# Thesis Title

by **Author Name**

Thesis submitted in fulfilment of the requirements for the degree of
*Doctor of Philosophy*
under the supervision of SUPERVISOR NAME

School of Electrical and Data Engineering
Faculty of Engineering and IT
University of Technology Sydney
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# Certificate of Authorship / Originality

I, Author Name, declare that this thesis is submitted in fulfilment of the requirements for the award of Doctor of Philosophy, in the School of Electrical and Data Engineering at the University of Technology Sydney.

This thesis is wholly my own work unless otherwise referenced or acknowledged. In addition, I certify that all information sources and literature used are indicated in the thesis. This document has not been submitted for qualifications at any other academic institution.

This research is supported by the Australian Government Research Training Program.

Signature: Signature

Date: October 22, 2023

# Abstract

This is the text of the abstract, providing a brief summary of the context, research problem, main contributions and conclusions of this work.

# Dedication

For (name of person - this is optional!)

# Acknowledgements

I would like to thank (supervisor, family, research collaborators, anyone else who significantly helped you with this work).

Author Name
October 22, 2023
Sydney, Australia

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# Chapter 1

# Introduction

The start of the introduction provides some context and brief background.

## 1.1 Research Objectives and Overview

The research question which this Thesis aims to answer is...

The specific research objectives of this Thesis are:

1. Objective 1
2. Objective 2

Chapter 2 provides a comprehensive review of literature which is relevant to the overall aim. This includes ...

Chapter 3 aims to ...

This chapter resulted in the following publications:

- D. R. Franklin and K. J. Wilson, "A LaTeX Thesis Template for the School of Electrical and Data Engineering," *IEEE Transactions on LaTeX Thesis Templates*, vol. 1, no. 1, Oct. 2021
- D. R. Franklin and K. J. Wilson, "A LaTeX Thesis Template for the School of Electrical and Data Engineering," *IEEE Transactions on LaTeX Thesis Templates*, vol. 1, no. 1, Oct. 2021

Chapter 4 aims to ...

This chapter resulted in the following publications:

- D. R. Franklin and K. J. Wilson, "A LaTeX Thesis Template for the School of Electrical and Data Engineering," *IEEE Transactions on LaTeX Thesis Templates*, vol. 1, no. 1, Oct. 2021

Chapter 5 aims to ...

This chapter resulted in the following publications:

- D. R. Franklin and K. J. Wilson, “A LaTeX Thesis Template for the School of Electrical and Data Engineering,” *IEEE Transactions on LaTeX Thesis Templates*, vol. 1, no. 1, Oct. 2021

Finally, Chapter 6 summarises the results and implications of this work, and provides recommended directions for continuation of this work in the future.

### 1.1.1 Additional Research Contributions

A number of additional research publications and presentations are listed below:

- xxx

Test [1]

# Chapter 2

# Literature Review

## 2.1 Introduction

Some contextual overview of what we are going to discuss.

Section 2.2 discusses Theme 1. Section 2.3 discusses Theme 2....

## 2.2 Theme 1

### 2.2.1 Subtopic A

### 2.2.2 Subtopic B

### 2.2.3 Subtopic C

## 2.3 Theme 2

Etc. etc.

## 2.4 Conclusion

# Chapter 3

# Contribution 1

## 3.1 Introduction

In this Chapter, XXX is presented. Note: to cross-reference to other parts of the document you do so like this - see Section 4.2.2.

Section 3.2 discusses Theme 1. Section 3.3 discusses Theme 2....

## 3.2 Theme 1

Figure 3.1 illustrates the key elements of Thesis synthesis. LaTeX will tend to place figures where it wants (they 'float' - generally they should be at the top or bottom of a page); you can override the default behaviour if you want, but you probably don't want to bother doing that until after your content is pretty much done. Instead, keep the figures as close as possible to the text; you can tweak this afterwards if you want by adding an option:

```
\begin{figure}[htb]
```

This says put it here, if you can, otherwise at the top, or otherwise at the bottom. BUT I strongly suggest not using **h** as it looks terrible.

Maybe you need an inline URL at this point: Here's one! `https://en.wikibooks.org/wiki/LaTeX`.

Mathematics can be inline, for example $x = \int_0^\infty y^2 dy$, or can be in display mode, as shown in (3.1):

$$F(x, y, z) = \sqrt{x^{y^z}} \tag{3.1}$$

You probably want to show some of your results in a table, such as Table 3.1.

The basic creative process is shown in Figure 3.2. Specifically, Figure 3.2a shows the first step, while Figure 3.2b and 3.2c show the remaining key steps of the procedure.

### 3.2.1 Subtopic A

Algorithm 1 shows a classical algorithm typeset in LaTeX. This is also a float.

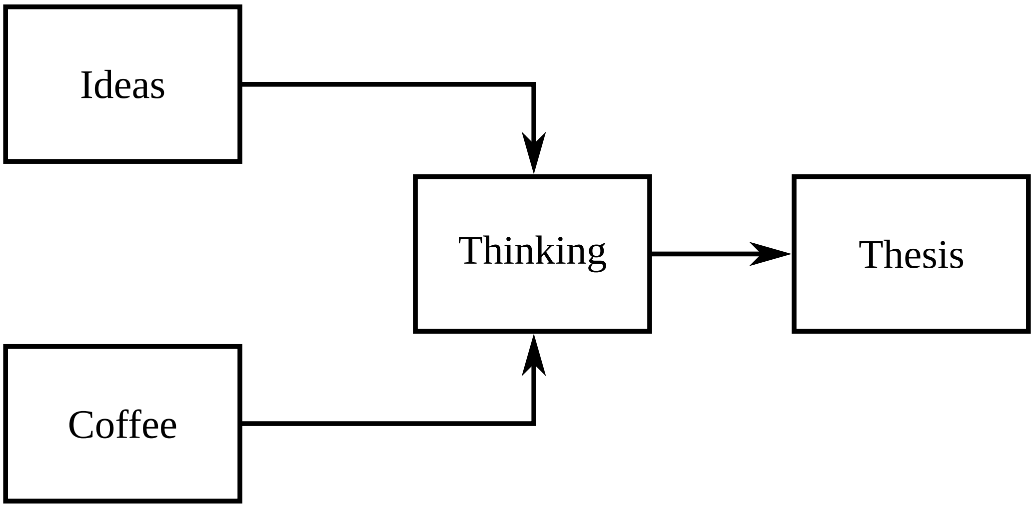


Figure 3.1: This the approximate process for producing a Thesis. It typically takes 3-4 years.

Table 3.1: Table captions normally go at the top.

| **Left column** | $\mu$ | $\sigma$ |
|---|---|---|
| Item 1 | 0.3 | 0.5 |
| Item 2 | 0.9 | 0.4 |

(a) Confusion (b) Work (c) Exhaustion

Figure 3.2: The stages of the creative process.

**Algorithm 1** An algorithm with caption

**Require:** $n \geq 0$
**Ensure:** $y = x^n$
  $y \leftarrow 1$
  $X \leftarrow x$
  $N \leftarrow n$
  **while** $N \neq 0$ **do**
    **if** $N$ is even **then**
      $X \leftarrow X \times X$
      $N \leftarrow \frac{N}{2}$ ▷ This is a comment
    **else if** $N$ is odd **then**
      $y \leftarrow y \times X$
      $N \leftarrow N - 1$
    **end if**
  **end while**

### 3.2.2 Subtopic B

### 3.2.3 Subtopic C

## 3.3 Theme 2

Etc. etc.

## 3.4 Introduction

## 3.5 Kahn-Kalai Conjecture

### 3.5.1 Thresholds

Let $n \in \mathbb{N}$ and $0 \leq p \leq 1$. The random graph $G(n, p)$ is a probability space over the set of graphs on $n$ labeled vertices determined by

$$\Pr[\{i, j\} \in G] = p$$

with these events mutually independent [2]. Given a graph theoretic property $A$, there is a probability that $G(n, p)$ satisfies $A$, which we write as $\Pr[G(n, p) \vDash A]$.

**Definition 3.5.1.** $r(n)$ is a threshold function for a graph theoretic property $A$ if

1. When $p(n) \in o(r(n))$, $\lim_{n\to\infty} \Pr[G(n, p(n)) \vDash A] = 0$,
2. When $r(n) \in o(p(n))$, $\lim_{n\to\infty} \Pr[G(n, p(n)) \vDash A] = 1$,

or vice versa. [2]

We give an example of a threshold function which illustrates a common method for proving that a function is a threshold.

**Threshold function for having isolated vertices**

Let $G$ be a graph on $n$ labeled vertices. An isolated vertex of $G$ is a vertex which does not belong to any of the edges of $G$. Let $A$ be the property that $G$ contains an isolated vertex. We will prove that $r(n) = \dfrac{\ln n}{n}$ is a threshold for $A$.

For each vertex $i$ in $G$ define the variable

$$X_i = \begin{cases} 1 & \text{if } i \text{ is an isolated vertex,} \\ 0 & \text{if } i \text{ is not an isolated vertex.} \end{cases}$$

Now, the probability that a vertex $i$ is isolated is $(1-p)^{n-1}$ since it is the probability that none of the other $n-1$ vertices is connected to $i$. Let $X = \sum_{i=1}^{n} X_i$, then the expected number of isolated vertices is

$$E[X] = \sum_{i=1}^{n} E[X_i] = \sum_{i=1}^{n} \Pr[X_i] = n(1-p)^{n-1}.$$

Let $p = k\dfrac{\ln n}{n}$ for $k \in \mathbb{R}_{>0}$. Then

$$\begin{aligned} \lim_{n\to\infty} E[X] &= \lim_{n\to\infty} n\left(1 - k\frac{\ln n}{n}\right)^{n-1} \\ &= ne^{-k\ln n} = n^{1-k}. \end{aligned}$$

Therefore, $\lim_{n\to\infty} E[X] = 0$ if $k > 1$. Since $E[X] \geq \Pr[X > 0]$, we conclude that

$$\lim_{n\to\infty} \Pr[G(n,p) \vDash A] = \lim_{n\to\infty} \Pr[X > 0] = 0.$$

Now, for $k < 1$, the fact that $\lim_{n\to\infty} E[X] = \infty$ is not enough to conclude that $\lim_{n\to\infty} \Pr[G(n,p) \vDash A] = 1$. We have to use the second moment method.

**Theorem.** *If $E[X] \to \infty$ and $Var[X] = o(E[X]^2)$, then $\lim_{n\to\infty} Pr[X > 0] = 1$. [2]*

**Proof.** We will prove that, in this case, $Var[X] = o(E[X]^2)$. First,

$$\begin{aligned} \sum_{i\neq j} E[X_iX_j] &= \sum_{i\neq j} \Pr[X_i = X_j = 1] \\ &= n(n-1)(1-p)^{n-1}(1-p)^{n-2} \\ &= n(n-1)(1-p)^{2n-3}, \end{aligned}$$

for if $i$ is an isolated vertex, then there is no edge between $i$ and $j$ so we only have to account for the remaining $n-2$ edges that contain $j$.

Thus, since $\sum_{i=1}^{n} E[X_i^2] = \sum_{i=1}^{n} E[X_i] = E[X]$ and $\lim_{n\to\infty} p = 0$,

$$\begin{aligned} \lim_{n\to\infty} \frac{\mathrm{Var}[X]}{E[X]^2} &= \lim_{n\to\infty} \frac{E(X^2) - E[X]^2}{E[X]^2} = \lim_{n\to\infty} \frac{\sum_{i=1}^{n} E[X_i^2] + \sum_{i\neq j} E[X_iX_j]}{E[X]^2} - 1 \\ &= 0 + \lim_{n\to\infty} \frac{n(n-1)(1-p)^{2n-3}}{n^2(1-p)^{2n-2}} - 1 = \lim_{n\to\infty} \frac{1}{1-p} - 1 = 0. \end{aligned}$$

We conclude that $\mathrm{Var}[X] \in o(E[X]^2)$ and so, if $k < 1$,

$$\lim_{n\to\infty} \Pr[G(n,p) \vDash A] = \lim_{n\to\infty} \Pr[X > 0] = 1.$$

Therefore, $r(n) = \frac{\ln n}{n}$ is a threshold function for property $A$.

### 3.5.2 The expectation threshold

Let $X$ be a set and $p \in [0,1]$. For $A \subset X$, let $P(A) = p^{|A|}(1-p)^{|X \setminus A|}$. A class $\mathcal{F}$ of subsets of $X$ is increasing if for $A \in \mathcal{F}$, $A \subset B$ implies $B \in \mathcal{F}$. If $\mathcal{F} \neq \emptyset$, $P(\mathcal{F}) = \sum_{A\in\mathcal{F}} P(A)$ is a strictly increasing function of $p$.

**TODO** Define increasing class and general definition of threshold [3]

**TODO** Define expectation threshold and show inequalities. [4]

**Theorem 3.5.1** (Park Theorem [3], originally Kahn-Kalai Conjecture)**.** ***TODO***

**TODO** How to prove something is p-small

#### An application of the Park Theorem

**TODO** Prove the threshold for perfect matchings for $G(n,p)$

## 3.6 Numerical Semigroups

A *numerical semigroup* is a subset $S \subset \mathbb{N}_0$ which is closed under addition, i.e. $a, b \in S$ implies $a + b \in S$. For instance, $\mathbb{N}_0$, $\mathbb{N}_0 \setminus \{0\}$, $2\mathbb{N}_0$ are all numerical semigroups, but $\mathbb{N}_0 \setminus \{2\}$ is not. Some literature requires that a semigroup has a finite complement in $\mathbb{Z}_{\geq 0}$ [5], but we prefer the more general definition.

**Example 3.6.1. TODO** McNugget Semigroup

As seen in the previous example, we can describe a numerical semigroup with a set of *generators*:

$$S = \langle a_1, ..., a_n \rangle := \left\{ \sum_{i=1}^{n} k_i a_i : k_1, ..., k_n \in \mathbb{N}_0 \right\}.$$

**TODO** give some examples.

The *Frobenius number* $F(S)$ of a numerical semigroup $S = \langle a_1, ..., a_n \rangle$ is defined as the largest integer divisible by $d(S) := \gcd(a_1, ..., a_n)$ which does not belong to $S$. In other terms,

$$F(S) = \max(d\mathbb{Z} \setminus S).$$

## 3.7 Probability Spaces over Numerical Semigroups

We generate a random numerical semigroup with a model similar to the Ërdos-Renyi model for random graphs.

**Definition 3.7.1.** For $p \in [0,1]$ and $M \in \mathbb{N}$, a random numerical semigroup $S(M,p)$ is a probability space over the set of semigroups $S = \langle \mathcal{A} \rangle$ with $\mathcal{A} \subset \{1, ..., M\}$, determined by

$$\Pr[n \in \mathcal{A}] = p,$$

with these events mutually independent.

## 3.8 Expected Frobenius Number

We prove a Theorem found in [6] without the use of the simplicial complex.

**Theorem 3.8.1.** *Let $S \sim S(M,p)$, where $p = p(M)$ is a monotone decreasing function of $M$. If $\frac{1}{M} \ll p \ll 1$, then $S$ is cofinite, i.e., the set of gaps is finite, a.a.s and*

$$\lim_{M\to\infty} E[e(S)] = \lim_{M\to\infty} E[g(S)] = \lim_{M\to\infty} E[F(S)] = \infty.$$

**Proof.** Let $X := \min(S \setminus \{0\})$ be a random variable. Then, for $0 < n \leq M$,

$$\Pr[X = n] = p(1-p)^{n-1},$$

and so

$$\begin{aligned} E[X] &= \sum_{n=0}^{\infty} n\Pr[X = n] = \sum_{n=0}^{M} np(1-p)^{n-1} = p\frac{d}{dp}\left[-\sum_{n=0}^{M}(1-p)^n\right] \\ &= p\frac{d}{dp}\frac{(1-p)^{M+1}-1}{p} = p\frac{1-(1-p)^{M+1}-(M+1)(1-p)^M p}{p^2} \\ &= \frac{1-(1-p)^M - M(1-p)^M p}{p} \geq \frac{1-e^{-Mp}-Mpe^{-Mp}}{p}. \end{aligned}$$

Thus, since $\lim_{M\to\infty} Mp = \infty$, then $\lim_{M\to\infty} Mpe^{-Mp} = \lim_{M\to\infty} e^{-Mp} = 0$, which implies that

$$\lim_{M\to\infty} E[X] = \lim_{M\to\infty} \frac{1-e^{-Mp}-Mpe^{-Mp}}{p} = \infty.$$

Also, note that if $p = \frac{c}{M}$, $c \in \mathbb{R}_+$ $(0 < e^{-c} + ce^{-c} < 1)$,

$$\lim_{M\to\infty} E[X] = \lim_{M\to\infty} \frac{1-e^{-c}-ce^{-c}}{p} = \infty.$$

**Proof.** Fix $a \in \mathbb{N}$ such that $a > 11$ and let $A = \{1, \dots, \lfloor \frac{a}{p} \rfloor\}$. Since $\frac{1}{M} \ll p$, we have that $\lfloor \frac{a}{p} \rfloor \leq M$ for large enough $M$. Consider the following events:

- $E_1$: No generator selected is less than $\frac{1}{ap}$.

  Let $X_1$ be the number of generators selected from $\{1, \dots, \lfloor \frac{1}{ap} \rfloor\}$. Then

  $$\Pr[\overline{E_1}] = \Pr[X_1 > 0] \leq E[X_1] \leq p \cdot \frac{1}{ap} = \frac{1}{a}.$$

- $E_2$ : At most $\frac{3a}{2}$ generators are selected from $A$.

  Let $X_2$ be the number of generators selected in $A$, then $X_2$ is a binomial random variable with $n = \frac{a}{p}$ and we can use the bound (Feller I can add this to the appendix)

  $$\Pr[\overline{E_2}] = \Pr\left[X_2 > \frac{3a}{2}\right] \leq \frac{\frac{3a}{2}(1-p)}{(\frac{3a}{2} - a)^2} \leq \frac{6}{a}.$$

  Also, note that by the union bound

  $$\Pr[E_1 \wedge E_2] \leq 1 - \frac{1}{a} - \frac{6}{a} = 1 - \frac{7}{a}.$$

- $E_3$ : At least $\frac{a}{2}$ generators are selected from $A$.

  Similarly, we can use the bound for the other tail of the distribution so that

  $$\Pr[\overline{E_3}] = \Pr\left[X_2 < \frac{a}{2}\right] \leq \frac{(n - \frac{a}{2})p}{(np - \frac{a}{2})^2} = \frac{a - (\frac{a}{2})p}{(\frac{a}{2})^2} \leq \frac{4}{a}.$$

- $E_4$ : The generators selected from $A$ are minimal.

  Let $Y_{(1)}, Y_{(2)}, \dots, Y_{(k)}$ denote the first $k$ generators selected in $A$. Assume $E_1$ and $E_2$. We have that $E_1$ implies $Y_{(1)} \geq \frac{1}{ap}$ and $E_2$ implies $k \leq \frac{3a}{2}$.

  First we bound for the probability that, given $E_1$ and $E_2$, $b \in A$ is selected as a generator. By conditional probability

  $$\begin{aligned}\Pr[b \text{ is selected}] = &\Pr[b \text{ is selected}|E_1 \wedge E_2]\Pr[E_1 \wedge E_2] \\ &+ \Pr[b \text{ is selected}|\overline{E_1 \wedge E_2}]\Pr[\overline{E_1 \wedge E_2}],\end{aligned}$$

  and so

  $$\Pr[b \text{ is selected}|E_1 \wedge E_2] \leq \frac{\Pr[b \text{ is selected}]}{\Pr[E_1 \wedge E_2]} \leq \frac{p}{1 - \frac{7}{a}}.$$

  Now, note that $Y_{(2)}$ is not minimal if a multiple of $Y_{(1)}$ is selected in $A$. Thus, if we fix $Y_{(1)} = y_1 \geq \frac{1}{ap}$, $Y_{(1)}$ is not minimal if $b \in \{2y_1, 3y_1, \dots, c_1 y_1\}$ is selected, where $c_1 y_1$ is the largest multiple of $y_1$ which does not exceed $\frac{a}{p}$. Since $y_1 \geq \frac{1}{ap}$, we have that $c_1 \leq a^2$. Then, using the union bound,

  $$\Pr[Y_{(2)} \text{ is not minimal}|E_1 \wedge E_2 \wedge Y_{(1)} = y_1] \leq \frac{pa^2}{1 - \frac{7}{a}}.$$

If we sum over all possible $y_1$, we get that

$$\Pr[Y_{(2)} \text{ is not minimal} | E_1 \wedge E_2] \leq \frac{pa^2}{1-\frac{7}{a}}.$$

Similarly, for $2 \leq t \leq k$ and fixed $Y_{(1)} = y_1, \ldots, Y_{(t-1)} = y_{t-1}$, $Y_{(t)}$ is not minimal if the first $t-1$ numbers selected from $A$ can generate $Y_{(t)}$. For the possible numbers generated by the first $t$ numbers selected, there are at most $a^2$ choices for each coefficient, so there are at most $a^{2t}$ such linear combinations. Then

$$\Pr[Y_{(t)} \text{ is not minimal} | E_1 \wedge E_2] \leq \frac{pa^{2t}}{1-\frac{7}{a}}.$$

Therefore, since $Y_{(1)}$ is always minimal, we can use the union bound and $k \leq \frac{3a}{2}$ to conclude that

$$\Pr[E_4 | E_1 \wedge E_2] \geq 1 - \frac{p}{1-\frac{7}{a}} \sum_{t=1}^{\frac{3a}{2}-1} a^{2t} = 1 - o(1).$$

Thus,

$$\Pr[E_4] = \Pr[E_4 | E_1 \wedge E_2] \Pr[E_1 \wedge E_2] \geq 1 - \frac{7}{a} - o(1).$$

Finally, note that by union bound,

$$\Pr[E_4 \wedge E_3] \geq 1 - \frac{11}{a} - o(1).$$

Therefore, for every $N \in \mathbb{N}$ and $\varepsilon > 0$, there exists $K$ such that $M \geq K$ implies

$$\Pr[f(S) > N],\ \Pr[g(S) > N],\ \Pr[e(S) > N] > 1 - \varepsilon.$$

.

## 3.9 Conclusion

# Chapter 4

# Contribution 2

## 4.1 Introduction

In this Chapter, XXX is presented.

Section 4.2 discusses Theme 1. Section 4.3 discusses Theme 2....

## 4.2 Theme 1

### 4.2.1 Subtopic A

### 4.2.2 Subtopic B

### 4.2.3 Subtopic C

## 4.3 Theme 2

Etc. etc.

## 4.4 Conclusion

# Chapter 5

# Contribution 3

## 5.1 Introduction

In this Chapter, XXX is presented.

Section 5.2 discusses Theme 1. Section 5.3 discusses Theme 2....

## 5.2 Theme 1

### 5.2.1 Subtopic A

### 5.2.2 Subtopic B

### 5.2.3 Subtopic C

## 5.3 Theme 2

Etc. etc.

## 5.4 Conclusion

# Chapter 6

# Conclusions and Future Work

## 6.1 Summary of Outcomes

## 6.2 Recommendations & Future Work

## 6.3 Concluding Remarks

In summary, ...

# Bibliography


[1] D. R. Franklin and K. J. Wilson, "A LaTeX Thesis Template for the School of Electrical and Data Engineering," *IEEE Transactions on LaTeX Thesis Templates*, vol. 1, no. 1, Oct. 2021.

[2] N. Alon and J. H. Spencer, *The probabilistic method.* John Wiley & Sons, 2016.

[3] J. Park and H. T. Pham, "A proof of the kahn-kalai conjecture," *arXiv e-prints*, arXiv–2203, 2022.

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[5] S. Chapman, R. Garcia, and C. O'Neill, "Beyond coins, stamps, and chicken mcnuggets: An invitation to numerical semigroups," *A Project-Based Guide to Undergraduate Research in Mathematics: Starting and Sustaining Accessible Undergraduate Research*, pp. 177–202, 2020.

[6] J. De Loera, C. O'Neill, and D. Wilburne, "Random numerical semigroups and a simplicial complex of irreducible semigroups," *arXiv preprint arXiv:1710.00979*, 2017.

# Appendix A

# Example Appendix

Here you might present some additional results, derivations, proofs etc. that were not included in the main text.

# Appendix B

# Software Documentation

Here's an example source code listing, where the code is read in from an external file:

```
% Function to create a nice rotating animated GIF of 3D volumetric data V

function animation (V)

h = volshow (V, 'BackgroundColor', [0 0 0], 'Renderer', 'MaximumIntensityProjection', 'CameraPosition', [2 2 0], 'CameraUpVector', [1 0 0], 'ColorMap', jet);

camproj ('perspective');

N = 500;

filename = 'animation.gif';
vec = linspace(0, 4 * pi(), N)';
myPosition = 2 * [zeros(size(vec)) cos(vec) sin(vec)];

for idx = 1:N
% Update current view.
    h.CameraPosition = myPosition(idx, :);
% Use getframe to capture image.
    I = getframe(gcf);

    [indI, cm] = rgb2ind (I.cdata,256);
% Write frame to the GIF File.
    if idx == 1
        imwrite(indI, cm, filename, 'gif', 'Loopcount', inf, 'DelayTime', 0.05);
    else
        imwrite(indI, cm, filename, 'gif', 'WriteMode', 'append', 'DelayTime', 0.05);
    end
end
```

## B.1 Code Availability

All scripts and source code used for simulation and analysis of the ... are available here:

`https://bitbucket.org/username/gitrepo.git`

## B.2 Software Requirements

- MATLAB code is confirmed working with version XXXX;
- Simulations require the use of gcc version XXX or llvm/clang version YYYY

## B.3 Simulation Code - How to Run