DSML Lab Assignment № 1

Problem 1: The Predator-Prey Model

The Lotka-Volterra model describes the interaction between two species in an isolated environment: one acting as a predator and the other as prey. The model is given by the following system of differential equations:

$$\frac{dx}{dt} = \alpha x - \beta xy$$
$$\frac{dy}{dt} = -\gamma y + \delta xy$$

where:

- $x(t)$ is the number of preys (e.g., rabbits) at time t.
- $y(t)$ is the number of predators (e.g., foxes) at time t.
- $\alpha, \beta, \gamma,$ and δ are positive constants that describe the interaction between the two species.

(a) Find and classify all equilibrium points of the system.

(b) Linearize the system around each equilibrium point and determine the type and stability of each equilibrium using the Jacobian matrix.

(c) Discuss the biological interpretation of each equilibrium point and its stability. For instance, what does it mean for the ecosystem if one of the equilibrium points is stable?

(d) Plot the phase portrait of the system. Describe the trajectories for a few initial conditions.

(e) Introduce a carrying capacity K for the prey, such that their growth rate slows as their population approaches K. Modify the differential equation for $x(t)$ to incorporate this effect and discuss its implications for the stability of the system.

Problem 2: The Lorenz System

The Lorenz system is a set of differential equations originally developed as a model for atmospheric convection. It is given by:

$$\frac{dx}{dt} = \sigma(y - x)$$
$$\frac{dy}{dt} = x(\rho - z) - y$$
$$\frac{dz}{dt} = xy - \beta z$$

where $x(t)$, $y(t)$, and $z(t)$ are the state variables, and σ, ρ, and β are positive constants. This system exhibits chaotic behavior for certain parameter values.

(a) Find the equilibrium points of the Lorenz system.

(b) For the parameter values $\sigma = 10$, $\beta = \frac{8}{3}$, and $\rho = 28$, linearize the system around each equilibrium point. Determine the nature of each equilibrium point (saddle point, spiral source, etc.) by examining the eigenvalues of the Jacobian matrix.

(c) Plot a trajectory of the Lorenz system for the above parameter values starting from an initial condition near one of the equilibrium points. Describe the nature of the trajectory. Is the motion predictable or chaotic?

(d) Experiment by varying the parameter ρ while keeping σ and β fixed. Discuss how the behavior of the system changes as ρ is varied.

(e) The Lorenz system is known for its "butterfly" attractor. Illustrate and explain this phenomenon based on the trajectory plot from part (c).

Problem 3*: Hopfield Neural Network

Consider a continuous-time Hopfield network of N neurons. The dynamics of the i^{th} neuron is described by the following differential equation:

$$\tau \frac{dV_i}{dt} = -V_i + \sum_{j=1}^{N} W_{ij} \tanh(V_j)$$

Where:

- τ is the time constant.
- V_i is the potential of the i^{th} neuron.
- W_{ij} is the synaptic weight between the i^{th} and j^{th} neuron.

(a) Discuss the stability of the trivial equilibrium point $V_i = 0$ for all i. Does the network remain near this equilibrium point for small initial deviations? Assume that the matrix W is symmetric (i.e., $W_{ij} = W_{ji}$) and has zero diagonal elements (i.e., $W_{ii} = 0$).

(b) Consider the case where $N = 2$. Choose some representative weights W_{ij} and plot the trajectories of V_1 and V_2 starting from various initial conditions. Identify and discuss any stable or unstable equilibrium points you observe.

(c) Increase the size of the network to $N = 3$ and choose another set of representative weights for W_{ij}. Examine the system for multistability (existence of several stable equilibrium points) by choosing different initial conditions. Report your observations.

(d) Using the configuration from (c), introduce a small periodic perturbation to one of the neuron's potentials. Observe and describe the dynamics. Do you see evidence of complex or possibly chaotic behavior in the network's response? You may also consider asymmetric weights.

(e) General Discussion: Reflect on the implications of stability, multistability, and chaos in neural networks, especially in the context of associative memory retrieval and pattern recognition. How might the observed dynamics be beneficial or detrimental in real-world applications of neural networks?