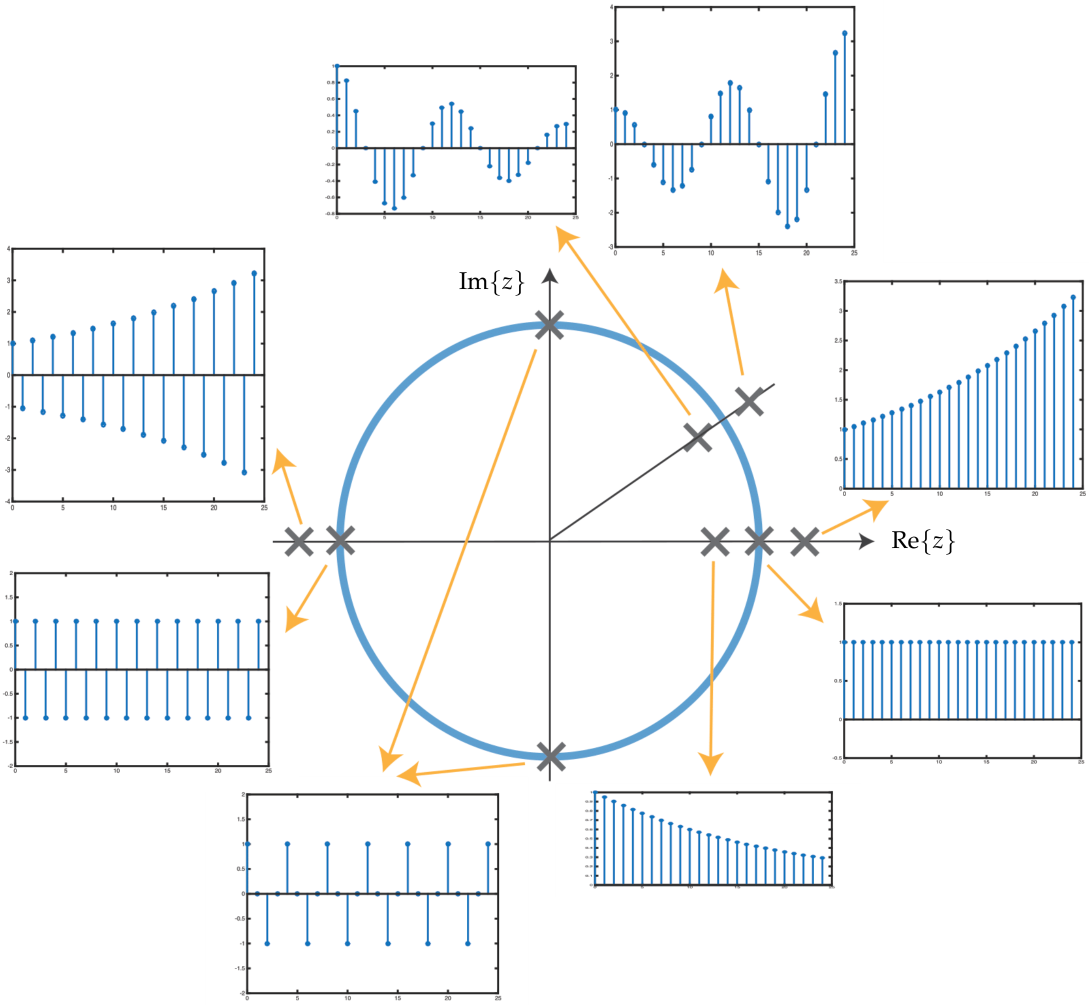


**Figure 3:** The real part of $z^t$ for various values of $z$ in the complex plane. It grows unbounded when $|z| > 1$, decays to zero when $|z| < 1$, and has constant amplitude when $z$ is on the unit circle ($|z| = 1$).

So when $|z| > 1$ then the first term "blows up", and we can see that this is an indicator of instability.

We can also consider the continuous-time and do a similar analysis. In the scalar case, the Continuous-Time LTI Differential Equation Model no Input has state trajectory

$$x(t) = \mathrm{e}^{at} x_0 + \int_0^t \mathrm{e}^{a(t-\tau)} b u(\tau) \, \mathrm{d}\tau \,. \tag{7}$$

Looking again at only the first term, we can see whether it "blows up". If the first term "blows up", we know that the system is unstable, because we can feed in $u(\tau) = 0$ for all $\tau$, and let the state "blow up". If the first term does not "blow up", then we would need to show that the second term does not "blow up" either. For now, it is important to get an idea of what is happening with the first term, and in particular the behavior of $\mathrm{e}^{st}$ for a complex number $s$. More formally, suppose $s := \alpha + \mathrm{j}\omega \in \mathbb{C}$ is a complex number. Then

$$\mathrm{e}^{st} = \mathrm{e}^{(\alpha + \mathrm{j}\omega)t} = \mathrm{e}^{\alpha t} \mathrm{e}^{\mathrm{j}\omega t} = \mathrm{e}^{\alpha t} \cos(\omega t) + \mathrm{j}\mathrm{e}^{\alpha t} \sin(\omega t). \tag{8}$$

 

The idea is that $\alpha$ controls the rate of growth of $|e^{st}|$, and $\omega$ controls any oscillatory behavior.

- When $\text{Re}\{s\} < 0$, the envelope $e^{\alpha t} \to 0$, so $e^{st}$ decays to 0, although if $\omega \neq 0$ then it also has oscillatory behavior due to the sine/cosine.
- When $\text{Re}\{s\} = 0$, the envelope $e^{\alpha t} = 1$, so if $\omega \neq 0$ then $e^{st} = \cos(\omega t) + j\sin(\omega t)$ oscillates around the complex unit circle $\{z \in \mathbb{C} : |z| = 1\}$.
- When $\text{Re}\{s\} > 0$, the envelope $e^{\alpha t} \to \infty$, so $e^{st}$ blows up to $\infty$, although if $\omega \neq 0$ it also has oscillatory behavior due to the sine/cosine.

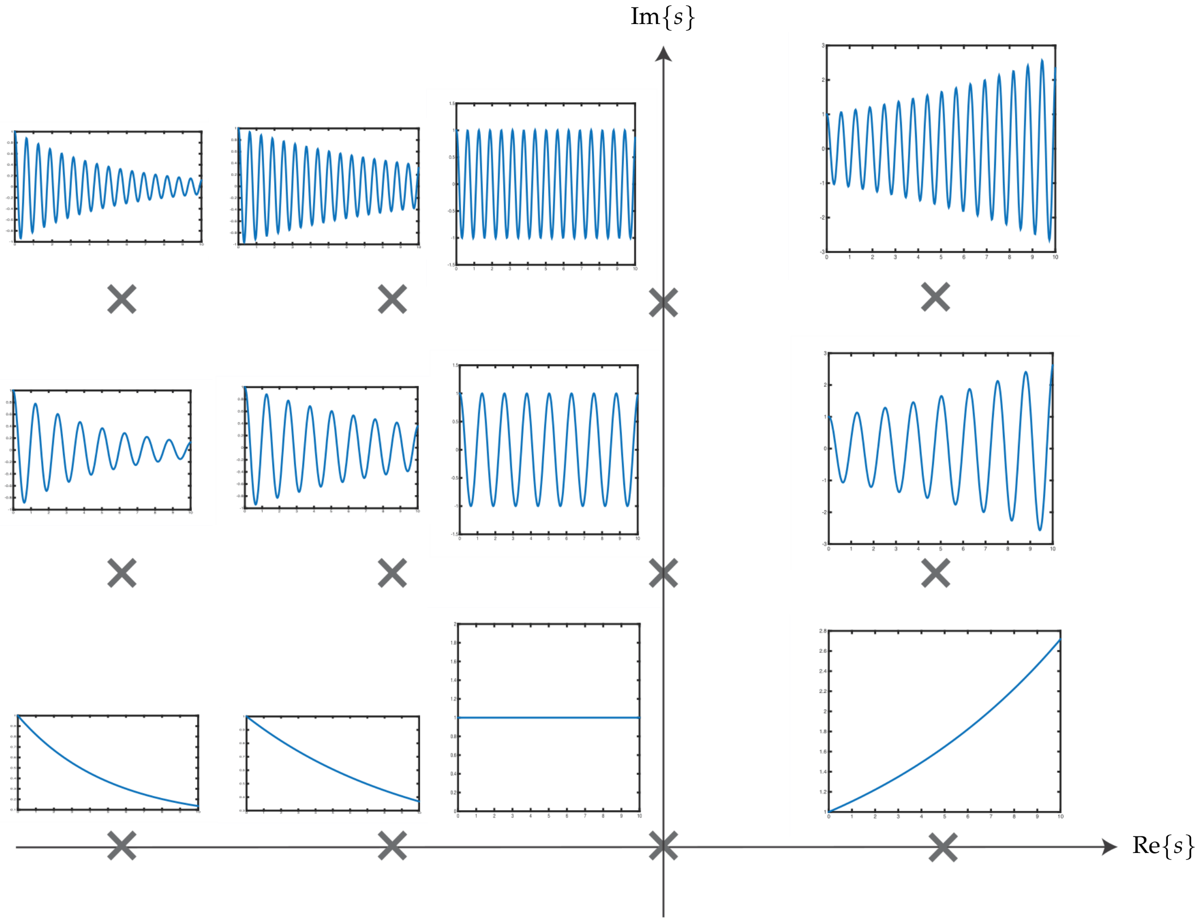


**Figure 4:** The real part of $e^{st}$ for various values of $s$ in the complex plane. Note that $e^{st}$ is oscillatory when $s$ has an imaginary component. It grows unboundedly when $\text{Re}\{s\} > 0$, decays to 0 when $\text{Re}\{s\} < 0$, and has constant amplitude when $\text{Re}\{s\} = 0$.

When $\text{Re}\{s\} > 0$ then the first term "blows up", and this is an indicator of instability.

 