Portofoliu matematica

clasele IX-XII

Asimptote

1. Verticale:

Ex: $f(-\infty, 0) \cup (1, \infty) \to \mathbb{R}$

- Pentru a determina asimptota verticala facem limitele la capetele finite ($\neq \infty$ / $-\infty$):

$$\left.\begin{array}{l}\lim\limits_{x \to 0} f(x) \\ \lim\limits_{x \to 1} f(x)\end{array}\right\}$$

Daca sunt egale cu ∞ sau $-\infty$ atunci spunem ca x = 0 respectiv x = 1 sunt asimptote verticale la stanga respectiv dreapta

Ecuatia asimptotei: $(y = x)$

2. Orizontale:

Ex: $f(-\infty, 0) \cup (1, \infty) \to \mathbb{R}$

- Pentru a determina asimptota orizontala facem limitele la capetele infinite (= ∞ / $-\infty$):

$$\left.\begin{array}{l}\lim\limits_{x \to \infty} f(x) \\ \lim\limits_{x \to -\infty} f(x)\end{array}\right\}$$

Daca sunt finite atunci spunem ca y = $\lim\limits_{x \to \infty} f(x)$ respectiv y = $\lim\limits_{x \to -\infty} f(x)$ sunt asimptote orizontale

Ecuatia asimptotei: $(y = \lim f(x))$

Daca sunt asymptote orizontale nu exista asymptote oblice **!!!**

3. Oblice:

Ex: $f(-\infty, 0) \cup (1, \infty) \to \mathbb{R}$

- Pentru a determina asimptota oblica procedam astfel:

$$m = \lim_{x \to \infty} \frac{f(x)}{x} \qquad\qquad n = \lim_{x \to \infty} (f(x) - mx)$$

Daca n este finit atunci spunem ca exista asimptote oblice

Ecuatia asimptotei: $(y = mx + n)$

Combinatorica

I.Permutari:

$P_n = n!$ $0! = 1$

II.Aranjamente:

$A_n^k = \frac{n!}{(n-k)!}$ ➡ ORDONATE (conteaza ordinea)

$n \geq k$

III.Combinari:

$C_n^k = \frac{n!}{k!(n-k)!}$ ➡ NEORDONATE (nu conteaza ordinea)

$C_n^1 = n$ $C_n^0 = 1$

Proprietati:

- $C_n^k = C_n^{n-k}$ $- combinari\ complementare$
- $C_n^k = C_{n-1}^k + C_{n-1}^{k-1}$ $- formula\ de\ recurenta$
- $C_n^0 + C_n^1 + C_n^2 + \cdots + C_n^{n-1} + C_n^n = 2^n$
- $C_n^1 + C_n^2 + \cdots + C_n^{n-1} + C_n^n = 2^n - 1$
- $C_n^0 + C_n^2 + C_n^4 + \cdots + C_n^{n-2} + C_n^n = 2^{n-1}(n - par)$
- $C_n^1 + C_n^3 + C_n^5 + \cdots + C_n^{n-3} + C_n^{n-1} = 2^{n-1}$
- $C_n^1 + 2C_n^2 + 3C_n^3 \ldots + nC_n^n = n * 2^n - 1$
- $C_n^k = \frac{n}{k} * C_{n-1}^{k-1}$

IV.Binomul lui Newton:

$$(a+b)^n = \sum_{k=0}^{n} C_n^k * a^{n-k} * b^k$$

- Termenul general:
  - $T_{k+1} = C_n^k * a^{n-k} * b^k$

!!! (Al treilea termen = T_3)

- Sa fie termen rational:
  - Faci T_{k+1}
  - Se pune conditie la radicalii de putere k

- Cel mai mare termen al dezvoltarii ($(a+b)^n$):
  - $\frac{T_{k+2}}{T_{k+1}} = \frac{n-k}{k+1} * \frac{b}{a}$
  - Scriem: $\frac{n-k}{k+1} * \frac{b}{a} > 1$
  - Gasim:

$$k \geq a+1 \rightarrow T_{k+2} < T_{k+1}$$
$$sau$$
$$k \geq a \quad\quad \rightarrow T_{k+2} > T_{k+1}$$

  - T_{a+2} = cel mai mare termen

Compunerea functiilor

Fie $f: A \to B,\ g: B \to C$ *functia* $gof: A \to C$

$(gof)(x) = g(f(x))$ - s.n. *“g compus cu f”*

Derivare si Continuitate

Continuitatea:

$$\lim_{\substack{x \to x_0 \\ x > x_0}} f(x) = \lim_{\substack{x \to x_0 \\ x < x_0}} f(x) = f(x_0) \;\; => f \; continua \; in \; x_0$$

(Daca 2 ↑ sunt diferite si finite => discontinuitate de speta I)

(Daca una din limite e ∞ sau ∄ => discontinuitate de speta II)

Continuitatea pe o multime:

$$f - cont \; in \; A \;\; \Leftrightarrow f \; cont \; in \; \forall \; x \in A$$

!!! Toate functiile elementare sunt continue pe domeniul lor (Inclusiv functiile de gradul I si II)

$$f \; continua \to f \; are \; proprietatea \; lui \; Darboux$$

$$f(a) * f(b) < 0 => \exists \; c \in (a, b) \;\; a.i. \;\; f(c) = 0$$

Derivabilitatea in punctul x_0:

$$\lim_{\substack{x \to x_0 \\ x > x_0}} \frac{f(x) - f(x_0)}{x - x_0} \quad si \quad \lim_{\substack{x \to x_0 \\ x < x_0}} \frac{f(x) - f(x_0)}{x - x_0}$$

sa ∃ si sa fie FINITE si daca sunt egale

f este derivabila in x_0

sau

$$\lim_{\substack{x \to x_0 \\ x > x_0}} f'(x) \quad si \quad \lim_{\substack{x \to x_0 \\ x < x_0}} f'(x)$$

sa ∃ si sa fie FINITE si daca sunt egale ($f's(x_0) = f'd(x_0)$)

Teorema lui Lagrange:

$$f:[a,b] \to \mathbb{R}$$

$$\left.\begin{matrix} a)\ f-continua\ pe\ [a,b] \\ b)\ f-derivabila\ pe\ (a,b) \end{matrix}\right\} => \exists\ c \in (a,b);$$

$$f(b) - f(a) = f'(c) * (b-a)$$

Derivatele Functiilor elementare:

- $c - constanta \qquad c' = 0$
- $(x^n)' = n * x^{n-1}$
- $\left(\sqrt{x}\right)' = \frac{1}{2\sqrt{x}}$
- $\left(\sqrt[n]{x}\right)' = \frac{1}{n\sqrt[n]{x^{n-1}}}$
- $(\ln x)' = \frac{1}{x}$
- $(a^x)' = a^x * \ln a \qquad (e^x)' = e^x$
- $\left(\frac{1}{x}\right)' = -\frac{1}{x^2}$
- $\sin' x = \cos x \qquad \cos' x = -\sin x$
- $tg'x = \frac{1}{cos^2 x} = 1 + tg^2 x$
- $ctg'x = -\frac{1}{sin^2 x} = -1 - ctg^2 x$
- $(\arcsin x)' = \frac{1}{\sqrt{1-x^2}}$
- $(\arccos x)' = -\frac{1}{\sqrt{1-x^2}}$
- $(\mathrm{arctg}\, x)' = \frac{1}{1+x^2}$
- $(\mathrm{arcctg}\, x)' = -\frac{1}{1+x^2}$
- Suma

$$(f+g)'(x) = f'(x) + g'(x)$$

- Produs

$$(f*g)'(x) = f'(x) * g(x) + f(x) * g'(x)$$

- Impartire

$$\left(\frac{f}{g}\right)'(x) = \frac{f'(x) * g(x) - f(x) * g'(x)}{g^2(x)}$$

$$f', f'', f''', \ldots, f^{(n)} - derivate\ de\ ordin\ superior$$

- $f''(x) = f'(f'(x))$

Formula lui Leibniz (derivarea unui produs de fct)

$$(f * g)^{(n)} = \sum_{k=0}^{n} C_n^k * f^{(n-k)} * g^k$$

Teorema lui Rolle:

$$\begin{matrix} functia\ este\ continua\ pe\ interval\ (si\ in\ x_0) \\ functia\ este\ derivabila\ in\ x_0 \end{matrix} \overset{T.R.}{\Longrightarrow} \exists\ c \in D$$

$$a.i.\ \ f'(c) = 0$$

Sirul lui Rolle:

- *se foloseste pentru numarearea solutiilor reale ale unei ecuatii de tip: f(x)=0*

Pasi:

1. $se\ identifica\ f\ si\ D_f$
2. $se\ calculeaza\ f'(x)\ si\ se\ stabileste\ D_{f'}$
3. $se\ rezolva\ f'(x) = 0\ cu\ sol\ x_1, x_2, \ldots\ , x_n$
4. $se\ calculeaza$: $\begin{matrix} f(x_1), f(x_2), \ldots, f(x_n) \\ limitele\ la\ catpetele\ D_f \end{matrix}$
5. $se\ face\ tabelul\ cu\ toate\ rez\ de\ la\ punctul\ anterior$
6. $se\ interpreteza\ tabelul$:
   - *Unde apar doua semne alaturate diferite (-/+) avem o solutie pe acel interval*
   - *Unde in loc de +/- avem 0 acolo avem o solutie dubla (doua solutii egale)*

Teorema lui Fermat:

$f: [a, b] \to \mathbb{R}\ si\ x_0 \in (a, b)\ , x_0 \to pct. de\ extrem\ al$

$functiei\ f \overset{T.F.}{\Longrightarrow} f'(x_0) = 0\ (derivata\ in\ pct\ de\ extrem\ e\ 0)$

Aplicare:

Se da o inegalitate. Se cere determinarea unei constante:

1. Se trece totul intr-o parte, se noteaza totul cu f si se reformuleaza ca o problema de max si min
2. Se identifica punctele de max si min (extrem)
3. Se aplica T.F. (egalam f derivat de pct de extrem cu 0)
4. Se afla constanta

Rolul derivatei a doua:

$$f: [a, b] \to \mathbb{R}, a < b:$$

$$f - continua\ pe\ [a, b]$$

$$f - derivabila\ de\ doua\ ori\ pe\ (a, b)$$

$$=> \begin{cases} f'' \geq 0 \quad => functia\ f\ este\ convexa\ pe\ [a, b] \\ f'' \leq 0 \quad => functia\ f\ este\ concava\ pe\ [a, b] \end{cases}$$

Drepte în plan

Panta unei drepte: $m_d = \quad -\frac{a}{b}$

Ecuatia unei drepte:

$$d\colon \; ax + by + c = 0$$

$$d\colon \; y = mx + n$$

Panta

$d_1 || d_2 \;\; => \;\; m_{d_1} = m_{d_2}$

$d_1 \perp d_2 \;\; => \;\; m_{d_1} * m_{d_2} = -1$

1) Ecuatia dreptei care trece prin punctele $A(x_A, y_A)$ si $B(x_B, y_B)$:

$$AB\colon \; \frac{x - x_A}{x_B - x_A} = \frac{y - y_A}{y_B - y_A}$$

$$m_{AB} = \frac{y_b - y_a}{x_b - x_a}$$

Valoarea Dreaptei:

$$AB = \sqrt{(x_B - x_A)^2 + (y_B - y_A)^2}$$

2) Ecuatia dreptei determinata de un punct $A(x_A, y_A)$ si panta m_d:

$$d_1(A, m_d) : \quad y - y_A = m_d(x - x_A)$$

3) Centrul de greutate al unui ΔABC:

$c.gr\ \Delta ABC = \{G\}$

$$x_G = \frac{x_A + x_B + x_C}{3} \qquad y_G = \frac{y_A + y_B + y_C}{3}$$

4) Mijlocul unui segment *AB* :

$$mijl[AB] = \{M\}$$

$$x_M = \frac{x_A + x_B}{2} \qquad y_M = \frac{y_A + y_B}{2}$$

5) Intersectia a doua drepte:

Faci sistem de ecuatii $\begin{cases} d_1: \ a_1 * x + b_1 * y + c_1 \\ d_2: \ a_2 * x + b_2 * y + c_2 \end{cases}$

6) Distanta de la un punct la o dreapta:

$$d: a * x + b * y + c \qquad A(x_A, y_A)$$

$$d'(d, A) = \frac{|a*x_A+b*y_A+c|}{\sqrt{a^2+b^2}}$$

7) Simetricul unui punct fata de un punct:

$$C = sim_B A$$

$$x_C = 2 * x_B - x_A \qquad y_C = 2 * y_B - y_A$$

8) Simetricul unui punct fata de o dreapta(se cunosc ec. dreptei si coordonatele punctului A):

$$B = sim_d A$$

Calcule: $\begin{cases} -Ec.\, dreptei\ AM \\ -\{M\} = d \cap AM \\ -M = mijl.\, [AB] \end{cases}$

9) Simetricul unei drepte fata de un punct:

$$d' = sim_A d \qquad se\ cunosc\ ecuatia\ dreptei\ d\ si\ coordonatele\ punctului\ A$$

Calcule: $\begin{cases} -Alegem\ B(x_B,y_B)\in d \\ -determinam\ C=sim_A B \\ -m*d'=m*d \quad (d'||d) \\ -scriem\ ecuatia\ d'(cu\ ajutorul\ c,m*d') \end{cases}$

10) Simetricul unei drepte fata de o dreapta:

$$d_2 = sim_d d_1 \quad se\ cunosc\ ec.\ dreptelor\ d, d_1$$

Calcule: $\begin{cases} -determinam\ \{P\}=d_1\cap d_2 \\ alegem\ A\in d_1 \\ -determinam\ A'=sim_{d_1} A \\ -scriem\ ecuatia\ PA'(d_2) \end{cases}$

11)Distanta de la o dreapta la o dreapta (d_1, d_2):

$$M(x_0, y_0) \in d_1\ sa\ verifice\ ec\ lui\ d_1$$
$$apoi\ calculam\ d(M, d_2)$$

12)Tangenta dintre doua drepte:

$$tg\left(\widehat{d_1, d_2}\right) = \frac{m_1 - m_2}{1 + m_1 m_2}$$

Ec. Exponențiale – Formule

I. De tipul $a^{f(x)} + a^{g(x)} = b$:

Se da factor comun $a^{k\,x}$

II. De tipul $a^{f(x)} = b$:

=>$f(x) = \log_a b$

III. De tipul $A\,a^{2f(x)} + B\,a^{f(x)} + C = 0$:

Notam $a^{f(x)} = t\,, t > 0$
$A\,t^2 + B\,t + C = 0$
…

IV. De tipul $A\,a^{2f(x)} + B\,(a\,b)^{f(x)} + C\,b^{2f(x)} = 0$:
Impartim cu $\Big| : b^{2f(x)}$
$A\,(\frac{a}{b})^{2f(x)} + B\,(\frac{a}{b})^{f(x)} + C = 0$
Notam $(\frac{a}{b})^{f(x)} = t\,, t > 0$
…

V. De tipul $A\left(a + b\sqrt{d}\right)^{f(x)} + B\left(a - b\sqrt{d}\right)^{f(x)} = C$:

$\left(a + b\sqrt{d}\right)\left(a - b\sqrt{d}\right) = 1$
Notam $(a + b\sqrt{d})^{f(x)} = t\,, t > 0$
=> $\left(a - b\sqrt{d}\right)^{f(x)} = \frac{1}{t}$
…

VI. Ec. cu descompunere in factori

VII. De tipul $f(x)^{g(x)} = f(x)^{h(x)}$:

I. $g(x) = h(x)\,, f(x) > 0$
II. $f(x) = 1$
III. $f(x) = 0\,, g(x) > 0\,, h(x) > 0$
IV. $f(x) = -1\,, Cu\ verificare!!!$

VIII. Ec. cu solutie unica:

- *Verificam sa nu existe termini cu semn negativ* $(-)$
- *Aducem exponentialele la accelasi exponent*
- *Impartim cu cel mai mare* $a^{f(x)}$
- *Identificam solutia unica* (x)

!!! $x^2 + x + 1 = 0$

$$=> x^3 - 1 = (x-1)\underbrace{(x^2 + x + 1)}_{=\,0}$$

$$=> x^3 = 1$$

$$x^n \begin{cases} x^0, n\ de\ forma\ 3k + 0 \\ x^1, n\ de\ forma\ 3k + 1 \\ x^2, n\ de\ forma\ 3k + 2 \end{cases}$$

<u>Schema lui Horner :</u> *(pentru polinoamele cu grad ≥ 3)*

$$ax^3 + bx^2 + cx + d = 0$$

Valorile lui x cu care incercam	x^3	x^3	x^3	x^3	Trebuie sa dea 0
	a	b	c	d	
x_1		a	$(x_1 * a) + b$	...	

!!! $e^x \geq x + 1 \quad \forall\, x \in \mathbb{R}$

Ec. Irationale – Formule

$\sqrt[n]{a}$ radicalul este **definit** pentru **a>=0** (daca **n** este **par**)

$\sqrt{a^2}$=|a| $\sqrt[n]{a*b}=\sqrt[n]{a}*\sqrt[n]{b}$ $\sqrt[n]{\frac{a}{b}}=\frac{\sqrt[n]{a}}{\sqrt[n]{b}}$ $\sqrt[n]{a^m}=a^{\frac{m}{n}}$

$\sqrt[m]{\sqrt[n]{a}}=\sqrt[m*n]{a}$ $(\sqrt[n]{a})^m=\sqrt[n]{a^m}$

Formula radicalilor compusi:

$$\sqrt{a+\sqrt{b}}=\sqrt{\frac{a+\sqrt{a^2-b}}{2}}+\sqrt{\frac{a-\sqrt{a^2-b}}{2}}$$

$$\sqrt{a-\sqrt{b}}=\sqrt{\frac{a+\sqrt{a^2-b}}{2}}-\sqrt{\frac{a-\sqrt{a^2-b}}{2}}$$

Numitorul	Formula utilizata	Cu ce amplificam	Num. dupa amplificare
$\sqrt{a}+/-\sqrt{b}$	$x^2-y^2=(x-y)(x+y)$	$\sqrt{a}-/+\sqrt{b}$	a-b
$\sqrt[n]{a^k}$	-	$\sqrt[n]{a^{n-k}}$	a
$\sqrt[3]{a}+\sqrt[3]{b}$	$x^3+y^3=(x+y)(x^2-xy+y^2)$	$\sqrt[3]{a^2}-\sqrt[3]{ab}+\sqrt[3]{b^2}$	a+b
$\sqrt[3]{a^2}-\sqrt[3]{ab}+\sqrt[3]{b^2}$	$x^3+y^3=(x+y)(x^2-xy+y^2)$	$\sqrt[3]{a}+\sqrt[3]{b}$	a+b
$\sqrt[3]{a}-\sqrt[3]{b}$	$x^3-y^3=(x-y)(x^2+xy+y^2)$	$\sqrt[3]{a^2}+\sqrt[3]{ab}+\sqrt[3]{b^2}$	a-b
$\sqrt[3]{a^2}+\sqrt[3]{ab}+\sqrt[3]{b^2}$	$x^3-y^3=(x-y)(x^2+xy+y^2)$	$\sqrt[3]{a}-\sqrt[3]{b}$	a-b
$\sqrt[n]{a}-\sqrt[n]{b}$	$x^n-y^n=(x-y)(x^{n-1}+x^{n-2}y+\ldots+y^{n-1})$	$\sqrt[n]{a^{n-1}}+\ldots+\sqrt[n]{b^{n-1}}$	a-b

Ec. Logaritmice – Formule

argument

$\log_a b = x \quad \Longleftrightarrow \quad a^x = b$

bază

$\log_a x + \log_a y = \log_a(xy)$

$\log_a x - \log_a y = \log_a(\frac{x}{y})$

$\log_a(x^n) = n \log_a x$!!!

$\log_a x = \frac{\log_b x}{\log_b a}$

$a^{\log_b c} = c^{\log_b a}$

!!! *Un logaritm este definit dacă:*

$$\log_a b \qquad Conditii \begin{cases} a > 0 \\ a \neq 1 \\ b > 0 \end{cases}$$

I. *De tipul $log_{f(x)}\, g(x) = a$:*

=>$g(x) = f(x)^a$

II. *De tipul $log_{f(x)}\, g(x) = log_{f(x)}\, h(x)$:*

=>$g(x) = h(x)$

III. *Ec cu logaritmi in baze diferite:*

Se aduc la acceiasi bază

IV. *Ec cu logaritmi in baze diferite:*

V. *Ec exponențial-logaritmice:*

VI. *Ec cu logaritmi cu soluție unică:*

Primul logaritm se notează cu a;

Al doilea logaritm se notează cu b;

Se exprima x-ul in funcție de a si de b;

…

Ec. Trigonometrice – Formule

I. Fundamentale:

sin x = a
cos x = a
$\Big\}\ a \in [-1,1]$
tg x = a
ctg x = a

sin(x) = a $=>$ x=$(-1)^k \arcsin(a) + k\pi$
cos(x) = a $=>$ x=$\pm \arccos(a) + 2k\pi$
tg(x) = a $=>$ x=$\text{arctg}(a) + k\pi$
ctg(x) = a $=>$ x=$\text{arcctg}(a) + k$

!!! - sin(x) = sin(-x)
- cos(x) = cos($\pi \pm x$)

arcsin(-x) = -arcsinx
arccos(-x) = π-arccosx
arctg(-x) = -arctgx
arcctg(-x) = π-arcctgx

Ecuatia cos intr-un triunghi:

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

Ecuatia sin intr-un triunghi:

$$\sin A = \frac{a}{2R} \qquad R - raza\ cercului\ format\ de\ mijloacele\ laturilor$$

II. De forma: sin f(x)=sin g(x), cos f(x)=cos g(x), tg f(x)=tg g(x), …

sin(x) = sin(a) $=>$ x=$(-1)^k$a $+ k\pi$
cos(x) = cos(a) $=>$ x=$\pm$ a $+ 2k\pi$
tg(x) = tg(a) $=>$ x= a $+ k\pi$
ctg(x) = ctg(a) $=>$ x= a $+ k\pi$

III. Care se reduc la **gradul 2** sau **3**:

cos 2x = $cos^2 x - sin^2 x$
= $2cos^2 x - 1$
=$1 - 2sin^2 x$

sin 2x = $2 \sin x\ cos\ x$
sin 3x = $3 \sin x - 4sin^3 x$
cos 3x = $4cos^3 x - 3 \cos x$

$tg\ 2x = \frac{2tg\ x}{1-tg^2x}$

$tg\ (x \pm y) = \frac{tg\ x \pm tg\ y}{1 \pm tg\ x\ tg\ y}$

$ctg\ x = \frac{1}{tg\ x}$

$tg\left(\frac{\pi}{2} + y\right) = -\frac{1}{tg\ y}$

IV. Omogene:

$$a\ sin^2x + b \sin x \cos x + c\ cos^2x = 0 \qquad :cos^2x$$

$$a\ tg^2x + b\ tg\ x + c = 0$$

...

V. Aproape omogene:

$$a\ sin^2x + b \sin x \cos x + c\ cos^2x = d$$

$$d * 1 = d * (sin^2x + cos^2x)$$

⇨ ECUATIE omogena

VI. Liniare:

<u>M1:</u> $a \sin x + b \cos x = c \qquad : \sqrt{a^2 + b^2}$

<u>M2:</u> $\sin x = \frac{2\ tg\frac{x}{2}}{1+tg^2\frac{x}{2}} \qquad \cos x = \frac{1-tg^2\frac{x}{2}}{1+tg^2\frac{x}{2}}$

!!! La M2 verificam la final daca $x = (2k+1)\pi$ este solutie

$$\sin(2k+1)\pi = 0$$

$$\cos(2k+1)\pi = -1$$

VII. Simetrice:

$$a(\sin x + \cos x) + b \sin x \cos x + c = 0$$

$$\sin x + \cos x = t \quad /^2$$

...

VIII. Ce contin: $sin^{2n}x + cos^{2n}x$

$$sin^4x + cos^4x = 1 - 2(\frac{\sin 2x}{2})^2$$

$$sin^6x + cos^6x = 1 - 3(\frac{\sin 2x}{2})^2$$

IX. Ce contin **patrate** de sin si cos:

$$sin^2x = \frac{1-\cos 2x}{2} \qquad cos^2x = \frac{\cos 2x+1}{2}$$

$$\cos a + \cos b = 2\cos\frac{a+b}{2}\cos\frac{a-b}{2}$$

$$\cos a - \cos b == 2\sin\frac{a+b}{2}\sin\frac{a-b}{2}$$

$$\sin a + \sin b = 2\sin\frac{a+b}{2}\cos\frac{a-b}{2}$$

$$\sin a - \sin b = 2\sin\frac{a-b}{2}\cos\frac{a+b}{2}$$

X. Ce contin **produse** de sin si cos:

$$\sin a * \cos b = \frac{\sin(a+b) + \sin(a-b)}{2}$$

$$\cos a * \cos b = \frac{\cos(a+b) + \cos(a-b)}{2}$$

$$\sin a * \sin b = \frac{\cos(a-b) - \cos(a+b)}{2}$$

$$\sin(a+b) = \sin a\cos b + \sin b\cos a \qquad \sin x = \cos(\frac{\pi}{2} - x)$$

$$\sin(a-b) = \sin a\cos b - \sin b\cos a \qquad \cos x = \sin(\frac{\pi}{2} - x)$$

$\cos(a+b) = \cos a \cos b - \sin a \sin b$ $\qquad$ $sin^2 x + cos^2 x = 1$

$\cos(a-b) = \cos a \cos b + \sin a \sin b$ $\qquad$ $ctg\ x = \frac{1}{tg\ x}$

VIII. Suma de sin:

$$S = \sin x + \sin 2x + \sin 3x + \cdots + \sin nx$$

- Inmultim cu sin de jumatate de ratie $(\frac{x}{2})$

$$\sin\frac{x}{2} S = \sin\frac{x}{2}\sin x + \cdots$$

- Scriem cu formula de inmultire de sin
- ...

- Arcsin / sin :

$\arcsin(-x) = -\arcsin(x)$

0	1	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$
0	$\frac{\pi}{2}$	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$

$$\sin : \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \to [-1,1]$$

$$\arcsin: [-1,1] \to \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

$\sin(\arcsin x) = x$ $\qquad$ $\arcsin(\sin x) = x$

- Arccos / cos :

$\arccos(-x) = \pi - \arccos(x)$

0	1	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$
$\boldsymbol{\frac{\pi}{2}}$	0	$\frac{\pi}{3}$	$\frac{\pi}{4}$	$\frac{\pi}{6}$

$$\cos:[0,\pi] \rightarrow [-1,1]$$

$$\arccos: [-1,1] \rightarrow [0,\pi]$$

$\cos(\arccos x) = x$ $\qquad$ $\arccos(\cos x) = x$

$arctg(\infty) = \frac{\pi}{2}$ $\qquad$ $arctg(-\infty) = -\frac{\pi}{2}$

$arcctg(\infty) = 0$ $\qquad$ $arcctg(-\infty) = \pi$

- Teorema medianei:

$$m_A{}^2 = \frac{2(b^2 + c^2) - a^2}{4}$$

Forma trigonometrica

$$z = r(\cos t + i \sin t)$$

Forma algebrica → Forma trigonometrica

$z = x + iy$ $\qquad$ $r = \sqrt{x^2 + y^2}$

$$t = arctg\frac{y}{x} + k\pi\ , k = \begin{cases} 0, r \in C1 \\ 1, r \in C2, C3 \\ 2, r \in C4 \end{cases}$$

Operatii cu forma trigonometrica:

1. Inmultire:

$z_1 = r_1(\cos t_1 + i \sin t_1)$

$z_2 = r_2(\cos t_2 + i \sin t_2)$

$z_1 * z_2 = r_1 * r_2(\cos t_1 * \cos t_2 - \sin t_1 \sin t_2) + i(\cos t_1 * \cos t_2 + \sin t_1 \sin t_2)$

2. Inversul:

$z' = \frac{1}{z} = \frac{1}{r}(\cos(-t) + i \sin(-t))$

3. Impartire:

$\frac{z_1}{z_2} = \frac{r_1}{r_2}(\cos(t_1 - t_2) + i \sin(t_1 - t_2))$

4. Ridicare la putere:

$z^n = r^n(\cos(n * t) + i \sin(n * t))$

Functii surjective și injective

$$f : A \to B$$

Domeniu   Codomeniu

I. Injectivitate:

M1:

$\exists\, x_1, x_2 \in A \quad \text{a.î.} \quad f(x_1) \neq f(x_2) \;=>\; x_1 \neq x_2$

Arătăm că: $f(x_1) = f(x_2) \;=>\; x_1 = x_2$

"Fals" => f este injectivă

M2:

Aratam ca functia este strict monotona (cu ajutorul derivatei)

II. Surjectivitate:

M1:

Orice paralela la Ox dusa printr-un punct al co-domeniului taie graficul in **cel putin un punct**

M2:

$\forall\, y \in B(Codomeniu)\;;\; \exists\, x \in A(Domeniu)\;; f(x) = y$

Se exprimă ***x*** în funcție de ***y*** , și se verifică dacă îi aparține lui **A** (Domeniului).

M3:

Facem tabelul cu f(x) x si f'(x). Apoi stabilim Imf dupa tabel facnd limitele la capete si f(x) unde derivate e 0 si daca Imf e egala cu domeniul atunci functia este surj

III. Bijectivitate:

Cand o funcție este atat injectivă cat și surjectivă !

La funcțiile pe ramuri:

$$f: A \cup B \to C \qquad f(x)\begin{cases} f_1(x), x \in A \\ f_2(x), x \in B \end{cases}$$

$$f - surj \Longleftarrow Imf_1 \cup Imf_2 = C$$

$$f - inj \Longleftarrow \begin{cases} f_1, f_2 - inj \\ Imf_1 \cap Imf_2 = \emptyset \end{cases}$$

Functia de gradul doi

$$f(x) = ax^2 + bx + c$$

- <u>In tabel :</u>

semnul lui a 0 semnul contrar lui a 0 semnul lui a

- <u>Parabola (varful):</u>

$$V\left(-\frac{b}{2a}, -\frac{\Delta}{4a}\right) \qquad V(x_v, y_v)$$

- <u>Valoarea minima:</u>

$$a > 0 \; si \; este \; -\frac{\Delta}{4a} \; pentru \;\; x = -\frac{b}{2a}$$

- <u>Valoarea maxima:</u>

$$a < 0 \;\; si \; este \; -\frac{\Delta}{4a} \; pentru \;\; x = -\frac{b}{2a}$$

- <u>Monotonia functiei de gradul doi:</u>

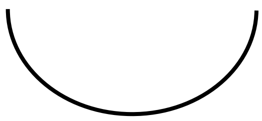
$$\begin{cases} \Delta > 0 \;\; => strict \;\downarrow \\ \quad \Delta \geq 0 \;\; => \;\downarrow \\ \Delta < 0 \;\; => strict \;\uparrow \\ \quad \Delta \leq 0 \;\; => \;\uparrow \end{cases}$$

- !!!Pentru ca functia sa se anuleze intr-un singur punct $\Delta = 0$
- Vf parabolelor sa fie pe:
  - O dreapta (y_v sa fie functie de gradul 1)
  - O parabola (y_v sa fie functie de gradul 2)
- Vf parabolelor se afla pe:
  - Prima bisectoare $x_v = y_v$
  - A doua bisectoare $x_v = -y_v$
  - Dreapta … (Inlocuim x si y cu x_v si y_v)
- Parabolele trec prin cel putin un punct fix

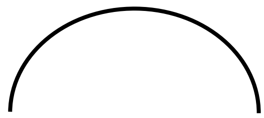
- Il scoatem pe *a* factor comun si coeficientii acestuia ii egalam cu *0*

- Sa se determine curba pe care se gasesc varfurile parabolelor
  - Exprimam a in functie de x si il inlocuim in ecuatia lui y
- Parbolele a doua functii:
  - sa se interecteze in doua puncta distincte $\begin{cases} f(x) = g(x) \\ \Delta\ la\ \uparrow > 0 \end{cases}$
  - au un singur punct comun $\begin{cases} f(x) = g(x) \\ \Delta\ la\ \uparrow = 0 \end{cases}$
- Grafic:

a>0



a<0



- Marginirea:
  - Inferior => $\exists\ m\ ;\ \ m \leq f(x)$
  - Superior => $\exists\ M\ ;\ \ M \geq f(x)$

Inductia matematica

$p(n)$:

$Presupunem\ ca\ p(n)\ este$ "Adevarat"

$si\ demonstram\ ca\ p(n+1)\ este\ "Adevarat"$

$p(n)$:

$p(n+1)$:

$(-)/(\div)$

Integrale

$$F'(x) = f(x) \qquad\qquad F(x) = \int f(x)$$

$F - primitiva\ lui\ f\ (antiderivata)$

$f - derivata\ lui\ F$

!!! *O functie sa admita primitive => functia sa fie continua pe D (sa nu aibe discontinuitatii de speta 1)*

1) Integrare prin parti:

$$\int f'(x) * g(x)dx = f(x) * g(x) - \int f(x) * (g(x))'dx$$

$$(f * g)' = f'(x) * g(x) + f(x) * g'(x)$$

$$\left(\frac{f}{g}\right)' = \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)}$$

2) Formule de recurenta:

I =

Trebuie sa ajungem sa scriem I_n in functie de $I_{n-1}, I_{n+2}, \ldots$

3) Integrare prin schimbare de variabila:

Notam cu u = f(x) astfel incat (f(x))' sa ne apara in ecuatie si scriem I(x) in fuctie de u I(u)

x = t |'

dx = dt

4) Integrarea functiilor rationale oarecare:

Ex: $\frac{3x^2+x+1}{2x+1} = \frac{A}{x-2} + \frac{B}{(x-2)^2} + \frac{C}{2x+1}$

Le aflam pe cele cu puterea ce mai mare: $(x-2)^2$ *si* $(2x+1)$

$B = \frac{3x^2+x+1}{2x+1}$ *si inlocuim cu x din ecuatia* $(x-2)^2 = 0 \;\; => \;\; x = 2$

...

5) Impartirea polinoamelor:

Ex: $f = 2x^5 - 3x^4 + 7x^3 - x + 2$

$g = x^2 - x - 1$

f/g:

1. Ne uitam la puterea cea mai mare a lui x

$$
\begin{array}{r|l}
2x^5 - 3x^4 + 7x^3 - x + 2 & x^2 - x - 1 \\ \cline{2-2}
-2x^5 + 2x^4 + 2x^3 & 2x^3 - x^2 + 8x + 7 \\ \hline
-x^4 + 9x^3 - x - 2 & \\
\dots & \\ \hline
14x + 9 &
\end{array}
$$

2. Alegem cu ce inmultim

Nr. crt.	Funcția	Mulțimea primitivelor (integrala nedefinită)
1.	$f:\mathbb{R}\to\mathbb{R},\ f(x)=x^n,\ n\in\mathbb{N}$	$\int x^n dx=\frac{x^{n+1}}{n+1}+\mathscr{C}$
2.	$f:I\to\mathbb{R},\ f(x)=x^r,\ I\subset(0,+\infty)$, $r\in\mathbb{R}\setminus\{-1\}$	$\int x^r dx=\frac{x^{r+1}}{r+1}+\mathscr{C}$
3.	$f:\mathbb{R}\to\mathbb{R},\ f(x)=a^x,\ a>0,\ a\neq 1$	$\int a^x dx=\frac{a^x}{\ln a}+\mathscr{C}$
4.	$f:I\to\mathbb{R},\ f(x)=\frac{1}{x},\ I\subset\mathbb{R}^*$	$\int\frac{1}{x}dx=\ln\lvert x\rvert+\mathscr{C}$
5.	$f:I\to\mathbb{R},\ f(x)=\frac{1}{x^2-a^2}$, $I\subset\mathbb{R}\setminus\{\pm a\},\ a\neq 0$	$\int\frac{1}{x^2-a^2}dx=\frac{1}{2a}\ln\left\lvert\frac{x-a}{x+a}\right\rvert+\mathscr{C}$
6.	$f:\mathbb{R}\to\mathbb{R},\ f(x)=\frac{1}{x^2+a^2},\ a\neq 0$	$\int\frac{1}{x^2+a^2}dx=\frac{1}{a}\operatorname{arctg}\frac{x}{a}+\mathscr{C}$
7.	$f:\mathbb{R}\to\mathbb{R},\ f(x)=\sin x$	$\int\sin x\,dx=-\cos x+\mathscr{C}$
8.	$f:\mathbb{R}\to\mathbb{R},\ f(x)=\cos x$	$\int\cos x\,dx=\sin x+\mathscr{C}$
9.	$f:I\to\mathbb{R},\ f(x)=\operatorname{tg}x$, $I\subset\mathbb{R}\setminus\left\{(2k+1)\frac{\pi}{2}\,\middle\vert\,k\in\mathbb{Z}\right\}$	$\int\operatorname{tg}x\,dx=-\ln\lvert\cos x\rvert+\mathscr{C}$
10.	$f:I\to\mathbb{R},\ f(x)=\operatorname{ctg}x$, $I\subset\mathbb{R}\setminus\{k\pi\mid k\in\mathbb{Z}\}$	$\int\operatorname{ctg}x\,dx=\ln\lvert\sin x\rvert+\mathscr{C}$
11.	$f:I\to\mathbb{R},\ f(x)=\frac{1}{\cos^2x}$, $I\subset\mathbb{R}\setminus\left\{(2k+1)\frac{\pi}{2}\,\middle\vert\,k\in\mathbb{Z}\right\}$	$\int\frac{1}{\cos^2x}dx=\operatorname{tg}x+\mathscr{C}$
12.	$f:I\to\mathbb{R},\ f(x)=\frac{1}{\sin^2x}$, $I\subset\mathbb{R}\setminus\{k\pi\mid k\in\mathbb{Z}\}$	$\int\frac{1}{\sin^2x}dx=-\operatorname{ctg}x+\mathscr{C}$
13.	$f:I\to\mathbb{R},\ f(x)=\frac{1}{\sqrt{a^2-x^2}}$, $I\subset(-a,a),\ a>0$	$\int\frac{1}{\sqrt{a^2-x^2}}dx=\arcsin\frac{x}{a}+\mathscr{C}$
14.	$f:I\to\mathbb{R},f(x)=\frac{1}{\sqrt{x^2-a^2}}$, $I\subset(-\infty,-a)$ sau $I\subset(a,+\infty)$	$\int\frac{1}{\sqrt{x^2-a^2}}dx=\ln\left\lvert x+\sqrt{x^2-a^2}\right\rvert+\mathscr{C}$
15.	$f:\mathbb{R}\to\mathbb{R},\ f(x)=\frac{1}{\sqrt{x^2+a^2}}$, $a\neq 0$	$\int\frac{1}{\sqrt{x^2+a^2}}dx=\ln\left(x+\sqrt{x^2+a^2}\right)+\mathscr{C}$

Legi de compozitie

$$(G, \text{"-"} sau \text{"+"} sau \text{"*"} sau \text{"o"})$$

1. Parte stabila:

$$\forall\, x, y \in G => (x + sau * \ldots y) \in G$$

2. Comutativitate:

$$x * y = y * x \quad \forall\, x, y \in G$$

3. Asociativitate:

$$(x * y) * z = x * (y * z) \quad \forall\, x, y, z \in G$$

$$(La\ matrici\ rezulta\ din\ propr.inmultirii\ matricelor)$$

4. Elementul neutru:

$$\exists\, e \in G \quad a.i. \quad x * e = e * x = x \quad \forall\, x \in G$$

$$(Daca\ avem\ matrice\ el\ neutru\ il\ cautam\ I_n)$$

5. Elemental simetrizabil:

$$\forall\, x \in G\,, \exists\, x' \in G \quad x * x' = x' * x = e$$

$$(Simetricul\ este\ unic)$$

Lege sa fie:

i. MONOID:
- Legea sa fie asociativa
- ∃ element neutru

ii. GRUP:
- Legea este asociativa
- ∃ element neutru
- ∃ element simetrizabil

iii. GRUP ABELIAN (GRUP COMUTATIV):
- Legea este comutativa

- Legea este asociativa
- ∃ element neutru
- ∃ element simetrizabil

Morfisme si izomorfisme:

$$(G_1,*)\,,(G_2,+) - morfism$$

$$f(x*y) = f(x) + f(y)$$

Tabla unei legi de compozitie:

x	a_1	a_2	...	a_n
a_1				
a_2			$a_i * a_j$	
...				
a_n				

Concluzii din tabla:

- *Parte stabila: toate rezultatele din table sunt elemente ale lui G*
- *Comutativitate: table este simetrica fata de diagonala principala*
- *Elementul neutru: unde linia $a_1, a_2, \ldots\ , a_n$ se regaseste in table*
- *Elementul simetrizabil: daca gasim pe linia a_i elementul neutru*

Limite

Exemplu: $x = \frac{2n+1}{3n+1}$ => $\lim x = \frac{2}{3}$

Cazuri de NEDETERMINARE: $\begin{cases} 0 * \pm\infty \\ +\infty + (-\infty) \\ 1^{\infty} \\ 0^{0} \\ \infty^{0} \\ \frac{0}{0} \\ \frac{\pm\infty}{\pm\infty} \end{cases}$

1) Limita unui polinom:

$$\lim(a_k n^k + a_{k-1} n^{k-1} + \cdots + a_1 n + a_0) = \lim a_k n^k$$

n la puterea cea mai mare

2) Limita unui cat de polinom:

$$\lim \frac{(a_k n^k + a_{k-1} n^{k-1} + \cdots + a_1 n + a_0)}{(b_l n^l + b_{l-1} n^{l-1} + \cdots + b_1 n + b_0)} = \lim \frac{a_k n^k}{b_l n^l}$$

$$= \lim \frac{a_k}{b_l} * n^{k-l} \begin{cases} 0 & , k < l \\ \frac{a_k}{b_l} & , k = l \\ \frac{a_k}{b_l} (\pm\infty)^{k-l} & , k > l \end{cases}$$

3) Limita unei puteri:

$$\lim a^n = \begin{cases} \infty, & a > 1 \\ 1, & a = 1 \\ 0, & a \in (-1,1) \\ \nexists, & a = -1 \\ \nexists, & a < -1 \end{cases}$$

Scoti factor pe a^n, a cel mai mare

Daca avem $a^n + 7^n$ atunci luam pe cazuri $\begin{cases} a > 7 \\ a < 7 \\ a = 7 \end{cases}$

4) Limitele unor sume/produse:

Cu formulele:

$$\sum_{k=1}^{n} k = 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$$

$$\sum_{k=1}^{n} k^2 = 1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{k=1}^{n} k^3 = 1^3 + 2^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}$$

5) Limitele unor siruri cu radicali:

Daca avem:

$\lim\left(\sqrt{a} - b\right) \to$ rationalizam

$\lim\left(\sqrt{a} + b\right) \to$ scoatem factor comun

!!! $\lim\left(\sqrt[3]{n^3 + \cdots} - \sqrt[2]{n^2 + \cdots}\right) = \lim\left(\sqrt[3]{n^3 + \cdots} - n\right) + \lim\left(n - \sqrt[2]{n^2 + \cdots}\right)$

6) Limite remarcabile:

$$\lim_{n\to\infty}\frac{\sin x_n}{x_n}=1\ \ ,x_n\to tinde\ la\ 0$$

$$\lim_{n\to\infty}\frac{\arcsin x_n}{x_n}=1\ \ ,x_n\to tinde\ la\ 0$$

$$\lim_{n\to\infty}\frac{\operatorname{tg} x_n}{x_n}=1\ \ ,x_n\to tinde\ la\ 0$$

$$\lim_{n\to\infty}\frac{\operatorname{arctg} x_n}{x_n}=1\ \ ,x_n\to tinde\ la\ 0$$

Numarul e:

$$\lim_{n\to\infty}(1+x_n)^{\frac{1}{x_n}}=e\quad ,\qquad (1+x_n)^{\frac{1}{x_n}}\to tinde\ la\ 1^{\infty}$$

$$\lim_{n\to\infty}\frac{a^{x_n-1}}{x_n}=\ln a\quad sau\quad \lim_{n\to\infty}\frac{a^{\frac{1}{x_n}-1}}{\frac{1}{x_n}}=\ln a$$

$$\lim_{n\to\infty}\frac{\ln(1+x_n)}{x_n}=1\qquad\qquad \lim_{n\to\infty}\frac{\ln y_n}{y_n-1}=0$$

$$\lim_{n\to\infty}\frac{n^k}{a^k}=0\ \ ,\ \ a>1\quad k\in\mathbb{N}$$

$$(\ln 1=0)$$

7) Criteriul Clestelui:

La o suma:

- Incardam fiecare termen al sumei intre cel mai mare si cel mai mic termen al sumei:

Cel mai mic (T1) < T1 < Cel mai mare (Tn)

Cel mai mic < T2 < Cel mai mare

...

Cel mai mic < Tn < Cel mai mare

- Facem suma
- Obtinem:

Cel mai mic *n < suma < Cel mai mare *n

$\boxed{\text{Tinde la ...}}$

!!! Partea intreaga o incadram intre:

$$x - 1 < [x] \leq x$$

8) Cesaro-Stolz:

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{a_{n+1} - a_n}{b_{n+1} - b_n} \quad Doar\ daca\ b_n \rightarrow> 0, nemarg.superior$$

9) Cauchy-d'Alembert (criteriul radicalului):

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \frac{a_n + 1}{a_n} \qquad a_n > 0$$

10) Criteriul raportului:

$$\lim_{n\to\infty} \frac{a_n + 1}{a_n} = l \qquad \begin{array}{l} l > 1 \ => \ \lim = \infty \\ l \in [0,1] => \ \lim = 0 \end{array}$$

11) L'Hospital:

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} \overset{\frac{0}{0}}{\underset{\frac{\infty}{\infty}}{=}} \lim_{x \to x_0} \frac{f'(x)}{g'(x)}$$

Cazuri de nedeterminare:

- $0 * \infty = f * g = \frac{f}{\frac{1}{g}}\ sau\ \frac{g}{\frac{1}{f}} + l'hospital$
- $1^{\infty}, 0^{0}, \infty^{0} = f^{g} = e^{g \ln f} + \lim g \ln f$
- $\infty - \infty = f - g =$ $\begin{matrix} Calcule\ daca\ merg \\ sau \\ f\left(1 - \frac{g}{f}\right) + \lim \frac{g}{f} \end{matrix}$

!!!

$$a_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \ln n \Rightarrow \lim_{n \to \infty} a_n = c = 0{,}47$$

Constanta lui Euler

Limite de functii

$$\lim\big(f(x) + g(x)\big) = \lim f(x) + \lim\ g(x)$$

$$\lim\big(f(x) * g(x)\big) = \lim f(x) * \lim\ g(x)$$

$$\lim\left(\frac{f(x)}{g(x)}\right) = \frac{\lim f(x)}{\lim\ g(x)}$$

$$\lim\left(f(x)^{g(x)}\right) = \lim f(x)^{\lim\ g(x)}$$

$$\lim |f(x)| = |\lim f(x)|$$

- Daca un sir are limita:
  - $finita \Rightarrow sirul\ este\ \boldsymbol{CONVERGENT}$
  - $infinita \Rightarrow sirul\ este\ \boldsymbol{DIVERGENT}$

Matrice

$$A = \begin{matrix} a_{11} & a_{12} & \dots & a_{n1} \\ a_{21} & a_{22} & \dots & a_{n2} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{matrix}$$

$$1 \leq i \leq n$$
$$1 \leq j \leq n$$

1) $m = 1 => Matricea\ linie$

$$A = (a_{11}\ a_{12} \dots\ a_{1n})$$

2) $n = 1 => Matricea\ coloana$

$$A = \begin{matrix} a_{11} \\ a_{21} \\ \dots \\ a_{m1} \end{matrix}$$

3) $m = n => matrice\ patratica$

Adunare:

$$0 + A = A + 0 = A \qquad A + (-A) = -A + A = 0$$

Inmultire:

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad B = \begin{pmatrix} e & f \\ g & h \end{pmatrix}$$

$$A * B = \begin{pmatrix} a * e + b * g & a * f + b * h \\ c * e + d * g & c * f + d * h \end{pmatrix}$$

$$A * B \neq B * A \qquad (A * B) * C = A * (B * C)$$

El.neutru:

$$I_n = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 \end{pmatrix} \qquad A * I = I * A$$

Calculul lui A^n:

$A^n = se\ calculeaza\ A^1, A^2, A^3, \dots$ *pana se observam o regula*

$$Daca\ A = \begin{pmatrix} \cos x & \pm \sin x \\ \mp \sin x & \cos x \end{pmatrix} => \ A^n = \begin{pmatrix} \cos nx & \pm \sin nx \\ \mp \sin nx & \cos nx \end{pmatrix}$$

sau cu ***BINOMUL LUI NEWTON***: $\quad A = I + B$

$A^n = (I + B)^n = \sum_{k=0}^{n} C_n^k I^{n-k} B^k$

Relatia Canley-Hamilton:

$Tr(x) = a + d \qquad\qquad !\ (Tr(x))^2 = -a \ => \ Tr(x) = \pm i\sqrt{a}$

$\det(x) = a * d - b * c$

$A = B \quad => \quad \det A = \det B$

$$x^2 - Tr(x) * x + det\ x * I_2 = O_2$$

Determinanti:

1. de ordin 2:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = a * d - b * c$$

2. de ordin 3:

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a * e * i + b * f * g + d * h * c - g * e * c -$$

$$-a * f * h - d * b * i$$

3. de ordin n:

-*are* $\boldsymbol{n}!$ *termeni*

-*face zerouri prin adunarea si scaderea L si C*

-*dezvoltarea dupa o linie / o coloana*:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix} = a_{21}(-1)^{2+1} * \begin{vmatrix} a_{12} & a_{13} & a_{14} \\ a_{32} & a_{33} & a_{34} \\ a_{42} & a_{43} & a_{44} \end{vmatrix}$$

Transpusa unei matrici:

$$A = \begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix} \qquad {}^tA = \begin{pmatrix} a & d \\ b & e \\ c & f \end{pmatrix} \qquad a_{ij} \rightarrow a_{ji}$$

Modulul

Proprietatiile modulului:

- $|x| = |-x| = x$ $\qquad\qquad$ $|x| = 0 => x = 0$
- $|x * y| = |x| * |y|$ $\qquad\qquad$ $\left|\frac{x}{y}\right| = \frac{|x|}{|y|}$
- $\big||x| - |y|\big| \leq |x + y| \leq |x| + |y|$

Explicitarea modulului:

$$|x| = \begin{cases} x \;, & x \geq 0 \\ -x \;, & x < 0 \end{cases}$$

Numere complexe – Formule

$$z = a + bi = (a, b)$$

Conjugatul: $\bar{z} = a - bi$

- $i^2 = -i$

  $z = 0 => a = 0, b = 0$

  $z_1 = z_2 => a_1 = a, b_1 = b_2$

  $\overline{z} = z => z \in \mathbb{R}$

  $\overline{z} = -z => z \in i\mathbb{R}$

  $\overline{\overline{z}} = z$

- $|z| = \sqrt{a^2 + b^2}$

  $|z| = 1 => \overline{z} = \frac{1}{z}$

  $|z|^2 = z * \overline{z}$

- $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$

  $\overline{z_1 * z_2} = \overline{z_1} * \overline{z_2}$

  $\overline{z_1 - z_2} = \overline{z_1} - \overline{z_2}$

  $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z_1}}{\overline{z_2}}$

- $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2(|z_1|^2 + |z_2|^2)$

  $|z_1 + z_2|^2 = (z_1 + z_2)(\overline{z_1} + \overline{z_2})$

- Ecuația: $\underbrace{i + i^2 + i^3 + i^4}_{=0} + i^5 + \cdots + i^{100} \quad = \mathbf{0}$

  $i * i^2 * i^3 * i^4 * i^5 * \ldots * i^{100} \quad = \mathbf{-1}$

- Ecuația de gradul doi:

  $Dacă\ \Delta < 0 \quad => \quad x_{1,2} = \dfrac{-b \pm i\sqrt{-\Delta}}{2a}$

Ecuația de gradul doi cu coeficienți complexși

$$\Delta = u^2 \qquad x_{1,2} = \frac{-b \pm u}{2a}$$

Adunare: $(a, b) + (c, d) = (a + c, b + d)$

- comutativa
- asociativa
- el neutru (0,0)
- opus $(a, b) + (-a, -b) = (0,0)$

Inmultire: $(a, b) * (c, d) = (ac - bd, ad + bc)$

- comutativa
- asociativa
- el neutru (1,0)
- opus $(a, b) * (a', b') = (1,0)$

Radaciniile de ordinal *n* ale unui numar complex:

$$z^n = a = r(\cos t + i \sin t)$$

$$z_k = \sqrt[n]{r}\left(\cos\frac{t+2k\pi}{n} + i\sin\frac{t+2k\pi}{n}\right) \quad ; \quad k = \overline{0, n-1}$$

EX: $\sqrt[3]{8} = 2$

Parte intreaga, parte fractionara

Partea intreaga:

$$[x] - cel\ mai\ mare\ numar\ INTREG\ mai\ mic\ decat\ x$$

$$[x] \leq x < [x] + 1$$

Partea fractionara:

$$\{x\} = x - [x]$$

Proprietati:

- $[x + n] = [x] + n \qquad \forall\, n \in \mathbb{Z}$
- $[x] + \left[x + \frac{1}{2}\right] = [2x]$
- $[x] + \left[x + \frac{1}{n}\right] + \left[x + \frac{2}{n}\right] + \cdots + \left[x + \frac{n-1}{n}\right] = [nx] \qquad \forall\, n \in \mathbb{N}^*$

Progresii

1. Aritmetice:

$$a_n = a_{n-1} + r$$

Termenul general: $a_n = a_1 + (n-1) * r$

Suma: $S_n = \frac{(a_n + a_1) * n}{2}$

!!! $(1 + 3 + 5 + \cdots + x)$ => $x = a_n$

1) $a_1, a_2, a_3 \in \div \Rightarrow a_2 = \frac{a_1 + a_3}{2}$

2) Ex: $S_n = \div$? (n>=2)

- Notam $a_n = S_n - S_{n-1}$
- Scriem S_n in functie de n
- Aflam a_n apoi a_1
- Verificam daca a_1, a_2, a_3 sunt in $\div$
- Verificam daca $a_n - a_{n-1}$ este constant

3) $a_1 + a_n = a_2 + a_{n-1} = \cdots$

Progresii aritmetice – tipuri de exercitii

1. "Sa se determine primul termen si ratia"

Ex: $\begin{cases} a_4 - a_2 = 6 \\ a_3 - a_1 = 3 \end{cases}$

- Se noteaza fiecare termen cu formula termenului general si apoi se scad sau se aduna cele doua ecuatii

2. "O progresie aritmetica verifica relatiile"

Ex: $\begin{cases} a_3 + a_{10} + a_{11} = 15 \\ a_9 * a_{10} * a_{11} = 120 \end{cases}$

- Se noteaza a_9 si a_{11} in functie de a_{10}
- Afli r si a_{10} si apoi a_1

"Calculati suma primilor 20 de termeni"

- Se calculeaza S_{20} folosind r si a_1

3. *"Fie a,b,c termeni de rang l,m,n al unei p.a. Sa se calculeze E = … "*

Ex: $E = (m - n)a + (n - l)b + (l - m)c$

- Se noteaza a,b si c in functie de termenul general
- In ecuatie ii dam factori pe m,n si l
- Ne folosim de a,b si c si le inlocuim in parantezele din ecuatie

4. *"Pentru ce valori este sirul o progresie aritmetica"*

Ex: $S_n = an^3 + bn^2 + cn + d$

- Calculam $a_n = S_n - S_{n-1}$
- Punem conditia $a_n - a_{n-1}$ sa fie constant (coeficientul la n sa fie *0*)

5. *"Demonstrati ca numerele nu pot fi termeni ai unei progresii aritmetice"*

Ex: $\sqrt{3}, 6, \sqrt{35}$

- Notam numerele ca termenii a_m, a_n si a_k (m,n,k naturale)
- Scadem ecuatiile intre ele si apoi le impartim sa obtinem o fractie = cu o fractie fiind de naturi diferite ($\in Q\ = \notin Q$)

6. *"a_n progresie aritmetica, calculati sumele"*

Ex: $S = a_1a_2 + a_2a_3 + \cdots + a_{n-1}a_n$

$$S = \frac{1}{a_1a_2} + \frac{1}{a_2a_3} + \cdots + \frac{1}{a_{n-1}a_n}$$

$$S = S_1 + S_2 + S_3 + \cdots + S_n$$

- Notam $a_n a_{n+1}$ *(n>=2)* ca

$$\frac{a_{n-1}a_n a_{n+1} - a_n a_{n+1} a_{n+2}}{-3r\ (care\ este\ defapt\ a_{n-1} - a_{n+2} = -3r\ )}$$

- Notam $\frac{1}{a_n a_{n+1}}$ (n>=1) ca $\left(\frac{1}{a_n} - \frac{1}{a_{n+1}}\right)\frac{1}{r}$

7. "*Demonstrati ca numerele nu pot fi termeni ai unei progresii aritmetice*"

Ex: $\sqrt{3}, 6, \sqrt{35}$

2. Geometrice:

$$b_n = b_{n-1} * q$$

Termenul general: $b_n = b_1 * q^{n-1}$

Progresia: $P_n = b_1 * \frac{q^n - 1}{q - 1}$

1) $b_1, b_2, b_3 \in$ ÷÷ $\Rightarrow \begin{matrix} {b_2}^2 = b_1 * b_3 \\ |b_2| = \sqrt{b_1 * b_3} \end{matrix}$

2) Ex: $P_n =$ ÷÷ ? (n>=2)
   - Notam $b_n = P_n - P_{n-1}$
   - Aflam b_n apoi b_1
   - Verificam daca b_1, b_2, b_3 sunt in ÷÷
   - Verificam daca b_n / b_{n-1} este constant

3) $b_1 * b_n = b_2 * b_{n-1} = \cdots$

S = $\frac{1}{a} + \frac{1}{a^2} + \frac{1}{a^3} + \frac{1}{a^4} + \cdots + \frac{1}{a^n}$

$\frac{1}{a} S = \cdots$

Progresii geometrice – tipuri de exercitii

1. *“Sa se determine primul termen si ratia”*

   Ex: $\begin{cases} a_4 - a_2 = 6 \\ a_3 - a_1 = 3 \end{cases}$

   - Se noteaza fiecare termen cu formula termenului general
   - Se impart ecuatiile

2. *“Fie doua progresii una aritmetica una geometrica, fiecare cu cate*

4 termeni, ai adunand termenii de accelasi rang sa se obtina ca suma 18,18,26,58. Aflati ratia progresiei geometrice”

3. *“Aratati ca un sir este progresie geometrica”*

   Aratam ca $\frac{b_{n+1}}{b_n}$ este constanta $\forall n \geq 1$

Inegalitatea Cauchy - Bunikovski - Schwarz:

$$(a_1^2 + a_2^2 + \cdots + a_n^2)(b_1^2 + b_2^2 + \cdots + b_n^2) \geq (a_1 b_1 + a_2 b_2 + \cdots + a_n b_n)^2$$

Daca: $\frac{a_1}{b_1} = \frac{a_2}{b_2} = \cdots = \frac{a_n}{b_n}$

Sisteme de ecuatii in Z_n

$$ex:\begin{cases} 1.\ x+\hat{3}y+z=\hat{0} \\ 2.\ x+y+\hat{2}z=\hat{4} \\ 3.\ \hat{2}x+\hat{3}y+\hat{2}z=\hat{0} \end{cases} \qquad \mathbb{Z}_5$$

$$\Delta=\begin{vmatrix} \hat{1} & \hat{3} & \hat{1} \\ \hat{1} & \hat{1} & \hat{2} \\ \hat{2} & \hat{3} & \hat{2} \end{vmatrix}=a \quad , \qquad daca\ \Delta\ este\ inversabil$$

$$\Delta_x=\begin{vmatrix} \hat{0} & \hat{3} & \hat{1} \\ \hat{4} & \hat{1} & \hat{2} \\ \hat{0} & \hat{3} & \hat{2} \end{vmatrix}=c \quad , \quad x=c*a^{-1}$$

$$\Delta_y=\cdots$$

$$\Delta_z=\cdots$$

Vectori în plan

$$\vec{v} = x\vec{i} + y\vec{j} \qquad sau \qquad \vec{v}(x, y)$$

$\vec{u} + \vec{v} = (x_1 + x_2)\vec{i} + (y_1 + y_2)\vec{j}$

$\vec{u} - \vec{v} = (x_1 - x_2)\vec{i} + (y_1 - y_2)\vec{j}$

$\vec{u} * \vec{v} = |\vec{u}| * |\vec{v}| * \cos{(\vec{u}, \vec{v})}$

sau

$\vec{u} * \vec{v} = (x_1 * x_2) + (y_1 * y_2)$

$|\vec{u}| = \sqrt{x_1^2 + y_1^2}$

$\alpha * \vec{u} = (\alpha * x_1)\vec{i} + (\alpha * y_1)\vec{j}$

$\overrightarrow{AB}(x_B - x_A, y_B - y_A) \qquad \left|\overrightarrow{AB}\right| = \sqrt{(x_B - x_A)^2 + (y_B - y_A)^2}$

1) Unghiul dintre doi vectori:

$\vec{v} = xi + yj \qquad \vec{u} = x'i + y'j$

$$\cos(\vec{u}, \vec{v}) = \frac{xx' + yy'}{\sqrt{x^2 + y^2} * \sqrt{x'^2 + y'^2}}$$

!!! $\cos(\vec{u}, \vec{v}) < 0 => unghiul\ este\ obtuz$

$\cos(\vec{u}, \vec{v}) > 0 => unghiul\ este\ ascutit$

2) 2 vectori sa fie colineari:

$$\vec{u}\ si\ \vec{v} <=> \frac{x}{x'} = \frac{y}{y'}$$

3) 2 vectori sa fie perpendicular (ortogonali):

$$\vec{u} \perp \vec{v} <=> x * x' + y * y' = 0$$

4) Coordonatele punctului care imparte un segment intr-un raport dat k:

$$\overrightarrow{AB} = k * \overrightarrow{MB} => \begin{cases} x_M = \dfrac{x_A + k * x_B}{1 + k} \\ y_M = \dfrac{y_A + k * y_B}{1 + k} \end{cases}$$