

Quantum chaos meets quantum channels

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1 José Alfredo

1.1 Model

The spin chain we are interested in studying first is that studied by Mirkin and Wisniacki in Ref. [1]:

$$H = \sum_{i=1}^L (h_x \sigma_i^x + h_z \sigma_i^z) - \sum_{i=1}^{L-1} J_z \sigma_i^z \sigma_{i+1}^z. \quad \text{eq:H:wisniacki:ising:chain} \quad (1)$$

1.2 Mean level spacing ratio

The level spacing ratio \tilde{r}_n is defined as:

$$\tilde{r}_n = \frac{\min(s_n, s_{n-1})}{\max(s_n, s_{n-1})}, \quad \text{eq:level:spacing:ratio} \quad (2)$$

where $s_n = E_{n+1} - E_n$. The mean level spacing ratio $\langle \tilde{r}_n \rangle$ is known to attain the value $\langle \tilde{r}_n \rangle \approx 0.5207$ when the level spacing distribution $P(s)$ is Wigner-Dyson and $\langle \tilde{r}_n \rangle \approx 0.386$ when it is Poisson.

1.3 Spectral form factor

The spectral form factor $K(t)$ is defined as:

$$K(t) = \frac{1}{2L} \left\langle |\text{Tr } U(t)|^2 \right\rangle = \frac{1}{2L} \left\langle \sum_{i,j} e^{i(E_i - E_j)t} \right\rangle, \quad \text{eq:sff} \quad (3)$$

where $\langle \cdot \rangle$ denotes the ensemble-average over statistically-similar systems.

1.4 Chaometer's quantum channel

The reduced dynamics of the chaometer is described by the quantum channel:

$$\mathcal{E}(\rho) = \text{Tr}_E \left(e^{-iHt} \rho \otimes \left| \psi_0^{(E)} \right\rangle \left\langle \psi_0^{(E)} \right| e^{iHt} \right), \quad \text{eq:chaometer:channel} \quad (4)$$

where H is that of eq. (28), $\left| \psi_0^{(E)} \right\rangle$ the initial state of all spins except the chaometer, and ρ the initial state of the chaometer.

The chaometer's quantum channel \mathcal{E} , in general, is divisible into:

1. A unitary operation rotating the Bloch's sphere.
2. A quantum channel that deforms the Bloch's sphere and translates its origin.

Both operations do not commute.

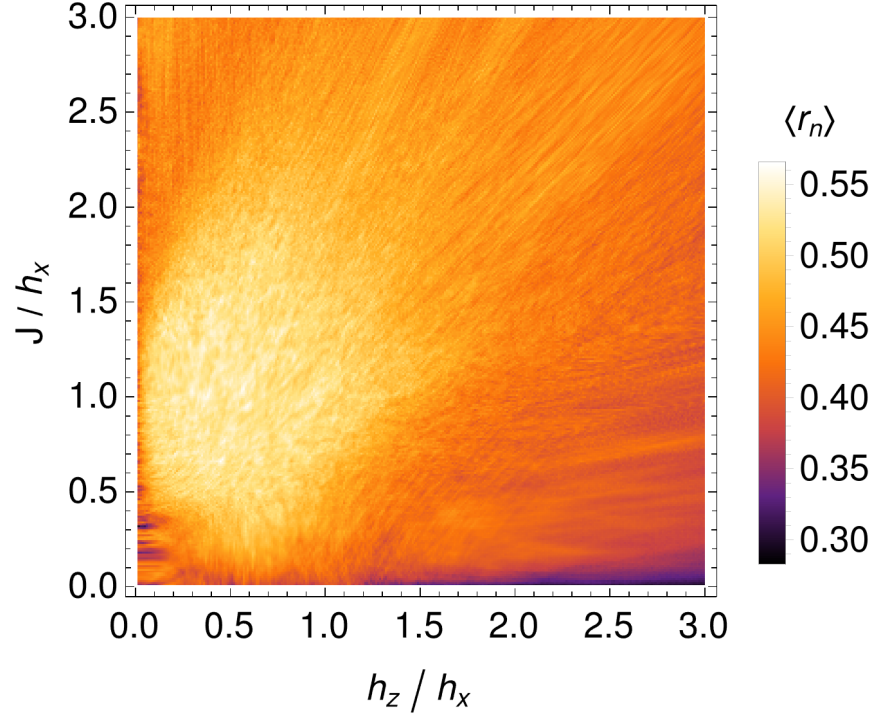


Figure 1: Mean level spacing ratio $\langle \tilde{r}_n \rangle$ [c.f. eq. (2)] of the Ising chain with Hamiltonian (28) as a function of ratios h_z/h_x and J/h_x . We assume $J_z = J \forall k$. fig:mean:level:spacing:ratio

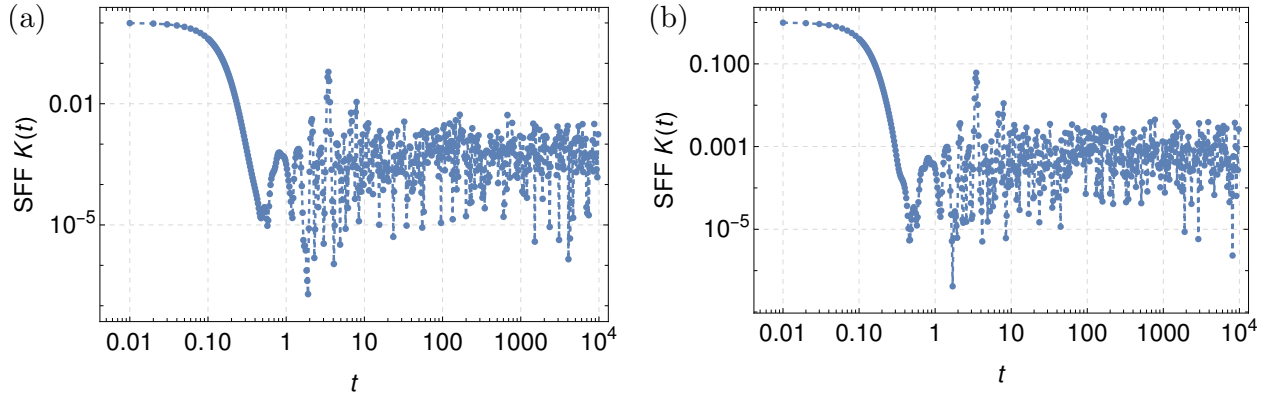


Figure 2: Spectral form factor (SFF) [c.f. (3)] in regular region: $h_z/h_x = 2.5$ and $J/h_x = 1$. **(a)** Whole spectrum. **(b)** Even-parity subspace spectrum. fig:sff:regular

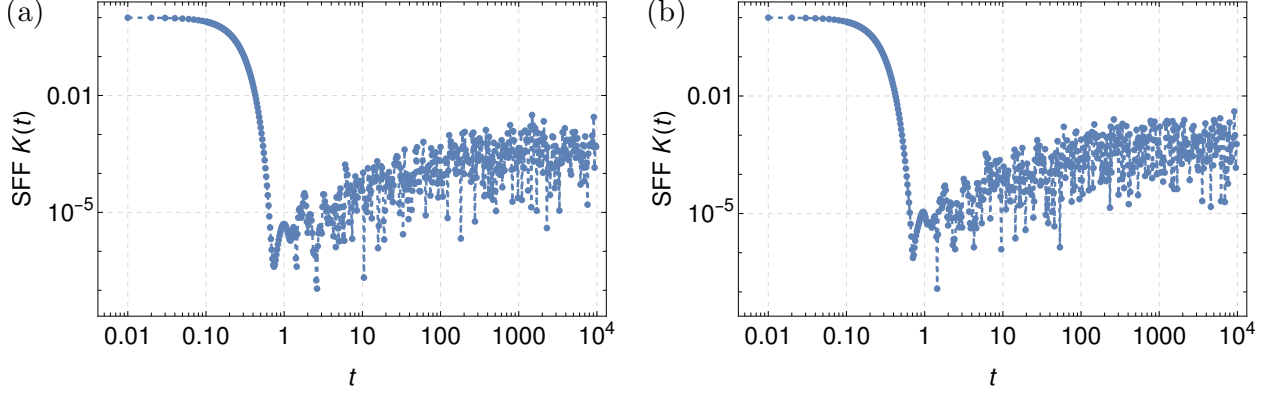


Figure 3: Spectral form factor (SFF) [c.f. (3)] in chaotic region: $h_z/h_x = 0.5$ and $J/h_x = 1$. (a) Whole spectrum. (b) Even-parity subspace spectrum. fig:sff:chaotic

1.5 Purity of the chaometer

Averaged purity \mathcal{P} is defined in Ref. [1] as:

$$\overline{\mathcal{P}} = \frac{1}{N} \sum_{i=1}^N \left(\frac{1}{T} \int_0^T \text{Tr} [\rho_i^2(t)] \right) \quad \text{eq:avg:purity} \quad (5)$$

where:

- $\rho_i(t)$: chaometer's density matrix.
- N : number of different random initial states of the whole chain. $N = 50$ in Mirkin and Wisniacki [1].
- T : maximum time. $T = 50$ in Mirkin and Wisniacki [1].

Also, the normalized averaged purity is defined as

$$\overline{\mathcal{P}}_{\text{Norm}} = \frac{\overline{\mathcal{P}} - \min(\overline{\mathcal{P}})}{\max(\overline{\mathcal{P}}) - \min(\overline{\mathcal{P}})}, \quad \text{eq:avg:norm:purity} \quad (6)$$

where $\max(\overline{\mathcal{P}})$ and $\min(\overline{\mathcal{P}})$ are the minimum and maximum value obtained when sweeping the parameter range (h_z in their case).

In Fig. 4 we plot one realization of the dynamics of purity of the chaometer. We compare in Fig. 5 our results and those of Mirkin and Wisniacki [1].

1.6 Purity of Choi-Jamiolkowski matrix

We investigate the purity of Choi-Jamiolkowski matrix of quantum channel $\mathcal{E}(t)$ of the chaometer [c.f. eq. (4)] in Fig. 6.

To compute the Choi-Jamiolkowski matrix $\mathcal{D}(t)$ of the chaometer's quantum channel in eq. (4) we use the definition $(\mathcal{E} \otimes \mathbb{1})[|\phi^+\rangle\langle\phi^+|]$, with $|\phi^+\rangle = 1/\sqrt{2}(|0,0\rangle + |1,1\rangle)$ is the maximally entangled state between two spins, and obtain

$$\mathcal{D}(t) = \frac{1}{2} \sum_{i,j,p,q} \langle j, \psi_{0E} | U^\dagger(t) (|p\rangle\langle q| \otimes \mathbb{1}_E) U(t) | i, \psi_{0E} \rangle | q, i \rangle \langle p, j |. \quad \text{eq:choi:chaometer} \quad (7)$$

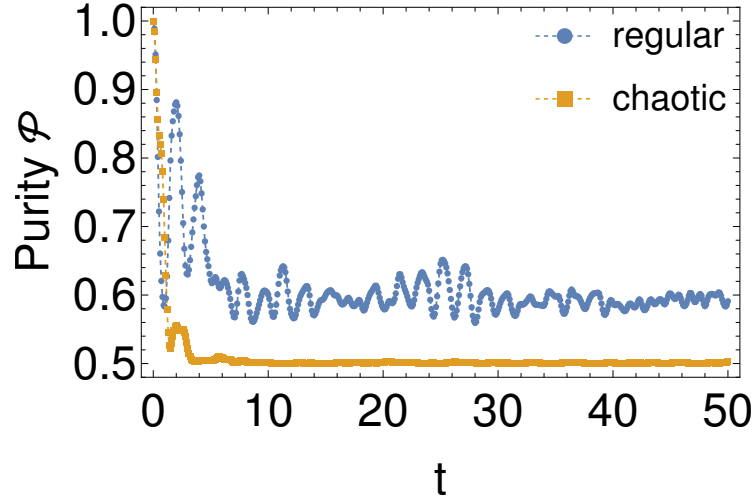


Figure 4: Dynamics of chaometer's purity of a single realization with the same initial state of the environment of the first chaometer's quantum channel of the videos. fig:purity:one:realization

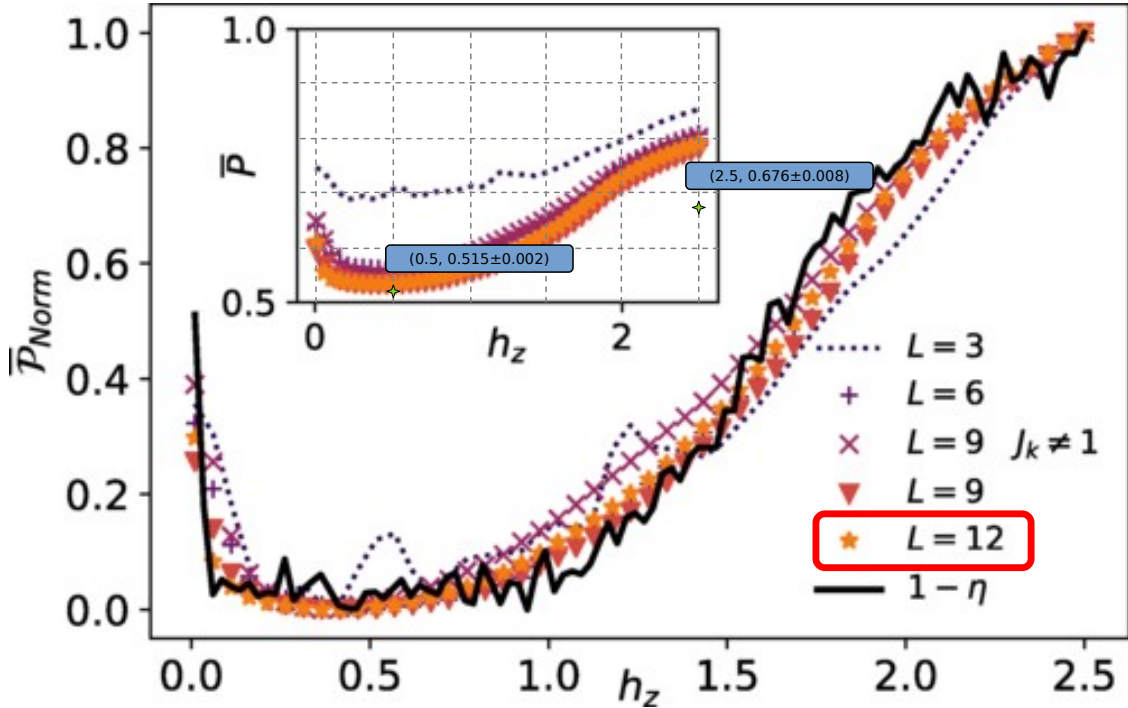


Figure 5: Averaged \bar{P} and averaged normalized purities \bar{P}_{Norm} [c.f. eqs. (5) and (6).] for $N = 50$ random initial states of the chaometer for each quantum channel showed in the videos. JA: Tendría más sentido sacar la pureza del canal. Lo pienso Taken and modified from Ref. [1]. fig:mirkin2021:fig2:modified

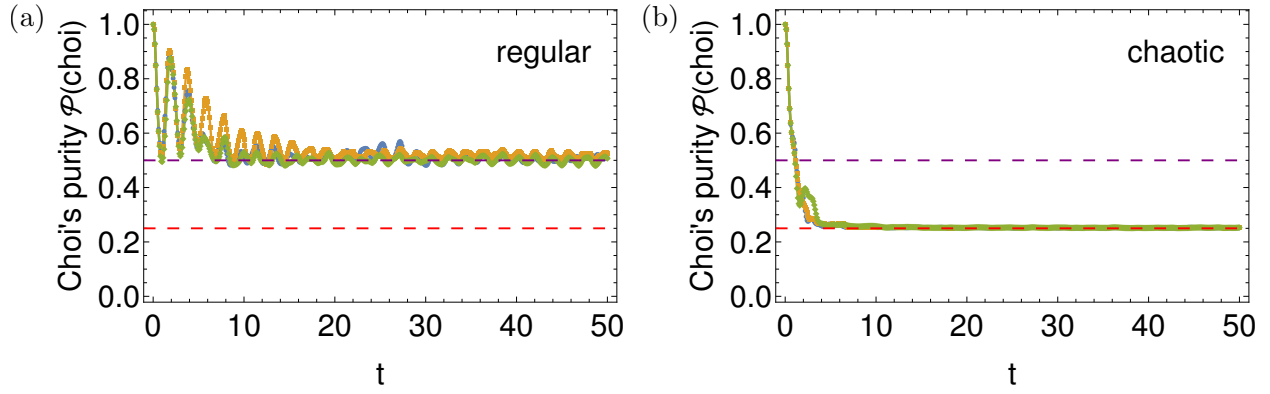


Figure 6: Purity of Choi-Jamiolkowski matrix in (a) regular ($h_z = 2.5$) and (b) chaotic ($h_z = 0.5$) for the three random initial states showed in the video. fig:choi:purity

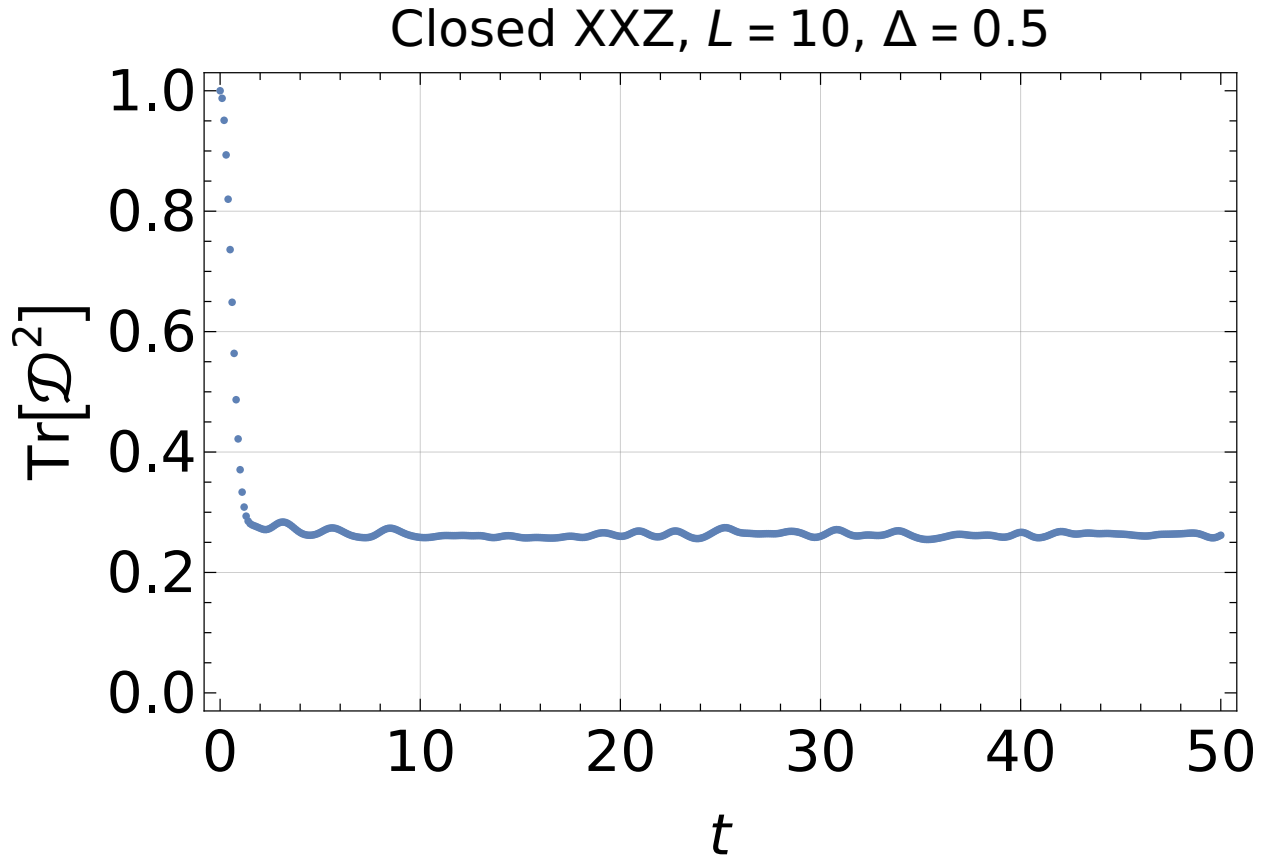


Figure 7:

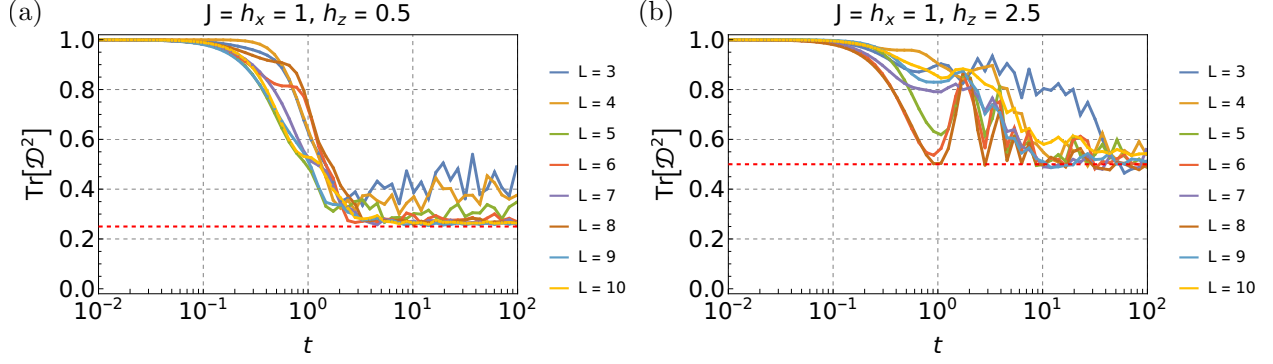


Figure 8: Dynamics of purity of the Choi-Jamiolkowski matrix for various lengths L of the spin chain. fig:choi-purity:dynamics

Let us write $\mathcal{E}(\rho)$ in computational basis. From eq. (4) we write

$$\mathcal{E}(\rho) = \sum_{k,l} e^{-i(E_k - E_l)t} \langle E_k | (\rho \otimes |\psi_E\rangle\langle\psi_E|) | E_l \rangle \text{Tr}_E (|E_k\rangle\langle E_l|) \quad (8)$$

$$= \sum_{k,l,p,p',\vec{q},\vec{q}'} e^{-i(E_k - E_l)t} \langle E_k | (\rho \otimes |\psi_E\rangle\langle\psi_E|) | E_l \rangle \text{Tr}_E (|p,\vec{q}\rangle\langle p,\vec{q}'| E_k \rangle \langle E_l | p',\vec{q}' \rangle \langle p',\vec{q}'|), \quad (9)$$

where \vec{q} and \vec{q}' are indices of elements of the basis of the environment,

$$= \sum_{p,p',\vec{q}} \langle p,\vec{q} | U(t) (\rho \otimes |\psi_E\rangle\langle\psi_E|) U^\dagger(t) | p',\vec{q} \rangle | p \rangle \langle p' | \quad (10)$$

$$= \sum_{p,p'} \langle p | \text{Tr}_E [U(t) (\rho \otimes |\psi_E\rangle\langle\psi_E|) U^\dagger(t)] | p' \rangle | p \rangle \langle p' |. \quad (11)$$

To compute the Choi-Jamiolkowski matrix $\mathcal{D}(t)$ of the chaometer's quantum channel in eq. (4) we use the definition $(\mathcal{E} \otimes \mathbb{1})[|\phi^+\rangle\langle\phi^+|]$, with $|\phi^+\rangle = 1/\sqrt{2}(|0,0\rangle + |1,1\rangle)$ is the maximally entangled state between two spins, and eq. (11) to obtain

$$\mathcal{D}(t) = \frac{1}{2} \sum_{i,j,p,p',\vec{q}} \langle p,\vec{q} | U(t) (|i,\psi_E\rangle\langle j,\psi_E|) U^\dagger(t) | p',\vec{q} \rangle | p,i \rangle \langle p',j | \quad (12)$$

$$= \frac{1}{2} \sum_{i,j,p,p'} \langle j,\psi_{0E} | U^\dagger(t) (|p'\rangle\langle p| \otimes \mathbb{1}_E) U(t) | i,\psi_{0E} \rangle | p,i \rangle \langle p',j |, \quad (13)$$

where we have only reordered the factors and used the resolution of identity over \mathcal{H}_E .

We compute the purity of $\mathcal{D}(t)$. From eq. (7) it is straightforward to obtain

$$\text{Tr}[\mathcal{D}^2(t)] = \frac{1}{4} \sum_{i,j,p,q} |\langle i,\psi_{0E} | U^\dagger(t) (|p\rangle\langle q| \otimes \mathbb{1}_E) U(t) | j,\psi_{0E} \rangle|^2. \quad (14)$$

Moreover, reordering this expression an interesting interpretation of the purity of the Choi-Jamiolkowski matrix is revealed. Let us conveniently rewrite eq. (14) as

$$\text{Tr}[\mathcal{D}^2(t)] = \langle \psi_{0E} | \text{Tr}_S \left\{ U^\dagger(t) \sum_{p,q} \left[\left(\frac{|p\rangle\langle q|}{\sqrt{2}} \otimes \mathbb{1}_E \right) U(t) \left(\frac{\mathbb{1}_S}{2} \otimes |\psi_{0E}\rangle\langle\psi_{0E}| \right) U^\dagger(t) \left(\frac{|q\rangle\langle p|}{\sqrt{2}} \otimes \mathbb{1}_E \right) \right] U(t) \right\} | \psi_{0E} \rangle. \quad (15)$$

- First, a forward evolution $\rho(t) = U(t)\rho_0 U^\dagger(t)$ occurs, with $\rho_0 = (\frac{\mathbb{1}_S}{2} \otimes |\psi_{0E}\rangle\langle\psi_{0E}|)$.

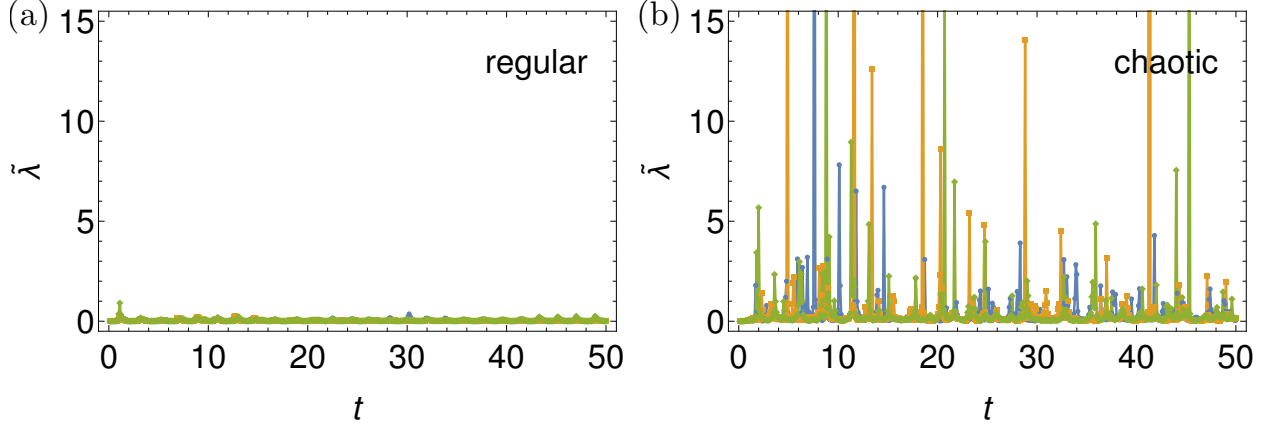


Figure 9: Most negative eigenvalue $\tilde{\lambda}$ of map $\Lambda(t, s)/2$, with $s = 0.1$ [c.f. [eqs. \(17\) and \(18\)](#)]. [eq:lambda:tilde](#) “jami-”

- Second, backwards evolution of $\rho(t)$ is followed, $\tilde{\rho}_0 = U^\dagger(t) \sum_{p,q} \left[\left(\frac{|p\rangle\langle q|}{\sqrt{2}} \otimes \mathbb{1}_E \right) \rho(t) \left(\frac{|q\rangle\langle p|}{\sqrt{2}} \otimes \mathbb{1}_E \right) \right] U(t)$, where before evolving backwards with $U^\dagger(t)$ a completely depolarizing quantum channel acts over the system.
- Finally, $\text{Tr}[\mathcal{D}^2(t)]$ is the probability of finding the environment in its initial state.

[JA: esta es una especie de echo de Loschmidt!!!](#) [Cálculos en el apéndice A](#) In other words, the purity of $\mathcal{D}(t)$ represents the probability that the environment remains in its initial state after the following process: First, the system and environment are initialized in a product state, with the system in a maximally mixed state. Next, the combined system evolves. Then, a completely depolarizing channel is applied to the system. Finally, the system and environment undergo reverse evolution.

The Loschmidt echo is defined as

$$M(t) = \left| \langle \psi_0 | e^{-iH_2 t/\hbar} e^{-iH_1 t/\hbar} | \psi_0 \rangle \right|^2 \quad \text{eq:loschmidt:echo} \quad (16)$$

1.7 Non complete-positiveness of $\Lambda(t, s)$

Any quantum channel $\mathcal{E}(t)$ can be composed as

$$\mathcal{E}(t) = \Lambda(t, s) \circ \mathcal{E}(s, 0), \quad \text{eq:Lambda} \quad (17)$$

nonetheless, $\Lambda(t, s)$ is not in general completely positive. A way to quantify how far is $\Lambda(t, s)$ from being completely positive is through $\tilde{\lambda}$: [eq:lambda:tilde](#)

$$\tilde{\lambda} = |\min(0, \lambda_{\text{smallest}})|, \quad (18)$$

where $\lambda_{\text{smallest}}$ is the smallest eigenvalue of $\Lambda^R(t, s)/2$. We have added the factor $1/2$ just so $\text{Tr}[\Lambda^R(t, s)/2] = 1$.

Let us fix $s = 0.1$ and investigate the complete positiveness of $\Lambda(t, s)$, see Fig. 9.

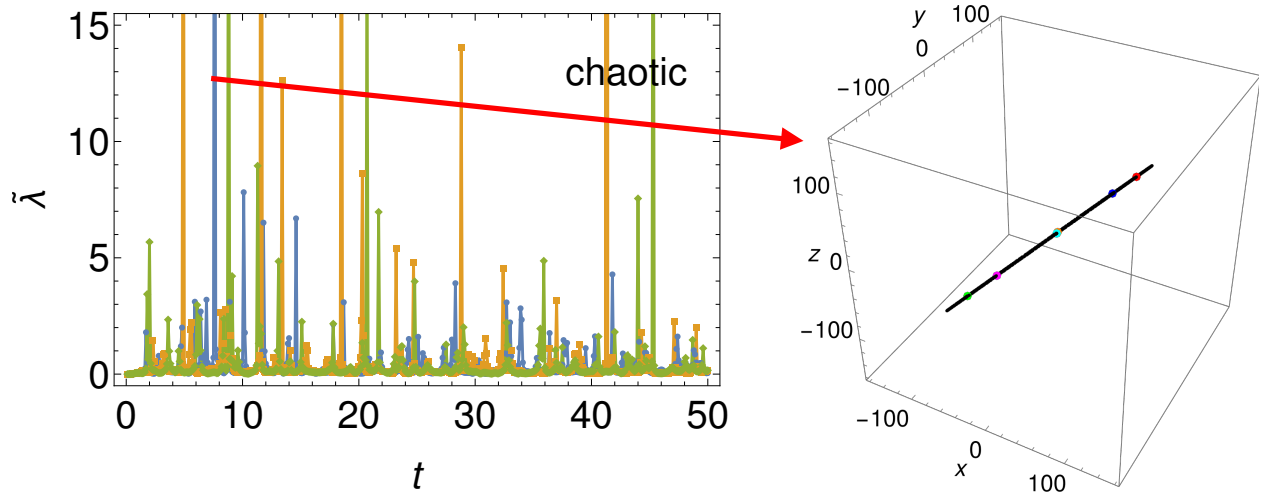


Figure 10: Burst of the Bloch sphere at $t = 0.5$ s.

fig:burst

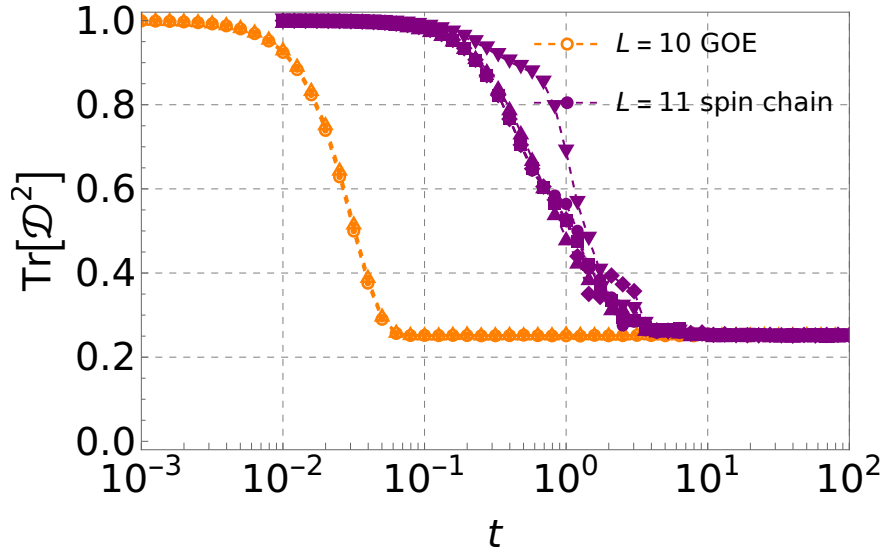


Figure 11: Purity of the Choi-Jamiolkowski matrix of chaometer's quantum channel and another channel from a Hamiltonian taken from the GOE ensemble.

fig:choi:purity:spin:chain:vs:goe

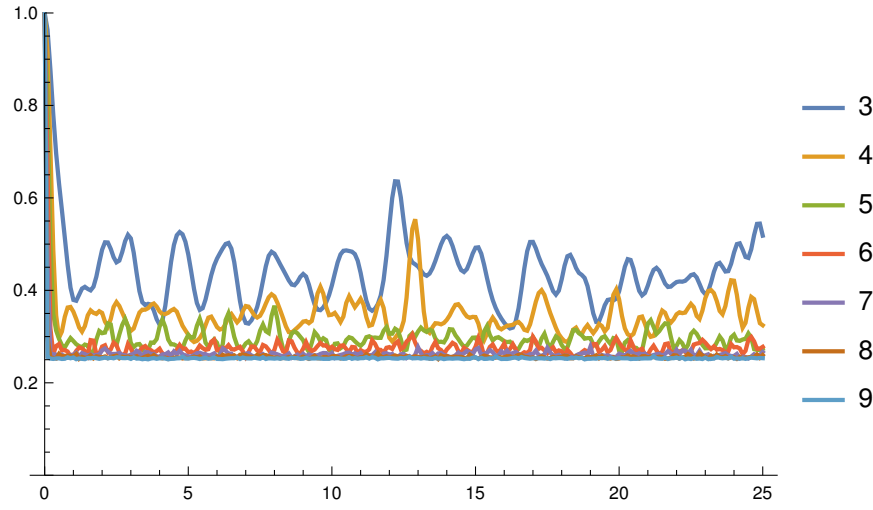


Figure 12: JA: La pureza de la matriz de Choi de una matriz U de COE se comporta así variando el número de espines L desde 3 hasta 9. Parece que para L muy grande el límite asintótico tiende a 0.25, Se podrá probar analíticamente?

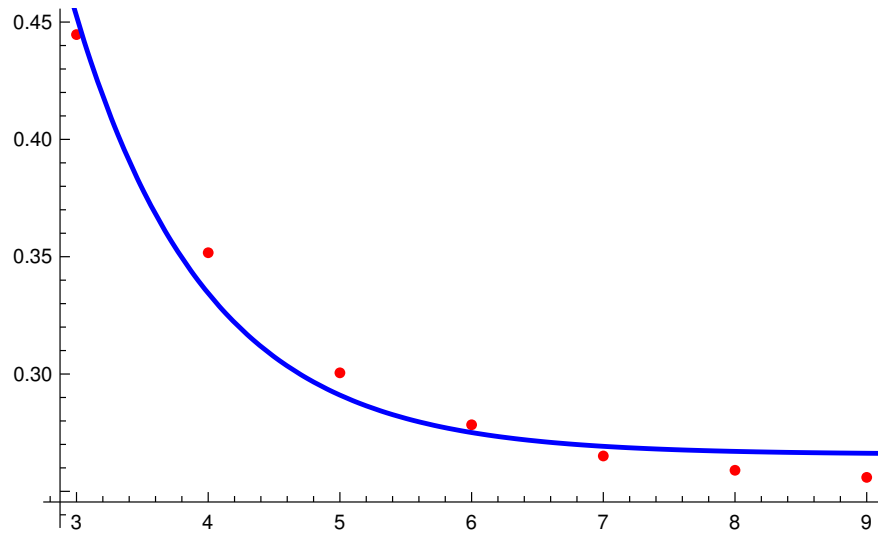


Figure 13: JA: $\text{Tr}[\mathcal{D}_\infty^2]$ vs L . Creo que se podría conseguir un mejor ajuste teniendo estadística para cada punto.

1.8 Purity of Choi-Jamiołkowski matrix

2 Charles

3 Miguel

4 Viko

A Choi

app:choi

$$\begin{aligned}
 \text{Tr}[\mathcal{D}^2(t)] &= \frac{1}{4} \left(\sum_{i,j,p} |\langle i, \psi_{0E} | U^\dagger(t) (|p\rangle\langle p| \otimes \mathbb{1}_E) U(t) | j, \psi_{0E} \rangle|^2 + \sum_{i,j,p \neq q} |\langle i, \psi_{0E} | U^\dagger(t) (|p\rangle\langle q| \otimes \mathbb{1}_E) U(t) | j, \psi_{0E} \rangle|^2 \right) \\
 &= \frac{1}{4} \left(\sum_{i,j} |\langle i, \psi_{0E} | U^\dagger(t) (\mathbb{1}_S \otimes \mathbb{1}_E) U(t) | j, \psi_{0E} \rangle|^2 \right. \\
 &\quad - 2 \sum_{i,j} \text{Re}(\langle i, \psi_{0E} | U^\dagger(t) (|0\rangle\langle 0| \otimes \mathbb{1}_E) U(t) | j, \psi_{0E} \rangle \langle j, \psi_{0E} | U^\dagger(t) (|1\rangle\langle 1| \otimes \mathbb{1}_E) U(t) | i, \psi_{0E} \rangle) \\
 &\quad \left. + \sum_{i,j,p \neq q} |\langle i, \psi_{0E} | U^\dagger(t) (|p\rangle\langle q| \otimes \mathbb{1}_E) U(t) | j, \psi_{0E} \rangle|^2 \right) \tag{20} \\
 &= \frac{1}{2} \left(1 - \sum_{i,j} \left(\text{Re}(\langle i, \psi_{0E} | U^\dagger(t) (|0\rangle\langle 0| \otimes \mathbb{1}_E) U(t) | j, \psi_{0E} \rangle \langle j, \psi_{0E} | U^\dagger(t) (|1\rangle\langle 1| \otimes \mathbb{1}_E) U(t) | i, \psi_{0E} \rangle) \right. \right. \\
 &\quad \left. \left. + |\langle i, \psi_{0E} | U^\dagger(t) (|0\rangle\langle 1| \otimes \mathbb{1}_E) U(t) | j, \psi_{0E} \rangle|^2 \right) \right) \tag{21} \\
 &= \frac{1}{2} \left(1 - \sum_{i,j} \left(\langle i, \psi_{0E} | U^\dagger(t) (|0\rangle\langle 0| \otimes \mathbb{1}_E) U(t) | j, \psi_{0E} \rangle \langle j, \psi_{0E} | U^\dagger(t) (|1\rangle\langle 1| \otimes \mathbb{1}_E) U(t) | i, \psi_{0E} \rangle \right. \right. \\
 &\quad \left. \left. + |\langle i, \psi_{0E} | U^\dagger(t) (|0\rangle\langle 1| \otimes \mathbb{1}_E) U(t) | j, \psi_{0E} \rangle|^2 \right) \right) \tag{22}
 \end{aligned}$$

eq:appendix:A:4

JA: trabajando en `analytic_choi_check.nb` eq. (15)

- regular: Segundo y tercer término aproximadamente cero para $t \gg 1$
- caótico: tercer término igual a cero para $t \gg 1$.

JA: numéricamente parece que podemos hacer esto:

$$\begin{aligned} \text{Tr}[\mathcal{D}^2(t)] = & \frac{1}{2} \left(1 - \sum_i \underbrace{\langle i, \psi_{0E} | U^\dagger(t) (|0\rangle\langle 0| \otimes \mathbb{1}_E) U(t) | i, \psi_{0E} \rangle \langle i, \psi_{0E} | U^\dagger(t) (|1\rangle\langle 1| \otimes \mathbb{1}_E) U(t) | i, \psi_{0E} \rangle}_{(1-r_z^2)/4} \right. \\ & - 2 \langle 0, \psi_{0E} | U^\dagger(t) (|0\rangle\langle 0| \otimes \mathbb{1}_E) U(t) | 1, \psi_{0E} \rangle \langle 1, \psi_{0E} | U^\dagger(t) (|1\rangle\langle 1| \otimes \mathbb{1}_E) U(t) | 0, \psi_{0E} \rangle \\ & \left. + \sum_{i,j} \left| \langle i, \psi_{0E} | U^\dagger(t) (|0\rangle\langle 1| \otimes \mathbb{1}_E) U(t) | j, \psi_{0E} \rangle \right|^2 \right) \end{aligned} \quad (23)$$

$$\begin{aligned} = & \frac{1}{2} \left(1 - \sum_i \underbrace{\langle i, \psi_{0E} | U^\dagger(t) (|0\rangle\langle 0| \otimes \mathbb{1}_E) U(t) | i, \psi_{0E} \rangle \langle i, \psi_{0E} | U^\dagger(t) (|1\rangle\langle 1| \otimes \mathbb{1}_E) U(t) | i, \psi_{0E} \rangle}_{(1-r_z^2)/4} \right. \\ & \left. + 2 \left| \langle 0, \psi_{0E} | U^\dagger(t) (|0\rangle\langle 0| \otimes \mathbb{1}_E) U(t) | 1, \psi_{0E} \rangle \right|^2 + \sum_{i,j} \left| \langle i, \psi_{0E} | U^\dagger(t) (|0\rangle\langle 1| \otimes \mathbb{1}_E) U(t) | j, \psi_{0E} \rangle \right|^2 \right) \end{aligned} \quad (24)$$

JA: notas

- para $t \gg 1$, $(1 - r_z^2)/4 \approx 1/4$ en el régimen caótico ($r_z \approx 0$)
- para $t \gg 1$ el tercer término se hace approx 0 para el régimen caótico y distinto de 0 para regular. De hecho, todos los términos, excepto, el $1/2$, son comparables a tiempos largos para el integrable
- a tiempos cortos el término que más contribuye es el término dentro la sumatoria del último término con índices $i = 0$ y $j = 1$

B Purity of Choi matrix

The quantum channel acting over the first ($i = 1$) spin is defined as:

$$\mathcal{E}(\rho) = \text{Tr}_E (U^\dagger(t) \rho \otimes |\psi_0\rangle\langle\psi_0| U(t)), \quad (25)$$

where $|\psi_0\rangle = \bigotimes_{i=2}^L |\phi_i\rangle$, with $|\phi_i\rangle$ a random state of a single particle, is the initial state of the environment.

To compute the Choi-Jamiołkowski matrix $\mathcal{D}(t)$ of the quantum channel in eq. (25) we use the definition $(\mathcal{E} \otimes \mathbb{1})[|\phi^+\rangle\langle\phi^+|]$, with $|\phi^+\rangle = 1/\sqrt{2}(|0,0\rangle + |1,1\rangle)$ the maximally entangled state between two spins, and obtain

$$\mathcal{D}(t) = \frac{1}{2} \sum_{i,j,p,q} \langle j, \psi_{0E} | U^\dagger(t) (|p\rangle\langle q| \otimes \mathbb{1}_E) U(t) | i, \psi_{0E} \rangle | q, i \rangle \langle p, j |. \quad (26)$$

After some algebraic step one may show that the purity of the Choi matrix is written as:

$$\text{Tr}[\mathcal{D}^2(t)] = \text{Tr}(\text{Tr}_S[U(t)(\mathbb{1}_S \otimes |\psi_0\rangle\langle\psi_0|)U^\dagger(t)]^2). \quad (27)$$

I believe that when the quantum channel acts over a d -dimensional system, and $U(t) = e^{-iHt}$, with H a chaotic Hamiltonian (a matrix that can be taken from an appropriate RMT ensemble), the purity of the Choi matrix is equal to $1/d^2$ for $t \gg 1$ in the chaotic regime, and $1/d$ in the regular regime.

In particular, I would like to consider the spin chain Hamiltonian

$$H = \sum_{i=1}^L (h_x \sigma_i^x + h_z \sigma_i^z) - \sum_{i=1}^{L-1} J_z \sigma_i^z \sigma_{i+1}^z, \quad (28)$$

for $h_x = j_i = 1$, $\forall i$, $h_z = 0.5$ (chaotic) and $h_z = 2.5$ (regular).

References

- [1] Nicolás Mirkin and Diego Wisniacki. Quantum chaos, equilibration, and control in extremely short spin chains. *Phys. Rev. E*, 103:L020201, Feb 2021.