Very stable *G*-Higgs bundles with regular nilpotent Higgs field

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G-Higgs bundles

G complex semisimple Lie group $(SL_n\mathbb{C}, PGL_n\mathbb{C}, Sp_{2n}\mathbb{C}, SO_n\mathbb{C}, G_2, ...)$, Lie algebra \mathfrak{g}

C smooth projective complex curve, $g\geqslant 2$, canonical $\mathcal{K}_{\mathcal{C}}$

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A *G*-**Higgs bundle** (E, φ) over *C*:

- E a principal G-bundle over C
- φ a section of $E(\mathfrak{g}) \otimes K_C$ (**Higgs field**)

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Examples:

• $G = \operatorname{SL}_n \mathbb{C}$. Then $V := E(\mathbb{C}^n)$ a vector bundle of rank n. det $V = \mathcal{O}_C$. Since $\mathfrak{g} = \operatorname{End}_0(\mathbb{C}^n)$, we get $\varphi : V \to V \otimes K_C$ traceless.

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- $G = SO_n \mathbb{C}$. $V := E(\mathbb{C}^n)$ a vector bundle of rank n with a non-degenerate symmetric form $Q : V \xrightarrow{\sim} V^*$. This time $\varphi : V \to V \otimes K_C$ with $\varphi^t = -\varphi$.

Very stable *G*-Higgs bundles

Moduli space $\mathcal{M}(G)$ of isomorphism classes of polystable *G*-Higgs bundles.

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The **upward flow** of the fixed point $(E, \varphi) \in \mathcal{M}(G)^{\mathbb{C}^{\times}}$:

$$W_{(E,\varphi)}^+ := \left\{ (E',\varphi') : \lim_{\lambda \to 0} (E',\lambda \varphi') = (E,\varphi) \right\} \subseteq \mathcal{M}(G)$$

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Definition (Hausel-Hitchin, 2022)

Stable fixed point (E,φ) is **very stable** if $W_{(E,\varphi)}^+\subseteq \mathcal{M}(G)$ is closed



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- $\mathfrak{g}_1 := \bigoplus \mathfrak{g}_{\alpha_i}$

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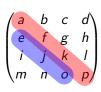
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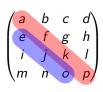
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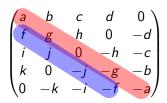


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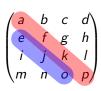


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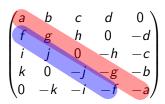
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We take (E, φ) such that

- E reduces to a T-bundle.
- φ section of $E(\mathfrak{g}_1) \otimes K_C$.



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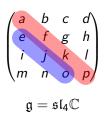


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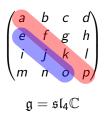


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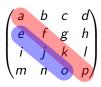


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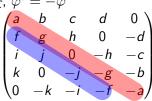
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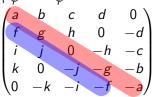
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Multiplicities

 $\mathfrak{g}_1 = \bigoplus \mathfrak{g}_{\alpha_i}$, projection:

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Definition

The multiplicity coweight at $c \in C$ is

$$\mu_c := \sum_{i=1}^r (\operatorname{ord}_c \varphi_i) \cdot \omega_i^{\vee}$$

where $\omega_1^{\vee}, \dots, \omega_r^{\vee} \in \mathfrak{t}$ are the fundamental coweights.

Classification

Multiplicity coweight $\mu_c := \sum_{i=1}^r (\operatorname{ord}_c \varphi_i) \cdot \omega_i^{\vee}$

(Dominant) coweight μ is **minuscule** if $\alpha(\mu) \in \{0, \pm 1\}$ for all roots $\alpha \in \Delta$.

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Main technique is the action of (subspaces of) the **affine Grassmannian** Gr_G on G-Higgs bundles by **Hecke transformations**.

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Minuscule coweights: $0, \omega_1^\vee, \omega_2^\vee, \omega_3^\vee$ Very stable $\iff \varphi_1, \varphi_2, \varphi_3$ no common zeroes, and simple (Hausel–Hitchin, 2022)

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• $G = G_2, F_4, E_8$. Minuscule coweight: 0 Very stable: "Hitchin section"



Thank you!