Quantum chaos meets quantum channels

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1.1 Model

The spin chain we are interested in studying first is that studied by Mirkin and Wisniacki in Ref. [1]:

eq:H:wisniacki:ising:chain

$$H = \sum_{i=1}^{L} (h_x \sigma_i^x + h_z \sigma_i^z) - \sum_{i=1}^{L-1} J_z \sigma_i^z \sigma_{i+1}^z. \tag{1}$$

1.2 Mean level spacing ratio

The level spacing ratio $\tilde{r}_n$ is defined as:

eq:level:spacing:ratio

$$\tilde{r}_n = \frac{\min(s_n, s_{n-1})}{\max(s_n, s_{n-1})}, \tag{2}$$

where $s_n = E_{n+1} - E_n$. The mean level spacing ratio $\langle \tilde{r}_n \rangle$ is known to attain the value $\langle \tilde{r}_n \rangle \approx 0.5207$ when the level spacing distribution $P(s)$ is Wigner-Dyson and $\langle \tilde{r}_n \rangle \approx 0.386$ when it is Poisson.

1.3 Spectral form factor

The spectral form factor $K(t)$ is defined as:

eq:sff

$$K(t) = \frac{1}{2^L} \left\langle \left| \mathrm{Tr}\, U(t) \right|^2 \right\rangle = \frac{1}{2^L} \left\langle \sum_{i,j} e^{i(E_i - E_j)t} \right\rangle, \tag{3}$$

where $\langle \cdot \rangle$ denotes the ensemble-average over statistically-similar systems.

1.4 Chaometer's quantum channel

The reduced dyanmics of the chaometer is described by the quantum channel:

eq:chaometer:channel

$$\mathcal{E}(\rho) = \mathrm{Tr}_E \left(e^{-iHt} \rho \otimes \left| \psi_0^{(E)} \right\rangle\!\left\langle \psi_0^{(E)} \right| e^{iHt} \right), \tag{4}$$

where H is that of eq. (28), $\left| \psi_0^{(E)} \right\rangle$ the initial state of all spins except the chaometer, and ρ the initial state of the chaometer.

The chaometer's quantum channel $\mathcal{E}$, in general, is divisible into:

1. A unitary operation rotating the Bloch's sphere.

2. A quantum channel that deforms the Bloch's sphere and translates its origin.

Both operations do not commute.

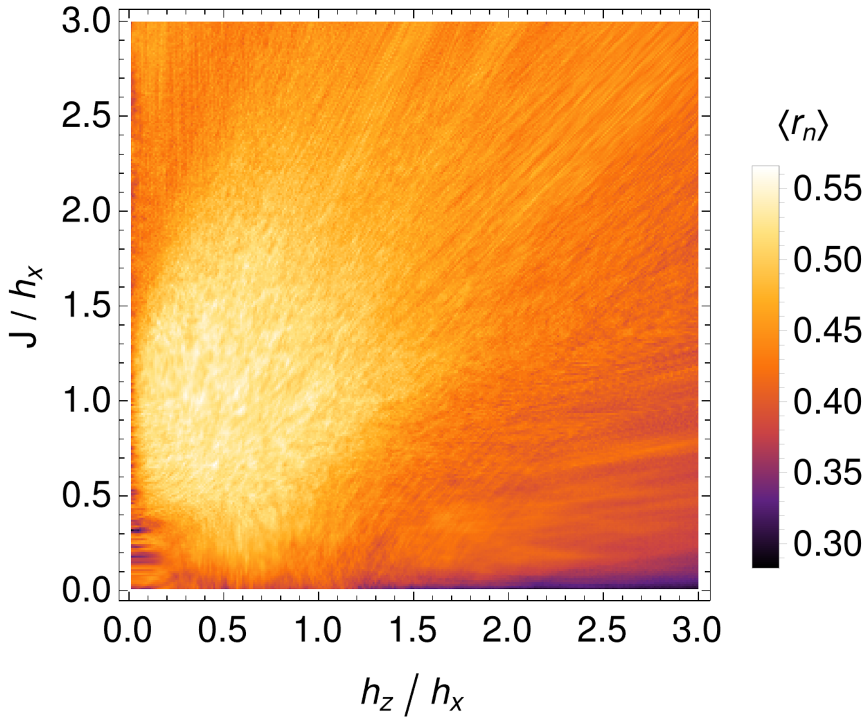

Figure 1: Mean level spacing ratio $\langle \tilde{r}_n \rangle$ [c.f. eq. (2)] of the Ising chain with Hamiltonian (28) as a function of ratios h_z/h_x and J/h_x. We assume $J_z = J \, \forall \, k$. *fig:mean:level:spacing:ratio*

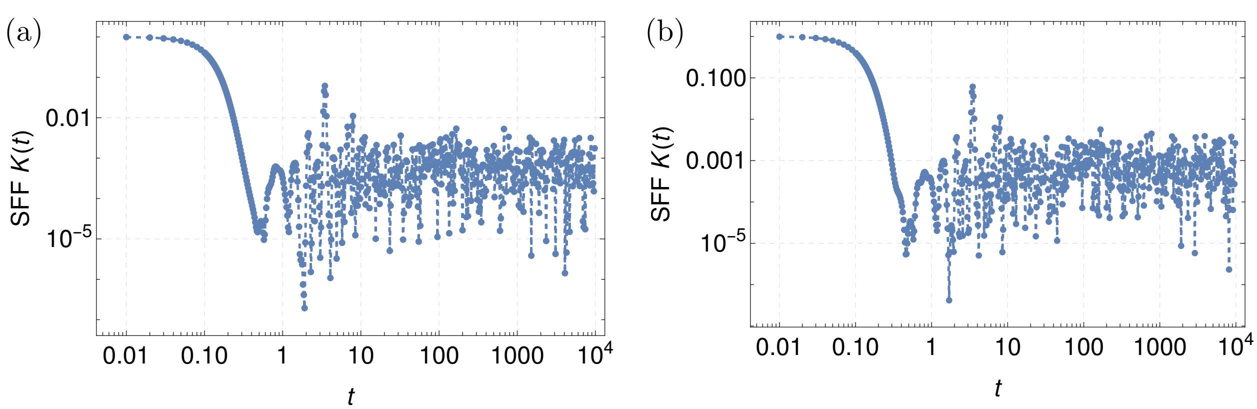

Figure 2: Spectral form factor (SFF) [c.f. (3)] in regular region: $h_z/h_x = 2.5$ and $J/h_x = 1$. **(a)** Whole spectrum. **(b)** Even-parity subspace spectrum. *fig:sff:regular*

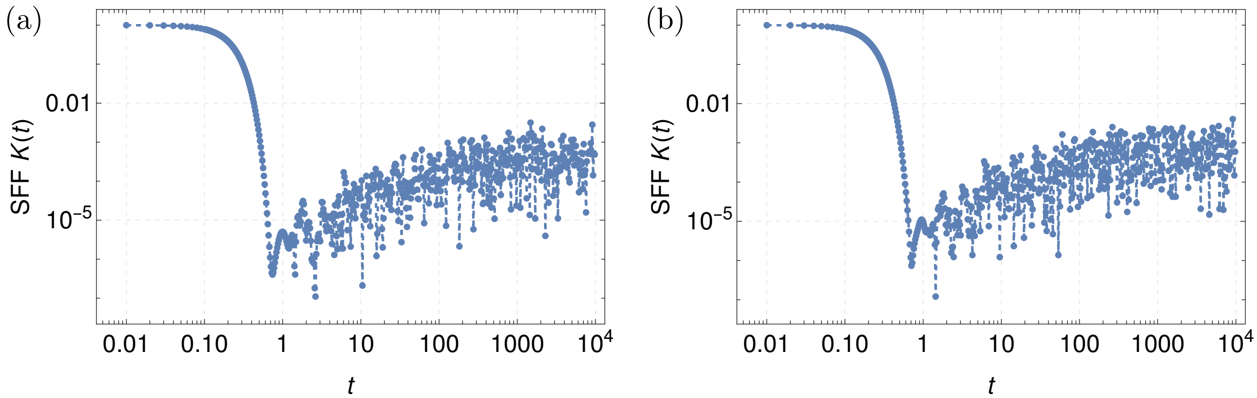

Figure 3: Spectral form factor (SFF) [c.f. (3)] in chaotic region: $h_z/h_x = 0.5$ and $J/h_x = 1$. **(a)** Whole spectrum. **(b)** Even-parity subspace spectrum.

fig:sff:chaotic

1.5 Purity of the chaometer

Averaged purity $\mathcal{P}$ is defined in Ref. [1] as:

eq:avg:purity

$$\overline{\mathcal{P}} = \frac{1}{N}\sum_{i=1}^{N}\left(\frac{1}{T}\int_0^T \mathrm{Tr}\left[\rho_i^2(t)\right]\right) \tag{5}$$

where:

- $\rho_i(t)$: chaometer's density matrix.
- N: number of different random initial states of the whole chain. $N = 50$ in Mirkin and Wisniacki [1].
- T: maximum time. $T = 50$ in Mirkin and Wisniacki [1].

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Also, the normalized averaged purity is defined as

eq:avg:norm:purity

$$\overline{P}_{Norm} = \frac{\overline{P} - \min(\overline{P})}{\max(\overline{P}) - \min(\overline{P})}, \tag{6}$$

where $\max(\overline{P})$ and $\min(\overline{P})$ are the minimum and maximum value obtained when sweeping the parameter range (h_z in their case).

In Fig. 4 we plot one realization of the dynamics of purity of the chaometer. We compare in Fig. 5 our results and those of Mirkin and Wisniacki [1].

1.6 Purity of Choi-Jamiołkowski matrix

We investigate the purity of Choi-Jamiołkowski matrix of quantum channel $\mathcal{E}(t)$ of the chaometer [c.f. eq. (4)] in Fig. 6.

To compute the Choi-Jamiołkowski matrix $\mathcal{D}(t)$ of the chaometer's quantum channel in eq. (4) we use the definition $(\mathcal{E}\otimes\mathbb{1})[|\phi^+\rangle\langle\phi^+|]$, with $|\phi^+\rangle = 1/\sqrt{2}(|0,0\rangle + |1,1\rangle)$ is the maximally entangled state between two spins, and obtain

eq:choi:chaometer

$$\mathcal{D}(t) = \frac{1}{2}\sum_{i,j,p,q}\langle j,\psi_{0E}|\,U^\dagger(t)(|p\rangle\langle q|\otimes\mathbb{1}_E)U(t)\,|i,\psi_{0E}\rangle\,|q,i\rangle\langle p,j|\,. \tag{7}$$

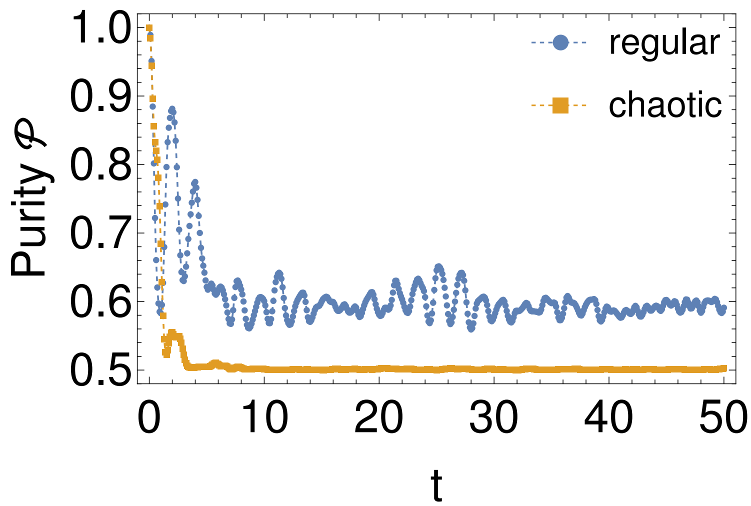

Figure 4: Dynamics of chaometer's purity of a single realization with the same initial state of the environment of the first chaometer's quantum channel of the videos. *fig:purity:one:realization*

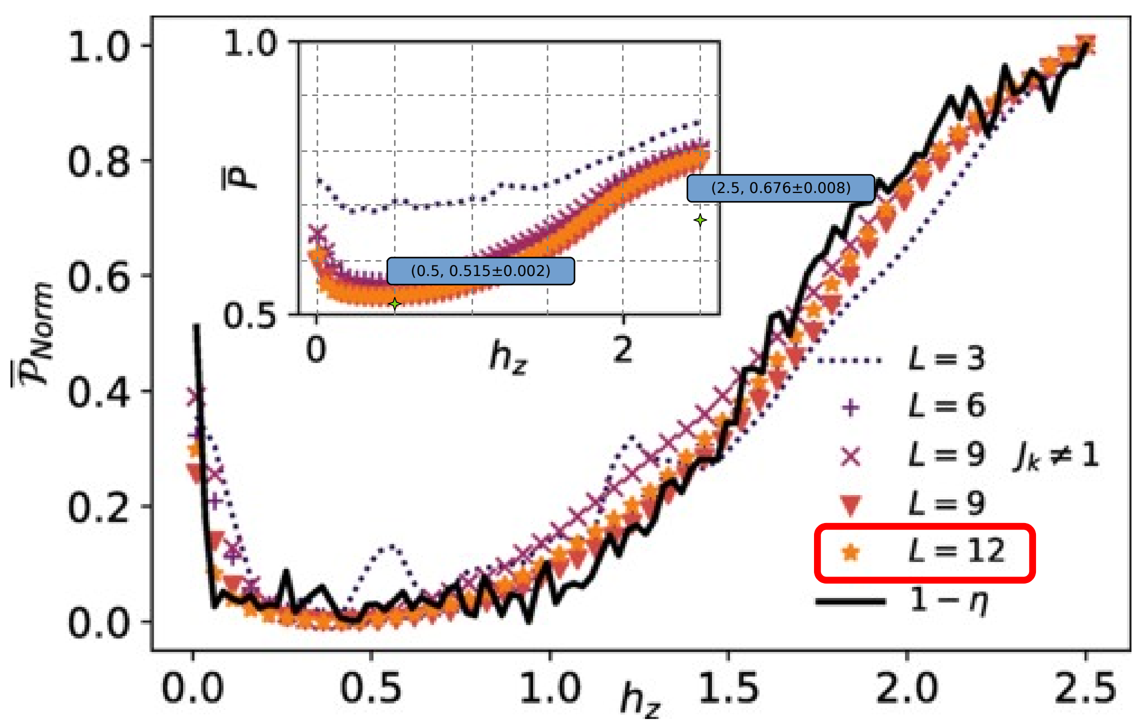

Figure 5: Averaged $\overline{P}$ and averaged normalized purities $\overline{P}_{Norm}$ [c.f. eqs. (5) and (6).] for $N = 50$ random initial states of the chaometer for each quantum channel showed in the videos. JA: Tendría más sentido sacar la pureza del canal. Lo pienso Taken and modified from Ref. [1]. *fig:mirkin2021:fig2:modified*

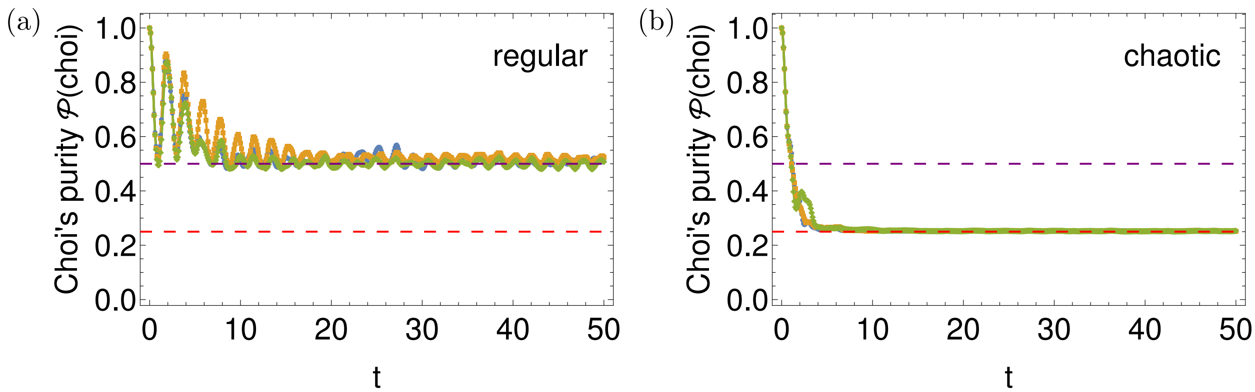

Figure 6: Purity of Choi-Jamiołkowski matrix in (a) regular ($h_z = 2.5$) and (b) chaotic ($h_z = 0.5$) for the three random initial states showed in the video. fig:choi:purity

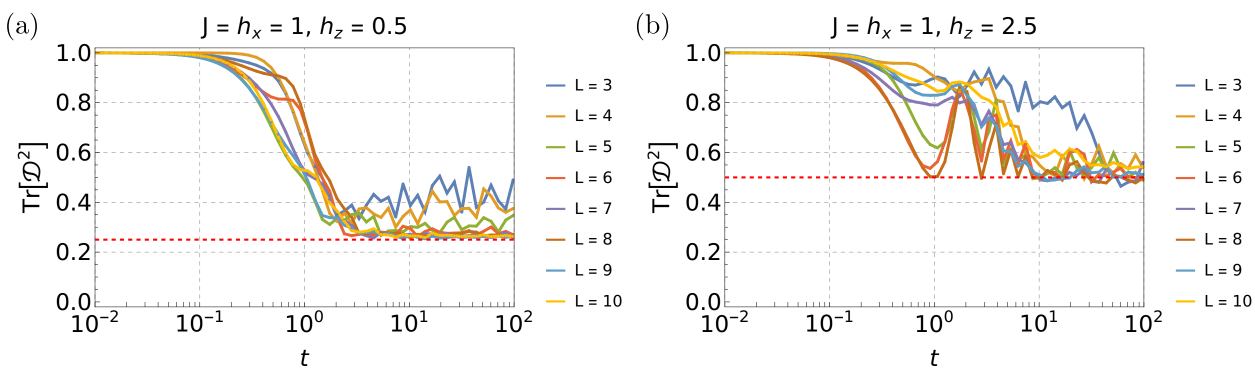

Figure 7: Dynamics of purity of the Choi-Jamiołkowski matrix for various lengths L of the spin chain. fig:choi:purity:dynamics

Let us write $\mathcal{E}(\rho)$ in computational basis. From eq. (4) we write

$$\mathcal{E}(\rho) = \sum_{k,l} e^{-i(E_k - E_l)t} \langle E_k| \left(\rho \otimes |\psi_E\rangle\!\langle\psi_E|\right) |E_l\rangle \operatorname{Tr}_E \left(|E_k\rangle\!\langle E_l|\right) \tag{8}$$

$$= \sum_{k,l,p,p',\vec{q},\vec{q}'} e^{-i(E_k - E_l)t} \langle E_k| \left(\rho \otimes |\psi_E\rangle\!\langle\psi_E|\right) |E_l\rangle \operatorname{Tr}_E \left(|p,\vec{q}\rangle \langle p,\vec{q}|E_k\rangle \langle E_l|p',\vec{q}'\rangle \langle p',\vec{q}'|\right), \tag{9}$$

where $\vec{q}$ and $\vec{q}'$ are indices of elements of the basis of the environment,

$$= \sum_{p,p',\vec{q}} \langle p,\vec{q}| U(t)(\rho \otimes |\psi_E\rangle\!\langle\psi_E|)U^\dagger(t) |p',\vec{q}\rangle |p\rangle\!\langle p'| \tag{10}$$

eq:chaometer:channel:2

$$= \sum_{p,p'} \langle p| \operatorname{Tr}_E[U(t)(\rho \otimes |\psi_E\rangle\!\langle\psi_E|)U^\dagger(t)] |p'\rangle |p\rangle\!\langle p'| \,. \tag{11}$$

To compute the Choi-Jamiołkowski matrix $\mathcal{D}(t)$ of the chaometer's quantum channel in eq. (4) we use the definition $(\mathcal{E} \otimes \mathbb{1})[|\phi^+\rangle\!\langle\phi^+|]$, with $|\phi^+\rangle = 1/\sqrt{2}(|0,0\rangle + |1,1\rangle)$ is the maximally entangled state between two spins, and eq. (11) to obtain

$$\mathcal{D}(t) = \frac{1}{2} \sum_{i,j,p,p',\vec{q}} \langle p,\vec{q}| U(t)(|i,\psi_E\rangle\!\langle j,\psi_E|)U^\dagger(t) |p',\vec{q}\rangle |p,i\rangle\!\langle p',j| \tag{12}$$

eq:choi:chaometer:computational:basis

$$= \frac{1}{2} \sum_{i,j,p,p'} \langle j,\psi_{0E}| U^\dagger(t)(|p'\rangle\!\langle p| \otimes \mathbb{1}_E)U(t) |i,\psi_{0E}\rangle |p,i\rangle\!\langle p',j| \,, \tag{13}$$

where we have only reordered the factors and used the resolution of identity over $\mathcal{H}_E$.

We compute the purity of $\mathcal{D}(t)$. From eq. (7) it is straightforward to obtain

eq:choi:chaometer:purity:1

$$\operatorname{Tr}\left[\mathcal{D}^2(t)\right] = \frac{1}{4} \sum_{i,j,p,q} \left| \langle i,\psi_{0E}|U^\dagger(t)\big(\,|p\rangle\!\langle q| \otimes \mathbb{1}_E\big)U(t)|j,\psi_{0E}\rangle\right|^2. \tag{14}$$

Moreover, reordering this expression an interesting interpretation of the purity of the Choi-Jamiołkowski matrix is revealed. Let us conveniently rewrite eq. (14) as

$$\operatorname{Tr}\left[\mathcal{D}^2(t)\right] = \langle\psi_{0E}| \operatorname{Tr}_S \left\{ U^\dagger(t) \sum_{p,q} \left[\left(\frac{|p\rangle\!\langle q|}{\sqrt{2}} \otimes \mathbb{1}_E \right) U(t) \left(\frac{\mathbb{1}_S}{2} \otimes |\psi_{0E}\rangle\!\langle\psi_{0E}| \right) U^\dagger(t) \left(\frac{|q\rangle\!\langle p|}{\sqrt{2}} \otimes \mathbb{1}_E \right) \right] U(t) \right\} |\psi_{0E}\rangle \,. \tag{15}$$

eq:choi:chaometer:purity:2

- First, a forward evolution $\rho(t) = U(t)\rho_0 U^\dagger(t)$ occurs, with $\rho_0 = \left(\frac{\mathbb{1}_S}{2} \otimes |\psi_{0E}\rangle\!\langle\psi_{0E}|\right)$.
- Second, backwards evolution of $\rho(t)$ is followed, $\tilde{\rho}_0 = U^\dagger(t) \sum_{p,q} \left[\left(\frac{|p\rangle\langle q|}{\sqrt{2}} \otimes \mathbb{1}_E \right) \rho(t) \left(\frac{|q\rangle\langle p|}{\sqrt{2}} \otimes \mathbb{1}_E \right) \right] U(t)$, where before evolving backwards with $U^\dagger(t)$ a completely depolarizing quantum channel acts over the system.
- Finally, $\operatorname{Tr}\left[\mathcal{D}^2(t)\right]$ is the probability of finding the environment in its initial state.

JA: esta es una especie de echo de Loschmidt!!! Cálculos en el apéndice A In other words, the purity of $\mathcal{D}(t)$ represents the probability that the environment remains in its initial state after the following process: First, the system and environment are initialized in a product state, with the system in a maximally mixed state. Next, the combined system evolves. Then, a completely depolarizing channel is applied to the system. Finally, the system and environment undergo reverse evolution.

The Loschmidt echo is defined as

eq:loschmidt:echo

$$M(t) = \left| \langle\psi_0| e^{-iH_2 t/\hbar} e^{-iH_1 t/\hbar} |\psi_0\rangle \right|^2 \tag{16}$$

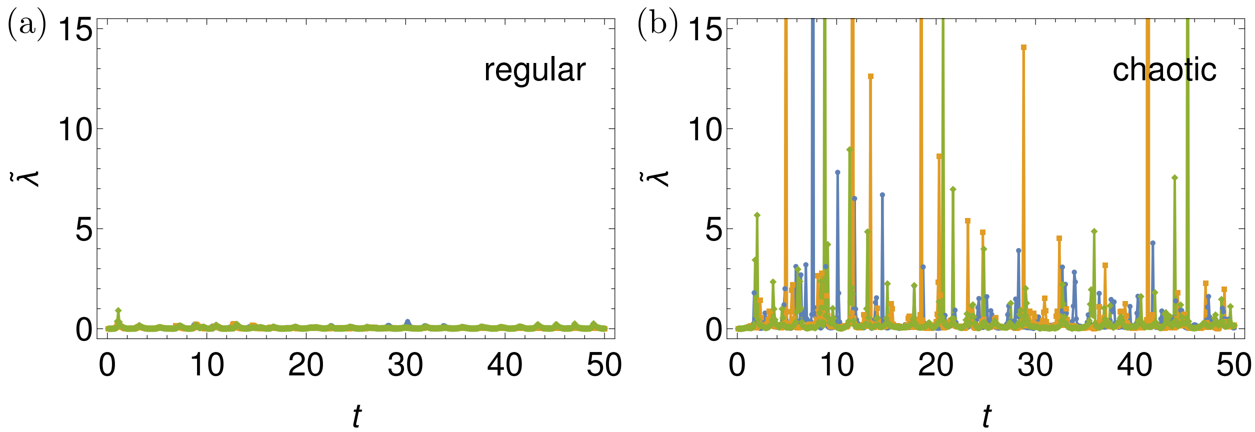

Figure 8: Most negative eigenvalue $\tilde{\lambda}$ of map $\Lambda(t,s)/2$, with $s = 0.1$ [c.f. eqs. (17) and (18)]. fig:lambda:tilde:Choi-“jami–”

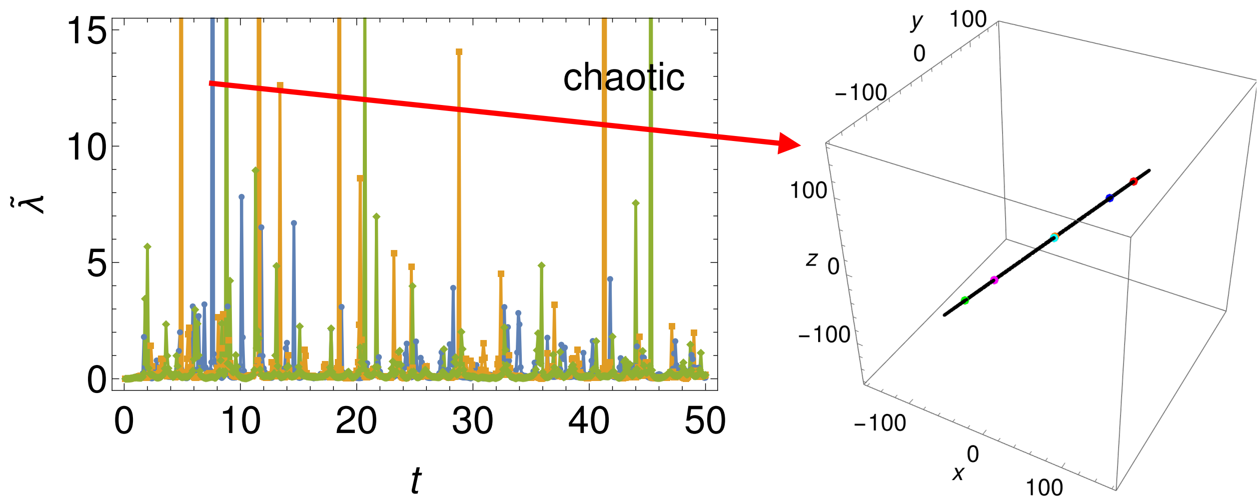

Figure 9: Burst of the Bloch sphere at $t = 0.5$ s. fig:burst

1.7 Non complete-positiveness of $\Lambda(t,s)$

Any quantum channel $\mathcal{E}(t)$ can composed as

eq:Lambda

$$\mathcal{E}(t) = \Lambda(t,s) \circ \mathcal{E}(s,0), \tag{17}$$

nonetheless, $\Lambda(t,s)$ is not in general completely positive. A way to quantify how far is $\Lambda(t,s)$ from being completely positive is through $\tilde{\lambda}$:

eq:lambda:tilde

$$\tilde{\lambda} = |\min(0, \lambda_{\text{smallest}})|, \tag{18}$$

where $\lambda_{\text{smallest}}$ is the smallest eigenvalue of $\Lambda^R(t,s)/2$. We have added the factor $1/2$ just so $\mathrm{Tr}\left[\Lambda^R(t,s)/2\right] = 1$.

Let us fix $s = 0.1$ and investigate the complete positiviness of $\Lambda(t,s)$, see Fig. 8.

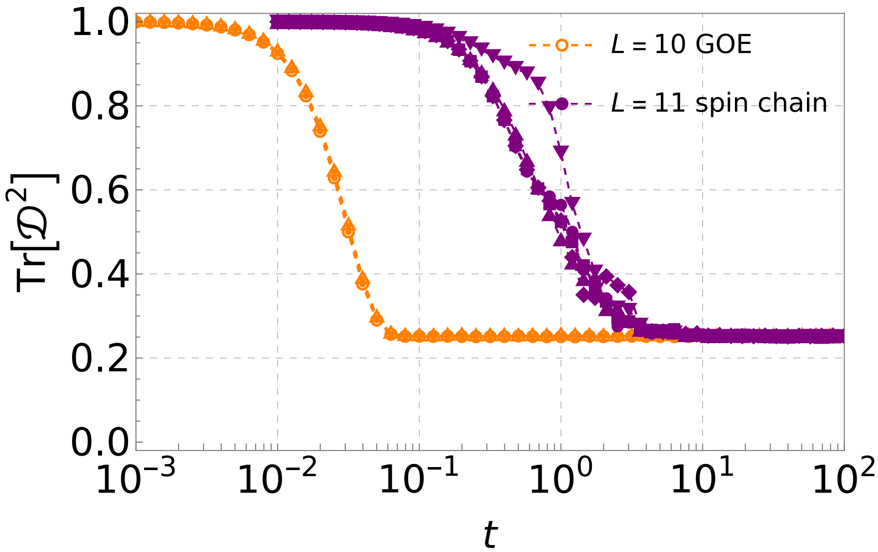

Figure 10: Purity of the Choi-Jamiołkowski matrix of chaometer's quantum channel and another channel from a Hamiltonian taken from the GOE ensemble. fig:choi:purity:spin:chain:vs:goe

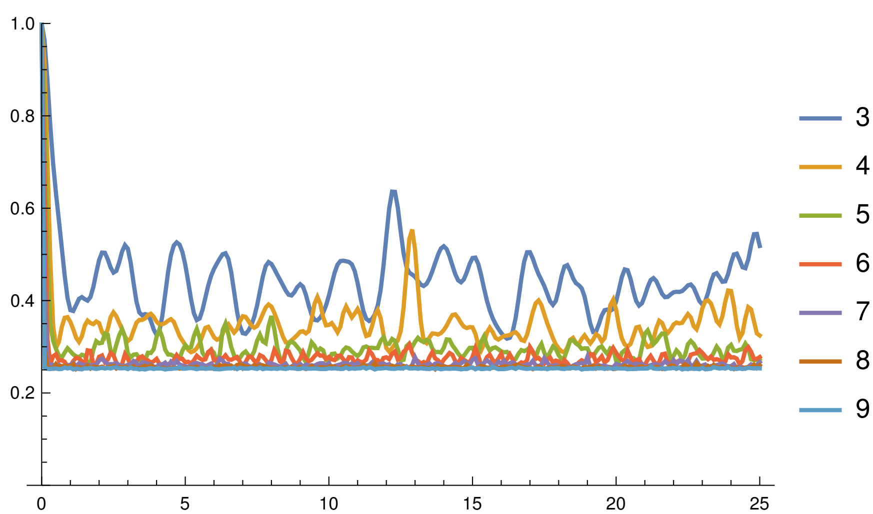

Figure 11: JA: La pureza de la matriz de Choi de una matriz U de COE se comporta así variando el número de espines L desde 3 hasta 9. Parece que para L muy grande el límite asintótico tiende a 0.25, Se podrá probar analíticamente?

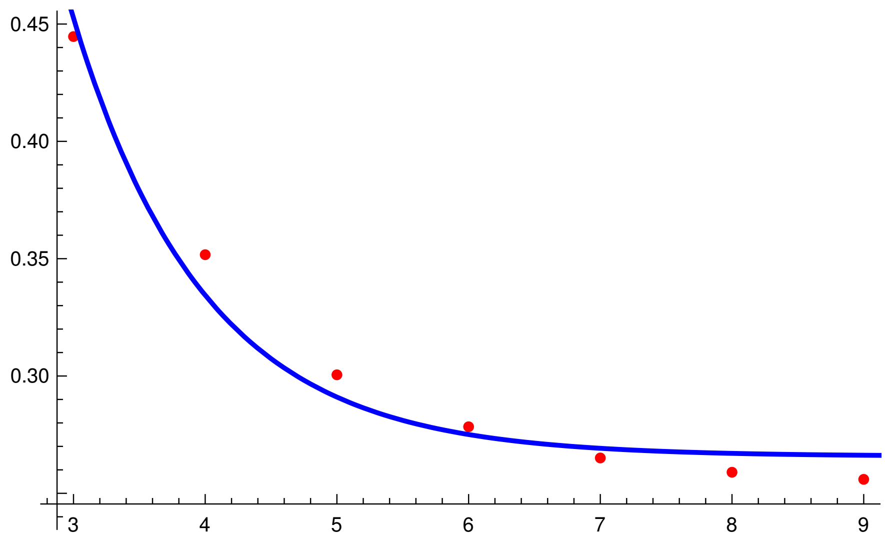

Figure 12: JA: $\mathrm{Tr}\left[\mathcal{D}_\infty^2\right]$ vs L. Creo que se podría conseguir un mejor ajusto teniendo estadística para cada punto.

1.8 Purity of Choi-Jamiołkowski matrix

2 Charles

3 Miguel

4 Viko

A Choi

app:choi

$$\mathrm{Tr}\big[\mathcal{D}^2(t)\big] = \frac{1}{4}\left(\sum_{i,j,p} \big|\, \langle i, \psi_{0E}| U^\dagger(t) \big(|p\rangle\langle p| \otimes \mathbb{1}_E \big) U(t) |j, \psi_{0E}\rangle \big|^2 + \sum_{i,j,p\neq q} \big|\, \langle i, \psi_{0E}| U^\dagger(t) \big(|p\rangle\langle q| \otimes \mathbb{1}_E \big) U(t) |j, \psi_{0E}\rangle \big|^2 \right) \tag{19}$$

eq:appendix:A:1

$$\begin{aligned} &= \frac{1}{4}\left(\sum_{i,j} \big|\, \langle i, \psi_{0E}| U^\dagger(t) \big(\mathbb{1}_S \otimes \mathbb{1}_E \big) U(t) |j, \psi_{0E}\rangle \big|^2 \right. \\ &\quad - 2 \sum_{i,j} \mathrm{Re}\big(\langle i, \psi_{0E}| U^\dagger(t) \big(|0\rangle\langle 0| \otimes \mathbb{1}_E \big) U(t) |j, \psi_{0E}\rangle \; \langle j, \psi_{0E}| U^\dagger(t) \big(|1\rangle\langle 1| \otimes \mathbb{1}_E \big) U(t) |i, \psi_{0E}\rangle \big) \\ &\quad \left. + \sum_{i,j,p\neq q} \big|\, \langle i, \psi_{0E}| U^\dagger(t) \big(|p\rangle\langle q| \otimes \mathbb{1}_E \big) U(t) |j, \psi_{0E}\rangle \big|^2 \right) \end{aligned} \tag{20}$$

eq:appendix:A:2

$$\begin{aligned} &= \frac{1}{2}\left(1 - \sum_{i,j} \Big(\mathrm{Re}\big(\langle i, \psi_{0E}| U^\dagger(t) \big(|0\rangle\langle 0| \otimes \mathbb{1}_E \big) U(t) |j, \psi_{0E}\rangle \; \langle j, \psi_{0E}| U^\dagger(t) \big(|1\rangle\langle 1| \otimes \mathbb{1}_E \big) U(t) |i, \psi_{0E}\rangle \big) \right. \\ &\quad \left. + \big|\, \langle i, \psi_{0E}| U^\dagger(t) \big(|0\rangle\langle 1| \otimes \mathbb{1}_E \big) U(t) |j, \psi_{0E}\rangle \big|^2 \Big) \right) \end{aligned} \tag{21}$$

eq:appendix:A:3

$$\begin{aligned} &= \frac{1}{2}\left(1 - \sum_{i,j} \Big(\langle i, \psi_{0E}| U^\dagger(t) \big(|0\rangle\langle 0| \otimes \mathbb{1}_E \big) U(t) |j, \psi_{0E}\rangle \; \langle j, \psi_{0E}| U^\dagger(t) \big(|1\rangle\langle 1| \otimes \mathbb{1}_E \big) U(t) |i, \psi_{0E}\rangle \right. \\ &\quad \left. + \big|\, \langle i, \psi_{0E}| U^\dagger(t) \big(|0\rangle\langle 1| \otimes \mathbb{1}_E \big) U(t) |j, \psi_{0E}\rangle \big|^2 \Big) \right) \end{aligned} \tag{22}$$

eq:appendix:A:4

JA: trabajando en analytic_choi_check.nb eq. (15)

- regular: Segundo y tercer término aproximadamente cero para $t \gg 1$
- caótico: tercer término igual a cero para $t \gg 1$.

JA: numéricamente parece que podemos hacer esto:

$$\begin{aligned}
\mathrm{Tr}\big[\mathcal{D}^2(t)\big] = \frac{1}{2}\Bigg(1 - \sum_i \underbrace{\langle i,\psi_{0E}|U^\dagger(t)\big(\,|0\rangle\langle 0|\otimes \mathbb{1}_E\big)U(t)|i,\psi_{0E}\rangle\ \langle i,\psi_{0E}|U^\dagger(t)\big(\,|1\rangle\langle 1|\otimes \mathbb{1}_E\big)U(t)|i,\psi_{0E}\rangle}_{(1-r_z^2)/4} \\
- 2\,\langle 0,\psi_{0E}|U^\dagger(t)\big(\,|0\rangle\langle 0|\otimes \mathbb{1}_E\big)U(t)|1,\psi_{0E}\rangle\ \langle 1,\psi_{0E}|U^\dagger(t)\big(\,|1\rangle\langle 1|\otimes \mathbb{1}_E\big)U(t)|0,\psi_{0E}\rangle \\
+ \sum_{i,j}\big|\,\langle i,\psi_{0E}|U^\dagger(t)\big(\,|0\rangle\langle 1|\otimes \mathbb{1}_E\big)U(t)|j,\psi_{0E}\rangle\big|^2\Bigg) \qquad (23)
\end{aligned}$$

$$\begin{aligned}
= \frac{1}{2}\Bigg(1 - \sum_i \underbrace{\langle i,\psi_{0E}|U^\dagger(t)\big(\,|0\rangle\langle 0|\otimes \mathbb{1}_E\big)U(t)|i,\psi_{0E}\rangle\ \langle i,\psi_{0E}|U^\dagger(t)\big(\,|1\rangle\langle 1|\otimes \mathbb{1}_E\big)U(t)|i,\psi_{0E}\rangle}_{(1-r_z^2)/4} \\
+ 2\big|\,\langle 0,\psi_{0E}|U^\dagger(t)\big(\,|0\rangle\langle 0|\otimes \mathbb{1}_E\big)U(t)|1,\psi_{0E}\rangle\big|^2 + \sum_{i,j}\big|\,\langle i,\psi_{0E}|U^\dagger(t)\big(\,|0\rangle\langle 1|\otimes \mathbb{1}_E\big)U(t)|j,\psi_{0E}\rangle\big|^2\Bigg) \qquad (24)
\end{aligned}$$

JA: notas

- para $t \gg 1$, $(1-r_z^2)/4 \approx 1/4$ en el régimen caótico ($r_z \approx 0$)
- para $t \gg 1$ el tercer término se hace approx 0 para el régimen caótico y distinto de 0 para regular. De hecho, todos los términos, excepto, el 1/2, son comparables a tiempos largos para el integrable
- a tiempos cortos el término que más contribuye es el término dentro la sumatoria del último término con índices $i = 0$ y $j = 1$

B Purity of Choi matrix

The quantum channel acting over the first ($i = 1$) spin is defined as:

eq:choi:chaometer:channel:2

$$\mathcal{E}(\rho) = \mathrm{Tr}_E \left(U^\dagger(t) \rho \otimes |\psi_0\rangle\langle\psi_0| \, U(t) \right), \tag{25}$$

where $|\psi_0\rangle = \bigotimes_{i=2}^{L} |\phi_i\rangle$, with $|\phi_i\rangle$ a random state of a single particle, is the initial state of the environment.

To compute the Choi-Jamiołkowski matrix $\mathcal{D}(t)$ of the quantum channel in eq. (25) we use the definition $(\mathcal{E} \otimes \mathbb{1})[|\phi^+\rangle\langle\phi^+|]$, with $|\phi^+\rangle = 1/\sqrt{2}(|0,0\rangle + |1,1\rangle)$ the maximally entangled state between two spins, and obtain

eq:choi:chaometer:2

$$\mathcal{D}(t) = \frac{1}{2} \sum_{i,j,p,q} \langle j, \psi_{0E}| \, U^\dagger(t) (|p\rangle\langle q| \otimes \mathbb{1}_E) U(t) \, |i, \psi_{0E}\rangle \, |q, i\rangle\langle p, j| \, . \tag{26}$$

After some algebraic step one may show that the purity of the Choi matrix is written as:

$$\mathrm{Tr}\left[\mathcal{D}^2(t)\right] = \mathrm{Tr}\left(\mathrm{Tr}_S[U(t)(\mathbb{1}_S \otimes |\psi_0\rangle\langle\psi_0|) U^\dagger(t)]^2\right). \tag{27}$$

I believe that when the quantum channel acts over a d-dimensional system, and $U(t) = e^{-iHt}$, with H a chaotic Hamiltonian (a matrix that can be taken from an appropriate RMT ensemble), the purity of the Choi matrix is equal to $1/d^2$ for $t \gg 1$ in the chaotic regime, and $1/d$ in the regular regime.

In particular, I would like to consider the spin chain Hamiltonian

eq:H:wisniacki:ising:chain

$$H = \sum_{i=1}^{L} (h_x \sigma_i^x + h_z \sigma_i^z) - \sum_{i=1}^{L-1} J_z \sigma_i^z \sigma_{i+1}^z, \tag{28}$$

for $h_x = j_i = 1$, $\forall i$, $h_z = 0.5$ (chaotic) and $h_z = 2.5$ (regular).

References

[1] Nicolás Mirkin and Diego Wisniacki. Quantum chaos, equilibration, and control in extremely short spin chains. *Phys. Rev. E*, 103:L020201, Feb 2021.