DSML Lab Assignment № 2

Problem 1: Tent map

For $c \in (0, 2]$, consider the tent map $T_c : [0, 1] \to [0, 1]$ defined by

$$T_c(x) = \begin{cases} cx & , \ 0 \leq x < \frac{1}{2} \\ c(1-x) & , \ \frac{1}{2} \leq x < 1 \end{cases}$$

a. For $c = 2$, sketch graphical iterations (cobweb diagram) for the initial conditions $x_0 \in \{\frac{1}{4}, \frac{1}{6}, \frac{5}{7}, \frac{1}{19}\}$. Find the fixed points and the periodic cycles of period 2, 3, and 4. Give a formula for the number of periodic points of period N.

b. For $c = \frac{3}{2}$, plot the graphs of $T(x)$, $T^2(x) = (T \circ T)(x)$ and $T^3(x) = (T \circ T \circ T)(x)$ and determine the number of fixed points and periodic points of period 2 and 3.

c. Plot a bifurcation diagram for the tent map, considering the bifurcation parameter $c \in (0, 2]$.

Problem 2: Logistic map

Consider the logistic map defined by

$$f_\mu(x) = \mu x(1-x).$$

a. For $\mu = 4$, plot the graphs of the functions $f_\mu(x)$, $f_\mu^2(x)$, $f_\mu^3(x)$ and $f_\mu^4(x)$. Determine the number of fixed points and periodic points of period 2, 3 and 4.

b. Plot a bifurcation diagram for the logistic map, with the bifurcation parameter $\mu \in (0, 4]$.

c. Consider the dynamical system $x_{n+1} = f_\mu(x_n)$ which may be used to model the population of a certain species of insect. Given that the population size periodically alternates between two distinct values, determine a value of μ that would be consistent with this behavior. Determine an equation that gives the points of period two for a general μ value.

Problem 3: Henon map

Consider the Henon map in $\mathbb{R}^2$ given by

$$\begin{cases} x_{n+1} = 1 - \alpha x_n^2 + y_n \\ y_{n+1} = \beta x_n \end{cases}$$

where $\alpha > 0$ and $|\beta| < 1$.

a. Determine the fixed points and investigate if they are attractive, repulsive, or saddle points.

b. Show that the Henon map undergoes a bifurcation from a fixed point to period-two cycle behavior exactly when $\alpha = \frac{3}{4}(\beta - 1)^2$, for fixed β.

c. Investigate the bifurcation diagrams for the Henon map by plotting the x_n values as a function of α for $\beta = 0.4$.

Problem 4: Mandelbrot sets

Plot the Mandelbrot sets for the mappings

a. $z_{n+1} = z_n^2 + c$

b. $z_{n+1} = z_n^3 + c$

c. $z_{n+1} = z_n^4 + c$

Problem 5*: Fractal dimension

The goal of this problem is to estimate the fractal dimension of a high-dimensional dataset.
Step 1. Generate High-Dimensional Synthetic Data: Create a synthetic dataset with 1000 points in 10-dimensional space, where each feature is drawn from a Gaussian distribution.
Python hint: You can use NumPy's `np.random.randn` and `np.random.choice` for this.
Step 2. Box-Counting Method:

- Implement the Box-Counting method to estimate the fractal dimension of your synthetic dataset.
- Vary the size of the "box" (i.e., the range within which you count points as belonging to a single box) and plot how the number of boxes needed changes as a function of the box size.
- Estimate the fractal dimension using the relation:

$$\mathrm{dim}_B(D) = \lim_{\delta \to 0} \frac{\log N(\delta)}{\log(1/\delta)}$$

Compute the fractal dimension of a data set of your choice.