Simulating Shocks

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Waves

Two of the fundamental processes in nature are diffusion and wave propagation. But... what is a wave? A wave is defined as an identifiable signal or disturbance in a medium that is propagated in time, carrying energy with it. A few familiar examples are electromagnetic waves, waves on the surface of water, sound waves, and stress waves in solids, as occur in earthquakes.

A shock wave is strong pressure wave in any elastic medium such as air, water, or a solid substance, produced by supersonic aircraft, explosions, lightning, or other phenomena that create violent changes in pressure. Shock waves differ from sound waves in that the wave front, in which compression takes place, is a region of sudden and violent change in stress, density and temperature.

Because of this, shock waves propagate in a manner different from that of ordinary acoustic waves. Shock waves travel faster than sound, and their speed increases as the amplitude is raised; but the intensity of a shock wave also decreases faster than does that of a sound wave, because some of the energy of the shock wave is expended to heat the medium in which it travels. The amplitude of a strong shock wave, as created in air by an explosion, decreases almost as the inverse square of the distance until the wave has become so weak that it obeys the laws of acoustic waves. Shock waves alter the mechanical, electrical, and thermal properties of solids and, thus, can be used to study the equation of state (a relation between pressure, temperature, and volume) of any material.

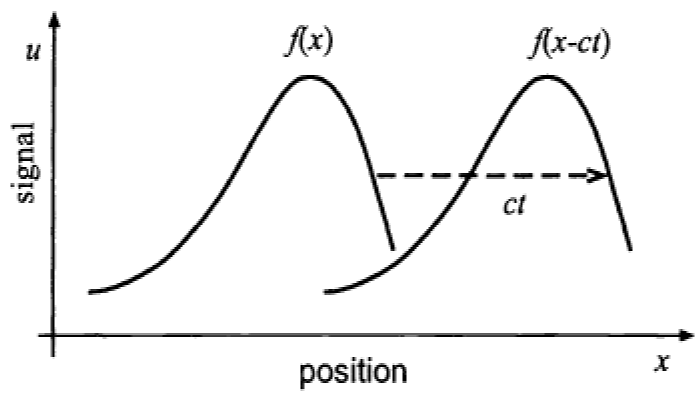


Figure 7.1 Right traveling wave.

For example, one simple mathematical model of a wave is the function:

$$u(x,t) = f(x - ct), \tag{1}$$

which represents an undistorted right-traveling wave moving at constant velocity c.

x - position, t - time, u - strength of the disturbance

it is characteristic of linear waves, or wave profiles that are solutions to linear partial differential equations. On
the other hand, waves that distort, and possibly break, are characteristic of nonlinear processes. To find a model that has (7.1) as the solution we compute ut and ux to get

$$Ut = -cf'(x - ct),\ ux = f'(x - ct),$$

Hence

$$Ut + Cu = 0. \tag{2}$$

Equation (2) is a first-order linear partial differential equation that, in the sense just described, is the simplest wave equation. It is called the **advection equation** and its general solution is (1), where f is an arbitrary differentiable function. the advection equation describes how a quantity is carried along with the bulk motion of a medium. Similarly, a traveling wave of the form u = f (x + ct) is a left moving wave and is a solution of the partial differential equation ut - cu. = 0.

Other waves of interest in many physical problems are periodic, or sinusoidal waves. These traveling waves are represented by expressions of the form

$$u(x, t) = A \cos(kx - wt)$$

The positive number A is the amplitude, k is the wave number (the number of oscillations in 27r units of space, observed at a fixed time), and w the angular frequency (the number of oscillations in 2ir units of time, observed at a fixed location x). The number A = 2-7r/k is the wavelength and P = 2ir/w is the time period. The wavelength measures the distance between successive crests and the time period is the smallest time for an observer located at a fixed position x to see a repeat pattern.

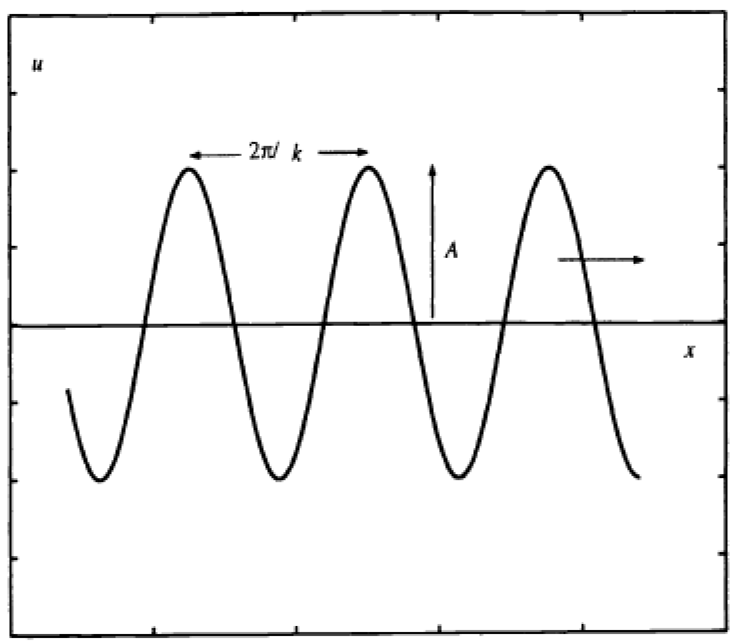


Figure 7.2 Plane wave.

Linear Transport Equations

The transport equation models the concentration of a substance flowing in a fluid at a constant rate.

We consider the one-dimensional version of the linear transport equation,

$$U_t + a(x,t)U_x = 0 \qquad \forall\ (x,t) \in \mathbb{R} \times \mathbb{R}_+.$$

The simplest case of the scalar transport equation arises when the velocity field is constant, that is, $a(x,t) \equiv a$. The resulting transport equation is

$$U_t + aU_x = 0.$$

The rather simple equation (2.2) has served as a crucible for designing highly efficient schemes for much more complicated systems of equations. We concentrate on it for the rest of this chapter.

Method of characteristics

The initial value problem (or Cauchy problem) for (2.1) consists of finding a solution of (2.1) that also satisfies the initial condition

$$U(x,0) = U_0(x) \qquad \forall\ x \in \mathbb{R}. \tag{2.3}$$

It is well known that the solution of the initial value problem can be constructed by using the *method of characteristics*. The idea underlying this method is to reduce a PDE like (2.1) to an ODE by utilizing the structure of the solutions. As an ansatz, assume that we are given some curve $x(t)$, along which the solution U is constant. This means that

$$\begin{aligned} 0 &= \frac{d}{dt}U(x(t),t) && \text{(as U is constant along } x(t)\text{)} \\ &= U_t(x(t),t) + U_x(x(t),t)x'(t) && \text{(chain rule).} \end{aligned}$$

We also know that $U_t(x(t),t) + U_x(x(t),t)a(x(t),t) = 0$, since U is assumed to be a solution of (2.1). Therefore, if $x(t)$ satisfies the ODE

$$\begin{aligned} x'(t) &= a(x(t),t) \\ x(0) &= x_0, \end{aligned} \tag{2.4}$$

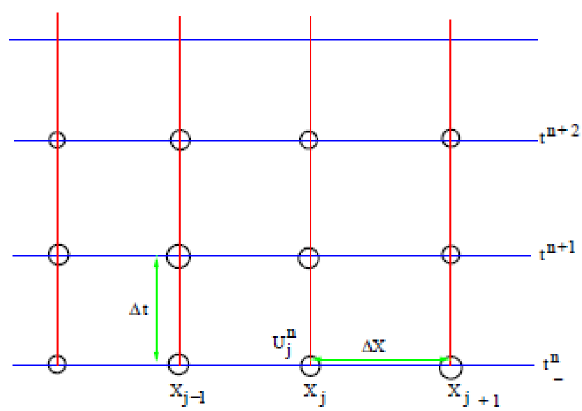
Finite difference schemes for the transport equation

It may not be possible to obtain an explicit formula for the solution of the characteristic equation. For example, the velocity field a(x; t) might have a complicated nonlinear expression. Hence, we have to devise numerical methods for approximating the solutions. For simplicity, we consider a(x; t) =a > 0 and solve. It is rather straightforward to extend the schemes to the case of a more general velocity field.

Discretization of the domain. The first step in any numerical method is to discretize both the spatial and temporal parts of the domain. Since $\mathbb{R}$ is unbounded, we have to truncate the domain to some bounded domain $[x_l, x_r]$. This truncation implies that suitable boundary conditions need to be imposed. We discuss the problem of boundary conditions later on.

For the sake of simplicity, the domain $[x_l, x_r]$ is discretized uniformly with a mesh size Δx into a sequence of $N+1$ points x_j such that $x_0 = x_l$, $x_N = x_r$ and $x_{j+1} - x_j = \Delta x$ for all j. A non-uniform discretization can readily be considered.

For the temporal discretization, we choose some terminal time T and divide $[0, T]$ into M points $t^n = n\Delta t$ $(n = 0, \ldots, M)$. The space-time mesh is shown in Figure 2.2. Our aim is obtain an approximation of the form $U_j^n \approx U(x_j, t^n)$. To get from the initial time step t^0 to the terminal time step t^M, we first set the initial data $U_j^0 = U_0(x_0)$ for all j. Then the solution U_j^1 at the next time step is computed



using some update formula, again for all j. This process is reiterated until we arrive at the final time step $t^M = T$ with our final solution U_j^M.

A simple centered finite difference scheme. On the mesh, we need to approximate the transport equation (2.2). We do so by replacing both the spatial and temporal derivatives by finite differences. The time derivative is replaced with a forward difference and the spatial derivative with a central difference. This combination is standard (see schemes for the heat equation in standard textbooks like [7]). The resulting scheme is

$$\frac{U_j^{n+1} - U_j^n}{\Delta t} + \frac{a(U_{j+1}^n - U_{j-1}^n)}{2\Delta x} = 0 \qquad \text{for } j = 1, \dots, N-1. \tag{2.7}$$

Some special care must be taken when defining the boundary values. We have a consistent discretization of (2.2) that is very simple to implement. We test it on the following numerical example.

A numerical example. Consider the linear transport equation (2.2) in the domain $[0, 1]$ with initial data

$$U_0(x) = \sin(2\pi x). \tag{2.8}$$

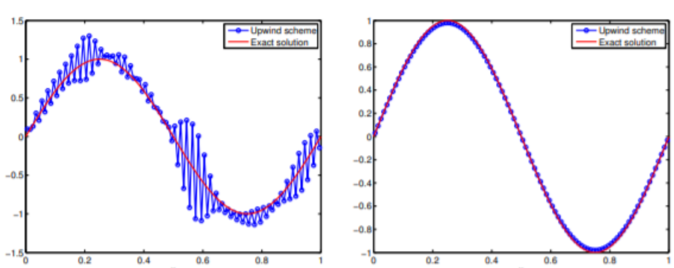
Since the data is periodic, it is natural to assume periodic boundary conditions. We implement this numerically by letting

$$U_0^n = U_{N-1}^n, \quad U_N^n = U_1^n.$$

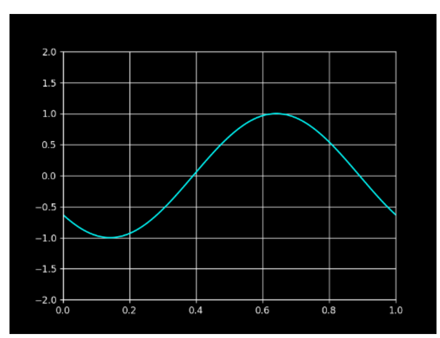
The exact solution is calculated by (2.5) as $U(x, t) = \sin(2\pi(x - at))$. We set $a = 1$ and compute the solutions with the central scheme (2.7) with 500 mesh points, and plot the solution at time $t = 0.3$ in Figure 2.3. The figure clearly shows that, despite being a consistent approximation, the scheme is unstable, with very large oscillations.

Test Case for the numerical example above:

Input data for program : c = 1 # wave speed; xmin = 0; ; xmax = 1; # spatial domain; CFL = 0.9; dt = 0.9*dx; np.sin(2*pi*x) # initial u (frontier condition); Plotted on a x=-2,2 y=0,1 graph



(a) $\Delta t = 1.3\Delta x$ (b) $\Delta t = 0.9\Delta x$



Expected results

Our results (like in test case 1 with rugne-kuta)

Burger Equation

In the previous section, we considered the scalar transport equation

$$U_t + a(x,t)U_x = 0.$$

This equation is linear as the velocity field a is a given function. However, most natural phenomena are nonlinear. In such models, the linear velocity field must be replaced with a field that depends on the solution itself. The simplest example of such a field is

$$a(x,t) = U(x,t).$$

Hence, the transport equation (3.1) becomes

$$U_t + UU_x = 0.$$

The transport equation (3.2) can be written in the *conservative form*

$$U_t + \left(\frac{U^2}{2}\right)_x = 0.$$

This is the inviscid *Burgers* equation. It serves as a prototype for *scalar conservation laws*, which in general take the form

$$U_t + f(U)_x = 0,$$

where U is the unknown and f is the flux function. Apart from Burgers' equation, scalar conservation laws arise in a wide variety of models. We consider a couple of examples below.

We start with Burgers' equation (3.3) and attempt to construct solutions to the initial value problem associated with it. As for the linear transport equation (3.1), we will use the method of characteristics for this purpose. Since (3.2) and (3.3) are equivalent whenever U is smooth, the characteristics $x(t)$ for Burgers' equation are given by

$$\begin{aligned} x'(t) &= U(x(t),t) \\ x(0) &= x_0. \end{aligned} \tag{3.12}$$

Note that these characteristics are different from the linear case (2.4) in that the velocity depends on the solution. We consider initial data

$$U_0(x) = \begin{cases} U_l & \text{if } x < 0 \\ U_r & \text{if } x > 0. \end{cases} \tag{3.13}$$

Data of this form is quite simple and consists of constants separated by a discontinuity at the origin. The initial value problem for a conservation law (3.4) with initial data of the form (3.13) is called a *Riemann problem.*

By definition, the solution U is constant along characteristics, that is, $U(x(t),t) = U_0(x_0)$. Therefore, the solution of (3.12), (3.13) in constant parts of U_0 is

$$x(t) = U_0(x_0)t + x_0.$$

We have gathered some test data from a paper about modeling neurological shock-waves using transport equation (see references)

Test Case 2:

Sinusoidal solution of u(x)=3415 cos(6.412x) with ω≅6.412, of a shock-wave solution curve, spatial domain 0 ≤ x ≤ 0.176 m up to a time period of 1 s.

Input Data for program: c = 1 # wave speed

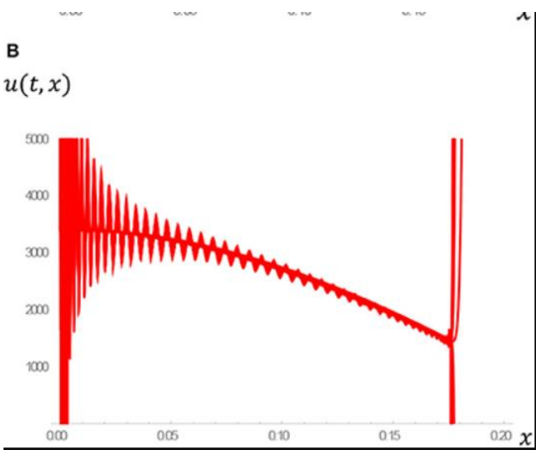
xmin = 0 # spatial domain

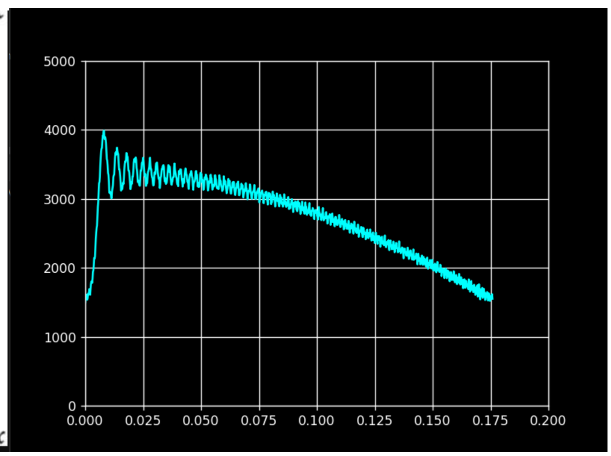
xmax = 0.176

CFL = 0.3

3415*np.cos(6.412*x) # initial u (frontier condition)

Plotted on a x=0,0.20 y=0,5000 graph



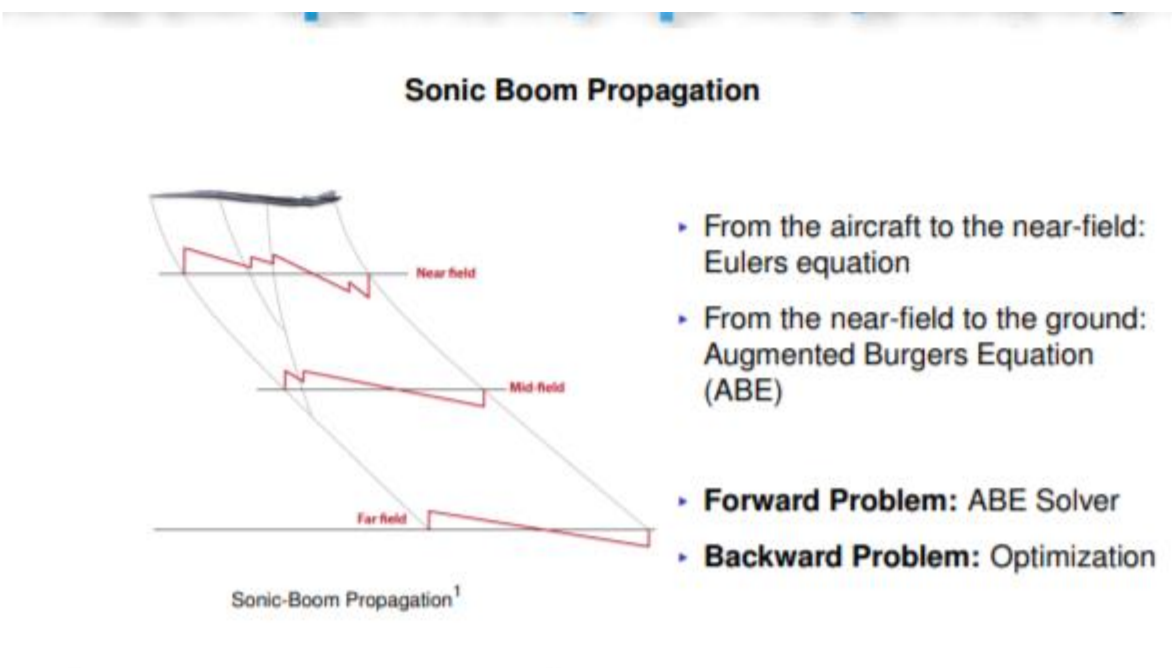


The first figure represents the expected resulted graph for the shock wave (from the paper's website) and the second figure is the shock-wave graph resulted after running our own script using the runge-kuta numeric method for spatial discretization.

Practical Example:

Sonic boom propagation in a quiet) stratified) lossy atmosphere is the subject of this dissertation. Two questions are considered in detail: (1) Does waveform freezing occur? (2) Are sonic booms shocks in steady state? Both assumptions have been invoked in the past to predict sonic boom waveforms at the ground.

To answer the second question we solve the full Burgers equation and for this purpose develop a new computer code, THOR. The code is based on an algorithm by Lee and Hamilton (J. Acoust. Soc. Am. 97, 906-917, 1995) and has the novel feature that all its calculations are done in the time domain, including absorption and dispersion. Results from the code compare very well with analytical solutions. In a NASA exercise to compare sonic boom computer programs, THOR gave results that agree well with those of other participants and ran faster. We show that sonic booms are not steady state waves because they travel through a varying medium, suffer spreading, and fail to approximate step shocks closely enough. Although developed to predict sonic boom propagation, THOR can solve other problems for which the extended Burgers equation is a good propagation model.



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