$H = H_S + H_E + H_{SE}$ $\qquad H = \sum_{i=1}^{N} h_z \sigma_i^z + h_x \sigma_i^x + \sum_{i=1}^{N} J \sigma_i^z \sigma_{i+1}^z$

$H_S = h_z \sigma_1^z + h_x \sigma_1^x$ $\qquad H_E = \sum_{i=2}^{N} h_z \sigma_i^z + h_x \sigma_i^x + \sum_{i=2}^{N} J \sigma_i^z \sigma_{i+1}^z$

$H_{SE} = J \sigma_1^z \sigma_2^z$

$U = e^{-i(H_S + H_E + H_{SE})t} \approx \underbrace{\left(e^{-i(H_S+H_E)\Delta t} \, e^{-i H_{SE} \Delta t} \right)^m}_{\tilde{U}}$, $t = m\Delta t$

$\tilde{U}|\phi\rangle|\psi\rangle = e^{-iH_{SE}\Delta t} \underbrace{e^{-iH_S\Delta t}|\phi\rangle}_{|\phi'\rangle} \underbrace{e^{-iH_E\Delta t}|\psi\rangle}_{|\psi'\rangle}$

$= e^{iH_{SE}\Delta t} |\phi'\rangle|\psi'\rangle$ $\qquad |\phi'\rangle = \alpha_1|0\rangle + \beta_1|1\rangle$

$= e^{-iJ\sigma_1^z\sigma_2^z\Delta t} |\phi'\rangle|\psi'\rangle$ $\qquad |\psi'\rangle = \alpha_2|0\rangle_2|\psi''\rangle + \beta_2|1\rangle_2|\chi\rangle$

$= e^{-iJ\sigma_1^z\sigma_2^z\Delta t} \left(\alpha_1\alpha_2|00\rangle|\psi''\rangle + \alpha_1\beta_1|01\rangle|\chi\rangle + \beta_1\alpha_2|10\rangle|\psi''\rangle + \beta_1\beta_2|11\rangle|\chi\rangle \right)$

$= \alpha_1\alpha_2 e^{-iJ\Delta t}|00\rangle|\psi''\rangle + \alpha_1\beta_1 e^{iJ\Delta t}|01\rangle|\chi\rangle + \beta_1\alpha_2 e^{iJ\Delta t}|10\rangle|\psi''\rangle + \beta_1\beta_2 e^{-iJ\Delta t}|11\rangle|\chi\rangle$

$= \left(\alpha_1\alpha_2 e^{-iJ\Delta t}|0\rangle + \beta_1\alpha_2 e^{iJ\Delta t}|1\rangle \right)|0\rangle|\psi''\rangle + \left(\alpha_1\beta_1 e^{iJ\Delta t}|0\rangle + \beta_1\beta_2 e^{-iJ\Delta t}|1\rangle \right)|1\rangle|\chi\rangle$