Angle to close a rhodonea curve

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Abstract

This paper finds and proves the minimum angle necessary to close a rhodonea curve. A rhodonea could be described as a closed curve, that resembles the shape of a flower, it was defined by the Italian mathematician Guido Grandi back in the 18th century, who also established this name to refer these curves. It is written after a little thought on the problem and the willing to find the most efficient way to plot them. It also aims to prove or perhaps refuse a hypothesis that was made in relation to this topic.

1 Wording

To define a rhodonea the most straightforward way is through polar coordinates, where the following equation defines it:

$$r(\theta) = \cos(k\theta) \tag{1}$$

Note that k is the value that changes the flower, and through our plotting it would be considered constant, as the working with polar coordinates could turn out a little complex, a Cartesian approach is considered, and the rhodonea can be restated as the curve whose coordinates are defined by the following equations:

$$x = \cos(k\theta)\cos(\theta) \tag{2}$$

$$y = \cos(k\theta)\sin(\theta) \tag{3}$$

2 First Considerations

In order to get the rhodonea closed it is necessary to find the points in the same exact position as the initial conditions, and as it can be easily calculated the starting point will always be $(1, 0)$, thus:

$$\begin{cases} cos\,(k\theta)cos(\theta)=1 \\ cos\,(k\theta)sin(\theta)=0 \end{cases}$$

We have considered that the best way to handle this problem might be by working on the different cases that may arise, in our case, having the fractions in its lowest terms, it can be identified a total of five cases, which are listed below:

- $k = a$ where $a_{(2)} = 0$
- $k = a$ where $a_{(2)} = 1$
- $k = \frac{a}{b}$ where $a_{(2)} = 0, b_{(2)} = 1$
- $k = \frac{a}{b}$ where $a_{(2)} = 1, b_{(2)} = 0$
- $k = \frac{a}{b}$ where $a_{(2)} = 1, b_{(2)} = 1$

3 First Case

In this first case the situation where $k = a$ and $a_{(2)} = 0$ is studied. So k can be rewritten as $2^n k'$, being $k' = \frac{k}{2^n}$ with $n \in \mathbb{N}$. Please note that $k'_{(2)} = 1$. Making use of all these considerations the x-equation could be displayed as:

$$\cos(2^n k'\theta)\cos(\theta) = 1 \tag{4}$$

As it can be observed this equation would require of an angle of 0 to be satisfied, but as the 0 is the one giving the initial conditions, the following one should be taken. This following value is 2π, which can be proven through:

$$\cos(2^n k'\theta) = \cos(2^n k' 2\pi) = \cos(2^{n+1} k\pi) = \cos(2\pi) = 1$$

And :

$$\cos(\theta) = \cos(2\pi) = 1$$

And hence:

$$1 \cdot 1 = 1$$

Having this simple proof it can now be claimed that this expression is going to be certain every 2π (considering the starting angle on 0). The following step to take is to find which the very first value that satisfies the y-equation, it can be seen that the second part would be cancelled by computing 2π:

$$\cos(2^n k'\theta)\sin(\theta) = \cos(2^n k' 2\pi)\sin(2\pi) = \cos(2^{n+1}k'\pi)\sin(2\pi) = 0 \tag{5}$$

So it has been determined that for completely closing the rhodonea a rotation of a minimum of 2π radians has to be applied when $k_{(2)} = 0$.

4 Second Case

The second case that's been considered is where $k_{(2)} = 1$. As k is odd, no further simplification can be applied without fully knowing the exact value of k. Taking all this into account the x-equation of the rhodonea could be written as:

$$\cos(k\theta)\cos(\theta) = 1 \tag{6}$$

It may seem obvious that making a rotation of 2π would of course satisfy the equation such as in the previous section, but there's a possibility within this angle, and it is to apply a rotation of π, and this can be shown as:

$$\cos(k\theta) = \cos(k\pi) = -1$$

And:

$$\cos(\theta) = \cos(\pi) = -1$$

And hence:

$$(-1)\cdot(-1) = 1$$

Through this two simple equations it has been proved that when k is odd, the rhodonea's x-coordinate has a value of 1 every π rotation. In order to establish this rotation as the minimum angle needed to close the rhodonea it is necessary that it satisfies the y-equation as well:

$$\cos(k\theta)\sin(\theta) = \cos(k\pi)\sin(\pi) = (-1)\cdot 0 = 0 \tag{7}$$

As it has been successfully computed for both equations, it can now be claimed that a minimum rotation of π radians is necessary to close a rhodonea whenever $k_{(2)} = 1$.

By having proved both first and second case it can be now determined the minimum angle of a rhodonea whenever the conditions of $k \in \mathbb{N}$ is satisfied. But as the wording states, $k \in |\mathbb{R}|$, so third and fourth case do still have to be proven. This is going to be done in the two following sections.

5 Third Case

It has now reached the point where the k no longer is a whole number. In this third case is stated that $k = \frac{a}{b}$ where $a_{(2)} = 0, b_{(2)} = 1$. It is assumed the fraction is in its lowest terms. Therefore, the fraction could be rewritten by factorizing a, which turns to $2^n a'$. Note that $a'_{(2)} = 1$. Having considered all this, the x-equation could be written as:

$$\cos\left(\frac{2^n a'}{b}\theta\right)\cos(\theta) = 1 \tag{8}$$

As it can be seen, no fraction is going to satisfy this equality, it is necessary to get rid of the denominator, which can be done by multiplying the whole expression by itself, which would turn as:

$$\cos(b \cdot 2^n a'\theta)\cos(\theta) = 1 \tag{9}$$

This is the exact same equation as #4, but noting that the angle is scaled by b. As a result there is no need to repeat the whole process, and it can surely be stated that the angle to close a rhodonea where k is a fraction whose numerator is even and denominator odd, a total of $b \cdot 2\pi$ radians is needed to close it, being b the denominator.

6 Fourth Case

In this section the fourth case the one where $k = \frac{a}{b}$ and $a_{(2)} = 1, b_{(2)} = 0$ is going to be studied. Due to the similarities among this and the former case the same process of resolution can be followed, and after the factorization of b the x-equation would be written as:

$$\cos\left(\frac{a}{2^n b'}\theta\right)\cos(\theta) = 1 \tag{10}$$

As it has been done in the previous section, here the denominator is not going to allow this equation to become true, therefore it can be rewritten as:

$$\cos(2^n b' a\theta)\cos(\theta) = 1 \tag{11}$$

It can be seen that equation #11 and equation #9, and as a result it has to be solved as the former. This means that for a rhodonea where the k is a fraction on its lowest terms whose numerator is odd and denominator even, the minimum angle that has to be applied to close it needs to be $b2\pi$ being b the denominator.

Note that in both cases third and fourth, the equation could not be satisfied by π, as it would be giving a value of -1 in the x-equation.

7 Fifth Case

In this last case it has to be found the angle for $k = \frac{a}{b}$ where both a and b are odd. Because of this no further simplification can be done to any of the terms, so the x-equation could be expressed as:

$$\cos\left(\frac{a}{b}\theta\right)\cos(\theta) = 1 \tag{12}$$

As happened in the two prior cases is necessary to work with whole numbers and in order to achieve that the multiplication by the denominator is necessary. The x-expression could be denoted as:

$$\cos(b \cdot a\theta)\cos(\theta) = 1 \tag{13}$$

Remembering that $a_{(2)}, b_{(2)} = 1$, and that the product of two odds is always an odd number, it can be observed that equations #13 and #6 are equal, having #13 scaled by b. And as a consequence both may be solved using the same method, and hence the angle necessary to close a rhodonea of those features is $b\pi$, being b the denominator of k.

8 Conclusion

So it has been shown the angle needed for all possible scenarios that one may encounter when working with a rhodonea, and we can state the following.
INTEGERS

- Odd: An angle of π is necessary to close the curve
- Even: An angle of 2π is necessary to close the curve

FRACTIONS

- Odd over even: An angle of $2\pi den$ is necessary (den stands for denominator)
- Even over odd: An angle of $2\pi den$ is necessary (den stands for denominator)
- Odd over odd: An angle of πden is necessary (den stands for denominator), others may prefer $2\pi\frac{den}{2}$

Please note that a whole number could also be expressed as a fraction of denominator 1, and it would still satisfy the claims made for the fractions.