Graduation Project

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February 20, 2023

1 Introduction

2 Kahn-Kalai Conjecture

2.1 Thresholds

Let $n \in \mathbb{N}$ and $0 \leq p \leq 1$. The random graph $G(n, p)$ is a probability space over the set of graphs on n labeled vertices determined by

$$\Pr[\{i, j\} \in G] = p$$

with these events mutually independent [1]. Given a graph theoretic property A, there is a probability that $G(n, p)$ satisfies A, which we write as $\Pr[G(n, p) \vDash A]$.

Definition 2.1. $r(n)$ is a threshold function for a graph theoretic property A if

1. When $p(n) \in o(r(n)),\ \lim_{n\to\infty} \Pr[G(n, p(n)) \vDash A] = 0$,
2. When $r(n) \in o(p(n)),\ \lim_{n\to\infty} \Pr[G(n, p(n)) \vDash A] = 1$,

or vice versa. [1]

We give an example of a threshold function which illustrates a common method for proving that a function is a threshold.

2.1.1 Threshold function for having isolated vertices

Let G be a graph on n labeled vertices. An isolated vertex of G is a vertex which does not belong to any of the edges of G. Let A be the property that G contains an isolated vertex. We will prove that $r(n) = \dfrac{\ln n}{n}$ is a threshold for A.

For each vertex i in G define the variable

$$X_i = \begin{cases} 1 & \text{if } i \text{ is an isolated vertex,} \\ 0 & \text{if } i \text{ is not an isolated vertex.} \end{cases}$$

Now, the probability that a vertex i is isolated $(1-p)^{n-1}$ since it is the probability that none of the other $n-1$ vertices is connected to i. Let $X = \sum_{i=1}^{n} X_i$, then the expected number of isolated vertices is thus

$$E[X] = \sum_{i=1}^{n} E[X_i] = \sum_{i=1}^{n} \Pr[X_i] = n(1-p)^{n-1}.$$

Let $p = k\dfrac{\ln n}{n}$ for $k \in \mathbb{R}_{>0}$. Then

$$\begin{aligned} \lim_{n\to\infty} E[X] &= \lim_{n\to\infty} n\left(1 - k\frac{\ln n}{n}\right)^{n-1} \\ &= ne^{-k\ln n} = n^{1-k}. \end{aligned}$$

Therefore, $\lim_{n\to\infty} E[X] = 0$ if $k > 1$. Since $E[X] \geq \Pr[X > 0]$, we conclude that

$$\lim_{n\to\infty} \Pr[G(n,p) \vDash A] = \lim_{n\to\infty} \Pr[X > 0] = 0.$$

Now, for $k < 1$, the fact that $\lim_{n\to\infty} E[X] = \infty$ is not enough to conclude that $\lim_{n\to\infty} \Pr[G(n,p) \vDash A] = 1$. We have to use the second moment method.

Theorem. *If $E[X] \to \infty$ and $Var[X] = o(E[X]^2)$, then $\lim_{n\to\infty} Pr[X > 0] = 1$. [1]*

We will prove that, in this case, $Var[X] = o(E[X]^2)$. First,

$$\begin{aligned} \sum_{i\neq j} E[X_i X_j] &= \sum_{i\neq j} \Pr[X_i = X_j = 1] \\ &= n(n-1)(1-p)^{n-1}(1-p)^{n-2} \\ &= n(n-1)(1-p)^{2n-3}, \end{aligned}$$

for if i is an isolated vertex, then there is no edge between i and j so we only have to account for the remaining $n-2$ edges that contain j.

Thus, since $\sum_{i=1}^n E[X_i^2] = \sum_{i=1}^n E[X_i] = E[X]$ and $\lim_{n\to\infty} p = 0$,

$$\lim_{n\to\infty} \frac{\mathrm{Var}[X]}{E[X]^2} = \lim_{n\to\infty} \frac{E(X^2) - E[X]^2}{E[X]^2} = \lim_{n\to\infty} \frac{\sum_{i=1}^n E[X_i^2] + \sum_{i\neq j} E[X_i X_j]}{E[X]^2} - 1$$
$$= 0 + \lim_{n\to\infty} \frac{n(n-1)(1-p)^{2n-3}}{n^2(1-p)^{2n-2}} - 1 = \lim_{n\to\infty} \frac{1}{1-p} - 1 = 0.$$

We conclude that $\mathrm{Var}[X] \in o(E[X]^2)$ and so, if $k < 1$,

$$\lim_{n\to\infty} \Pr[G(n,p) \vDash A] = \lim_{n\to\infty} \Pr[X > 0] = 1.$$

Therefore, $r(n) = \frac{\ln n}{n}$ is a threshold function for property A.

2.2 The expectation threshold

TODO Define increasing class and general definition of threshold [2]
TODO Define expectation threshold and show inequalities. [3]

Example 2.1. H_1 example

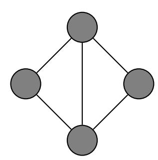

Figure 1: H_1

Example 2.2. H_2 example

Theorem 2.1 (Park Theorem [2], originally Kahn-Kalai Conjecture)**.** ***TODO***

TODO How to prove something is p-small

2.2.1 An application of the Park Theorem

TODO Prove the threshold for perfect matchings for $G(n,p)$

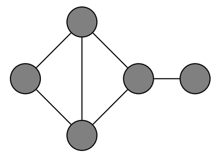

Figure 2: H_2

3 Numerical Semigroups

4 Probabilistic Spaces over Numerical Semigroups

5 Expected Frobenious Number

References

[1] N. Alon and J. H. Spencer, *The probabilistic method.* John Wiley & Sons, 2016.

[2] J. Park and H. T. Pham, "A proof of the kahn-kalai conjecture," *arXiv e-prints*, pp. arXiv–2203, 2022.

[3] K. Frankston, J. Kahn, B. Narayanan, and J. Park, "Thresholds versus fractional expectation-thresholds," *Annals of Mathematics*, vol. 194, no. 2, pp. 475–495, 2021.