Graduation Project

Santiago Morales - 201913369

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1 Introduction

2 Kahn-Kalai Conjecture

2.1 Thresholds

Let $n \in \mathbb{N}$ and $0 \leq p \leq 1$. The random graph $G(n,p)$ is a probability space over the set of graphs on n labeled vertices determined by

$$\Pr[\{i,j\} \in G] = p$$

with these events mutually independent [1]. Given a graph theoretic property A, there is a probability that $G(n,p)$ satisfies A, which we write as $\Pr[G(n,p) \vDash A]$.

Definition 2.1. $r(n)$ is a threshold function for a graph theoretic property A if

1. When $p(n) \in o(r(n))$, $\lim_{n\to\infty} \Pr[G(n,p(n)) \vDash A] = 0$,

2. When $r(n) \in o(p(n))$, $\lim_{n\to\infty} \Pr[G(n,p(n)) \vDash A] = 1$,

or vice versa. [1]

We give an example of a threshold function which illustrates a common method for proving that a function is a threshold.

2.1.1 Threshold function for having isolated vertices

Let G be a graph on n labeled vertices. An isolated vertex of G is a vertex which does not belong to any of the edges of G. Let A be the property that G contains an isolated vertex. We will prove that $r(n) = \dfrac{\ln n}{n}$ is a threshold for A.

For each vertex i in G define the variable

$$X_i = \begin{cases} 1 & \text{if } i \text{ is an isolated vertex,} \\ 0 & \text{if } i \text{ is not an isolated vertex.} \end{cases}$$

Now, the probability that a vertex i is isolated is $(1-p)^{n-1}$ since it is the probability that none of the other $n-1$ vertices is connected to i. Let $X = \sum_{i=1}^{n} X_i$, then the expected number of isolated vertices is

$$E[X] = \sum_{i=1}^{n} E[X_i] = \sum_{i=1}^{n} \Pr[X_i] = n(1-p)^{n-1}.$$

Let $p = k\dfrac{\ln n}{n}$ for $k \in \mathbb{R}_{>0}$. Then

$$\begin{aligned} \lim_{n\to\infty} E[X] &= \lim_{n\to\infty} n\left(1 - k\frac{\ln n}{n}\right)^{n-1} \\ &= ne^{-k\ln n} = n^{1-k}. \end{aligned}$$

Therefore, $\lim_{n\to\infty} E[X] = 0$ if $k > 1$. Since $E[X] \geq \Pr[X > 0]$, we conclude that

$$\lim_{n\to\infty} \Pr[G(n,p) \vDash A] = \lim_{n\to\infty} \Pr[X > 0] = 0.$$

Now, for $k < 1$, the fact that $\lim_{n\to\infty} E[X] = \infty$ is not enough to conclude that $\lim_{n\to\infty} \Pr[G(n,p) \vDash A] = 1$. We have to use the second moment method.

Theorem. *If $E[X] \to \infty$ and $Var[X] = o(E[X]^2)$, then $\lim_{n\to\infty} Pr[X > 0] = 1$. [1]*

Proof. We will prove that, in this case, $Var[X] = o(E[X]^2)$. First,

$$\begin{aligned} \sum_{i\neq j} E[X_i X_j] &= \sum_{i\neq j} \Pr[X_i = X_j = 1] \\ &= n(n-1)(1-p)^{n-1}(1-p)^{n-2} \\ &= n(n-1)(1-p)^{2n-3}, \end{aligned}$$

for if i is an isolated vertex, then there is no edge between i and j so we only have to account for the remaining $n-2$ edges that contain j.

Thus, since $\sum_{i=1}^n E[X_i^2] = \sum_{i=1}^n E[X_i] = E[X]$ and $\lim_{n\to\infty} p = 0$,

$$\lim_{n\to\infty} \frac{\mathrm{Var}[X]}{E[X]^2} = \lim_{n\to\infty} \frac{E(X^2) - E[X]^2}{E[X]^2} = \lim_{n\to\infty} \frac{\sum_{i=1}^n E[X_i^2] + \sum_{i\neq j} E[X_i X_j]}{E[X]^2} - 1$$
$$= 0 + \lim_{n\to\infty} \frac{n(n-1)(1-p)^{2n-3}}{n^2(1-p)^{2n-2}} - 1 = \lim_{n\to\infty} \frac{1}{1-p} - 1 = 0.$$

We conclude that $\mathrm{Var}[X] \in o(E[X]^2)$ and so, if $k < 1$,

$$\lim_{n\to\infty} \Pr[G(n,p) \vDash A] = \lim_{n\to\infty} \Pr[X > 0] = 1.$$

Therefore, $r(n) = \frac{\ln n}{n}$ is a threshold function for property A.

2.2 The expectation threshold

Let X be a set and $p \in [0,1]$. For $A \subset X$, let $P(A) = p^{|A|}(1-p)^{|X\setminus A|}$. A class $\mathcal{F}$ of subsets of X is increasing if for $A \in \mathcal{F}$, $A \subset B$ implies $B \in \mathcal{F}$. If $\mathcal{F} \neq \emptyset$, $P(\mathcal{F}) = \sum_{A\in\mathcal{F}} P(A)$ is a strictly increasing function of p.

TODO Define increasing class and general definition of threshold [2]
TODO Define expectation threshold and show inequalities. [3]

Example 2.1. H_1 example

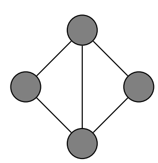

Figure 1: H_1

Example 2.2. H_2 example

Theorem 2.1 (Park Theorem [2], originally Kahn-Kalai Conjecture)**.** ***TODO***

TODO How to prove something is p-small

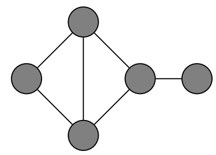

Figure 2: H_2

2.2.1 An application of the Park Theorem

TODO Prove the threshold for perfect matchings for $G(n,p)$

3 Numerical Semigroups

A *numerical semigroup* is a subset $S \subset \mathbb{N}_0$ which is closed under addition, i.e. $a, b \in S$ implies $a + b \in S$. For instance, $\mathbb{N}_0$, $\mathbb{N}_0 \setminus \{0\}$, $2\mathbb{N}_0$ are all numerical semigroups, but $\mathbb{N}_0 \setminus \{2\}$ is not. Some literature requires that a semigroup has a finite complement in $\mathbb{Z}_{\geq 0}$ [4], but we prefer the more general definition.

Example 3.1. TODO McNugget Semigroup

As seen in the previous example, we can describe a numerical semigroup with a set of *generators*:

$$S = \langle a_1, ..., a_n \rangle := \left\{ \sum_{i=1}^{n} k_i a_i : k_1, ..., k_n \in \mathbb{N}_0 \right\}.$$

TODO give some examples.

The *Frobenius number* $F(S)$ of a numerical semigroup $S = \langle a_1, ..., a_n \rangle$ is defined as the largest integer divisible by $d(S) := \gcd(a_1, ..., a_n)$ which does not belong to S. In other terms,

$$F(S) = \max(d\mathbb{Z} \setminus S).$$

4 Probability Spaces over Numerical Semigroups

We generate a random numerical semigroup with a model similar to the Ërdos-Renyi model for random graphs.

Definition 4.1. For $p \in [0,1]$ and $M \in \mathbb{N}$, a random numerical semigroup $S(M,p)$ is a probability space over the set of semigroups $S = \langle \mathcal{A} \rangle$ with $\mathcal{A} \subset \{1,...,M\}$, determined by

$$\Pr[n \in \mathcal{A}] = p,$$

with these events mutually independent.

5 Expected Frobenius Number

We prove a Theorem found in [5] without the use of the simplicial complex.

Theorem 5.1. *Let $S \sim S(M,p)$, where $p = p(M)$ is a monotone decreasing function of M. If $\frac{1}{M} \ll p \ll 1$, then S is cofinite, i.e., the set of gaps is finite, a.a.s and*

$$\lim_{M\to\infty} E[e(S)] = \lim_{M\to\infty} E[g(S)] = \lim_{M\to\infty} E[F(S)] = \infty.$$

Proof. Let $X := \min(S \setminus \{0\})$ be a random variable. Then, for $0 < n \leq M$,

$$P[X = n] = p(1-p)^{n-1},$$

and so

$$\begin{aligned} E[X] &= \sum_{n=0}^{\infty} nP[X=n] = \sum_{n=0}^{M} np(1-p)^{n-1} = p\frac{d}{dp}\left[-\sum_{n=0}^{M}(1-p)^n\right] \\ &= p\frac{d}{dp}\frac{(1-p)^{M+1}-1}{p} = p\frac{1-(1-p)^{M+1}-(M+1)(1-p)^M p}{p^2} \\ &= \frac{1-(1-p)^M - M(1-p)^M p}{p} \geq \frac{1-e^{-Mp}-Mpe^{-Mp}}{p}. \end{aligned}$$

Thus, since $\lim_{M\to\infty} Mp = \infty$, then $\lim_{M\to\infty} Mpe^{-Mp} = \lim_{M\to\infty} e^{-Mp} = 0$, which implies that

$$\lim_{M\to\infty} E[X] = \lim_{M\to\infty} \frac{1-e^{-Mp}-Mpe^{-Mp}}{p} = \infty.$$

Also, note that if $p = \frac{c}{M}$, $c \in \mathbb{R}_+$ $(0 < e^{-c} + ce^{-c} < 1)$,

$$\lim_{M\to\infty} E[X] = \lim_{M\to\infty} \frac{1-e^{-c}-ce^{-c}}{p} = \infty.$$

Proof 2. Fix $a \in \mathbb{N}$. Consider the following events:

- E_1: No number less than $\frac{1}{ap}$ is selected as a generator.

  Let X_1 be the number of generators which are less than or equal than $\frac{1}{ap}$, then

  $$P[X_1 > 0] \leq E[X_1] = p \cdot \frac{1}{ap} = \frac{1}{a},$$

  and so

  $$P[E_1] = P[X_1 = 0] \geq 1 - \frac{1}{a}.$$

- E_2 : At least $\frac{a}{2}$ generators between $\frac{1}{ap}$ and $\frac{a}{p} + \frac{1}{ap}$ are selected. Let X_2 be the number of generators between $\frac{1}{ap}$ and $\frac{a}{p} + \frac{1}{ap}$, then X_2 is a binomial random variable with $n = \frac{a}{p}$ and we can use the bound (Feller I can add this to the appendix)

  $$P\left[X_2 \leq \frac{a}{2}\right] \leq \frac{(n - a/2)p}{(np - a/2)^2} = \frac{a - (a/2)p}{(a/2)^2} \leq \frac{4}{a},$$

  and so

  $$P[E_2] = P\left[X_2 \geq \frac{a}{2}\right] \geq 1 - \frac{4}{a}.$$

- E_3 : At most $\frac{3a}{2}$ generators between $\frac{1}{ap}$ and $\frac{a}{p} + \frac{1}{ap}$ are selected. We can use the bound (Feller I can add this to the appendix)

  $$P\left[X_2 \geq \frac{3a}{2}\right] \leq \frac{3a/2(1-p)}{(3a/2 - a)^2} \leq \frac{6}{a},$$

  and so

  $$P[E_3] = P\left[X_2 \leq \frac{3a}{2}\right] \geq 1 - \frac{6}{a}.$$

- E_4 : All generators less than $\frac{a}{p} + \frac{1}{ap}$ are minimal. A generator is minimal if and only if it cannot be expressed as a sum of other generators. Given E_1, E_2 and E_3, the minimum generator is greater than $\frac{1}{ap}$ and there are less than $\frac{3a}{2}$ generators less than $\frac{a}{p} + \frac{1}{ap}$. Since $\left(\frac{a}{p} + \frac{1}{ap}\right) / \frac{1}{ap} = a^2 + 1$, every sum of less than $a^2 + 1$ of these generators is greater than $\frac{a}{p} + \frac{1}{ap}$. We construct the generators in order.

  Select the first generator, then there are at most $a^2 + 1$ multiples of this generator less than $\frac{a}{p} + \frac{1}{ap}$, so the probability for the second generator to be minimal is greater than $(1-p)^{2^{a^2+1}}$. Similarly, the probability that the third generator is minimal is greater than $(1-p)^{3^{a^2+1}}$, since we have to avoid every the combinations of the first two generators. If we continue, using the union bound, we get that the probability of the generators not being minimal is less than

$$S(M) = \sum_{i=1}^{\frac{3a}{2}} 1 - (1-p)^{(i+1)^{a^2+1}},$$

which tends to 0 as M tends to infinity, as $\lim_{M\to\infty} p = 0$.

Finally,

$$P[E_4] = P[E_4|E_1E_2E_3]P[E_1E_2E_3] \geq S(M)\left(1 - \frac{11}{a}\right).$$

Therefore, for every $N \in \mathbb{N}$ and $\varepsilon > 0$, there exists M and $a > N$ for which $P[E_4] > 1 - \varepsilon$.

References

[1] N. Alon and J. H. Spencer, *The probabilistic method.* John Wiley & Sons, 2016.

[2] J. Park and H. T. Pham, "A proof of the kahn-kalai conjecture," *arXiv e-prints*, pp. arXiv–2203, 2022.

[3] K. Frankston, J. Kahn, B. Narayanan, and J. Park, "Thresholds versus fractional expectation-thresholds," *Annals of Mathematics*, vol. 194, no. 2, pp. 475–495, 2021.

[4] S. Chapman, R. Garcia, and C. O'Neill, "Beyond coins, stamps, and chicken mcnuggets: An invitation to numerical semigroups," *A Project-Based Guide to Undergraduate Research in Mathematics: Starting and Sustaining Accessible Undergraduate Research*, pp. 177–202, 2020.

[5] J. De Loera, C. O'Neill, and D. Wilburne, "Random numerical semigroups and a simplicial complex of irreducible semigroups," *arXiv preprint arXiv:1710.00979*, 2017.