# Quantum chaos meets quantum channels

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## 1.1 Model

The spin chain we are interested in studying first is that studied by Mirkin and Wisniacki in Ref. [1]:

$$H = \sum_{i=1}^{L} \left( h_x \sigma_i^x + h_z \sigma_i^z \right) - \sum_{i=1}^{L-1} J_z \sigma_i^z \sigma_{i+1}^z. \tag{1}$$

## 1.2 Mean level spacing ratio

The level spacing ratio $\tilde{r}_n$ is defined as:

$$\tilde{r}_n = \frac{\min(s_n, s_{n-1})}{\max(s_n, s_{n-1})}, \tag{2}$$

where $s_n = E_{n+1} - E_n$. The mean level spacing ratio $\langle \tilde{r}_n \rangle$ is known to attain the value $\langle \tilde{r}_n \rangle \approx 0.5207$ when the level spacing distribution $P(s)$ is Wigner-Dyson and $\langle \tilde{r}_n \rangle \approx 0.386$ when it is Poisson.

## 1.3 Spectral form factor

The spectral form factor $K(t)$ is defined as:

$$K(t) = \frac{1}{2^L} \left\langle |\mathrm{Tr}\, U(t)|^2 \right\rangle = \frac{1}{2^L} \left\langle \sum_{i,j} e^{i(E_i - E_j)t} \right\rangle, \tag{3}$$

where $\langle \cdot \rangle$ denotes the ensemble-average over statistically-similar systems.

## 1.4 Chaometer's quantum channel

The reduced dyanmics of the chaometer is described by the quantum channel:

$$\mathcal{E}(\rho) = \mathrm{Tr}_E \left( e^{-iHt} \rho \otimes \left| \psi_0^{(E)} \right\rangle \left\langle \psi_0^{(E)} \right| e^{iHt} \right), \tag{4}$$

where $H$ is that of eq. (28), $\left| \psi_0^{(E)} \right\rangle$ the initial state of all spins except the chaometer, and $\rho$ the initial state of the chaometer.

The chaometer's quantum channel $\mathcal{E}$, in general, is divisible into:

1. A unitary operation rotating the Bloch's sphere.

2. A quantum channel that deforms the Bloch's sphere and translates its origin.
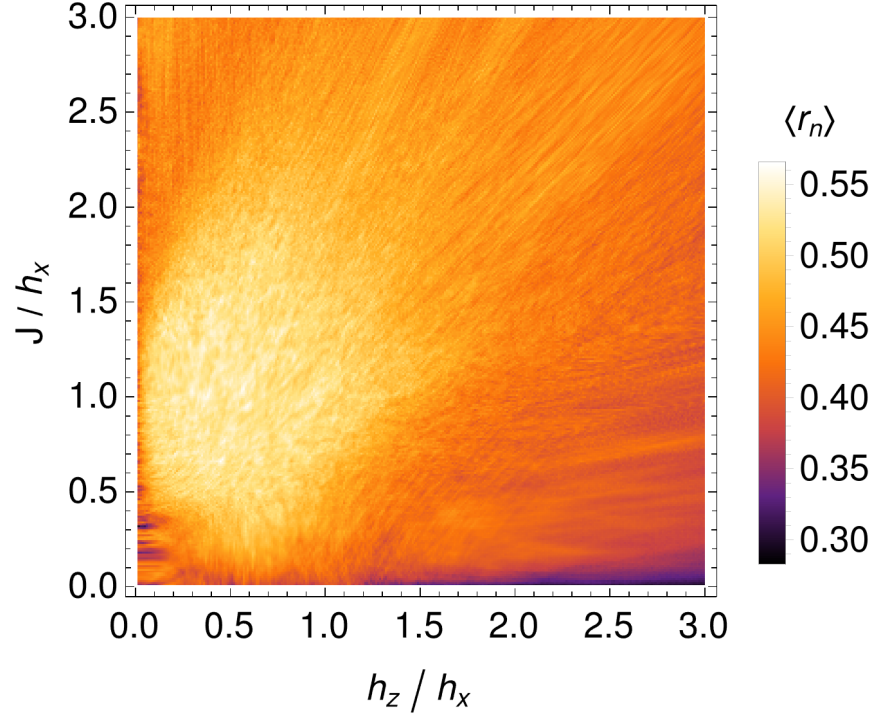
Both operations do not commute.

Figure 1: Mean level spacing ratio $\langle \tilde{r}_n \rangle$ [c.f. eq. (2)] of the Ising chain with Hamiltonian (28) as a function of ratios $h_z/h_x$ and $J/h_x$. We assume $J_z = J \, \forall \, k$.

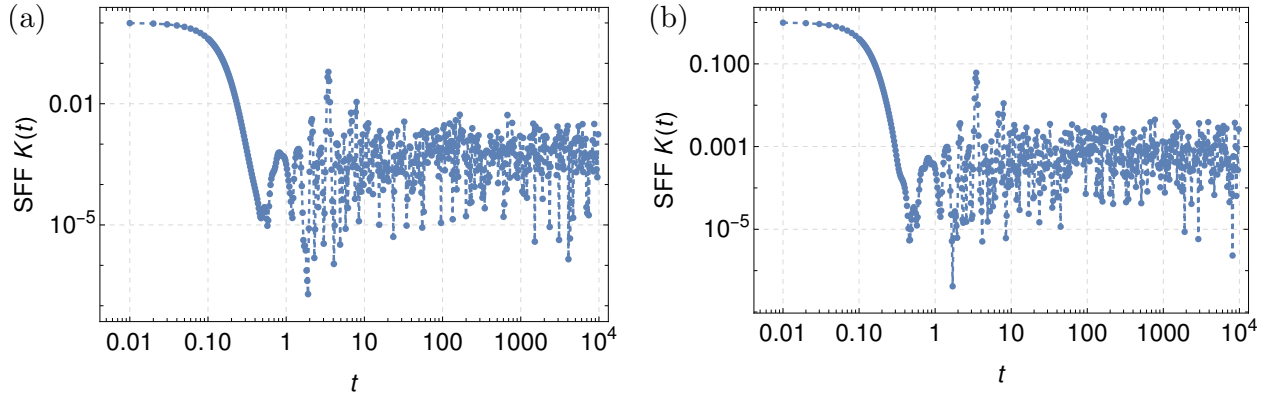<span style="color:red">fig:mean:level:spacing:ratio</span>



Figure 2: Spectral form factor (SFF) [c.f. (3)] in regular region: $h_z/h_x = 2.5$ and $J/h_x = 1$. **(a)** Whole spectrum. **(b)** Even-parity subspace spectrum.

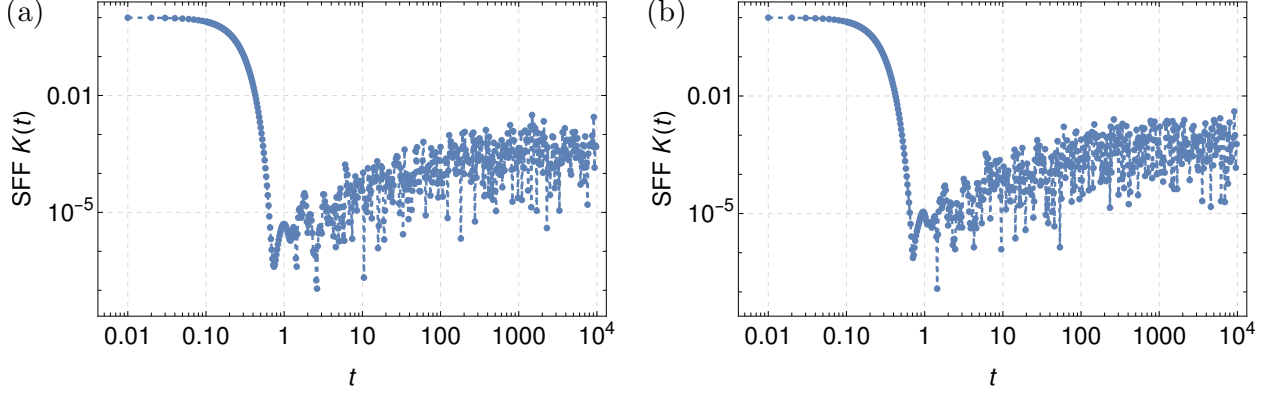<span style="color:red">fig:sff:regular</span>

Figure 3: Spectral form factor (SFF) [c.f. (3)] in chaotic region: $h_z/h_x = 0.5$ and $J/h_x = 1$. **(a)** Whole spectrum. **(b)** Even-parity subspace spectrum.

<span style="color:red">fig:sff:chaotic</span>

## 1.5  Purity of the chaometer

Averaged purity $\mathcal{P}$ is defined in Ref. [1] as:

<span style="color:red">eq:avg:purity</span>
$$\overline{\mathcal{P}} = \frac{1}{N} \sum_{i=1}^{N} \left( \frac{1}{T} \int_0^T \mathrm{Tr}\left[\rho_i^2(t)\right] \right) \tag{5}$$

where:

- $\rho_i(t)$: chaometer's density matrix.

- $N$: number of different random initial states of the whole chain. $N = 50$ in Mirkin and Wisniacki [1].

- $T$: maximum time. $T = 50$ in Mirkin and Wisniacki [1].

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Also, the normalized averaged purity is defined as

<span style="color:red">eq:avg:norm:purity</span>
$$\overline{P}_{Norm} = \frac{\overline{P} - \min(\overline{P})}{\max(\overline{P}) - \min(\overline{P})}, \tag{6}$$

where $\max(\overline{P})$ and $\min(\overline{P})$ are the minimum and maximum value obtained when sweeping the parameter range ($h_z$ in their case).

In Fig. 4 we plot one realization of the dynamics of purity of the chaometer. We compare in Fig. 5 our results and those of Mirkin and Wisniacki [1].

## 1.6  Purity of Choi-Jamiołkowski matrix

We investigate the purity of Choi-Jamiołkowski matrix of quantum channel $\mathcal{E}(t)$ of the chaometer [c.f. eq. (4)] in Fig. 6.

To compute the Choi-Jamiołkowski matrix $\mathcal{D}(t)$ of the chaometer's quantum channel in eq. (4) we use the definition $(\mathcal{E} \otimes \mathbb{1})[|\phi^+\rangle\langle\phi^+|]$, with $|\phi^+\rangle = 1/\sqrt{2}(|0,0\rangle + |1,1\rangle)$ is the maximally entangled state between two spins, and obtain

<span style="color:red">eq:choi:chaometer</span>
$$\mathcal{D}(t) = \frac{1}{2} \sum_{i,j,p,q} \langle j, \psi_{0E}| \, U^\dagger(t)(|p\rangle\langle q| \otimes \mathbb{1}_E)U(t) \, |i, \psi_{0E}\rangle \, |q,i\rangle\langle p,j| . \tag{7}$$
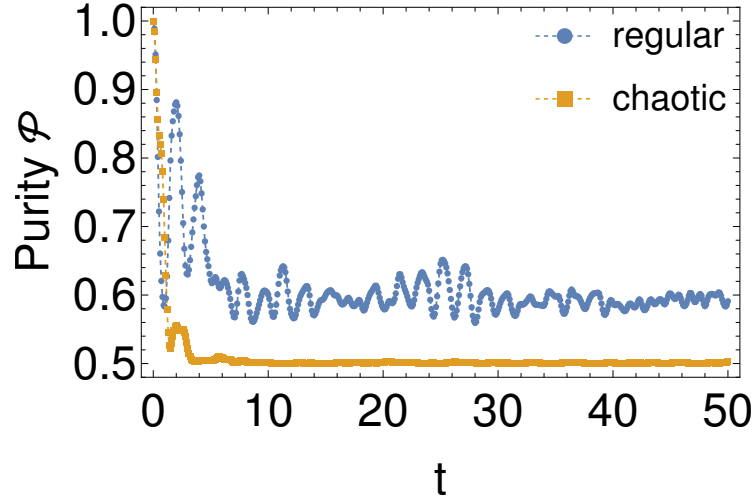
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Figure 4: Dynamics of chaometer's purity of a single realization with the same initial state of the environment of the first chaometer's quantum channel of the videos.

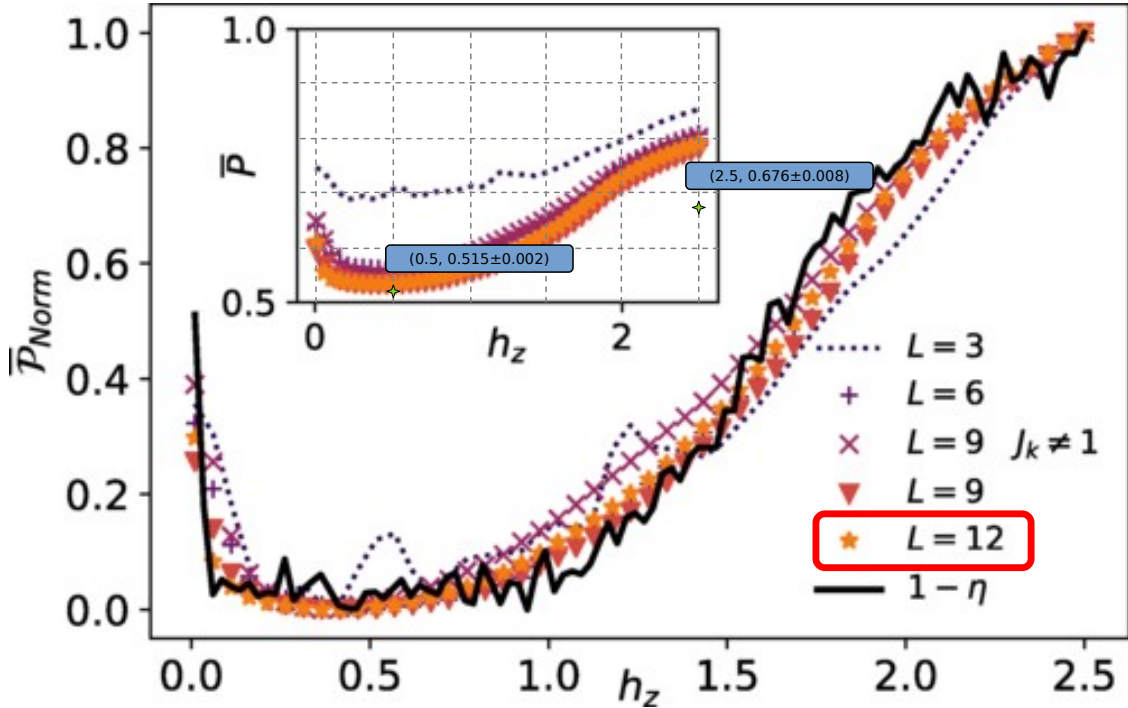Figure 5: Averaged $\overline{P}$ and averaged normalized purities $\overline{P}_{Norm}$ [c.f. eqs. (5) and (6).] for $N = 50$ random initial states of the chaometer for each quantum channel showed in the videos. JA: Tendría más sentido sacar la pureza del canal. Lo pienso Taken and modified from Ref. [1].
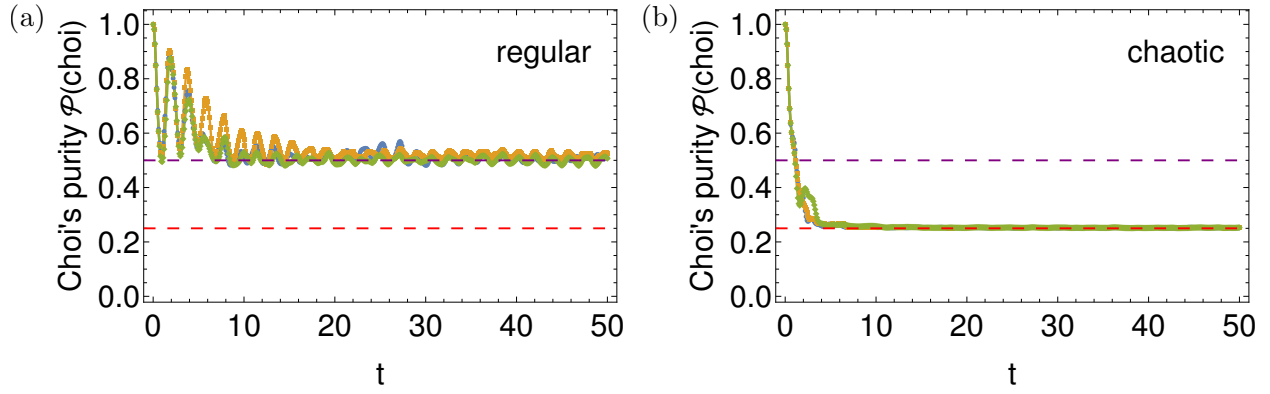
Figure 6: Purity of Choi-Jamiołkowski matrix in (a) regular ($h_z = 2.5$) and (b) chaotic ($h_z = 0.5$) for the three random initial states showed in the video. *fig:choi:purity*
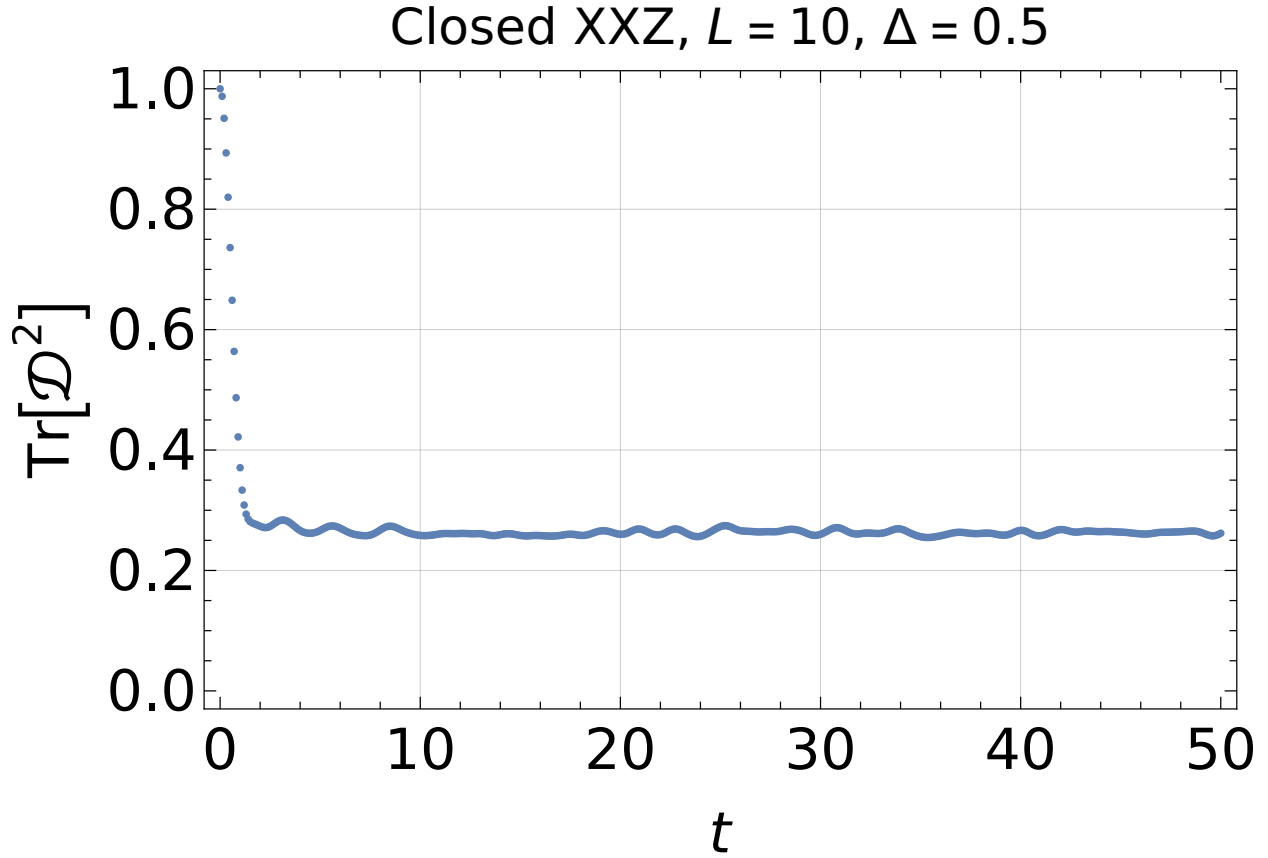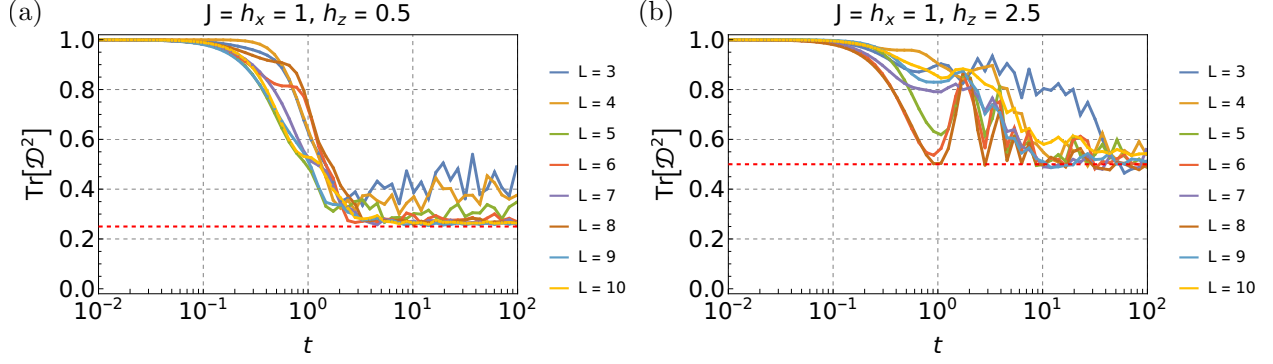


Figure 7:

Figure 8: Dynamics of purity of the Choi-Jamiołkowski matrix for various lengths $L$ of the spin chain.

Let us write $\mathcal{E}(\rho)$ in computational basis. From eq. (4) we write

$$\mathcal{E}(\rho) = \sum_{k,l} e^{-i(E_k - E_l)t} \langle E_k | (\rho \otimes |\psi_E\rangle\langle\psi_E|) |E_l\rangle \operatorname{Tr}_E (|E_k\rangle\langle E_l|) \tag{8}$$

$$= \sum_{k,l,p,p',\vec{q},\vec{q}'} e^{-i(E_k - E_l)t} \langle E_k | (\rho \otimes |\psi_E\rangle\langle\psi_E|) |E_l\rangle \operatorname{Tr}_E (|p,\vec{q}\rangle \langle p,\vec{q}|E_k\rangle \langle E_l|p',\vec{q}'\rangle \langle p',\vec{q}'|), \tag{9}$$

where $\vec{q}$ and $\vec{q}'$ are indices of elements of the basis of the environment,

$$= \sum_{p,p',\vec{q}} \langle p,\vec{q}| U(t)(\rho \otimes |\psi_E\rangle\langle\psi_E|)U^\dagger(t) |p',\vec{q}\rangle |p\rangle\langle p'| \tag{10}$$

$$\textit{eq:chaometer:channel:2}$$

$$= \sum_{p,p'} \langle p| \operatorname{Tr}_E[U(t)(\rho \otimes |\psi_E\rangle\langle\psi_E|)U^\dagger(t)] |p'\rangle |p\rangle\langle p'| . \tag{11}$$

To compute the Choi-Jamiołkowski matrix $\mathcal{D}(t)$ of the chaometer's quantum channel in eq. (4) we use the definition $(\mathcal{E} \otimes \mathbb{1})[|\phi^+\rangle\langle\phi^+|]$, with $|\phi^+\rangle = 1/\sqrt{2}(|0,0\rangle + |1,1\rangle)$ is the maximally entangled state between two spins, and eq. (11) to obtain

$$\mathcal{D}(t) = \frac{1}{2} \sum_{i,j,p,p',\vec{q}} \langle p,\vec{q}| U(t)(|i,\psi_E\rangle\langle j,\psi_E|)U^\dagger(t) |p',\vec{q}\rangle |p,i\rangle\langle p',j| \tag{12}$$

$$\textit{eq:choi:chaometer:computational:basis}$$

$$= \frac{1}{2} \sum_{i,j,p,p'} \langle j,\psi_{0E}| U^\dagger(t)(|p'\rangle\langle p| \otimes \mathbb{1}_E)U(t) |i,\psi_{0E}\rangle |p,i\rangle\langle p',j| , \tag{13}$$

where we have only reordered the factors and used the resolution of identity over $\mathcal{H}_E$.

We compute the purity of $\mathcal{D}(t)$. From eq. (7) it is straightforward to obtain

$$\textit{eq:choi:chaometer:purity:1}$$

$$\operatorname{Tr}\big[\mathcal{D}^2(t)\big] = \frac{1}{4} \sum_{i,j,p,q} \big| \langle i,\psi_{0E}|U^\dagger(t)\big(|p\rangle\langle q| \otimes \mathbb{1}_E\big)U(t)|j,\psi_{0E}\rangle \big|^2 . \tag{14}$$

Moreover, reordering this expression an interesting interpretation of the purity of the Choi-Jamiołkowski matrix is revealed. Let us conveniently rewrite eq. (14) as

$$\operatorname{Tr}\big[\mathcal{D}^2(t)\big] = \langle\psi_{0E}| \operatorname{Tr}_S \left\{ U^\dagger(t) \sum_{p,q} \left[ \left( \frac{|p\rangle\langle q|}{\sqrt{2}} \otimes \mathbb{1}_E \right) U(t) \left( \frac{\mathbb{1}_S}{2} \otimes |\psi_{0E}\rangle\langle\psi_{0E}| \right) U^\dagger(t) \left( \frac{|q\rangle\langle p|}{\sqrt{2}} \otimes \mathbb{1}_E \right) \right] U(t) \right\} |\psi_{0E}\rangle .$$

$$\textit{eq:choi:chaometer:purity:2}$$

$$\tag{15}$$

- First, a forward evolution $\rho(t) = U(t)\rho_0 U^\dagger(t)$ occurs, with $\rho_0 = \left( \frac{\mathbb{1}_S}{2} \otimes |\psi_{0E}\rangle\langle\psi_{0E}| \right)$.
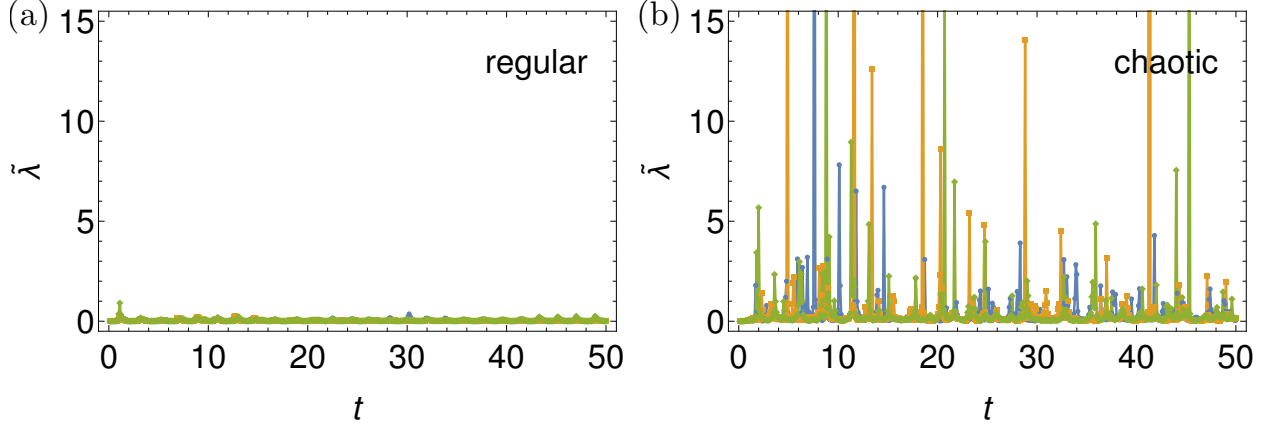
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Figure 9: Most negative eigenvalue $\tilde{\lambda}$ of map $\Lambda(t,s)/2$, with $s = 0.1$ [c.f. eqs. (17) and (18)]. fig:lambda.tilde(end- "jami−"

- Second, backwards evolution of $\rho(t)$ is followed, $\tilde{\rho}_0 = U^\dagger(t) \sum_{p,q} \left[ \left( \frac{|p\rangle\langle q|}{\sqrt{2}} \otimes \mathbb{1}_E \right) \rho(t) \left( \frac{|q\rangle\langle p|}{\sqrt{2}} \otimes \mathbb{1}_E \right) \right] U(t)$, where before evolving backwards with $U^\dagger(t)$ a completely depolarizing quantum channel acts over the system.

- Finally, $\text{Tr}\left[\mathcal{D}^2(t)\right]$ is the probability of finding the environment in its initial state.

JA: esta es una especie de echo de Loschmidt!!! Cálculos en el apéndice A In other words, the purity of $\mathcal{D}(t)$ represents the probability that the environment remains in its initial state after the following process: First, the system and environment are initialized in a product state, with the system in a maximally mixed state. Next, the combined system evolves. Then, a completely depolarizing channel is applied to the system. Finally, the system and environment undergo reverse evolution.

The Loschmidt echo is defined as

eq:loschmidt:echo
$$M(t) = \left| \langle \psi_0 | e^{-iH_2 t/\hbar} e^{-iH_1 t/\hbar} | \psi_0 \rangle \right|^2 \tag{16}$$

## 1.7   Non complete-positiveness of $\Lambda(t,s)$

Any quantum channel $\mathcal{E}(t)$ can composed as

eq:Lambda
$$\mathcal{E}(t) = \Lambda(t,s) \circ \mathcal{E}(s,0), \tag{17}$$

nonetheless, $\Lambda(t,s)$ is not in general completely positive. A way to quantify how far is $\Lambda(t,s)$ from being completely positive is through $\tilde{\lambda}$:

eq:lambda:tilde
$$\tilde{\lambda} = |\min(0, \lambda_{\text{smallest}})|, \tag{18}$$

where $\lambda_{\text{smallest}}$ is the smallest eigenvalue of $\Lambda^R(t,s)/2$. We have added the factor $1/2$ just so $\text{Tr}\left[\Lambda^R(t,s)/2\right] = 1$.

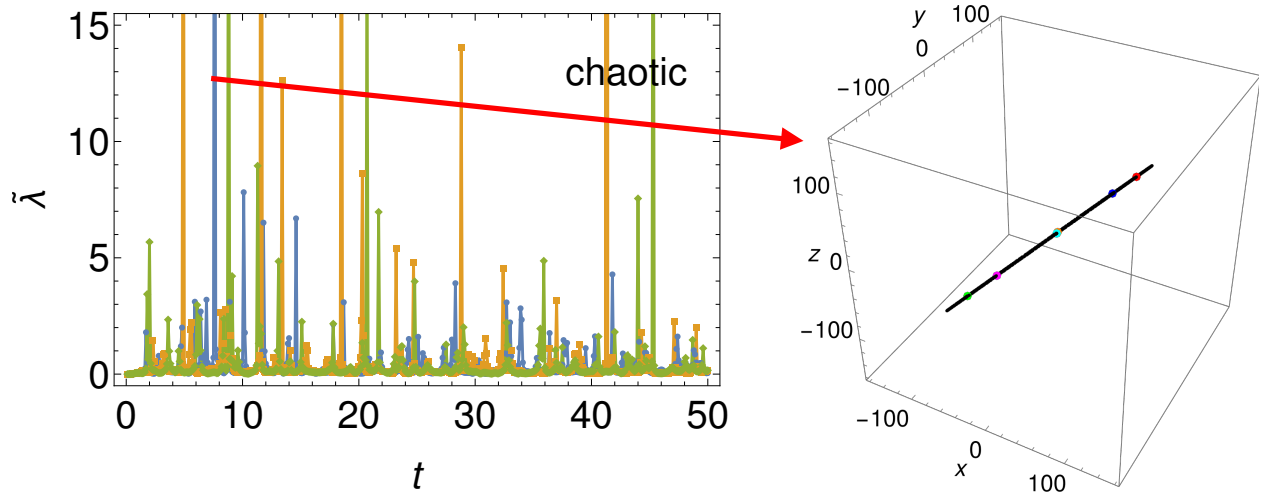Let us fix $s = 0.1$ and investigate the complete positiviness of $\Lambda(t,s)$, see Fig. 9.

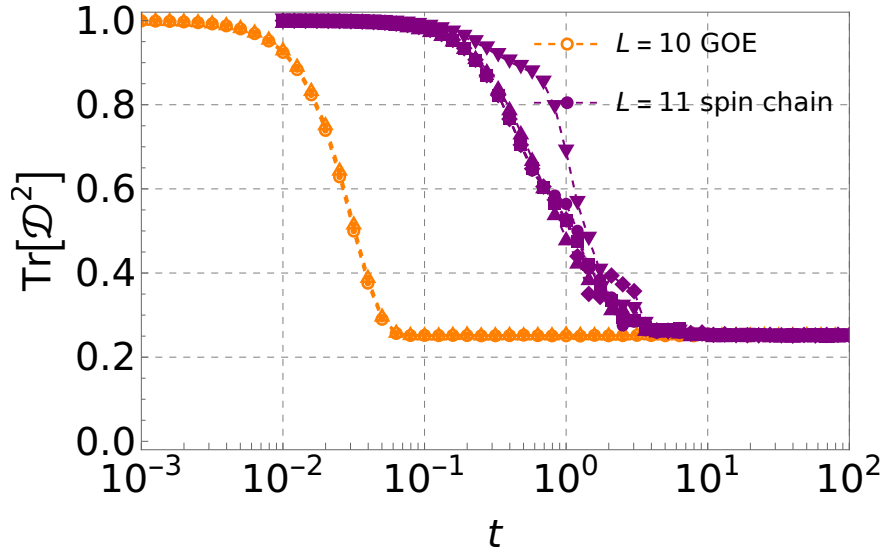Figure 10: Burst of the Bloch sphere at $t = 0.5$ s.

Figure 11: Purity of the Choi-Jamiołkowski matrix of chaometer's quantum channel and another channel from a Hamiltonian taken from the GOE ensemble.
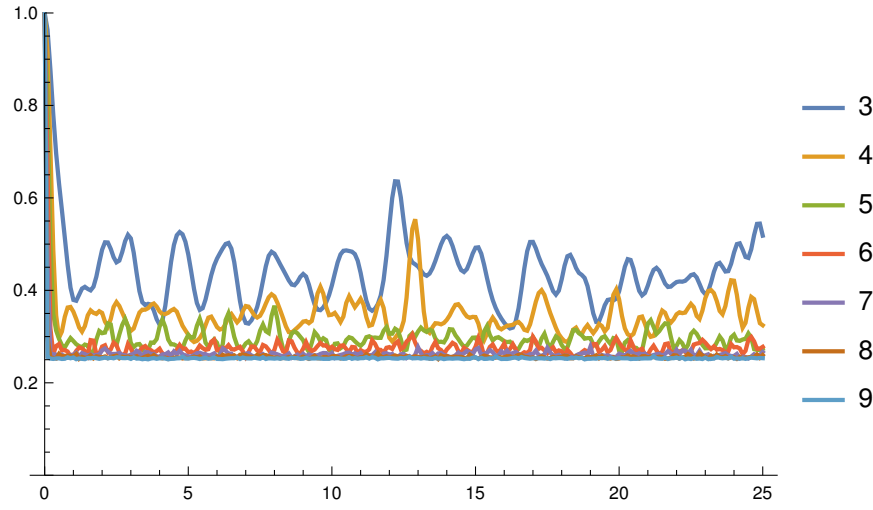
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Figure 12: JA: La pureza de la matriz de Choi de una matriz U de COE se comporta así variando el número de espines $L$ desde 3 hasta 9. Parece que para $L$ muy grande el límite asintótico tiende a 0.25, Se podrá probar analíticamente?
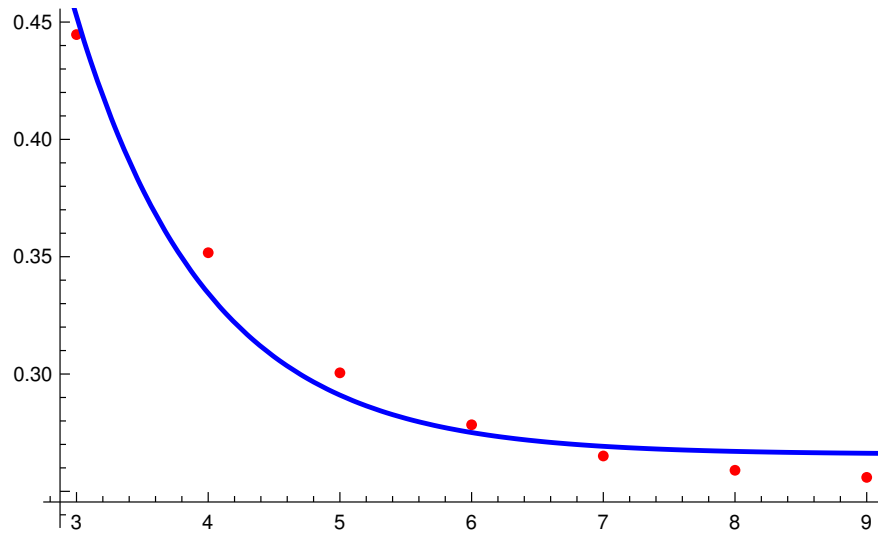


Figure 13: JA: $\mathrm{Tr}\left[\mathcal{D}_\infty^2\right]$ vs $L$. Creo que se podría conseguir un mejor ajuste teniendo estadística para cada punto.

85 **2 Charles**

86 **3 Miguel**

87 **4 Viko**

88 **A Choi**

$$
89 \quad \mathrm{Tr}\big[\mathcal{D}^2(t)\big] = \frac{1}{4}\left( \sum_{i,j,p} \big| \langle i,\psi_{0E}|U^\dagger(t)\big(|p\rangle\langle p| \otimes \mathbb{1}_E\big)U(t)|j,\psi_{0E}\rangle \big|^2 + \sum_{i,j,p\neq q} \big| \langle i,\psi_{0E}|U^\dagger(t)\big(|p\rangle\langle q| \otimes \mathbb{1}_E\big)U(t)|j,\psi_{0E}\rangle \big|^2 \right)
$$
$$(19)$$

$$
90 \quad = \frac{1}{4}\Bigg( \sum_{i,j} \big| \langle i,\psi_{0E}|U^\dagger(t)\big(\mathbb{1}_S \otimes \mathbb{1}_E\big)U(t)|j,\psi_{0E}\rangle \big|^2
$$

$$
91 \quad - 2\sum_{i,j} \mathrm{Re}\big( \langle i,\psi_{0E}|U^\dagger(t)\big(|0\rangle\langle 0| \otimes \mathbb{1}_E\big)U(t)|j,\psi_{0E}\rangle \; \langle j,\psi_{0E}|U^\dagger(t)\big(|1\rangle\langle 1| \otimes \mathbb{1}_E\big)U(t)|i,\psi_{0E}\rangle\big)
$$

$$
92 \quad + \sum_{i,j,p\neq q} \big| \langle i,\psi_{0E}|U^\dagger(t)\big(|p\rangle\langle q| \otimes \mathbb{1}_E\big)U(t)|j,\psi_{0E}\rangle \big|^2 \Bigg)
$$
$$(20)$$

$$
93 \quad = \frac{1}{2}\Bigg( 1 - \sum_{i,j} \Big( \mathrm{Re}\big( \langle i,\psi_{0E}|U^\dagger(t)\big(|0\rangle\langle 0| \otimes \mathbb{1}_E\big)U(t)|j,\psi_{0E}\rangle \; \langle j,\psi_{0E}|U^\dagger(t)\big(|1\rangle\langle 1| \otimes \mathbb{1}_E\big)U(t)|i,\psi_{0E}\rangle\big)
$$

$$
94 \quad + \big| \langle i,\psi_{0E}|U^\dagger(t)\big(|0\rangle\langle 1| \otimes \mathbb{1}_E\big)U(t)|j,\psi_{0E}\rangle \big|^2 \Big)\Bigg)
$$
$$(21)$$

$$
95 \quad = \frac{1}{2}\Bigg( 1 - \sum_{i,j} \Big( \langle i,\psi_{0E}|U^\dagger(t)\big(|0\rangle\langle 0| \otimes \mathbb{1}_E\big)U(t)|j,\psi_{0E}\rangle \; \langle j,\psi_{0E}|U^\dagger(t)\big(|1\rangle\langle 1| \otimes \mathbb{1}_E\big)U(t)|i,\psi_{0E}\rangle
$$

$$
96 \quad + \big| \langle i,\psi_{0E}|U^\dagger(t)\big(|0\rangle\langle 1| \otimes \mathbb{1}_E\big)U(t)|j,\psi_{0E}\rangle \big|^2 \Big)\Bigg)
$$
$$(22)$$

97 JA: trabajando en analytic_choi_check.nb eq. (15)

98 • regular: Segundo y tercer término aproximadamente cero para $t \gg 1$

99 • caótico: tercer término igual a cero para $t \gg 1$.

101 $\text{Tr}\big[\mathcal{D}^2(t)\big] = \dfrac{1}{2}\Bigg(1 - \sum_i \underbrace{\langle i,\psi_{0E}|U^\dagger(t)\big(|0\rangle\langle 0|\otimes \mathbb{1}_E\big)U(t)|i,\psi_{0E}\rangle\,\langle i,\psi_{0E}|U^\dagger(t)\big(|1\rangle\langle 1|\otimes \mathbb{1}_E\big)U(t)|i,\psi_{0E}\rangle}_{(1-r_z^2)/4}$

102 $\hspace{3cm} - 2\,\langle 0,\psi_{0E}|U^\dagger(t)\big(|0\rangle\langle 0|\otimes \mathbb{1}_E\big)U(t)|1,\psi_{0E}\rangle\,\langle 1,\psi_{0E}|U^\dagger(t)\big(|1\rangle\langle 1|\otimes \mathbb{1}_E\big)U(t)|0,\psi_{0E}\rangle$

103 $\hspace{3cm} + \sum_{i,j}\big|\,\langle i,\psi_{0E}|U^\dagger(t)\big(|0\rangle\langle 1|\otimes \mathbb{1}_E\big)U(t)|j,\psi_{0E}\rangle\big|^2\Bigg)$       (23)

104 $\hspace{1.2cm} = \dfrac{1}{2}\Bigg(1 - \sum_i \underbrace{\langle i,\psi_{0E}|U^\dagger(t)\big(|0\rangle\langle 0|\otimes \mathbb{1}_E\big)U(t)|i,\psi_{0E}\rangle\,\langle i,\psi_{0E}|U^\dagger(t)\big(|1\rangle\langle 1|\otimes \mathbb{1}_E\big)U(t)|i,\psi_{0E}\rangle}_{(1-r_z^2)/4}$

105 $\hspace{2cm} + 2\big|\,\langle 0,\psi_{0E}|U^\dagger(t)\big(|0\rangle\langle 0|\otimes \mathbb{1}_E\big)U(t)|1,\psi_{0E}\rangle\big|^2 + \sum_{i,j}\big|\,\langle i,\psi_{0E}|U^\dagger(t)\big(|0\rangle\langle 1|\otimes \mathbb{1}_E\big)U(t)|j,\psi_{0E}\rangle\big|^2\Bigg)$

$\hspace{13cm}$ (24)

107 • para $t \gg 1$, $(1-r_z^2)/4 \approx 1/4$ en el régimen caótico ($r_z \approx 0$)

108 • para $t \gg 1$ el tercer término se hace approx 0 para el régimen caótico y distinto de 0 para regular. De
109    hecho, todos los términos, excepto, el $1/2$, son comparables a tiempos largos para el integrable

110 • a tiempos cortos el término que más contribuye es el término dentro la sumatoria del último término
111    con índices $i = 0$ y $j = 1$

## B Purity of Choi matrix

The quantum channel acting over the first ($i = 1$) spin is defined as:

$$\mathcal{E}(\rho) = \text{Tr}_E\left(U^\dagger(t)\rho \otimes |\psi_0\rangle\langle\psi_0| U(t)\right), \tag{25}$$

where $|\psi_0\rangle = \bigotimes_{i=2}^{L} |\phi_i\rangle$, with $|\phi_i\rangle$ a random state of a single particle, is the initial state of the environment.

To compute the Choi-Jamiołkowski matrix $\mathcal{D}(t)$ of the quantum channel in eq. (25) we use the definition $(\mathcal{E} \otimes \mathbb{1})[|\phi^+\rangle\langle\phi^+|]$, with $|\phi^+\rangle = 1/\sqrt{2}(|0,0\rangle + |1,1\rangle)$ the maximally entangled state between two spins, and obtain

$$\mathcal{D}(t) = \frac{1}{2} \sum_{i,j,p,q} \langle j, \psi_{0E}| U^\dagger(t)(|p\rangle\langle q| \otimes \mathbb{1}_E)U(t) |i, \psi_{0E}\rangle |q,i\rangle\langle p,j|. \tag{26}$$

After some algebraic step one may show that the purity of the Choi matrix is written as:

$$\text{Tr}\left[\mathcal{D}^2(t)\right] = \text{Tr}\left(\text{Tr}_S[U(t)(\mathbb{1}_S \otimes |\psi_0\rangle\langle\psi_0|)U^\dagger(t)]^2\right). \tag{27}$$

I believe that when the quantum channel acts over a $d$-dimensional system, and $U(t) = e^{-iHt}$, with $H$ a chaotic Hamiltonian (a matrix that can be taken from an appropriate RMT ensemble), the purity of the Choi matrix is equal to $1/d^2$ for $t \gg 1$ in the chaotic regime, and $1/d$ in the regular regime.

In particular, I would like to consider the spin chain Hamiltonian

$$H = \sum_{i=1}^{L} (h_x\sigma_i^x + h_z\sigma_i^z) - \sum_{i=1}^{L-1} J_z\sigma_i^z\sigma_{i+1}^z, \tag{28}$$

for $h_x = j_i = 1$, $\forall i$, $h_z = 0.5$ (chaotic) and $h_z = 2.5$ (regular).

# References

[1] Nicolás Mirkin and Diego Wisniacki. Quantum chaos, equilibration, and control in extremely short spin chains. *Phys. Rev. E*, 103:L020201, Feb 2021.