Very stable G-Higgs bundles with regular nilpotent Higgs field

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G-Higgs bundles

G complex semisimple Lie group ($\mathrm{SL}_n\,\mathbb{C}, \mathrm{PGL}_n\mathbb{C}, \mathrm{Sp}_{2n}\,\mathbb{C}$, $\mathrm{SO}_n\,\mathbb{C}, \mathrm{G}_2, \dots$), Lie algebra $\mathfrak{g}$

C smooth projective complex curve, $g \geqslant 2$, canonical K_C

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A G-**Higgs bundle** (E, φ) over C:

- E a principal G-bundle over C
- φ a section of $E(\mathfrak{g}) \otimes K_C$ (**Higgs field**)

Examples

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Examples:

- $G = \mathrm{SL}_n \mathbb{C}$. Then $V := E(\mathbb{C}^n)$ a **vector bundle of rank** n. $\det V = \mathcal{O}_C$. Since $\mathfrak{g} = \mathrm{End}_0(\mathbb{C}^n)$, we get $\varphi : V \to V \otimes K_C$ traceless.

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- $G = \mathrm{SO}_n\,\mathbb{C}$. $V := E(\mathbb{C}^n)$ a **vector bundle of rank** n with a **non-degenerate symmetric form** $Q : V \xrightarrow{\sim} V^*$. This time $\varphi : V \to V \otimes K_C$ with $\varphi^t = -\varphi$.

Very stable G-Higgs bundles

Moduli space $\mathcal{M}(G)$ of isomorphism classes of polystable G-Higgs bundles.
Natural $\mathbb{C}^\times$-action:

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The **upward flow** of the fixed point $(E, \varphi) \in \mathcal{M}(G)^{\mathbb{C}^\times}$:

$$W^+_{(E,\varphi)} := \left\{ (E', \varphi') : \lim_{\lambda \to 0} (E', \lambda\varphi') = (E, \varphi) \right\} \subseteq \mathcal{M}(G)$$

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Definition (Hausel–Hitchin, 2022)

Stable fixed point (E, φ) is **very stable** if $W^+_{(E,\varphi)} \subseteq \mathcal{M}(G)$ is closed

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Fixed points of **Borel type**:
(Regular nilpotent Higgs field)
We will need:

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We take (E, φ) such that

- E reduces to a T-bundle.
- φ section of $E(\mathfrak{g}_1) \otimes K_C$.

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Multiplicities

$\mathfrak{g}_1 = \bigoplus \mathfrak{g}_{\alpha_i}$, projection:

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Higgs field gives **components** φ_i, sections of each $E(\mathfrak{g}_{\alpha_i}) \otimes K_C$

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Definition

The **multiplicity coweight** at $c \in C$ is

$$\mu_c := \sum_{i=1}^{r} (\mathrm{ord}_c \varphi_i) \cdot \omega_i^\vee$$

where $\omega_1^\vee, \dots, \omega_r^\vee \in \mathfrak{t}$ are the the fundamental coweights.

Classification

Multiplicity coweight $\mu_c := \sum_{i=1}^{r} (\mathrm{ord}_c \varphi_i) \cdot \omega_i^\vee$

(Dominant) coweight μ is **minuscule** if $\alpha(\mu) \in \{0, \pm 1\}$ for all roots $\alpha \in \Delta$.

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Main technique is the action of (subspaces of) the **affine Grassmannian** Gr_G on G-Higgs bundles by **Hecke transformations**.

Examples and consequences

- $G = \mathrm{SL}_4\,\mathbb{C} \rightsquigarrow V = L_0 \oplus L_1 \oplus L_2 \oplus L_3$

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Minuscule coweights: $0, \omega_1^\vee, \omega_2^\vee, \omega_3^\vee$
Very stable $\iff$ $\varphi_1, \varphi_2, \varphi_3$ no common zeroes, and simple
(Hausel–Hitchin, 2022)

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- $G = G_2, F_4, E_8$. Minuscule coweight: 0
  Very stable: **"Hitchin section"**

Thank you!