equivalent to the fact that $\underline{\mathfrak{B}}(G, M, I)$ can be order-embedded in the power-set lattice of an n-element set.

$2 \Rightarrow 1$: If φ is an order embedding in a power-set lattice $\mathfrak{P}(T)$, then by $\alpha g := \varphi\gamma g$, $\beta m := T \setminus \varphi\mu m$ we obtain a complementary set representation of (G, M, I), since

$$gIm \iff \gamma g \leq \mu m \iff \varphi\gamma g \subseteq \varphi\mu m \iff \varphi\gamma g \cap (T \setminus \varphi\mu m) = \emptyset.$$

□

We know already from the observations following Proposition 102 that in order to determine the Ferrers dimension of a doubly founded context $\underline{\mathfrak{B}}(G, M, I)$, it suffices to cover those pairs (g, m) of the complement of I for which $g \swarrow m$ or $g \nearrow m$ holds. In fact, we can even restrict ourselves to the double arrows, since they play the role for the Ferrers dimension which the *critical pairs* known from order theory play for the order dimension.

Theorem 48. *A doubly founded context (G, M, I) has Ferrers dimension $\leq n$ if and only if there are n Ferrers relations $F_t \subseteq G \times M \setminus I$ whose union contains all pairs (g, m) with $g \swarrow\!\!\!\nearrow m$.*

If, in this connection, all F_t are at most $(k-1)$-step, then

$$\mathrm{fdim}_k(G, M, I) \leq n$$

also holds.

Proof. First we describe the possibility of suitably extending a given Ferrers relation F. The basic idea is that the Ferrers condition is not affected if we double some row or column of the context. Formally, this can be described as follows: For a Ferrers relation $F \subseteq G \times M$ and objects $g, h \in G$,

$$F \cup \{(g, m) \mid (h, m) \in F\}$$

is also a Ferrers relation. Moreover, if $F \cap I = \emptyset$, then from $g' \subseteq h'$ it follows that $F \cup \{(g, m) \mid (h, m) \in F\}$ is also disjoint to I. The number of steps does not increase. The corresponding is true when we copy column n into column m $(m, n \in M)$, i.e., when we pass from F to

$$F \cup \{(g, m) \mid (g, n) \in F\}.$$

If, therefore, we define for a Ferrers relation $F \subseteq G \times M \setminus I$

$$\begin{aligned} \widetilde{F} &:= F \cup \bigcup_{g' \subseteq h'} \{(g, m) \mid (h, m) \in F\} \\ \text{and thus} \quad \overline{F} &:= \widetilde{F} \cup \bigcup_{m' \subseteq n'} \{(g, m) \mid (g, n) \in F\}, \end{aligned}$$

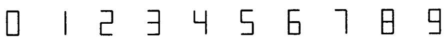
we obtain again a Ferrers relation $\overline{F} \subseteq G \times M \setminus I$. If F is at most k-step, then $\overline{F}$ is at most k-step.

Now, assume that for $t \in T$ Ferrers relations $F_t \subseteq G \times M \setminus I$ are given which together cover all double arrows. We claim that the relations $\overline{F}_t, t \in T$ then completely cover $G \times M \setminus I$:

If $(g, m) \notin I$, then because of doubly foundedness, there is an attribute n with $g \nearrow n$ and $m' \subseteq n'$, and furthermore an object h with $h \swarrow\!\!\nearrow n$ and $g' \subseteq h'$. If $(h, n) \in F_t$, then $(g, n) \in \widetilde{F}_t$ and $(g, m) \in \overline{F}_t$. □

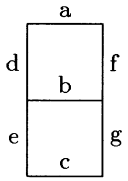
From Definition 85 it follows immediately that a set representation of (G, M, I) is a complementary set representation of the complementary context $(G, M, G \times M \setminus I)$, and vice versa. The determination of the **set dimension** of $\mathbb{K}$, i.e., the smallest possible cardinality of a set representation, is therefore equivalent to the determination of the 2-dimension of $\mathbb{K}^c$ (and difficult, since it is also $\mathcal{NP}$-complete). According to the above-stated results, the task consists in filling up the relation I with as few as possible 1-step Ferrers relations, i.e., with as few as possible "rectangular" subrelations. For such squares (A, B) with $A \times B \subseteq I$ we had earlier introduced the term *preconcept* of the context (G, M, I). Hence the set dimension is equal to the smallest number of preconcepts (more precisely: sets $A \times B$, where (A, B) is a preconcept) whose union fills up I. Since, however, every preconcept can be extended to a concept, the set dimension is also equal to the smallest number of concepts whose union is I. We give an example:

Example 15. We want to find out whether the digits of the seven-segment display



can be represented as the unions of less than seven parts.

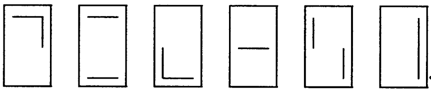
For this purpose we consider the context in Figure 6.5, whose objects are those digits and whose attributes are the segments of the display; the incidence is explained in the obvious way. The intents naturally correspond to partial figures of the display; a set of concepts fills out I if and only if every object intent can be represented as the union of (concept) intents from this



	a	b	c	d	e	f	g
0	×		×	⊗	×	×	×
1						⊗	×
2	×	×	×		⊗	×	
3	×	×	⊗			×	×
4		⊗		×		×	×
5	×	×	×	×			×
6	×	×	×	×	×		×
7	⊗					×	×
8	×	×	×	×	×	×	×
9	×	×	×	×		×	×

Figure 6.5 The seven-segment display and the context of the digits. The crosses surrounded by circles mark a blocking set.

set. Thus, the task of getting along with as few partial figures as possible is equivalent to the determination of the set dimension. In fact, six intents are sufficient, namely

.

The fact that a smaller number cannot suffice is evidenced by the "blocking set" of incidences marked in Figure 6.5: No two of these crosses can belong to a common preconcept.

Embeddings in direct products of chains are also of interest for the drawing of lattice diagrams. In this context, $\bigvee$-embeddings, i.e., injective, $\bigvee$-preserving maps, have proved helpful, and we are faced with the question of the existence of such embeddings. The theoretical background of this question is elaborated in Chapter 7 in a different context. We give the following result without proof, because it can be obtained as a special case of the Propositions 119 (p. 260) and 121 (p. 263).

Theorem 49. *There exists a $\bigvee$-embedding of a finite lattice $\boldsymbol{V}$ in the direct product of n chains of the lengths $l_1, \dots, l_n$ if and only if $M(\boldsymbol{V})$ can be covered by n chains of cardinality $m_1, \dots, m_n$, where $m_i \leq l_i$ holds for $i \in \{1, \dots, n\}$.* □

As an immediate consequence, we can determine two parameters for this problem of embedding: If we define the **$\bigvee$-dimension** to be the smallest number of chains in whose product the lattice can be embedded $\bigvee$-preservingly and the **$\bigvee$-rank** to be the least length of such a product, the theorem gives us satisfactory information on those numbers, at least for finite lattices. For this purpose we use a well known result of Dilworth ([29], p. 3), which says that the smallest number of chains by which a finite ordered set can be covered is equal to the width of this ordered set.

Corollary 106. *In the case of a finite lattice $\mathbf{V}$, the $\bigvee$-dimension is equal to the width of $M(\mathbf{V})$ and the $\bigvee$-rank is equal to the cardinality of $M(\mathbf{V})$.* □

6.5 Hints and References

The lattice properties treated in this chapter, namely distributivity, its generalizations, and modularity, as well as questions of dimension, are basic topics of the mathematical theories of lattices and order. We did not intend to offer a concept-analytic representation of the general state of the research; rather, we want to show that these subjects can be included without difficulty in our language and can thus be used. Naturally, our summary lays no claim to completeness. We refer to the text books on order and lattice theory which were mentioned in the beginning and to current discussion in the respective scientific journals, e.g. ORDER. A detailed comparison of many properties of finite lattices has been elaborated by Reeg and Weiß [139]. Algorithms and complexity estimations can be found in the doctoral thesis [157] of Skorsky.

6.1 A detailed examination of the distributivity of concept lattices is attributable to Erné. His study [47] contains many results which go beyond the contents of this chapter. Furthermore, he was the first to work out that the different variants of the distributive law coincide in the doubly founded case. A connection is established between Theorem 40 and topological statements. This theorem was taken from [197] and goes back to ideas of Raney [138]. Finite distributive concept lattices are described in [196]. That distributivity can be described by means of the arrow relations has been known for quite a long time in the version of Theorem 41(9). The elegant concise version in Theorem 41(3,4) also goes back to Erné.

6.2 There is a book by Stern [162] on semimodular lattices. The characterization of semimodularity by means of the arrow relations was derived by Skorsky from a result of Faigle and Herrmann [51].

6.3 The literature on the generalizations of the distributive laws is so extensive that we have to refer to the relevant text books, in particular to Crawley & Dilworth [29]. Semidistributivity is of great importance in connection with the examination of free lattices, as Jónsson and Kiefer [90] have demonstrated; see also Nation [130]. Day [33] already uses the characterization by means of the double arrow relations. There are generalizations of semidistributivity which can also be described by means of the arrow relations; see Day, Nation & Tschantz [35] as well as Geyer [70].

Locally distributive lattices have been introduced by Dilworth [40] and appear in different connections as natural structures. A concentrated overview over the development of this term is given by Monjardet [127]. The different parts of Theorem 44 have been compiled from publications by Green and Markowsky [78], Jamison [88] and Edelman [46].

The comparison of lattice properties in Figure 6.1 again goes back to Skorsky; see also R. Schmidt [153]. Skorsky also has worked on neighbourhood-preserving embeddings into distributive lattices [156], since this is interesting for the automatic generation of diagrams. Wild [185] had solved the problem for power-set lattices.

6.4 The fact that the order dimension of $(P, \leq)$ is equal to that of the Dedekind-MacNeille-completion has been known for quite a long time, compare [5], [151]. Baldy & Mitas [4] have generalized this.

The terms *Ferrers relation* and *-dimension* go back to Riguet [145] and Cogis [27], [28], The works of the latter already contain parts of the content of Theorem 46. Bouchet [23] proves the theorem for ordered sets, the generalized version can be found in [204]. With regard to the Ferrers dimension see also Doignon, Ducamp & Falmagne [41] and Koppen [102]. Applications can be found in Reuter [141], [142], Ganter, Nevermann, Reuter and Stahl [61]. Closely related to the Ferrers relations are the **interval orders**. These are ordered sets $(P, \leq)$ in which  cannot be embedded. Formally, the condition reads:

$$u < v, \quad x < y, \quad u \not< y \quad \Rightarrow \quad x < v.$$

By means of Definition 83 we recognize that this is precisely equivalent to the fact that $<$ is a Ferrers relation. Hence examinations of the "interval dimension" are closely related to those of the Ferrers dimension. It is possible to generalize this considerably by considering the order of the intervals of an arbitrary ordered set. This was done by Mitas [126].

The example of the seven-segment display goes back to Stahl and Wille [159]. With respect to set representations see also Markowsky [124].

7. Context Comparison and Conceptual Measurability

Maps between concept lattices that can be used for structure comparison are above all the complete homomorphisms. In Section 3.2 we have worked out the connection between compatible subcontexts and complete congruences, i.e., the kernels of complete homomorphisms. A further approach consists in coupling the lattice homomorphisms with context homomorphisms. In this connection, it seems reasonable to use pairs of maps, i.e., to map the objects and the attributes separately. Those pairs can be treated like maps. We do so without further ado and write, for instance,

$$(\alpha, \beta) : (G, M, I) \to (H, N, J),$$

if we mean a pair of maps $\alpha : G \to H$, $\beta : M \to N$, using the usual notations for maps by analogy. This does not present any problems, since in the case that $G \cap M = \emptyset = H \cap N$ we can replace such a pair of maps (α, β) by the map

$$\alpha \cup \beta : G \,\dot{\cup}\, M \to H \,\dot{\cup}\, N.$$

First we treat automorphisms of contexts in order to describe lattice automorphisms with their help. In this context, the algorithmic questions are of interest. We show that it is possible to generate the automorphisms of a context with the same algorithm that we have already used in the second chapter in order to generate the concepts.

If we are looking for a duality that maps the complete homomorphisms between concept lattices and suitable morphisms between the corresponding contexts onto each other, then very simple examples show that this cannot be realized without restrictions. The reason is that very different contexts can have isomorphic concept lattices. We already know that the structure of the concept lattice does not change if we clarify or reduce the context.

There are different possibilities to get by with. One of them consists in using set-valued maps. Another approach considers the concept lattices only up to isomorphism and describes the homomorphisms by morphisms between suitable contexts. This will be explained in the second section.

When *scaling* a many-valued context as defined in the first chapter, we have the possibility of choosing the scales. In order to be able to use this instrument more purposefully, we need methods of comparing scales. Is it possible to scale an attribute *coarser* or *finer*? And what are the consequences to the concept lattice? In order to answer these questions, we shall introduce the suitable morphisms, namely the *scale measures.*

The following is a task which occurs frequently: Data are given in the form of a one-valued context, but in reality we suspect the data to be of a many-valued nature. We can try to reverse the procedure of scaling and ask whether the conceptual structure of the given one-valued context can be explained entirely or partly by the introduction of many-valued attributes with a given scaling. This can also be approached with the help of the scale measures. It leads to the development of the approach of a *concept-analytic measurement theory.*

7.1 Automorphisms of Contexts

Definition 86. An **isomorphism between contexts** $\mathbb{K}_1 := (G, M, I)$ and $\mathbb{K}_2 := (H, N, J)$ is a pair (α, β) of bijective maps $\alpha : G \to H$, $\beta : M \to N$ with

$$gIm \iff \alpha(g)J\beta(m).$$

In the case $\mathbb{K}_1 = \mathbb{K}_2$ we call this an **automorphism**; the group of automorphisms of a context $\mathbb{K}$ is denoted by $\mathrm{Aut}(\mathbb{K})$. ◇

Isomorphic contexts have isomorphic concept lattices, since every context isomorphism (α, β) through

$$(A, B) \mapsto (\alpha(A), \beta(B)) \qquad \text{for } (A, B) \in \underline{\mathfrak{B}}(\mathbb{K}_1)$$

induces a lattice isomorphism of $\underline{\mathfrak{B}}(\mathbb{K}_1)$ onto $\underline{\mathfrak{B}}(\mathbb{K}_2)$. If both contexts are reduced, then every lattice isomorphism is induced by one (and only one) context isomorphism. More generally: If the contexts are clarified, a lattice isomorphism φ is induced by a context isomorphism if and only if φ surjectively maps object concepts onto object concepts and attribute concepts onto attribute concepts.

An observation by W. Xia shows that in turn we can interpret the isomorphisms themselves as concepts of a suitable context. For this purpose we define:

Definition 87. For contexts $\mathbb{K}_1 := (G, M, I)$ and $\mathbb{K}_2 := (H, N, J)$ we have

$$\mathbb{K}_1 \mathbin{\tilde{\times}} \mathbb{K}_2 := (G \times H, M \times N, \sim)$$

with

$$(g, h) \sim (m, n) \quad :\iff \quad (gIm \iff hJn). \qquad ◇$$

Theorem 50. *If $\mathbb{K}_2$ is clarified and if $\alpha \subseteq G \times H$, $\beta \subseteq M \times N$ are bijective maps between G and H or M and N, respectively, then the following conditions are equivalent:*

1. *(α, β) is a context isomorphism of* $\mathbb{K}_1$ *onto* $\mathbb{K}_2$
2. $(\alpha, \beta) \in \underline{\mathfrak{B}}(\mathbb{K}_1 \mathbin{\tilde{\times}} \mathbb{K}_2)$.

Proof. For $A \subseteq G \times H$, $B \subseteq M \times N$ it holds that

$$A^{\sim} = \{(m,n) \mid gIm \iff hJn \text{ for all } (g,h) \in A\}$$

as well as

$$B^{\sim} = \{(g,h) \mid gIm \iff hJn \text{ for all } (m,n) \in B\}.$$

Hence, a pair (α, β) of bijective maps is an isomorphism of $\mathbb{K}_1$ onto $\mathbb{K}_2$ if and only if $\alpha \subseteq \beta^{\sim}$ or, equivalently, if $\beta \subseteq \alpha^{\sim}$. This, however, also implies that $\alpha = \beta^{\sim}$ and $\beta = \alpha^{\sim}$, since, if $(g,h) \in \beta^{\sim}$, then

$$hJ\beta m \iff gIm \iff \alpha g J \beta m$$

for all $m \in M$, from which we infer that $h^J = (\alpha g)^J$. If $\mathbb{K}_2$ is clarified, this implies $h = \alpha g$ and we obtain $\alpha = \beta^{\sim}$. □

Furthermore, we learn from the theorem that, in the case of clarified contexts, an isomorphism (α, β) is determined already by each of its two components.

The theorem permits a useful tightening. Frequently, we know already that a certain object g of $\mathbb{K}_1$ cannot be mapped onto $h \in \mathbb{K}_2$ by an isomorphism, for instance, because g does not have the same number of attributes as h. Hence, the object (g,h) of $\mathbb{K}_1 \mathbin{\tilde{\times}} \mathbb{K}_2$ cannot belong to an isomorphism and is superfluous in this respect. Indeed, such object or attribute pairs can be omitted without changing the content of the theorem. This is formulated in the following theorem, for which we define that

$$\tilde{G} := \bigcup\{\alpha \mid (\alpha, \beta) \text{ is an isomorphism of } \mathbb{K}_1 \text{ onto } \mathbb{K}_2\},$$

$$\widetilde{M} := \bigcup\{\beta \mid (\alpha, \beta) \text{ is an isomorphism of } \mathbb{K}_1 \text{ onto } \mathbb{K}_2\}.$$

Corollary 107. *Let X and Y be sets with $\tilde{G} \subseteq X \subseteq G \times H$ and $\widetilde{M} \subseteq Y \subseteq M \times N$. Let $\mathbb{K}_2$ again be clarified. Then for bijective maps $\alpha : G \to H$, $\beta : M \to N$ the following conditions are equivalent:*

1. *(α, β) is an isomorphism of* $\mathbb{K}_1$ *onto* $\mathbb{K}_2$
2. $(\alpha, \beta) \in \underline{\mathfrak{B}}(X, Y, \sim \cap X \times Y)$.

Proof. Every isomorphism is a concept of $\underline{\mathfrak{B}}(\mathbb{K}_1 \mathbin{\tilde{\times}} \mathbb{K}_2)$ whose extent and intent are entirely contained in the subcontext, i.e., a concept of the subcontext. If, conversely, (α, β) is a concept of the subcontext, then certainly $\alpha \subseteq \beta^{\sim}$. In the proof of the preceding theorem we have already shown that, in the case of bijective maps, this implies that (α, β) is an isomorphism. □

We can combine this theorem (as well as its corollary) with the algorithms from Section 2.1 and thereby obtain for instance a method for the generation of all automorphisms of a (finite) context $\mathbb{K} := (G, M, I)$. However, we shall want to avoid working in the context $\mathbb{K} \tilde{\times} \mathbb{K}$, since this context is considerably larger than $\mathbb{K}$. Indeed, the necessary calculations can be reduced to calculations in $\mathbb{K}$. This is prepared by the following definition:

Definition 88. A **box relation** on a (finite) set S is a subset $R \subseteq S \times S$ of the form

$$R = A_1 \times B_1 \cup A_2 \times B_2 \cup \ldots \cup A_r \times B_r,$$

with $A_1, \ldots, A_r$ being pairwise disjoint non-empty subsets of S and $B_1, \ldots, B_r$ as well. A box relation is called **regular** if $|A_i| = |B_i|$ for all $i \in \{1, \ldots, r\}$ and if, furthermore, $\bigcup A_i = S = \bigcup B_i$. A box relation is a **partial permutation** if $|A_i| = 1 = |B_i|$ for $i \in \{1, \ldots, r\}$. $\diamond$

The box relations on S form a closure system. Their role for the automorphisms is illustrated by the following proposition:

Proposition 108. *All extents of* $\mathbb{K} \tilde{\times} \mathbb{K}$ *are box relations on* G *and all intents are box relations on* M.

If (α, β) *is an automorphism of* $\mathbb{K}$ *and* (A, B) *is a concept of* $\mathbb{K} \tilde{\times} \mathbb{K}$ *with* $(A, B) \leq (\alpha, \beta)$, *then* A *is a partial permutation and* B *is regular.*

Proof. Because of

$$(g, h)^{\sim} = g' \times h' \cup (M \setminus g') \times (N \setminus h'),$$

every object intent of $\mathbb{K} \tilde{\times} \mathbb{K}$ is a box relation, and thus so is every intersection of object intents, i.e., every concept intent. The corresponding is true for extents.

$(A, B) \leq (\alpha, \beta)$ is equivalent to $A \subseteq \alpha$ and $\beta \subseteq B$. If α and β are permutations, then trivially A is a partial permutation. The fact that B must be regular can be seen from the following (trivial) proposition. $\square$

Proposition 109. *If a box relation contains a regular box relation, it is regular itself.* $\square$

For practical work it is useful to represent a box relation R by means of two maps $\rho_1 : S \to \mathbb{N}_0$, $\rho_2 : S \to \mathbb{N}_0$ with

$$(s, t) \in R \iff \rho_1(s) = \rho_2(t) > 0.$$

With this representation it is for example easy to find the intersection of box relations or to carry out the simpler context operations. If, for example,

$$R = A_1 \times B_1 \cup A_2 \times B_2 \cup \ldots \cup A_r \times B_r$$

is an intent of $\mathbb{K} \tilde{\times} \mathbb{K}$ and $(g, h) \in G \times G$, then

$$(g,h) \in R^{\sim} \iff \begin{cases} g' = \bigcup\{A_i \mid B_i \subseteq h'\} \\ h' = \bigcup\{B_i \mid A_i \subseteq g'\} \end{cases}.$$

We now want to state an algorithm generating all automorphisms of a clarified context $\mathbb{K}$. In principle, we can use the results of Section 2.1 for this purpose: there we have introduced an algorithm for the generation of all concepts of a context, and by Theorem 50 the automorphisms are particular concepts. Hence, we could obtain the automorphisms by generating the concepts of $\mathbb{K} \tilde{\times} \mathbb{K}$ and rejecting those which are not automorphisms. However, this would generally require an absurd amount of work, we would have to check huge numbers of concepts in order to find some automorphisms. Therefore, we shall try to use Theorem 6 (p. 68) and search for a sub-family $\mathcal{F}$ of the set of all extents of $\mathbb{K} \tilde{\times} \mathbb{K}$ which satisfies the prerequisites of the theorem and supplies all automorphisms, but few other extents.

However, this is also unrealistic. Considerations concerning complexity show that we cannot expect that there is an algorithm which can easily find the automorphisms for any context $\mathbb{K}$. The set $\mathcal{F}$ we use therefore contains in addition to the automorphisms further extents, in the worst case even many of them.

In order to apply Theorem 6 (p. 68), we have to order the object set linearly. For this purpose we choose an arbitrary linear order on G and set

$$(g_1,h_1) < (g_2,h_2) :\iff \begin{cases} g_1 < g_2 & \text{or} \\ g_1 = g_2 & \text{and } h_1 < h_2. \end{cases}$$

If $\alpha \subseteq G \times G$ is a partial permutation and $g \in G$ is an object, then we say that α is **undefined** for g if there is no $h \in G$ with $(g,h) \in \alpha$. We call α **flush left** if the following holds true: If α is undefined for g, then $\alpha = \alpha \cap \{(g_1,g_2) \mid (g_1,g_2) < (g,g)\}$.

Proposition 110. *The set $\mathcal{F}$ of all extents of concepts $(\alpha,\beta) \in \underline{\mathfrak{B}}(\mathbb{K} \tilde{\times} \mathbb{K})$ for which it holds that*

- *α is a flush left partial permutation,*
- *β is a regular box relation,*

satisfies the prerequisites of Theorem 6 (p. 68).

Proof. Let (α,β) be such a concept and let $(i,j) \in G \times G$ be arbitrary. Then

$$\alpha_0 := (\alpha \cap \{(g,h) \mid (g,h) < (i,j)\})''$$

is the extent of a subconcept (α_0,β_0) of (α,β), in particular, we have $\alpha_0 \subseteq \alpha$ and $\beta \subseteq \beta_0$. Trivially α_0 is a partial permutation and, by Proposition 109, β_0 is regular. Hence, we only have to prove that α_0 is flush left. Let g be undefined for α_0. If $g \geq i$, there is nothing to prove, hence, let $g < i$. Then, however, α is undefined for g, from which the assertion follows. □

The set of concepts described in the proposition contains all automorphisms. They are precisely those elements (α, β) of $\mathcal{F}$ for which α is a (complete) permutation. Hence, the algorithm consists of scanning the set $\mathcal{F}$ in the manner described in Theorem 6 (p. 68) and of counting those elements α which are not undefined for any object as hits.

A further application of Theorem 6 allows us to calculate a concept lattice "modulo automorphisms". This means: If $\Gamma \leq \mathrm{Aut}(G, M, I)$ is a group of context automorphisms, then $\underline{\mathfrak{B}}(G, M, I)$ divides into orbits under Γ. We select exactly one concept from each such orbit, namely the one with the lectically largest extent. Such an extent is called **orbit-maximal.** We want to avoid calculating first all extents and then determining the orbit-maximal ones among them. The following theorem shows that the algorithm from Section 2.1 can be suitably modified such that it only generates the desired extents.

Theorem 51. *The smallest orbit-maximal concept extent lectically greater than a set $A \subseteq G$ is*

$$A^+ := A \oplus i,$$

i being the largest element of G for which $A <_i A \oplus i$ and, at the same time, $\alpha(A \oplus i) \leq A \oplus i$ for all $(\alpha, \beta) \in \Gamma$.

Proof. What we have to show is that the system of orbit-maximal extents satisfies the prerequisites of Theorem 6 (p. 68). We prove that, more generally, the following holds true: *If B is an extent and $A = (B \cap \{1, \ldots, i-1\})''$, then*

$$A < \alpha(A) \Rightarrow B < \alpha(B)$$

holds for every automorphism (α, β) of (G, M, I).

$A < \alpha(A)$ means $A <_j \alpha(A)$ for some $j \in G$. If $i \leq j$, this would mean that

$$A \cap \{1, \ldots, i-1\} = \alpha(A) \cap \{1, \ldots, i-1\},$$

which, because of

$$A = (A \cap \{1, \ldots, i-1\})'',$$

would imply that $A \subseteq \alpha(A)$ and thus $A = \alpha(A)$, contradictory to $A < \alpha(A)$. Hence, $j < i$ must hold. But then we have

$$B \cap \{1, \ldots, j-1\} = A \cap \{1, \ldots, j-1\} = \alpha(A) \cap \{1, \ldots, j-1\} \subseteq \alpha(B)$$

and $j \in \alpha(B) \setminus B$, from which it follows that $B < \alpha(B)$. □

In the case of lattices with many automorphisms a diagram of the orbits is often easier to read than a diagram representing all elements of the lattice. As an example, we show in Figure 7.1 the lattice of the 59 subgroups of the alternating group A_5. Represented are the nine orbits of the automorphism group Γ, which is isomorphic to S_5. We have stated one representative of each subgroup orbit. The orbits are ordered according to the following rule:

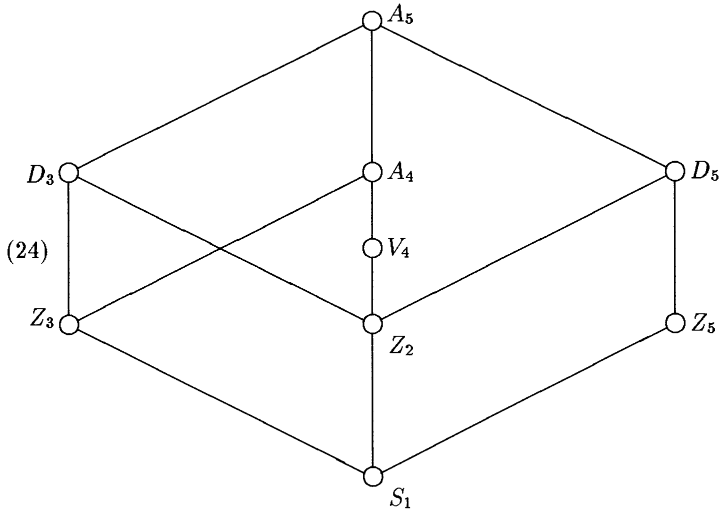


Figure 7.1 The lattice of subgroups of the alternating group A_5 "modulo automorphisms". The representatives of the subgroup orbits are: $S_1 := \{id\}$, $Z_2 := \{id, (01)(23)\}$, $Z_3 := \{id, (012), (021)\}$, $V_4 := \{id, (01)(23), (03)(12), (02)(13)\}$, $Z_5 := \langle(01234)\rangle$, $D_3 := \langle(01)(23), (014)\rangle$, $D_5 := \langle(01)(23), (01234)\rangle$, $A_4 := \langle(01)(23), (012)\rangle$, $A_5 := \langle(012), (01234)\rangle$.

The orbit a^Γ is less than or equal to the orbit b^Γ, if and only if there is an automorphism $\gamma \in \Gamma$ with the property that γa is a subgroup of b. This does not necessarily imply that a is also a subgroup of b. In Figure 7.1 for example, D_3 does not contain the group Z_3 but the group $(24)Z_3(24)$, which is obtained from Z_3 through conjugation. This is the reason why the element (24) is entered at the corresponding edge of the line diagram.

Such diagrams "modulo automorphisms" are used in group theory, but can be used more generally. M. Zickwolff (from whose publications [218], [219] the example has been taken) has worked out which information has to be entered into the diagram of the orbits of a (finite) lattice $\boldsymbol{V}$ with automorphism group Γ, so that we can reconstruct $\boldsymbol{V}$ from it. For this purpose, we first choose an arbitrary system of representatives R of the orbits (for example the set of the orbit maximal elements in Theorem 51). For every representative $a \in R$ we note down the stabilizer Γ_a and enter it into the orbit diagram at the corresponding element. It is often sufficient, as in Figure 7.1, to enter the representative a and to calculate the stabilizer Γ_a when needed. For each two elements $a, b \in R$ we determine the set

$$\{\gamma \in \Gamma \mid \gamma a \prec b\}$$

of those automorphisms which map a onto a lower neighbour of b. This set divides into (disjoint) double cosets of the form

$$\Gamma_b \alpha \Gamma_a.$$

Therefore, it is sufficient to note down a system of representatives $\beta(a,b)$ for each of these classes. Provided that it is non-empty, $\beta(a,b)$ is also entered into the orbit diagram, namely at the edge between a and b. The complication may arise that $\beta(a,b) \neq \emptyset$, but that a^Γ is not a lower neighbour of b^Γ (in the orbit diagram). In this case we add an edge from a to b to the diagram. For reasons of simplicity we agree that an unlabelled edge from a to b shall symbolize $\beta(a,b) = \{id\}$. Figure 7.2 shows a further simple example: on the left, a lattice whose automorphism group is apparently isomorphic to the cyclic group Z_3 and on the right, the orbit diagram with the necessary information.

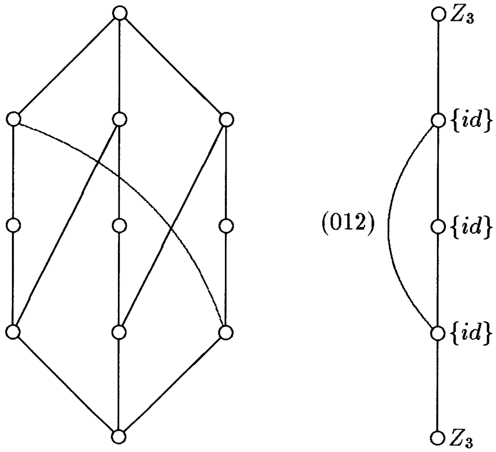


Figure 7.2 The diagram of orbits (on the right) has an additional edge.

7.2 Morphisms and Bonds

Definition 89. If $\mathbb{K}_1 := (G, M, I)$ and $\mathbb{K}_2 := (H, N, J)$ are contexts, we call a map $\alpha : G \to H$

- **extensionally continuous**, if for every extent U of $\mathbb{K}_2$ the pre-image $\alpha^{-1}(U)$ is an extent of $\mathbb{K}_1$.
- **extensionally closed**, if the image $\alpha(U)$ of an extent U of $\mathbb{K}_1$ is always an extent of $\mathbb{K}_2$.

The extensionally continuous maps are also called **scale measures**; they will be examined in detail in the next section. Dually, we will explain in which cases a map is $\beta : M \to N$ **intensionally continuous** and **intensionally closed**, respectively. A pair of maps $(\alpha, \beta) : \mathbb{K}_1 \to \mathbb{K}_2$ is called

- **incidence preserving** if

$$gIm \Rightarrow \alpha(g)J\beta(m)$$

  for all $g \in G, m \in M$,
- **incidence reflecting** if

$$gIm \Leftarrow \alpha(g)J\beta(m)$$

  for all $g \in G, m \in M$,
- **continuous** if α is extensionally continuous and β is intensionally continuous,
- **concept preserving** if, for every concept $(A,B) \in \underline{\mathfrak{B}}(\mathbb{K}_1)$, the pair $(\beta(B)', \alpha(A)')$ is a concept of $\mathbb{K}_2$, and
- **concept faithful** if it is at the same time concept preserving and continuous.

$\diamond$

Even if (α, β) is incidence-preserving and -reflecting, the two maps need not be injective. However, in this case from $\alpha(g) = \alpha(h)$ follows that $g' = h'$ and dually; i.e., this kind of map is "injective up to clarification".

From Proposition 33 (p. 99) it follows that to the incidence-preserving maps correspond certain order-preserving maps between the concept lattices. Concept-preserving maps are necessarily incidence-preserving, the map stated in the definition being order-preserving. These maps, however, need not be lattice homomorphisms. It is different in the case of the concept faithful maps:

Theorem 52. *If* $(\alpha, \beta) : \mathbb{K}_1 \to \mathbb{K}_2$ *is a concept faithful map, then*

$$(A, B) \mapsto (\beta(A)', \alpha(A)')$$

is a complete homomorphism from $\underline{\mathfrak{B}}(\mathbb{K}_1)$ *to* $\underline{\mathfrak{B}}(\mathbb{K}_2)$.

A complete homomorphism is induced by a concept faithful map if and only if it maps object concepts onto object concepts and attribute concepts onto attribute concepts. $\square$

Erné has proved this theorem with the aim of representing lattice homomorphisms through concept faithful maps; the restriction made in the theorem disappears if we consider the lattices only up to an isomorphism: According to the Basic Theorem, every lattice $\boldsymbol{V}$ is isomorphic to $\underline{\mathfrak{B}}(V, V, \leq)$ and for this context every concept is an object and an attribute concept, and consequently, every complete homomorphism between such concept lattices is induced by a concept faithful map. Erné furthermore gives a simple characterization of the concept faithful maps, which we add:

Proposition 111. *An incidence-preserving map*

$$(\alpha, \beta) : (G, M, I) \to (H, N, J)$$

is concept faithful if and only if the following condition is satisfied for all $h \in H$, $n \in N$:

$$(h, n) \notin J \quad \Rightarrow \quad \exists_{(g,m) \notin I}\ \alpha^{-1}(n') \subseteq m' \text{ and } \beta^{-1}(h') \subseteq g'. \qquad \square$$

The notions of morphism for contexts which we have treated so far have the disadvantage that they do not represent all lattice homomorphisms, as long as the context may not be changed as described above. Various attempts have been made to overcome this difficulty through the introduction of *set-valued* maps. We choose a different way, which however makes it easy to achieve these results. The question for a context description for the $\bigvee$-morphisms leads us back to the notion of a *bond* from a context (G, M, I) to a context (H, N, J), which we have already examined in connection with the subdirect products. By this we understand (Definition 69, p. 185) a relation $R \subseteq G \times N$ with the property that the set g^R of elements being in a relation with an object $g \in G$ always forms an intent of (H, N, J) and that dually for every $n \in N$ the set n^R forms an extent of (G, M, I). In order to make it easier to take up previously obtained results, we modify this definition by dualizing the target context:

Definition 90. A **dual bond** from (G, M, I) to (H, N, J) is a bond between (G, M, I) and the context dual to (H, N, J), i.e., a relation $R \subseteq G \times H$ for which it holds that:

– for every object $g \in G$, g^R is an extent of (H, N, J) and
– for every object $h \in H$, h^R is an extent of (G, M, I).

$\diamond$

The notion of the dual bond is symmetric: if R is a dual bond from $\mathbb{K}_1$ to $\mathbb{K}_2$, then R^{-1} is a dual bond from $\mathbb{K}_2$ to $\mathbb{K}_1$.

Theorem 53. *For every dual bond $R \subseteq G \times H$,*

$$\varphi_R(X, X^I) := (X^R, X^{RJ}), \qquad \psi_R(Y, Y^I) := (Y^R, Y^{RJ})$$

defines a Galois connection (φ_R, ψ_R) between $\underline{\mathfrak{B}}(G, M, I)$ and $\underline{\mathfrak{B}}(H, N, J)$. Conversely, for every Galois connection (φ, ψ),

$$R_{(\varphi,\psi)} := \{(g, h) \mid \gamma g \leq \psi\gamma h\} = \{(g, h) \mid \gamma h \leq \varphi\gamma g\}$$

is a dual bond, and we have

$$\varphi_{R_{(\varphi,\psi)}} = \varphi, \quad \psi_{R_{(\varphi,\psi)}} = \psi \quad \text{as well as} \quad R_{(\varphi_R,\psi_R)} = R.$$

Proof. Because of $X \subseteq Y^R \iff Y \subseteq X^R$, Proposition 4 (p. 11) can be used to prove that (φ_R, ψ_R) is a Galois connection. Besides, the same proposition shows that the two sets specified in order to define $R_{(\varphi,\psi)}$ are equal. Therefore, for every $g \in G$, the set $g^{R_{(\varphi,\psi)}}$ is equal to the extent of $\varphi\gamma g$ and correspondingly, for $h \in H$, $h^{R_{(\varphi,\psi)}}$ is the extent of $\psi\gamma h$. Therefore, $R_{(\varphi,\psi)}$ is a dual bond.

For every dual bond R we have $g^R = g^{IIR}$, since for every extent U of (G, M, I) and thus for every set of the form $U = h^R, h \in H$, holds

$$g \in U \iff g'' \subseteq U.$$

This also holds for $R_{(\varphi,\psi)}$ and therefore we infer that

$$\begin{aligned}
\varphi_{R_{(\varphi,\psi)}}\gamma g &= \varphi_{R_{(\varphi,\psi)}}(g'', g') \\
&= (g''^{R_{(\varphi,\psi)}}, \cdots) \\
&= (g^{R_{(\varphi,\psi)}}, \cdots) \\
&= (\{h \mid (g,h) \in R_{(\varphi,\psi)}\}, \cdots) \\
&= \bigvee\{\gamma h \mid (g,h) \in R_{(\varphi,\psi)}\} \\
&= \bigvee\{\gamma h \mid \gamma h \leq \varphi\gamma g\} \\
&= \varphi\gamma g \quad \text{(since } \gamma(H) \bigvee\text{–dense)},
\end{aligned}$$

and thus that, for an arbitrary concept (X, X'),

$$\begin{aligned}
\varphi(X, X') &= \varphi \bigvee_{g \in X} \gamma g \\
&= \bigwedge_{g \in X} \varphi\gamma g \qquad \text{(according to Proposition 7 (p. 12))} \\
&= \bigwedge_{g \in X} \varphi_{R_{(\varphi,\psi)}}\gamma g \\
&= \varphi_{R_{(\varphi,\psi)}} \bigvee_{g \in X} \gamma g \\
&= \varphi_{R_{(\varphi,\psi)}}(X, X').
\end{aligned}$$

□

Hence, the dual bonds correspond to the Galois connections between the concept lattices, and the latter, according to Proposition 7 (p. 12), correspond to certain morphisms. If we reverse the dualization, i.e., if we go back to a bond, the condition from the proposition becomes easier to use: In this case we obtain a residuated pair of maps, as described in Proposition 9 (p. 14). The two maps mutually determine each other, one of them is a $\bigvee$-morphism, the other is a $\bigwedge$-morphism. This is summarized in the following corollary:

Corollary 112. *For every bond $R \subseteq G \times N$ between contexts (G, M, I) and (H, N, J)*

$$\varphi_R(A, B) := (A^{RJ}, A^R)$$

defines a $\bigvee$-morphism $\varphi_R : \underline{\mathfrak{B}}(G, M, I) \to \underline{\mathfrak{B}}(H, N, J)$, and

$$\psi_R(A, B) := (B^R, B^{RI})$$

defines a $\bigwedge$-morphism $\psi_R : \underline{\mathfrak{B}}(H, N, J) \to \underline{\mathfrak{B}}(G, M, I)$. Conversely, from every $\bigvee$-morphism $\varphi : \underline{\mathfrak{B}}(G, M, I) \to \underline{\mathfrak{B}}(H, N, J)$ results a bond R_φ by virtue of

$$R_\varphi := \{(g, n) \mid \varphi\gamma g \leq \mu n\}$$

and from every $\bigwedge$-morphism $\psi : \underline{\mathfrak{B}}(H, N, J) \to \underline{\mathfrak{B}}(G, M, I)$ results a bond R^ψ by virtue of

$$R^\psi := \{(g, n) \mid \gamma g \leq \psi\mu n\}.$$

We have $R_\varphi = R^\psi$ if and only if ψ is residual to φ. □

In Proposition 83 (p. 185) we introduced a product of bonds. It is possible to show (and we shall do so in the following proposition) that this product corresponds to the concatenation of $\bigwedge$-morphisms. For this purpose, we again use the notations which we used the Propositions 83 and 84.

Proposition 113. *If J_{rs} is a bond from $\mathbb{K}_r$ to $\mathbb{K}_s$ and if J_{st} is a bond from $\mathbb{K}_s$ to $\mathbb{K}_t$, then*

$$\psi_{J_{rs} \circ J_{st}} = \psi_{J_{rs}} \circ \psi_{J_{rs}}$$

holds for the corresponding $\bigwedge$-morphisms.

Proof. We have

$$R^{\psi_{J_{rs}} \circ \psi_{J_{st}}} = \{(g, m) \mid \gamma g \leq \psi_{J_{rs}} \circ \psi_{J_{st}}(\mu m)\}$$

and

$$\psi_{J_{st}}(\mu m) = (m^s, m^{ss}),$$
$$\psi_{J_{rs}}(m^s, m^{ss}) = (m^{ssr}, m^{ssrr}),$$

and therefore

$$(g, m) \in R^{\psi_{J_{rs}} \circ \psi_{J_{st}}} \iff g \in m^{ssr} \iff (g, m) \in J_{rs} \circ J_{st}.$$ □

Complete homomorphisms are those maps which are $\bigvee$-morphisms as well as $\bigwedge$-morphisms. Therefore, the above corollary also yields a characterization of the complete homomorphisms:

Corollary 114. *For every complete homomorphism*

$$\varphi : \underline{\mathfrak{B}}(G, M, I) \to \underline{\mathfrak{B}}(H, N, J),$$

$$R := R_\varphi = \{(g, n) \mid \varphi\gamma g \leq \mu n\} \subseteq G \times N$$

defines a bond from (G, M, I) *to* (H, N, J) *and*

$$S := R^{\varphi} = \{(h, m) \mid \gamma h \leq \varphi\mu m\} \subseteq H \times M$$

defines a bond from (H, N, J) *to* (G, M, I), *and*

$$A^{RJ} = B^S, \qquad B^{SJ} = A^R$$

for all $(A, B) \in \underline{\mathfrak{B}}(G, M, I)$.

If, conversely, $R \subseteq G \times N$ *and* $S \subseteq H \times M$ *are bonds satisfying this condition, then*

$$\varphi(A, B) := (B^S, A^R)$$

defines a complete homomorphism from $\underline{\mathfrak{B}}(G, M, I)$ *to* $\underline{\mathfrak{B}}(H, N, J)$. □

	M	N
G	I	$R := R_{\varphi}$
H	$S := R^{\varphi}$	J

Hence, Corollary 114 shows that complete homomorphisms between concept lattices can be satisfactorily described by means of suitable pairs of bonds between the contexts. Now we have different possibilities of turning these bonds into set-valued maps. For example, we can assign maps

$$\alpha : G \to \mathfrak{P}(N), \quad \beta : M \to \mathfrak{P}(H)$$

to each homomorphism

$$\varphi : \underline{\mathfrak{B}}(G, M, I) \to \underline{\mathfrak{B}}(H, N, J)$$

through

$$\begin{aligned} \alpha g &:= g^R = \{n \mid \varphi\gamma g \leq \mu n\} \\ \beta m &:= m^S = \{h \mid \gamma h \leq \varphi\mu m\}. \end{aligned}$$

The homomorphism φ can be reconstructed from (α, β) through

$$\varphi(A, B) = \Big(\bigcap_{m \in B} \beta m, \bigcap_{g \in A} \alpha g\Big).$$

It is not difficult to characterize the pairs of maps which result in this way from complete homomorphisms by means of the conditions stated in Corollary 114.

The symmetrical situation in Corollary 114 permits further variations. We can also describe the bonds through maps in the other direction, i.e., $\alpha : H \to \mathfrak{P}(M)$, $\beta : N \to \mathfrak{P}(G)$, and so on. We shall give only one further example of this kind:

Proposition 115. *For every complete homomorphism*

$$\varphi : \underline{\mathfrak{B}}(G, M, I) \to \underline{\mathfrak{B}}(H, N, J)$$

the definition

$$g \in \alpha h : \Longleftrightarrow \gamma h \leq \varphi\gamma g, \quad m \in \beta n : \Longleftrightarrow \varphi\mu m \leq \mu n$$

yields a pair of maps

$$\alpha : H \to \mathfrak{P}(G), \quad \beta : N \to \mathfrak{P}(M)$$

with

$$\varphi(A, B) = (\{h \in H \mid \alpha h \subseteq A\}, \{n \in N \mid \beta n \subseteq B\}).$$

If, conversely, $\alpha : H \to \mathfrak{P}(G)$ *and* $\beta : N \to \mathfrak{P}(M)$ *are maps with the property that, for every concept* $(A, B) \in \underline{\mathfrak{B}}(G, M, I)$,

$$(\{h \in H \mid \alpha h \subseteq A\}, \{n \in N \mid \beta n \subseteq B\})$$

is a concept of (H, N, J)*, then the map*

$$(A, B) \mapsto (\{h \in H \mid \alpha h \subseteq A\}, \{n \in N \mid \beta n \subseteq B\})$$

is a complete homomorphism of $\underline{\mathfrak{B}}(G, M, I)$ *to* $\underline{\mathfrak{B}}(H, N, J)$. □

7.3 Scale Measures

Definition 91. Let $\mathbb{K} := (G, M, I)$ be a context and let $\mathbb{S} := (G_\mathbb{S}, M_\mathbb{S}, I_\mathbb{S})$ be a scale. A map $\sigma : G \to G_\mathbb{S}$ is called an $\mathbb{S}$-**measure** if the pre-image $\sigma^{-1}(U)$ of every extent of $\mathbb{S}$ is an extent of $\mathbb{K}$. An $\mathbb{S}$-measure σ is called **full** if every extent of $\mathbb{K}$ is at the same time the pre-image of an extent of $\mathbb{S}$. ◇

In order to visualize this definition, we imagine a new context $\mathbb{K}_\sigma$ whose objects are the objects of $\mathbb{K}$ and whose attributes are the attributes of $\mathbb{S}$. Into the g-row of this context, we enter the $\sigma(g)$-row of the scale. This means that formally we define $\mathbb{K}_\sigma := (G, M_\mathbb{S}, I_\sigma)$ by

$$gI_\sigma m :\Leftrightarrow \sigma(g) I_\mathbb{S} m.$$

Now, the definition says that σ is an $\mathbb{S}$-measure if and only if every extent of $\mathbb{K}_\sigma$ is also an extent of $\mathbb{K}$; σ is full if and only if $\mathbb{K}$ and $\mathbb{K}_\sigma$ have the same extents. Since the context $\mathbb{K}_\sigma$ is defined on the same object set as $\mathbb{K}$, we can imagine the two contexts joined together to form the apposition

$$\mathbb{K} \mid \mathbb{K}_\sigma \ ,$$

whose extents are the same as those of $\mathbb{K}$, provided that σ is a measure. In this way a $\mathbb{S}$-measure is understood as the possibility of extending the given context by attributes from the scale $\mathbb{S}$ without changing the extents. σ is full if the new attributes render the old ones dispensable.

Proposition 116. *For a map $\sigma : G \to G_{\mathbb{S}}$ the following conditions are equivalent:*

1. *σ is an $\mathbb{S}$-measure.*
2. *For all subsets $A \subseteq G$ it holds that $\sigma(A'') \subseteq \sigma(A)''$.*
3. *For all subsets $A, B \subseteq G$ it holds that $A \to B \Rightarrow \sigma(A) \to \sigma(B)$ (In accordance with Section 2.3, $A \to B$ is used as an abbreviation of $B \subseteq A''$).*

σ is full if and only if the inverse implication also holds in (3).

Proof. (1) $\Rightarrow$ (2): Every scale measure satisfies condition (2), since, for every $A \subseteq G$, $\sigma^{-1}(\sigma(A)'')$ is an extent containing A and thus also A''.
(2) $\Rightarrow$ (3): $A \to B \iff B \subseteq A'' \Rightarrow \sigma(B) \subseteq \sigma(A'') \subseteq \sigma(A)'' \Rightarrow \sigma(A) \to \sigma(B)$.
(3) $\Rightarrow$ (1): If U is an extent of $\mathbb{S}$ and g is an arbitrary object from $(\sigma^{-1}(U))$, i.e., with $\sigma^{-1}(U) \to g$, this yields $U \to \sigma g$, i.e. $\sigma g \in U$ and consequently $g \in \sigma^{-1}(U)$, hence this set must be an extent of $\mathbb{K}$.

σ is full if and only if for $A \subseteq G$ it always holds that $A'' = \sigma^{-1}(\sigma(A)'')$; this is however equivalent to

$$g \in A'' \iff \sigma(g) \in \sigma(A)'',$$

i.e., to

$$A \to g \iff \sigma(A) \to \sigma g.$$

□

We do not really have to check the definition of the $\mathbb{S}$-measure for all extents. It suffices that the pre-images of the column extents of $\mathbb{S}$ are extents of $\mathbb{K}$ (since the pre-image of an intersection of sets is the intersection of the pre-images). Likewise, an $\mathbb{S}$-measure is already full if every column extent of $\mathbb{K}$ is the pre-image of an extent of $\mathbb{S}$. If we call a subset T of the attribute set of a context $\mathbb{K}$ **dense** in the case that the set $\{\mu m \mid m \in T\}$ is infimum-dense in $\underline{\mathfrak{B}}(\mathbb{K})$, then we can continue as follows: A scale measure is full, if and only if the set

$$\{m \in M \mid m' \text{ is the pre-image of an extent}\}$$

is dense.

A surjective $\mathbb{S}$-measure is not automatically full. Indeed, every scale measure σ can be replaced by a surjective one if we switch to a subscale (by a **subscale** of a scale $\mathbb{S}$ we understand a subcontext $(T, M_{\mathbb{S}}, I_{\mathbb{S}} \cap (T \times M_{\mathbb{S}}))$ with $T \subseteq G_{\mathbb{S}}$). This is the content of the following proposition:

Proposition 117. *For every subscale $(T, M_{\mathbb{S}}, I_{\mathbb{S}} \cap (T \times M_{\mathbb{S}}))$ of $\mathbb{S}$, the identical map is a full $\mathbb{S}$-measure. For a context (G, M, I), $\sigma : G \to G_{\mathbb{S}}$ is an $\mathbb{S}$-measure if and only if σ is a $(\sigma(G), M_{\mathbb{S}}, I_{\mathbb{S}} \cap (\sigma(G) \times M_{\mathbb{S}}))$-measure.*

Proof. The extents of the subscale are precisely the sets of the form $U \cap \sigma(G)$, with U being an extent of $\mathbb{S}$. □

From a scale measure σ we obtain two maps between the concept lattices of the corresponding contexts $\mathbb{K}$ and $\mathbb{S}$, i.e., one in each direction. This is described by the following two propositions. From the circumstance that the set mapping σ^{-1} is intersection-preserving we immediately infer:

Proposition 118. *If σ is an $\mathbb{S}$-measure of $\mathbb{K}$,*

$$(A, A') \mapsto (\sigma^{-1}(A), \sigma^{-1}(A)')$$

defines a $\bigwedge$-preserving map of $\underline{\mathfrak{B}}(\mathbb{S})$ to $\underline{\mathfrak{B}}(\mathbb{K})$. If σ is surjective, this map is injective. $\square$

(We denote this map also by σ^{-1}.)

In other words: If σ is an $\mathbb{S}$-measure, we can rediscover a copy of the subscale $(\sigma(G), M_{\mathbb{S}}, I_{\mathbb{S}} \cap (\sigma(G) \times M_{\mathbb{S}}))$ in the system of the extents of $\mathbb{K}$.

We shall now show that it is possible to assign to every $\mathbb{S}$-measure of $\mathbb{K}$ a $\bigvee$-preserving map of $\underline{\mathfrak{B}}(\mathbb{K})$ to $\underline{\mathfrak{B}}(\mathbb{S})$ in a unique manner.

Proposition 119. *For every $\mathbb{S}$-measure σ of $\mathbb{K}$,*

$$\tilde{\sigma}(A, A') := (\sigma(A)'', \sigma(A)')$$

defines a $\bigvee$-morphism

$$\tilde{\sigma} : \underline{\mathfrak{B}}(\mathbb{K}) \to \underline{\mathfrak{B}}(\mathbb{S}).$$

$\tilde{\sigma}$ maps the object concepts of $\mathbb{K}$ onto object concepts of $\mathbb{S}$. If $\mathbb{S}$ is a scale in which $g \neq h$ always implies $g' \neq h'$ (for all $g, h \in G_{\mathbb{S}}$), then, conversely, every $\bigvee$-preserving map of $\underline{\mathfrak{B}}(\mathbb{K})$ to $\underline{\mathfrak{B}}(\mathbb{S})$ with this property results from an $\mathbb{S}$-measure in the manner specified. σ and $\tilde{\sigma}$ uniquely determine each other. σ is full if and only if $\tilde{\sigma}$ is injective.

Proof. Let ψ be the map residual to σ^{-1} and let X be an extent of $\mathbb{K}$. We have

$$\begin{aligned}
\psi(X, X') &= \bigwedge\{(Y'', Y') \mid X \subseteq \sigma^{-1}(Y'')\} \\
&= \bigwedge\{(Y'', Y') \mid \sigma(X) \subseteq Y''\} \\
&= \left(\bigcap\{Y'' \mid \sigma(X) \subseteq Y''\}, (\bigcap\{\ldots\})'\right) \\
&= (\sigma(X)'', \sigma(X)') = \tilde{\sigma}(X, X').
\end{aligned}$$

Hence, $\tilde{\sigma}$ is residual to σ^{-1} and thus $\bigvee$-preserving.

Next we show that $\tilde{\sigma}$ maps object concepts onto object concepts: Let $x \in G$ be an object and let $g := \sigma(x)$ be the image of x in $G_{\mathbb{S}}$. $\sigma^{-1}(g'')$ is an extent of $\mathbb{K}$ containing x and thus also x'', hence, $\sigma(x'') \subseteq g''$, which implies $\sigma(x'')'' \subseteq g''$. However, since $g \in \sigma(x'')$, equality must hold, i.e., $\sigma(x'')'' = g''$, and thus $\tilde{\sigma}(x'', x') = (g'', g')$.

It remains to be shown that every $\bigvee$-preserving map that maps object concepts to object concepts goes back to a measure. Hence, let ψ be such a map. We define $\sigma : G \to G_{\mathbb{S}}$ through

$$\sigma(x) = g \quad :\Leftrightarrow \quad \psi(x'', x') = (g'', g').$$

(Here the assumption $g' = h' \Rightarrow g = h$ is used.) Then, we have

$$\begin{aligned}
\psi(X, X') &= \psi(\bigvee\{(x'', x') \mid (x \in X\}) \\
&= \bigvee\{\psi(x'', x') \mid x \in X\} \\
&= \bigvee\{(\sigma(x)'', \sigma(x)') \mid x \in X\} \\
&= \Big((\bigcup\{\sigma(x)'' \mid x \in X\})'', (\ldots)'\Big) \\
&= (\sigma(X)'', \sigma(X)').
\end{aligned}$$

Hence, every map of this kind is induced by a measure. The remaining statements follow from Proposition 9 (p. 14). □

Definition 92. If $\mathbb{S}_1$ and $\mathbb{S}_2$ are scales with the same scale values, i.e., with $G_{\mathbb{S}_1} = G_{\mathbb{S}_2}$, we call $\mathbb{S}_1$ **finer** than $\mathbb{S}_2$ if every extent of $\mathbb{S}_2$ is also an extent of $\mathbb{S}_1$. $\mathbb{S}_2$ then is **coarser** than $\mathbb{S}_1$. $\mathbb{S}_1$ and $\mathbb{S}_2$ are called **(scale-)equivalent** if there is a full bijective $\mathbb{S}_2$-measure of $\mathbb{S}_1$. ◇

If σ is a bijective full measure, so is σ^{-1}. Hence, the equivalence of scales is symmetrical. Since the concatenation of (full) scale measures again yields a (full) scale measure, the equivalence of scales is in fact an equivalence relation. If $\mathbb{S}_1$ is simultaneously finer and coarser than $\mathbb{S}_2$, the two scales are equivalent.

The possibility of scaling variably fine is very useful for data analysis. The fact that $\mathbb{S}_1$ is a finer scale than $\mathbb{S}_2$ means that $\mathbb{S}_1$ (up to equivalence) can be written as an apposition of $\mathbb{S}_2$ with another context. For example, $\mathbb{S}_1$ is equivalent to $\mathbb{S}_2 \mid \mathbb{S}_1$. In the case of plain scaling, this is inherited by the derived one-valued context: finer scaling yields a finer derived context. If we use finer scales, the derived context simply has "some more columns", i.e., it can be written as an apposition of two contexts, one of which is the derived context with respect to the coarser scaling. From Section 2.2 we know that the concept lattice of an apposition can be adequately represented by a nested line diagram, the concept lattice of the coarser scale then represents a rough structure which is further differentiated by the attributes added through the finer scale.

An example: Questionnaires are often formulated in such a way that they present opinions to which the participants can express approval or rejection, offering an alternative with intermediate values, more or less in the following form:

agree □ □ □ □ □ □ □ *do not agree*

In Section 1.4 we have suggested the interordinal scale for the scaling of such attributes, however, it is usual first to scale coarser, for example with the **threshold scale** (displayed on the right), which only uses two elements of the interordinal scale. In this way we obtain an approximate impression of the results. However, this coarsening is quite correct: According to Proposition 119, the concept lattice obtained in this way is an image (under a $\bigvee$-morphism) of the concept lattice which is the result of interordinal scaling.

	≤ 2	≥ 6
1	×	
2	×	
3		
4		
5		
6		×
7		×

The scales $\mathbb{N}_n$, $\mathbb{O}_n$ and $\mathbb{M}_{a,n-a}$ are also (equivalent to) coarsenings of the interordinal scale $\mathbb{I}_n$. The finest scale with n values is the contranominal scale $\mathbb{N}_n^c$.

In order to describe the role of the scale measures in the case of plain scaling, we first convince ourselves that the *semiproduct*

$$\mathop{\mathsf{X}}_{j\in J} \mathbb{S}_j := \left(\mathop{\times}_{j\in J} G_j, \bigcup_{j\in J} \dot{M}_j, \nabla \right)$$

with

$$(g_j)_{j\in J} \nabla (k,m) :\Longleftrightarrow g_k I_k m$$

of scales introduced in 1.4 is also a product in the sense of category theory, namely in the category of the scales with the scale measures as morphisms. For this purpose we check that the projections

$$\pi_k : \mathop{\times}_{j\in J} G_j \to G_k$$

$$\text{with} \qquad \pi_k((g_j)_{j\in J}) := g_k$$

are surjective $\mathbb{S}_k$-measures of $\mathsf{X}_{j\in J}\, \mathbb{S}_j$. Furthermore, we have to show that the product map is a scale. The property claimed follows from the fact that the product in the category of sets is the Cartesian product.

Definition 93. If $\mathbb{K}$ is a context and if for every $j \in J$ the map σ_j is an $\mathbb{S}_j$-measure of $\mathbb{K}$, then the **product measure**

$$\sigma : \mathbb{K} \to \mathsf{X}_{j\in J}\, \mathbb{S}_j$$

is defined by

$$\sigma(g) := (\sigma_j(g))_{j\in J}. \qquad \diamond$$

Proposition 120. *The product measure σ is a $\mathsf{X}_{j\in J}\, \mathbb{S}_j$-measure with $\pi_k \circ \sigma = \sigma_k$.*

Proof. The extents of the semiproduct are precisely the products $\times_{j\in J} U_j$ of extents of the individual scales. The pre-image of such an extent with respect to σ is given by

$$\sigma^{-1}(\bigtimes_{j\in J} U_j) = \bigcap_{j\in J} \sigma_j^{-1}(U_j).$$

Hence, the pre-images under σ are precisely such subsets that are intersections of pre-images under the measures σ_j. Each set of this kind is an extent of $\mathbb{K}$, hence σ is a scale measure. □

We can slightly refine the argument in this proof: Every extent is the intersection of attribute extents. Hence, the pre-images of extents of the product scale under σ are precisely the intersections of pre-images of the attribute extents of $\mathbb{S}_j, j \in J$. This leads to the following observation:

Proposition 121. *The product measure is full if and only if the set of concepts of the form*

$$(\sigma_j^{-1}(m^{I_j}), (\sigma_j^{-1}(m^{I_j})'), j \in J, m \in M_j$$

is infimum-dense in $\underline{\mathfrak{B}}(\mathbb{K})$. □

If all attributes of $\mathbb{K}$ are irreducible, the following holds true: The product measure is full if and only if every attribute extent of $\mathbb{K}$ is the pre-image of an attribute extent under one of the scale measures σ_j, i.e., if for every attribute m of $\mathbb{K}$ there exists some $j \in J$ and some attribute $m_j \in M_j$ with $m' = \sigma_j^{-1}(m_j^{I_j})$.

Finally, we can use the notion of the product measure to give an alternative definition of the derived context with respect to plain scaling:

Proposition 122. *Let (G, M, W, I) be a complete many-valued context and let $\mathbb{S}_m$, $m \in M$, be scales for the attributes of M. Furthermore, let $\mathbb{K}$ be the derived context with respect to plain scaling. Then, for every many-valued attribute $m \in M$, the map*

$$g \mapsto m(g)$$

is an $\mathbb{S}_m$-measure of $\mathbb{K}$, and $\mathbb{K}$ is isomorphic to the subscale of the semiproduct of the $\mathbb{S}_m$ which is the image of the product measure of those scales.

The *proof* results immediately from the definitions. □

7.4 Measurability Theorems

Proposition 122 has shown that full scale measures into semiproducts of scales can be understood as a kind of inversion of plain scaling. Now, we can try to recognize derived contexts, i.e., to decide in the case of a given one-valued context whether it could have been derived from a many-valued context through scaling with given scales. Hence, the question is which contexts can be fully measured in a semiproduct of nominal scales, ordinal scales etc. and, if so, which size the necessary semiproduct must have. Proposition 120 is very useful in this context, since it can be used to split up the problem. Therefore, we first examine how we can recognize whether a given context allows a measure in one of the standard scales. If this is the case, some of the attributes of the context can be combined to form a many-valued attribute (with given scaling) and a repeated implementation of this procedure according to Proposition 120 finally yields a full measure into the semiproduct.

Since every scale measure is surjective onto a subscale, it is useful to know the subscales of the standard scales. In many cases they belong to the same family of scales.

Proposition 123. *The following families of scales have the property that every subscale of a scale belonging to the family is equivalent to a scale of the same family:*

a) nominal scales,
b) one-dimensional ordinal scales,
c) one-dimensional interordinal scales,
d) multiordinal scales,
e) contranominal scales,
f) contraordinal scales,
g) convex-ordinal scales.

We shall omit the *proof.* □

Theorem 54. *The context* $\mathbb{K} := (G, M, I)$ *allows a surjective* $\mathbb{S}$*-measure for*

a) $\mathbb{S} = \mathbb{N}_n$*, if and only if there is a partition of the object set* G *into* n *extents.*

b) $\mathbb{S} = \mathbb{O}_n$*, if and only if there is a chain* $U_1 \subset U_2 \subset \ldots \subset U_n$ *of* n *non-empty extents.*

c) $\mathbb{S} = \mathbb{I}_n$*, if and only if there is a chain of* n *non-empty extents of* $\mathbb{K}$ *whose complements are also extents.*

d) $\mathbb{S} = \mathbb{M}_{n_1,\ldots,n_k}$*, if and only if there are* k *chains, each made up of* n_i *non-empty extents, whose largest elements form a partition of* G*.*

e) $\mathbb{S} = \mathbb{N}_n^c$*, if and only if there is a partition of* G *into* n *extents whose unions are also extents.*

f) $\mathbb{S} = \mathbb{O}_{\mathbf{P}}^{cd}$*, if and only if there is a set* $\mathcal{P}$ *of extents with the following properties:*
- *The set* $\mathcal{P}$*, ordered by set inclusion* $\subseteq$*, is isomorphic to* $\mathbf{P}$*.*
- *Every union of extents from* $\mathcal{P}$ *is an extent.*
- *For every object* $g \in G$ *there is a largest extent* $U_g \in \mathcal{P}$ *which does not contain* g*.*
- $\mathcal{P} = \{U_g \mid g \in G\}$*.*

g) $\mathbb{S} = \mathbb{C}_{\mathbf{P}}$*, if and only if there is a set* $\mathcal{P}$ *of extents which satisfy the conditions under f) and for which additionally the following statement is true:*
- *The complements of extents from* $\mathcal{P}$ *and the unions of such complements are also extents.*

Proof. From Proposition 118 we know that $\mathbb{K}$ allows a surjective $\mathbb{S}$-measure if and only if there is a family $\mathcal{U}_{\mathbb{S}}$ of extents of $\mathbb{K}$ and a map $\sigma : G \longrightarrow G_{\mathbb{S}}$ such that the (attribute) extents of $\mathbb{S}$ are precisely the images of $\mathcal{U}_{\mathbb{S}}$ under σ. In other words: In the system of the extents of $\mathbb{K}$, those attribute extents of a scale must occur which are isomorphic to $\mathbb{S}$ after clarification.

This makes a), b), c) and d) obvious. In order to show e), we first convince ourselves of the fact that the complementary nominal scale has the property specified: For every scale value $g \in G_{\mathbb{S}}$, $\{g\}$ is an extent, but also $G_{\mathbb{S}} \setminus \{g\}$. Hence, the pre-image sets of the scale values under a surjective $\mathbb{S}$-measure form a partition with the property specified in the Proposition. However, the converse is also true: Given such a partition, a map of G onto $G_{\mathbb{S}}$ mapping the classes of the partition onto the values of the scale is an $\mathbb{S}$-measure.

For f) we argue similarly: According to the definition, the extents of a contraordinal scale are precisely the order ideals of $\mathbf{P}$, the attribute extents are precisely the complements of principal filters. Hence, the system of attribute extents of $\mathbb{S} := \mathbb{O}^{cd}_{\mathbf{P}}$ satisfies the conditions specified in f), and so do the pre-images under an $\mathbb{S}$-measure. If, conversely, a system of such extents of a context is given, and if φ is the order isomorphism of this system onto $\mathbf{P}$, then we obtain an $\mathbb{S}$-measure through $\sigma(g) := \varphi(U_g)$ for all $g \in G$. Under these premises, the pre-image of the attribute extent $\{x \in \mathbf{P} \mid x \not\geq p\}$ is equal to

$$\sigma^{-1}(p') = \{g \mid \varphi(U_g) \not\geq p\} = \{g \mid U_g \not\supseteq U_{\bar{p}}\},$$

for any scale attribute $p \in \mathbf{P}$, provided that $\bar{p}$ is an object of $\mathbb{K}$ with $\varphi(U_{\bar{p}}) = p$. Since U_g is the largest among the selected extents which does not contain g, $U_g \not\supseteq U_{\bar{p}}$ is equivalent to $g \in U_{\bar{p}}$, i.e., we get

$$\sigma^{-1}(p') = \{g \mid U_g \not\supseteq U_{\bar{p}}\} = U_{\bar{p}}.$$

The pre-image of every column extent is an extent: σ is a measure.

g): The convex-ordinal scale is the apposition of two contraordinal scales, therefore a $\mathbb{C}_{\mathbf{P}}$-measure is in particular a $\mathbb{O}^{cd}_{\mathbf{P}}$-measure and has to satisfy the conditions under f). The convex-ordinal scale even satisfies the condition additionally required under g), which in this particular case demands that the unions of principal filters are convex sets. Hence this condition is necessary. It remains to be shown that it is also sufficient.

Hence, let $\mathcal{P}$ be a system of extents of $\mathbb{K}$ which satisfies the conditions under f), and let σ be the $\mathbb{O}^{cd}_{\mathbf{P}}$-measure constructed in the proof of f). We shall prove that under the additional condition the same map σ is also an $\mathbb{O}^{c}_{\mathbf{P}}$-measure (from which the statement follows).

For this purpose we define a set system $\mathcal{Q} := \{V_g \mid g \in G\}$ through

$$V_g := \bigcup \{G \setminus U \mid U \in \mathcal{P}, g \in U\},$$

and show that $\mathcal{Q}$ satisfies the conditions specified under f), namely for the order dual to $\mathbf{P}$. The additional condition in g) guarantees that every V_g

and all unions of such sets are extents. According to the definition, we have furthermore

$$\begin{aligned} V_g \subseteq V_h &\iff \{U \in \mathcal{P} \mid g \in U\} \subseteq \{U \in \mathcal{P} \mid h \in U\} \\ &\iff \{U \in \mathcal{P} \mid g \notin U\} \supseteq \{U \in \mathcal{P} \mid h \notin U\} \\ &\iff \{U \in \mathcal{P} \mid U \subseteq U_g\} \supseteq \{U \in \mathcal{P} \mid U \subseteq U_h\} \\ &\iff U_g \supseteq U_h. \end{aligned}$$

Hence, $\mathcal{Q}$ is order-isomorphic to $\mathbf{P}^d$ by means of the isomorphism $\psi V_g := \varphi U_g$. In the proof of f) we have shown that in this case the map $g \mapsto \psi V_g$ is an $\mathbb{O}^{cd}_{\mathbf{P}^d}$-measure from which, because of $\sigma g = \varphi U_g = \psi V_g$ and $\mathbb{O}^{cd}_{\mathbf{P}^d} = \mathbb{O}^{c}_{\mathbf{P}}$, everything else follows. □

Now we can turn our attention to the question we asked in the beginning: Let $\mathcal{S}$ be a family of scales, for example the family of nominal scales or that of ordinal scales. We want to characterize those derived contexts which result from many-valued contexts through plain scaling with scales from $\mathcal{S}$. According to Proposition 121, these are precisely the contexts which are equivalent to a subscale of the semiproduct of scales from $\mathcal{S}$. We coin a shorter name for this:

Definition 94. Let $\mathbb{K}$ be a context and let $\mathcal{S}$ be a family of scales. We say that $\mathbb{K}$ is **fully $\mathcal{S}$-measurable** if $\mathbb{K}$ can be fully measured into a semiproduct of scales from $\mathcal{S}$.

If, in particular, $\mathcal{S}$ is the family of the nominal scales, we say **fully nominally measurable** instead of "fully $\mathcal{S}$-measurable". A fully $\{\mathbb{N}_n\}$-measurable context is called **fully n-valued nominally measurable**, in the special case that $n = 2$ it is also called **fully dichotomially measurable.** ◇

Proposition 124. *For every family $\mathcal{S}$ of scales, one of the following alternatives holds:*

1. *Every context is fully $\mathcal{S}$-measurable.*
2. *Every fully $\mathcal{S}$-measurable context is fully nominally measurable.*

Proof. First, we show that every context $\mathbb{K}$ is fully ordinally measurable, even fully $\{\mathbb{O}_2\}$-measurable. This follows immediately from Proposition 120: If we define for every attribute m of $\mathbb{K}$ an $\mathbb{O}_2$-measure σ_m through

$$\sigma_m(g) := \begin{cases} 1 \text{ if } gIm \\ 2 \text{ if } g\not{I}m \end{cases},$$

then, because of $\sigma_m^{-1}(1) = m'$, the product measure is full.

In this argument we have only made use of the fact that in $\mathbb{O}_2$ there are two objects g, h with $g' \subset h'$ and $g' \neq h'$, i.e., that the context is not **atomistic** in the sense of the definition on Page 47. Hence, the first alternative only does not occur in the case that all scales in $\mathcal{S}$ are atomistic. Therefore, it only remains to be shown that every atomistic scale is fully nominally measurable itself. This follows from Proposition 125 below. □

Concept lattices of atomistic contexts are atomistic, and every atomistic complete lattice is isomorphic to the concept lattice of an atomistic context. The context property "atomistic" means precisely that the extents of the object concepts form a partition of the object set. It is inherited by semiproducts and by the pre-images under full scale measures: A context which can be fully measured into a semiproduct of atomistic scales has to be atomistic itself. A reduced context is atomistic if and only if from $g \not{I} m$, $g \swarrow m$ always follows.

Definition 95. An extent U of a context $\mathbb{K} = (G, M, I)$ is called n**-valent** if $G \setminus U$ is the disjoint union of $n-1$, but not of fewer extents. If $\mathbb{K}$ is atomistic, every extent has a valence. In this case we define the valence of a set of extents as the supremum of the valences involved. The **attribute-valence** $V_M(\mathbb{K})$ of an atomistic context is the valence of the set of attribute extents, provided that $\mathbb{K}$ is reduced. In the general case we say that an atomistic context $\mathbb{K}$ has attribute-valence $V_M(\mathbb{K}) \leq n$ if and only if there is an infimum-dense set of concepts of $\mathbb{K}$ whose extents all have a valence $\leq n$. ◇

Proposition 125. *A context is fully n-valued nominally measurable if and only if it has attribute-valence $\leq n$. Every atomistic context is fully nominally measurable.*

Proof. Every $\mathbb{N}_n$-measure of $\mathbb{K}$ induces a partition of the object set into no more than n extents, i.e., all those extents have a valence $\leq n$. Hence, according to Proposition 120 we find, for every full measure into a semiproduct of $\mathbb{N}_n$-scales, an infimum-dense set of concepts whose extents all have a valence $\leq n$. If, on the other hand, $\mathbb{K}$ has such a set of concepts, it is possible, by means of the same proposition, to construct a product measure with the desired properties: An $\mathbb{N}_n$-measure can be assigned to every partition of the object set in at most $\leq n$ classes by mapping the classes of the extent partition onto different objects of the nominal scale.

An atomistic context (G, M, I) has an attribute-valence $\leq |G|$, i.e., it is certainly fully $\mathbb{N}_{|G|}$-measurable. □

There has been little investigation as to which "measurability classes" there are within the class of the atomistic scales. A first clue can be obtained if we also define a valence for objects $g \in G$: g has **valence** n, if there are $n-1$ objects $g_1, g_2, \ldots, g_{n-1}$ (but no more) with the property that $g, g_1, \ldots, g_{n-1}$ generate the same concept pairwise: $\{g, g_1\}'' = \{g, g_2\}'' \ldots = \{g_1, g_2\}'' \ldots = \{g_{n-2}, g_{n-1}\}'' \neq g''$. $V_G(\mathbb{K})$ denotes the supremum of the valences of objects of $\mathbb{K}$.

Proposition 126. *If $\mathcal{S}$ consists of atomistic scales and if n is a natural number, the following statement is true: $\mathbb{N}_n$ is fully $\mathcal{S}$-measurable if and only if $\mathcal{S}$ contains a scale $\mathbb{S}$ of the object-valence $V_G(\mathbb{S}) \geq n$.*

Proof. Every $\bigvee$-preserving map of $\underline{\mathfrak{B}}(\mathbb{N}_n)$, $n > 2$, into a lattice which is not injective maps two atoms onto comparable elements. Therefore, by Proposition 119, every non-trivial measure of $\mathbb{N}_n$, $n > 2$, into an atomistic context is injective and thus full. If $\mathbb{N}_2$ is measured fully into a semiproduct of atomistic scales, then at least one of the factors must separate the two objects, and we have: If $\mathbb{N}_n$ is fully $\mathcal{S}$-measurable, then there is a scale $\mathbb{S} \in \mathcal{S}$, such that $\mathbb{N}_n$ is fully $\mathbb{S}$-measurable. Again by Proposition 119, then there must be a $\bigvee$-embedding of $\underline{\mathfrak{B}}(\mathbb{N}_n)$ in $\underline{\mathfrak{B}}(\mathbb{S})$ in the case of which object concepts are mapped to object concepts, i.e., $\mathbb{S}$ has an object-valence $\geq n$.

Conversely, in a scale which has an object-valence $\geq n$, we also find an object of the valence n and thus a measure mapping $\mathbb{N}_n$ injectively and fully onto $\mathbb{S}$. □

From the last two propositions we draw a simple conclusion for a special case:

Proposition 127. *If $\mathcal{S}$ consists of atomistic scales with an attribute-valence $\leq n$ and if $\mathcal{S}$ contains a scale of the object-valence n ($n \in \mathbb{N}$), then the following holds true: A context $\mathbb{K}$ is fully $\mathcal{S}$-measurable if it is fully n-valued nominally measurable.* □

This already suffices to provide us with an overview over the measurability classes with respect to the atomistic standard scales. We have for all $n \geq 2$

$$V_G(\mathbb{N}_n) = V_M(\mathbb{N}_n) = n,$$
$$V_G(\mathbb{I}_n) = V_M(\mathbb{I}_n) = 2,$$
$$V_G(\mathbb{N}_n^c) = V_M(\mathbb{N}_n^c) = 2.$$

From this follows:

Proposition 128. *For a context $\mathbb{K}$ the following statements are equivalent:*

1. *$\mathbb{K}$ is fully dichotomially measurable.*
2. *$\mathbb{K}$ is fully interordinally measurable.*
3. *$\mathbb{K}$ is fully contranominally measurable.*

□

The proposition also makes it possible to characterize the concept lattices of many-valued contexts which are scaled plainly by means of elementary scales. For this purpose, we need another definition: We say that an element x of an atomistic complete lattice $\boldsymbol{V}$ has **valence** $\leq n$ if there is an n-element subset T of V which contains x and which has the property that each atom of V is less than or equal to precisely one element of T. If, in the special case $n = 2$, the set $T = \{x, y\}$ has the property specified, then we call y a **pseudo-complement** of x.

Theorem 55. *Every complete lattice is isomorphic to the concept lattice of an ordinally scaled many-valued context.*
A complete lattice is isomorphic to the concept lattice of a nominally scaled complete many-valued context, if and only if it is atomistic.
A complete lattice is isomorphic to the concept lattice of a nominally scaled complete n-valued context if and only if it is atomistic and contains an infimum-dense set of elements of a valence $\leq n$.

For the special case $n = 2$, we obtain:
A complete lattice is isomorphic to the concept lattice of a nominally scaled complete 2-valued context if and only if it is atomistic and contains an infimum-dense set of elements with a pseudo-complement. This at the same time characterizes the concept lattices of interordinally scaled many-valued contexts as well as the concept lattices of complementary nominally scaled many-valued contexts. □

7.5 Hints and References

7.1 Not only isomorphisms but also other classes of maps can be represented as concepts, this has been worked out by W. Xia [216]. Theorem 51 has been taken from [59], compare also Ganter and Reuter [62].

With regard to the group-theoretical background of Figure 7.1, extensive information can be found in Kerber ([95], in particular Chapter 3).

7.2 Definition 89 follows - with slight modifications - the article [50] by Erné, from which we have also taken Theorem 52 and Proposition 111 and which contains a lot of additional information concerning this subject. The question to which extent lattice morphisms can be represented by context maps so that a duality is created has also been examined by G. Hartung [84].

7.3 Conceptual scales and conceptual measuring have first been discussed in [63]. Many results of this section can be found in different formulations in books such as that of Blyth and Janowitz [16]. Otherwise, this section and the following one make use of results taken from [65].

7.4 We again refer to [65]. Parts of Theorem 55 are contained in [191].

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