Prac for 2023 honours multivariate CCA section

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Question One (8)

Hirschey and Wichern (reference in Johnson and Wichern) investigate the consistency, determinants and uses of accounting and market-value measures of profitability.

The correlation matrix of accounting historical, accounting replacement and market-value measures of profitability for a sample of firms operating in 1977 is as follows:

Variable	HRA	HRE	HRS	RRA	RRE	RRS	Q	REV
Historical return on assets (HRA)	1.000							
Historical return on equity (HRE)	0.738	1.000						
Historical return on sales (HRS)	0.731	0.520	1.0000					
Replacement return on assets (RA)	0.828	0.688	0.6520	1.000				
Replacement return on equity (RRE)	0.681	0.831	0.5113	0.887	1.000			
Replacement return on sales (RRS)	0.712	0.543	0.8260	0.867	0.692	1.000		
Market Q ratio (Q)	0.625	0.322	0.5790	0.639	0.419	0.608	1.000	
Market relative excess value (REV)	0.604	0.303	0.6170	0.563	0.352	0.610	0.937	1

Let $\mathbf{X}^{(1)} = \left[X_1^{(1)}, X_2^{(1)}, \dots, X_6^{(1)}\right]$ be the vector of variables representing accounting measures of profitability (HRA to RRS). Let $\mathbf{X}^{(2)}$ be the vector of variables representing market-value measures of profitability (Q and REV). Partition the sample correlation matrix accordingly, and perform a canonical correlation analysis. Specifically, without using anything more than the `eigen` function in `R` (e.g. not the `CCA` package):

a Find the estimated coefficient vector for the accounting measures of profitability in the first canonical variate (i.e. find $\hat{\mathbf{a}}_1$ in $\hat{U}_1 = \hat{\mathbf{a}}_1' \mathbf{X}^{(1)}$). (2)

b Find $\hat{\mathbf{b}}_1$ in $\hat{V}_1 = \hat{\mathbf{b}}_1' \mathbf{X}^{(2)}$ directly using result from part (a). (1)

c Find the first canonical correlation coefficient. (1)

d Interpret these canonical variables using the coefficient vectors. (2)

e Interpret these canonical variables using the correlation matrices $\boldsymbol{\rho}_{U_1,\mathbf{X}^{(1)}}$ and $\boldsymbol{\rho}_{V_1,\mathbf{X}^{(2)}}$. (3)

The correlation matrix can be read into `R` using the following code:

```
corr_mat <- structure(
  c(
    1, 0.738, 0.731, 0.828, 0.681, 0.712, 0.625, 0.604,
    0.738, 1, 0.52, 0.688, 0.831, 0.543, 0.322, 0.303, 0.731, 0.52,
    1, 0.652, 0.5113, 0.826, 0.579, 0.617, 0.828, 0.688, 0.652, 1,
    0.887, 0.867, 0.639, 0.563, 0.681, 0.831, 0.5113, 0.887, 1, 0.692,
    0.419, 0.352, 0.712, 0.543, 0.826, 0.867, 0.692, 1, 0.608, 0.61,
    0.625, 0.322, 0.579, 0.639, 0.419, 0.608, 1, 0.937, 0.604, 0.303,
    0.617, 0.563, 0.352, 0.61, 0.937, 1
    ),
  dim = c(8L, 8L)
  )
```

Question Two

Suppose that we perform CCA on the random vectors $\mathbf{X}^{(1)} : 2 \times 1$ and $\mathbf{X}^{(2)} : 2 \times 1$ where

$$\boldsymbol{\Sigma} = \text{Cov}\left[\begin{pmatrix} \mathbf{X}^{(1)} \\ \mathbf{X}^{(2)} \end{pmatrix}\right] = \begin{bmatrix} 100 & 0 & 0 & 0 \\ 0 & 1 & 0.95 & 0 \\ 0 & 0.95 & 1 & 0 \\ 0 & 0 & 0 & 100 \end{bmatrix}$$

The first pair of canonical variates are given by $U_1 = X_2^{(1)}$ and $V_1 = X_1^{(2)}$.

a Despite that $\text{Var}[X_1^{(1)}] > \text{Var}[X_2^{(1)}]$ and $\text{Var}[X_2^{(2)}] > \text{Var}[X_1^{(1)}]$, only $X_2^{(1)}$ and $X_1^{(2)}$ were included in the first canonical variate pair. Why? (1)

b Suppose that the correlation between $X_2^{(1)}$ and $X_1^{(2)}$ is reduced to 0. What is the correlation between the first canonical variables? Why? (2)

c Suppose that the $\text{Var}[X_2^{(1)}]$ and $\text{Var}[X_1^{(2)}]$ are each doubled. What happens to the first canonical correlation? Why? (2)

  a Bonus: How are the lengths of the first canonical coefficient vectors affected, i.e. do they decrease, increase or stay the same? Why? (2) (note that question three would help here)

In parts c-d, you do not need to demonstrate your claim using R code or provide a "proof". Just describe why it makes sense. If you want to give an R demonstration or a formal proof, you can! Note that R code does not address "why".

Without even attempting to answer why, you may well not get any marks for giving an answer as it could just be a random guess or copying. A bad explanation does not look like copying, for the same reason that "All happy families are alike; each unhappy family is unhappy in its own way" (Dostoyevsky).

Question Three (2)

For random vectors $\mathbf{X}^{(1)}$ and $\mathbf{X}^{(2)}$, the first canonical coefficient vectors $\mathbf{a}$, $\mathbf{b}$ maximise

$$\text{Cor}(\mathbf{a}'\mathbf{X}^{(1)}, \mathbf{b}'\mathbf{X}^{(2)}) = \mathbf{a}'\boldsymbol{\rho}_{12}\mathbf{b}$$

where $\boldsymbol{\rho}_{12}$ is the correlation matrix between $\mathbf{X}^{(1)}$ and $\mathbf{X}^{(2)}$.

Suppose that we created standardised variables $\mathbf{Z}^{(1)} = \mathbf{V}_{11}^{-1}\mathbf{X}^{(1)}$ and $\mathbf{Z}^{(2)} = \mathbf{V}_{22}^{-1}\mathbf{X}^{(2)}$, where $\mathbf{V}_{ii}$ is the matrix of diagonal elements of the covariance matrix for $\mathbf{X}^{(i)}$. The corresponding first canonical coefficient vectors $\mathbf{a}^z$, $\mathbf{b}^z$ maximise exactly the same function:

$$\text{Cor}(\mathbf{a}^{z\prime}\mathbf{X}^{(1)}, \mathbf{b}^{z\prime}\mathbf{X}^{(2)}) = \mathbf{a}^{z\prime}\boldsymbol{\rho}_{12}^{z}\mathbf{b}^{z} = \mathbf{a}^{z\prime}\boldsymbol{\rho}_{12}\mathbf{b}^{z},$$

since the correlation matrix is identical for standardised variables ($\boldsymbol{\rho}_{12}^{z}$) as unstandardised variables ($\boldsymbol{\rho}_{12}$).

However, the coefficient vectors are not in general the same, as $\mathbf{a}^z = \mathbf{V}_{11}\mathbf{a}$ and $\mathbf{b}^z = \mathbf{V}_{22}\mathbf{b}$.

Why?

Hint: The underlying reason is on the slide titled "Properties of canonical variates".