# COIM calculations

Miguel Sarraf

## Contents

# 1 Definitions

For each kind of applied constrain, we need to determine which variables will be predicted (usually they will be some sort of calculation between the original variables). Moreover, we need to extract the formula to calculated the error for the original variables given the error for the predicted model.

For this sake, the following standard is defined:

- Capital letters: constants
- Small letters: variables
- $\Delta x$: the error related to variable $x$

It will be used the general error propagation formula:
$\Delta f(x_0, x_1, ...) = \sqrt{(\sum_{i=0}^{N} (\frac{\partial f}{\partial x_i})^2 \Delta x_i^2)}$

# 2 Simple operations

## 2.1 Addition by scalar

Constrain: $b = a + K$

Both variables have the same variance, so either of them will work similarly.

$a' = a$

Henceforth, the variables to be predicted are $a'$.

To retrieve the original values, we re-apply the constrain formulas:

$a = a'$

$b = a' + K$

Finally, we calculate the propagated error formula with the other formulas:

$\frac{\partial a}{\partial a'} = 1$

$\frac{\partial b}{\partial a'} = 1$

$\Delta a = \sqrt{1 \cdot \Delta a'^2} = \Delta a'$

$\Delta b = \sqrt{1 \cdot \Delta a'^2} = \Delta a'$

## 2.2 Multiplication by scalar

Constrain: $b = Ka$

For a better prediction, the value must not be too varying, so we choose, between the 2 variables, the one with less variance.

$a' = \begin{cases} b & |K| \leq 1 \\ a & |K| > 1 \end{cases}$

Henceforth, the variables to be predicted are $a'$.

To retrieve the original values, we re-apply the derived formulas:

$$\begin{cases} b = a' \;\; a = \frac{a'}{K} & |K| \leq 1 \\ a = a' \;\; b = a'K & |K| > 1 \end{cases}$$

Finally, we calculate the propagated error formula with the other formulas:

$\frac{\partial a}{\partial a'} = \begin{cases} \frac{1}{K} & |K| \leq 1 \\ 1 & |K| > 1 \end{cases}$

$\frac{\partial b}{\partial a'} = \begin{cases} 1 & |K| \leq 1 \\ K & |K| > 1 \end{cases}$

$\Delta a = \begin{cases} \sqrt{\frac{1}{K^2} \cdot \Delta a'^2} = \frac{\Delta a'}{K} & |K| \leq 1 \\ \sqrt{1 \cdot \Delta a'^2} = \Delta a' & |K| > 1 \end{cases}$

$\Delta b = \begin{cases} \sqrt{1 \cdot \Delta a'^2} = \Delta a' & |K| \leq 1 \\ \sqrt{K^2 \cdot \Delta a'^2} = K\Delta a' & |K| > 1 \end{cases}$

# 3 Groupings

## 3.1 Constant sum

Constrain: $\sum_{i=0}^{n} a_i = K$

For a better prediction, the value must not be too varying, so we divide them by their sum. Also, it is interesting to predict the relation between the variables, instead of the values themselves, so we divide all of the variables by the least varying one (called $a_0$).

$\sum_{i=0}^{N} a_i = K$
$a_0 + \sum_{i=1}^{N} a_i = K$
$\frac{1}{K} + \sum_{i=1}^{N} \frac{a_i}{a_0 K} = \frac{1}{a_0}$
$a'_i = \frac{a_i}{a_0 K}$

Henceforth, the variables to be predicted are $a'_i$.

To retrieve the original values, we re-apply the derived formulas:

$\frac{1}{K} + \sum_{i=1}^{N} a'_i = \frac{1}{a_0} \rightarrow a_0 = \frac{1}{\frac{1}{K} + \sum_{i=1}^{N} a'_i} = \frac{K}{1 + K \sum_{i=1}^{N} a'_i}$
$a'_i = \frac{a_i}{a_0 K} \rightarrow a_i = a'_i a_0 K$

Finally, we calculate the propagated error formula with the other formulas:
$\frac{\partial a_0}{\partial a'_i} = \frac{-K^2}{(1+K\sum_{j=1}^{N} a'_j)^2} = -(\frac{K}{1+K\sum_{j=1}^{N} a'_j})^2 = -a_0^2$
$\frac{\partial a_i}{\partial a_0} = K((-\frac{a_i}{a_0^2 K})a_0 + (a'_i)1) = K(-\frac{a_i}{a_0 K} + \frac{a_i}{a_0 K}) = 0$
$\frac{\partial a_i}{\partial a'_i} = K(1(a_0) + a'_i(-a_0^2)) = K(a_0 - \frac{a_i}{a_0 K} a_0^2) = a_0(K - a_i)$
$\Delta a_0 = \sqrt{\sum_{j=1}^{N} a_0^4 \cdot \Delta a_j'^2} = a_0^2 \sqrt{\sum_{j=1}^{N} \Delta a_j'^2}$
$\Delta a_i = \sqrt{a_0^2 (K - a_i)^2 \cdot \Delta a_i'^2 + 0 \cdot \Delta a_0^2} = a_0(K - a_i) \Delta a'_i$

### 3.2 Variable sum

Constrain: $\sum_{i=0}^{N} a_i = b$

For a better prediction, the value must not be too varying, so we divide them by their sum. Also, it is interesting to predict the relation between the variables, instead of the values themselves, so we divide all of the variables by the least varying one (called $a_0$).

$\sum_{i=0}^{N} a_i = b$

$a_0 + \sum_{i=1}^{N} a_i = b$

$\frac{1}{b} + \sum_{i=1}^{N} \frac{a_i}{a_0 b} = \frac{1}{a_0}$

$a'_i = \frac{a_i}{a_0 b}$

Henceforth, the variables to be predicted are $a'_i, b$

To retrieve the original values, we re-apply the derived formulas:

$\frac{1}{b} + \sum_{i=1}^{N} a'_i = \frac{1}{a_0} \rightarrow a_0 = \frac{b}{1+b\sum_{i=1}^{N} a'_i}$

$a'_i = \frac{a_i}{a_0 b} \rightarrow a_i = a'_i a_0 b$

Finally, we calculate the propagated error formula with the other formulas:

$\frac{\partial a_0}{\partial a'_i} = \frac{-b^2}{(1+b\sum_{j=1}^{N} a'_j)^2} = -(\frac{b}{1+b\sum_{j=1}^{N} a'_j})^2 = -a_0^2$

$\frac{\partial a_0}{\partial b} = \frac{1(1+b\sum_{j=1}^{N} a'_j) - b(\sum_{j=1}^{N} a'_j)}{(1+b\sum_{j=1}^{N} a'_j)^2} = \frac{1}{(1+b\sum_{j=1}^{N} a'_j)^2} = (\frac{a_0}{b})^2$

$\frac{\partial a_i}{\partial a'_i} = 1(a_0 b) + (-a_0)^2(a'_i b) + \frac{-a_i}{a_0 a_i'^2}(a'_i a_0) = a_0 b - a_0 a_i - a_0 b = -a_0 a_i$

$\frac{\partial a_i}{\partial a_0} = \frac{-a_i}{a_0^2 b}(a_0 b) + 1(a'_i b) + (\frac{b}{a_0})^2(a'_i a_0) = -\frac{a_i}{a_0} + \frac{a_i}{a_0} + \frac{b^2}{a_0^2}\frac{a_i}{b} = \frac{a_i b}{a_0^2}$

$\frac{\partial a_i}{\partial b} = \frac{-a_i}{a_0 b^2}(a_0 b) + (\frac{a_0}{b})^2(a'_i b) + 1(a'_i a_0) = -\frac{a_i}{b} + \frac{a_0^2}{b^2}\frac{a_i}{a_0} + \frac{a_i}{b} = \frac{a_i a_0}{b^2}$

$\Delta a_0 = \sqrt{\sum_{j=1}^{N} a_0^4 \Delta a_j'^2 + (\frac{a_0}{b})^4 \Delta b^2} = (\frac{a_0}{b})^2 \sqrt{b^4 \sum_{j=1}^{N} \Delta a_j'^2 + \Delta b^2}$

$\Delta a_i = \sqrt{a_0^2 a_i^2 \Delta a_i'^2 + \frac{a_i^2 b^2}{a_0^4}\Delta a_0^2 + \frac{a_i^2 a_0^2}{b^4}\Delta b^2} =$

$= \sqrt{a_0^2 a_i^2 \Delta a_i'^2 + a_i^2 b^2 \Delta a_i'^2 + \frac{a_i^2}{b^2}\Delta b^2 + \frac{a_i^2 a_0^2}{b^4}\Delta b^2} =$

$= \sqrt{a_i^2(a_0^2 + b^2)\Delta a_i'^2 + \frac{a_i^2}{b^4}(b^2 + a_0^2)\Delta b^2} =$

$= \frac{a_i}{b^2}\sqrt{b^2 + a_0^2}\sqrt{b^4 \Delta a_i'^2 + \Delta b^2}$

# 4 Conditionals

## 4.1 Constant condition and constant implication

Constrain: $b = \begin{cases} V & a < L \\ \bar{V} & a \geq L \end{cases}$

Reworking this formula using Heaviside's step function $H$ and Dirac's $\delta$ unit function:

$b = V + (\bar{V} - V) \cdot H(a - L)$

As $b$ is a categorical variable, $a$ is probably a better behaved one. Therefore, we chose to predict the $H$'s argument:

$a' = a - L$

Henceforth, the variables to be predicted are $a'$.

To retrieve the original values, we re-apply the constrain formulas:

$a = a' + L$

$b = V + (\bar{V} - V) \cdot H(a')$

Finally, we calculate the propagated error formula with the other formulas:

$\frac{\partial a}{\partial a'} = 1$

$\frac{\partial b}{\partial a'} = (\bar{V} - V) \cdot \delta(a')$

Where $\delta$ is modelled as a Gaussian normal distribution with 0 mean and $\Delta a'$ standard deviation.

$\delta \sim \mathcal{N}(0, \Delta a'^2)$

$\Delta a = \sqrt{1 \cdot \Delta a'^2} = \Delta a'$

$\Delta b = \sqrt{(\bar{V} - V)^2 \cdot \delta(a')^2 \cdot \Delta a'^2} = (\bar{V} - V) \cdot \delta(a') \cdot \Delta a'$

### 4.2 Variable condition and constant implication

Constrain: $b = \begin{cases} V & a < l \\ \bar{V} & a \geq l \end{cases}$

Reworking this formula using Heaviside's step function $H$ and Dirac's $\delta$ unit function:

$b = V + (\bar{V} - V) \cdot H(a - l)$

As $b$ is a categorical variable, $a$ is probably a better behaved one. Therefore, we chose to predict the $H$'s argument and the limit's value:

$l' = l$

$d' = a - l$

Henceforth, the variables to be predicted are $l', d'$.

To retrieve the original values, we re-apply the constrain formulas:

$l = l'$

$a = l' + d'$

$b = V + (\bar{V} - V) \cdot H(d')$

Finally, we calculate the propagated error formula with the other formulas:

$\frac{\partial l}{\partial d'} = 0$

$\frac{\partial l}{\partial l'} = 1$

$\frac{\partial a}{\partial l'} = \frac{\partial a}{\partial d'} = 1$

$\frac{\partial b}{\partial l'} = 0$

$\frac{\partial b}{\partial d'} = (\bar{V} - V) \cdot \delta(d')$

Where $\delta$ is modelled as a Gaussian normal distribution with 0 mean and $\Delta a'$ standard deviation.

$\delta \sim \mathcal{N}(0, \Delta a'^2)$

$\Delta l = \sqrt{0 \cdot \Delta d'^2 + 1 \cdot \Delta l'^2} = \Delta l'$

$\Delta a = \sqrt{1 \cdot \Delta d'^2 + 1 \cdot \Delta l'^2} = \sqrt{\Delta d'^2 + \Delta l'^2}$

$\Delta b = \sqrt{(\bar{V} - V)^2 \cdot \delta(d')^2 \cdot \Delta d'^2 + 0 \cdot \Delta l'^2} = (\bar{V} - V) \cdot \delta(d') \cdot \Delta d'$

### 4.3 Constant condition and variable implication

Constrain: $b = \begin{cases} v & a < L \\ \bar{v} & a \geq L \end{cases}$

Reworking this formula using Heaviside's step function $H$ and Dirac's $\delta$ unit function:

$b = v + (\bar{v} - v) \cdot H(a - L)$

As $b$ is a categorical variable, $a$ is probably a better behaved one. Therefore, we chose to predict the $H$'s argument:

$a' = a - L$

Also, the difference between the categorical values are better to be predicted than both values, therefore:

$v' = v$

$h' = \bar{v} - v$

Henceforth, the variables to be predicted are $a', v', h'$.

To retrieve the original values, we re-apply the constrain formulas:

$a = a' + L$

$v = v'$

$\bar{v} = v' + h'$

$b = v' + h' \cdot H(a')$

Finally, we calculate the propagated error formula with the other formulas:

$\frac{\partial a}{\partial a'} = 1$

$\frac{\partial a}{\partial v'} = \frac{\partial a}{\partial h'} = 0$

$\frac{\partial v}{\partial v'} = 1$

$\frac{\partial v}{\partial a'} = \frac{\partial v}{\partial h'} = 0$

$\frac{\partial \bar{v}}{\partial v'} = \frac{\partial \bar{v}}{\partial h'} = 1$

$\frac{\partial \bar{v}}{\partial a'} = 0$

$\frac{\partial b}{\partial a'} = h' \cdot \delta(a')$

$\frac{\partial b}{\partial v'} = 1$

$\frac{\partial b}{\partial h'} = H(a')$

Where $\delta$ is modelled as a Gaussian normal distribution with 0 mean and $\Delta a'$ standard deviation and $H$ it's definite integral from $-\infty$ to $x$.

$\delta \sim \mathcal{N}(0, \Delta a'^2)$

$\Delta a = \sqrt{1 \cdot \Delta a'^2 + 0 \cdot \Delta v'^2 + 0 \cdot \Delta h'^2} = \Delta a'$

$\Delta v = \sqrt{0 \cdot \Delta a'^2 + 1 \cdot \Delta v'^2 + 0 \cdot \Delta h'^2} = \Delta v'$

$\Delta \bar{v} = \sqrt{0 \cdot \Delta a'^2 + 1 \cdot \Delta v'^2 + 1 \cdot \Delta h'^2} = \sqrt{\Delta v'^2 + \Delta h'^2}$

$\Delta b = \sqrt{h'^2 \cdot \delta(a')^2 \cdot \Delta a'^2 + 1 \cdot \Delta v'^2 + H(a')^2 \cdot \Delta h'^2} =$

$= \sqrt{h'^2 \cdot \delta(a')^2 \cdot \Delta a'^2 + \Delta v'^2 + H(a')^2 \cdot \Delta h'^2}$

### 4.4 Variable condition and variable implication

Constrain: $b = \begin{cases} v & a < l \\ \bar{v} & a \geq l \end{cases}$

Reworking this formula using Heaviside's step function $H$ and Dirac's $\delta$ unit function:

$b = v + (\bar{v} - v) \cdot H(a - l)$

As $b$ is a categorical variable, $a$ is probably a better behaved one. Therefore, we chose to predict the $H$'s argument and the limit's value:

$l' = l$

$d' = a - l$

Also, the difference between the categorical values are better to be predicted than both values, therefore:

$v' = v$

$h' = \bar{v} - v$

Henceforth, the variables to be predicted are $l', d', v', h'$.

To retrieve the original values, we re-apply the constrain formulas:

$l = l'$

$a = d' + l'$

$v = v'$

$\bar{v} = v' + h'$

$b = v' + h' \cdot H(d')$

Finally, we calculate the propagated error formula with the other formulas:

$\frac{\partial l}{\partial l'} = 1$

$\frac{\partial l}{\partial d'} = \frac{\partial l}{\partial v'} = \frac{\partial l}{\partial h'} = 0$

$\frac{\partial a}{\partial l'} = \frac{\partial a}{\partial d'} = 1$

$\frac{\partial a}{\partial v'} = \frac{\partial a}{\partial h'} = 0$

$\frac{\partial v}{\partial v'} = 1$

$\frac{\partial v}{\partial d'} = \frac{\partial v}{\partial l'} = \frac{\partial v}{\partial h'} = 0$

$\frac{\partial \bar{v}}{\partial v'} = \frac{\partial \bar{v}}{\partial h'} = 1$

$\frac{\partial \bar{v}}{\partial d'} = \frac{\partial \bar{v}}{\partial l'} = 0$

$\frac{\partial b}{\partial l'} = 0$

$\frac{\partial b}{\partial d'} = h' \cdot \delta(d')$

$\frac{\partial b}{\partial v'} = 1$

$\frac{\partial b}{\partial h'} = H(d')$

Where $\delta$ is modelled as a Gaussian normal distribution with 0 mean and $\Delta a'$ standard deviation and $H$ it's definite integral from $-\infty$ to $x$.

$\delta \sim \mathcal{N}(0, \Delta a'^2)$

$\Delta l = \sqrt{1 \cdot \Delta l'^2 + 0 \cdot \Delta d'^2 + 0 \cdot \Delta v'^2 + 0 \cdot \Delta h'^2} = \Delta l'$

$\Delta a = \sqrt{1 \cdot \Delta l'^2 + 1 \cdot \Delta d'^2 + 0 \cdot \Delta v'^2 + 0 \cdot \Delta h'^2} = \sqrt{\Delta l'^2 + \Delta d'^2}$

$\Delta v = \sqrt{0 \cdot \Delta l'^2 + 0 \cdot \Delta d'^2 + 1 \cdot \Delta v'^2 + 0 \cdot \Delta h'^2} = \Delta v'$

$\Delta \bar{v} = \sqrt{0 \cdot \Delta l'^2 + 0 \cdot \Delta d'^2 + 1 \cdot \Delta v'^2 + 1 \cdot \Delta h'^2} = \sqrt{\Delta v'^2 + \Delta h'^2}$

$\Delta b = \sqrt{0 \cdot \Delta l'^2 + h'^2 \cdot \delta(d')^2 \cdot \Delta d'^2 + 1 \cdot \Delta v'^2 + H(d')^2 \cdot \Delta h'^2} =$
$= \sqrt{h'^2 \cdot \delta(d')^2 \cdot \Delta d'^2 + \Delta v'^2 + H(d')^2 \cdot \Delta h'^2}$