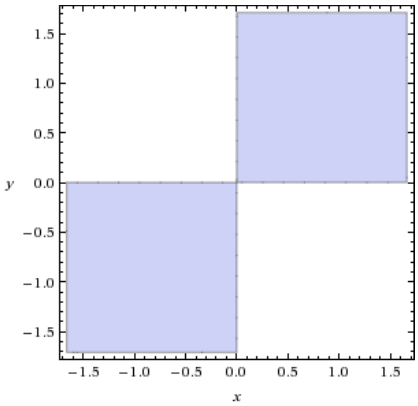
1. Encontre o domínio D de $f(x,y)=\sqrt{xy}$ e as curvas de nível.

$D_f=\{(x,y)\in\mathbb{R}^2 : xy\geq 0\}$ é o primeiro e terceiro quadrantes

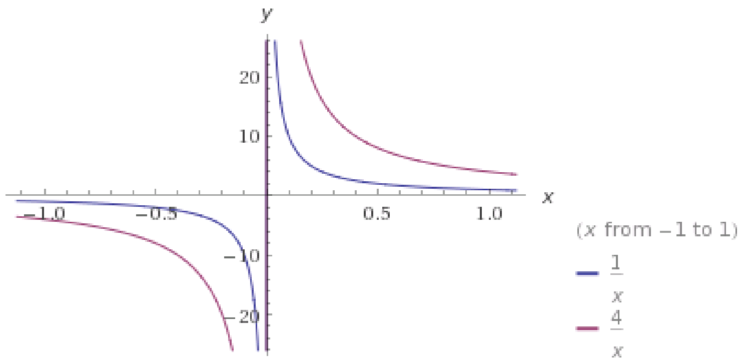


Curvas de nível: $f(x,y)=K \Leftrightarrow \sqrt{xy}=K \Leftrightarrow xy=K^2 \Leftrightarrow y=\frac{K^2}{x}$

As curvas de nível são hipérboles

$$K=1\rightarrow \frac{1}{x}$$

$$K=2\rightarrow\ y=\frac{4}{x}$$



2. $u=x^2\sin y+\ y^2\sin x$

$$\frac{\partial^4 u}{\partial x^2 y^2}=\frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}\left(\frac{\partial(x^2\sin y+\ y^2\sin x)}{\partial y}\right)\right)\right)=\frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}\left(\frac{\partial(x^2\cos y+\ 2y\sin x)}{\partial y}\right)\right)=$$

$$=\frac{\partial}{\partial x}\left(\frac{\partial(-x^2\sin y+2\sin x)}{\partial x}\right)=\frac{\partial(-2x\sin y+2\cos x)}{\partial x}=-2\sin y-2\sin x.$$

3. Mostre que numa vizinhança de (0,1,1,0), o seguinte sistema de equações define implicitamente u e v como funções contínuas e diferenciáveis de x e y. Determine a matriz jacobiana e o jacobiano nesse ponto.

$$\begin{cases} xu-yv=0\\ yu+xv=1\end{cases}$$

$$\begin{cases} xu-yv=0\\ yu+xv=1\end{cases}\Leftrightarrow\begin{cases} xu-yv=0\\ yu+xv-1=0\end{cases}\longrightarrow\begin{cases} F_1=xu-yv\\ F_2=yu+xv-1\end{cases}$$

O sistema define implicitamente u e v em função de x e y, perto de (x,y,u,v)=(0,1,1,0) se

$$\left|\begin{matrix}\frac{\partial F_1}{\partial u} & \frac{\partial F_1}{\partial v}\\ \frac{\partial F_2}{\partial u} & \frac{\partial F_2}{\partial v}\end{matrix}\right|_{(0,1,1,0)}\neq 0\Leftrightarrow\left|\begin{matrix} x & -y\\ y & x\end{matrix}\right|_{(0,1,1,0)}\neq 0\Leftrightarrow\left|\begin{matrix} 0 & -1\\ 1 & 0\end{matrix}\right|\neq 0\Leftrightarrow 1\neq 0\ \checkmark$$

As derivadas … a matriz jacobiana é dada por

$$\begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{bmatrix}_{(0,1,1,0)} = \begin{bmatrix} \frac{\partial F_1}{\partial u} & \frac{\partial F_1}{\partial v} \\ \frac{\partial F_2}{\partial u} & \frac{\partial F_2}{\partial v} \end{bmatrix}^{-1}_{(0,1,1,0)} \times \begin{bmatrix} \frac{\partial F_1}{\partial x} & \frac{\partial F_1}{\partial y} \\ \frac{\partial F_2}{\partial x} & \frac{\partial F_2}{\partial y} \end{bmatrix}_{(0,1,1,0)}$$

$$\begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{bmatrix}_{(0,1,1,0)} = \begin{bmatrix} x & -y \\ y & x \end{bmatrix}^{-1}_{(0,1,1,0)} \times \begin{bmatrix} u & -v \\ v & u \end{bmatrix}_{(0,1,1,0)}$$

$$\begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{bmatrix}_{(0,1,1,0)} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}^{-1} \times \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

A inversa de uma matriz 2x2 é dada por

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{determinante} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Assim $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}^{-1} = \frac{1}{1} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

Finalmente

$$\begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{bmatrix}_{(0,1,1,0)} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \times \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

Esta é a matriz jacobiana. O jacobiano é o seu determinante que é 1.

4. Encontre os pontos extremos de $f(x,y) = \frac{1}{x} + \frac{1}{y}$ sujeito a $\frac{1}{x^2} + \frac{1}{y^2} = \frac{1}{2}$.

A função de Lagrange é $\mathcal{L}(x,y,\lambda) = \frac{1}{x} + \frac{1}{y} + \lambda\left(\frac{1}{x^2} + \frac{1}{y^2} - \frac{1}{2}\right)$. Os pontos críticos são…

$$\begin{cases} \frac{\partial \mathcal{L}}{\partial x} = 0 \\ \frac{\partial \mathcal{L}}{\partial y} = 0 \\ \frac{\partial \mathcal{L}}{\partial \lambda} = 0 \end{cases} \Leftrightarrow \begin{cases} \frac{-1}{x^2} - \frac{2\lambda}{x^3} = 0 \\ \frac{-1}{y^2} - \frac{2\lambda}{y^3} = 0 \\ \frac{1}{x^2} + \frac{1}{y^2} = \frac{1}{2} \end{cases} \Leftrightarrow \begin{cases} \lambda = -\frac{x}{2} \\ \lambda = -\frac{y}{2} \\ \frac{1}{x^2} + \frac{1}{y^2} = \frac{1}{2} \end{cases} \Leftrightarrow \begin{cases} \lambda = -\frac{x}{2} \\ x = y \\ \frac{2}{x^2} = \frac{1}{2} \end{cases} \Leftrightarrow \begin{cases} \lambda = \mp 1 \\ y = \pm 2 \\ x = \pm 2 \end{cases}$$

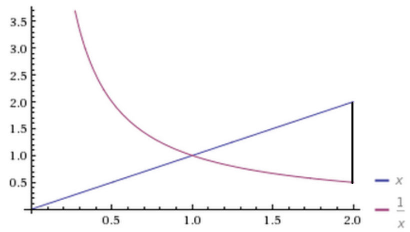
Pontos críticos $(2,2,-1)$ e $(-2,-2,1)$.

$$f(2,2) = \frac{1}{2} + \frac{1}{2} = 1 \ \ máximo$$

$$f(-2,-2) = -\frac{1}{2} - \frac{1}{2} = -1 \ mínimo$$

5. Calcule o integral $\iint_D \frac{x^2}{y^2}$, $\quad D \to x = 2, y = x, xy = 1$

A região D é dada por $\begin{cases} 1 \le x \le 2 \\ 1/x \le y \le x \end{cases}$



$$\iint_D \frac{x^2}{y^2} = \int_1^2 \int_{\frac{1}{x}}^{x} \frac{x^2}{y^2} dydx = \int_1^2 \left[\frac{-x^2}{y}\right]_{y=\frac{1}{x}}^{y=x} dx = \int_1^2 \frac{-x^2}{x} + \frac{x^2}{\frac{1}{x}} dx = \int_1^2 -x + x^3 dx = \left[\frac{-x^2}{2} + \frac{x^4}{4}\right]_{x=1}^{x=2}$$

$$= -2 + 4 - \left(-\frac{1}{2} + \frac{1}{4}\right) = 2 + \frac{1}{4} = \frac{9}{4}.$$

6. Calcule a massa de um sólido com densidade $\rho(x,y,z) = \sqrt{x^2+y^2}$, definido pela coroa esférica

$$\{(x,y,z) \in \mathbb{R}^3 : a^2 \le x^2 + y^2 + z^2 \le b^2, a < b\}$$

$$Massa = \iiint densidade$$

A coroa esférica em coordenadas esféricas fica

$$\begin{cases} a \le \rho \le b \\ 0 \le \theta \le 2\pi \\ 0 \le \varphi \le \pi \end{cases}$$

$$Massa = \int_a^b d\rho \int_0^{2\pi} d\theta \int_0^{\pi} \sqrt{\rho^2 \cos^2\theta \sin^2\varphi + \rho^2 \sin^2\theta \sin^2\varphi} \times \rho^2 \sin\varphi \, d\varphi =$$

$$= \int_a^b d\rho \int_0^{2\pi} d\theta \int_0^{\pi} \sqrt{\rho^2 \sin^2\varphi} \times \rho^2 \sin\varphi \, d\varphi = \int_a^b d\rho \int_0^{2\pi} d\theta \int_0^{\pi} \rho^3 \sin^2\varphi \, d\varphi =$$

$$= \int_a^b \rho^3 d\rho \int_0^{2\pi} \left[\frac{\varphi}{2} - \frac{\sin\varphi\cos\varphi}{2}\right]_{\varphi=0}^{\varphi=\pi} d\theta = \int_a^b \rho^3 d\rho \int_0^{2\pi} \frac{\pi}{2} d\theta = \int_a^b \rho^3 \frac{\pi}{2}(2\pi - 0) d\rho =$$

$$= \pi^2 \left[\frac{\rho^4}{4}\right]_{\rho=a}^{\rho=b} = \pi^2 \left(\frac{b^4}{4} - \frac{a^4}{4}\right).$$

7. Aplicando o Teorema de Green, calcule o integral de linha...

$$\oint_C \left(y + e^{\sqrt{x}}\right) dx + (2x + \cos y^3) dy$$

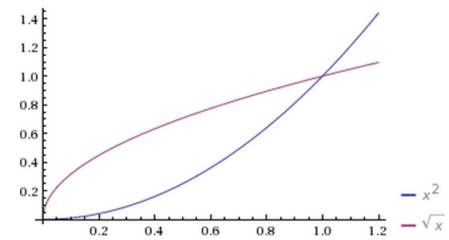
Onde C é a curva fechada que delimita a região situada entre $x = y^2, y = x^2$, percorrida no sentido anti-horário.

Chamando $P = \left(y + e^{\sqrt{x}}\right), Q = (2x + \cos y^3)$, o Teorema de Green diz que

$$\oint_C Pdx + Qdy = \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$$

Onde D é a região delimitada pelas curvas indicadas

$$\begin{cases} 0 \le x \le 1 \\ x^2 \le y \le \sqrt{x} \end{cases}$$



$$\oint_C \left(y+e^{\sqrt{x}}\right)dx + (2x+\cos y^3)dy = \int_0^1 dx \int_{x^2}^{\sqrt{x}} 2-1dy = \int_0^1 \sqrt{x}-x^2dx = \left[\frac{x^{3/2}}{3/2}-\frac{x^3}{3}\right]_{x=0}^{x=1} = \frac{2}{3}-\frac{1}{3}=\frac{1}{3}.$$

8. Encontre, se possível, a função u tal que o seu diferencial é dado por:

(a) $du = (x+\cos y)dx + (x\sin y - e^y)dy$

Quando se tem $du = Pdx + Qdy$, existe a função u se $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$.
Neste caso $\frac{\partial P}{\partial y} = -\sin y$ e $\frac{\partial Q}{\partial x} = \sin y$… que são diferentes. Assim não existe a função u.

(b) $du = (2x\cos y - y^2\sin x)dx + (2y\cos x - x^2\sin y)dy$

Neste caso $\frac{\partial P}{\partial y} = -2x\sin y - 2y\sin x$ e $\frac{\partial Q}{\partial x} = -2y\sin x - 2x\sin y$… que são iguais. Existe a função u.

$$\begin{cases}\dfrac{\partial u}{\partial x} = P\\ \dfrac{\partial u}{\partial y} = Q\end{cases} \Leftrightarrow \begin{cases}\dfrac{\partial u}{\partial x} = 2x\cos y - y^2\sin x\\ \dfrac{\partial u}{\partial y} = 2y\cos x - x^2\sin y\end{cases} \Leftrightarrow \begin{cases}u = \displaystyle\int 2x\cos y - y^2\sin x\,dx\\ u = \displaystyle\int 2y\cos x - x^2\sin y\,dy\end{cases}$$

$$\Leftrightarrow \begin{cases}u = x^2\cos y + y^2\cos x + A(y)\\ u = y^2\cos x + x^2\cos y + B(x)\end{cases} \Longrightarrow u(x,y) = x^2\cos y + y^2\cos x + C.$$