

Integración

Problema 22.a

Se tienen los siguientes datos:

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$\vec{r}' = a\cos\phi\hat{i} - a\sin\phi\hat{j}$$

$$dl' = ad\phi$$

$$L_0 = 0$$

$$L = 2\pi$$

$$\lambda = \frac{Q}{2\pi a}$$

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int_{L_0}^L \frac{\lambda(\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3} dl'$$

Se desarrollan las ecuaciones:

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0} \frac{Q}{2\pi a} \int_0^{2\pi} \frac{(\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3} a d\phi = \frac{1}{4\pi\epsilon_0} \frac{Q}{2\pi} \int_0^{2\pi} \frac{(x\hat{i} + y\hat{j} + z\hat{k} - (a\cos\phi\hat{i} - a\sin\phi\hat{j}))}{|x\hat{i} + y\hat{j} + z\hat{k} - a\cos\phi\hat{i} + a\sin\phi\hat{j}|^3} d\phi$$

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0} \frac{Q}{2\pi} \int_0^{2\pi} \frac{(x\hat{i} + y\hat{j} + z\hat{k} - (a\cos\phi\hat{i} - a\sin\phi\hat{j}))}{(\sqrt{(x - a\cos\phi)^2 + (y + a\cos\phi)^2 + (z)^2})^3} d\phi$$

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0} \frac{Q}{2\pi} \int_0^{2\pi} \frac{(x\hat{i} + y\hat{j} + z\hat{k} - (a\cos\phi\hat{i} - a\sin\phi\hat{j}))}{(x^2 + y^2 + z^2 - 2ax\sin\phi - 2ay\cos\phi + a^2\cos^2\phi + a^2\sin^2\phi)^{3/2}} d\phi$$

Se utiliza la identidad pitagórica:

$$a^2 \cos^2 \phi + a^2 \sin^2 \phi = a^2 (\sin^2 \phi + \cos^2 \phi) = a^2$$

Luego, se hace la integral para cada componente de $\vec{E}(\vec{r})$:

$$\begin{aligned}\vec{E}(x, y, z) = & \frac{1}{4\pi\epsilon_0} \frac{Q}{2\pi} \left(\int_0^{2\pi} \frac{(x - a \cos \phi)}{(x^2 + y^2 + z^2 + a^2 - 2ax \sin \phi - 2ax \cos \phi)^{3/2}} \hat{i} d\phi \right. \\ & + \int_0^{2\pi} \frac{(y - a \sin \phi)}{(x^2 + y^2 + z^2 + a^2 - 2ax \sin \phi - 2ax \cos \phi)^{3/2}} \hat{j} d\phi \\ & \left. + \int_0^{2\pi} \frac{z}{(x^2 + y^2 + z^2 + a^2 - 2ax \sin \phi - 2ax \cos \phi)^{3/2}} \hat{k} d\phi \right)\end{aligned}$$

$$\vec{E}_x(x, y, z) = \frac{1}{4\pi\epsilon_0} \frac{Q}{2\pi} \int_0^{2\pi} \frac{(x - a \cos \phi)}{(x^2 + y^2 + z^2 + a^2 - 2ax \sin \phi - 2ax \cos \phi)^{3/2}} d\phi$$

$$\vec{E}_y(x, y, z) = \frac{1}{4\pi\epsilon_0} \frac{Q}{2\pi} \int_0^{2\pi} \frac{(y - a \sin \phi)}{(x^2 + y^2 + z^2 + a^2 - 2ax \sin \phi - 2ax \cos \phi)^{3/2}} d\phi$$

$$\vec{E}_z(x, y, z) = \frac{1}{4\pi\epsilon_0} \frac{Q}{2\pi} \int_0^{2\pi} \frac{z}{(x^2 + y^2 + z^2 + a^2 - 2ax \sin \phi - 2ax \cos \phi)^{3/2}} d\phi$$